Non-stationary spectral analysis

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January 25, 2024

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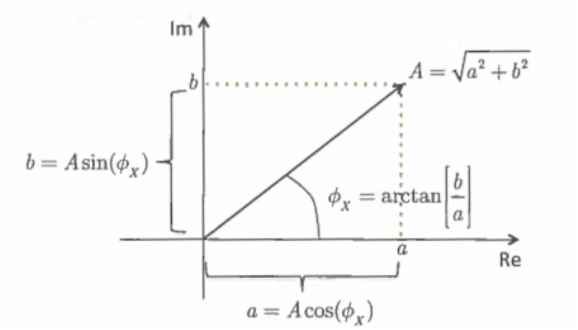
Complex numbers

A complex number $a + ib$ is a vector in the complex plane.



- a and b are the real and imaginary parts, respectively.
- A is the magnitude.
- ϕ_X is the phase.
- using the **Euler's formula** $a + ib = A(\cos(\phi_X) + i\sin(\phi_X)) = Ae^{i\phi_X}$.

Four types of Fourier transforms

FT[1]	continuous	$x(t) = \frac{1}{2\pi} \int x(j\Omega)\ e^{j\Omega t}\ d\Omega$	continuous	$x(j\Omega) = \int_{-\infty}^{\infty} x(t)\ e^{-j\Omega t}\ dt$
FS[2]	periodic (T)	$x(t) = \sum_{k=-\infty}^{\infty} X[k]\ e^{j\frac{2\pi}{T}kt}$	discrete (inf)	$X[k] = \int_{-T/2}^{T/2} x(t)\ e^{-j\frac{2\pi}{T}kt}\ dt$
DTFT[3]	discrete (inf)	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\ e^{j\omega n}\ d\omega$	periodic (2π)	$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\ e^{-j\omega n}$
DFT[4]	discrete (finite)	$x[n] = \sum_{k=0}^{N-1} X[k]\ e^{j\frac{2\pi}{N}kn}$	discrete (finite)	$X[k] = \sum_{n=0}^{N-1} x[n]\ e^{-j\frac{2\pi}{N}nk}$

- how does ω of the DTFT relates to Ω of the FT?

$$x[n] \sim x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$$

$$X_s(j\Omega) = \int_{-\infty}^{\infty} x_s(t)e^{j\Omega t}dt = \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)\right)e^{j\Omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left(\int_{-\infty}^{\infty}\delta(t-nT)e^{j\Omega t}dt\right) = \sum_{n=-\infty}^{\infty} x[n]e^{j\Omega nT} = X_s(\omega)\big|_{\omega=\Omega T}$$

Four types of Fourier transforms

FT[5]	continuous	$x(t) = \frac{1}{2\pi} \int x(j\Omega)\ e^{j\Omega t}\ d\Omega$	continuous	$x(j\Omega) = \int_{-\infty}^{\infty} x(t)\ e^{-j\Omega t}\ dt$
FS[6]	periodic (T)	$x(t) = \sum_{k=-\infty}^{\infty} X[k]\ e^{j\frac{2\pi}{T}kt}$	discrete (inf)	$X[k] = \int_{-T/2}^{T/2} x(t)\ e^{-j\frac{2\pi}{T}kt}\ dt$
DTFT[7]	discrete (inf)	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\ e^{j\omega n}\ d\omega$	periodic (2π)	$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\ e^{-j\omega n}$
DFT[8]	discrete (finite)	$x[n] = \sum_{k=0}^{N-1} X[k]\ e^{j\frac{2\pi}{N}kn}$	discrete (finite)	$X[k] = \sum_{n=0}^{N-1} x[n]\ e^{-j\frac{2\pi}{N}nk}$

- how to find the frequency in Hz corresponding to a DFT coefficient k?

$$\omega = \frac{2\pi}{N} k$$
$$\Omega = \frac{\omega}{T} = 2\pi \frac{k}{NT}$$

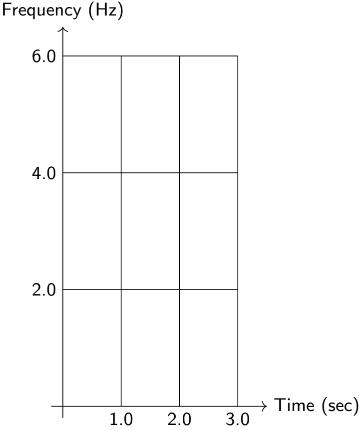
  Thus, the coefficient k corresponds to the frequency $\frac{k}{NT}$ Hz.

- what is the **frequency resolution** of a Fourier transform?

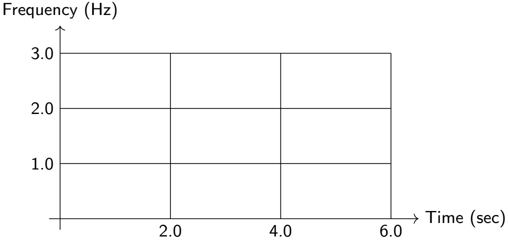
  It is the distance in Hz between two neighboring frequencies, i.e., frequency resolution $= \frac{1}{NT} = \frac{1}{\text{signal duration}}$.

Time-frequency uncertainty

$$\text{frequency resolution} = \frac{1}{NT} = \frac{1}{\text{signal duration}}$$



(a) High temporal resolution



(b) High frequency resolution

Tradeoff between frequency and time resolution.

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Spectral measures for multiple time series

- The **cross-power** is

$$
\begin{aligned}
S_{XY}(f) &= \sum_{\tau=-\infty}^{\infty} R_{XY}(\tau)e^{-i2\pi f\tau} \\
&= X(f)Y^*(f)
\end{aligned}
$$

- The **multi-trial spectral coherence** is

$$
C_{XY}(f) = \frac{|< S_{XY,k}(f) >|^2}{< S_{XX}(f) >< S_{YY}(f) >}
$$

- The **spectral coherence** is

$$
C_{XY}(f) = \frac{|S_{XY}(f)|^2}{S_{XX}(f)S_{YY}(f)}
$$

Multi-trial spectral coherence: intuition

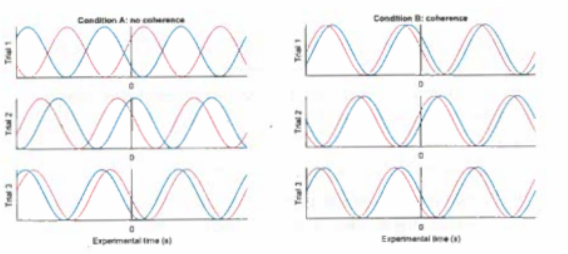


Figure 2: Spectral coherence measures constant phase difference between two times series at a given frequency.

Multi-trial spectral coherence: interpretation

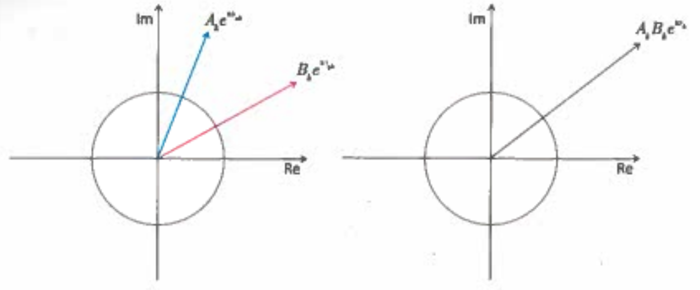
- define

$$
\begin{aligned}
X(f) &= A(f)e^{j\phi_X(f)} \\
Y(f) &= B(f)e^{j\phi_Y(f)}
\end{aligned}
$$

- then

$$
\begin{aligned}
S_{XY,k}(f) &= X_k Y_k^* \\
&= A_k e^{j\phi_{xk}} \left(B_k e^{j\phi_{yk}}\right)^* \\
&= A_k e^{j\phi_{xk}} \left(B_k e^{-j\phi_{yk}}\right) \\
&= A_k B_k e^{j(\phi_{xk}-\phi_{yk})}
\end{aligned}
$$

Multi-trial spectral coherence: interpretation



Left: $X(f)$ and $Y(f)$. **Right:** Cross-power: $X(f) \times Y(f)^*$.

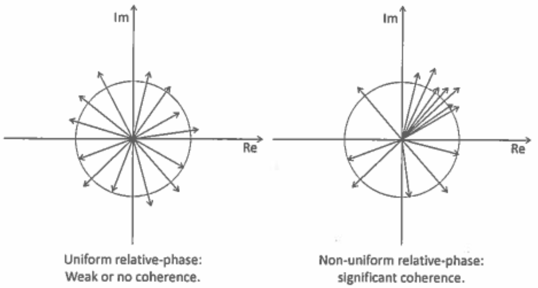
$$\phi_k = \phi_{X,k} - \phi_{Y,k}.$$

Multi-trial spectral coherence: interpretation

Given its definition, **multi-trial spectral coherence**

$$C_{XY}(f) = \frac{|< S_{XY,k}(f) >|^2}{< S_{XX}(f) >< S_{YY}(f) >}$$

corresponds to averaging the cross-power vectors and normalizing the result by the corresponding power spectrum terms.



Multi-trial spectral coherence: interpretation

Multi-trial spectral coherence at frequency f is large when the phase difference at frequency f is approximately constant across trials.

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Understanding the plots we will generate in the next worksheet

Please refer to the plots in the next worksheet.

Summary

Bibliography