Ruin with Stochastic Investments

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Our objective is to study an extension of the classical ruin problem, where the insurance company invests its surplus capital and earns stochastic returns on these investments. As in the Cramér-Lundberg model, we suppose that the capital growth from the insurance business is modeled as

$$X_t = ct - \sum_{i=1}^{N_t} Y_i,$$

where c denotes the premiums income rate, $\{Y_i\}$ denotes the sizes of the individual claims, and N_t denotes the number of claims occuring in the interval $[0,t]$. It is assumed that $\{Y_i\}$ is and i.i.d. sequence, independent of $\{N_t\}$, and that $\{N_t\}$ is a Poisson process.

For simplicity, we will study the problem in discrete time. For this purpose, set

$$L_i = -(X_i - X_{i-1}), \quad i = 1, 2, \ldots;$$

that is, L_i denotes the capital *loss* incurred by the insurance business in the ith discrete time interval. Moreover, during the ith interval, we assume that the surplus capital available at time $(i-1)$ has been invested and earns the stochastic return R_i on these investments, where it is assumed that $\{R_i\}$ is an i.i.d. sequence of random variables and $\mathbf{E}[\log R_i] > 0$. If we let $\mathscr{C}_n$ denote the total capital of the insurance company at time n, then it follows from these definitions that

$$\mathscr{C}_n = R_n \mathscr{C}_{n-1} - L_n, \quad n = 1, 2, \ldots; \quad \mathscr{C}_0 = u; \tag{1}$$

where u denotes the company's initial capital.

The recursive sequence (1) may be solved to obtain an explicit expression for the company's capital at time n, expressed in terms of the given quantities $\{R_i\}$ and $\{L_i\}$. Namely, observe that

$$\begin{aligned} \mathscr{C}_n &= R_n \mathscr{C}_{n-1} - L_n = R_n \left(R_{n-1} \mathscr{C}_{n-2} - L_{n-1} \right) - L_n \\ &= \cdots = \left(R_n \cdots R_1 \right) \mathscr{C}_0 - \left(R_n \cdots R_2 \right) L_1 - \cdots - R_n L_{n-1} - L_n. \end{aligned} \tag{2}$$

Now set $A_i = 1/R_i$; this quantity is called the "discount factor." Then multiplying (2) by $A_1 \cdots A_n$, we obtain that

$$(A_1 \cdots A_n) \mathscr{C}_n = \mathscr{C}_0 - A_1 L_1 - \cdots - (A_1 \cdots A_n) L_n.$$

Since $\mathscr{C}_0 = u$, it follows that

$$\begin{aligned} \mathscr{C}_n < 0 &\Longleftrightarrow u - A_1 L_1 - \cdots - (A_1 \cdots A_n) L_n < 0 \\ &\Longleftrightarrow A_1 L_1 + \cdots + (A_1 \cdots A_n) L_n > u. \end{aligned} \tag{3}$$

Now define the probability of ruin as

$$\psi(u) = \mathbf{P} \left\{ \mathscr{C}_n < 0, \text{ for some } n = 1, 2, \ldots \right\},$$

and set

$$Y_n = A_1 L_1 + \cdots + (A_1 \cdots A_n) L_n, \quad n = 1, 2, \ldots. \tag{4}$$

Then it follows from (3) that

$$\psi(u) = \mathbf{P} \left\{ Y_n > u, \text{ some } n = 1, 2, \ldots \right\} = \mathbf{P} \left\{ \sup_{n \geq 1} Y_n > u \right\}. \tag{5}$$

The sequence $\{Y_n\}$ is called a "perpetuity sequence." Note that Y_n represents the *discounted* losses that accumulate in the time interval $[0,n]$.

To determine the probability of ruin, we will first show that $Y := \sup_{n \geq 1} Y_n$ satisfied a *stochastic fixed point equation.*

Definition. We say that a random variable V satisfies a *stochastic fixed point equation* (SFPE) if

$$V \stackrel{d}{=} \Phi(V), \tag{6}$$

for some known random function Φ.

[In (6), $\stackrel{d}{=}$ means equality in distribution; that is, the probability distribution on the left-hand side of this equation is the same as the probability distribution of the quantity on the right-hand side.]

A stochastic fixed point equation. The next objective is to find a stochastic fixed point equation satisfied by $Y = \sup_{n \geq 1} Y_n$. To this end, observe by definition that

$$Y_n = A_1 L_1 + A_1 \left(A_2 L_2 + \cdots + A_2 \cdots A_n L_n \right). \tag{7}$$

Next, define a new sequence of random variables $\{Y_n^{(1)}\}$ by setting

$$Y_n^{(1)} = A_2 L_2 + \cdots + (A_2 \cdots A_{n+1}) L_{n+1}, \quad n = 1, 2, \ldots; \qquad Y_0^{(1)} = 0. \tag{8}$$

Note that, in the definition of $Y_n^{(1)}$, the input sequence $(A_1, \ldots, A_n, L_1, \ldots, L_n)$ has been replaced with $(A_2, \ldots, A_{n+1}, L_2, \ldots, L_{n+1})$, i.e., the elements of this sequence have been *shifted* by one unit in time; cf. (4). Also observe that the distribution of $Y_n^{(1)}$ is the same as the distribution of Y_n for all $n \geq 1$. Substituting (8) into (7) yields

$$Y_n = A_1 L_1 + A_1 L_{n-1}^{(1)}, \quad n = 1, 2, \ldots.$$

Hence

$$\begin{aligned} Y :=& \sup_{n \geq 1} Y_n = \sup_{n \geq 1} \left\{ A_1 L_1 + A_1 Y_{n-1}^{(1)} \right\} \\ =& A_1 L_1 + A_1 \sup_{n \geq 1} Y_{n-1}^{(1)} = A_1 L_1 + A_1 \sup_{n \geq 0} Y_n^{(1)} \\ =& A_1 L_1 + A_1 \max \left\{ \sup_{n \geq 1} Y_n^{(1)}, 0 \right\} := A_1 L_1 + A_1 \max \left\{ Y^{(1)}, 0 \right\}. \end{aligned} \tag{9}$$

Finally set $(A, L) \stackrel{d}{=} (A_1, L_1)$ and independent of $\{Y_n\}$. Then we obtain that

$$Y \stackrel{d}{=} AL + A \max\{Y, 0\} := \Phi(Y), \tag{10}$$

where

$$\Phi(y) := AL + A \max\{y, 0\},$$

i.e., Φ is a *known* random function determined by the input random variables (A, L), and our goal is to ascertain the distribution of Y based on the SFPE (10).

Stochastic fixed point equations arising in related problems.

1. GARCH processes. The typical objective in these financial models is to determine the fluctuations of a stock (or related financial process, such as an interest rate process) using a more realistic model than the Black-Scholes model. We first recall that under the Black-Scholes model, the price of a stock at time t is modeled according to the stochastic differential equation

$$dS_t = S_t \left(r dt + \sigma dW_t \right),$$

where $\{W_t\}_{t \geq 0}$ is standard Brownian motion and the volatility, σ, is assumed to be constant. Integrating this stochastic differential equation, using Ito's formula, we obtain that

$$S_T = S_t \exp\left\{ (r - \sigma^2/2)(T - t) + \sigma(W_T - W_t) \right\}, \quad 0 \leq t \leq T.$$

Then the *logarithmic returns* over the nth discrete time interval are given by

$$\widetilde{\mathscr{R}}_n := \log\left(\frac{S_n}{S_{n-1}}\right) = \left(r - \frac{\sigma^2}{2}\right) + \sigma Z_n, \quad n = 1, 2, \ldots,$$

where $\{Z_i\}$ is an i.i.d. sequence of standard Normal random variables. Normalizing this sequence so that the elements each have mean zero, we obtain that the normalized logarithmic returns are given by

$$\mathscr{R}_n = \widetilde{R}_n - \left(r - \frac{\sigma^2}{2}\right) = \sigma Z_n. \tag{11}$$

In reality, the assumption of a constant volatility in (11) is not particularly realistic, and various alternative models have been suggested, where the volatility is allowed to be *stochastic*. The ARCH and GARCH models are particular examples of this type. Specifically, in the ARCH(1) model, proposed in the famous paper of Engle (1982), one assumes that

$$\mathscr{R}_n = \sigma_n Z_n, \quad n = 1, 2, \ldots, \tag{12}$$

where $\{Z_i\}$ is an i.i.d. sequence of standard Normal random variables, and $\{\sigma_n\}$ satisfies the recursive equation

$$\sigma_n^2 = a + b\mathscr{R}_{n-1}^2, \quad n = 1, 2, \ldots, \tag{13}$$

for certain positive constants a and b. This model is motivated by the observation that the logarithmic returns display correlation with respect to the squared returns, but not with respect to the actual returns (i.e., with respect to $\mathscr{R}_{n-1}^2$, although not with respect to $\mathscr{R}_{n-1}$). Continuing in this manner, the GARCH(1,1) model of Bollerslev (1986) suggested that we model

$$\mathscr{R}_n = \sigma_n Z_n, \quad n = 1, 2, \ldots, \tag{14}$$

where $\{Z_i\}$ is an i.i.d. sequence of standard Normal random variables, and $\{\sigma_n\}$ satisfies the recursive equation

$$\sigma_n^2 = a_0 + b_1\sigma_{n-1}^2 + a_1\mathscr{R}_{n-1}^2, \quad n = 1, 2, \ldots. \tag{15}$$

In these models, one typically would like to determine the extremal behavior of the given process in *steady state*, i.e.,

$$\mathbf{P}\{\mathscr{R} > u\}, \tag{16}$$

where $\mathscr{R} := \lim_{n\to\infty} \mathscr{R}_n$ and u is large. Since Z_n is independent of σ_n, it follows that $\mathscr{R} \stackrel{d}{=} (\lim_{n\to\infty} \sigma_n) Z$, where Z is a standard Normal random variable, independent of $\lim_{n\to\infty} \sigma_n$. Thus, to determine (16), it is sufficient to characterize $\lim_{n\to\infty} \sigma_n$.

To this end, observe that in the GARCH(1,1) model, we obtain upon substituting (14) into (15) that

$$\begin{aligned} \sigma_n^2 &= a_0 + b_1\sigma_{n-1}^2 + a_1\left(\sigma_{n-1}^2 Z_{n-1}^2\right) \\ &= (b_1 + a_1 Z_{n-1}^2)\sigma_{n-1}^2 + a_0; \end{aligned}$$

that is,

$$\sigma_n^2 = A_n\sigma_{n-1}^2 + B_n, \quad \text{where} \quad A_n := b_1 + a_1 Z_{n-1}^2 \quad \text{and} \quad B_n := a_0. \tag{17}$$

Iterating (17) yields that

$$\begin{aligned} \sigma_n^2 &= A_n\sigma_{n-1}^2 + B_n = A_n\left(A_{n-1}\sigma_{n-2}^2 + B_{n-1}\right) + B_n \\ &= \cdots = (A_n \cdots A_1)\sigma_0^2 + (A_n \cdots A_2)B_1 + \cdots + A_n B_{n-1} + B_n \\ &\stackrel{d}{=} (A_1 \cdots A_n)\sigma_0^2 + (A_1 \cdots A_{n-1})B_n + \cdots + A_1 B_2 + B_1, \end{aligned} \tag{18}$$

where the last step follows since

$$(A_1, B_1; \ldots; A_n, B_n) \stackrel{d}{=} (A_n, B_n; \ldots; A_1, B_1).$$

Then

$$\sigma_n^2 \stackrel{d}{=} B_1 + A_1\left(A_2 \cdots A_n \sigma_0^2 + A_2 \cdots A_{n-1} B_n + \cdots + B_2\right). \tag{19}$$

Set

$$W_n = (A_1 \cdots A_n)\sigma_0^2 + (A_1 \cdots A_{n-1})B_n + \cdots + A_1 B_2 + B_1, \quad \text{and}$$
$$W_n^{(1)} = (A_2 \cdots A_{n+1})\sigma_0^2 + (A_2 \cdots A_n)B_{n+1} + \cdots + A_2 B_3 + B_2, \quad n = 1, 2, \ldots.$$

[Thus, as before, $W_n^{(1)}$ denotes that the input sequence (A_i, B_i) has been shifted forward by one time unit.] Note by the last line in (18) that $W_n \stackrel{d}{=} \sigma_n^2$, and hence $W_n^{(1)} \stackrel{d}{=} \sigma_n^2$. Hence, substituting into (18), we obtain that

$$\sigma^2 := \lim_{n\to\infty} \sigma_n^2 = B_1 + A_1 \lim_{n\to\infty} W_n^{(1)} \stackrel{d}{=} B + A\sigma^2, \tag{20}$$

where $(A, B) \stackrel{d}{=} (A_1, B_1)$ and independent of $\{\sigma_n^2\}$. Then

$$\sigma^2 \stackrel{d}{=} \Phi(\sigma^2), \quad \text{where} \quad \Phi(w) = Aw + B. \tag{21}$$

Moreover, for this model, it is typically assumed that $a_0 + a_1 + b_1 < 1$, and in that case one can show using (18) that σ^2 is finite a.s., i.e., $\lim_{n\to\infty} \sigma_n^2$ converges to a *proper* random variable.

A similar reasoning can be applied to the ARCH(1) model. In that case, one obtains that

$$\mathscr{R}^2 \stackrel{d}{=} A\mathscr{R}^2 + B, \tag{22}$$

where $A \stackrel{d}{=} bZ_1^2$, $B_n \stackrel{d}{=} aZ_n^2$, and $\mathscr{R} = \lim_{n\to\infty} \mathscr{R}_n$, which has the same form as the SFPE obtained for the GARCH(1,1) process. This SFPE is called the "affine" SFPE and arises in various other contexts as well.

2. The classical ruin problem. As a final example, we revisit the classical ruin problem. Recall from the random walk representation that the probability of ruin is given by

$$\psi(u) = \mathbf{P}\left\{S_n < -u, \ \text{for some } n = 0, 1, \ldots\right\} \tag{23}$$

for

$$S_n = \mathscr{Z}_1 + \cdots + \mathscr{Z}_n, \quad n = 1, 2, \ldots,$$

where

$$\mathscr{Z}_i = c\tau_i - Y_i, \quad i = 1, 2, \ldots,$$

and c denotes the premiums income rate, τ_i denotes the ith interarrival time (i.e., the time between the $(i-1)$st claim and the ith claim), and Y_i denotes the size of the ith claim. Furthermore, according to the net profit condition, it is assumed that $\mathbf{E}[\mathscr{Z}_i] > 0$.

To derive a corresponding stochastic fixed point equation for this problem, we need to first introduce the so-called *dual process*. Namely, let $W_0 = 0$ and set

$$W_n = (W_{n-1} - \mathscr{Z}_n)^+ \equiv \max\left\{0, W_{n-1} - \mathscr{Z}_n\right\}. \tag{24}$$

The process $\{W_n\}$ is called *reflected* random walk, since, whenever this process would become negative, it is reflected to the level zero.

Applying (24) twice, we obtain that

$$\begin{aligned} W_n &= \max\left\{0, W_{n-1} - \mathscr{Z}_n\right\} \\ &= \max\left\{0, \max\{0, W_{n-2} - \mathscr{Z}_{n-1}\} - \mathscr{Z}_n\right\} \\ &= \max\left\{0, W_{n-2} - \mathscr{Z}_{n-1} - \mathscr{Z}_n, -\mathscr{Z}_n\right\}. \end{aligned} \tag{25}$$

Now continue in this fashion; namely, use (24) to substitute for W_{n-2} in (25), and continue until obtaining $W_0 = 0$. This yields

$$W_n = \max\left\{0, -(\mathscr{Z}_1 + \cdots + \mathscr{Z}_n), \ldots, -(\mathscr{Z}_{n-1} + \mathscr{Z}_n), -\mathscr{Z}_n\right\}. \tag{26}$$

Moreover, since

$$(\mathscr{Z}_1, \ldots, \mathscr{Z}_n) \stackrel{d}{=} (\mathscr{Z}_n, \ldots, \mathscr{Z}_1),$$

it follows from (26) that

$$W_n \stackrel{d}{=} \max\left\{0, -(\mathscr{Z}_n + \cdots + \mathscr{Z}_1), \ldots, -(\mathscr{Z}_2 + \mathscr{Z}_1), -\mathscr{Z}_1\right\} \qquad = \max_{0 \le k \le n} (-S_k) = -\min_{0 \le k \le n} S_k. \tag{27}$$

From this last equation, we conclude that

$$\begin{aligned}\mathbf{P}\left\{W_n > u\right\} &= \mathbf{P}\left\{\min_{0 \le k \le n} S_k < -u\right\} \\ &= \mathbf{P}\left\{S_k < -u, \text{ for some } k = 0, \ldots, n\right\},\end{aligned}$$

and hence, upon setting $W = \lim_{n\to\infty} W_n$ and letting $n \to \infty$, we obtain that

$$\begin{aligned}\mathbf{P}\left\{W > u\right\} := \mathbf{P}\left\{\lim_{n\to\infty} W_n > u\right\} &= \lim_{n\to\infty} \mathbf{P}\left\{W_n > u\right\} \\ &= \lim_{n\to\infty} \mathbf{P}\left\{S_k < -u, \text{ for some } k = 0, \ldots, n\right\} \\ &= \mathbf{P}\left\{S_k < -u, \text{ for some } k = 0, 1, \ldots\right\} := \psi(u).\end{aligned}$$

Thus, if the insurance company begins with an initial capital of u, then the the probability of ruin may be equated to $\mathbf{P}\left\{W > u\right\}$, where $\{W_n\}$ denotes reflected random walk and $W = \lim_{n\to\infty} W_n$.

We now derive a stochastic fixed point equation satisfied by $V := e^W$. To this end, set $V_n = e^{W_n}$ and observe that

$$\begin{aligned}V_n := e^{W_n} &:= \exp\left\{\max\{0, W_{n-1} - \mathscr{Z}_n\}\right\} \\ &= \max\left\{1, e^{W_{n-1} - \mathscr{Z}_n}\right\} = \max\left\{1, A_n V_{n-1}\right\},\end{aligned} \tag{28}$$

where $A_n := e^{-\mathscr{Z}_n}$. Now if A is a random variable with $A \stackrel{d}{=} A_n$, but where A is independent of the process $\{V_n\}$, then it follows from (28) that

$$V_n \stackrel{d}{=} \max\left\{1, AV_{n-1}\right\}.$$

Then letting $n \to \infty$ yields

$$V \stackrel{d}{=} \lim_{n\to\infty} \max\left\{1, AV_{n-1}\right\} = \max\left\{1, AV\right\};$$

that is,

$$V \stackrel{d}{=} \Phi(V), \quad \text{where} \quad \Phi(v) = \max\left\{1, Av\right\}. \tag{29}$$

Solving stochastic fixed point equations: tail estimates. The previous examples illustrate a general problem; namely, we would like to determine the tail probabilities of a random variable V (that is, $\mathbf{P}\left\{V > u\right\}$ for large u), and what is known about V is that it satisfies the stochastic fixed point equation

$$V \stackrel{d}{=} \Phi(V),$$

where Φ is a known random function. Moreover, in each of these examples, we have that $\Phi(v) = Av(1 + o(v))$ as $v \to \infty$. In other words, if v is large, then $\Phi(v) \approx Av$, so that, roughly speaking, Φ

involves multiplication by a random variable A, and thus resembles the process $M_n := (A_n \cdots A_1)M_0$ when not too close to the origin. The process $\{M_n\}$ is called *multiplicative random walk*, and its large exceedance probabilities can be studied using methods from renewal theory. Thus, it is natural to employ a generalization of the classical renewal theorem to study the tail probability $\mathbf{P}\{V > u\}$ as $u \to \infty$.

Assumptions:

1. There exists a random variable $A > 0$ such that $\Phi(v) = Av(1 + o(v))$ as $v \to \infty$.
2. The random variable $\log A$ is nonarithmetic, $\mathbf{E}[\log A] < 0$, and
$$\Lambda(\alpha) := \log \mathbf{E}\left[e^{\alpha \log A}\right]$$
is finite for some $\alpha > 0$. Moreover there is a *positive* solution to the equation $\Lambda(\mathfrak{R}) = 0$, where $\Lambda'(\mathfrak{R}) < \infty$. In analogy with the classical ruin problem, we call the constant $\mathfrak{R}$ the *adjustment coefficient.*
3. We have
$$\mathbf{E}\left[\left(\Phi(V)^+\right)^{\mathfrak{R}} - (AV^+)^{\mathfrak{R}}\right] < \infty. \tag{30}$$

This last condition states, in a more explicit mathematical sense, that $\Phi(V)$ behaves similarly to AV for large values of V.

The main result of this section is the following theorem, first established by Kesten (1973) and Goldie (1991).

Theorem 1.1. *Let V satisfy the stochastic fixed point equation $V \stackrel{d}{=} \Phi(V)$, and suppose that the above assumptions are satisfied. Then*
$$\mathbf{P}\{V > u\} \sim C u^{-\mathfrak{R}} \quad \text{as} \quad u \to \infty, \tag{31}$$
where
$$C = \frac{1}{\mathfrak{R}\Lambda'(\mathfrak{R})}\mathbf{E}\left[\left(\Phi(V)^+\right)^{\mathfrak{R}} - \left(AV^+\right)^{\mathfrak{R}}\right] \in [0, \infty). \tag{32}$$

To obtain a rough idea of the proof, we assume that $V \geq 0$ and then observe that for any $v \in \mathbb{R}$,
$$\begin{aligned}\mathbf{P}\{V > e^v\} &= \mathbf{P}\{\Phi(V) > e^v\} \\ &= \Big(\mathbf{P}\{\Phi(V) > e^v\} - \mathbf{P}\{AV > e^v\}\Big) + \mathbf{P}\{AV > e^v\}.\end{aligned} \tag{33}$$

Moreover, letting F denote the distribution function of $\log A$, we also have
$$\mathbf{P}\{AV > e^v\} = \mathbf{P}\left\{V > e^{v - \log A}\right\} = \int_{-\infty}^{\infty} \mathbf{P}\left\{V > e^{v-x}\right\} dF(x). \tag{34}$$

Substituting (34) into (33) yields
$$\mathbf{P}\{V > e^v\} = \Big(\mathbf{P}\{\Phi(V) > e^v\} - \mathbf{P}\{AV > e^v\}\Big) + \int_{-\infty}^{\infty} \mathbf{P}\left\{V > e^{v-x}\right\} dF(x), \tag{35}$$

or
$$Z(v) = z(v) + \int_{-\infty}^{\infty} Z(v - x) dF(x), \tag{36}$$

where
$$Z(v) := \mathbf{P}\{V > e^v\} \quad \text{and} \quad z(v) := \mathbf{P}\{\Phi(V) > e^v\} - \mathbf{P}\{AV > e^v\}.$$

Eq. (36) can be recognized as "nearly" the renewal equation. One issue is that F denotes the distribution function of the random variable $\log A$, and $\mathbf{E}[\log A] < 0$, whereas in the classical renewal equation, F would denote the distribution function of a *nonnegative* random variable, and hence the mean of this random variable would be positive. We can partly remedy this problem by introducing a change of measure. Namely, multiply (36) by $e^{\Re v}$ to obtain that

$$e^{\Re v}Z(v) = e^{\Re v}z(v) + \int_{-\infty}^{\infty} e^{\Re(v-x)}Z(v-x)e^{\Re x}dF(x),$$

or

$$Z_{\Re}(v) = z_{\Re}(v) + \int_{-\infty}^{\infty} Z_{\Re}(v-x)dF_{\Re}(x), \tag{37}$$

where

$$Z_{\Re}(v) = e^{\Re v}Z(v), \quad z_{\Re}(v) = e^{\Re v}z(v), \quad \text{and} \quad dF_{\Re}(x) = e^{\Re x}dF(x).$$

It can be checked that $F_{\Re}$ is a probability measure for a random variable with mean $\Lambda'(\Re)$, which is *positive.*

While (37) is "closer" to the true renewal equation, it is still not a proper renewal equation since (i) the function $z_{\Re}$ is *unknown*, so we have no way of knowing whether it is directly Riemann integrable, as would be needed to apply the renewal theorem, and (ii) the integral is over the range $(-\infty, \infty)$, not $[0, \infty)$.

Nonetheless, it was shown by Goldie (1991) (following earlier work by Kesten) that these difficulties can be circumvented using "implicit" renewal theory, where the function $z_{\Re}$ is "smoothed" to assure that it is directly Riemann integrable, and the range of the integrand is handled by a nonclassical argument. In the end, the conclusion is the same as we would expect by applying the renewal theorem to (37); namely, for an appropriate constant C,

$$Z_{\Re}(v) \to C \quad \text{as} \quad v \to \infty,$$

or

$$Z(v) \sim Ce^{-\Re v} \quad \text{as} \quad v \to \infty.$$

Finally recall that $Z(v) := \mathbf{P}\{V > e^v\}$. Thus, setting $u = e^v$, we conclude from the previous equation that

$$\mathbf{P}\{V > u\} \sim Cu^{-\Re} \quad \text{as} \quad u \to \infty.$$

Examples. We now return to the three motivating examples stated at the beginning.

1. Ruin with stochastic investments. We conclude from the main theorem that the ruin probability decays as

$$\psi(u) \sim Cu^{-\Re} \quad \text{as} \quad u \to \infty,$$

where $\Lambda(\Re) := \log \mathbf{E}\left[e^{\Re \log A}\right] = 0$, or $\mathbf{E}\left[A^{\Re}\right] = 1$. Note that the decay rate $\Re$ is *determined entirely by the investment returns*, and not by the insurance business itself. In place of the net profit condition, we assume that $\mathbf{E}[\log A] < 0$, or $\mathbf{E}[\log R] > 0$, where $R \stackrel{d}{=} R_1$, i.e. $\log R$ describes the logarithmic returns on the investments, which must be *positive.* While the insurance business does not affect the decay constant $\Re$, it nonetheless *does* influence the constant C.

Another interesting observation is that the decay is polynomial rather than exponential. This results from the multiplicative effect of the investment returns. Thus, in this problem, the probability of ruin will always resemble ruin for heavy-tailed claims, although the reason for the polynomial decay is different, resulting from the investment returns which, incidentally, need not be themselves heavy-tailed.

2. The GARCH(1,1) process. Here we conclude that

$$\mathbf{P}\left\{\sigma^2 > u\right\} \sim Cu^{-\Re}$$

for certain constants C and $\Re$, which shows that σ^2 (and hence $\mathscr{R} \stackrel{d}{=} \sigma^2 Z$) has polynomially-decaying tails.

3. The classical ruin problem. Based on our arguments above, the probability of ruin is given by

$$\psi(u) = \mathbf{P}\{W > u\} = \mathbf{P}\{\log V > u\},$$

where

$$V \stackrel{d}{=} \max\{1, AV\} \quad \text{and} \quad A = e^{-\mathscr{Z}}$$

(for $\mathscr{Z} \stackrel{d}{=} \mathscr{Z}_1$ and $S_n = \mathscr{Z}_1 + \cdots + \mathscr{Z}_n$). Then by the Goldie-Kesten theorem stated above, we obtain that

$$\mathbf{P}\{V > v\} \sim Cv^{-\mathfrak{R}} \quad \text{as} \quad v \to \infty,$$

and hence

$$\mathbf{P}\{W > u\} = \mathbf{P}\{\log V > u\} = \mathbf{P}\{V > e^u\} \sim Ce^{-\mathfrak{R}u} \quad \text{as} \quad u \to \infty.$$

We conclude by verifying that this rate of decay is consistent with the rate of decay we have already obtained in our previous discussion of the classical Cramér-Lundberg estimate. To this end, observe that the constant $\mathfrak{R}$ must satisfy

$$\Lambda(\mathfrak{R}) = \log \mathbf{E}\left[e^{\mathfrak{R}\log A}\right] = 0,$$

or $\mathbf{E}\left[A^{\mathfrak{R}}\right] = 1$. Since $A = e^{-\mathscr{Z}}$, this leads to the requirement that

$$\mathbf{E}\left[e^{-\mathfrak{R}\mathscr{Z}}\right] = 1,$$

i.e., $\mathfrak{R}$ is the positive solution to the equation

$$\widetilde{\Lambda}(\alpha) := \log \mathbf{E}\left[e^{\alpha\mathscr{Z}}\right] = 0.$$

This is the same requirement we imposed for the adjustment coefficient in the classical ruin problem. Thus, the Kesten-Goldie theorem is seen to be a generalization of the classical Cramér-Lundberg theorem to a wider class of problems. Note, however, that the constant C is no longer explicit and calculable, as it was in the classical theory. In particular, the formula given for C in the Kesten-Goldie theorem (cf. (32)) involves the unknown quantity V, and thus does not yield a useful expression for this constant. However, using an alternative approach based on Markov chain theory and nonlinear renewal theory, recent work has shown that this constant can be computed in a variety of examples, such as the three examples considered here.