Solution for the worksheet temporal time series analysis

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February 10, 2025

1. The random process in Eq. 1 of the worksheet is an autoregressive model of order one, AR(1).

   When $|\phi| < 1$, as show in the lecture slide **Mean function** containing **Example (Mean function of the autoregressive model of order 1)**, the mean of this AR(1) model is zero.

   When $|\phi| < 1$, as show in the lecture slide **Autovariance function** containing **Example (Autocovariance function of AR(1))**, the covariance of an AR(1) random process is $\gamma(t+h,t) = \phi^h\sigma_w^2\frac{1}{1-\phi^2}$. Therefore, the variance of this AR(1) random process is $\sigma^2 = \sigma_w^2\frac{1}{1-\phi^2}$.

   When $|\phi| < 1$, because (1) its mean is constant, (2) its variance is also constant and (3) its covariance does not depend on t, the AR(1) random process in wide-sense stationary (WSS).

   When $|\phi| \geq 1$, assume $\{x_t\}$ is WSS, then, because the variance of the sum of indepenndent random variables is the sum of their variances, and because x_{t-1} and w_t are uncorrelated Gaussian random variables, and therefore independent random variables, $\sigma_x^2 = var\{x_t\} = \phi^2 var\{x_{t-1}\} + var\{w_t\} = \phi^2\sigma_x^2 + \sigma_w^2$ or $\sigma_x^2 = \frac{\sigma_w^2}{1-\phi^2}$. The last equality states that $\sigma_x^2 < 0$, because $|\phi| \geq 1$, which cannot be. This wrong result arose from our assumption that, when $|\phi| \geq 1$, $\{x_t\}$ was WSS. Therefore, when $|\phi| \geq 1$, $\{x_t\}$ cannot be WSS.

2. The code to generate the figures in the lecture slide titled **Analytical and estimated autocovariance function for AR(1)** and the generated figures are available here.

3. For the random walk with drift model:

   (a) The covariance function $\gamma(s,t)$ is

$$
\begin{aligned}
\gamma(t,s) = E\{(x_t-\mu_t)(x_s-\mu_s)\} &= E\left\{\left(\left(\delta t + \sum_{i=0}^{t} w_i\right) - \delta t\right)\left(\left(\delta s + \sum_{j=0}^{s} w_j\right) - \delta s\right)\right\} \\
&= E\left\{\left(\sum_{i=0}^{t} w_i\right)\left(\sum_{j=0}^{s} w_j\right)\right\} = E\left\{\sum_{i=0}^{t}\sum_{j=0}^{s} w_i w_j\right\} \\
&= \sum_{i=0}^{t}\sum_{j=0}^{s} E\{w_i w_j\} = \sum_{k=0}^{\min(t,s)} E\{w_k^2\} = \sum_{k=0}^{\min(t,s)} \sigma_w^2 = \min(t,s)\ \sigma_w^2
\end{aligned}
$$

   Then the variance function $var(t)$ is

$$
var(t) = \gamma(t,t) = t\sigma_w^2
$$

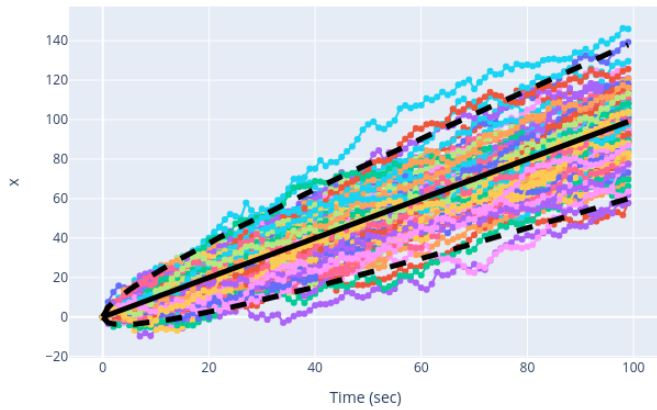

Figure 1: One hundred samples of a random walk with drift process (colour traces). The solid line is the mean of the random process and the dotted lines mark the 95% confidence interval. At any time point 95% of the samples (i.e., 5 samples) should lie above or below the dotted lines. Click on the figure to see its interactive version.

(b) Figure 1 plots 100 samples of a random walk with drift model, its mean function and its 95% confidence bounds. That approximately 5 samples escape the confidence bounds at any time point suggest that the calculated variance function is correct.

(c) The random walk with drift process is not wide-sense stationary because (1) when $\delta \neq 0$ the mean is not constant, (2) the variance is neither constant, and (3) the autocovariance function does not depend only on the time separation of its arguments.