Exam-prep (FinKont)

Joakim Bilyk

2023-01-23

Contents

Exam sets (2017/18-2021/22)

In progress

Problem 2

Consider a standard Black-Scholes model, that is, a model consisting of a bank account B_t with P-dynamics given by

$$dB_t = rB_t\ dt,\ B_0 = 1$$

and a stock S_t with P-dynamics given by

$$dS_t = \alpha S_t\ dt + \sigma S_t\ d\overline{W}_t,\ S_0 = s > 0$$

where $r, \alpha \in \mathbb{R}$ and $\sigma > 0$ are constants and $\overline{W}_t$ is a P-Brownian motion. Let $T > 0$ be a given and fixed date.

Consider the derivative that at time T pays

$$X = \max\left\{\min\left\{S_T, K_2\right\}, K_1\right\},$$

where $0 < K_1 < K_2$ are constants.

a. Determine the arbitrage free price of derivative X at time $t < T$.

Consider a new derivative that at time T pays

$$Y = (S_T^2 - K^2)^+ - (K^2 - S_T^2)^+.$$

b. i. Determine the arbitrage free price of derivative Y at time $t < T$.

ii. Find a hedging portfolio for derivative Y.

Solution (a).

We see that the derivative is the bull spread given by the payout function

$$X = \begin{cases} K_2 & \text{if } S_T > K_2, \\ S_T & \text{if } K_1 \leq S_T \leq K_2, \\ K_1 & \text{if } S_T < K_1. \end{cases}$$

We know from exercise 10.3 that this can be replicated by holding K_1 bonds, one call option with strike K_1 and a short on a call with strike K_2. The arbitrage free price of the derivative is then the value process of the mentioned portfolio i.e.

$$\Pi_t[X] = K_1 e^{-r(T-t)} + c(K_1; t, T) - c(K_2; t, T),$$

where c denotes the pricing function for a European call option (non-instructive parameters supressed). □

Solution (b).

(i): We start by seeing that the derivative pays out

$$Y = \begin{cases} S_T^2 - K^2 & \text{if } S_T^2 \geq K^2, \\ -(K^2 - S_T^2) & \text{if } S_T^2 < K^2. \end{cases}$$

hence the payout is $Y = S_T^2 - K^2 = \Phi(S_T)$ where $\Phi(s) = s^2 - K^2$. That is Y is in fact a simple claim. We have from the risk neutral valueation formula 7.11 that

$$\begin{aligned} \Pi_t[Y] &= e^{-r(T-t)} E_{t,s}^Q[S_T^2 - K^2] \\ &= e^{-r(T-t)} E_{t,s}^Q[S_T^2] - e^{-r(T-t)} K^2. \end{aligned}$$

Recall that under the martingale measure Q we have that S_t is a GBM hence

$$S_t = s \cdot \exp\left\{ \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W_T^Q - W_t^Q\right) \right\}$$

then

$$S_T^2 = s^2 \cdot \exp\left\{ 2\left(r - \frac{1}{2}\sigma^2\right)(T-t) + 2\sigma\left(W_T^Q - W_t^Q\right) \right\}.$$

Inserting this into the risk neutral valuation formula we get

$$\begin{aligned} \Pi_t[Y] &= e^{-r(T-t)} E_{t,s}^Q[S_T^2] - e^{-r(T-t)} K^2 \\ &= e^{-r(T-t)} s^2 e^{2\left(r - \frac{1}{2}\sigma^2\right)(T-t)} E^Q\left[\exp\left\{2\sigma\left(W_T^Q - W_t^Q\right)\right\}\right] - e^{-r(T-t)} K^2 \\ &= e^{-r(T-t)} s^2 e^{2\left(r - \frac{1}{2}\sigma^2\right)(T-t)} e^{\frac{1}{2}4\sigma^2(T-t)} - e^{-r(T-t)} K^2 \\ &= e^{-r(T-t)} \left(s^2 e^{(2r - \sigma^2)(T-t) + \frac{1}{2}4\sigma^2(T-t)} - K^2\right) \\ &= e^{-r(T-t)} \left(s^2 e^{(2r + \sigma^2)(T-t)} - K^2\right). \end{aligned}$$

The arbitrage free price of the derivative is then given above. □

(ii):

Solution (c).

Solution (d).

Problem 3

Solution (a).

Solution (b).

Solution (c).

Exam 2017/18

Problem 1

Let W_t denote a Brownian motion and let

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma(\{W_s \mid 0 \leq s \leq t\}).$$

Let $T > 0$ be a given and fixed time.

Let $f(t)$ be a bounded deterministic continuous function. Define the two processes

$$\begin{cases} X_t = \int_0^t f(u)\ dW_u, \\ M_t^{(\lambda)} = \exp\left\{\lambda X_t - \frac{\lambda^2}{2}\int_0^t f^2(u)\ du\right\}, \end{cases}$$

where $\lambda \in \mathbb{R}$ is a constant.

a. Show that $M^{(\lambda)}$ is a martingale with $E[M_t^{(\lambda)}] = 1$.

Let $0 < s < t$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ be given and fixed.

b. i. Show that

$$\begin{aligned} M_s^{(\lambda_1)} &= E\left[\frac{M_s^{(\lambda_1)} M_t^{(\lambda_2)}}{M_s^{(\lambda_2)}}\ \middle|\ \mathcal{F}_s\right] \\ &= E\left[\exp\left\{\lambda_1 X_s + \lambda_2(X_t - X_s) - \frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du - \frac{1}{2}\lambda_2^2 \int_s^t f^2(u)\ du\right\}\ \middle|\ \mathcal{F}_s\right] \end{aligned}$$

ii. Show that X_s and $X_t - X_s$ are normally distributed and independent.

c. Compute the mean value of $M_T^{(\lambda)} \log(M_T^{(\lambda)})$.

Solution (a).

First, we see that since X_t is on integral form we know that

$$\begin{cases} dX_t = f(t)\ dW_t \\ X_0 = 0. \end{cases}$$

Hence we may represent M as $M_t^{(\lambda)} = g(t, X_t, Y_t)$ given by

$$g(t,x,y) = \exp\left\{\lambda x - \frac{\lambda^2}{2} y\right\},$$

where $Y_t = \int_0^t f^2(u)\ du$ with dynamics

$$\begin{cases} dY_t = f^2(t)\ dt \\ Y_0 = 0. \end{cases}$$

Hence by the multidimensional Ito's formula we have the dynamics of M given by

$$\begin{aligned} dM_t^{(\lambda)} &= g_t\ dt + g_x\ dX_t + g_y\ dY_t + \frac{1}{2} g_{yy}\ (dY_t)^2 + \frac{1}{2} g_{xx}\ (dX_t)^2 + f_{xy}(dX_t)(dY_t) \\ &= 0 + \lambda g\ dX_t - \frac{\lambda^2}{2} g\ dY_t + 0 + \frac{1}{2}\lambda^2 g\ (dX_t)^2 + 0 \\ &= \lambda M_t^{(\lambda)} f(t)\ dW_t - \frac{1}{2}\lambda^2 M_t^{(\lambda)} f^2(t)\ dt + \frac{1}{2}\lambda M_t^{(\lambda)} f^2(t)\ dt \\ &= \lambda f(t) M_t^{(\lambda)}\ dW_t, \end{aligned}$$

And so we see that M is a martingale as it only has dynamics wrt. the Brownian motion W (assuming $\lambda f_t M_t^{(\lambda)} \in \mathcal{L}^2$). Furthermore we have that

$$M_0^{(\lambda)} = g(0, X_0, Y_0) = \exp\left\{\lambda X_0 - \frac{1}{2}\lambda^2 Y_0\right\} = e^0 = 1$$

and so we have $E[M_t^{(\lambda)}] = M_0^{(\lambda)} = 1$ as desired. $\square$

Solution (b).

"(i)" We have from the previous exercise

$$\begin{aligned} &\frac{M_s^{(\lambda_1)} M_t^{(\lambda_2)}}{M_s^{(\lambda_2)}} \\ &= \exp\left\{\lambda_1 X_s - \frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du\right\} \exp\left\{\lambda_2 X_t - \frac{1}{2}\lambda_2^2 \int_0^t f^2(u)\ du\right\} \exp\left\{\frac{1}{2}\lambda_2^2 \int_0^s f^2(u)\ du - \lambda_2 X_s\right\} \\ &= \exp\left\{\lambda_1 X_s - \frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du + \lambda_2 X_t - \frac{1}{2}\lambda_2^2 \int_0^t f^2(u)\ du + \frac{1}{2}\lambda_2^2 \int_0^s f^2(u)\ du - \lambda_2 X_s\right\} \\ &= \exp\left\{\lambda_1 X_s + \lambda_2 (X_t - X_s) - \frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du - \frac{1}{2}\lambda_2^2 \int_s^t f^2(u)\ du\right\} \end{aligned}$$

and so the conclusion follows. $\square$

"(ii)" We have that from lemma 4.18 that

$$X_s = \int_0^s f(u)\ dW_u \sim \mathcal{N}\left(0, \int_0^s f^2(u)\ dW_u\right)$$

furthermore we have that

$$X_t - X_s = \int_s^t f(u)\ dW_u \sim \mathcal{N}\left(0, \int_s^t f^2(u)\ dW_u\right).$$

In regard to the independence claim we could check identity below

$$E[e^{t_1 X} e^{t_2 Y}] = E[e^{t_1 X}] E[e^{t_2 Y}]$$

where X, Y are independent random variables. The above identity holds if and only if X and Y are independent. From above we have that

$$M_s^{(\lambda_1)} = E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)} \mid \mathcal{F}_s] e^{-\frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du - \frac{1}{2}\lambda_2^2 \int_s^t f^2(u)\ du}$$

and so taking expectation we have

$$1 = E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)}] e^{-\frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du - \frac{1}{2}\lambda_2^2 \int_s^t f^2(u)\ du}$$

Which the gives

$$E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)}] = e^{\frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du + \frac{1}{2}\lambda_2^2 \int_s^t f^2(u)\ du} = E[e^{\lambda_1 X_s}] E[e^{\lambda_2 (X_t - X_s)}]$$

and so the conclusion is that X_s and $X_t - X_s$ are independent. $\square$

Solution (c).

We recall the definition of $M_t^{(\lambda)}$ and observe that

$$\log M_t^{(\lambda)} = \lambda X_t - \frac{1}{2}\lambda^2 \int_0^t f^2(u)\ du.$$

Furthermore we have the dynamics of $M^{(\lambda)}$ given by the differential form

$$dM_t^{(\lambda)} = \lambda f(t) M_t^{(\lambda)}\ dW_t.$$

with $M_0^{(\lambda)} = 1$. Since we know that $M_t^{(\lambda)}$ is a martingale we have

$$E^P[M_T^{(\lambda)}] = E^P[M_0^{(\lambda)}] = 1,$$

and so we may define a new probability measure as

$$d\tilde{P} = M_T^{(\lambda)}\ dP$$

on $\mathcal{F}_T$. We then have a new Brownian motion $\tilde{W}$ such that

$$dW_t = \lambda f(t)\ dt + d\tilde{W}_t.$$

We can then see

$$
\begin{aligned}
E^P[M_T^{(\lambda)} \log M_T^{(\lambda)}] &= \int M_T^{(\lambda)} \log M_T^{(\lambda)} \; dP = \int M_T^{(\lambda)} \log M_T^{(\lambda)} \frac{1}{M_T^{(\lambda)}} \; d\tilde{P} \\
&= \int \log M_T^{(\lambda)} \; d\tilde{P} = E^{\tilde{P}}[\log M_T^{(\lambda)}].
\end{aligned}
$$

Then we can evaluate the mean value by seeing the X has representation wrt. $\tilde{P}$ by

$$
X_t = \int_0^t f(u) \; (\lambda f(u) \; du + d\tilde{W}_u) = \lambda \int_0^t f^2(u) \; du + \int_0^t f(u) \; d\tilde{W}_u.
$$

Giving that

$$
\begin{aligned}
E^P[M_T^{(\lambda)} \log M_T^{(\lambda)}] &= E^{\tilde{P}}[\log M_T^{(\lambda)}] \\
&= E^{\tilde{P}} \left[\lambda X_T - \frac{1}{2}\lambda^2 \int_0^T f^2(u) \; du \right] \\
&= E^{\tilde{P}} \left[\lambda^2 \int_0^T f^2(u) \; du + \lambda \int_0^T f(u) \; d\tilde{W}_u - \frac{1}{2}\lambda^2 \int_0^T f^2(u) \; du \right] \\
&= \lambda E^{\tilde{P}} \left[\frac{1}{2}\lambda \int_0^T f^2(u) \; du + \int_0^T f(u) \; d\tilde{W}_u \right] \\
&= \frac{1}{2}\lambda^2 \int_0^T f^2(u) \; du + \lambda E^{\tilde{P}} \left[\int_0^T f(u) \; d\tilde{W}_u \right] \\
&= \frac{1}{2}\lambda^2 \int_0^T f^2(u) \; du
\end{aligned}
$$

Since

$$
\tilde{X}_T = \int_0^T f(u) \; d\tilde{W}_u,
$$

is a $\tilde{P}$-martingale. $\square$

Exam 2018/19

Some content

Exam 2019/20

Some content

Exam 2020/21

Some content

Exam 2021/22

Some content

Exam 2022/23

Some content