Worksheet: hypothesis tests

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1. **Detailed hypothesis test for example 2 in discussion notes**

   Identify the null hypothesis $\mathcal{H}_0$: the mean peak visual ERP in medicated subjects is 2 mV

   Identify the alternative hypothesis $\mathcal{H}_a$: the mean peak visual ERP in medicated subjects is different from 2 mV.

   Select a test statistic: standarized sample mean Z.

   Calculate the observed value of the test statistic: $z = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{1.3-2}{2.6/\sqrt{50}} = -1.9$

   Calculate the p-value: p_value $= P(-|z| > Z) + P(Z > |z|) = 2P(Z > |z|) = 0.057$.

   Draw our conclusion: p_value $= 0.057 > 0.05$ then **do not reject** $\mathcal{H}_0$.

   A Python script to solve this exercise can be found here and a shell script with the corresponding parameters can be found here.

2. (a) $\mathcal{H}_0$**:** the population mean is $\mu_0 = 2.3$
   $\mathcal{H}_a$**:** the population mean is $\mu_0 > 2.3$

   (b) Because $n > 30$ it is reasonable to assume that $Z \sim \mathcal{N}(0,1)$. Then the rejection region is $z > z_\alpha$, with $\alpha = 0.05$.

   (c) Roughly, for $Y \sim \mathcal{N}(\mu, \sigma^2)$ there is a considerable probability of obtaining a sample in the range $[\mu, \mu + 2\sigma]$. Because under the null hypothesis $\bar{X} \sim \mathcal{N}(\mu_0, s/\sqrt{n})$, there is a considerable probabability of obtaining a sample of $\bar{X}$ in the range $[\mu_0, \mu_0 + 2s/\sqrt{n}]$. $s/\sqrt{n} \sim 0.3/6 \sim 0.05$. Thus, there is a considerable probability of obtaining by chance a sample of $\bar{X}$ in the range $[2.3, 2.3{+}2\ 0.05] = [2.3, 2.4]$. Because 2.4 is in the boundary of this interval, it is not obvious if a hypothesis test will reject or not the null hypothesis. Lets do the test. We first compute the observed test statistic:

   $$z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{2.4 - 2.3}{0.29/\sqrt{35}} = 2.04$$

   The p-value corresponding to this observed statistic is $p = 0.02$, so we reject the null hypothesis with a confidence level $\alpha = 0.05$.

   A Python script to solve this exercise can be found here and a shell script with the corresponding parameters can be found here.

3. **Potency of an antibiotic**

   (a) $\mathcal{H}_0$: the mean potency of the antibiotic is $\mu_0 = 80\%$.

   (b) $\mathcal{H}_a$: the mean potency of the antibiotic is $\mu_0 \neq 80\%$.

   (c) because $n = 100$ it is sensible to assume $Z \sim \mathcal{N}(0,1)$. I will perform a two-tailed z-test with $\bar{x} = 79.7\%, \mu_0 = 80.0\%, n = 100, s = 0.8$ and $\alpha = .05$. Lets compute the observed statistic.

   $$z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{29.7 - 80}{0.8/\sqrt{100}} = -3.75$$

   The p-value corresponding to this observed statistic is $p < 0.0002$, so we reject the null hypothesis with a confidence level $\alpha = 0.05$.

   A Python script to solve this exercise can be found here and a shell script with the corresponding parameters can be found here.

4. **Smoking and lung capacity** Because $n = 20$ it is not safe to assume $\bar{Z}\ \mathcal{N}(0,1)$. We will perform a right-tailed t-test instead. Lets compute the observed statistic.

   $$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{89.85 - 100}{14.53/\sqrt{20}} = -3.12$$

   The p-value corresponding to this observed statistic is $p < 0.003$, so we reject the null hypothesis with a confidence level $\alpha = 0.01$.

   A Python script to solve this exercise can be found here and a shell script with the corresponding parameters can be found here.

5. **Power of a test**

   The power of a statistical test is $1 - \beta$, where β is the probability of type II error (i.e., the probability that the null hypothesis is not rejected given that an alternative hypothesis is true).

   To calculate β for a right-tailed test, I first find the critical value of the sample mean, $\bar{x}_c$, such that $P(\bar{X} > \bar{x}_c | \mathcal{H}_0) = \alpha$ (note that for a right-tailed test $\alpha = P(Z > z_\alpha | \mathcal{H}_0) = P(\frac{\bar{X} - \mu_0}{s/\sqrt{n}} > z_\alpha | \mathcal{H}_0) = P(\bar{X} > \mu_0 + z_\alpha\ s/\sqrt{n} | \mathcal{H}_0)$, so that $\bar{x}_c = \mu_0 + z_\alpha\ s/\sqrt{n}$). Then $\beta = P(\bar{X} < \bar{x}_c | \mathcal{H}_a) = P(\frac{\bar{X} - \mu_a}{s/\sqrt{n}} < \frac{\bar{x}_c - \mu_a}{s/\sqrt{n}} | \mathcal{H}_a) = P(Z < \frac{\bar{x}_c - \mu_a}{s/\sqrt{n}}) = \Phi(\frac{\bar{x}_c - \mu_a}{s/\sqrt{n}})$, where $\Phi(x) = P(Z < x)$ is the standard normal cummulative distribution function. The power of the test is $1 - \beta$. β is the the red area under in Figure 2 of the discusion slides. Code to compute β is given in the listing below.

   ```
   def compute_beta(mu_Ha, critical_value, ste_mean):
       beta = scipy.stats.norm.cdf((critical_value - mu_Ha) / ste_mean)
       return beta
   ```

   (a) the calculated powers are:

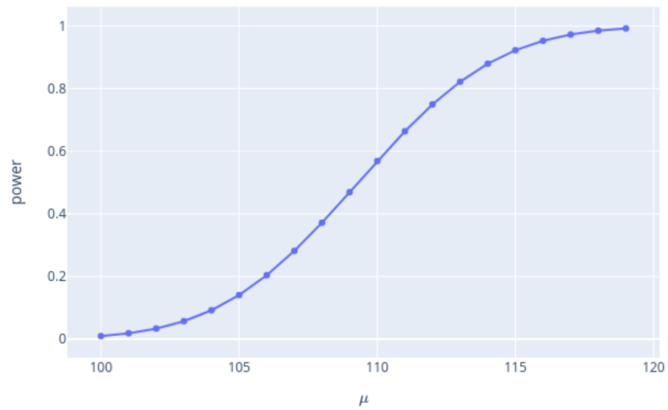

Figure 1: Power versus effect size.

$\mu_{\mathbf{Ha}} = 108$**:** power=0.37

$\mu_{\mathbf{Ha}} = 112$**:** power=0.75

$\mu_{\mathbf{Ha}} = 116$**:** power=0.95

Power versus effect size is shown in Figure 1.

A Python script solving this item appears here.

(b) Figure 2 shows power plots for $\alpha = 0.01$ and $\alpha = 0.05$.

A Python script solving this item appears here.

(c) Figure 3 shows power plots for $n = 16$ and $n = 65$.

A Python script solving this item appears here.

(d) following the exercise hints I first derived an expression for the critical value of the sample mean only considering the type I error, as we did above.

$$\alpha = P(Z > z_\alpha|\mathcal{H}_0) = P(\frac{\bar{X} - \mu_0}{s/\sqrt{n}} > z_\alpha|\mathcal{H}_0) = P(\bar{X} > \mu_0 + z_\alpha\ s/\sqrt{n}|\mathcal{H}_0)$$
$$\text{then} \quad \bar{x}_{c0} = \mu_0 + z_\alpha\ s/\sqrt{n} \tag{1}$$

then I derived another expression for the critical value of the sample mean only considering the type II error

$$\beta = P(Z < -z_\beta|\mathcal{H}_a) = P(\frac{\bar{X} - \mu_a}{s/\sqrt{n}} < -z_\beta|\mathcal{H}_a) = P(\bar{X} < \mu_a - z_\beta\ s/\sqrt{n}|\mathcal{H}_a)$$
$$\text{then} \quad \bar{x}_{ca} = \mu_a - z_\beta\ s/\sqrt{n} \tag{2}$$

Finally I equate $\bar{x}_{c0}$ in Eq. 1 with $\bar{x}_{ca}$ in Eq. 2 and solve for n

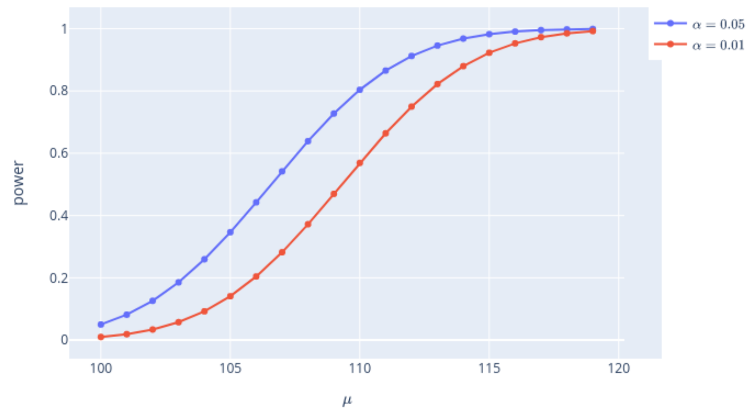

Figure 2: Power plots for significance levels $\alpha = 0.01$ and $\alpha = 0.05$.

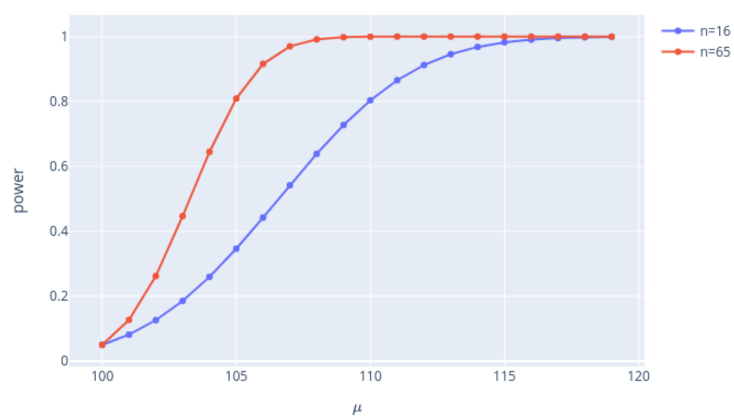

Figure 3: Power plots for $n = 16$ and $n = 65$.

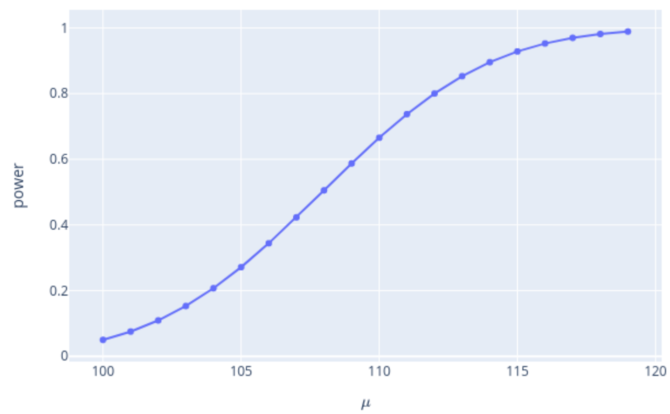

Figure 4: Power plots for $n = 11$, $\alpha = 0.05$, and $\beta = 0.2$. As required for $\mu_a = 112$ the power of the test is $1 - \beta = 0.8$.

$$
\begin{aligned}
\bar{x}_{c0} = \bar{x}_{ca} \quad &\text{iff} \\
\mu_0 + z_\alpha \; s/\sqrt{n} = \mu_a - z_\beta \; s/\sqrt{n} \quad &\text{iff} \\
n = \left(s \frac{z_\beta + z_\alpha}{\mu_a - \mu_0} \right)^2
\end{aligned}
$$

the minimum sample size to achieve significance level of $\alpha = 0.05$ and power of $1 - \beta = 0.8$ for $\mathcal{H}_a : \mu_a = 112$ is $n = 11$. Figure 4 shows the power plot for this sample size. As required for $\mu_a = 112$ the power of the test is $1 - \beta = 0.8$.

A Python script solving this item appears here.