

Spectral analysis

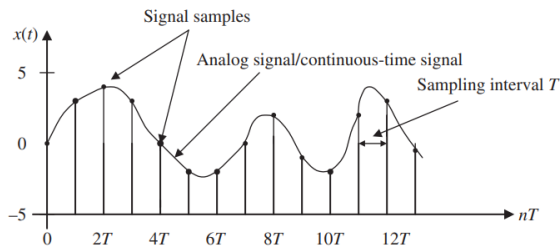
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January 21, 2024

Motivation

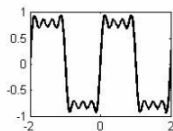
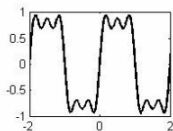
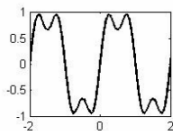
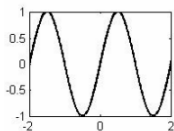
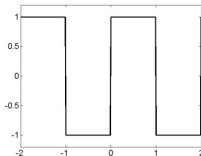
- We are monitoring **continuous** signal.
- We are saving **discrete** samples of this signal in our computer.
- Under what conditions, and how, can we recover the continuous signal $x(t)$ from the saved samples?



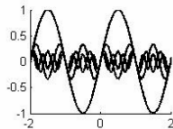
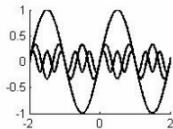
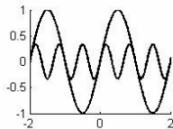
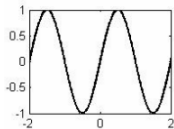
The Fourier Transform

The Fourier transform allows to represent a continuous signal as a linear combination of sinusoids.

SQUARE WAVE



summed
waveform



component
sine waves

The Fourier Transform

$$\begin{aligned}x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) (\cos(\Omega t) + j \sin(\Omega t)) d\Omega \\&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) \cos(\Omega t) d\Omega + j \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) \sin(\Omega t) d\Omega\end{aligned}$$

where

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Sampled signal

We sample values of a continuous function $x(t)$ at regular times nT , where T is called the **sampling interval** and its inverse $1/T$ is called the **sampling frequency**.

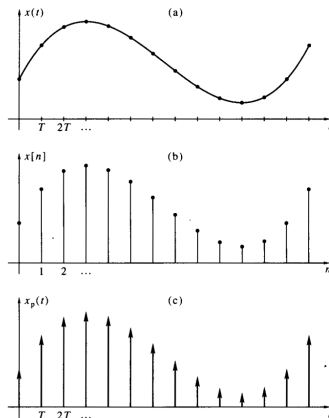


Figure 3.1 The sampling operation: (a) the continuous-time signal $x(t)$; (b) the point-sampled sequence $x[n]$; (c) the impulse-sampled signal $x_p(t)$.

Continuous representation of a sampled signal

- Dirac delta function ($\delta(t)$):

$$\delta(t - t_0) = 0 \quad t \neq t_0$$

$$\int_{t_1}^{t_2} f(t) \delta(t - t_0) dt = f(t_0) \quad t_1 < t_0 < t_2$$

- Thus, a sampled signal (with sample interval T) can be represented with the function:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)$$

- $x_c(t) = 0 \quad t \neq nT$
- $\int_{nT-\Delta}^{nT+\Delta} x_c(t) dt = x(nT)$

Sampling theorem

Theorem (Sampling theorem (Shannon, 1948))

Let $x_s(t)$ be a sampled signal, with sampling period T , of a continuous signal $x(t)$.

Let $X(j\Omega)$ be the Fourier transform of $x(t)$.

Then the Fourier transform of $x_s(t)$ is $X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\Omega - \frac{2\pi k}{T}))$.

Sampling theorem

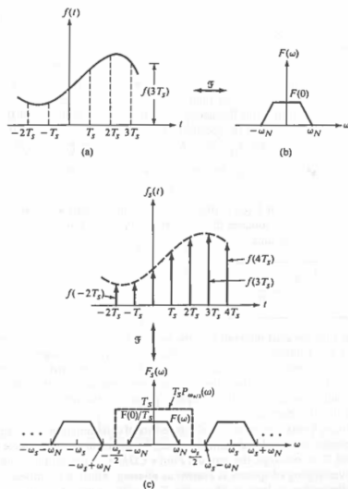
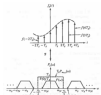


Figure 6.4 Illustrations of the Sampling Theorem

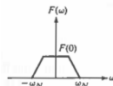
$$\omega_s = \frac{2\pi}{T_s}$$

Reconstruction procedure

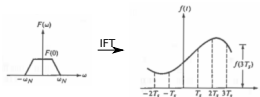
- 1 Fourier transform the sampled signal $x_s(t)$, yielding $X_s(j\Omega)$.



- 2 low-pass filter $X_s(j\Omega)$ between frequencies $\Omega = -\frac{\pi}{T}$ and $\Omega = \frac{\pi}{T}$, yielding $X_s^{LP}(j\Omega)$.

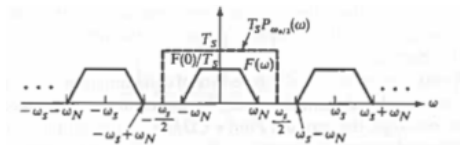


- 3 the desired continuous signal is the inverse Fourier transform of $X_s^{LP}(j\Omega)$ (i.e., $x(t) = \mathcal{IFT}\{X_s^{LP}(j\Omega)\}$).

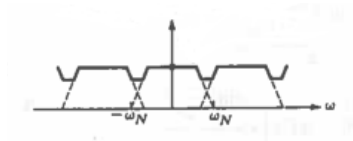


Reconstruction requirements

- we want



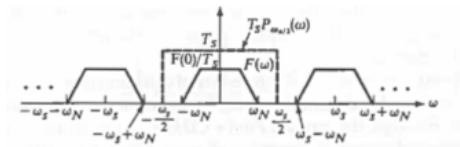
- we do NOT want



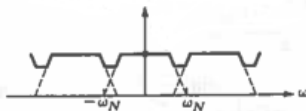
- we want $\frac{\omega_s}{2} > \omega_N$, or $\frac{2\pi}{T} \frac{1}{2} = \frac{\omega_s}{2} > \omega_N = 2\pi f_N$, or $\frac{1}{T} \frac{1}{2} > f_N$, or $f_s = \frac{1}{T} > 2f_N$, or $f_s > 2f_N$.

Reconstruction requirements

- we want



- we do NOT want



- we want $\frac{\omega_s}{2} > \omega_N$, or $\frac{2\pi}{T} \frac{1}{2} = \frac{\omega_s}{2} > \omega_N = 2\pi f_N$, or $\frac{1}{T} \frac{1}{2} > f_N$, or $f_s = \frac{1}{T} > 2f_N$, or $f_s > 2f_N$.

To reconstruct without error a continuous signal from its samples we need to sample it at a frequency larger than twice its maximal frequency.

Reconstruction example

- continuous signal in time domain

$$x(t) = (\text{sinc}(f_0 t))^2$$
$$\text{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

- continuous signal in frequency domain

$$x(j\Omega) = \begin{cases} \frac{1}{f_0} \left(1 - \frac{|\Omega|}{2\pi f_0}\right) & \Omega \leq 2\pi f_0 \\ 0 & \Omega > \pi f_0 \end{cases}$$

$\omega_N = 2\pi f_0$, $f_N = f_0$ and by the sampling theorem $f_s > 2f_0$.

Reconstruction example: sampling below the Nyquist rate

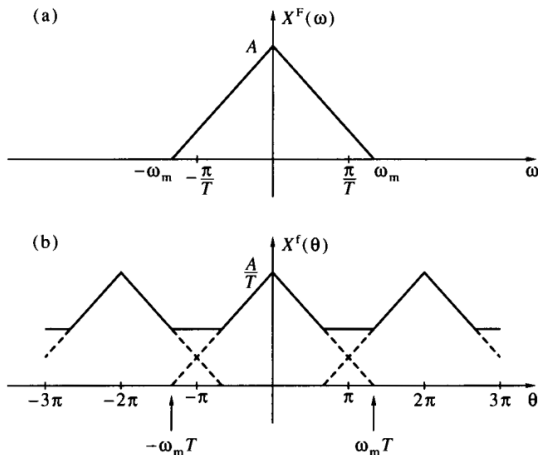
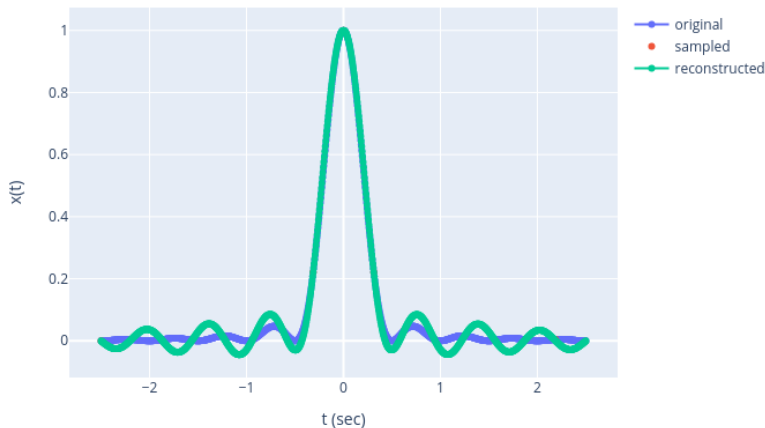


Figure 3.4 Sampling of a band-limited signal below the Nyquist rate: (a) Fourier transform of the continuous-time signal; (b) Fourier transform of the sampled signal.

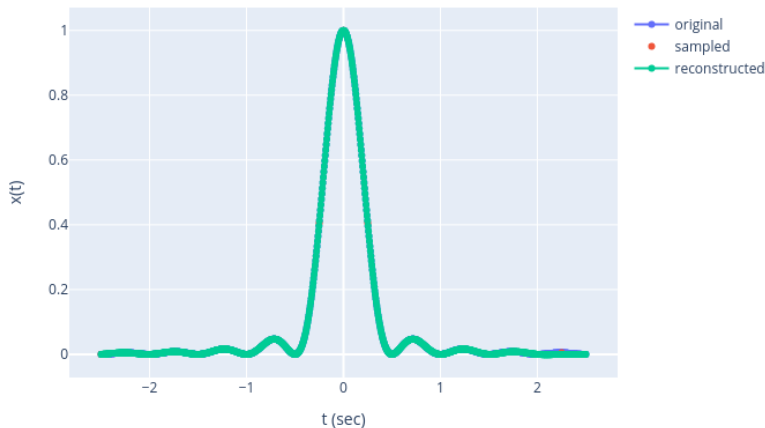
Reconstruction example: sampling below the Nyquist rate

$f_s = 0.8 \times \text{Nyquist rate}$



Reconstruction example: sampling at the Nyquist rate

$f_s = 1.0 \times \text{Nyquist rate}$



Summary

Shannon, C. E. (1948). A mathematical theory of communication. *The Bell system technical journal*, 27(3):379–423.