Introduction and hypothesis testing

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Contents

Contents

Course logistics

Contents

Working examples

Example 1

We know that the average running speed of control mice is 1 cm/sec. The sample average running speed of a cohort (n=100) of transgenic mice is $\bar{x} = 2.7\,\text{cm/sec}$ and the sample standard deviation is $s = 10$ cm/sec. Is the average running speed of mice in the transgenic cohort larger than that of control mice? Test with a confidence level $\alpha = 0.01$.

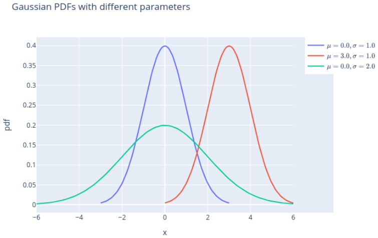
Working examples

Example 2

We want to study the effect of a new drug on visual electrophysiology in humans. We know that the mean peak evoked response potential (ERP) over V1 during the first 200 ms after stimuli presentation is 2 mV. The sample mean peak ERP for a group of 50 medicated subjects is $\bar{x} = 1.3\,\text{mV}$ and the sample standard deviation is $s = 2.6$ mV. Does taking the new drug change the mean evoked ERP over V1? Provide your test p-value.

Statistical remarks

1. Random data (e.g., observed data) is characterized using probability distributions. For example a:
   - Normal distribution with parameters mean μ and variance σ^2, $\mathcal{N}(\mu, \sigma^2)$,



   - Exponential distribution with rate parameter λ, $\mathcal{E}(\lambda)$,
   - Poisson distribution with expected rate parameter λ, $\mathcal{P}(\lambda)$,
   - Binomial distribution with number of observation parameter n and with a success probability parameter p, $\mathcal{B}(n, p)$.

Statistical remarks

2. One branch of statistics, **estimation theory**, provides tools to estimate parameters of distributions from observations.
3. Another branch of statistics, **hypothesis testing**, provides tools to make statistically-informed decisions about values of parameters of distributions.

Statistical remarks

4. To estimate parameters, or to make decisions about them, we use observations, $x_1, \ldots, x_N$, that are **independent and identically distributed**.

Statistical remarks

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Example 1

An observation is the average speed of a transgentic mouse during an experimetal session. We assume that the average speeds of all mice are samples from a common probability density function (identically distributed) and that average speeds are independent across mice (independent).

Statistical remarks

6. To estimate parameters, or to make decisions about them, we use observations, $x_1, \ldots, x_N$, that are **independent and identically distributed**.

Example 1

An observation is the average speed of a transgentic mouse during an experimetal session. We assume that the average speeds of all mice are samples from a common probability density function (identically distributed) and that average speeds are independent across mice (independent).

Example 2

An observation is the peak ERP of a medicated cohort subject. We assume tha these ERPs are samples from the same probability density function (identically distributed) and that these ERPs are independent across subjects (independent).

Statistical remarks

7. A goal of statistics is to **infer properties of the population** (e.g., the effect of the genetic manipulation on the running speed of mice) from **properties of the sample** (e.g., the effect of the manipulation on the running speed of the 100 sampled mice).

Statistical remarks

Theorem (Central Limit Theorem)

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Let n be large. Then the sample mean

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{N} X_i \tag{1}$$

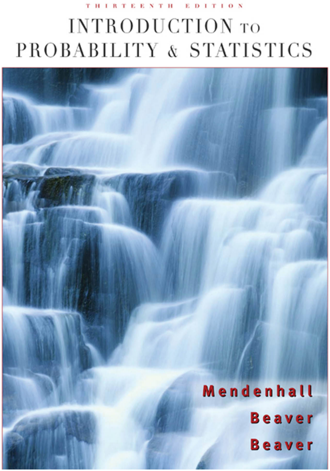
is distributed as $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$.

Note: if σ^2 is unknown, we can estimate σ^2 with the sample variance $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$. Then, for large n, $\bar{X}$ is approximately distributed as $\mathcal{N}(\mu, \frac{s^2}{n})$.

Contents

Main source

Chapter 9 "Large-sample test of hypothesis" and chapter 10 "Inference from small samples" from



Null and alternative hypothesis

- In hypothesis testing we work with a **null hypothesis**, $\mathcal{H}_0$, and an **alternative hypothesis**, $\mathcal{H}_a$, collect a sample of data $x_1, \ldots, x_N$, and test if this data provides sufficient statistical evidence in favor of the alternative hypothesis. If this happens we reject the null hypothesis.
- However, if the collected data does not provide sufficient statistical evicence in favor of the aternative hypothesis, we do not accept the null hypothesis, but we say that we failed to reject it. **Hypothesis tests do not prove null hypothesis, they only provide statistical evidence to reject it, or fail to reject it.**

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The hypothesis that we aim to prove should be the alternative one.

Null and alternative hypothesis

Example 1

Is the average running speed of the transgenic cohort larger than that of the control cohort (i.e., 2 cm/sec)?

$\mathcal{H}_0$: the average running speed of the trangenic cohort is 2 cm/sec.

$\mathcal{H}_a$: the average running speed of the transgenic cohort is lager than 2 cm/sec.

Null and alternative hypothesis

Example 2

Is the mean peak visual ERP in the first 200 ms post stimuli different in medicated than in control subjects (i.e., 2 mV)?

$\mathcal{H}_0$: the mean peak visual ERP in medicated subjects is 2 mV.

$\mathcal{H}_a$: the mean peak visual ERP in medicated subjects is different from 2 mV.

One- and two-tailed tests of hypothesis

One-tailed test of hypothesis: directionality is suggested by the alternative hypothesis.

Example 1

It is a one-tailed hypothesis test because the alternative hypothesis requires that the mean speed of the transgenic mice be larger (directionality) than that of the control mice.

One- and two-tailed tests of hypothesis

One-tailed test of hypothesis directionality is suggested by the alternative hypothesis.

Example 1

It is a one-tailed hypothesis test because the alternative hypothesis requires that the mean speed of the transgenic mice be larger (directionality) than that of the control mice.

Two-tailed test of hypothesis directionality is not suggested by the alterntive hypothesis.

Example 2

It is a two-tailed hypothesis test because the alternative hypothesis requires that the visual ERP of the medicated subjects be different (no directionality) than that of the control subjects.

Test statistic and its sampling distribution

- To perform a hypothesis test we propose a **test statistic**, a function of the sample data, like the sample mean in Eq. 1.
- Because the sample data is random, the test statistic is also random. To perform hypothesis tests we need to know the distribution of the test statistic, which is called the **sampling distribution**.

Test statistic and its sampling distribution for the working examples

Because both examples are tests for the population mean, μ, because the sample mean, $\bar{x}$, is a good estimator of the population mean, and because both examples use a large number of samples, we will use the standardized sample mean as our test statistic:

$$Z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

From the central limit theorem, we know the sampling distribution of this test statistic under the null hypothesis:

$$Z \sim \mathcal{N}(0, 1)$$

Rejection region

The **reject region** is a region of low probability under the null hypohesis, which is consitent with alternative hypothesis. The **non-reject region**, is a region of large probability under the null hypothesis, that is inconsistent with the alternative hypothesis.

Rejection region

Example 1

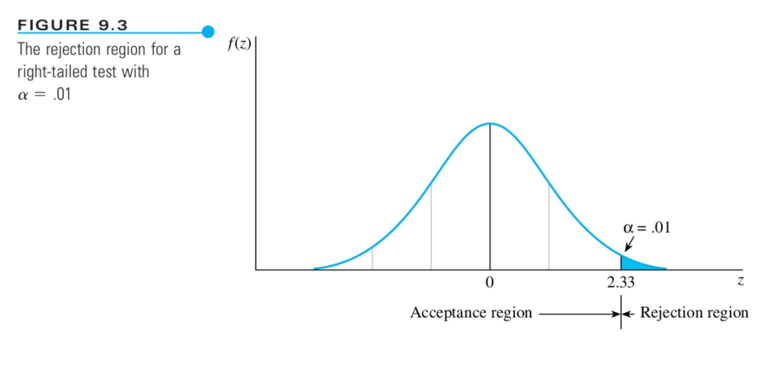
We want to reject the null hypothesis that the average running speed of the trasgenic mice equals 2 cm/sec if their sample mean speed is much larger than 2 cm/sec. That is, we want to reject the null hypothesis if z is large and positive. How large is large?

To answer this question we define the **Type I error** of a test, as the error of rejecting the null hypothesis when it is valid. We also define the **significance level of the hypothesis test**, α, as the probability of Type I error admitted by the test.

When designing a test we first decide on its significane level α. We then reject the null hypothesis if z is larger than the value z_α that leaves α probability to its right, We call this value the **critical value** of the test (see figure on next slide).

Rejection region

Example 1



Mendenhall et al., 2009

Rejection region

Example 2

We want to reject the null hypothesis that the drug has not effect on the average visually evoked ERP if the sample mean visually evoked ERP of medicated subjects is much smaller or much larger than the mean visually evoked ERP of control subjects (2 mV). How small is small and how large is large?

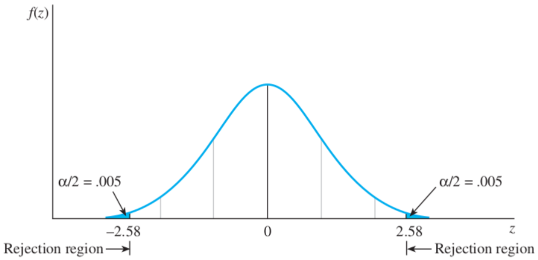
We will reject the null hypothesis if the standarized mean, z, is larger than the value $z_{\alpha/2}$ that leaves $\alpha/2$ probability to its right or smaller than the value $-z_{\alpha/2}$ that leaves $\alpha/2$ probability to its left (see figure on next slide).

Rejection region

Example 1

FIGURE 9.4

The rejection region for a two-tailed test with $\alpha = .01$



Mendenhall et al., 2009

Large-sample hypothesis test for the mean

LARGE-SAMPLE STATISTICAL TEST FOR μ

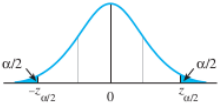
1. Null hypothesis: $H_0 : \mu = \mu_0$
2. Alternative hypothesis:

One-Tailed Test	Two-Tailed Test
$H_a : \mu > \mu_0$ (or, $H_a : \mu < \mu_0$)	$H_a : \mu \neq \mu_0$

3. Test statistic: $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ estimated as $z = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$

4. Rejection region: Reject H_0 when

One-Tailed Test	Two-Tailed Test
$z > z_\alpha$ (or $z < -z_\alpha$ when the alternative hypothesis is $H_a : \mu < \mu_0$)	$z > z_{\alpha/2}$ or $z < -z_{\alpha/2}$



Mendenhall et al., 2009

Complete hypothesis test for example 1

Example 1

We know that the average running speed of control mice is 1 cm/sec. The sample average running speed of a cohort (n=100) of transgenic mice is $\bar{x} = 2.7\,\text{cm/sec}$ and the sample standard deviation is $s = 10$ cm/sec. Is the average running speed of mice in the transgenic cohort larger than that of control mice? Test with a confidence level $\alpha = 0.01$.

Relevant quantities: $\mu_0 = 1$, $n = 100$, $\bar{x} = 2.7$, $s = 10$, $\alpha = 0.01$.

1. identify the null hypothesis H_0: the average running speed of the trangenic cohort is 2 cm/sec.
2. identify the alternative hypothesis H_a: the average running speed of the trangenic cohort is larger than 2 cm/sec.
3. select a test statistic: standarized sample mean Z.

Complete hypothesis test for example 1

Example 1

4. set the rejection region: right-tailed hypothesis test, with critical value $z_{0.01} = 2.3263$.
5. calculate the observed value of the test statistic.

$$z_{\text{obs}} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{2.7 - 1}{10/\sqrt{100}} = 1.7$$

6. draw your conclusion: $z_{\text{obs}} = 1.7 < 2.3263 = z_{0.01}$. Thus, there is not enough statistical evidence to reject the null hypothesis with a confidence level $\alpha = 0.01$.

Would you reject the null hypothesis with confidence level $\alpha = 0.05$?

p-value

We should not only report the conclusion "reject/not reject" of our hypothesis test. We should provide a measure of how much our data disagrees with the null hypothesis. One such measure is the p-value:

Definition (p-value)

The p-value (or observed significance level of a statistical test) is the probability that the test statistic is larger than the observed test statistic value under the null hypothesis.

Notes:

- small p-values indicate that the probability of obtaining a test statistic equal or larger than the observed test statistic value is small, showing that our data support the alternative hypothesis.
- large p-values show that our data does not support the alternative hypothesis.

Taxonomy of hypothesis tests for the the mean

- The previous tests worked well because the sample statistic for the sample mean was approximately normal since
    1. samples were Gaussian, with known variance, with any sample size.
    2. the number of samples was sufficiently large (e.g., $n > 30$) and samples were iid from any distribution with mean μ and finite variance σ^2, known or estimated by the sample variance s^2 (central limit theorem).
- but what if samples are Gaussian, with unknown variance, and a small number n of samples?
  Then the sample distribution of the mean follows a student-t distribution with n-1 degrees of freedom.

Small-sample hypothesis test for the mean (t-test)

SMALL-SAMPLE HYPOTHESIS TEST FOR μ

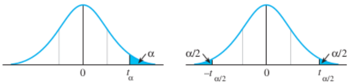
1. Null hypothesis: $H_0 : \mu = \mu_0$
2. Alternative hypothesis:

One-Tailed Test	**Two-Tailed Test**
$H_a : \mu > \mu_0$ (or, $H_a : \mu < \mu_0$)	$H_a : \mu \neq \mu_0$

3. Test statistic: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$
4. Rejection region: Reject H_0 when

One-Tailed Test	**Two-Tailed Test**
$t > t_\alpha$ (or $t < -t_\alpha$ when the alternative hypothesis is $H_a : \mu < \mu_0$)	$t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$

or when p-value $< \alpha$



The critical values of t, t_α, and $t_{\alpha/2}$ are based on $(n - 1)$ degrees of freedom. These tabulated values can be found using Table 4 of Appendix I or the **Student's *t* Probabilities** applet.

Assumption: The sample is randomly selected from a normally distributed population.

Mendenhall et al., 2009

Rejection region

Type I error : reject the null hypothesis when it is true.

Type II error : not reject the null hypothesis when the alternative one is true.

Some statistical tests are designed to constrain the probability of type I error, called the *confidence level,* α of the test. Most commonly $\alpha = 0.05$. Other tests compute the observed value of the test statistic, t_{obs} and calculate the probability that the test statistic is larger or equal than its observed value. This probability is called the **p-value** of the test.

Steps to perform a hypothesis test

We start assuming that we know the distribution of the experimental samples and we agree on a sample size n (in Section **??** we discuss how to select and optimal sample size). We set the confidence level α of the test (i.e., the probability that the test statistic falls in the reject region given that the null hypothesis is valid).

1. collect an experimental sample $x_1, \ldots, x_N$.
2. compute the value of the test statistic corresponding to the collected experimental sample (e.g., sample mean, Eq. 1)
3. calculate the sample statistic, t_{obs}.
4. for testing based on confidence level:
   a. divide the space of all possible values of the test statistic on a reject and a non-reject regons at a confidence level α.
   b. if the value of the test statistic falls in the reject region, reject the null hypothesis at a confidence α. If it does not, do not reject the null hypothesis at this confidence.
5. for testing based on p-value:
   a. calculate the p-value (i.e., the probability that the test statistic is larger than the observed one.

Central limit theorem