Spectral Time Series Analysis

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Oscillations in nature

Today we will characterise oscillations in neural time series. But oscillations are everywhere in nature. For example:

circadian rhythm the 24 hour cycle that governs sleep, hormone release, body temperature and other physiological processes.

menstrual cycle a roughly 28-day hormonal cycle regulating ovulation and fertility in females, driven by oscillations in estrogen and progesterone levels.

ultradian rhythms of physical performance these are cycles shorter than a day that governs periods of focus and fatigue. Athletes often perform better at specific times of the day due to oscillatory patterns in physical strength and endurance.

animal migration many animal migrations are governed by oscillatory patterns linked to seasonal changes (e.g., birds migrating during spring and autumn).

Oscillations in nature

population dynamics: predator-prey systems exhibit oscillatory behaviour. When prey populations rise, predator populations grow, leading to a decline in prey, and then a subsequent decline in predators.

lunar cycles: the moon's orbit around the Earth (roughly 28 days) governs the tides, influencing ecosystems like coral spawning and animal behaviours.

Oscillations in the brain

Oscillations, measured in different frequency bands (e.g., delta, theta, alpha, beta, gamma), play a crucial role in various cognitive functions, including:

- Synchronisation and Communication:

  Binding Oscillations enable the brain to synchronise the activity of neurons across different regions, allowing them to work together as a cohesive unit. This is particularly important for tasks like perception, where information from different senses needs to be integrated.

  Information Transfer Oscillations can act as a "clock" to coordinate the timing of neuronal firing, facilitating efficient communication between brain areas.

Oscillations in the brain

- Cognitive Functions:
  - Attention: Different frequency bands are associated with different attentional states. For example, alpha waves are prominent during relaxed wakefulness, while beta waves are associated with focused attention.
  - Memory: Oscillations have been implicated in memory processes, particularly in the consolidation of information from short-term to long-term memory.
- Sensory Processing:
  - Sensory Perception: Oscillations help to filter out irrelevant sensory information and enhance the processing of relevant stimuli.
  - Motor Control: Oscillations play a role in coordinating the timing of muscle movements, ensuring smooth and efficient motor actions.

Oscillations in the brain

- Sleep:

  Defining Sleep Stages Different sleep stages (NREM and REM) are characterised by distinct patterns of brain oscillations.

  - **NREM Sleep** Dominated by slow waves (delta) and sleep spindles.
  - **REM Sleep** Characterised by fast, low-amplitude waves resembling wakefulness.

  Sleep Functions

  - **Memory Consolidation** Slow waves and sleep spindles are crucial for strengthening memories.
  - **Synaptic Plasticity** Sleep oscillations contribute to the strengthening and weakening of connections between neurons.
  - **Brain Rest and Recovery** Slow waves may help the brain recover from the demands of wakefulness.

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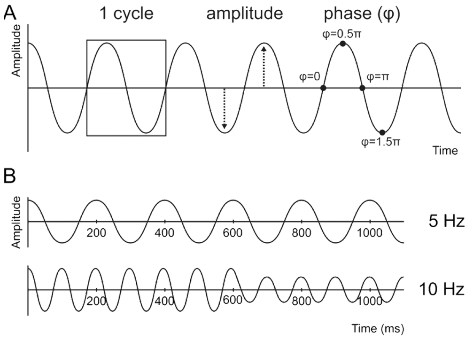
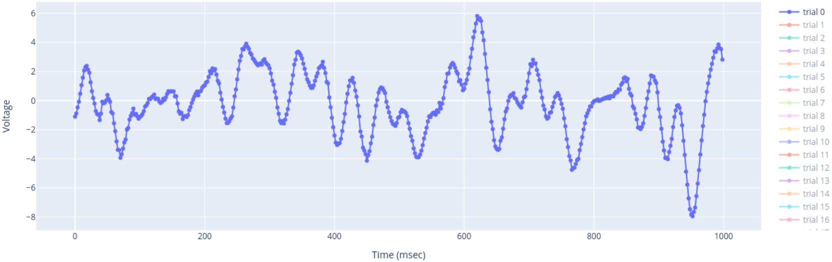
Describing oscillations

$$x(t) = A\cos(2\pi f\, t + \phi)$$

- A amplitude (μV)
- f frequency (Hertz – 1/sec)
- ϕ phase (radians)



Example EEG time series



Liu et al. (2024)

Complex exponential

We will use complex exponentials to represent oscillations.

$$e^{j\Omega t} = \cos(\Omega t) + j\sin(\Omega t) \quad \text{with}\, j^2 = -1$$

Fourier transforms for deterministic signals (1/4)

Name Fourier integral

Time domain type continuous aperiodic

Frequency domain type continuous aperiodic

Formulas

$$x(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

Existence (sufficient condition)

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

Fourier transforms for deterministic signals (2/4)

Name Fourier series

Time domain type continuous periodic

Frequency domain type discrete infinite

Formulas

$$x[k] = \int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi}{T}kt} dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{\frac{2\pi}{T}kt}$$

Fourier transforms for deterministic signals (3/4)

Name Discrete Fourier transform

Time domain type discrete infinite

Frequency domain type continuous periodic

Formulas

$$x(j\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\omega) e^{j\omega n} d\omega$$

Fourier transforms for deterministic signals (4/4)

Name Discrete Time Fourier transform

Time domain type discrete finite

Frequency domain type discrete finite

Formulas

$$x[k] = \sum_{n=1}^{N} x[n] e^{-j\frac{2\pi}{N}nk}$$

$$x[n] = \sum_{k=1}^{N} x[k] e^{j\frac{2\pi}{N}nk}$$

Spectral density function

Definition 1 (spectral density function)

A function f is the **spectral density** of a stationary time series $\{X_t\}$ with autocovariance function $\gamma(\cdot)$ if

1. $f(\omega) \geq 0$ for all $\omega \in (-\pi, \pi]$, and
2. $\gamma(h) = \int_{-\pi}^{\pi} f(\omega) e^{j\omega h} d\omega$, for all integers h.

Note: $\gamma(h)$ is the inverse Fourier transform of $f(\omega)$. Thus, $f(\omega)$ is the Fourier transform of $\gamma(h)$.

Spectral density function

Claim 1 (Interpretation of the spectral density function)

Given a WSS process $X(t)$, consider the finite support random process

$$X_T(t) \triangleq X(t) I[-T, T](t)$$

where $I[-T, T](t)$ is the indicator function equal to one if $-T \leq t \leq T$ and equal to 0 otherwise. Then

$$f(\omega) = \lim_{T \to \infty} \frac{1}{2T} E\left\{ |X(j\omega)|^2 \right\}$$

Estimator of spectral density function

An **estimator of the spectral density function** is $\hat{f}(\omega) = |X(j\omega)|^2 = X(j\omega) X^*(j\omega)$.

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Estimation of the spectral density function

Please refer to the lecture The Power Spectrum (Part 1) in the repository Case Studies in Neural Data Analysis.

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Sampling theorem

Please refer to the lecture Sampling Theorem.

Liu, Q., Jia, S., Tu, N., Zhao, T., Lyu, Q., Liu, Y., Song, X., Wang, S., Zhang, W., Xiong, F., et al. (2024). Open access eeg dataset of repeated measurements from a single subject for microstate analysis. *Scientific Data*, 11(1):379.