

UNIVERSITY OF COPENHAGEN

CONTINUOUS TIME FINANCE (FINKONT)

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## Exam Papers

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JANUARY 27, 2023



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## Exam sets

### In progress

#### Exam 2017/18

##### Problem 1

Let  $W_t$  denote a Brownian motion and let

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma(\{W_s \mid 0 \leq s \leq t\}).$$

Let  $T > 0$  be a given and fixed time.

Let  $f(t)$  be a bounded deterministic continuous function. Define the two processes

$$\begin{cases} X_t = \int_0^t f(u) dW_u, \\ M_t^{(\lambda)} = \exp \left\{ \lambda X_t - \frac{\lambda^2}{2} \int_0^t f^2(u) du \right\}, \end{cases}$$

where  $\lambda \in \mathbb{R}$  is a constant.

a. Show that  $M^{(\lambda)}$  is a martingale with  $E[M_t^{(\lambda)}] = 1$ .

Let  $0 < s < t$  and  $\lambda_1, \lambda_2 \in \mathbb{R}$  be given and fixed.

b. i. Show that

$$\begin{aligned} M_s^{(\lambda_1)} &= E \left[ \frac{M_s^{(\lambda_1)} M_t^{(\lambda_2)}}{M_s^{(\lambda_2)}} \mid \mathcal{F}_s \right] \\ &= E \left[ \exp \left\{ \lambda_1 X_s + \lambda_2 (X_t - X_s) - \frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du - \frac{1}{2} \lambda_2^2 \int_s^t f^2(u) du \right\} \mid \mathcal{F}_s \right] \end{aligned}$$

ii. Show that  $X_s$  and  $X_t - X_s$  are normally distributed and independent.

c. Compute the mean value of  $M_T^{(\lambda)} \log(M_T^{(\lambda)})$ .

#### Solution (a).

First, we see that since  $X_t$  is on integral form we know that

$$\begin{cases} dX_t = f(t) dW_t \\ X_0 = 0. \end{cases}$$

Hence we may represent  $M$  as  $M_t^{(\lambda)} = g(t, X_t, Y_t)$  given by

$$g(t, x, y) = \exp \left\{ \lambda x - \frac{\lambda^2}{2} y \right\},$$

where  $Y_t = \int_0^t f^2(u) du$  with dynamics

$$\begin{cases} dY_t = f^2(t) dt \\ Y_0 = 0. \end{cases}$$

Hence by the multidimensional Ito's formula we have the dynamics of  $M$  given by

$$\begin{aligned}
dM_t^{(\lambda)} &= g_t dt + g_x dX_t + g_y dY_t + \frac{1}{2}g_{yy} (dY_t)^2 + \frac{1}{2}g_{xx} (dX_t)^2 + f_{xy}(dX_t)(dY_t) \\
&= 0 + \lambda g dX_t - \frac{\lambda^2}{2}g dY_t + 0 + \frac{1}{2}\lambda^2 g (dX_t)^2 + 0 \\
&= \lambda M_t^{(\lambda)} f(t) dW_t - \frac{1}{2}\lambda^2 M_t^{(\lambda)} f^2(t) dt + \frac{1}{2}\lambda M_t^{(\lambda)} f^2(t) dt \\
&= \lambda f(t) M_t^{(\lambda)} dW_t,
\end{aligned}$$

And so we see that  $M$  is a martingale as it only has dynamics wrt. the Brownian motion  $W$  (assuming  $\lambda f_t M_t^{(\lambda)} \in \mathcal{L}^2$ ). Furthermore we have that

$$M_0^{(\lambda)} = g(0, X_0, Y_0) = \exp \left\{ \lambda X_0 - \frac{1}{2} \lambda^2 Y_0 \right\} = e^0 = 1$$

and so we have  $E[M_t^{(\lambda)}] = M_0^{(\lambda)} = 1$  as desired.  $\square$

**Solution (b).**

“(i)” We have from the previous exercise

$$\begin{aligned}
&\frac{M_s^{(\lambda_1)} M_t^{(\lambda_2)}}{M_s^{(\lambda_2)}} \\
&= \exp \left\{ \lambda_1 X_s - \frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du \right\} \exp \left\{ \lambda_2 X_t - \frac{1}{2} \lambda_2^2 \int_0^t f^2(u) du \right\} \exp \left\{ \frac{1}{2} \lambda_2^2 \int_0^s f^2(u) du - \lambda_2 X_s \right\} \\
&= \exp \left\{ \lambda_1 X_s - \frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du + \lambda_2 X_t - \frac{1}{2} \lambda_2^2 \int_0^t f^2(u) du + \frac{1}{2} \lambda_2^2 \int_0^s f^2(u) du - \lambda_2 X_s \right\} \\
&= \exp \left\{ \lambda_1 X_s + \lambda_2 (X_t - X_s) - \frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du - \frac{1}{2} \lambda_2^2 \int_s^t f^2(u) du \right\}
\end{aligned}$$

and so the conclusion follows.  $\square$

“(ii)” We have that from lemma 4.18 that

$$X_s = \int_0^s f(u) dW_u \sim \mathcal{N} \left( 0, \int_0^s f^2(u) dW_u \right)$$

furthermore we have that

$$X_t - X_s = \int_s^t f(u) dW_u \sim \mathcal{N} \left( 0, \int_s^t f^2(u) dW_u \right).$$

In regard to the independence claim we could check identity below

$$E[e^{t_1 X} e^{t_2 Y}] = E[e^{t_1 X}] E[e^{t_2 Y}]$$

where  $X, Y$  are independent random variables. The above identity holds if and only if  $X$  and  $Y$  are independent. From above we have that

$$M_s^{(\lambda_1)} = E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)} \mid \mathcal{F}_s] e^{-\frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du - \frac{1}{2} \lambda_2^2 \int_s^t f^2(u) du}$$

and so taking expectation we have

$$1 = E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)}] e^{-\frac{1}{2}\lambda_1^2 \int_0^s f^2(u) du - \frac{1}{2}\lambda_2^2 \int_s^t f^2(u) du}$$

Which the gives

$$E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)}] = e^{\frac{1}{2}\lambda_1^2 \int_0^s f^2(u) du + \frac{1}{2}\lambda_2^2 \int_s^t f^2(u) du} = E[e^{\lambda_1 X_s}] E[e^{\lambda_2 (X_t - X_s)}]$$

and so the conclusion is that  $X_s$  and  $X_t - X_s$  are independent.  $\square$

**Solution (c).**

We recall the definition of  $M_t^{(\lambda)}$  and observe that

$$\log M_t^{(\lambda)} = \lambda X_t - \frac{1}{2}\lambda^2 \int_0^t f^2(u) du.$$

Furthermore we have the dynamics of  $M^{(\lambda)}$  given by the differential form

$$dM_t^{(\lambda)} = \lambda f(t) M_t^{(\lambda)} dW_t.$$

with  $M_0^{(\lambda)} = 1$ . Since we know that  $M_t^{(\lambda)}$  is a martingale we have

$$E^P[M_T^{(\lambda)}] = E^P[M_0^{(\lambda)}] = 1,$$

and so we may define a new probability measure as

$$d\tilde{P} = M_T^{(\lambda)} dP$$

on  $\mathcal{F}_T$ . We then have a new Brownian motion  $\tilde{W}$  such that

$$dW_t = \lambda f(t) dt + d\tilde{W}_t.$$

We can then see

$$\begin{aligned} E^P[M_T^{(\lambda)} \log M_T^{(\lambda)}] &= \int M_T^{(\lambda)} \log M_T^{(\lambda)} dP = \int M_T^{(\lambda)} \log M_T^{(\lambda)} \frac{1}{M_T^{(\lambda)}} d\tilde{P} \\ &= \int \log M_T^{(\lambda)} d\tilde{P} = E^{\tilde{P}}[\log M_T^{(\lambda)}]. \end{aligned}$$

Then we can evaluate the mean value by seeing the  $X$  has representation wrt.  $\tilde{P}$  by

$$X_t = \int_0^t f(u) (\lambda f(u) du + d\tilde{W}_u) = \lambda \int_0^t f^2(u) du + \int_0^t f(u) d\tilde{W}_u.$$

Giving that

$$\begin{aligned}
E^P[M_T^{(\lambda)} \log M_T^{(\lambda)}] &= E^{\tilde{P}}[\log M_T^{(\lambda)}] \\
&= E^{\tilde{P}} \left[ \lambda X_T - \frac{1}{2} \lambda^2 \int_0^T f^2(u) \, du \right] \\
&= E^{\tilde{P}} \left[ \lambda^2 \int_0^T f^2(u) \, du + \lambda \int_0^T f(u) \, d\tilde{W}_u - \frac{1}{2} \lambda^2 \int_0^T f^2(u) \, du \right] \\
&= \lambda E^{\tilde{P}} \left[ \frac{1}{2} \lambda \int_0^T f^2(u) \, du + \int_0^T f(u) \, d\tilde{W}_u \right] \\
&= \frac{1}{2} \lambda^2 \int_0^T f^2(u) \, du + \lambda E^{\tilde{P}} \left[ \int_0^T f(u) \, d\tilde{W}_u \right] \\
&= \frac{1}{2} \lambda^2 \int_0^T f^2(u) \, du
\end{aligned}$$

Since

$$\tilde{X}_T = \int_0^T f(u) \, d\tilde{W}_u,$$

is a  $\tilde{P}$ -martingale.  $\square$

### Problem 2

Consider a standard Black-Scholes model, that is, a model consisting of a bank account  $B_t$  with  $P$ -dynamics given by

$$dB_t = rB_t \, dt, \quad B_0 = 1$$

and a stock  $S_t$  with  $P$ -dynamics given by

$$dS_t = \alpha S_t \, dt + \sigma S_t \, d\bar{W}_t, \quad S_0 = s > 0$$

where  $r, \alpha \in \mathbb{R}$  and  $\sigma > 0$  are constants and  $\bar{W}_t$  is a  $P$ -Brownian motion. Let  $T > 0$  be a given and fixed date.

Consider the derivative that at time  $T$  pays

$$X = \max \{ \min \{ S_T, K_2 \}, K_1 \},$$

where  $0 < K_1 < K_2$  are constants.

- a. Determine the arbitrage free price of derivative  $X$  at time  $t < T$ .

Consider a new derivative that at time  $T$  pays

$$Y = (S_T^2 - K^2)^+ - (K^2 - S_T^2)^+.$$

- b.
  - i. Determine the arbitrage free price of derivative  $Y$  at time  $t < T$ .
  - ii. Find a hedging portfolio for derivative  $Y$ .

Let  $h(t) = (h_0(t), h_1(t))$  be a portfolio where

$$h_0(t) = -e^{r(T-2t)+\sigma^2(T-t)} S^2(t)$$

is the number of units in the bank account at time  $t$  and

$$h_1(t) = 2e^{(r+\sigma^2)(T-t)} S(t)$$

is the number of shares in the stock at time  $t$ . Let  $V^h(t)$  denote the associated value process.

- c. Determine whether the portfolio  $h$  is self-financing or not.
- d. Compute  $V^h(T)$ .

**Solution (a).**

We see that the derivative is the bull spread given by the payout function

$$X = \begin{cases} K_2 & \text{if } S_T > K_2, \\ S_T & \text{if } K_1 \leq S_T \leq K_2, \\ K_1 & \text{if } S_T < K_1. \end{cases}$$

We know from exercise 10.3 that this can be replicated by holding  $K_1$  bonds, one call option with strike  $K_1$  and a short on a call with strike  $K_2$ . The arbitrage free price of the derivative is then the value process of the mentioned portfolio i.e.

$$\Pi_t[X] = K_1 e^{-r(T-t)} + c(K_1; t, T) - c(K_2; t, T),$$

where  $c$  denotes the pricing function for a European call option (non-instructive parameters suppressed).  $\square$

**Solution (b).**

(i): We start by seeing that the derivative pays out

$$Y = \begin{cases} S_T^2 - K^2 & \text{if } S_T^2 \geq K^2, \\ -(K^2 - S_T^2) & \text{if } S_T^2 < K^2. \end{cases}$$

hence the payout is  $Y = S_T^2 - K^2 = \Phi(S_T)$  where  $\Phi(s) = s^2 - K^2$ . That is  $Y$  is in fact a simple claim. We have from the risk neutral valuation formula 7.11 that

$$\begin{aligned} \Pi_t[Y] &= e^{-r(T-t)} E_{t,s}^Q[S_T^2 - K^2] \\ &= e^{-r(T-t)} E_{t,s}^Q[S_T^2] - e^{-r(T-t)} K^2. \end{aligned}$$

Recall that under the martingale measure  $Q$  we have that  $S_t$  is a GBM hence

$$S_t = s \cdot \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) (T-t) + \sigma (W_T^Q - W_t^Q) \right\}$$

then

$$S_T^2 = s^2 \cdot \exp \left\{ 2 \left( r - \frac{1}{2} \sigma^2 \right) (T-t) + 2\sigma (W_T^Q - W_t^Q) \right\}.$$

Inserting this into the risk neutral valuation formula we get

$$\begin{aligned}
\Pi_t[Y] &= e^{-r(T-t)} E_{t,s}^Q[S_T^2] - e^{-r(T-t)} K^2 \\
&= e^{-r(T-t)} s^2 e^{2(r-\frac{1}{2}\sigma^2)(T-t)} E^Q \left[ \exp \left\{ 2\sigma \left( W_T^Q - W_t^Q \right) \right\} \right] - e^{-r(T-t)} K^2 \\
&= e^{-r(T-t)} s^2 e^{2(r-\frac{1}{2}\sigma^2)(T-t)} e^{\frac{1}{2}4\sigma^2(T-t)} - e^{-r(T-t)} K^2 \\
&= e^{-r(T-t)} \left( s^2 e^{(2r-\sigma^2)(T-t)+\frac{1}{2}4\sigma^2(T-t)} - K^2 \right) \\
&= e^{-r(T-t)} \left( s^2 e^{(2r+\sigma^2)(T-t)} - K^2 \right).
\end{aligned}$$

The arbitrage free price of the derivative is then given above.  $\square$

(ii): From theorem 8.5 we can determine a hedging portfolio with weightings

$$\begin{aligned}
w_t^B &= \frac{\Pi_t - S_t \frac{\partial \Pi}{\partial s}}{\Pi_t} \\
&= 1 - \frac{S_t 2S_t e^{-r(T-t)} e^{(2r+\sigma^2)(T-t)}}{e^{-r(T-t)} (S_t^2 e^{(2r+\sigma^2)(T-t)} - K^2)} \\
&= 1 - \frac{2S_t^2 e^{(2r+\sigma^2)(T-t)}}{S_t^2 e^{(2r+\sigma^2)(T-t)} - K^2} \\
&= 1 - \frac{2}{1 - K^2 S_t^{-2} e^{(2r+\sigma^2)(t-T)}} \\
w_t^S &= \frac{2}{1 - K^2 S_t^{-2} e^{(2r+\sigma^2)(t-T)}}.
\end{aligned}$$

In absolute terms we will hold the portfolio

$$\begin{aligned}
h_t^S &= 2S_t e^{-r(T-t)} e^{(2r+\sigma^2)(T-t)} \\
h_t^B &= \frac{e^{-r(T-t)} \left( s^2 e^{(2r+\sigma^2)(T-t)} - K^2 \right) - S_t h_t^S}{B_t} \\
&= \frac{e^{-r(T-t)} \left( s^2 e^{(2r+\sigma^2)(T-t)} - K^2 \right) - S_t h_t^S}{e^{rt}} \\
&= e^{-rT} s^2 e^{(2r+\sigma^2)(T-t)} - e^{-rT} K^2 - e^{-rt} S_t h_t^S.
\end{aligned}$$

The portfolio above will hedge  $Y$  with probability one.  $\square$

### Solution (c).

We assume no dividends and no consumption that is  $c_t = 0$  and  $dD_t^i = 0$  for  $i = 0, 1$ . Then the portfolio is self-financing if and only if the value process has dynamics.

$$h_0(t) dB_t + h_1(t) dS_t = 0$$

This is given in lemma 6.12.

**THE BELOW IS IN WORKS AND NOT CORRECT!**



Now we have that the value process is given by

$$V_t^h = h_0(t)B_t + h_1(t)S_t.$$

Using the representation  $V_t^h = f(h_0(t), B_t) + f(h_1(t), S_t)$  given by  $f(x, y) = xy$  we have

$$dV_t^h = df(h_0(t), B_t) + df(h_1(t), S_t).$$

Using Ito's formula on each term we have

$$\begin{aligned} df(h_0(t), B_t) &= B_t dh_0(t) + h_0(t) dB_t + (dB_t)(dh_0(t)), \\ df(h_1(t), S_t) &= S_t dh_1(t) + h_1(t) dS_t + (dS_t)(dh_1(t)), \end{aligned}$$

since of cause  $f_{xx} = f_{yy} = 0$ . We can the determine the dynamics of the portfolio by

$$\begin{aligned} dh_0(t) &= -(-2t - \sigma^2)S_t^2 e^{r(T-2t) + \sigma^2(T-t)} dt \\ &\quad - 2S_t e^{r(T-2t) + \sigma^2(T-t)} dS_t \\ &\quad - \frac{1}{2} 2e^{r(T-2t) + \sigma^2(T-t)} (dS_t)^2 \\ &= (-2t - \sigma^2)h_0(t) dt + \frac{2}{S_t} h_0(t) (\mu S_t dt + \sigma S_t dW_t) + \frac{1}{S_t^2} h_0(t) \sigma^2 S_t^2 dt \\ &= (\mu - 1)2h_0(t) dt + 2\sigma h_0(t) dW_t \end{aligned}$$

and

$$\begin{aligned} dh_1(t) &= (-r - \sigma^2)2e^{(r+\sigma^2)(T-t)} S_t dt \\ &\quad + 2e^{(r+\sigma^2)(T-t)} dS_t + 0 \\ &= (-r - \sigma^2)h_1(t) dt + \frac{1}{S_t} h_1(t) (\mu S_t dt + \sigma S_t dW_t) \\ &= (-r - \sigma^2 + \mu)h_1(t) dt + h_1(t) \sigma dW_t \end{aligned}$$

And so in total

$$\begin{aligned}
dV_t^h &= df(h_0(t), B_t) + df(h_1(t), S_t) \\
&= B_t dh_0(t) + h_0(t) dB_t + (dB_t)(dh_0(t)) \\
&\quad + S_t dh_1(t) + h_1(t) dS_t + (dS_t)(dh_1(t)) \\
&= B_t ((\mu - 1)2h_0(t) dt + 2\sigma h_0(t) dW_t) + h_0(t) rB_t dt + 0 \\
&\quad + S_t ((-r - \sigma^2 + \mu)h_1(t) dt + h_1(t)\sigma dW_t) + h_1(t) (\mu S_t dt + \sigma S_t dW_t) + \sigma^2 S_t h_1(t) dt \\
&= [B_t(\mu - 1)2h_0(t) + h_0(t)rB_t + S_t(-r - \sigma^2 + \mu)h_1(t) + h_1(t)\mu S_t + \sigma^2 S_t h_1(t)] dt \\
&\quad + [B_t 2\sigma h_0(t) + S_t h_1 \sigma + h_1 \sigma S_t] dW_t \\
&= [(2\mu - 2 + r)B_t h_0(t) + (-r + 2\mu)S_t h_1(t)] dt \\
&\quad + [B_t h_0(t) + h_1 S_t] 2\sigma dW_t \\
&= V_t^h 2\mu dt + V_t^h dW_t
\end{aligned}$$

**Solution (d).**

We compute  $V_T^h$  easily by inserting  $h_0$  and  $h_1$  below

$$\begin{aligned}
V_T^h &= B_T h_0(T) + S_T h_1(T) \\
&= B_T \left( -e^{r(T-2T)+\sigma^2(T-T)} S_T^2 \right) + S_T \left( 2e^{(r+\sigma^2)(T-T)} S_T \right) \\
&= -S_T^2 + 2S_T^2 = S_T^2.
\end{aligned}$$

and so  $h$  hedge the payout  $\Phi(S_T) = S_T^2$ .  $\square$

### Problem 3

Consider a two-dimensional model. The market model consist of three assets: A bank account  $B_t$  and two stocks  $S_1$  and  $S_2$ . The  $P$ -dynamics of  $B_t$  is

$$dB_t = rB_t dt, \quad B_0 = 1,$$

where  $r \in \mathbb{R}$  is a constant interest rate. The  $P$ -dynamics of  $S_1$  and  $S_2$  are given by

$$\begin{aligned}
dS_1(t) &= \alpha_1 S_1(t) dt + \sigma_1 S_1(t) d\bar{W}_1(t), & S_1(0) &= s_1 > 0, \\
dS_2(t) &= \alpha_2 S_2(t) dt + \sigma_2 S_2(t) d\bar{W}_2(t), & S_2(0) &= s_2 > 0,
\end{aligned}$$

where  $\alpha_1, \alpha_2 \in \mathbb{R}$  are constants. Moreover,  $\sigma_1 > 0$  is a constant and  $\sigma_2(t) = \sigma_0 e^{-\gamma t}$  where  $\sigma_0 > 0$  and  $\gamma > 0$  are constants and  $\bar{W}_1(t)$  and  $\bar{W}_2(t)$  are two independent  $P$ -Brownian motions. The filtration is the one generated by the two Brownian motions, that is,  $\mathcal{F}_t = \sigma(\bar{W}_1(s), \bar{W}_2(s) \mid 0 \leq s \leq t)$ . Let  $T > 0$  be a given and fixed (expiry) date.

- a. i. Is the model arbitrage free?
- ii. Is the model complete?

Consider the derivative that at time  $T$  pays  $X = S_1(T)S_2(T)$  and let  $F(t, s_1, s_2)$  be the pricing function of the derivative.

- b. i. Determine the arbitrage free price of derivative  $X$  at time  $t = 0$ .
- ii. Determine the equation satisfied by the pricing function  $F(t, s_1, s_2)$ .

Consider a new derivative that at time  $T$  pays  $Y = \log(S_2(T))$ .

c. Determine the arbitrage free price of derivative  $Y$  at time  $t < T$ .

**Solution (a).**

(i): We know that the model is arbitrage free if and only if there exist a martingale measure  $Q$ . This is equivalent with finding a likelihood process  $L$  with Radon-Nikodym derivative  $\varphi$  given by the solution to the equation

$$\sigma_t \varphi_t = r_t - \mu_t.$$

We see that

$$\sigma_t = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_0 e^{-\gamma t} \end{bmatrix} \Rightarrow \sigma_t^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 \\ 0 & e^{\gamma t}/\sigma_0 \end{bmatrix}.$$

Hence we trivially have a **solution** given by

$$\varphi_t = \begin{bmatrix} 1/\sigma_1 & 0 \\ 0 & e^{\gamma t}/\sigma_0 \end{bmatrix} \begin{bmatrix} r - \alpha_1 \\ r - \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{r - \alpha_1}{\sigma_1} \\ \frac{r - \alpha_2}{\sigma_0} e^{\gamma t} \end{bmatrix}.$$

Proposition 14.1 gives now that if  $L$ , given by

$$dL_t = \varphi_t^\top L_t dW_t, \quad L_0 = 1,$$

is a martingale then the market is arbitrage free. This is true if the Novikov condition is satisfied. We have

$$E^P \left[ e^{\frac{1}{2} \int_0^T \|\varphi_t\|^2 dt} \right] = e^{\frac{1}{2} \int_0^T \|\varphi_t\|^2 dt} = e^{\frac{1}{2} \int_0^T \left( \frac{r - \alpha_1}{\sigma_1} \right)^2 + \left( \frac{r - \alpha_2}{\sigma_0} e^{\gamma t} \right)^2 dt} < \infty$$

since of cause

$$\int_0^T \left( \frac{r - \alpha_1}{\sigma_1} \right)^2 + \left( \frac{r - \alpha_2}{\sigma_0} e^{\gamma t} \right)^2 dt = T \left( \frac{r - \alpha_1}{\sigma_1} \right)^2 + \left( \frac{r - \alpha_2}{\sigma_0} \right)^2 \int_0^T e^{2\gamma t} dt < \infty$$

for all  $T \geq 0$ . Then the Novikov condition is satisfied and  $L$  is martingale with  $E[L_T] = 1$ .  $\square$

(ii): The model is complete if the martingale measure is unique. This is equivalent with  $\text{Ker}[\sigma_t] = \{0\}$  and since  $\sigma_t$  is invertible (diagonal) we have that the model is complete.  $\square$

**Solution (b).**

(i): We may determine the price of the derivative using the risk neutral valuation formula

$$\Pi_t[X] = E^Q \left[ e^{-\int_t^T r(u) du} X \mid \mathcal{F}_t \right]$$

Hence we have for  $t = 0$  and  $S_1(0) = s_1$  and  $S_2(0) = s_2$  that

$$\Pi_0[X] = E^Q \left[ e^{-\int_0^T r(u) du} X \mid \mathcal{F}_0 \right] = e^{-rT} E^Q [S_1(T)S_2(T) \mid \mathcal{F}_0],$$

Since we have that  $S_1$  and  $S_2$  have dynamics wrt. two independent Brownian motions we know that the price processes are independent. If we multiply by  $B(T)/B(t)$  we obtain two martingale processes under the measure  $Q$ :

$$\begin{aligned}\Pi_0[X] &= e^{-rT} B(T)^2 E^Q \left[ \frac{S_1(T)}{B(T)} \mid \mathcal{F}_0 \right] E^Q \left[ \frac{S_2(T)}{B(T)} \mid \mathcal{F}_0 \right] \\ &= e^{-rT} e^{2rT} s_1(0) s_2(0) = e^{rT} s_1(0) s_2(0),\end{aligned}$$

and so the arbitrage free price is given above.  $\square$

(ii): We have from Bjork (14.31) that  $\Pi$  satisfies the PDE below

$$\begin{cases} F_t + \sum_{i=1}^2 r s_i F_{s_i} + \frac{1}{2} \text{tr}[\sigma_t^\top D(S) F_{ss} D(S) \sigma_t] - rF = 0 \\ F(T, s_1, s_2) = \Phi(s_1, s_2) \end{cases}$$

The PDE is in detail

$$\begin{aligned}0 &+ rS_1(t)S_2(t) + rS_2S_1 + \frac{1}{2} \text{tr} \begin{bmatrix} S_1\sigma_1 & 0 \\ 0 & S_2\sigma_0 e^{-\gamma t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S_1\sigma_1 & 0 \\ 0 & S_2\sigma_0 e^{-\gamma t} \end{bmatrix} - r\Pi_t \\ &= 2rS_1(t)S_2(t) + \frac{1}{2} \text{tr} \begin{bmatrix} S_1\sigma_1 & 0 \\ 0 & S_2\sigma_0 e^{-\gamma t} \end{bmatrix} \begin{bmatrix} 0 & S_2\sigma_0 e^{-\gamma t} \\ S_1\sigma_1 & 0 \end{bmatrix} \\ &= 2rS_1(t)S_2(t) + \frac{1}{2} \text{tr} \begin{bmatrix} 0 & S_1S_2\sigma_0\sigma_1 e^{-\gamma t} \\ S_1S_2\sigma_0\sigma_1 e^{-\gamma t} & 0 \end{bmatrix} - r\Pi_t \\ &= 2rS_1(t)S_2(t) - r\Pi_t = 0.\end{aligned}$$

or

$$F(t, s_1, s_2) = 2s_1s_2, \quad F(T, s_1, s_2) = s_1s_2$$

this ends the question.  $\square$

**Solution (c).**

We have the derivative  $Y = \log(S_2(T))$ . By the risk neutral valuation formula we have that the arbitrage free price is given by

$$\Pi_t[Y] = E^Q \left[ e^{-\int_t^T r(u) du} Y \mid \mathcal{F}_t \right] = e^{-r(T-t)} E^Q [\log(S_2(T)) \mid \mathcal{F}_t].$$

Under the measure  $Q$  the dynamics of  $S_2$  is that of a GBM hence

$$d \log(S_2(t)) = \left( r - \frac{1}{2} \sigma_0^2 e^{-2\gamma t} \right) dt + \sigma_0^2 e^{-2\gamma t} dW_t^Q,$$

and so with the knowledge that  $S_2(t) = s_2$  we have

$$\begin{aligned}
\Pi_t[Y] &= e^{-r(T-t)} E^Q \left[ \log(s_2) + \int_t^T \left( r - \frac{1}{2} \sigma_0^2 e^{-2\gamma s} \right) ds + \int_t^T \sigma_0^2 e^{-2\gamma t} dW_t^Q \middle| \mathcal{F}_t \right] \\
&= e^{-r(T-t)} \left[ \log(s_2) + \int_t^T \left( r - \frac{1}{2} \sigma_0^2 e^{-2\gamma s} \right) ds \right] \\
&= e^{-r(T-t)} \left[ \log(s_2) + r(T-t) - \frac{1}{2} \sigma_0^2 \int_t^T e^{-2\gamma s} ds \right] \\
&= e^{-r(T-t)} \left[ \log(s_2) + r(T-t) + \frac{1}{4\gamma} \sigma_0^2 [e^{-2\gamma s}]_t^T \right] \\
&= e^{-r(T-t)} \left[ \log(s_2) + r(T-t) + \frac{1}{4\gamma} \sigma_0^2 (e^{-2\gamma T} - e^{-2\gamma t}) \right].
\end{aligned}$$

The arbitrage free price of the derivative is then given above.  $\square$

## Exam 2018/19

### Problem 1

Let  $W(t)$  denote a Brownian motion and let  $\mathcal{F}_t = \mathcal{F}_t^W$ . Let  $T > 0$  be a given and fixed time.

Consider the stochastic differential equation

$$dX(t) = \alpha dt + \sqrt{X(t)} dW(t),$$

and  $X(0) = x > 0$  where  $\alpha \in \mathbb{R}$ .

- a.
  - i. Compute the mean value of  $X(T)$ .
  - ii. Compute the variance of  $X(T)$ .
- b. Find the solution of the partial differential equation

$$\begin{aligned}
4F_t(t, x) + 8x^2 F_{xx}(t, x) + 3x F_x(t, x) &= 5F(t, x) \text{ for } t < T \text{ and } x > 0. \\
F(T, x) &= x^3.
\end{aligned}$$

Let  $\widetilde{W}(t)$  be another Brownian motion such that  $W(t)$  and  $\widetilde{W}(t)$  are two independent Brownian motions. Let  $Y(t)$  and  $Z(t)$  be two martingales given by the following dynamics

$$\begin{aligned}
dY(t) &= W(t) dW(t) + \widetilde{W}(t) d\widetilde{W}(t), \\
dZ(t) &= \widetilde{W}(t) dW(t) - W(t) d\widetilde{W}(t).
\end{aligned}$$

with  $Y(0) = Z(0) = 0$ .

- c. Show that  $M(t) = Y(t)Z(t)$  is a martingale.

### Solution (a).

(i): We start by writing  $X$  on integral form given as

$$X(t) = x + \int_0^t \alpha ds + \int_0^t \sqrt{X(s)} dW(s).$$

Taking expectation yields.

$$E[X(t)] = E \left[ x + \alpha t + \int_0^t \sqrt{X(s)} dW(s) \right] = x + \alpha t,$$

since

$$E \left[ \int_0^t \sqrt{X(s)} dW(s) \right] = E \left[ \int_0^0 \sqrt{X(s)} dW(s) \right] = 0.$$

This result follows from lemma 4.10 as the process  $M_t = \int_0^t \sqrt{X(s)} dW(s)$  is a martingale. From this we have that  $E[X(T)] = x + \alpha T$ .  $\square$

(ii): We have that the variance is given by

$$\text{Var}(X(t)) = E(X^2(t)) - (E(X(t)))^2.$$

and so we have

$$\begin{aligned} \text{Var}(X(t)) + E(X(t))^2 &= E \left[ \left( x + \alpha t + \int_0^t \sqrt{X(s)} dW(s) \right)^2 \right] \\ &= (x + \alpha t)^2 + E \left[ \left( \int_0^t \sqrt{X(s)} dW(s) \right)^2 \right] + 2(x + \alpha t) E \left[ \int_0^t \sqrt{X(s)} dW(s) \right] \\ &= (x + \alpha t)^2 + E \left[ \left( \int_0^t \sqrt{X(s)} dW(s) \right)^2 \right]. \end{aligned}$$

Now by setting  $Z(t) = \int_0^t \sqrt{X(s)} dW(s)$  we see that  $Z$  has dynamics  $dZ(t) = \sqrt{X(t)} dW(t)$  with  $Z(0) = 0$ . By Ito's formula on the variable  $f(t, Z(t))$  with  $f(t, z) = z^2$  we have

$$\begin{aligned} df(t, Z(t)) &= 0 dt + 2Z(t) dZ(t) + \frac{1}{2} 2 (dZ(t))^2 \\ &= 2Z(t) \sqrt{X(t)} dW(t) + X(t) dt. \end{aligned}$$

Obviously when taking expectation on  $f(t, Z(t))$  we see that the integral part related to the Brownian motion is a martingale with mean 0 and then

$$E[f(t, Z(t))] = E \left[ \int_0^t X(s) ds \right].$$

In total we have

$$\text{Var}(X(t)) = (x + \alpha t)^2 + E \left[ \int_0^t X(s) ds \right] - (x + \alpha t)^2 = E \left[ \int_0^t X(s) ds \right].$$

Moving the expectation inside the integral then gives

$$\text{Var}(X(t)) = \int_0^t (x + \alpha s) ds = xt + \frac{1}{2} \alpha t^2.$$

Inserting  $t = T$  gives the desired result.  $\square$

**Solution (b).**

We see by dividing by 4 we have the PDE given by

$$F_t + 2x^2 F_{xx} + \frac{3}{4}x F_x = \frac{5}{4}F$$

hence by setting  $r = 5/4$ ,  $\mu = 3x/4$  and  $\sigma^2 = 4x^2$  we have the boundary value problem

$$\begin{cases} F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx} - rF = 0, \\ F(T, x) = x^3. \end{cases}$$

From Feymann-Kac we know this has solution on  $[0, T] \times \mathbb{R}$  given by the stochastic representation

$$F(t, x) = e^{-r(T-t)} E_{t,x}[X_T^3],$$

where  $X$  satisfies the SDE

$$dX_t = \frac{3}{4}X_t dt + 2X_t dW_t.$$

Giving that  $X(t) = x$  and  $X$  is a GBM we have

$$X_T = x \cdot e^{(r - \frac{1}{2}2^2)(T-t) + 2(W_T - W_t)} = x \cdot e^{\frac{-5}{4}(T-t) + 2(W_T - W_t)}.$$

The relevant mean value is then

$$\begin{aligned} F(t, x) &= e^{-\frac{5}{4}(T-t)} E \left[ x^3 \cdot e^{\frac{-15}{4}(T-t) + 6(W_T - W_t)} \right] \\ &= e^{-\frac{5}{4}(T-t)} x^3 e^{\frac{-15}{4}(T-t)} E \left[ e^{6(W_T - W_t)} \right] \\ &= x^3 e^{\frac{-20}{4}(T-t)} e^{\frac{1}{2}6^2(T-t)} = x^3 e^{-5(T-t) + 18(T-t)} \\ &= x^3 e^{13(T-t)}. \end{aligned}$$

The solution is the given above.  $\square$

**Solution (c).**

We show that  $M$  has dynamics solely given in terms of Brownian motions. We have that  $M(t) = f(t, Y(t), Z(t))$  for  $f(t, y, z) = yz$  the dynamics given by Ito's formula:

$$dM(t) = 0 dt + Z(t) dY(t) + Y(t) dZ(t) + (dY(t))(dZ(t))$$

since the only second derivative not zero is  $f_{yz} = f_{zy} = 1$ . The product  $(dY(t))(dZ(t))$  is computed first

$$\begin{aligned} (dY(t))(dZ(t)) &= (W(t) dW(t) + \widetilde{W}(t) d\widetilde{W}(t)) \cdot (\widetilde{W}(t) dW(t) - W(t) d\widetilde{W}(t)) \\ &= W(t)\widetilde{W}(t) dt - \widetilde{W}(t)W(t) dt = 0, \end{aligned}$$

where we use that  $dW(t)d\widetilde{W}(t) = dt$  is the only non-zero term. Then we obviously have

$$\begin{aligned} dM(t) &= Z(t) dY(t) + Y(t) dZ(t) \\ &= Z(t)W(t) dW(t) + Z(t)\widetilde{W}(t) d\widetilde{W}(t) + Y(t)\widetilde{W}(t) dW(t) - Y(t)W(t) d\widetilde{W}(t). \end{aligned}$$

Giving that  $M(t)$  is a martingale. (lemma 4.11)

## Problem 2

Consider a standard Black-Scholes model, that is, a model consisting of a bank account  $B(t)$  with  $P$ -dynamics given by

$$dB(t) = rB(t) dt,$$

with  $B(0) = 1$  and a stock  $S(t)$  with  $P$ -dynamics given by

$$dS(t) = \alpha S(t) dt + \sigma S(t) d\overline{W}(t),$$

with  $S(0) = s > 0$  and where  $r, \alpha \in \mathbb{R}$  and  $\sigma > 0$  are constants and  $\overline{W}(t)$  is a  $P$ -Brownian motion. Let  $T > 0$  be a given fixed (expiry) date.

Let  $h(t) = (h_0(t), h_1(t))$  be a portfolio where

$$h_0(t) = \exp\left(\frac{1}{2}\sigma\overline{W}(t) + \left(\frac{\alpha - r}{2} - \frac{1}{8}\sigma^2\right)t\right)$$

is the number of units in the bank account at time  $t$  and

$$h_1(t) = \frac{1}{s} \exp\left(-\frac{1}{2}\sigma\overline{W}(t) + \left(\frac{r - \alpha}{2} - \frac{3}{8}\sigma^2\right)t\right)$$

is the number of shares in the stock at time  $t$ . Let  $V^h(t)$  denote the associated value process and let  $u(t) = (u_0(t), u_1(t))$  denote the relative portfolio.

- a.
  - i. Determine whether the portfolio  $h$  is self-financing or not.
  - ii. Compute  $u_1(t)$ .

Consider two derivatives that at time  $T$  pay  $X_1 = \Phi_1(S(T))$  and  $X_2 = \Phi_2(S(T))$ . For  $i = 1, 2$ , the arbitrage free price of derivative  $X_i$  is given by  $\pi_i(t) = F_i(t, S(t))$  where  $F_i(t, s)$  is the pricing function of the derivative. Assume that  $\pi_i(t) > 0$ . The price process  $\pi_i(t)$  has dynamics (under the  $P$ -measure) given by

$$d\pi_i(t) = \alpha_i(t)\pi_i(t) dt + \sigma_i(t)\pi_i(t) d\overline{W}(t).$$

- b.
  - i. Determine  $\alpha_i(t)$  and  $\sigma_i(t)$  for  $i = 1, 2$ .
  - ii. Show that

$$\frac{r - \alpha_1(t)}{\sigma_1(t)} = \frac{r - \alpha_2(t)}{\sigma_2(t)}.$$

Let  $C(t, s; K, T)$  denote the Black-Scholes price at time  $t$  of an European call option with strike  $K$  and expiry date  $T$  when the current price of the underlying is  $s$ . Similarly, let  $P(t, s; K, T)$  denote the Black-Scholes price at time  $t$  of an European put option with strike  $K$  and expiry date  $T$  when the current price of the underlying is  $s$ . Consider a new derivative that at time  $T$  pays



$$Y = \max \{C(T, S(T); K, T_1), P(T, S(T); K, T_1)\}$$

where  $T < T_1$  is a fixed date.

c. Determine the arbitrage free price of derivative  $Y$  at time  $t < T$ . (Hint: recall  $\max(x, y) = (x + y)^+ y$ )

Assume that the call option and the put option do not have the same strike prices, that is, a derivative that at time  $T$  pays

$$\tilde{Y} = \max \{C(T, S(T); K_1, T_1), P(T, S(T); K_2, T_1)\}$$

where the strike prices  $K_1 \neq K_2$ . Let  $F(t, s)$  be the pricing function of the derivative.

d. Determine the equation satisfied by the pricing function  $F(t, s)$ .

**Solution (a).**

We have that  $h$  is self-financing if and only if the equation

$$dV^h(t) = h_0(t) dB(t) + h_1(t) dS(t)$$

is satisfied. And so, we start by determining the dynamics of the number of assets denoted by  $h_0$  and  $h_1$ . From Ito's formula we can conclude that

$$\begin{aligned} dh_0(t) &= \left( \frac{\alpha - r}{2} - \frac{1}{8}\sigma^2 \right) h_0(t) dt + \frac{1}{2}\sigma h_0(t) dW(t) + \frac{1}{2}\frac{1}{2}\sigma\frac{1}{2}\sigma h_0 (dW(t))^2 \\ &= \left( \frac{\alpha - r}{2} - \frac{1}{8}\sigma^2 \right) h_0(t) dt + \frac{1}{2}\sigma h_0(t) dW(t) + \frac{1}{2^3}\sigma^2 h_0 dt \\ &= \left( \frac{\alpha - r}{2} - \frac{1}{8}\sigma^2 + \frac{1}{8}\sigma^2 \right) h_0(t) dt + \frac{1}{2}\sigma h_0(t) dW(t) \\ &= \frac{\alpha - r}{2} h_0(t) dt + \frac{1}{2}\sigma h_0(t) dW(t). \end{aligned}$$

For the number of stocks we have

$$\begin{aligned} dh_1(t) &= \left( \frac{r - \alpha}{2} + \frac{3}{8}\sigma^2 \right) h_1(t) dt - \frac{1}{2}\sigma h_1(t) dW(t) + \frac{1}{2}\frac{1}{2}\sigma\frac{1}{2}\sigma h_1(t) (dW(t))^2 \\ &= \left( \frac{r - \alpha}{2} + \frac{1}{2}\sigma^2 \right) h_1(t) dt - \frac{1}{2}\sigma h_1(t) dW(t). \end{aligned}$$

We may derive the dynamics of the portfolio as

$$\begin{aligned} dV^h(t) &= d(h_0(t)B(t) + h_1(t)S(t)) \\ &= B(t) dh_0(t) + h_0(t) dB(t) + (dh_0(t))(dB(t)) \\ &\quad + S(t) dh_1(t) + h_1(t) dS(t) + (dh_1(t))(dS(t)) \end{aligned}$$

and so we want that

$$(*) = B(t) dh_0(t) + (dh_0(t))(dB(t)) + S(t) dh_1(t) + (dh_1(t))(dS(t)) = 0.$$

Inserting the dynamics given and portfolio dynamics above we have

$$\begin{aligned}
 (*) &= B(t) \frac{\alpha - r}{2} h_0(t) dt + B(t) \frac{1}{2} \sigma h_0(t) dW(t) + 0 \\
 &\quad + S(t) \left( \frac{r - \alpha}{2} + \frac{1}{2} \sigma^2 \right) h_1(t) dt - S(t) \frac{1}{2} \sigma h_1(t) dW(t) \\
 &\quad - \frac{1}{2} \sigma h_1(t) \sigma S(t) dt \\
 &= \left( B(t) \frac{\alpha - r}{2} h_0(t) - \frac{\alpha - r}{2} S(t) h_1(t) \right) dt \\
 &\quad + \left( \frac{1}{2} B(t) \sigma h_0(t) - \frac{1}{2} S(t) \sigma h_1(t) \right) dW(t)
 \end{aligned}$$

We see that this is zero if  $h_0(t)B(t) = h_1(t)S(t)$ . First we have

$$\begin{aligned}
 h_0(t)B(t) &= \exp \left( \frac{1}{2} \sigma \bar{W}(t) + \left( \frac{\alpha - r}{2} - \frac{1}{8} \sigma^2 \right) t \right) B(t) \\
 &= \exp \left( \frac{1}{2} \left( \alpha - r - \frac{1}{4} \sigma^2 \right) t + \frac{1}{2} \sigma \bar{W}(t) \right) B(t) \\
 &= \exp \left( \frac{1}{2} \left( \alpha - \frac{1}{2} \sigma^2 \right) t + \frac{1}{2} \sigma \bar{W}(t) \right) \exp \left( \frac{1}{2} \left( -r + \frac{1}{4} \sigma^2 \right) t \right) B(t) \\
 &= (S(t))^{1/2} \exp \left( \left( -\frac{r}{2} + \frac{1}{8} \sigma^2 \right) t \right) B(t),
 \end{aligned}$$

and

$$\begin{aligned}
 h_1(t)S(t) &= \frac{1}{s} \exp \left( -\frac{1}{2} \sigma \bar{W}(t) + \left( \frac{r - \alpha}{2} - \frac{3}{8} \sigma^2 \right) t \right) s \cdot \exp \left( \left( \alpha - \frac{1}{2} \sigma^2 \right) t + \sigma \bar{W}(t) \right) \\
 &= \exp \left( \frac{1}{2} \sigma \bar{W}(t) + \left( \frac{r + \alpha}{2} - \frac{7}{8} \sigma^2 \right) t \right) \\
 &= \exp \left( \frac{1}{2} \sigma \bar{W}(t) + \frac{1}{2} \left( \alpha - \frac{2}{4} \sigma^2 \right) t \right) \exp \left( \frac{1}{2} \left( r - \frac{5}{4} \sigma^2 \right) t \right) \\
 &= (S(t))^{1/2} \exp \left( \left( -\frac{5}{8} \sigma^2 \right) t \right) B(t)
 \end{aligned}$$

Which does not hold. **THIS EXERCISE SHOULD BE ABLE TO BE SOLVED..**  $\square$

(ii): We have that

$$u_1(t) = \frac{h_1(t)S(t)}{V^h(t)}.$$

Using that  $S$  is a GBM and  $B(t) = e^{rt}$  we have

$$\begin{aligned}
u_1(t) &= \frac{h_1(t)S(t)}{h_1(t)S(t) + h_0(t)B(t)} \\
&= \frac{e^{-\frac{1}{2}\sigma\bar{W}(t) + (\frac{r-\alpha}{2} - \frac{3}{8}\sigma^2)t} e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma\bar{W}_t}}{e^{-\frac{1}{2}\sigma\bar{W}(t) + (\frac{r-\alpha}{2} - \frac{3}{8}\sigma^2)t} e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma\bar{W}_t} + e^{\frac{1}{2}\sigma\bar{W}(t) + (\frac{\alpha-r}{2} - \frac{1}{8}\sigma^2)t} e^{rt}} \\
&= \frac{se^{(\frac{r-\alpha}{2} - \frac{3}{8}\sigma^2 + \alpha - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma\bar{W}_t}}{e^{(\frac{r-\alpha}{2} - \frac{3}{8}\sigma^2 + \alpha - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma\bar{W}_t} + e^{\frac{1}{2}\sigma\bar{W}(t) + (\frac{\alpha-r}{2} - \frac{1}{8}\sigma^2 + r)t}} \\
&= \frac{e^{(\frac{r+\alpha}{2} - \frac{7}{8}\sigma^2)t + \frac{1}{2}\sigma\bar{W}_t}}{e^{(\frac{r+\alpha}{2} - \frac{7}{8}\sigma^2)t + \frac{1}{2}\sigma\bar{W}_t} + e^{\frac{1}{2}\sigma\bar{W}(t) + (\frac{\alpha+r}{2} - \frac{1}{8}\sigma^2)t}} \\
&= \frac{e^{-\frac{7}{8}\sigma^2 t}}{e^{-\frac{7}{8}\sigma^2 t} + e^{-\frac{1}{8}\sigma^2 t}} = \frac{e^{-\frac{6}{8}\sigma^2 t}}{e^{-\frac{6}{8}\sigma^2 t} + 1}.
\end{aligned}$$

**OBVIOUSLY** had the previous exercise been done correct we would have  $h_1(t)S(t) = h_0(t)B(t)$  i.e.  $u_1(t) = \frac{1}{2}$ .  
□

**Solution (b).**

(i): We know that  $\pi_i(t) = F_i(t, S(t))$  and so from Ito's formula we have the dynamics (we suppress the argument  $(t, S(t))$  in the derivatives):

$$\begin{aligned}
d\pi_i(t) &= \frac{\partial F_i}{\partial t} dt + \frac{\partial F_i}{\partial s} dS(t) + \frac{1}{2} \frac{\partial^2 F_i}{\partial s^2} (dS(t))^2 \\
&= \frac{\partial F_i}{\partial t} dt + \frac{\partial F_i}{\partial s} (\alpha S(t) dt + \sigma S(t) d\bar{W}(t)) + \frac{1}{2} \frac{\partial^2 F_i}{\partial s^2} \sigma^2 S(t)^2 dt \\
&= \left( \frac{\partial F_i}{\partial t} + \frac{\partial F_i}{\partial s} \alpha S(t) + \frac{1}{2} \frac{\partial^2 F_i}{\partial s^2} \sigma^2 S(t)^2 \right) dt + \frac{\partial F_i}{\partial s} \sigma S(t) d\bar{W}(t) \\
&= \underbrace{\left( \frac{\partial F_i}{\partial t} + \frac{\partial F_i}{\partial s} \alpha S(t) + \frac{1}{2} \frac{\partial^2 F_i}{\partial s^2} \sigma^2 S(t)^2 \right)}_{=\alpha_i(t)} \pi_i(t) dt + \underbrace{\frac{\partial F_i}{\partial s} \sigma S(t)}_{=\sigma_i(t)} \pi_i(t) d\bar{W}(t)
\end{aligned}$$

as desired. □

(ii): We have

$$\begin{aligned}
\frac{r - \alpha_i(t)}{\sigma_i(t)} &= \frac{r - \frac{\frac{\partial F_i}{\partial t} + \frac{\partial F_i}{\partial s} \alpha S(t) + \frac{1}{2} \frac{\partial^2 F_i}{\partial s^2} \sigma^2 S(t)^2}{\pi_i(t)}}{\frac{\frac{\partial F_i}{\partial s} \sigma S(t)}{\alpha_i(t)}} \\
&= \frac{r\pi_i(t) - \frac{\partial F_i}{\partial t} - \frac{\partial F_i}{\partial s} \alpha S(t) - \frac{1}{2} \frac{\partial^2 F_i}{\partial s^2} \sigma^2 S(t)^2}{\frac{\partial F_i}{\partial s} \sigma S(t)} \\
&= \frac{r\pi_i(t) - \frac{\partial F_i}{\partial t} - \frac{\partial F_i}{\partial s} rS(t) - \frac{1}{2} \frac{\partial^2 F_i}{\partial s^2} \sigma^2 S(t)^2 + \frac{\partial F_i}{\partial s} rS(t) - \frac{\partial F_i}{\partial s} \alpha S(t)}{\frac{\partial F_i}{\partial s} \sigma S(t)} \\
&= \frac{\frac{\partial F_i}{\partial s} rS(t) - \frac{\partial F_i}{\partial s} \alpha S(t)}{\frac{\partial F_i}{\partial s} \sigma S(t)} = \frac{r - \alpha}{\sigma},
\end{aligned}$$

where we used the Black-Scholes equation i.e.

$$r\pi_i(t) - \frac{\partial F_i}{\partial t} - \frac{\partial F_i}{\partial s} rS(t) - \frac{1}{2} \frac{\partial^2 F_i}{\partial s^2} \sigma^2 S(t)^2 = 0$$

for any derivative's arbitrage free pricing process. Since  $i$  is not included in the fraction above we have the desired result.  $\square$

**Solution (c).**

We follow the hint and see that the payout is

$$\begin{aligned} Y &= \max \{C(T, S(T); K, T_1), P(T, S(T); K, T_1)\} \\ &= \left( C(T, S(T); K, T_1) - P(T, S(T); K, T_1) \right)^+ + P(T, S(T); K, T_1) \\ &= \left( C(T, S(T); K, T_1) - Ke^{-r(T_1-T)} - C(T, S(T); K, T_1) + S(T) \right)^+ + P(T, S(T); K, T_1) \\ &= \left( S(T) - Ke^{-r(T_1-T)} \right)^+ + P(T, S(T); K, T_1) \\ &= C(T, S(T); Ke^{-r(T_1-T)}, T) + P(T, S(T); K, T_1) \end{aligned}$$

Hence we can hedge this payout with a call option with strike  $Ke^{-r(T_1-T)}$  at expiry  $T$  and a put with strike  $K$  at expiry  $T_1$ , that is

$$\Pi_t[Y] = C(t, S(t); Ke^{-r(T_1-T)}, T) + P(t, S(t); K, T_1)$$

as desired.  $\square$

**Solution (d).**

We have that the arbitrage free pricing function  $F(t, s)$  has to satisfy the Black-Scholes equation 7.10 i.e.

$$\begin{aligned} F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF &= 0 \\ F(T, s) &= \max \{C(T, s; K_1, T_1), P(T, s; K_2, T_1)\}. \end{aligned}$$

which may be written differently in terms of call options, stock price  $s$  and zero-coupon bonds.  $\square$

**Problem 3**

Consider a two-dimensional Black-Scholes model. The market model consist of three assets: A bank account  $B(t)$  and two stocks  $S_1(t)$  and  $S_2(t)$ . The  $P$ -dynamics of  $B(t)$  is

$$dB(t) = rB(t) dt$$

with  $B(0) = 1$  where  $r \in \mathbb{R}$  is a constant interest rate. The  $P$ -dynamics of  $S_1(t)$  and  $S_2(t)$  are given by

$$\begin{aligned} dS_1(t) &= \alpha_1 S_1(t) dt + \sigma S_1(t) d\bar{W}_1(t), \\ dS_2(t) &= \alpha_2 S_2(t) dt + \sigma S_2(t) (d\bar{W}_1(t) + d\bar{W}_2(t)), \end{aligned}$$

with  $S_1(0) = s_1 > 0$  and  $S_2(0) = s_2 > 0$  where  $\alpha_1, \alpha_2 \in \mathbb{R}$  and  $\sigma > 0$  are constants and  $\bar{W}_1(t)$  and  $\bar{W}_2(t)$  are two independent  $P$ -Brownian motions. The filtration is the one generated by the two Brownian motions. Let  $T > 0$  be a given and fixed (expiry) date.

- a. i. Is the model arbitrage free?  
 ii. Is the model complete?
- b. Compute the covariance of  $S_1(T)$  and  $S_2(T)$ . (Hint: recall  $cov(X, Y) = E[XY] - E[X]E[Y]$ ). Consider the derivative that at time  $T$  pays  $X = S_1(T_0) + S_2(T)$  where  $0 < T_0 < T$  is a fixed date.
- c. Find a hedge portfolio for derivative  $X$ .

**Solution (a).**

(i): The model is arbitrage free if and only if a martingale measure exists. That is if the equation

$$\sigma_t \varphi_t = r - \alpha$$

has at least one solution. We have the following market on matrix form

$$dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t$$

or written out in total

$$\begin{bmatrix} dS_1(t) \\ dS_2(t) \end{bmatrix} = \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} dt \\ dt \end{bmatrix} + \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ \sigma & \sigma \end{bmatrix} \begin{bmatrix} d\bar{W}_1(t) \\ d\bar{W}_2(t) \end{bmatrix}.$$

Hence we want to solve

$$\begin{bmatrix} \sigma & 0 \\ \sigma & \sigma \end{bmatrix} \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = \begin{bmatrix} r - \alpha_1 \\ r - \alpha_2 \end{bmatrix}.$$

This is easy if  $\sigma$  is invertible. We see that we in fact have that the inverse of  $\sigma$  is

$$\sigma_t^{-1} = \begin{bmatrix} 1/\sigma & 0 \\ -1/\sigma & 1/\sigma \end{bmatrix}$$

as we have

$$\begin{bmatrix} 1/\sigma & 0 \\ -1/\sigma & 1/\sigma \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ \sigma & \sigma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Then we clearly have the solution

$$\varphi_t = \begin{bmatrix} 1/\sigma & 0 \\ -1/\sigma & 1/\sigma \end{bmatrix} \begin{bmatrix} r - \alpha_1 \\ r - \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{r - \alpha_1}{\sigma} \\ -\frac{r - \alpha_1}{\sigma} + \frac{r - \alpha_2}{\sigma} \end{bmatrix} = \begin{bmatrix} \frac{r - \alpha_1}{\sigma} \\ \frac{\alpha_1 - \alpha_2}{\sigma} \end{bmatrix}.$$

By defining the likelihood process  $L_t$  as

$$dL_t = \varphi_t^\top L_t d\bar{W}_t, \quad L_0 = 1,$$

we know from the Novikov condition that if the integral  $E^P[e^{1/2 \int_0^T \|\varphi_t\|^2 dt}]$  is finite then  $L$  is a martingale with  $E^P[L_T] = 1$ . We see that

$$E^P \left[ e^{\frac{1}{2} \int_0^T \|\varphi_t\|^2 dt} \right] = E^P \left[ e^{\frac{1}{2} \int_0^T \left( \frac{r - \alpha_1}{\sigma} \right)^2 + \left( \frac{\alpha_1 - \alpha_2}{\sigma} \right)^2 dt} \right] = E^P \left[ e^{\frac{1}{2} T \left( \frac{r - \alpha_1}{\sigma} \right)^2 + \frac{1}{2} T \left( \frac{\alpha_1 - \alpha_2}{\sigma} \right)^2} \right] < \infty.$$

Hence we have found a martingale measure defined by the likelihood process  $L$  above. We conclude that the market is arbitrage free.  $\square$

(ii): The model is complete if the martingale measure is unique. This is equivalent with  $\text{Ker}[\sigma_t] = \{0\}$  and since  $\sigma_t$  is invertible (diagonal) we have from theorem 14.7 that the model is complete.  $\square$

**Solution (b).**

We have by definition:

$$\text{cov}(S_1(T), S_2(T)) = E[S_1(T)S_2(T)] - E[S_1(T)]E[S_2(T)].$$

Thus we set  $Z(t) = S_1(t)S_2(t)$  and evaluate the mean value of  $Z$ . By Ito's formula on  $f(s_1, s_2) = s_1s_2$  we have

$$\begin{aligned} dZ(t) &= df(S_1(t), S_2(t)) \\ &= S_2(t) dS_1(t) + S_1(t) dS_2(t) + \frac{1}{2}(dS_1(t))(dS_2(t)) \\ &= S_2(t)\alpha_1(t)S_1(t) dt + S_2(t)\sigma S_1(t) d\bar{W}_1(t) \\ &\quad + S_1(t)\alpha_2(t)S_2(t) dt + S_1(t)\sigma S_2(t) (d\bar{W}_1(t) + d\bar{W}_2(t)) \\ &\quad + \frac{1}{2}\sigma^2 S_1(t)S_2(t) d\bar{W}_1(t)(d\bar{W}_1(t) + d\bar{W}_2(t)) \\ &= (\alpha_1(t) + \alpha_2(t))S_1(t)S_2(t) dt + 2\sigma S_1(t)S_2(t) d\bar{W}_1(t) + \sigma S_1(t)S_2(t) d\bar{W}_2(t) \\ &\quad + \frac{1}{2}\sigma^2 S_1(t)S_2(t) dt \\ &= (\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2)Z(t) dt + 2\sigma Z(t) d\bar{W}_1(t) + \sigma Z(t) d\bar{W}_2(t). \end{aligned}$$

Thus we have that the terms involving the Brownian motions will vanish when taking expectation hence

$$\begin{aligned} E[Z(t)] &= Z(0) + E \left[ \int_0^t (\alpha_1(s) + \alpha_2(s) + \frac{1}{2}\sigma^2)Z(s) ds \right] \\ &= Z(0) + (\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2) \int_0^t E[Z(s)] ds. \end{aligned}$$

Then we have the dynamics of  $E[Z(t)]$  is given as

$$dE[Z(t)] = (\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2)E[Z(t)] dt.$$

We may then solve this using  $E[Z(0)] = Z(0) = s_1s_2$ :

$$E[Z(T)] = Z(0)e^{(\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2)T} = s_1s_2e^{(\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2)T}.$$

Inserting in the formula for covariance we arrive at

$$\begin{aligned}
\text{cov}(S_1(T), S_2(T)) &= E[S_1(T)S_2(T)] - E[S_1(T)]E[S_2(T)] \\
&= s_1 s_2 e^{(\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2)T} - E[S_1(T)]E[S_2(T)] \\
&= s_1 s_2 e^{(\alpha_1 + \alpha_2 + \frac{1}{2}\sigma^2)T} - s_1 e^{\alpha_1 T} s_2^{\alpha_2 T} \\
&= s_1 s_2 e^{(\alpha_1 + \alpha_2)T} \left( e^{\frac{1}{2}\sigma^2 T} - 1 \right).
\end{aligned}$$

as desired.  $\square$

### Solution (c).

We may look at this problem on two subintervals:  $[0, T_0]$  and  $(T_0, T]$ . On the latter we know that the portfolio should consist of  $S_1(T_0)$  zero coupon bonds with expiry  $T$  and one position in the second stock. Hence on the interval  $(T_0, T]$  the hedging portfolio is

$$h(t) = (h_0(t), h_1(t), h_2(t)) = (e^{-r(T-T_0)}S(T_0), 0, 1), \quad t > T_0.$$

Hence we on the interval  $[0, T_0]$  we want to replicate the derivative  $\tilde{X} = e^{-r(T-T_0)}S(T_0)$ . This is obviously easy since we should hold  $e^{-r(T-T_0)}$  of the first stock. Then we have

$$h(t) = \begin{cases} (0, e^{-r(T-T_0)}, 1) & \text{for } t \leq T_0, \\ (e^{-r(T-T_0)}S(T_0), 0, 1) & \text{for } t > T_0. \end{cases}$$

This then give a self-financing portfolio with value process

$$V^h(t) = \begin{cases} S_1(t)e^{-r(T-T_0)} + S_2(t) & \text{for } t \leq T_0, \\ e^{-r(T-t)}S(T_0) + S_2(t) & \text{for } t > T_0. \end{cases}$$

as desired.  $\square$

## Exam 2019/20

### Problem 1

Let  $W(t)$  denote a Brownian motion and let  $\mathcal{F}_t = \mathcal{F}_t^W$ . Let  $T > 0$  be a given and fixed time.

Consider the two dimensional stochastic differential equation

$$\begin{aligned}
dX(t) &= \frac{1}{2}X(t) dt + Y(t) dW(t), \\
dY(t) &= \frac{1}{2}Y(t) dt + X(t) dW(t),
\end{aligned}$$

with  $X(0) = 0$  and  $Y(1) = 1$ .

- Show that  $(X(t), Y(t)) = (\sinh(W(t)), \cosh(W(t)))$  solves the two-dimensional stochastic differential equation. (Hint: Recall that  $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$  and  $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ ).
- Show that  $M(t) = e^{-t/2}\cosh(W(t))$  is a martingale.
  - Find a constant  $z$  and a process  $h(t)$  such that

$$\cosh(W(T)) = z + \int_0^T h(t) dW(t).$$

Let  $L(t)$  be a Likelihood process and let  $dQ = L(T)dP$  be a new probability measure.

- c. Determine the Likelihood process  $L(t)$  such that  $\sinh(W(t))$  is a martingale under the probability measure  $Q$ .

**Solution (a).**

Assume that  $X(t) = \sinh(W(t))$  and  $Y(t) = \cosh(W(t))$ . The relevant derivatives is then

$$\frac{d}{dw}\sinh(w) = \frac{1}{2}e^w + \frac{1}{2}e^{-w} = \cosh(w), \quad \frac{d^2}{dw^2}\sinh(w) = \frac{d}{dw}\cosh(w) = \frac{1}{2}e^w - \frac{1}{2}e^{-w} = \sinh(w).$$

That is sinus and cosinus hyperbolic are their each others derivative. Then by Ito's formula we have

$$\begin{aligned} dX(t) &= \cosh(W(t)) dW(t) + \frac{1}{2}\sinh(W(t)) (dW(t))^2 \\ &= \frac{1}{2}\sinh(W(t)) dt + \cosh(W(t)) dW(t) \\ &= \frac{1}{2}X(t) dt + Y(t) dW(t). \end{aligned}$$

and

$$\begin{aligned} dY(t) &= \sinh(W(t)) dW(t) + \frac{1}{2}\cosh(W(t)) (dW(t))^2 \\ &= \frac{1}{2}\cosh(W(t)) dt + \sinh(W(t)) dW(t) \\ &= \frac{1}{2}Y(t) dt + X(t) dW(t). \end{aligned}$$

And thus the result has been proved.  $\square$

**Solution (b).**

(i): Consider the function  $f(z, y) = zy$ . Then we have that  $M(t) = f(Z(t), Y(t))$  for  $Z(t) = e^{-t/2}$  hence  $M$  has dynamics given by Ito's formula:

$$\begin{aligned} dM(t) &= df(Z(t), Y(t)) \\ &= Y(t) dZ(t) + Z(t) dY(t) + (dZ(t))(dY(t)) \\ &= Y(t) \left(-\frac{1}{2}Z(t) dt\right) + Z(t) \left(\frac{1}{2}Y(t) dt + X(t) dW(t)\right) + \left(-\frac{1}{2}Z(t) dt\right)\left(\frac{1}{2}Y(t) dt + X(t) dW(t)\right) \\ &= X(t)Z(t) dW(t). \end{aligned}$$

and so we see that pr. lemma 4.11  $M$  is a martingale.  $\square$

(ii): We have from above

$$M(T) = M(0) + \int_0^T X(t)Z(t) dW(t) = Z(T)\cosh(W(T))$$

Hence it follows that



$$\cosh(W(T)) = \frac{M(0)}{Z(T)} + \int_0^T \frac{X(t)Z(t)}{Z(T)} dW(t).$$

Using that the martingale has initial value

$$M(0) = e^{-0/2} \cosh(0) = 1$$

we have

$$\cosh(W(T)) = e^{T/2} + \int_0^T \sinh(W(t)) e^{(T-t)/2} dW(t).$$

In total we have  $z = e^{T/2}$  and  $h(t) = \sinh(W(t)) e^{(T-t)/2}$  as desired.  $\square$

**Solution (c).**

We have that under the measure  $Q$  the dynamics of  $W$  is given by the Girsanov Theorem

$$dW(t) = \varphi dt + dW_t^Q,$$

where  $\varphi$  is the Girsanov kernel associated with  $L$ . Then we know that  $X$  has dynamics under the  $Q$  measure:

$$\begin{aligned} dX(t) &= \frac{1}{2} X(t) dt + Y(t) (\varphi dt + dW_t^Q) \\ &= \left( \frac{1}{2} X(t) + \varphi Y(t) \right) dt + Y(t) dW_t^Q \end{aligned}$$

and so we would have that  $X$  is a martingale under  $Q$  if

$$\varphi_t = -\frac{1}{2} \frac{X(t)}{Y(t)} = -\frac{1}{2} \frac{\sinh(W(t))}{\cosh(W(t))} = -\frac{1}{2} \tanh(W(t)).$$

Then we can define a Likelihood process with initial condition  $L_0 = 1$  and dynamics  $dL_t = \varphi_t L_t dW(t)$  i.e.  $L$  is the solution

$$L_t = \exp \left\{ \int_0^t -\frac{1}{2} \tanh(W(s)) dW(s) - \frac{1}{2} \int_0^t \left( -\frac{1}{2} \tanh(W(s)) \right)^2 ds \right\} > 0.$$

We lastly show that the Novikov condition is satisfied i.e.

$$E^P \left[ e^{\frac{1}{2} \int_0^T \|\varphi_s\|^2 ds} \right] = E^P \left[ e^{\frac{1}{8} \int_0^T \tanh^2(W(s)) ds} \right] < \infty$$

and so  $L$  is a  $P$ -martingale and  $L$  is a Likelihood process. We thus have found a Likelihood process such that  $X$  is a martingale under the measure  $Q$  given by  $dQ = L_T dP$ .  $\square$

**Problem 2**

Consider a standard Black-Scholes model, that is, a model consisting of a bank account  $B(t)$  with  $P$ -dynamics given by

$$dB(t) = rB(t) dt,$$

with  $B(0) = 1$  and a stock  $S(t)$  with  $P$ -dynamics given by

$$dS(t) = \alpha S(t) dt + \sigma S(t) d\bar{W}(t),$$

with  $S(0) = s > 0$  where  $r, \alpha \in \mathbb{R}$  and  $\sigma > 0$  are constants and  $\bar{W}(t)$  is a  $P$ -Brownian motion. Let  $T > 0$  be a given and fixed (expiry) date.

Consider the derivative that at time  $T$  pays  $X = \min \left[ \max \left[ S(T), K_1 \right], K_2 \right]$  where  $0 < K_1 < K_2$  are constants. Let  $F(t, s)$  be the pricing function of the derivative.

- a.
  - i. Determine the equations satisfied by the pricing function  $F(t, s)$ .
  - ii. Find a hedging portfolio for the derivative  $X$ . (Hint: Draw a picture of the payoff function).

Let  $h(t) = (h_0(t), h_1(t))$  be a self-financing portfolio given by

$$h_0(t) = (1 - u) \frac{V^h(t)}{B(t)}, \quad h_1(t) = u \frac{V^h(t)}{S(t)}$$

where  $u$  is a constant and set  $V^h(0) = 1$ . Note that  $h_0(t)$  is the number of units of the bank account at time  $t$ , and  $h_1(t)$  is the number of shares in the stock at time  $t$ , and  $V^h(t)$  denotes the associated value process. Consider the derivative that at time  $T$  pays  $Y = \sqrt{V^h(T)}$ .

- b. Determine the arbitrage free price of derivative  $Y$  at time  $t = 0$ .

**Solution (a).**

(i): In the Black-Scholes model the derivatives pricing process must satisfy the boundary value problem

$$\begin{aligned} rF(t, s) &= F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2(t, s)F_{ss}(t, s), \\ F(T, s) &= \min \left[ \max \left[ s, K_1 \right], K_2 \right]. \end{aligned}$$

as desired.  $\square$

(ii): We see that the derivative is the so-called bull-spread given by the payout

$$X = \begin{cases} K_2 & S_T > K_2, \\ S_T & K_1 \leq S_T \leq K_2, \\ K_1 & S_T < K_1. \end{cases}$$

We can replicate this with a buy-and-hold strategy where we long one stock giving payout  $S_T$ , shorting a call option with strike  $K_2$  giving the payout  $K_2 - S_T$  on  $(S_T > K_2)$  and lastly longing a put option with strike  $K_1$  giving payout  $K_1 - S_T$  on  $(S_T < K_1)$ . Then we arrive at the desired payout.

Using the put-call parity we see that we can replicate this with the following portfolio:

- Long one stock,

- short one European call option with strike  $K_2$ ,
- $K_1$  zero-coupon bonds,
- long one European call option with strike  $K_1$  and
- short one stock.

Which obviously reduces to the portfolio

- short one European call option with strike  $K_2$ ,
- $K_1$  zero-coupon bonds and
- long one European call option with strike  $K_1$ .

Furthermore, we have the value process of the portfolio given by

$$V_t^h = K_1 e^{-r(T-t)} - c(K_2; t, T) + c(K_1; t, T)$$

as desired.  $\square$

**Solution (b).**

We have per assumption that  $h$  is self-financing. Then we know that

$$\begin{aligned} dV^h(t) &= h_0(t) dB(t) + h_1(t) dS(t) \\ &= h_0(t)rB(t) dt + h_1(t)\alpha S(t) dt + h_1(t)\sigma S(t) d\bar{W}(t) \\ &= ((1-u)V^h(t)r + uV^h(t)\alpha) dt + uV^h(t)\sigma d\bar{W}(t) \\ &= ((1-u)r + u\alpha) V^h(t) dt + u\sigma V^h(t) d\bar{W}(t). \end{aligned}$$

That is  $V^h$  is a GBM with initial condition  $V^h(0) = 1$  (per assumption). Under the martingale measure  $Q$  we have the dynamics as follows

$$dV^h(t) = rV^h(t) dt + u\sigma V^h(t) dW^Q(t)$$

This gives that  $V^h$  takes the representation

$$\begin{aligned} V^h(t) &= V^h(0) \cdot \exp \left\{ \left( r - \frac{1}{2}u^2\sigma^2 \right) t + u\sigma W^Q(t) \right\} \\ &= \exp \left\{ \left( r - \frac{1}{2}u^2\sigma^2 \right) t + u\sigma W^Q(t) \right\}. \end{aligned}$$

and so we have using the risk neutral valuation formula the price process

$$\begin{aligned} F(0, s) &= e^{-r(T-0)} E_{0,s}^Q \left[ \sqrt{V^h(0)} \exp \left\{ \frac{1}{2} \left( r - \frac{1}{2}u^2\sigma^2 \right) T + \frac{1}{2}u\sigma W^Q(T) \right\} \right] \\ &= e^{-rT} e^{\frac{1}{2}(r-\frac{1}{2}u^2\sigma^2)T} E^Q \left[ \exp \left\{ \frac{1}{2}u\sigma W^Q(T) \right\} \right] \\ &= e^{-rT} e^{\frac{1}{2}(r-\frac{1}{2}u^2\sigma^2)T} e^{\frac{1}{2}(T-0)\frac{1}{4}\sigma^2 u^2} \\ &= e^{-\frac{1}{2}(r+\frac{1}{4}u^2\sigma^2)T}. \end{aligned}$$

Hence the price of the derivative at time 0 is  $\Pi_0[Y] = e^{-\frac{1}{2}(r+\frac{1}{4}u^2\sigma^2)T}$ .  $\square$

**Problem 3**

Let  $W_1(t)$  and  $W_2(t)$  be two independent  $P$ -Brownian motions. Let the filtration  $\mathcal{F}_t$  be the one generated by the two Brownian motions.

Consider a market model with two assets: A bank account  $B(t)$  and a stock  $S_1(t)$ . The  $P$ -dynamics of  $B(t)$  is

$$dB(t) = rB(t) dt,$$

and  $B(0) = 1$  where  $r \in \mathbb{R}$  is a constant interest rate. The  $P$ -dynamics of  $S_1(t)$  is given by

$$dS_1(t) = \alpha_1 S_1(t) dt + \sigma_1 S_1(t) \left( \sqrt{1 - \rho^2} dW_1(t) + \rho dW_2(t) \right),$$

and  $S_1(0) = s_1 > 0$  where  $\alpha_1 \in \mathbb{R}$ ,  $\sigma > 0$  and  $-1 < \rho < 1$  are constants. Let  $T > 0$  be a given and fixed (expiry) date.

- a.
  - i. Is the model arbitrage free?
  - ii. Is the model complete?
- b.
  - i. Show that  $W_\rho(t) = \sqrt{1 - \rho^2} W_1(t) + \rho W_2(t)$  is a Brownian motion. (Hint: Levy Characterization of Brownian motion).
  - ii. Show that  $dW_2(t)dW_\rho(t) = \rho dt$ .

Consider the derivative that at time  $T$  pays  $X = W_\rho(T)$ .

- c. Show that the arbitrage free price of derivative  $X$  is unique, that is, all choices of equivalent martingale measures produce the same arbitrage free price of derivative  $X$ .

Consider the new derivative that at time  $T$  pays  $Y = W_2(T)$ .

- d. Show that different choices of equivalent martingale measures give different arbitrage free prices of derivative  $Y$ .

For the remainder of this problem, assume that the market model is extended to include a second stock with price process  $S_2(t)$ . The  $P$ -dynamics of  $S_2(t)$  is given by

$$dS_2(t) = \alpha_2 S_2(t) dt + \sigma_2 S_2(t) dW_2(t),$$

with  $S_2(0) = s_2 > 0$  where  $\alpha_2 \in \mathbb{R}$  and  $\sigma_2 > 0$  are constants. The new model  $(B, S_1, S_2)$  is a two-dimensional Black-Scholes model.

- e.
  - i. Show that the model is arbitrage free and complete.
  - ii. Find a hedging portfolio for derivative  $Y$ .

**Solution (a).**

(i): By the meta theorem we should have that the market is arbitrage free but not complete since the number of random sources are greater than the underlying risky assets. We will show this by seeing that we may define the market as in chapter 14:

$$dS_1(t) = S_t \alpha_1 dt + S_t \begin{bmatrix} \sigma_1 \sqrt{1 - \rho^2} & \sigma_1 \rho \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}$$

And so we are to solve the equation

$$\begin{bmatrix} \sigma_1 \sqrt{1 - \rho^2} & \sigma_1 \rho \end{bmatrix} \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = r - \alpha_1$$

which amounts to the system

$$\sqrt{1-\rho^2}\varphi_1(t) + \rho\varphi_2(t) = \frac{r-\alpha_1}{\sigma_1}$$

i.e.

$$\varphi_2(t) = -\frac{\sqrt{1-\rho^2}}{\rho}\varphi_1(t) + \frac{1}{\rho}\frac{r-\alpha_1}{\sigma_1}$$

i.e. we have that  $(\varphi_1, \varphi_2) = (\varphi, a\varphi + b)$  for some  $\varphi \in \mathbb{R}$  and  $a, b$  define as

$$a = -\frac{\sqrt{1-\rho^2}}{\rho}, \quad b = \frac{1}{\rho}\frac{r-\alpha_1}{\sigma_1}.$$

We are left to show the Novikov condition.

$$E^P \left[ e^{\frac{1}{2} \int_0^T \|\varphi(s)\|^2 ds} \right] = e^{\frac{1}{2} T(\varphi^2 + (a\varphi+b)^2)} < \infty$$

for any choice of  $\varphi$ . Hence we may define the martingale  $L_t$  as  $dL_t = L_t \varphi(t)^\top dW(t)$  with  $L_0 = 1$ . It follows that  $Q$  defined by  $dQ = L_T dP$  on  $\mathcal{F}_T$  is a martingale measure and so we have found a martingale. The conclusion from proposition 14.1 is that the model is arbitrage free.  $\square$

(ii): As  $\phi$  is chosen free we have that  $Q$  may be indexed with  $\varphi$  and so it is not unique. The conclusion from proposition 14.7 is that the model is not complete.  $\square$

### Solution (b).

(i): We know from Levy characterisation of Brownian motions that  $W$  is a Brownian motion if and only if  $W$  and  $W^2 - t$  is martingales. Per definition is  $W_\rho$  a weighted sum of two martingales and so itself is a martingale. Now we see that by defining  $X_t = \sqrt{1-\rho^2}W_1(t)$  and  $Y_t = \rho W_2(t)$  we have from Ito's formula on  $f(t, x, y) = (x+y)^2 - t = x^2 + y^2 + 2xy - t$  the dynamics of  $W_\rho^2 - t$  as

$$\begin{aligned} d(W_\rho^2 - t) &= df(t, X_t, Y_t) \\ &= -1 dt + (2X_t + 2Y_t) dX_t + (2Y_t + 2X_t) dY_t \\ &\quad + 2(dX_t)(dY_t) + \frac{1}{2}2(dX_t)^2 + \frac{1}{2}2(dY_t)^2 \\ &= -dt + (2X_t + 2Y_t)(dX_t + dY_t) + (dX_t)^2 + (dY_t)^2 \\ &= -dt + \left(2\sqrt{1-\rho^2}W_1(t) + 2\rho W_2(t)\right) \left(\sqrt{1-\rho^2}dW_1(t) + \rho dW_2(t)\right) + (1-\rho^2)dt + \rho^2 dt \\ &= \left(2\sqrt{1-\rho^2}W_1(t) + 2\rho W_2(t)\right) \left(\sqrt{1-\rho^2}dW_1(t) + \rho dW_2(t)\right). \end{aligned}$$

And so  $W_\rho^2 - t$  has only dynamics wrt. the Brownian motions. This implies that  $W_\rho^2 - t$  is a martingale. By Levy characterisation of Brownian motions we have that  $W_\rho$  is a Brownian motion.  $\square$

(ii): We compute the product of  $W_2$  and  $W_\rho$  dynamics:

$$\begin{aligned} dW_2(t)dW_\rho(t) &= dW_2(t) \left(\sqrt{1-\rho^2}dW_1(t) + \rho dW_2(t)\right) \\ &= \rho dt, \end{aligned}$$

as the only non-zero term is  $(dW_2(t))^2 = dt$ .  $\square$

**Solution (c).**

For a choice of Girsanov kernel written in question 3.(a) we have that the dynamics under the new measure  $Q_\varphi$  is given by

$$dW(t) = \varphi_t dt + dW^{Q_\varphi}(t)$$

Then it follows that

$$dW_\rho(t) = \begin{bmatrix} \sqrt{1-\rho^2} & \rho \end{bmatrix} \begin{bmatrix} \varphi dt + dW_1^{Q_\varphi}(t) \\ (a\varphi + b) dt + dW_2^{Q_\varphi}(t) \end{bmatrix}$$

Then we have that

$$\begin{aligned} W_\rho(T) &= \sqrt{1-\rho^2} \left( \varphi T + W_1^{Q_\varphi}(T) \right) + \rho \left( (a\varphi + b) T + dW_2^{Q_\varphi}(T) \right) \\ &= \sqrt{1-\rho^2} \left( \varphi T + W_1^{Q_\varphi}(T) \right) + \rho \left( \left( -\frac{\sqrt{1-\rho^2}}{\rho} \varphi + \frac{1}{\rho} \frac{r-\alpha_1}{\sigma_1} \right) T + dW_2^{Q_\varphi}(T) \right) \\ &= \sqrt{1-\rho^2} \varphi T + \rho \left( -\frac{\sqrt{1-\rho^2}}{\rho} \varphi + \frac{1}{\rho} \frac{r-\alpha_1}{\sigma_1} \right) T + \sqrt{1-\rho^2} W_1^{Q_\varphi}(T) + \rho dW_2^{Q_\varphi}(T) \\ &= \left( \sqrt{1-\rho^2} \varphi - \sqrt{1-\rho^2} \varphi + \frac{r-\alpha_1}{\sigma_1} \right) T + \sqrt{1-\rho^2} W_1^{Q_\varphi}(T) + \rho dW_2^{Q_\varphi}(T) \\ &= \left( \frac{r-\alpha_1}{\sigma_1} \right) T + \sqrt{1-\rho^2} W_1^{Q_\varphi}(T) + \rho dW_2^{Q_\varphi}(T). \end{aligned}$$

Hence under any choice of  $\varphi$  the expected value under the measure  $Q_\varphi$  is the same and so the arbitrage free price of the derivative  $X$  is unique.  $\square$

**Solution (d).**

We have that under the  $Q_\varphi$  the dynamics of  $W_2$  is given by

$$dW_2(t) = (a\varphi + b) dt + dW_2^{Q_\varphi}(t)$$

Then we have

$$W_2(T) = (a\varphi + b) T + W_2^{Q_\varphi}(T),$$

and so it clearly follows that the price does indeed depend on  $\varphi$  and so the derivative does not have an unique arbitrage free price.  $\square$

**Solution (e).**

(i): We follow along the lines of the question a. We have now the market with dynamics

$$dS(t) = \begin{bmatrix} dS_1(t) \\ dS_2(t) \end{bmatrix} = \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} dt + \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix} \begin{bmatrix} \sigma_1 \sqrt{1-\rho^2} & \sigma_1 \rho \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}.$$

Then the equation to solve is

$$\begin{bmatrix} \sigma_1 \sqrt{1-\rho^2} & \sigma_1 \rho \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = \begin{bmatrix} r - \alpha_1 \\ r - \alpha_2 \end{bmatrix}.$$

Clearly  $\sigma$  is invertible as

$$\det(\sigma) = \sigma_1 \sigma_2 \sqrt{1-\rho^2} \neq 0.$$

Furthermore, the inverse is given by

$$\sigma^{-1} = \begin{bmatrix} \sigma_1 \sqrt{1-\rho^2} & \sigma_1 \rho \\ 0 & \sigma_2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1 \sqrt{1-\rho^2}} & -\frac{\rho^2}{\sigma_2 \sqrt{1-\rho^2}} \\ 0 & \frac{1}{\sigma_2} \end{bmatrix}.$$

We then have that the Girsanov kernel is given by

$$\varphi = \begin{bmatrix} \frac{1}{\sigma_1 \sqrt{1-\rho^2}} & -\frac{\rho^2}{\sigma_2 \sqrt{1-\rho^2}} \\ 0 & \frac{1}{\sigma_2} \end{bmatrix} \begin{bmatrix} r - \alpha_1 \\ r - \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{r - \alpha_1}{\sigma_1 \sqrt{1-\rho^2}} - \frac{\rho^2(r - \alpha_1)}{\sigma_2 \sqrt{1-\rho^2}} \\ \frac{r - \alpha_2}{\sigma_2} \end{bmatrix}.$$

The Novikov condition is clearly satisfied and so the market is arbitrage free. Since the matrix  $\sigma$  is invertible we have that  $\ker(\sigma) = \{0\}$  and so the market is complete.  $\square$

(ii): Under the measure  $Q$  given by the Girsanov kernel in the above we have that

$$dW_2(t) = \frac{r - \alpha_2}{\sigma_2} dt + dW_2^Q(t)$$

and so the arbitrage free price is given by the risk neutral valuation formula.

$$\begin{aligned} \Pi_t[Y] &= e^{-r(T-t)} E_{t,s}^Q [W_2[T]] \\ &= e^{-r(T-t)} E^Q \left[ \frac{r - \alpha_2}{\sigma_2} T + W_2^Q[T] - W_2^Q[t] + W_2^Q[t] \right] \\ &= e^{-r(T-t)} \left( \frac{r - \alpha_2}{\sigma_2} T + W_2^Q(t) \right) \end{aligned}$$

On the assumption  $S_2(t) = s_2$  we have

$$s_2 = S_2(0) \exp \left\{ \left( r - \frac{1}{2} \sigma_2^2 \right) t + \sigma_2 W_2^Q(t) \right\} \iff W_2^Q(t) = \frac{1}{\sigma_2} \log \frac{s_2}{S_2(0)} - \frac{1}{\sigma_2} \left( r - \frac{1}{2} \sigma_2^2 \right) t$$

and so

$$\Pi_t[Y] = e^{-r(T-t)} \left( \frac{r - \alpha_2}{\sigma_2} T + \frac{1}{\sigma_2} \log \frac{s_2}{S_2(0)} - \frac{1}{\sigma_2} \left( r - \frac{1}{2} \sigma_2^2 \right) t \right)$$

And so we see that we have to hold the following portfolio to hedge the derivative

$$\begin{aligned}
h_1(t) &= \frac{\partial \Pi_t[Y]}{\partial s_1}(t, S_1, S_2) = 0, \\
h_2(t) &= \frac{\partial \Pi_t[Y]}{\partial s_2}(t, S_1, S_2) \\
&= e^{-r(T-t)} \frac{1}{\sigma_2 S_2} \\
h_0(t) &= \frac{1}{B(t)} \left( \Pi_t[Y] - S_1 \frac{\partial \Pi_t[Y]}{\partial s_1}(t, S_1, S_2) - S_2 \frac{\partial \Pi_t[Y]}{\partial s_2}(t, S_1, S_2) \right) \\
&= \frac{1}{B(t)} \left( e^{-r(T-t)} \left( \frac{r - \alpha_2}{\sigma_2} T + \frac{1}{\sigma_2} \log \frac{s_2}{S_2(0)} - \frac{1}{\sigma_2} \left( r - \frac{1}{2} \sigma_2^2 \right) t \right) - e^{-r(T-t)} \frac{1}{\sigma_2} \right) \\
&= \frac{1}{B(t)} e^{-r(T-t)} \frac{1}{\sigma_2} \left( (r - \alpha_2) T + \log \frac{s_2}{S_2(0)} - \left( r - \frac{1}{2} \sigma_2^2 \right) t - 1 \right).
\end{aligned}$$

as desired.  $\square$

## Exam 2020/21

### Problem 1

#### Solution (a).

We have from Ito's formula that the stochastic process  $M(t) = e^{-bX(t)} = f_b(t, X(t))$  with  $f_b(t, x) = e^{-bx}$  has dynamics

$$\begin{aligned}
dM(t) &= df(t, X(t)) \\
&= 0 \, dt - bM(t) \, dX(t) + \frac{1}{2} b^2 M(t) (dX(t))^2 \\
&= -bM(t) \mu X(t) \, dt - bM(t) \sigma \sqrt{X(t)} \, dW(t) + \frac{1}{2} b^2 M(t) \sigma^2 X(t) \, dt.
\end{aligned}$$

and so it clearly follow that  $M$  is a martingale if

$$-bM(t) \mu X(t) + \frac{1}{2} b^2 M(t) \sigma^2 X(t) = 0.$$

Isolating  $b$  yields

$$b \left( \frac{1}{2} b M(t) \sigma^2 X(t) - M(t) \mu X(t) \right) = 0 \iff b \in \left\{ 0, 2 \frac{M(t) \mu X(t)}{M(t) \sigma^2 X(t)} \right\} = \left\{ 0, 2 \frac{\mu}{\sigma^2} \right\}.$$

Hence the non-trivial choice of  $b$  is  $b = 2\mu/\sigma^2$ .  $\square$

#### Solution (b).

(i): We start by seeing that

$$X(t) = X(0) + \mu \int_0^t X(s) \, ds + \sigma \int_0^t \sqrt{X(s)} \, dW(s).$$

Using that  $X(0) = 1$  and that the last term is a martingale with mean 0 we have



$$E[X(t)] = 1 + \mu \int_0^t E[X(s)] ds,$$

that is,  $E[X(t)]$  satisfies the differential equation

$$dE[X(t)] = \mu E[X(t)] dt, \quad E[X(0)] = 1.$$

Then we have that the mean value is the solution to the above equation i.e.

$$E[X(t)] = E[X(0)]e^{\mu t} = e^{\mu t}$$

hence

$$E[X(T)] = e^{\mu T}$$

as desired.  $\square$

(ii): We assume that  $\mu = 0$  and so  $E[X(T)] = 1$ . Furthermore, by definition we have

$$\text{Var}[X(T)] = E[X(T)^2] - (E[X(T)])^2 = E[X(T)^2] - 1.$$

We follow along the lines of (i) when computing the second moment of  $X(T)$ . We have by Ito's formula on  $Z(t) = X(t)^2 = f(t, X(t))$  with  $f(t, x) = x^2$ :

$$\begin{aligned} dZ(t) &= df(t, X(t)) \\ &= 0 dt + 2X(t) dX(t) + \frac{1}{2} 2 (dX(t))^2 \\ &= 2\mu(X(t))^2 dt + \sigma(X(t))^{3/2} dW(t) + \sigma^2 X(t) dt \\ &= \left( 2\mu Z(t) + \sigma^2 \sqrt{Z(t)} \right) dt + \sigma(Z(t))^{3/4} dW(t) \\ &= \sigma^2 \sqrt{Z(t)} dt + \sigma(Z(t))^{3/4} dW(t) \end{aligned}$$

This implies that

$$Z(t) = Z(0) + \int_0^t \sigma^2 \sqrt{Z(s)} ds + \int_0^t \sigma(Z(s))^{3/4} dW(s).$$

Taking expectation and using  $Z(0) = X(0)^2 = 1$ :

$$\begin{aligned} E[Z(t)] &= 1 + \int_0^t \sigma^2 E[\sqrt{Z(s)}] ds = 1 + \int_0^t \sigma^2 E[X(s)] ds \\ &= 1 + \sigma^2 \int_0^t e^{\mu s} ds = 1 + \sigma^2 \int_0^t ds \\ &= 1 + \sigma^2 t \end{aligned}$$

hence we have the variance

$$\text{Var}[X(T)] = 1 + \sigma^2 T - 1 = \sigma^2 T$$

as desired.  $\square$

**Solution (c).**

If  $L$  is a Likelihood proces then  $dQ = L(T)dP$  determines a martingale measure under which

$$dW(t) = \varphi(t) dt + dW^Q(t).$$

We futhermore have that  $Z(t) = 1/Y(t)$  has  $P$  dynamics given by

$$\begin{aligned} dZ(t) &= d(e^{t/2 - W(t)}) \\ &= \frac{1}{2}Z(t) dt - Z(t) dW(t) + \frac{1}{2}Z(t) (dW(t))^2 \\ &= \frac{1}{2}Z(t) dt - Z(t) dW(t) + \frac{1}{2}Z(t) dt \\ &= Z(t) dt - Z(t) dW(t) \end{aligned}$$

and so under  $Q$  we have from Girsanov theorem

$$\begin{aligned} dZ(t) &= Z(t) dt - Z(t) (\varphi(t) dt + dW^Q(t)) \\ &= Z(t) dt - Z(t)\varphi(t) dt + Z(t) dW^Q(t) \\ &= (1 - \varphi(t))Z(t) dt + Z(t) dW^Q(t) \end{aligned}$$

and so we see that  $\varphi = 1$  if  $Z$  are to be a martingale under  $Q$  i.e.  $Q = P$ .  $\square$

**Problem 2**

**Solution (a).**

(i): We search for a replicating buy-and-hold portfolio that hedges  $X$ . We see that we have the following inequality

$$K < K + \Delta < K + 3\Delta$$

We have that the payout should be zero below the price  $K > 0$ . We start by cosidering a short position in a European call option with strike  $K$  giving the payout  $K - S_T$  on  $(S_T > K)$ . Then on the event  $(S_T > K + \Delta)$  we nee an additional

$$S_T - K - 2\Delta - (K - S_T) = 2S_T - 2K - 2\Delta = 2(S_T - (K + \Delta))$$

i.e. the payot of two European call options with strike  $K + \Delta$ . Then on the event  $(S_T > K + 3\Delta)$  we then need the additional payout

$$\Delta - (S_T - K - 2\Delta) = -S_T + K + 3\Delta = -(S_T - (K + 3\Delta))$$

Which can be replicated with one short position on a European call option with strike  $K + 3\Delta$ . Then the portfolio is in total

- Short one European call option with strike  $K$ ,
- Long two European call option with strike  $K + \Delta$ ,
- Short one European call option with strike  $K + 3\Delta$ .

We check that the portfolio indeed does replicate  $X$ .

$$X = \begin{cases} 0 + 0 + 0 = 0 & S_T \leq K, \\ -(S_T - K) + 0 + 0 = K - S_T & K < S_T \leq K + \Delta, \\ -(S_T - K) + 2(S_T - (K + \Delta)) + 0 = S_T - K - 2\Delta & K + \Delta < S_T \leq K + 3\Delta, \\ -(S_T - K) + 2(S_T - (K + \Delta)) - (S_T - (K + 3\Delta)) = \Delta & S_T > K + 3\Delta. \end{cases}$$

as desired. Then we have that the arbitrage free price is given by the value process of the above portfolio i.e.

$$\Pi_t[X] = V_t^h = -c(K; t, T) + 2c(K + \Delta; t, T) - c(K + 3\Delta; t, T)$$

where  $c$  denotes the pricing function for the European call option.  $\square$

(ii): We have from the Black-Scholes equation 7.10 that the pricing function must satisfy

$$rF(t, s) = F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2F_{ss}(t, s)$$

with condition  $F(T, s) = X$  where  $X$  is given in the problem formulation.  $\square$

**Solution (b).**

(i): We see that at time  $t = T_0$  the payout is a simple claim in that  $S(T_0)$  is  $\mathcal{F}_{T_0}$ -measurable. Furthermore, we see that the payout is the same as a European put option with strike  $K = S(T_0)$ . Such a derivative may be hedged with  $S(T_0)$  zero-coupon bonds, one call options with the same strike and a short position in the underlying stock. The price is then, in detail, given as.

$$\begin{aligned} \Pi_{T_0}[Y] &= S(T_0)e^{-r(T-T_0)} + c(S(T_0); T_0, T) - S(T_0) \\ &= S(T_0) \left( e^{-r(T-T_0)} - 1 \right) + S(T_0)N(d_1(T_0, S(T_0))) - e^{-r(T-T_0)}S(T_0)N(d_2(T_0, S(T_0))) \\ &= S(T_0) \left\{ e^{-r(T-T_0)} - 1 + N(d_1(T_0, S(T_0))) - e^{-r(T-T_0)}N(d_2(T_0, S(T_0))) \right\} \\ &= S(T_0)p. \end{aligned}$$

where we set

$$p = e^{-r(T-T_0)} - 1 + N(d_1(T_0, S(T_0))) - e^{-r(T-T_0)}N(d_2(T_0, S(T_0))).$$

We simply now need to show that this does indeed not depend on  $S(T_0)$ . We see that if the components  $d_1$  and  $d_2$  does not depend on  $S(T_0)$  we are done. We have

$$\begin{aligned} d_1(T_0, S(T_0)) &= \frac{1}{\sigma\sqrt{T-T_0}} \left( \log \left( \frac{S(T_0)}{S(T_0)} \right) + \left( r + \frac{1}{2}\sigma^2 \right) (T - T_0) \right) \\ &= \frac{1}{\sigma\sqrt{T-T_0}} \left( r + \frac{1}{2}\sigma^2 \right) (T - T_0). \end{aligned}$$

and so  $d_1$  does not depend on  $S(T_0)$ . This by extension means that  $d_2$  as well does not depend on  $S(T_0)$  as  $d_2(t, s) = d_1(t, s) - \sigma\sqrt{T-t}$ .  $\square$

(ii): As  $p$  does not depend on  $S(T_0)$  we would have that a portfolio consisting of  $p$  stocks would have the desired payout at time  $t = T_0$  to finance the portfolio on  $[T_0, T]$ . Hence we have the pricing process on  $t < T_0$  given as  $\Pi_t[Y] = pS(t)$ .  $\square$

**Solution (c).**

We hope that there exists a  $\tilde{p}$  such that  $\Pi_{T_0}[\tilde{Y}] = \tilde{p}X$  and  $\tilde{p}$  does not depend on  $X$ , where  $X$  is the payout of a buy-and-hold portfolio. We have that the payout is hedged by a call option with strike  $S(T_0)$  on  $[T_0, T]$  i.e.

$$\begin{aligned}\Pi_{T_0}[\tilde{Y}] &= S(T_0)N(d_1(T_0, S(T_0))) - e^{-r(T-T_0)}S(T_0)N(d_2(T_0, S(T_0))) \\ &= S(T_0) \left\{ N(d_1(T_0, S(T_0))) - e^{-r(T-T_0)}N(d_2(T_0, S(T_0))) \right\} \\ &= S(T_0)\tilde{p}.\end{aligned}$$

for  $t \in [T_0, T]$ . Per 2.a we know that  $\tilde{p}$  does not depend on  $S(T_0)$  and so we have the pricing process  $\Pi_t[\tilde{Y}] = \tilde{p}S(t)$  on  $t < T_0$ . Then we simply have

$$\Pi_t[Y] - \Pi_t[\tilde{Y}] = (p - \tilde{p})S(t)$$

by inserting  $p$  and  $\tilde{p}$  we have

$$\Pi_t[Y] - \Pi_t[\tilde{Y}] = (e^{-r(T-T_0)} - 1)S(t)$$

as desired.  $\square$

**Solution (d).**

**THE FOLLOWING WAS AN ATTEMPT AT CALCULATING THE INTEGRALS RIGOROUSLY.** It did not turn out well.. After “END OF TRY” below the solution is given..

We start by calculating the right hand side. We see that  $(t - s)$  is approximated on  $[0, t]$  with the following sequence of simple functions:

$$g^n(s) = \sum_{i=0}^{n-1} 1_{[i \cdot t/n, (i+1)t/n]}(s)(t - t_i).$$

Obviously, we have

$$\int_0^t (g^n(s) - (t - s))^2 ds = \int_0^t \sum_{i=0}^{n-1} 1_{[i \cdot t/n, (i+1)t/n]}(s)(s - t_i)^2 ds = \sum_{i=0}^{n-1} \int_{i \cdot t/n}^{(i+1) \cdot t/n} (s - t_i)^2 ds \rightarrow 0.$$

We then define the integral as follows

$$\begin{aligned}
\int_0^t (t-u) dW(u) &= \lim_{n \rightarrow \infty} \left\{ \int_0^t g^n(s) dW(s) \right\} \\
&= \lim_{n \rightarrow \infty} \left\{ \int_0^t \sum_{i=0}^{n-1} 1_{[i \cdot t/n, (i+1)t/n]}(s) (t - t_i) dW(s) \right\} \\
&= \lim_{n \rightarrow \infty} \left\{ \sum_{i=0}^{n-1} \int_{i \cdot t/n}^{(i+1) \cdot t/n} (t - t_i) dW(s) \right\} \\
&= \lim_{n \rightarrow \infty} \left\{ \sum_{i=0}^{n-1} (t - t_i) \left( W\left(\frac{(i+1) \cdot t}{n}\right) - W\left(\frac{i \cdot t}{n}\right) \right) \right\} \\
&= t - t \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} i \left( W\left(\frac{(i+1) \cdot t}{n}\right) - W\left(\frac{i \cdot t}{n}\right) \right) \right\}.
\end{aligned}$$

If we define the random variable inside lim as  $S_n$  we see that

$$\begin{aligned}
E[S_n] &= \frac{1}{n} \sum_{i=0}^{n-1} i E \left( W\left(\frac{(i+1) \cdot t}{n}\right) - W\left(\frac{i \cdot t}{n}\right) \right) \\
&= \frac{1}{n} \sum_{i=0}^{n-1} i \cdot 0 = 0.
\end{aligned}$$

and

$$\begin{aligned}
E[(S_n)^2] &= \frac{1}{n^2} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} ij E \left\{ \left( W\left(\frac{(i+1) \cdot t}{n}\right) - W\left(\frac{i \cdot t}{n}\right) \right) \left( W\left(\frac{(j+1) \cdot t}{n}\right) - W\left(\frac{j \cdot t}{n}\right) \right) \right\} \\
&= \frac{1}{n^2} \sum_{i=0}^{n-1} i^2 E \left\{ \left( W\left(\frac{(i+1) \cdot t}{n}\right) - W\left(\frac{i \cdot t}{n}\right) \right)^2 \right\} \\
&= \frac{1}{n^2} \sum_{i=0}^{n-1} i^2 \left\{ \frac{(i+1) \cdot t}{n} - \frac{i \cdot t}{n} \right\} \\
&= \frac{t}{n^3} \sum_{i=0}^{n-1} i^2 = \frac{t}{n^3} \frac{1}{6} (n-1)n(2n-1) \\
&= \frac{t}{6n^2} (n-1)(2n-1) = \frac{t}{6n^2} (2n^2 - n - 2n + 1) \\
&= \frac{t}{3} - \frac{t}{2n} + \frac{t}{6n^2} \rightarrow \frac{t}{3}
\end{aligned}$$

**END OF TRY.**

(i): We start by calculating the dynamics of  $X(u) = (t-u)W(u)$ . We use Ito's formula

$$dX(u) = -W(u) du + (t-u) dW(u)$$

and so

$$X(t) = 0 = - \int_0^t W(s) ds + \int_0^t (t-s) dW(s).$$

Rearranging give the desired result.  $\square$

(ii): We have that under the  $Q$ -measure  $S$  is a GBM and under assumption  $S(0) = s$  we have

$$S(t) = s \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) t - \sigma W^Q(t) \right\}$$

hence we see that

$$\log S(t) = \log(s) + \left( r - \frac{1}{2} \sigma^2 \right) t - \sigma W^Q(t).$$

Integrating from  $t = 0$  to  $t = T$  gives

$$\begin{aligned} \int_0^T \log S(t) dt &= \log(s)T + \left( r - \frac{1}{2} \sigma^2 \right) \int_0^T t dt - \sigma \int_0^T W^Q(t) dt \\ &= \log(s)T + \frac{1}{2} \left( r - \frac{1}{2} \sigma^2 \right) T^2 - \sigma \int_0^T (T-t) dW^Q(t) \end{aligned}$$

using 2.d(i). We recall that from lemma 4.18 that

$$\sigma \int_0^T (T-t) dW^Q(t) \sim \mathcal{N} \left( 0, \sigma^2 \int_s^T (T-t)^2 dt \right).$$

Using this we have that

$$\int_0^T \log S(t) dt \sim \mathcal{N} \left( \log(s)T + \frac{1}{2} \left( r - \frac{1}{2} \sigma^2 \right) T^2, \sigma^2 \int_s^T (T-t)^2 dt \right).$$

Using the risk neutral valuation formula we have

$$\begin{aligned} \Pi_0[Z] &= e^{-rT} E^Q \left( \left( \int_0^T \log S(t) dt \right)^2 \right) \\ &= e^{-rT} \left( \text{Var}^Q \left( \int_0^T \log S(t) dt \right) - E^Q \left( \int_0^T \log S(t) dt \right)^2 \right) \\ &= e^{-rT} \left( \sigma^2 \int_s^T (T-t)^2 dt - \log^2(s)T^2 - \frac{1}{4} \left( r - \frac{1}{2} \sigma^2 \right)^2 T^4 - \log(s)T \left( r - \frac{1}{2} \sigma^2 \right) T^2 \right) \\ &= e^{-rT} \left( \sigma^2 \frac{T^3}{3} - \log^2(s)T^2 - \frac{1}{4} \left( r - \frac{1}{2} \sigma^2 \right)^2 T^4 - \log(s)T \left( r - \frac{1}{2} \sigma^2 \right) T^2 \right) \end{aligned}$$

as desired.  $\square$

**Problem 3****Solution (a).**

We see that we are searching for a unique Girsanov kernel. This amounts to solving

$$\sigma_t \varphi_t = r - \mu$$

or on matrix form

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} \varphi_t = \begin{bmatrix} r - \mu_1 \\ r - \mu_2 \end{bmatrix}.$$

The linear equation system is then

$$\sigma_1 \varphi(t) = r - \mu_1, \quad \sigma_2 \varphi(t) = r - \mu_2$$

Hence

$$\varphi = \frac{r - \mu_1}{\sigma_1} = \frac{r - \mu_2}{\sigma_2}.$$

Which holds if and only if  $\sigma_2(r - \mu_1) = \sigma_1(r - \mu_2)$ . Then if this equation holds we have that the Girsanov kernel exists and is unique. The Novikov condition is furthermore satisfied and so it follows that the model is arbitrage free and unique.  $\square$

**Solution (b).**

We have that the portfolio is self-financing if and only if

$$dV_t^h = h_0(t) dB(t) + h_1(t) dS_1(t) + h_2(t) dS_2(t).$$

Calculating we have

$$\begin{aligned} dV^h(t) &= \frac{\sigma_1 - \sigma_2}{B(t)} dB(t) + \frac{\sigma_2}{S_1(t)} dS_1 + \frac{S_2(t) - \sigma_1}{S_2(t)} dS_2(t) \\ &= (\sigma_1 - \sigma_2)r dt + \sigma_2 \mu_1 dt + \sigma_1 \sigma_2 dW(t) + (S_2(t) - \sigma_1)\mu_2 dt + (S_2(t) - \sigma_1)\sigma_2 dW(t) \\ &= (\sigma_1 r - \sigma_2 r + \sigma_2 \mu_1 - \sigma_1 \mu_2) dt + S_2(t)\mu_2 dt + S_2(t)\sigma_2 dW(t) \\ &= \underbrace{(\sigma_2(\mu_1 - r) - \sigma_1(\mu_2 - r))}_{=0} dt + S_2(t)\mu_2 dt + S_2(t)\sigma_2 dW(t) \\ &= S_2(t)\mu_2 dt + S_2(t)\sigma_2 dW(t) = dS_2(t) \end{aligned}$$

Furthermore, we see that

$$\begin{aligned} V^h(t) &= \frac{\sigma_1 - \sigma_2}{B(t)} B(t) + \frac{\sigma_2}{S_1(t)} S_1 + \frac{S_2(t) - \sigma_1}{S_2(t)} S_2(t) \\ &= \sigma_1 - \sigma_2 + \sigma_2 + S_2(t) - \sigma_1 = S_2(t). \end{aligned}$$

This is consistent with the self-financing condition above as  $dV^h(t) = dS_2(t)$ . We conclude that  $h$  is self-financing.  $\square$

**Solution (c).**

We know that under the measure  $Q$  both  $S_1$  and  $S_2$  are GBM with

$$S_i(t) = s_i \exp \left\{ \left( r - \frac{1}{2} \sigma_i^2 \right) (T - t) + \sigma_i (W^Q(T) - W^Q(t)) \right\}$$

for  $i = 1, 2$  under assumption that  $S_1(t) = s_1$  and  $S_2(t) = s_2$ . Then we have that  $X$  pays out

$$X = \frac{s_1}{s_2} \exp \left\{ \frac{1}{2} (\sigma_2^2 - \sigma_1^2) (T - t) + (\sigma_1 - \sigma_2) (W^Q(T) - W^Q(t)) \right\}.$$

Using the risk neutral valuation formula we have the price

$$\begin{aligned} \Pi_t[X] &= e^{-r(T-t)} E^Q \left[ \frac{s_1}{s_2} \exp \left\{ \frac{1}{2} (\sigma_2^2 - \sigma_1^2) (T - t) + (\sigma_1 - \sigma_2) (W^Q(T) - W^Q(t)) \right\} \right] \\ &= e^{-r(T-t)} \frac{s_1}{s_2} e^{\frac{1}{2} (\sigma_2^2 - \sigma_1^2) (T-t)} E^Q [\exp \{ (\sigma_1 - \sigma_2) (W^Q(T) - W^Q(t)) \}] \\ &= \frac{s_1}{s_2} e^{-r(T-t) + \frac{1}{2} (\sigma_2^2 - \sigma_1^2) (T-t)} e^{\frac{1}{2} (T-t) (\sigma_1 - \sigma_2)^2} \\ &= \frac{s_1}{s_2} e^{-r(T-t) + \frac{1}{2} (\sigma_2^2 - \sigma_1^2) (T-t) + \frac{1}{2} (T-t) (\sigma_1 - \sigma_2)^2} \\ &= \frac{s_1}{s_2} e^{-r(T-t) + \frac{1}{2} (2\sigma_2^2 - 2\sigma_1\sigma_2) (T-t)}. \end{aligned}$$

We now may hedge this derivative with

$$\begin{aligned} h_1(t) &= \frac{\partial \Pi_t[Y]}{\partial s_1}(t, S_1(t), S_2(t)) \\ &= \frac{1}{S_2(t)} e^{-r(T-t) + \frac{1}{2} (2\sigma_2^2 - 2\sigma_1\sigma_2) (T-t)} \\ h_2(t) &= \frac{\partial \Pi_t[Y]}{\partial s_2}(t, S_1(t), S_2(t)) \\ &= -\frac{S_1(t)}{S_2(t)^2} e^{-r(T-t) + \frac{1}{2} (2\sigma_2^2 - 2\sigma_1\sigma_2) (T-t)} \\ h_0 &= \frac{1}{B(t)} (\Pi_t[Y] - S_1(t)h_1(t) - S_2(t)h_2(t)) \\ &= e^{-rt} e^{-r(T-t) + \frac{1}{2} (2\sigma_2^2 - 2\sigma_1\sigma_2) (T-t)} \left( \frac{S_1(t)}{S_2(t)} - S_1(t) \frac{1}{S_2(t)} + S_2(t) \frac{S_1(t)}{S_2(t)^2} \right) \\ &= e^{-rt} e^{-r(T-t) + \frac{1}{2} (2\sigma_2^2 - 2\sigma_1\sigma_2) (T-t)} \frac{S_1(t)}{S_2(t)} = e^{-rt} \Pi_t[Y]. \end{aligned}$$

as desired.  $\square$

**Exam 2021/22****Problem 1**

**Solution (a).**

**Solution (b).**

**Solution (c).**



**Problem 2**

**Solution (a).**

**Solution (b).**

**Solution (c).**

**Solution (d).**

**Solution (e).**

**Solution (f).**

**Solution (g).**

**Exam 2022/23**

Some content