TASK 4.2

To derive the basic function $\phi(x)$ from the kernel function $K(x,y)$, we express the kernel as an inner product in a higher-dimensional space. The kernel function provided is a second degree polynomial kernel given as

$$K(x,y) = (x_1y_1 + x_2y_2 + c)^2$$

using binomial expansion

$$K(x,y) = (x_1y_1)^2 + (x_2y_2)^2 + 2x_1y_1x_2y_2 + 2c(x_1y_1 + x_2y_2) + c^2$$

we express the kernel as an inner product between 2 feature vectors $\phi(x)$ and $\phi(y)$ such that

$$K(x,y) = \cancel{\phi(x)^T} \; \phi(x)^T\phi(y)$$

By comparing the expanded form of the kernel with standard form of an inner product, we can deduce what the feature map $\phi(x)$ should look like. The kernel contains terms involving the squares of x_1 and x_2, product of x_1 and x_2 and the linear term of x_1 and x_2 with constant.

Thus, the feature map $\phi(x)$ must include these terms to allow the kernel to be written as a dot product. the terms in the kernel suggest the following basic function $\phi(x)$

$$\phi(x) = [x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2c}\,x_1, \sqrt{2c}\,x_2 + c^2]^T$$

Now we verify that this feature map gives the correct kernel function. the inner product

$\phi(x)^T\phi(y)$ is

$$\phi(x)^T\phi(y) = (x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2c}\,x_1, \sqrt{2c}\,x_2, c^2) \cdot (y_1^2, y_2^2, \sqrt{2}y_1y_2, \sqrt{2c}\,y_1, \sqrt{2c}\,y_2\ c^2)$$

expanding the dot product gives

$$\phi(x)^T\phi(y) = x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2 + 2cx_1y_1 + 2cx_2y_2 + c^4$$

this expression correspond with the expanded form of the kernel function $K(x,y)$ which confirms the map is correct

Hence, the basic function that maps 2-d sample $x = [x_1, x_2]^T$ to a higher dimensional space is

$$\phi(x) = [x_1^2 \; x_2^2, \sqrt{2}x_1x_2, \sqrt{2C}x_1, \sqrt{2C}x_2, C^2]^T$$

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To determine the higher-dimensional space to which the 2-d samples should be mapped using the derived basis function, we start with the mapping defined by the ~~featu~~ feature map $\phi(x)$. The original samples will be represented in a higher-dimensional space formed by the combination of the original features, their square and additional constant terms

Recall that the basic function derived from the polynomial kernel $K(x,y) = (x_1y_1 + x_2y_2 + C)^2$ is

$$\phi(x) = [x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2C}x_1, \sqrt{2C}x_2, C^2]^T$$

Thus, the higher dimensional space is 6-dimensional

Given the XOR gate below

x_1	x_2	y
-1	-1	-1
-1	1	1
1	-1	1
1	1	-1

We transform into the 6-dimensional space as follow

Sample 1: $(-1, -1)$

$$\phi(-1,-1) = [(-1)^2, (-1)^2, \sqrt{2}(-1)(-1), \sqrt{2}C(-1), \sqrt{2}C(-1) + C^2]^T$$

$$= [1, 1, \sqrt{2}, -\sqrt{2}C, -2\sqrt{2}C, -\sqrt{2}C + C^2]^T$$

Sample 2: (-1, 1)

$$\phi(-1,1) = \left[(-1)^2, 1^2, -\sqrt{2}, -\sqrt{2}c, \sqrt{2}c\ c^2\right]^T$$

Sample 3 (1, -1)

$$\phi(1,-1) = \left[1^2, (-1)^2, -\sqrt{2}, \sqrt{2}c, -\sqrt{2}c, c^2\right]^T$$

Sample 4 (+1, +1)

$$\phi(1,1) = \left[1^2, 1^2, \sqrt{2}, \sqrt{2}c, \sqrt{2}c, c^2\right]^T$$