Practical: temporal time series analysis

Joaquin Rapela

January 17, 2025

1 Harmonic process

Given

$$x_t = \sum_{k=1}^{K} A_k \cos(w_k t + \phi_k)$$

with $K, \{A_k\}, \{w_k\}$ constants and $\{\phi_k\}$ independent random variables, uniformly distributed in the range $[-\pi, \pi]$

(a) Simulate and plot 100 samples of x_t. Does x_t look stationary? I suggest using a large sampling rate, much larger than twice the maximal frequency of the cosines.

(b) Calculate the mean, variance and covariance of x_t. Is x_t wide sense stationary?

(c) Add the 95% confidence band to the samples plotted in (a).

(d) Simulate a long time series from x_t and use it to estimate the autocovariance function. Plot this estimate and the analytical covariance computed in (b).

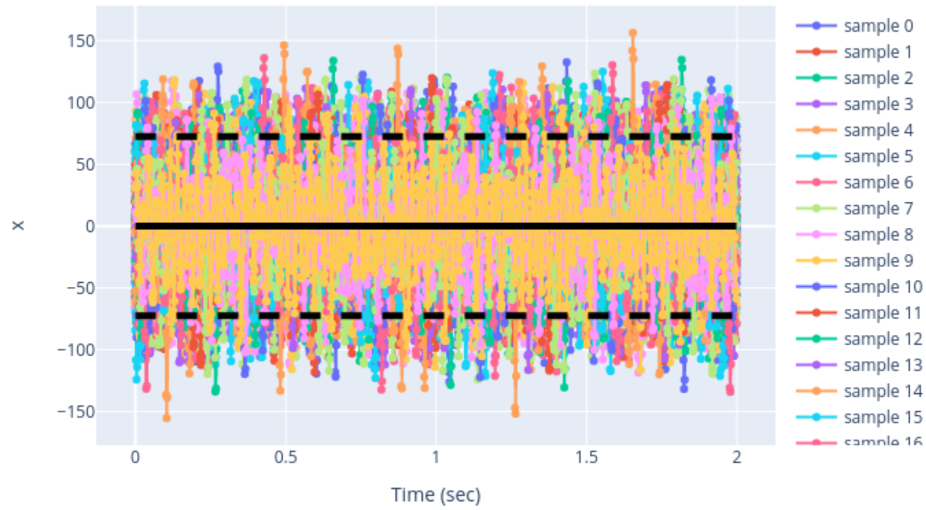

Figure 1: 100 samples from an harmonic process, mean (solid line) and 95% confidence interval (dotted lines).

Answer

(a) The code to simulate and plot 100 samples of x_t, and the generated plots, are given here. The mean of x_t appears constant and equal to zero. The variance of the samples also appears constant. It is not clear to my eye if the covariances only depend on the time lag. I guess that x_t is WSS.

(b) $\mu_t = 0$ and $\gamma(t, t-h) = \sum_{k=1}^{K} \frac{A_k^2}{2} \cos(w_k h)$.

Claim 1.

$$\mu_t = 0 \tag{1}$$

Proof.

$$\mu_t = E\{x_t\} = \sum_{k=1}^{K} A_k E\{\cos(w_k t + \phi_k)\} = \sum_{k=1}^{K} A_k 0 = 0 \tag{2}$$

□

Claim 2.

$$\gamma(t, t-h) = \sum_{i=0}^{K} \frac{A_i^2}{2} \cos(w_i h) \tag{3}$$

Proof.

$$\begin{aligned}
\gamma(t,t-h) =& \text{cov}(x_t, x_{t-h}) = E\{(x_t-\mu_t)(x_{t-h}-\mu_{t-h})\} = E\{x_t x_{t-h}\} && (4)\\
=& E\left\{\left(\sum_{i=0}^{K} A_i\cos(w_i t+\phi_i)\right)\left(\sum_{j=0}^{K} A_j\cos(w_j(t-h)+\phi_j)\right)\right\} && (5)\\
=& E\left\{\sum_{i=0}^{K}\sum_{j=0}^{K} A_iA_j\cos(w_i t+\phi_i)\cos(w_j(t-h)+\phi_j)\right\} && (6)\\
=& \sum_{i=0}^{K}\sum_{j=0}^{K} A_iA_jE\left\{\cos(w_i t+\phi_i)\cos(w_j(t-h)+\phi_j)\right\} && (7)\\
=& \sum_{i=0}^{K} A_i^2E\left\{\cos(w_i t+\phi_i)\cos(w_i(t-h)+\phi_i)\right\}+ && (8)\\
& \sum_{i=0}^{K}\sum_{j\neq i} A_iA_jE\left\{\cos(w_i t+\phi_i)\cos(w_j(t-h)+\phi_j)\right\} && (9)\\
=& \sum_{i=0}^{K} A_i^2E\left\{\cos(w_i t+\phi_i)\cos(w_i(t-h)+\phi_i)\right\}+ && (10)\\
& \sum_{i=0}^{K}\sum_{j\neq i} A_iA_jE\left\{\cos(w_i t+\phi_i)\right\}E\left\{\cos(w_j(t-h)+\phi_j)\right\} && (11)\\
=& \sum_{i=0}^{K} A_i^2E\left\{\cos(w_i t+\phi_i)\cos(w_i(t-h)+\phi_i)\right\}+ && (12)\\
& \sum_{i=0}^{K}\sum_{j\neq i} A_iA_j0\;0 && (13)\\
=& \sum_{i=0}^{K} A_i^2E\left\{\cos(w_i t+\phi_i)\cos(w_i(t-h)+\phi_i)\right\} && (14)\\
=& \sum_{i=0}^{K} \frac{A_i^2}{2}E\left\{\cos(2w_i t++w_ih+2\phi_i)+\cos(w_ih)\right\} && (15)\\
=& \sum_{i=0}^{K} \frac{A_i^2}{2}\cos(w_ih) && (16)
\end{aligned}$$

□

Thus $\sigma_t^2 = \sum_{k=1}^{K}\frac{A_k^2}{2}$. Hence x_t is WSS.

(c) See the code here and refer to Figure 1.

(d) See the code and generated figures here.

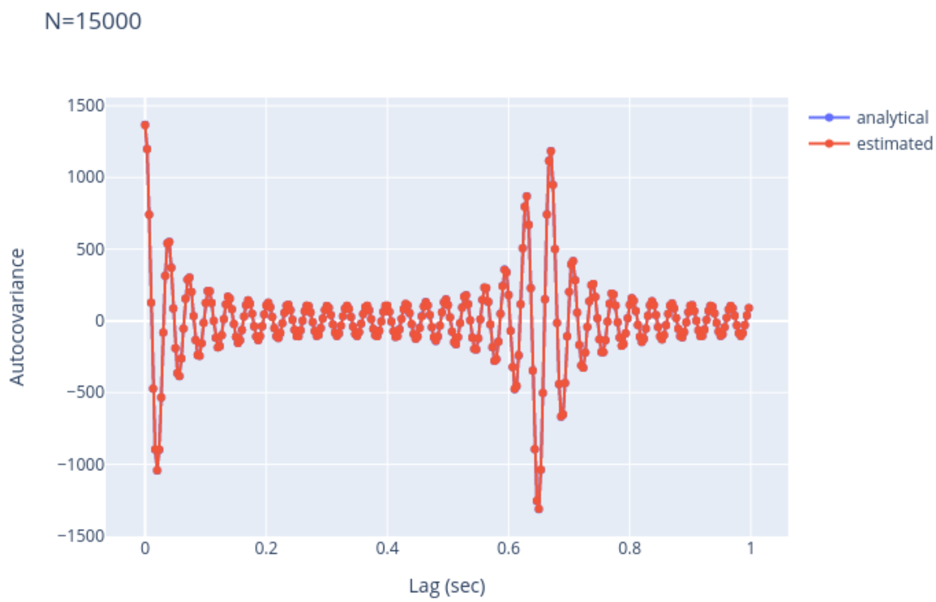

Figure 2: Analytical and estimated autocovariance of an harmonic process.