1. Is the x_t random process in Eq. 1 wide-sense stationary (WSS)?

$$x_t = \phi x_{t-1} + w_t \quad (1)$$

where $\{w_t\}$ is a white noise random process with variance σ_w^2. For this random process to be WSS, can the parameter ϕ take any value? Why or why not?

For (1) to be WSS:

i) μ has to be constant over time

ii) $\gamma(s,t)$ depends on s,t only through $|s-t|$ not individually, $|s-t| = l$

i) Applying linearity of $\mathbb{E}$:

$$\mathbb{E}[x_t] = \phi\,\mathbb{E}[x_{t-1}] + \mathbb{E}[w_t]$$

$\mathbb{E}[w_t] = 0$ by definition

$$\mathbb{E}[x_t] = \phi\,\mathbb{E}[x_{t-1}]$$

For μ to remain constant:

$$\mu = \phi\mu \Rightarrow \mu(1-\phi) = 0$$

$$\mu = 0, \quad \phi = 1$$

If $\phi = 1 \Rightarrow x_t = x_{t-1} + w_t$ ← random walk which doesn't have a constant mean, it increases/decreases over time

Can you prove this?

If μ constant, μ_t is too:

$$\mu_t = \phi\mu_{t-1} = \phi(\phi\mu_{t-2}) = \dots$$

$$\Rightarrow \mu_t = \phi^k \mu_{t-k}$$

If $|\phi| \geq 1$, ϕ^k grows without bound as $k \to \infty$ when $\phi > 1$ or oscillates when $\phi = -1$

For |\phi|<1, in the lecture we used the MA(\inty) representation of an AR(1) process to show that \mu=0

However, if $|\phi| < 1$, $\phi^k \to 0$ as $k \to \infty$

Thus the past values of the series affect the current value to a lesser extent → stationarity.

This is not the definition of stationarity that I know.

$\Leftrightarrow \mu = 0$ if $|\phi| < 1$

ii) $$\operatorname{cov}(X_s, X_t) = \mathbb{E}[X_s X_t] - \mathbb{E}[X_s]\,\mathbb{E}[X_t]$$

$\mathbb{E}[X_t] = \mu = 0$ from i)

$$\operatorname{cov}(X_s, X_t) = \mathbb{E}[X_s X_t]$$

In the lecture we also derived the autocovariance function of AR(1) random processes for |\phi|<1 using the MA(\infty) representation of an AR(1) random process.
Autogregressive models are defined for positive and negative indices. I have clarified this in the lecture notes. Thanks.
Thus, there is not an initial condition. However, one can use initial conditions for simulation.

Since $X_t = \phi X_{t-1} + w_t = \phi(\phi X_{t-2} + w_{t-1}) + w_t = \dots$

$$\Rightarrow X_t = \underbrace{\phi^t X_0}_{} + \sum_{k=0}^{t-1} \phi^k w_{t-k}$$

init. condition could be dropped assuming stationarity also it would vanish is applying the expectation in the next step.

$$\operatorname{cov}(X_s, X_t) = \mathbb{E}\left[\sum_{k=0}^{s-1} \phi^k w_{s-k} \sum_{j=0}^{t-1} \phi^j w_{t-j}\right]$$

White noise terms are independent:

$$= \sum_{k=0}^{(s-t)-1} \phi^k \phi^{|s-t|} \sigma_w^2 = \phi^{|s-t|} \frac{\sigma_w^2}{1-\phi^2},$$ thus only depends on $|s-t|$ under condition $|\phi| < 1$

This summation is not well defined if s<t

why?

Ans: $\mu = 0$ when $|\phi| < 1$.

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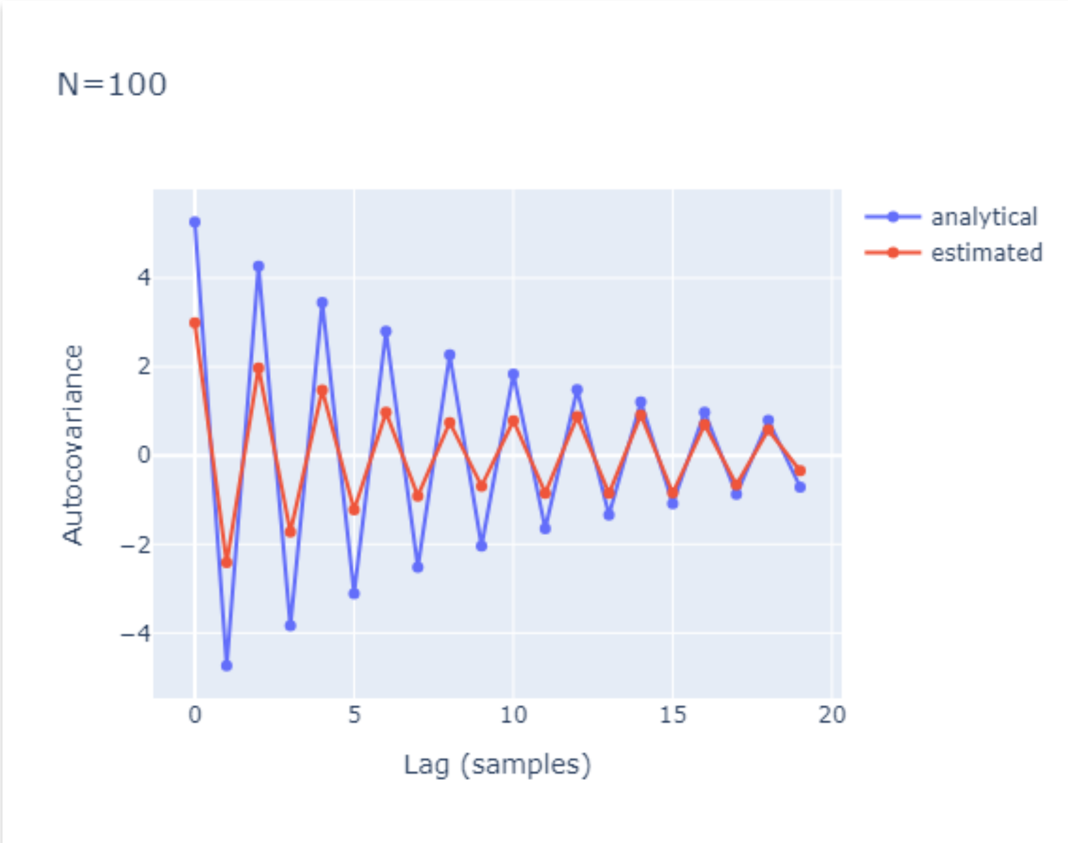
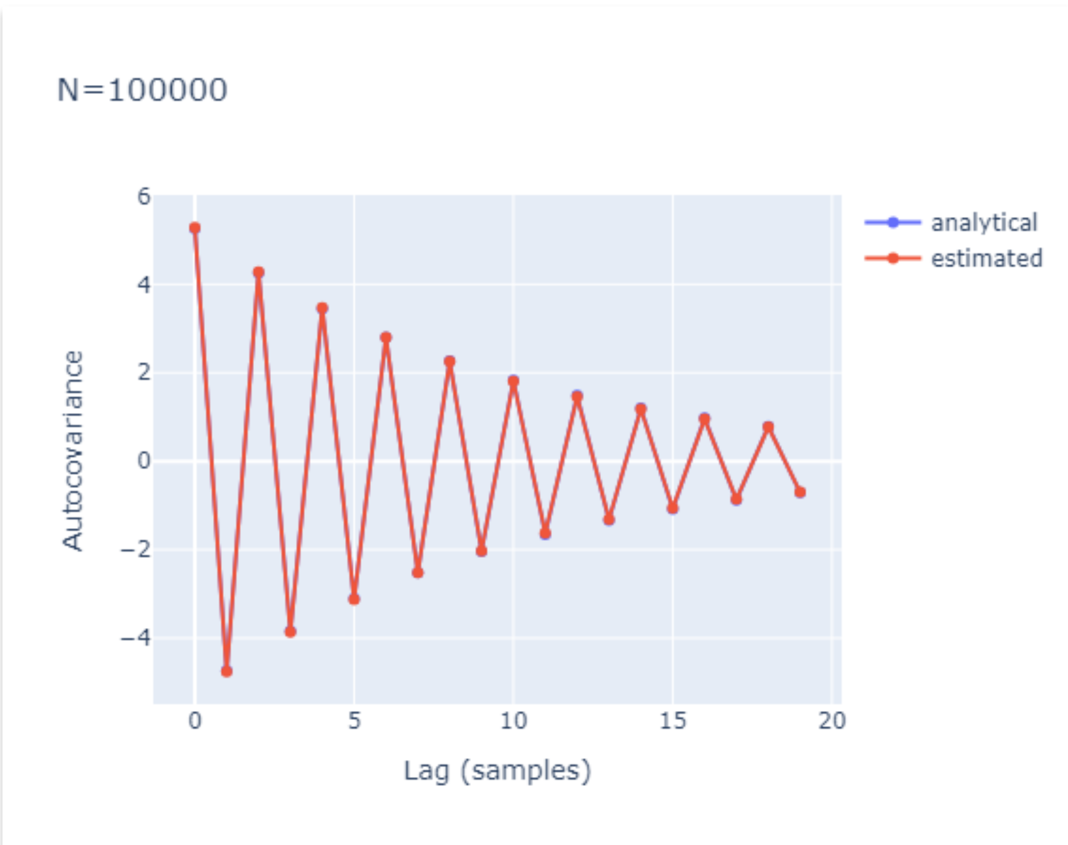
2. Write code to generate the figures in the lecture slide titled *Analytical and estimated autocovariance function for AR(1)*. Provide the code and the generated figures.

   Hint: you may want to modify the code in the solution of the lecture slide titled *Analytical and estimated autocovariance function for MA*.





```
"""
Simulate an AR(1) time series with N=100 and N=100,000 samples,
 φ =0.9 and σw =1.0. For each N, plot the analytical and estimated
 autocovariance function.
=================================================================================
"""

#%%
# Import requirements
# -------------------

import os
import numpy as np
import plotly.graph_objects as go

#%%
# Define variables
# -----------------

srate = 1
T = 10000
sigma_w = 1.0
phi=-0.9
lags = np.arange(20)

#%%
# Create white noise
# ------------------
#

time = np.arange(0, T, 1.0/srate)
N = len(time)
w = np.random.normal(loc=0, scale=sigma_w, size=N)

#%%
# Simulate the AR process
xi = np.zeros(T)
xi[0] = w[0]
for t in range(1, T):
    xi[t] = phi * xi[t-1] + w[t]

#%%
# Analytical autococovariance
# From 1. γ(xt)=phi**h*sigma_w**2/(1 - phi**2)

analytical_autocov = np.array([phi**h*(sigma_w**2/(1 - phi**2)) for h in lags])

#%%
# Estimated autocovariance
# Using the general formula for autocovariance
```

```
n = len(xi)
mean = np.mean(xi)
est_autocov = np.zeros(len(lags))

for i in range(len(lags)):
    est_autocov[i] = np.mean((xi[i:] - mean) * (xi[:n-i] - mean))


#%%
# Plot true and estimated autocovaraince for the AR process.
# -----------------------------------------------------------------
#

fig = go.Figure()
trace = go.Scatter(x=time, y=xi, mode="lines+markers")
fig.add_trace(trace)
fig.update_layout(xaxis=dict(title="Time (sec)"), yaxis=dict(title="x"))

if not os.path.exists("figures"):
    os.mkdir("figures")

fig.write_html(f"figures/whiteNoiseSamplesN{T}.html")
# fig.write_image(f"figures/whiteNoiseSamplesN{T}.png")


fig = go.Figure()
trace = go.Scatter(x=lags, y=analytical_autocov, mode="lines+markers",
name="analytical")
fig.add_trace(trace)
trace = go.Scatter(x=lags, y=est_autocov, mode="lines+markers", name="estimated")
fig.add_trace(trace)
fig.update_layout(title=f"N={T}", xaxis=dict(title="Lag (samples)"),
                  yaxis=dict(title="Autocovariance"))

if not os.path.exists("figures"):
    os.mkdir("figures")

fig.write_html(f"figures/ARAutoCovN{T}.html")
# fig.write_image(f"figures/whiteNoiseAutoCorN{T}.png")

fig

# %%
```

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3. (optional) For the random walk with drift model:

   (a) Calculate the covariance function $\gamma(s,t)$ and use it to derive the variance function $var(t)$.

   (b) To check that your variance function is correct, complete and execute this python script. It plots 100 samples of the random walk with drift model (coloured traces), with the mean (solid line) and 95% confidence bands (dashed lines). See Figure 1. At any time point (abscissa) you should observe that 95% of the traces (5 traces) are above the upper line or below the lower line.

   (c) Is the random walk with drift process wide-sense stationary? Why or why not?

a) $X_t = \theta + X_{t-1} + w_t$

$= \underbrace{\theta + \theta}_{2\theta} + X_{t-2} + w_{t-1} + w_t$

$\vdots$

$= t \cdot \theta + X_0 + w_1 + w_2 + \dots + w_t$

$= \theta t + X_0 + \sum_{i=0}^{t-1} w_{t-1}$

Likewise

$X_s = \theta s + X_0 + \sum_{i=0}^{s-1} w_{s-1}$

$\gamma(s,t) = E[X_s X_t] - E[X_s]E[X_t]$

$E[X_t] = \theta t + \cancel{X_0}$ ← presuming $X_0 = 0$

$E[X_s] = \theta s + \cancel{X_0}$

$X_s X_t = \left(\theta t + X_0 + \sum_{i=0}^{t-1} w_{t-1}\right)\left(\theta s + X_0 + \sum_{i=0}^{s-1} w_{s-1}\right)$

$E[w_t] = 0 \quad E[w_t w_s] = \sigma_w^2 \delta_{ts}$ (From properties) ← Kronecker delta

$E[X_s X_t] = \theta^2 st + E[X_0^2] + \sigma_w^2 \min(s,t)$

Ans.: $\gamma(s,t) = \cancel{\theta^2 st} + \underbrace{E[X_0^2]}_{=0} + \sigma_w^2 \min(s,t) - \cancel{(\theta s \cdot \theta t)}$

$= \sigma_w^2 \min(s,t)$

$var(t) = cov(t,t) = \sigma_w^2 \min(t,t) = \sigma_w^2 t$

c) For WSS:

i) The mean is constant and does not depend on time

ii) $\gamma(s,t)$ depends on s and t only through their difference

i) $E[X_t] = \theta_t + X_0$ – dependent on time! (linearly) Also μ is not constant across t (by visual inspect.)

ii) $\gamma(s,t) = \sigma_w^2 (s,t)$ – depends on individual values of s and t, not their abs. difference.

Ans.: No!

Excellent!

(b)?

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