

# Temporal Time Series Analysis

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- 1 Course notes
- 2 Time series analysis
  - Introduction to time series analysis
  - Generation of time series
  - Measures used to describe time series
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- Last Spring 2023 I helped in the discussion sessions of this course.
- Suggested to Klara Olofsdotter (SWC PhD program coordinator) and Sonja Hofer (SWC PhD program faculty coordinator) to ask SWC PhD students to take this course. They liked the idea.
- I volunteered to lead discussions and do grading with Gatsby Unit PhD students and postdoctoral scholars.

# A few motivations to run this course

- ① Gain more teaching experience.
- ② Provide SWC PhD students with essential neural data-analysis tools.
- ③ Contribute to better interactions between the SWC and the Gatsby Unit.

# Course structure

Week 01	Jan 11	The t-test and randomisation tests	Joaquin Rapela	tutorial
Week 02	Jan 18	Power spectra	Joaquin Rapela Yousef Mohammadi Joe Ziminski	tutorial
Week 03	Jan 25	Spectrograms and coherence	Joaquin Rapela Yousef Mohammadi Joe Ziminski	tutorial
Week 04	Feb 01	Circular statistics	Joaquin Rapela	tutorial
Week 05	Feb 08	Singular value decomposition	Will Dorrell	tutorial
Week 06	Feb 15 Feb 16	Linear regression	Lior Fox	lecture tutorial
Week 07	Feb 22  Feb 23	Linear dynamical systems	Aniruddh Galgali Joaquin Rapela	lecture  tutorial
Week 08	Feb 29	no class (CoSyNe)		
Week 09	Mar 07 Mar 08	Artificial neural networks	Erin Grant	lecture tutorial
Week 10	Mar 14  Mar 15	Experimental control with Bonsai	Goncalo Lopes Joaquin Rapela	lecture  tutorial
Week 11	Mar 21 Mar 22	Reinforcement learning		lecture tutorial
Week 12 Week 15	Mar 28 Apr 25	Project development		
Week 16	May 02	Project presentations		

Teaching assistants: Kira Dusterwald, Sihao (Daniel) Liu

Every Thursday we will assign you a worksheet that is due on the second Monday after the assignment.

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# What is time series analysis?

- Time series analysis characterizes **data that is correlated in time**.
- These correlations severely **restrict the applicability of conventional techniques** assuming data samples that are independent and identically distributed.
- These correlations allow to **forecast** future values of a time series based on present and past values.

# Relevance of time series analysis

**economics** daily stock market quotations, monthly unemployment figures.

**social scientists** birthrates, school enrollment.

**epidemiology** number of influenza cases observed over some time period.

**medicine** blood pressure measurements traced over time.

# Temporal vs spectral time series analysis

**temporal time series analysis** focuses on the analysis of lagged relationship (e.g., how does what happened today affect what will happen tomorrow?).

**spectral time series analysis** centers on the analysis of rhythms (e.g., can we observe rhythmic activity in local field potentials recorded from human brains?)

# An example time series

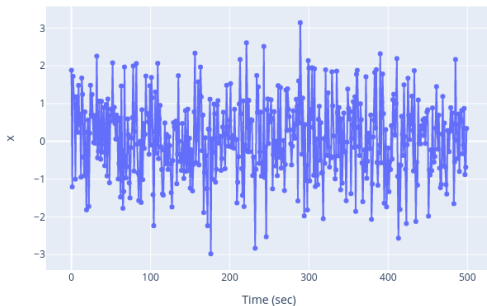
In the examples below we will use spontaneous EEG (i.e., no task) recorded from a human subject ([Liu et al., 2024](#)).

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# Generation of time series: white noise

The first step to generate time series is to generate **white noise** (i.e., independent Gaussian random variables with zero mean and fixed variance, **example**).

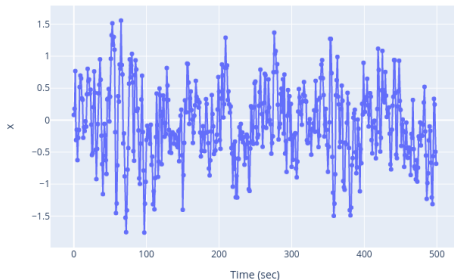
```
w = np.random.normal(loc=0, scale=sigma, size=N)
```



# Generation of time series: moving average model

In a white noise stochastic process  $x$ , for any pairs of time points,  $t_1$  and  $t_2$ , the random variables  $x_{t_1}$  and  $x_{t_2}$  are uncorrelated. The **moving average model** adds serial correlation to white noise (**example**).

$$\nu_t = \frac{1}{3}(w_{t-1} + w_t + w_{t+1}) \quad (1)$$

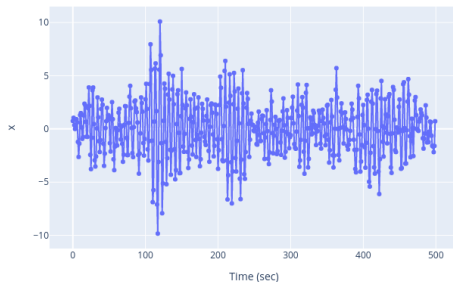




# Generation of time series: autoregressive model

Many neural time series, like local field potential recordings, exhibit oscillations of the type of sine waves. The **autoregressive model** generates oscillations (**example**). **Lecture on linear dynamical systems.**

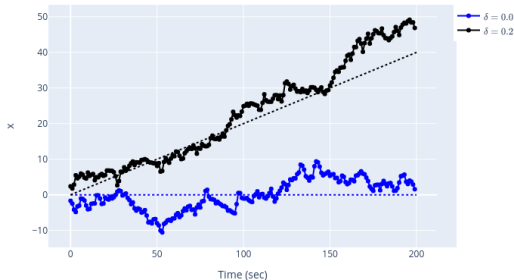
$$x_t = x_{t-1} - 0.9x_{t-2} + w_t$$



# Generation of time series: random walk with drift

The **random walk with noise** model is used to characterize trends in time series (**example**).

$$x_t = \delta + x_{t-1} + w_t \quad (2)$$



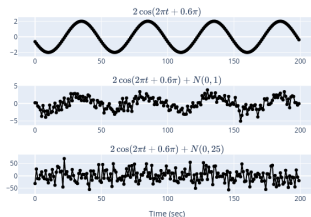
# Generation of time series: signal plus noise

Many realistic models of time series assume an underlying signal with a periodic variation contaminated by adding a random noise (**example**).

$$x_t = 2 \cos\left(2\pi \frac{t}{50} + 2\pi \frac{15}{50}\right) + w_t$$

$$A \cos(2\pi\omega t + \phi)$$

where  $A = 2, \omega = 1/50, \phi = 2\pi 15/50$ . **Lecture on spectral analysis of time series.**



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# Mean function

## Definition 1 (Mean function)

The mean function,  $\mu_t$ , is defined as  $\mu_t = E\{x_t\}$ .

## Example (Mean function of a moving average model)

Calculate the mean function of the moving average model in Eq. 5.

$$E\{\nu_t\} = \frac{1}{3}(E\{w_{t-1}\} + E\{w_t\} + E\{w_{t+1}\}) = \frac{1}{3}(0 + 0 + 0) = 0$$

## Example (Mean function of the autoregressive model of order 1)

Calculate the mean function of the autoregressive model of order 1, AR(1), in Eq. 3.

$$x_t = \phi x_{t-1} + w_t \quad (3)$$

An AR(1) model (Eq. 3) can be represented as a moving average of infinite order MA( $\infty$ ). See **MA( $\infty$ ) representation of AR(1) random process** in Appendix. Then

$$x_t = \sum_{i=0}^{\infty} \phi^i w_{t-i}$$

$$E\{x_t\} = \sum_{i=0}^{\infty} \phi^i E\{w_{t-i}\} = \sum_{i=0}^{\infty} \phi^i 0 = 0$$

## Example (Mean function of the random walk with drift model)

Calculate the mean function of the random noise with drift model, in Eq. 2.

The random noise with drift model in Eq. 2 can be represented as

$$x_t = t\delta + \sum_{i=0}^{\infty} w_{t-i}$$

$$E\{x_t\} = \delta t + \sum_{i=0}^{\infty} E\{w_{t-i}\} = \delta t + \sum_{i=0}^{\infty} 0 = \delta t$$

See the figure of samples of the random noise with drift random process.

# Autocovariance function

## Definition 2 (Autocovariance function)

The autocovariance function is defined as

$$\gamma(s, t) = \text{cov}(x_s, x_t) = E\{(x_s - \mu_s)(x_t - \mu_t)\}.$$

## Note

For  $s = t$  the covariance reduces to the variance, because

$$\gamma(t, t) = E\{(x_t - \mu_t)^2\} = \text{var}(x_t).$$



# Autocovariance function

## Example (Autocovariance function of moving average)

Calculate the autocovariance function of the moving average model in Eq. 5.

$$\gamma_\nu(s, t) = \text{cov}(\nu_s, \nu_t) = \text{cov}\left(\frac{1}{3}(w_{s-1} + w_s + w_{s+1}), \frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right)$$

If  $s=t$ :

$$\begin{aligned}\gamma_\nu(t, t) &= \text{cov}(\nu_t, \nu_t) = \text{cov}\left(\frac{1}{3}(w_{t-1} + w_t + w_{t+1}), \frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right) \\ &= \frac{1}{9} (\text{cov}(w_{t-1}, w_{t-1}) + \text{cov}(w_t, w_t) + \text{cov}(w_{t+1}, w_{t+1})) \\ &= \frac{1}{9} (\sigma_w^2 + \sigma_w^2 + \sigma_w^2) = \frac{3}{9} \sigma_w^2\end{aligned}$$

## Example (Autocovariance function of moving average)

If  $s=t+1$ :

$$\begin{aligned}\gamma_\nu(t+1, t) &= \text{cov}(\nu_{t+1}, \nu_t) \\ &= \text{cov}\left(\frac{1}{3}(w_t + w_{t+1} + w_{t+2}), \frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right) \\ &= \frac{1}{9} (\text{cov}(w_t, w_t) + \text{cov}(w_{t+1}, w_{t+1})) \\ &= \frac{1}{9} (\sigma_w^2 + \sigma_w^2) = \frac{2}{9}\sigma_w^2\end{aligned}$$

## Example (Autocovariance function of moving average)

If  $s=t+2$ :

$$\begin{aligned}\gamma_\nu(t+2, t) &= \text{cov}(\nu_{t+2}, \nu_t) \\ &= \text{cov}\left(\frac{1}{3}(w_{t+1} + w_{t+2} + w_{t+3}), \frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right) \\ &= \frac{1}{9} (\text{cov}(w_{t+1}, w_{t+1})) \\ &= \frac{1}{9} \sigma_w^2\end{aligned}$$

## Example (Autocovariance function of moving average)

$$\gamma_v(s, t) = \begin{cases} \frac{3}{9}\sigma_w^2 & \text{if } s = t, \\ \frac{2}{9}\sigma_w^2 & \text{if } |s - t| = 1, \\ \frac{1}{9}\sigma_w^2 & \text{if } |s - t| = 2, \\ 0 & \text{if } |s - t| > 2. \end{cases}$$

# Summary

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# MA( $\infty$ ) representation of AR(1) random process

## Claim 1

Let  $|\phi| < 1$ , then

$$x_t = \phi x_{t-1} + w_t \quad \text{if and only if} \quad (4)$$

$$x_t = \sum_{i=0}^{\infty} \phi^i w_{t-i} \quad (5)$$

# MA( $\infty$ ) representation of AR(1) random process

## Proof.

We first show that  $x_t$ , as defined in Eq. 5, satisfies Eq. 4.

$$\begin{aligned}\phi x_{t-1} &= \phi \sum_{i=0}^{\infty} \phi^i w_{t-1-i} = \phi \sum_{j=1}^{\infty} \phi^{j-1} w_{t-j} = \sum_{j=1}^{\infty} \phi^j w_{t-j} \\ \phi x_{t-1} + w_t &= \sum_{j=0}^{\infty} \phi^j w_{t-j} = x_t\end{aligned}$$



# MA( $\infty$ ) representation of AR(1) random process

## Proof.

We now show that Eq. 5 is the unique solution to Eq. 4. Suppose  $y_t$  is stationary and satisfies Eq. 4, then

$$\begin{aligned}y_t &= \phi y_{t-1} + w_t \\&= \phi(\phi y_{t-2} + w_{t-1}) + w_t = \phi^2 y_{t-2} + \phi w_{t-1} + w_t \\&= \phi^{t+1} y_{t-(t+1)} + \phi^t w_t + \dots + \phi w_{t-1} + w_t \\&= \phi^{k+1} y_{t-(k+1)} + \sum_{i=0}^k \phi^k w_{t-i}\end{aligned}$$

$$E \left\{ \left( y_t - \sum_{i=0}^k \phi^i w_{t-i} \right)^2 \right\} = \phi^{2k+2} E \{ y_{t-(k+1)}^2 \} = \phi^{2k+2} \sigma^2$$

$$E \left\{ \left( y_t - \sum_{i=0}^{\infty} \phi^i w_{t-i} \right)^2 \right\} = \lim_{k \rightarrow \infty} \phi^{2k+2} \sigma^2 = 0$$

# MA( $\infty$ ) representation of AR(1) random process

Proof.

Thus  $y_t$  equals  $\sum_{i=0}^{\infty} \phi^i w_{t-i}$  in the mean-squared sense. □

Liu, Q., Jia, S., Tu, N., Zhao, T., Lyu, Q., Liu, Y., Song, X., Wang, S., Zhang, W., Xiong, F., et al. (2024). Open access eeg dataset of repeated measurements from a single subject for microstate analysis. *Scientific Data*, 11(1):379.