Solution for the worksheet temporal time series analysis

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1. The random process in Eq. 1 of the worksheet is an autoregressive model of order one, AR(1).

   As show in the slide **Mean function** containing **Example (Mean function of the autoregressive model of order 1)**, the mean of this AR(1) model is zero.

   When $|\phi| < 1$, as show in the slide **Autovariance function** containing **Example (Autocovariance function of AR(1))**, the covariance of an AR(1) random process is $\gamma(t+h,t) = \phi^h \sigma_w^2 \frac{1}{1-\phi^2}$. When $|\phi| \geq 1$, the summation $\sum_{i=0}^{\infty} \phi^{2i}$ does not converge, and the covariance $\gamma(t+h,t)$ is not well defined. Therefore, only when $|\phi| < 1$, the variance of this AR(1) random process is $\sigma^2 = \sigma_w^2 \frac{1}{1-\phi^2}$.

   When $|\phi| < 1$, because (1) its mean is constant, (2) its variance is also constant and (3) its covariance does not depend on t, the AR(1) random process in wide-sense stationary.

2. The code to generate the figures in the lecture slide titled **Analytical and estimated autocovariance function for AR(1)**, and the generated figures are available here.

3. For the random walk with drift model:

   (a) The covariance function $\gamma(s,t)$ is

   $$\begin{aligned}\gamma(t,s) = E\{(x_t-\mu_t)(x_s-\mu_s)\} &= E\left\{\left(\left(\delta t + \sum_{i=0}^{t} w_i\right) - \delta t\right)\left(\left(\delta s + \sum_{j=0}^{s} w_j\right) - \delta s\right)\right\}\\ &= E\left\{\left(\sum_{i=0}^{t} w_i\right)\left(\sum_{j=0}^{s} w_j\right)\right\} = E\left\{\sum_{i=0}^{t}\sum_{j=0}^{s} w_i w_j\right\}\\ &= \sum_{i=0}^{t}\sum_{j=0}^{s} E\{w_i w_j\} = \sum_{k=0}^{\min(t,s)} E\{w_k^2\} = \sum_{k=0}^{\min(t,s)} \sigma_w^2 = \min(t,s)\ \sigma_w^2\end{aligned}$$

   Then the variance function $var(t)$ is

   $$var(t) = \gamma(t,t) = t\sigma_w^2$$

   (b) Figure 1 plots 100 samples of a random walk with drift model, its mean function and its 95% confidence bounds. That approximately 5 samples escape the confidence bounds at any time point suggest that the calculated variance function is correct.

   (c) The random walk with drift process not wide-sense stationary because (1) when $\delta \neq 0$ the mean is not constant, (2) the variance is neither constant, and (3) the autocovariance function does not depend only on the time separation of its arguments.

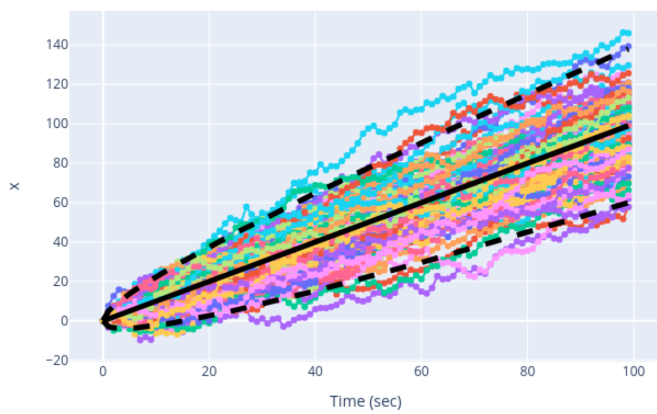

Figure 1: One hundred samples of a random walk with drift process (colour traces). The solid line is the mean of the random process and the dotted lines mark the 95% confidence interval. At any time point 95% of the samples (i.e., 5 samples) should lie above or below the dotted lines. Click on the figure to see its interactive version.