Remark to Black-Scholes formula

Using the set-up and notation from Chapter 7.6 in Björk, we have the following results.

Proposition. Let Y be a random variable that is standard normal distributed, that is, Y is normally distributed with mean 0 and variance 1. Let λ and c be real constants. Then

(i) $\mathbf{E}[e^{\lambda Y} 1_{\{Y>c\}}] = e^{\lambda^2/2}(1 - N[c-\lambda]) = e^{\lambda^2/2} N[\lambda - c]$

(ii) $\mathbf{E}[e^{\lambda Y} 1_{\{Y<c\}}] = e^{\lambda^2/2} N[c-\lambda]$

(iii) $\mathbf{E}[e^{\lambda Y}] = e^{\lambda^2/2}$

Proof. For (i), we have that

$$\mathbf{E}[e^{\lambda Y} 1_{\{Y>c\}}] = \int_c^\infty e^{\lambda y} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy.$$

Substituting $y = z + \lambda$ gives

$$\mathbf{E}[e^{\lambda Y} 1_{\{Y>c\}}] = \int_{c-\lambda}^\infty e^{\lambda(z+\lambda)} \frac{1}{\sqrt{2\pi}} e^{-(z+\lambda)^2/2}\, dz = e^{\lambda^2/2} \int_{c-\lambda}^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = e^{\lambda^2/2}(1 - N[c-\lambda]).$$

(ii) is derived by same arguments.
(iii) is obtained by adding (i) and (ii).

Black-Scholes formula for a call option. The price of a European call option with strike K and maturity T is given by $\Pi(t) = F(t, S(t))$ where

$$F(t,s) = sN[d_1(t,s)] - Ke^{-r(T-t)} N[d_2(t,s)]$$

and

$$d_{1,2}(t,s) = \frac{1}{\sigma\sqrt{T-t}} \Big(\log\Big(\frac{s}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)(T-t) \Big).$$

Proof. Under the martingale measure, the stock price $S(T)$ is a geometric Brownian motion with the explicit expression $S(T) = se^{\sigma(W(T)-W(t)) + (r-\sigma^2/2)(T-t)}$. If Y is standard normally distributed then $S(T)$ and $se^{\sigma\sqrt{T-t}Y + (r-\sigma^2/2)(T-t)}$ have the same distribution. Hence

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}[S(T)1_{\{S(T)>K\}}] &= se^{(r-\sigma^2/2)(T-t)} \mathbf{E}[e^{\sigma\sqrt{T-t}Y} 1_{\{Y > \frac{\log(K/s) - (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\}}] \\
&= se^{(r-\sigma^2/2)(T-t)} e^{\sigma^2(T-t)/2} N[\sigma\sqrt{T-t} - \frac{\log(K/s) - (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}] \\
&= se^{r(T-t)} N[d_1(t,s)]
\end{aligned}$$

and

$$\mathbf{E}^{\mathbf{Q}}[1_{\{S(T)>K\}}] = \mathbf{P}(Y > \frac{\log(K/s) - (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}) = \mathbf{P}(-Y < d_2(t,s)) = N[d_2(t,s)].$$

Using Risk neutral valuation formula we get that

$$\begin{aligned}
F(t,s) &= e^{-r(T-t)} \mathbf{E}^{\mathbf{Q}}[(S(T)-K)^+] \\
&= e^{-r(T-t)} (\mathbf{E}^{\mathbf{Q}}[S(T)1_{S(T)>K}] - K\mathbf{E}^{\mathbf{Q}}[1_{\{S(T)>K\}}]) \\
&= sN[d_1(t,s)] - Ke^{-r(T-t)} N[d_2(t,s)].
\end{aligned}$$