Worksheet: Reinforcement Learning

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1 Complete notebook

Please answer all questions in the notebook that we discuss in the last practical session.

2 Bayesian generalisation of the Rescorla-Wagner model

One objective of this exercise is to check if the code implementing the Bayesian generalisation of the Rescola-Wagner model in the previous notebook is correct.

On slide 23 of the practical session on linear dynamical systems we named, and assigned meaning to, all variables of the Kalman filtering algorithm. Another objective of this exercise is to assign a meaning to the variables in the function `kalman_filter_update` of the above notebook. For example, we want to know which variable in this function corresponds to the filtered mean or the filtered covariance.

On slide 7 of the practical session on linear dynamical systems we presented the equations that describe the linear dynamical system model.

$$\begin{aligned}
&\mathbf{x}_n = A\mathbf{x}_{n-1} + \mathbf{w}_n \quad && \mathbf{w}_n \sim N(\mathbf{w}_n|\mathbf{0}, Q) \quad && \mathbf{x}_n \in \mathbb{R}^M \\
&\mathbf{y}_n = C\mathbf{x}_n + \mathbf{v}_n \quad && \mathbf{v}_n \sim N(\mathbf{v}_n|\mathbf{0}, R) \quad && \mathbf{y}_n \in \mathbb{R}^P \quad n = 1 \dots N \\
&\mathbf{x}_0 \sim N(\mathbf{w}_n|\mathbf{m}_0, V_0)
\end{aligned} \tag{1}$$

On slide 8 of the reinforcement learning lesson we provided the equations that describe the Bayesian generalisation of the Rescorla-Wagner model. These equations can be rewritten as

$$\begin{aligned}
&\mathbf{w}_n = \mathbf{w}_{n-1} + \mathbf{p}_n \quad && \mathbf{p}_n \sim N(\mathbf{p}_n|\mathbf{0}, \tau^2 I_M) \quad && \mathbf{w}_n \in \mathbb{R}^M \\
&r_n = x_n^T\mathbf{w}_n + \mathbf{q}_n \quad && \mathbf{q}_n \sim N(\mathbf{q}_n|\mathbf{0}, \sigma_r^2) \quad && r_n \in \mathbb{R} \quad n = 1 \dots N \\
&\mathbf{w}_0 \sim N(\mathbf{w}_0|\mathbf{0}, \sigma^2 I_M)
\end{aligned} \tag{2}$$

Therefore we observe that:

1. $\mathbf{x}_n$ in Eq. 1 correspond to $\mathbf{w}_n$ in Eq. 2.
2. A in Eq. 1 correspond to I_M in Eq. 2.
3. $\mathbf{w}_n$ in Eq. 1 correspond to $\mathbf{p}_n$ in Eq. 2.

4. Q in Eq. 1 correspond to $\tau^2 I_M$ in Eq. 2.

5. ...

On slide 22 of the practical session on linear dynamical systems we introduced the prediction and filtering inference problems solved by the Kalman filter algorithm.

Prediction

$$P(\mathbf{x}_n|\mathbf{y}_1,\ldots,\mathbf{y}_{n-1}) = N(\mathbf{x}_n|\mathbf{x}_{n|n-1}, P_{n|n-1})$$

Filtering

$$P(\mathbf{x}_n|\mathbf{y}_1,\ldots,\mathbf{y}_n) = N(\mathbf{x}_n|\mathbf{x}_{n|n}, P_{n|n})$$

Similarly, for the Bayesian generalisation of the Rescorla-Wagner model we can define the following prediction and filtering problems

Prediction

$$P(\mathbf{w}_n|r_1,\ldots,r_{n-1}) = N(\mathbf{w}_n|\mathbf{w}_{n|n-1}, P_{n|n-1})$$

Filtering

$$P(\mathbf{w}_n|r_1,\ldots,r_n) = N(\mathbf{w}_n|\mathbf{w}_{n|n}, P_{n|n})$$

On slide 23 of the practical session on linear dynamical systems we provided the recursive equations used by the Kalman filter algorithm to estimate the prediction means and covariances, $\mathbf{x}_{n|n-1}, P_{n|n-1}$, and the filtering means and covariances, $\mathbf{x}_{n|n}, P_{n|n}$.

$$\begin{aligned}
\mathbf{x}_{0|0} &= \mathbf{m}_0 && \text{init filtered mean}\\
P_{0|0} &= V_0 && \text{init filtered covariance}\\
\mathbf{x}_{n+1|n} &= A\mathbf{x}_{n|n} && \text{prediction mean}\\
P_{n+1|n} &= AP_{n|n}A^\intercal + Q && \text{prediction covariance}\\
\mathbf{y}_{n|n-1} &= C\mathbf{x}_{n|n-1} && \text{predicted observation}\\
\tilde{\mathbf{y}}_n &= \mathbf{y}_n - \mathbf{y}_{n|n-1} && \text{residual}\\
S_n &= CP_{n|n-1}C^\intercal + R && \text{residual covariance}\\
\mathbf{x}_{n|n} &= \mathbf{x}_{n|n-1} + K_n\tilde{\mathbf{y}}_n && \text{filtering mean}\\
K_n &= P_{n|n-1}C^\intercal S_n^{-1} && \text{Kalman gain}\\
P_{n|n} &= (I_M - K_nC)P_{n|n-1} && \text{filtering covariance}
\end{aligned}$$

Exercises:

1. based on the previous correspondences and Kalman filter algorithm equations write down the equations of the Kalman filter algorithm to estimate the prediction means and covariances, $\mathbf{w}_{n|n-1}, P_{n|n-1}$, and the filtering means and covariances, $\mathbf{w}_{n|n}, P_{n|n}$, of the Bayesian extension of the Rescorla-Wagner model.

2. find out what each line is doing in the function `kalman_filter_update` of the notebook (e.g., the first line is computing the predictive covariance $P_{n|n}$)

3. determine what are the inputs and outputs of the function `kalman_filter_update` in terms of variables of the Kalman filter algorithm (e.g., the input `meanw` is the filtered mean at time $n-1$ and the output `meanw` is the filtered mean at time n).

4. is the function `kalman_filter_update` a correct implementation of the Kalman filter? Please explain in detail why the calculation of the Kalman gain is correct or incorrect.