

UNIVERSITY OF COPENHAGEN

MSC IN ACTUARIAL MATHEMATICS

Complete Theory

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Abstract

This document contain a comprehensive outline of theory on probability theory and mathematical statistics applied in finance, life insurance and non-life insurance.

Keywords: *probability theory, insurance mathematics, life insurance, non-life insurance, stochastic differential equations.*

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References

Below is given the abbreviations used when referencing to books:

Chapter	Abbreviation	Source
Finance	Bjork	<i>Arbitrage Theory in Continuous Time (Fourth edition)</i> by Thomas Björk, Oxford University Press (2019).
Measure Theory	Bjork	<i>Arbitrage Theory in Continuous Time (Fourth edition)</i> by Thomas Björk, Oxford University Press (2019).
Linear Algebra	Wiki	Wikipedia
Random Variables	Bjork	<i>Arbitrage Theory in Continuous Time (Fourth edition)</i> by Thomas Björk, Oxford University Press (2019).
	Hansen	<i>Stochastic Processes</i> (2. edition) by Ernst Hansen (2021).
Continuous Time Stochastic Processes	Bjork	<i>Arbitrage Theory in Continuous Time (Fourth edition)</i> by Thomas Björk, Oxford University Press (2019).
Discrete Time Stochastic Processes	Hansen	<i>Stochastic Processes</i> (2. edition) by Ernst Hansen (2021).

Finance

This topic revolves around the theory of the Brownian motion and martingale processes. Other main topics are the binomial model and an introduction to financial derivatives. Financial derivatives is contingent on the outcome of a stochastic process at some future time $t = T$ and often is a function Φ of some assets price S_t . As such the derivative will give a stochastic payout, at time $t = T$ of the size $X_T = \Phi(S_T)$. Naturally we want to say something about the *fair* price of the derivative in the form of

$$\Pi_t(X_T) = \mathbb{E}[\Phi(S_T) \mid \mathcal{F}_t],$$

where $\mathcal{F}_t \subset \mathcal{F}$ is the available information at time t . We will by default interpret the times $t = 0$ as *today* and $t = T$ as *tomorrow*. This indeed require some fundamental understanding of the behaviour of the asset price S_t . This lead us over to discussing the process in center of the *Black-Scholes* model: the Brownian motion.

Discrete time models

One-period time models

The study of this course is the **European call** option (and *put* option). This financial derivative is an agreement between two parties where the holder of the option has the right to “*exercise*” the derivative, at a future time $t = T$. Exercising means buying an asset at a certain agreed upon price-strike K . In the case of the put-option: the holder has the right (but not obligation) to sell the asset at the strike price K . As such the derivative has the payoff

$$\text{Call option: } \Phi(S_T) = (S_T - K)^+, \quad \text{Put option: } \Phi(S_T) = (K - S_T)^+.$$

Our objective is to understand when an arbitrage exist and to find the fair price of these derivative. The strategy in pricing is finding a replicating portfolio with the same payoff as the option (with probability one) and then price the derivative accordingly.

Model description In the one-period model we consider the simplest possible market. We have two distinct times $t = 0$ (today) and $t = 1$ (tomorrow) and we may buy any portfolio as a mixture of bonds and one stock. We denote the bonds price by B_t and the stocks price by S_t and we assume the following:

$$B_0 = 1, \quad B_1 = 1 + R, \quad S_0 = s, \quad S_1 = \begin{cases} s \cdot u, & \text{with probability } p_u. \\ s \cdot d, & \text{with probability } p_d. \end{cases}$$

We may introduce Z as the random variable

$$Z = u \cdot (I) + d \cdot (1 - I),$$

for an bernoulli variable I with succes probability p_u . Naturally, we assume $d \leq (1 + R) \leq u$ (this is imperative to ensure no arbitrage as we will see).

Portfolios and arbirtage We study any portfolio on the (B, S) market as a vector $h = (x, y)$ where x is the amount of bonds and y is the amount of stock held in the portfolio. Notice that we allow for shorting, that is $x < 0$ or $y < 0$. As such, we have that $h \in \mathbb{R}^2$. In this we have made some unrealistic, but attractable assumptions included in the assumptions:

- We allow short positions and fractional holding, i.e. $h \in \mathbb{R}^2$,
- We assume no spread between ask and bids,
- No transaction costs and
- A completely liquid market i.e. we may borrow and buy as much stock and bonds as wanted.

Given that we have chosen a portfolio h we may introduce the value process.

Definition 2.1. (Bjork) *The **value process** of the portfolio $h \in \mathbb{R}^2$ is the stochastic process*

$$V_t^h = xB_t + yS_t, \quad t = 0, 1.$$

Given this notation we may define what an arbitrage is.

Definition 2.2. (Bjork) *An **arbitrage** is a portfolio h with the properties: 1) $V_0^h = 0$, 2) $P(V_1^h \geq 0) = 1$ and 3) $P(V_1^h > 0) > 0$.*

That is h is an deterministic money-machine where we at least never loose any money. Granted the bonds give a deterministic non-negative return, but an arbitrage does not require any money out of pocket. With the notion of an arbitrage we will show the first proposition regarding the choice of R, u, d as defined above.

Proposition 2.3. (Bjork) *The one-period binomial model is arbitrage free if and only if the following inequality hold:*

$$d \leq (1 + R) \leq u. \quad (2.1)$$

Proof.

The statement is proofed by contradiction. Assume that $d > 1 + R$ holds. Then by definition $u > d > 1 + R$. Notice that any portfolio satisfying $V_0^h = 0$ must satisfy

$$0 = xB_0 + yS_0 = x + ys \iff x = -ys$$

That is for some choice y the only arbitrage candidate is the portfolio $h = (-ys, y)$. Calculating the value at time $t = 1$ we have

$$V_1^h = -ys \cdot (1 + R) + y \cdot s \cdot Z = ys(Z - 1 - R)$$

However since $Z \geq d$ we have $Z - (1 + R) \geq 0$ and therefore an arbitrage (for $y > 0$). The other inequality $1 + R > u$ follows analog steps. Simply choose some $y < 0$ and the result follows. ■

From inequality (2.1) we see that since $1 + R$ is between u and d we may find a pair $q_d, q_u \geq 0$ with $q_d + q_u = 1$ such that

$$1 + R = q_u \cdot u + q_d \cdot d.$$

This yields the important risk neutral valuation formula as summed up in the following definition

Definition 2.4. (Bjork) *A probability measure Q is called a **martingale measure** if the following condition holds:*

$$S_0 = \frac{1}{1 + R} E^Q[S_1].$$

The above measure Q is the measure $Q(Z = d) = q_d$ and $Q(Z = u) = q_u$ for the binomial model. This does in fact yield the risk neutral valuation formula:

$$\begin{aligned} S_0 &= \frac{1}{1+R} E^Q[S_1] = \frac{1}{1+R} (Q(Z = d) \cdot d \cdot s + Q(Z = u) \cdot u \cdot s) \\ &= s \frac{1}{1+R} (q_d \cdot d + q_u \cdot u) = s, \end{aligned}$$

where we simply use $1 + R = q_d \cdot d + q_u \cdot u$. We call this the risk neutral valuation formula because it in some sense gives an expected discounted value of the future stock price. We end this endeavour with reformulating the arbitrage proposition and determining the values of the Q -measure.

Proposition 2.5. (Bjork) *The one-period binomial model is arbitrage free if and only if there exists a martingale measure Q .*

Proposition 2.6. (Bjork) *The one-period binomial model has martingale probabilities given by:*

$$\begin{cases} q_u = \frac{(1+R)-d}{u-d}, \\ q_d = \frac{u-(1+R)}{u-d}. \end{cases}$$

Contingent Claims This chapter revolves around the financial derivative and we start by stating the definition of the financial derivative.

Definition 2.7. (Bjork) *A **contingent claim** (financial derivative) is any stochastic variable X of the form $\Phi(Z)$, where Z is the stochastic variable driving the stock price process.*

We may also call the function Φ the **contract function** as it states how the contract is resolved once the stochastic variable Z has been realised. Our objective is now to study, what a buyer of said contract would have to pay at any given time t . We call the fair price of X at time t : $\Pi_t[X]$. As such it is easy to see that the fair price at the time of maturity T is simply the payout X i.e. $\Pi_T[X] = X$. Our strategy is to find a replicating portfolio h and determine the price of said portfolio.

Definition 2.8. (Bjork) *A contingent claim X can be **replicated**, or said to be **reachable** if there exist a portfolio h such that*

$$V_1^h = X,$$

*with probability one. In that case, we say that the portfolio h is a **hedging** portfolio or a **replicationg** portfolio. If all claims can be replicated we say that the market is **complete**.*

Our pricing strategy is then to determine the value process of the replicating portfolio and then by the first pricing principle below we say that the price is imply the value of the replicating portfolio.

Pricing principle 1. If a claim X is reachable with replicating portfolio h , then the only reasonable price process for X is given by

$$\Pi_t[X] = V_t^h.$$

Notice, that this assumes that a replicating portfolio exist and even so we have a uniqueness statement to solve. We end this section by writing two important results.

Proposition 2.9. (Bjork) *Suppose that a claim X is reachable with replicating portfolio h . Then any price at time $t \geq 0$ of the claim X other than the value process of h will lead to an arbitrage on the extended market (B, S, X) .*

Proposition 2.10. (Bjork) *If the one-period binomial model is free of arbitrage, then it is also complete.*

The hedging portfolio in the one-period binomial model is given by the portfolio (x, y) below

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d}, \quad (2.2)$$

$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}. \quad (2.3)$$

Risk Neutral Valuation We see that since the one-period model is complete we can price any contingent claim and we see that

$$\begin{aligned} \Pi_0[X] &= \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d} + s \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d} \\ &= \frac{1}{1+R} \left\{ \frac{u\Phi(d) - d\Phi(u)}{u-d} + (1+R) \frac{\Phi(u) - \Phi(d)}{u-d} \right\} \\ &= \frac{1}{1+R} \left\{ \frac{(1+R) - d}{u-d} \Phi(u) + \frac{u - (1+R)}{u-d} \Phi(d) \right\} \\ &= \frac{1}{1+R} E^Q[X]. \end{aligned}$$

i.e. the price at time $t = 0$ should simply be the expected discounted payout according to the martingale measure. This leads to the important pricing proposition:

Proposition 2.11. (Bjork) *If the one-period binomial model is free of arbitrage, then the arbitrage free price of a contingent claim X is given by*

$$\Pi_0[X] = \frac{1}{1+R} E^Q[X]. \quad (2.4)$$

Here the martingale measure Q is uniquely determined by the relation

$$S_0 = \frac{1}{1+R} E^Q[S_1], \quad (2.5)$$

and the explicit expressions for q_u and q_d are given in proposition 2.6. Furthermore the claim X can be replicated using the portfolio

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d}, \quad (2.6)$$

$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}. \quad (2.7)$$

Multi-period model

The one-period binomial model can easily be extended to a multi-period model, by assuming that the bond and stock prices evolve by the processes:

$$t \geq 1 : B_t = (1 + R)B_{t-1} \quad \text{and} \quad B_0 = 1,$$

$$t \geq 1 : S_t = Z_{t-1}S_{t-1} \quad \text{and} \quad S_0 = s,$$

where we obviously have that $B_t = (1 + R)^t$ for $t \geq 0$. In the above Z_t is u with probability p_u and d with probability p_d . In this context, we need to define a portfolio in terms of a strategy.

Definition 2.13. (Bjork) A **portfolio strategy** is a stochastic process on $\{1, \dots, T\}$

$$h = \{h_t = (x_t, y_t); t = 1, \dots, T\}$$

such that h_t is a function of S_0, S_1, \dots, S_{t-1} . For a given portfolio strategy h we set $h_0 = h_1$ by convention. The associated **value process** corresponding to the portfolio h is defined by

$$V_t^h = x_t(1 + R) + y_tS_t.$$

Given this notation we may define what an arbitrage is, but first we introduce the notion of a self-financing portfolio. A self-financing portfolio in an intuitive sense is a portfolio that is not withdrawn from or deposited into.

Definition 2.14. (Bjork) A portfolio strategy h is said to be **self-financing** if the following condition holds for all $t = 0, \dots, T - 1$:

$$x_t(1 + R) + y_tS_t = x_{t+1} + y_{t+1}S_{t+1}.$$

The above equation says that the portfolio purchased at time t and held until $t + 1$ (x_{t+1}, y_{t+1}) can only be financed by the market value of the portfolio held from $[t - 1, t]$ i.e. (x_t, y_t) . We now define an arbitrage.

Definition 2.15. (Bjork) An **arbitrage** is a self-financing portfolio h with the properties: 1) $V_0^h = 0$, 2) $P(V_T^h \geq 0) = 1$ and 3) $P(V_T^h > 0) > 0$.

The multiperiod binomial model has an just like the oneperiod model a result regarding when an arbitrage exists.

Lemma 2.16. (Bjork) If $d \leq (1 + R) \leq u$ (eq. 2.8) then the multiperiod model is arbitrage-free.

As one can see, the multiperiod model is rather similar to the one period model. We will in the following summarise equivalent statements for the multiperiod model as the ones in the oneperiod model.

Definition 2.17. (Bjork) The martingale probabilities q_u and q_d are defined as the probabilities for which the relation below holds.

$$s = \frac{1}{1 + R} E^Q[S_{t+1} | S_t].$$

Proposition 2.18. (Bjork) The martingale probabilities q_u and q_d are given by

$$\begin{cases} q_u = \frac{(1+R)-d}{u-d}, \\ q_d = \frac{u-(1+R)}{u-d}. \end{cases}$$

Definition 2.19. (Bjork) A **contingent claim** is a stochastic variable X of the form

$$X = \Phi(S_T),$$

where the **contract function** Φ is some given real valued function.

Definition 2.20. (Bjork) A given contingent claim X is said to be **reachable** if there exists a self-financing portfolio h such that

$$V_T^h = X,$$

with probability one. In that case we say that the portfolio h is a **hedging** portfolio or a **replicating** portfolio. If all claims can be replicated we say that the market is (dynamically) **complete**.

Pricing principle 2. (Bjork) If a claim X is reachable with replicating portfolio h , then the only reasonable price process for X is given by

$$\Pi_t[X] = V_t^h, \quad t = 0, 1, \dots, T.$$

Proposition 2.21. (Bjork) Assume X is reachable by h , then any price other than V_t^h for some $t \geq 0$ leads to an arbitrage opportunity.

Proposition 2.22. (Bjork) The multiperiod model is complete, i.e. every claim can be replicated by a self-financing portfolio.

Proposition 2.24. (Bjork) (Binomial algorithm) Consider a T -claim $X = \Phi(S_T)$. Then this claim can be replicated using a self-financing portfolio. If $V_t(k)$ denotes the value of the portfolio at the node (t, k) (k referring to k amount of up-moves for the stock), then $V_t(k)$ can be computed recursively by the scheme

$$\begin{cases} V_t(k) = \frac{1}{1+R} \{q_u V_{t+1}(k+1) + q_d V_{t+1}(k)\}, \\ V_T(k) = \Phi(su^k d^{T-k}). \end{cases}$$

where the martingale probabilities q_u and q_d are given by

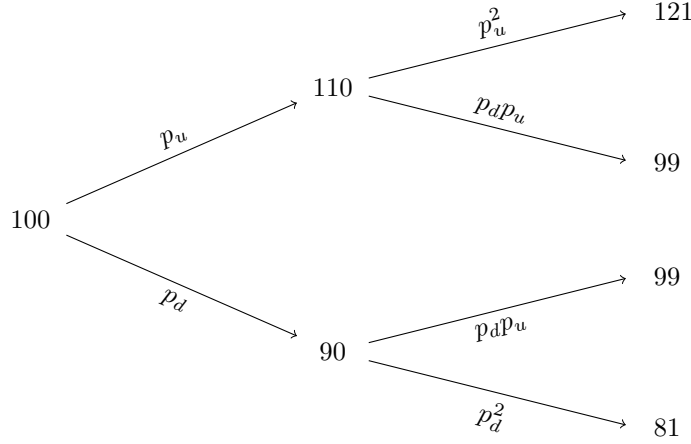
$$\begin{cases} q_u = \frac{(1+R)-d}{u-d}, \\ q_d = \frac{u-(1+R)}{u-d}. \end{cases}$$

With the notation as above, the hedging portfolio is given by

$$\begin{cases} x_t(k) = \frac{1}{1+R} \cdot \frac{uV_t(k)-dV_t(k+1)}{u-d}, \\ y_t(k) = \frac{1}{S_{t-1}} \cdot \frac{V_t(k+1)-V_t(k)}{u-d}. \end{cases}$$

In particular, the arbitrage free price of the claim at $t = 0$ is given by $V_0(0)$.

Example.



Consider $R = 0.04$, $s = 100$, $u = 1.1$, $d = 0.9$, $p_u = 0.6$ and $p_d = 0.4$. We consider a model of length $T = 2$ and we want to evaluate the price of the european call option with strike $K = 90$ that is the contingent claim

$$X = (S_T - K)^+, \quad \Phi(s) = (s - K)^+.$$

For each time t we know the replicating portfolio, if we know the payoff the following period. Therefore we start from the leaves of the tree and work towards the root. Since the strike price is $K = 90$ the end result will be the following payoffs:

$$\begin{aligned} u^2 : & \quad (121 - 90)^+ = 31 \\ ud : & \quad (99 - 90)^+ = 9 \\ du : & \quad (99 - 90)^+ = 9 \\ d^2 : & \quad (81 - 90)^+ = 0 \end{aligned}$$

Therefore by the risk neutral valuation formula with $q_u = \frac{(1+R)-d}{u-d} = 0.7$ and $q_d = \frac{u-(1+R)}{u-d} = 0.3$ we have that the cost of the replicating portfolio at time $t = 1$ is respectively

$$\begin{aligned} u : & \quad \frac{1}{1+R} \{31 \cdot q_u + 9 \cdot q_d\} \approx 23.46 \\ d : & \quad \frac{1}{1+R} \{9 \cdot q_u + 0 \cdot q_d\} \approx 6.06 \end{aligned}$$

To replicate this payoff at time $t = 1$ we can use the risk neutral valuation formula once more to find the base cost of the replicating portfolio i.e. the price of X at time $t = 0$

$$\frac{1}{1+R} \{23.46 \cdot q_u + 6.06 \cdot q_d\} \approx 17.54.$$

Working from the root to the leaves we can now calculate the hedging portfolio at time $t = 0, 1$ for each path. For time $t = 0$ we calculate

$$\begin{aligned} x &= \frac{1}{1+R} \cdot \frac{u \cdot 6.06 - d \cdot 23.46}{u - d} \approx -69.46, \\ y &= \frac{1}{s} \cdot \frac{23.46 - 6.06}{u - d} \approx 0.87 \end{aligned}$$

We see by calculations that this does indeed replicate the payoff at time $t = 1$:

$$\begin{aligned} u : & \quad V_1^h = (1+R) \cdot x + 110 \cdot y \approx 23.46, \\ d : & \quad V_1^h = (1+R) \cdot x + 90 \cdot y \approx 6.06. \end{aligned}$$

We also see by calculation that the initial portfolio does cost the expected 17.54 as

$$x \cdot 1 + y \cdot 100 = 87 - 69.46 = 17.54.$$

Following these steps at time $t = 1$ the portfolios $(-86.54, 1)$ (for the up-scenario) and $(-38.94, 0.5)$ (for the down-scenario) would arise. Notice when calculating y one has to use the current price $S_1 = S_0 \cdot Z$ not S_0 . One should also check by similar calculations as above, that these portfolios does indeed replicate the payoff of the contingent claim X . \square

Proposition 2.25. (Bjork) *The arbitrage free price at $t = 0$ of a T -claim X is given by*

$$\Pi_0[X] = \frac{1}{(1+R)^T} E^Q[X]$$

where Q denotes the martingale measure, or more explicitly

$$\Pi_0[X] = \frac{1}{(1+R)^T} \sum_{k=0}^T \binom{T}{k} q_u^k q_d^{T-k} \Phi(su^k d^{T-k}).$$

Example.

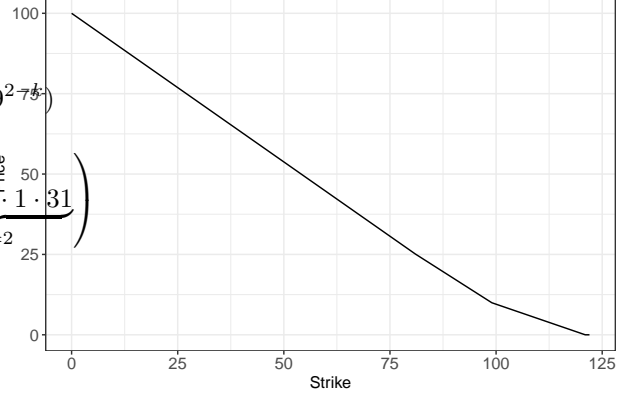
We follow an analog example as the one after proposition 2.24. Let $K = 90$ and we see that

$$\begin{aligned} \Pi_0[X] &= \frac{1}{(1+0.04)^2} \sum_{k=0}^2 \binom{2}{k} \cdot 0.7^k \cdot 0.3^{2-k} \cdot \Phi(100 \cdot 1.1^k \cdot 0.9^{2-k}) \\ &= 0.9245562 \cdot \left(\underbrace{1 \cdot 1 \cdot 0.09 \cdot 0}_{k=0} + \underbrace{2 \cdot 0.7 \cdot 0.3 \cdot 9}_{k=1} + \underbrace{1 \cdot 0.49 \cdot 1 \cdot 31}_{k=2} \right) \\ &= 0.9245562 \cdot (0 + 3.78 + 15.19) \\ &= 17.53883 \end{aligned}$$

Since we know that K must meaningfully range in $[0, 121]$ we could try to calculate the price of the contingent claim at time $t = 0$ for all integers in this interval. We see that the price range between S_0 and 0 as expected. One can also see that the price changes slope at the prices 99 and 121 as the function is linear in Φ and some realisations lose any effect on the price when the strike is higher than the outcome. \square

Proposition 2.26. (Bjork) *The condition $d < (1+R) < u$ is necessary and sufficient condition for absence of arbitrage.*

Price of european call option



Generalised one-period model

In the previous we had the simple model where we only had one stochastic asset S and only one stochastic variable Z determining the future stock price. Now we will generalise this model by introducing N assets and introducing some stochastic behaviour to the system.

Model specification We consider the market consisting of a collection of stochastic prices assets $i = 1, \dots, N$ with N -dimensional price process.

$$S_t = \begin{bmatrix} S_t^1 \\ \vdots \\ S_t^N \end{bmatrix}$$

We now assume that S_t is defined on a background space with finite sample space $\Omega = \{\omega_1, \dots, \omega_M\}$ with associated probabilities $p_j = P(\omega_j)$, $j = 1, \dots, M$. We can then for each time $t = 1, \dots, T$ define the $N \times M$ matrix D_t as such

$$D_t = \begin{bmatrix} S_t^1(\omega_1) & \cdots & S_t^1(\omega_M) \\ \vdots & \ddots & \vdots \\ S_t^N(\omega_1) & \cdots & S_t^N(\omega_M) \end{bmatrix}.$$

We will assume that $S_0^1 > 0$ and $S_1^1(\omega_j) > 0$, $j = 1, \dots, M$.

Absence of Arbitrage We now define a **portfolio** as an N -dimensional row vector

$$h = [h^1, \dots, h^N]$$

representing the amount of assets held at time $t = 0$ and held until $t = 1$. The **value process** is then

$$V_t^h = h \cdot S_t = \sum_{i=1}^N h^i S_t^i, \quad t = 0, 1. \quad (3.1)$$

For a given $\omega_j \in \Omega$ we have the realisation

$$V_t^h = h S_t(\omega_j) = h d_j = (h D)_j.$$

Definition 3.1. (Bjork) The portfolio h is an **arbitrage portfolio** if it satisfies the conditions: $V_0^h = 0$, $P(V_1^h \geq 0) = 1$ and $P(V_1^h > 0) > 0$.

Lemma 3.2. (Bjork) (Farkas' Lemma) Suppose that d_0, d_1, \dots, d_M are column vectors in \mathbb{R}^N . Then exactly one of the following problems possesses a solution.

- **Problem 1:** There exist $\lambda_1, \dots, \lambda_M \geq 0$ such that $d_0 = \sum_{j=1}^M \lambda_j d_j$.
- **Problem 2:** There exist $h \in \mathbb{R}^N$ such that $h^\top d_0 < 0$ and $h^\top d_j \geq 0$ for $j = 1, \dots, M$.

We now investigate this system for any possible arbitrage portfolios. However first we acknowledge that there exist a nominal price system S_t and a normalised price system Z_t . The latter we define as the nominal price under the numeraire S_t^1 that is

$$Z_t = \begin{bmatrix} S_t^1/S_t^1 \\ S_t^2/S_t^1 \\ \vdots \\ S_t^N/S_t^1 \end{bmatrix} = \begin{bmatrix} 1 \\ S_t^2/S_t^1 \\ \vdots \\ S_t^N/S_t^1 \end{bmatrix}.$$

The reason for introducing the normalized price system is that we can without much effort translate results in this system to the nominal system and the normalised system is easier to analyze. For this, however, we need a few results.

Lemma 3.3. (Bjork) *With notation as above, the following hold.*

1. The Z_t value process is related to the S_t value process by

$$V_t^{h,Z} = hZ_t = \frac{1}{S_t^1} V_t^h.$$

2. A portfolio is an arbitrage in the S_t system if and only if there is an arbitrage in the Z_t system.
3. In the Z_t price system, the numeraire asset Z^1 has unit constant prices i.e. $Z_t^1 = 1$ for all $t \geq 0$.

One of the reasons that the normalised system is attractive is that the numeraire asset is constant i.e. risk free in the normalised system. Let us formulate our first main result.

Proposition 3.4. (Bjork) *The market is arbitrage free if and only if there exists strictly positive real numbers $q_1, \dots, q_M \geq 0$ with $q_1 + \dots + q_M = 1$ (eq. 3.2) (probability vector) such that the following vector equality holds*

$$\begin{bmatrix} Z_0^1 \\ \vdots \\ Z_N^1 \end{bmatrix} = \begin{bmatrix} Z_1^1(\omega_1) \\ \vdots \\ Z_1^N(\omega_1) \end{bmatrix} q_1 + \dots + \begin{bmatrix} Z_1^1(\omega_M) \\ \vdots \\ Z_1^N(\omega_M) \end{bmatrix} q_M. \quad (3.3)$$

Martingale Measures **Definition 3.5. (Bjork)** *Given the objective probability measure P on (Ω, \mathcal{F}, P) , we say that another probability measure Q defined on Ω is **equivalent** to P if*

$$\forall A \in \mathcal{F} : P(A) = 0 \iff Q(A) = 0,$$

or equivalently

$$\forall A \in \mathcal{F} : P(A) = 1 \iff Q(A) = 1.$$

Definition 3.7. (Bjork) *Consider the market model above and set S^1 as the numeraire asset. We say that a probability measure Q defined on Ω is a **martingale measure** if it satisfies the following conditions:*

1. Q is equivalent to P , i.e. $Q \sim P$.
2. For every $i = 1, \dots, N$, the normalized asset price process

$$Z_t^i = \frac{S_t^i}{S_t^1},$$

is martingale under the measure Q .

Theorem 3.8. (Bjork) (First Fundamental Theorem) *Given a fixed numeraire, the market is free of arbitrage possibilities if and only if there exists a martingale measure Q .*

By assuming that the numeraire asset is risk free (i.e. does not depend on ω) then by scaling we can derive the short interest rate as

$$1 + R = \frac{S_1^1}{S_0^1}.$$

With this in mind we can formulate theorem 3.8 in its more widely used form.

Theorem 3.9. (Bjork) (First Fundamental Theorem) *Assume that there exist a risk free asset, and denote the corresponding risk free interest rate by R . Then the market is arbitrage free if and only if there exist a measure $Q \sim P$ such that*

$$S_0^i = \frac{1}{1+R} E^Q[S_1^i], \quad \text{for all } i = 1, \dots, N. \quad (3.9)$$

Martingale Pricing Moving forward we will assume that there exist a risk free asset and we will denote it by B_t ($B_t = S_t^1/S_0^1$).

Definition 3.10. (Bjork) *A **contingent claim** is any random variable X , defined on the sample space Ω .*

To ensure no arbitrage in the extended market containing the N assets and the contingent claim we can apply the first fundamental pricing theorem on the extended market.

Proposition 3.11. (Bjork) *Consider a given claim X . In order to avoid arbitrage, X must then be priced according to the formula*

$$\Pi_0[X] = \frac{1}{1+R} E^Q[X], \quad (3.10)$$

where Q is a martingale measure for the underlying market (Π, S^1, \dots, S^N) .

Completeness Given that a market is arbitrage-free we may run into a uniqueness issue when determining the price of a contingent claim. If a martingale measure exist we will very much like it to be unique as this will ensure that the price from the risk neutral valuation formula is unique. To this we need the market to be complete.

Definition 3.12. (Bjork) *Consider a contingent claim X . If there exists a portfolio h , based on the underlying assets, such that*

$$V_1^h = X, \text{ with probability } 1 \quad (3.11)$$

i.e.

$$V_1^h(\omega_j) = X(\omega_j), \quad j = 1, \dots, M, \quad (3.12)$$

then we say that X is **replicated**, or **hedged** by h . Such a portfolio h is called a replicating, or hedging portfolio. If every contingent claim can be replicated, we say that the market is **complete**.

We can now formulate a proposition on when the market is complete in terms of the matrix D .

Proposition 3.13. (Bjork) *The market is complete if and only if the rows of the matrix D span \mathbb{R}^M , i.e. if and only if D has rank M .*

Now we formulate the second fundamental pricing theorem in terms of the martingale measure Q .

Proposition 3.14. (Bjork) (Second Fundamental Theorem) *Assume that the model is arbitrage free i.e. Q exist. Then the market is unique if and only if the martingale measure is unique.*

Stochastic Discount Factors **Definition 3.16. (Bjork)** *The random variable L on Ω is defined by*

$$L(\omega_i) = \frac{q_i}{p_i}, \quad i = 1, \dots, M.$$

Definition 3.17. (Bjork) *Assume the absence of arbitrage, and fix a martingale measure Q . With notation as above, the **stochastic discount factor** (or “state price deflator”) is the random variable Λ on Ω by*

$$\mathbf{M}(\omega) = \frac{1}{1+R} \cdot L(\omega). \quad (3.19)$$

Proposition 3.18. (Bjork) *The arbitrage free price of any claim X is given by the formula*

$$\Pi_0[X] = E^P[\mathbf{M} \cdot X] \quad (3.20)$$

where \mathbf{M} is a stochastic discount factor.

Self-financing portfolios

We move forward in this chapter by first defining a self-financing portfolio in discrete time and then by letting the step length tend to zero obtain the continuous time analogue.

Discrete time SF portfolio

We consider N different adapted price processes S^1, \dots, S^N . We use the following definition.

Definition 6.1. (Bjork) *We use the following definitions.*

- S_n^i is the price of asset i at time n ,
- h_n^i is the number of units of asset i held during $[n, n+1)$, that is bought at time n ,
- d_n^i is the dividends from asset i in the time-interval $[n-1, n)$, that is received at time n ,
- h_n is the portfolio (h_n^1, \dots, h_n^N) held during $[n, n+1)$,
- c_n is the consumption i.e. withdrawal at time n (negative being deposits/saving),
- V_n is the value of the portfolio just before time n i.e. of the portfolio h_{n-1} at time n .

We are now ready to define the self-financing portfolio

Definition 6.2. (Bjork) *A self-financing portfolio supporting the consumption stream c is a portfolio adhering to the **budget constraint** given as*

$$h_{n+1}S_{n+1} + c_{n+1} = h_nS_{n+1} + h_nd_{n+1}.$$

The interpretation being, that we may only use funds obtained from selling the old portfolio h_n and received in dividends to buy the new portfolio h_{n+1} and consume the amount c_{n+1} .

Before studying the self-financing portfolio we define the operator Δ (in definition 6.3) as the increment $\Delta x_n = x_{n+1} - x_n$ of a countable sequence $(x_n)_{n \in \mathbb{N}_0}$. Notice that we define the increment forward so the increment n is the increment over the time period $[n, n+1)$ with the first increment being $[0, 1)$. Using this notation we can derive the lemma below.

Lemma 6.4. (Bjork) *For any pair of sequences of real numbers $(x_n)_{n \in \mathbb{N}_0}$ and $(y_n)_{n \in \mathbb{N}_0}$ we have the relations*

$$\Delta(xy)_n = x_n \Delta y_n + y_{n+1} \Delta x_n, \quad (6.5)$$

$$\Delta(xy)_n = y_n \Delta x_n + x_{n+1} \Delta y_n, \quad (6.6)$$

$$\Delta(xy)_n = x_n \Delta y_n + y_n \Delta x_n + \Delta x_n \Delta y_n. \quad (6.7)$$

This is also valid if the sequences are N -dimensional, where we interpret the products above as scalar products (xy^\top) .

Using these definitions and the lemma above we see that the dynamics of the self-financing portfolio is given below.

Proposition 6.6. (Bjork) *The dynamics of any self-financing portfolio supporting the consumption stream c are given by*

$$\Delta V_n = h_n \Delta S_n + h_n d_{n+1} - c_{n+1}, \quad (6.11)$$

or, in more detail

$$\Delta V_n = \sum_{i=1}^N h_n^i (\Delta S_n^i + d_{n+1}^i) - c_{n+1}. \quad (6.12)$$

We may rewrite the dividends as accumulating dividends $D_n^i = \sum_{k=1}^n d_k^i$ and see that $d_{n+1}^i = \Delta D_n^i$ and so the above condition is equivalent with.

Proposition 6.8. (Bjork) *The dynamics of any self-financing portfolio supporting the consumption stream c are given by*

$$\Delta V_n = h_n \Delta S_n + h_n \Delta D_n - c_{n+1}, \quad (6.15)$$

or, in more detail

$$\Delta V_n = \sum_{i=1}^N h_n^i (\Delta S_n^i + \Delta D_n^i) - c_{n+1}. \quad (6.16)$$

Continuous time SF portfolio

Formulating the dynamics of the self-financing portfolio in continuous time is easy work given the discrete setup above. However since we now are in continuous time we will change the n with a t and consider the behaviour $V_{t+dt} - V_t$ as we let $dt \rightarrow 0$. First we formulate some basic notation.

Definition 6.9. (Bjork) *We use the following definitions.*

- S_t^i is the price of asset i at time t ,
- h_t^i is the number of units of asset i held at time t ,
- D_t^i is the cumulative dividend process for asset i ,
- h_t is the portfolio (h_t^1, \dots, h_t^N) held at time t ,
- c_t is the consumption rate at time t (negative being deposits/saving),
- V_t is the value of the portfolio at time t i.e. of the portfolio h_t at time t .

Given these definitions we may define a portfolio strategy that is self-financing.

Definition 6.10. (Bjork) *Let S be an adapted N -dimensional price process. We define the following*

1. A **portfolio strategy** is any adapted N -dimensional process h .
2. The **value process** V^h corresponding to the portfolio h is given by

$$V_t^h = \sum_{i=1}^N h_t^i S_t^i. \quad (6.17)$$

3. A **consumption process** is any adapted one-dimensional process c .
4. A portfolio-consumption pair (h, c) is called **self-financing** if the value process V^h satisfies the condition

$$dV_t^h = \sum_{i=1}^N h_t^i (dS_t^i + dD_t^i) - c_t dt, \quad (6.18)$$

i.e. if

$$dV_t^h = h_t dS_t + h_t dD_t - c_t dt.$$

5. The **gain process** G is defined by

$$G_t = S_t + D_t \quad (6.19)$$

so we can write the self-financing condition as

$$dV_t = h_t dG_t - c_t dt. \quad (6.20)$$

6. The portfolio h is said to be **Markovian** if it is of the form

$$h_t = h(t, S_t),$$

for some function $h : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}^N$.

Portfolio weights

Definition 6.11. (Bjork) For a given portfolio h the corresponding **relative portfolio** or **portfolio weights** w are defined by

$$w_t^i = \frac{h_t^i S_t^i}{V_t^h}, \quad i = 1, \dots, N, \quad (6.21)$$

so, in particular, we have $\sum_{i=1}^N w_i = 1$.

Lemma 6.12. (Bjork) A portfolio-consumption pair (h, c) is self-financing if and only if

$$dV_t^h = V_t^h \sum_{i=1}^N w_t^i \frac{dS_t^i + dD_t^i}{S_t^i} - c_t \, dt \quad (6.22)$$

or equivalently with the absolute weights

$$dV_t^h = \sum_{i=1}^N h_t^i (dS_t^i + dD_t^i) - c_t \, dt.$$

Lemma 6.13. (Bjork) Consider the case with no dividends. Let c be a consumption process, and assume that there exist a scalar process Z and a vector process $q = (q^1, \dots, q^N)$ such that

$$dZ_t = Z_t \sum_{i=1}^N q_t^i \frac{dS_t^i}{S_t^i} - c_t \, dt, \quad (6.23)$$

and $\sum_{i=1}^N q q^i = 1$ (eq. 6.24). Now define a portfolio h by

$$h_t^i = \frac{q_t^i Z_t}{S_t^i}. \quad (6.25)$$

Then the value process V^h is given by $V^h = Z$, the pair (h, c) is self-financing, and the corresponding relative portfolio w is given by $w = q$.

Black-Scholes PDE

The Black-Scholes model revolves around SDE's as seen above. In this model we have two assets a risk free asset B and a stochastic priced asset S . We therefore start by defining what we mean by a quote-on-quote *risk free* asset.

Definition 7.1. (Bjork) *The price process B is the price of a **risk free** asset if it has the dynamics*

$$dB_t = r_t B_t dt, \quad (7.1)$$

where r is any \mathcal{F}_t adapted process.

We see from this definition that the meaning of “risk free” is the property, that B is priced locally deterministic in the sense that r is adapted and therefore known at time t and we therefore know the yield on a short term basis. This is also why we may call r the **short interest rate**. Given the dynamics above, we know that B in fact is represented by the process

$$B_t = B_0 e^{\int_0^t r_s ds},$$

for some B_0 initial value. We will moving forward assume that $B_0 = 1$. The stochastic asset S has dynamics.

$$dS_t = \mu(t, S_t) dt + \sigma(t, S_t) dW_t, \quad (7.2)$$

where as usual μ and σ are deterministic functions and W_t is a standard Brownian motion. Note that the risk free asset has a similarly process with $\sigma = 0$. We may now include this in the definition of the Black-Scholes model.

Definition 7.2. (Bjork) *The **Black-Scholes model** consists of two assets with dynamics given by*

$$dB_t = r B_t dt, \quad (7.3)$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (7.4)$$

where r, μ, σ are deterministic constants.

Definition 7.3. (Bjork) *A **zero coupon bond** with maturity T (henceforth “ T -bond”) is an asset which pays the holder the face value 1 dollar at time T . The price at time n of a T -bond is denoted by $p(n, T)$.*

Definition 7.4. *The (possible stochastic) discrete **short rate** r_n , for the period $[n, n + 1]$, is defined as*

$$p(n, n + 1) = \frac{1}{1 + r_n}. \quad (7.6)$$

From this short rate we may derive the dynamics of the bank account receiving zero-coupon rates for each distinct time interval.

Definition 7.5. (Bjork) *The dynamics of the bank account are given by*

$$\Delta B_n = r_n B_n. \quad (7.7)$$

Contingent Claims and Arbitrage

Definition 7.6. (Bjork) *A **European call option** with **exercise price** (or **strike price**) K and **time of maturity** (exercise date) T on the **underlying asset** S is a contract defined by the following clauses:*

- *The holder of the option has, at time T , the right to buy one share of the underlying stock at the price K dollars from the underwriter of the option.*

- The holder of the option is in no way obliged to buy the underlying stock.
- The right to buy the underlying stock at the price K can only be exercised at the precise time T .

Obviously, we also have the **European put** option which gives the owner the right to sell an asset at price K at time T . Let us formally define a contingent claim.

Definition 7.7. Consider a financial market with vector price process S . A **contingent claim** with **date of maturity** T , also called a T -claim, is any random variable $\mathcal{X} \in \mathcal{F}_T^S$. A contingent claim \mathcal{X} is called a **simple claim** if it is of the form $\mathcal{X} = \Phi(S_T)$. The function Φ is called the **contract function**.

Definition 7.8. (Bjork) An **arbitrage possibility** on a financial market is a self-financed portfolio h such that

$$V^h(0) = 0, \quad (7.13)$$

$$P(V_T^h \geq 0) = 1, \quad (7.14)$$

$$P(V_T^h > 0) > 0. \quad (7.15)$$

We say that the market is **arbitrage free** if there are no arbitrage possibilities.

Definition 7.9. (Bjork) Suppose that there exists a self-financing portfolio h , such that the value process V^h has the dynamics

$$dV_t^h = k_t V_t^h dt, \quad (7.16)$$

where k is an adapted process. Then it must hold that $k_t = r_t$ for all t , or there exists an arbitrage possibility.

Theorem 7.10. (Bjork) (Black-Scholes equation) Assume that the market is specified by the equations

$$dB_t = rB_t dt, \quad (7.18)$$

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \quad (7.19)$$

and that we want to price a contingent claim of the form $\mathcal{X} = \Phi(S_T)$ (eq. 7.20). Then the only pricing function of the form $\Pi_t[\Phi(S_T)] = F(t, S_t)$ (eq. 7.21) which is consistent with the absence of arbitrage in the market $[B_t, S_t, \Pi_t]$ is when F is the solution of the following boundary value problem in the domain $[0, T] \times \mathbb{R}_+$:

$$F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2(t, s)F_{ss}(t, s) - rF(t, s) = 0, \\ F(T, s) = \Phi(s).$$

Risk Neutral Valuation

Theorem 7.11. (Bjork) (Risk Neutral Valuation) The arbitrage free price of the claim $\Phi(S_T)$ is given by $\Pi_t[\Phi] = F(t, S_t)$, where F is given by the formula

$$F(t, s) = e^{-r(T-t)} E_{t,s}^Q[\Phi(S_T)], \quad (7.43)$$

where the Q -dynamics of S are those of

$$dS_t = rS_t dt + S_t\sigma(t, S_t) dW_t^Q. \quad (7.42)$$

Property 7.12. (Bjork) (The Martingale Property) In the Black-Scholes model, the price process Π_t for every traded asset, be it the underlying or derivate asset, has the property the the normalized price process

$$Z_t = \frac{\Pi_t}{B_t},$$

(including S_t/B_t) is a martingale under the measure Q .

Black-Scholes formula

This chapter will center on deriving the famous Black-Scholes formula. We start by laying out the assumptions of the model. We have a market consisting of two assets: a stochastic prices asset S and a risk free asset B . The prices processes have dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (7.45)$$

$$dB_t = rB_t dt, \quad (7.44)$$

where $S_0 = s$ and $B_0 = 1$ (by assumption). Now from Feymann-Kac and the definition of arbitrage we know that a simple claim $\Phi(S_t)$ has the arbitrage free price given by the risk neutral valuation formula.

$$F(t, s) = e^{-r(T-t)} E_{t,s}^Q[\Phi(S_T)], \quad (7.46)$$

where Q is a probability measure, namely a Martingale measure, such that the dynamics of S under this measure is

$$dS_t = rS_t dt + \sigma S_t dW_t^Q, \quad (7.47)$$

with W_t^Q being a Brownian motion wrt. to the probability measure Q (not P). The above still has the initial condition $S_0 = s$. Given these assumptions we may formulate the Black-Scholes formula.

Theorem 7.13. (Bjork) (Black-Scholes formula) *The price of the european call option with strikeprice K and maturity T (contract function $\Phi(S_t) = (S_t - K)^+$) takes the form $\Pi_t = F(t, s)$, where*

$$F(t, s) = sN(d_1(t, s)) - e^{-r(T-t)}KN(d_2(t, s)), \quad (7.52)$$

where N is the distribution-function for an $\mathcal{N}(0, 1)$ -distributed random variable and

$$d_1(t, s) = \frac{1}{\sigma\sqrt{T-t}} \left(\log\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right), \quad (7.53)$$

$$d_2(t, s) = d_1(t, s) - \sigma\sqrt{T-t}. \quad (7.54)$$

Proof.

We let the market be given in terms of the price processes S and B with dynamics.

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

$$dB_t = rB_t dt,$$

with $B_t = 1$ and $S_t = s$. We assume that μ, σ, r are deterministic real numbers. Consider the contingent claim

$$\Phi(S_t) = (S_t - K)^+,$$

that is the European call option. Let Q be a martingale measure such that the dynamics of S may be written as

$$dS_t = rS_t dt + \sigma S_t dW_t^Q,$$

then S_t is clearly a GBM wrt. the measure Q . Therefore we know the solution given in terms of the increment of the Brownian motion W^Q as follows

$$S_u = s \cdot \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) (u - t) + \sigma (W_u^Q - W_t^Q) \right\},$$

for some initial condition $S_t = s$. From theorem 7.10 we know that the only pricing function which takes the form

$$\Pi_t[\Phi(S_T)] = F(t, S_t),$$

can only be consistent with the absence of arbitrage if F is the solution the the boundary value problem

$$\begin{aligned} F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2F_{ss}(t, s) - rF(t, s) &= 0, \\ F(T, s) &= \Phi(s). \end{aligned}$$

From Feymann-Kac we then know that the stochastic representation of such a solution take the form

$$F(t, s) = e^{-r(T-t)} E_{t,s}^Q[\Phi(S_T)].$$

Here the superscript refers to taking mean value with respect to the measure Q . This gives the solution to the pricing function

$$F(t, s) = e^{-r(T-t)} \int \Phi(S_T) dQ.$$

Under the measure Q we have that for $u \geq t$:

$$Z_u = \log(S_u/s) \sim \mathcal{N} \left(\left(r - \frac{1}{2} \sigma^2 \right) (u - t), \sigma \sqrt{u - t} \right)$$

Hence we may set $u = T$ and observe that

$$\begin{aligned} F(t, s) &= e^{-r(T-t)} \int_{-\infty}^{\infty} \Phi(se^z) f(z) dz \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} (se^z - K)^+ f(z) dz \\ &= e^{-r(T-t)} \int_{\log(\frac{K}{s})}^{\infty} (se^z - K) f(z) dz \\ &= e^{-r(T-t)} \left(s \int_{\log(\frac{K}{s})}^{\infty} e^z f(z) dz - K \int_{\log(\frac{K}{s})}^{\infty} f(z) dz \right), \end{aligned}$$

where we used that f is the distribution function of a normal distributed random variable with mean $(r - \sigma^2/2)(T - t)$ and variance $\sigma\sqrt{T - t}$ and that

$$(se^z - K)^+ \geq 0 \iff se^z \geq K \iff z \geq \log \left(\frac{K}{s} \right)$$

Using that the MGF of a $X \sim \mathcal{N}(\alpha, \beta^2)$ variable is

$$E[e^{tX}] = e^{\alpha t + \frac{1}{2} \beta^2 t^2},$$

and the shorthand $N(t)$ for the distribution function of the standard normal distribution, we have

$$\begin{aligned}
F(t, s) &= e^{-r(T-t)} \left(sE \left[e^{Z_T} 1_{Z_T \geq \log(\frac{K}{s})} \right] - KP \left(Z_T \geq \log \left(\frac{K}{s} \right) \right) \right) \\
&= e^{-r(T-t)} s \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) (T-t) + \frac{1}{2} \sigma^2 (T-t) \right\} E \left[1_{Z_T \geq \log(\frac{K}{s})} \right] \\
&\quad - e^{-r(T-t)} KP \left(X \geq \frac{1}{\sigma \sqrt{T-t}} \left(\log \left(\frac{K}{s} \right) - (r - \sigma^2/2)(T-t) \right) \right) \\
&= sE \left[1_{Z_T \geq \log(\frac{K}{s})} \right] - e^{-r(T-t)} KP \left(X \leq \frac{1}{\sigma \sqrt{T-t}} \left(\log \left(\frac{s}{K} \right) + (r - \sigma^2/2)(T-t) \right) \right) \\
&= sN(d_1(s, t)) - e^{-r(T-t)} KN(d_2(s, t)),
\end{aligned}$$

as desired. ■

Completeness and Hedging

We derived the pricing function of the european call option above and introduced the theory around boundary value problems and Feymann-Kac solution to the partial differential stochastic equation. Now we want to see if a portfolio exists such that it gives the payout $\Phi(S_T)$ with probability one.

In order to do this, we return to the concept of hedge and replication.

Definition 8.1. (Bjork) We say that a T -claim \mathcal{X} can be **replicated**, alternatively the it is **reachable** or **hedgeable**, if there exists a self-financing portfolio h such that

$$V_T^h = \mathcal{X}, \quad P - \text{a.s.} \quad (8.1)$$

In this case we say that h is a **hedge** against \mathcal{X} . Alternatively, h is called a **replicating** or **hedging** portfolio. If every contingent claim is reachable we say that the market is **complete**.

If we can find a portfolio h that reaches \mathcal{X} in value over the time period $[t, T]$ it must mean, that holding the portfolio is equivalent with holding the contract itself. We therefore have the natural assumption that the price process must satisfy $\Pi_t[\mathcal{X}] = V_t^h$ for all $t \geq 0$. How this relates to the absence of arbitrage is given below.

Proposition 8.2. (Bjork) Suppose \mathcal{X} is hedged using the portfolio h . Then the only price process $\Pi_t[\mathcal{X}]$ which is consistent with no arbitrage is given by $\Pi_t[\mathcal{X}] = V_t^h$. Furthermore, if \mathcal{X} can be hedged by both h and g then $V_t^g = V_t^h$ for all t with probability one.

Completeness in Black-Scholes

The Black-Scholes model will be investigated in the following. We start by stating the important theorem.

Theorem 8.3. (Bjork) Consider the Black-Scholes model given by

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \quad (8.2)$$

$$dB_t = rB_t dt, \quad (8.3)$$

The model above is complete.

The following lemma gives us replicability of a **simple** claim (which we will restrict to).

Lemma 8.4. (Bjork) Suppose that there exist an adapted process V and an adapted process $w = [w^B, w^S]$ with $w_t^B + w_t^S = 1$ (eq. 8.4) for all $t \geq 0$, such that

$$dV_t = V_t(w_t^B r + w_t^S \mu(t, S_t)) dt + V_t w_t^S \sigma(t, S_t) dW_t, \quad (8.5)$$

$$V_t = \Phi(S_t). \quad (8.5)$$

Then the claim $\mathcal{X} = \Phi(S_T)$ can be replicated using w as the relative portfolio. The corresponding value process is given by the process V and the absolute portfolio h is given by

$$h_t^B = \frac{w_t^B V_t}{B_t}, \quad (8.6)$$

$$h_t^S = \frac{w_t^S V_t}{S_t}. \quad (8.7)$$

Doing some heuristics we come up with some clever weights, which turns on to adhere to the boundary value problem formulated in the Black-Scholes equation. Given that the weights gives rise to the desired value process, we have successfully found the portfolio weight (see lemma above).

Theorem 8.5. (Bjork) Consider the Black-Scholes model given in (8.3)-(8.4), and a simple contingent claim $\mathcal{X} = \Phi(S_t)$. Define F as the solution to the boundary value problem

$$F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2F_{ss}(t, s) - rF(t, s) = 0, \quad (8.17)$$

$$F(T, s) = \Phi(s). \quad (8.17)$$

Then \mathcal{X} can be replicated by the relative portfolio

$$w_t^B = \frac{F(t, S_t) - S_t F_s(t, S_t)}{F(t, S_t)}, \quad (8.18)$$

$$w_t^S = \frac{S_t F_s(t, S_t)}{F(t, S_t)}. \quad (8.19)$$

The corresponding absolute portfolio is given by

$$h_t^B = \frac{F(t, S_t) - S_t F_s(t, S_t)}{B_t}, \quad (8.20)$$

$$h_t^S = F_s(t, S_t), \quad (8.21)$$

and the value process V^h is given by

$$V_t^h = F(t, S_t). \quad (8.22)$$

Proposition 8.6. (Bjork) Consider the Black-Scholes model given in (8.3)-(8.4), and a contingent claim on the form $\mathcal{X} = \Phi(S_T, Z_T)$ (eq. 8.29). We define the process Z_t as

$$Z_t = \int_0^t g(u, S_u) du, \quad (8.30)$$

for some choice of the deterministic function g . Then \mathcal{X} can be replicated using a relative portfolio given by

$$w_t^B = \frac{F(t, S_t, Z_t) - S_t F_s(t, S_t, Z_t)}{F(t, S_t, Z_t)}, \quad (8.31)$$

$$w_t^S = \frac{S_t F_s(t, S_t, Z_t)}{F(t, S_t, Z_t)}. \quad (8.32)$$

where F is the solution to the boundary value problem

$$F_t(t, s, z) + rsF_s(t, s, z) + \frac{1}{2}s^2\sigma^2F_{ss}(t, s, z) - rF(t, s, z) = 0, \quad (8.33)$$

$$F(T, s, z) = \Phi(s, z). \quad (8.33)$$

The corresponding value process is given by $V_t = F(t, S_t, Z_t)$, and F has the stochastic representation

$$F(t, s, z) = e^{-r(T-t)} E_{t,s,z}^Q[\Phi(S_T, Z_T)], \quad (8.34)$$

where the Q -dynamics are given by

$$dS_u = rS_u du + S_u \sigma(u, S_u) dW_u^Q, \quad (8.35)$$

$$S_t = s, \quad (8.36)$$

$$dZ_u = g(u, S_u) du, \quad (8.37)$$

$$Z_t = z. \quad (8.38)$$

Absence of Arbitrage

In general we have conflicting forces when evaluating when a certain market is arbitrage free and/or complete. We have in simple terms the non-rigorous theorem below.

Meta-theorem 8.3.1. (Bjork) *Let N denote the number of underlying **traded** assets in the model **excluding** the risk free asset, and let R denote the number of random sources driving the price system. Generically we then have the following statements.*

- *The model is arbitrage free if and only if $N \leq R$.*
- *The model is complete if and only if $N \geq R$.*
- *The model is arbitrage free and complete if and only if $N = R$.*

Incomplete Markets

We assume a market with a risk free asset and one risky assets with dynamics

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t. \quad (9.1)$$

We want to find a unique price of a derivative on a functional form of the risky asset. We assume that we cannot invest in the asset representing the process X_t and so we can solely write contracts based on the observation X_T . The problem here is that we can only short or long the risk free asset and so no derivative is replicable.

The way we solve this problem is by having the market set the price of risk and universally price derivatives based on this given price process. We then have the assumptions

Assumption 9.2.1 *We have the market given with the only investable asset B with dynamics*

$$dB_t = rB_t dt. \quad (9.2)$$

*We furthermore, have an empirically observable stochastic process X which is **not** the price process of any traded asset. The P -dynamics of X is given by*

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t.$$

Assumption 9.2.2 *There is a liquid market for every contingent claim.*

Assumption 9.2.3 *We assume that*

- *There is a liquid, frictionless market for each of the contingent claims \mathcal{Y} and \mathcal{Z} .*
- *The market prices of the claims are of the form*

$$\Pi_t[\mathcal{Y}] = F(t, X_t),$$

$$\Pi_t[\mathcal{Z}] = G(t, X_t),$$

where F and G are smooth real valued function.

From Ito's formula we have the dynamics

$$dF = \mu_F F dt + \sigma_F F dW, \quad (9.4)$$

$$dG = \mu_G G dt + \sigma_G G dW. \quad (9.5)$$

Where the processes μ_F and σ_F are given by

$$\mu_F = \frac{F_t + \mu F_x + \frac{1}{2} \sigma^2 F_{xx}}{F},$$

$$\sigma_F = \frac{\sigma F_x}{F}.$$

By forming a portfolio of the two contracts we lead to the relation.

$$\frac{\mu_F - r}{\sigma_F} = \frac{\mu_G - r}{\sigma_G}.$$

This gives the important insight.

Proposition 9.1. (Bjork) *Assume that the market for derivatives is free of arbitrage. Then there exists a universal process $\lambda(t, X_t)$ such that, with probability one, and for all t , we have*

$$\frac{\mu_F(t, X_t) - r}{\sigma_F(t, X_t)} = \mu(t, X_t), \quad (9.7)$$

regardless of the specific choice of the derivative F .

Proposition 9.2. (Bjork) *Assume absence of arbitrage, the pricing function $F(t, x)$ of the T -claim $\Phi(X_T)$ solves the following boundary value problem.*

$$F_t(t, x) + \mathcal{A}F(t, x) - rF(t, x) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}, \quad (9.8)$$

$$F(T, x) = \Phi(x), \quad x \in \mathbb{R}, \quad (9.9)$$

where

$$\mathcal{A}F(t, x) = \{\mu(t, x) - \lambda(t, x)\sigma(t, x)\} F_x(t, x) + \frac{1}{2}\sigma^2(t, x)F_{xx}(t, x).$$

Proposition 9.3. (Bjork) (Risk neutral valuation) *Assuming absence of arbitrage, the pricing function $F(t, x)$ of the T -claim $\Phi(X_T)$ is given by the formula*

$$F(t, x) = e^{-r(T-t)} E_{t,x}^Q[\Phi(X_T)]. \quad (9.11)$$

The dynamics of X under the martingale measure Q are given by

$$dX_t = \{\mu(t, x) - \lambda(t, x)\sigma(t, x)\} F_x(t, x) + \sigma(t, x) dW_t^Q,$$

where W^Q is a Q -Brownian motion.

Parity relations

Put-call Parity

The notion of continuous rebalancing the replicating portfolio require leads to problems in the real world. Trading does cost some money (typical in fractions) and so continuous balancing would make the portfolio go to 0 rather quickly. Why? The Brownian motion has unbounded variation and so we would have to sell and buy the portfolio uncountable many time in any interval and the shift in weight is not negligible. Because of this we would like to see which claims we can replicate by buying and holding a combination of assets and derivatives.

Proposition 10.1. (Bjork) *Let Φ and Ψ be contract functions for the T -claims $\mathcal{X} = \Phi(S_T)$ and $\mathcal{Y} = \Psi(S_T)$. Then for any real numbers α and β we have the following price relation:*

$$\Pi_t[\alpha\Phi + \beta\Psi] = \alpha\Pi_t[\Phi] + \beta\Pi_t[\Psi]. \quad (10.1)$$

If we consider the basic contract functions

$$\Phi_S(x) = x, \quad (10.2)$$

$$\Phi_B(x) = 1, \quad (10.3)$$

$$\Phi_{C,K}(x) = (x - K)^+, \quad (10.4)$$

$$\Phi_{P,K}(x) = (K - x)^+.$$

That is a contract paying (respectively): the price of one stock, 1 dollar, one european call and one european put both with strike K . It is clear that the following prices are

$$\Pi_t[\Phi_S] = S_t, \quad (10.5)$$

$$\Pi_t[\Phi_B] = e^{-r(T-t)}, \quad (10.6)$$

$$\Pi_t[\Phi_{C,K}] = c(t, S_t; K, T), \quad (10.7)$$

$$\Pi_t[\Phi_{P,K}] = p(t, S_t; K, T).$$

Where $c(t, s, K, T, r, \sigma)$ and $p(t, s, K, T, r, \sigma)$ are the pricing function of the european call and put option. We see that we can replicate these payouts by: buying the stock today and selling at time T , buying a zero coupon T -bond with face value 1, buying the call and put option.

Then we can by choosing $\alpha, \beta, \gamma_1, \dots, \gamma_n$ form a portfolio consisting of α stocks, β T -bonds and γ_i call options with maturity T and strike K_i . The price is then a linear combination given the choice (see proposition 10.1).

The put option is not included in the above portfolio as we have the put-call parity below

Proposition 10.2. (Bjork) (Put-call parity) *Consider a European call and a European put, both with strike K and time of maturity T . Then we have the relation.*

$$p(t, s) = Ke^{-r(T-t)} + c(t, s) - s. \quad (10.11)$$

In particular the put option can be replicated by a constant portfolio consisting of K zero coupon T -bonds, a European call option and a single short position in the underlying stock.

We now have the pleasing proposition given the class of claims we can reach with the buy-and-hold portfolio with T -bonds, stock and call options

Proposition 10.3. (Bjork) *Fix an arbitrary continuous contract function Φ with compact support. Then the corresponding contract can be replicated with arbitrary precision (in sup-norm) using a constant portfolio consisting only of bonds, call options and the underlying stock.*

The Greeks

When holding a portfolio we may denote the pricing function by $P(t, s)$. Here we only have one **underlying** asset with price process S_t . We now have two types of risk:

- Price changes in the underlying asset.
- Misspecifications in the model parameters.

These two risk give rise to “the greeks” as defined below.

Definition 10.4. (Bjork) *The greeks of a portfolio is given by*

$$\Delta = \frac{\partial P}{\partial s}, \quad \Gamma = \frac{\partial^2 P}{\partial s^2}, \quad \rho = \frac{\partial P}{\partial r}, \quad \Theta = \frac{\partial P}{\partial t}, \quad \mathcal{V} = \frac{\partial P}{\partial s}.$$

For the call option in particular we have the following derivatives.

Proposition 10.5. (Bjork) *The greeks of a portfolio consisting of a single European call option with maturity T and strike price K have the following greeks (φ denoting the density function of a $\mathcal{N}(0, 1)$ -variable):*

$$\Delta = N(d_1), \tag{10.17}$$

$$\Gamma = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}, \tag{10.18}$$

$$\rho = K(T-t)e^{-r(T-t)}N(d_2), \tag{10.19}$$

$$\Theta = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2), \tag{10.20}$$

$$\mathcal{V} = s\varphi(d_1)\sqrt{T-t}. \tag{10.21}$$

Fundamental pricing theorem I and II

We start by stating the following theorem.

Theorem 11.1. (Bjork) *If at least one of the assets S^1, \dots, S^N has diffusion term which is non-zero at all times, and if naive portfolio strategies are admitted, then the model admits arbitrage.*

We will go as follows. Derive the fundamental pricing theorem 1 and 2 in a setting with zero interest rate. Then we will extend the result in general by choosing a simple numeraire. We start by defining some basic notation.

Definition 11.2. (Bjork) *Define the process h as*

$$h = [h^0, h^S] := [h^0, h^1, \dots, h^N]$$

We define the following.

- For a process h , its **value process** V_t^h is defined by

$$V_t^h = h_t^0 \cdot 1 + \sum_{i=1}^N h_t^i S_t^i, \quad (11.3)$$

or in compact form

$$V_t^h = h_t^0 \cdot 1 + h_t^S S_t^S \quad (11.4)$$

- An adapted process h^S is called **admissible** if there exists a non-negative real number α (which may depend on the choice of h^S) such that

$$\int_0^t h_u^S dS_u \geq -\alpha, \quad (11.5)$$

for all $t \in [0, T]$. A process h , is called an **admissible portfolio process** if h^S is admissible.

- An admissible portfolio is said to be **self-financing**, if

$$V_t^h = V_0^h + \int_0^t h_u^S dS_u, \quad (11.6)$$

i.e. if

$$dV_t^h = h_t^S dS_t. \quad (11.7)$$

Lemma 11.3. (Bjork) *For any adapted process h^S satisfying the admissibility condition above, and for any real number x , there exists a unique adapted process h^0 , such that:*

- The process h defined by $h = [h^0, h^S]$ is self-financing.
- The value process is given by

$$V_t^h = x + \int_0^t h_u^S dS_u. \quad (11.8)$$

In particular, the space \mathcal{K}_0 of portfolio values, reachable at time T by means of a self-financing portfolio with zero initial cost is given by

$$\mathcal{K}_0 = \left\{ \int_0^T h_u^S dS_u : h^S \text{ is admissible} \right\}. \quad (11.9)$$

Definition 11.4. (Bjork) *A probability measure Q on \mathcal{F}_T is called **equivalent martingale measure** for the market model, the numeraire S^0 , and the time interval $[0, T]$, if it has the following properties:*

- $Q \sim P$ on \mathcal{F}_T , so P and Q are equivalent.
- All price processes S^0, S^1, \dots, S^N are martingales under Q on the time interval $[0, T]$.

An equivalent martingale measure will often be referred to as just “a martingale measure” or as “an EMM”. If $Q \sim P$ has the property that S^0, S^1, \dots, S^N are local martingales, then Q is called a **local martingale measure**.

Theorem 11.5. (Bjork) (The First Fundamental Theorem) *The model is arbitrage free “essentially” if and only if there exists a (local) martingale measure Q .*

Definition 11.6. (Bjork) *With the notation above, we say that the model admits*

- **No Arbitrage (NA)** if

$$\mathcal{C} \cap L_+^\infty = \{0\}, \quad (11.21)$$

- **No Free Lunch with Vanishing Risk (NFLVR)** if

$$\tilde{\mathcal{C}} \cap L_+^\infty = \{0\}, \quad (11.22)$$

where $\tilde{\mathcal{C}}$ denotes the closure of \mathcal{C} in L^∞ .

Theorem 11.7. (Bjork) (Kreps-Yan Separation Theorem) *If \mathcal{C} is weak* closed, and if*

$$\mathcal{C} \cap L_+^\infty = \{0\},$$

then there exists a random variable $L \in L^1$ such that L is P almost surely strictly positive, and

$$E^P[L \cdot X] \leq 0,$$

for all $X \in \mathcal{C}$.

Proposition 11.8. (Bjork) *If the asset price processes are uniformly bounded, then the condition NFLVR implies that \mathcal{C} is weak* closed.*

Theorem 11.9. (Bjork) (First Fundamental Theorem) *Assume that the asset price process S is bounded. Then there exists an equivalent martingale measure if and only if the model satisfies NFLVR.*

Theorem 11.10. (Bjork) (First Fundamental Theorem) *Assume that the asset price process S is locally bounded. Then there exists an equivalent martingale measure if and only if the model satisfies NFLVR.*

Assumption 11.4.1. (Bjork) *We assume that $S_t^0 > 0$ P -a.s. for all $t \geq 0$.*

Definition 11.11. (Bjork) *The **normalized economy** (also referred to as the “Z-economy”) is defined by the price vector process Z , where*

$$Z_t = \frac{S_t}{S_t^0}.$$

Definition 11.12. (Bjork)

- A **portfolio strategy** is any adapted $(N + 1)$ -dimensional process

$$h_t = [h_t^0, h_t^1, \dots, h_t^N].$$

- The **S-value process** V_t^S corresponding to the portfolio h is $h_t S_t$.
- The **Z-value process** V_t^Z corresponding to the portfolio h is $h_t Z_t$.
- A portfolio is said to be **admissible** if it is admissible as an Z portfolio.
- An admissible portfolio is **S-self-balancing** if

$$dV_t^S = \sum_{i=0}^N h_t^i dS_t^i \quad (11.26)$$

- An admissible portfolio is **Z-self-balancing** if

$$dV_t^Z = \sum_{i=0}^N h_t^i dZ_t^i. \quad (11.28)$$

Lemma 11.13. (Bjork) (Invariance Lemma) *With assumptions as above, the following hold.*

- A portfolio h is S -self-financing if and only if it is Z -self-financing.
- The value processes V^S and V^Z are connected by

$$V_t^Z = \frac{1}{S_t^0} \cdot V_t^S.$$

- A claim \mathcal{Y} is S -replical if and only if the claim

$$\frac{\mathcal{Y}}{S_T^0}$$

is Z -replicable.

- The model is S arbitrage free if and only if it is Z arbitrage free.

Theorem 11.14. (Bjork) (The First Fundamental Theorem) *Consider the market model S^0, S^1, \dots, S^N where we assume that $S_t^0 > 0$, P -a.s. for all $t \geq 0$. Assume furthermore that S^0, S^1, \dots, S^N are locally bounded. Then the followin conditions are equivalent:*

- The model satisfies NFLVR.
- There exists a measure $Q \sim P$ such that the processes

$$Z^0, Z^1, \dots, Z^N,$$

are local martingales under Q .

Completeness

Lemma 11.15. (Bjork) *Consider a given T -claim X . Fix a martingale measure Q and assume that the normalized claim X/S_T^0 is integrable. If the Q -martingale M , defined by*

$$M_t = E^Q \left[\frac{X}{S_T^0} \middle| \mathcal{F}_t \right], \quad (11.34)$$

admits an integral representation of the form

$$M_t = x + \sum_{i=1}^N \int_0^t h_s^i dZ_s^i, \quad (11.35)$$

then X can be hedged in the S -economy. Furthermore, the replicating portfolio (h^0, h^1, \dots, h^N) is given by the above for h^i , $i = 1, \dots, N$ and $h_t^0 = M_t - \sum_{i=1}^N h_t^i Z_t^i$.

Theorem 11.16. (Bjork) (Jacod) *Let \mathcal{M} denote the convex set of equivalent martingale measures. Then, for any fixed $Q \in \mathcal{M}$, the following statements are equivalent:*

- Every Q local martingale M has dynamics of the form

$$dM_t = \sum_{i=1}^N h_s^i dZ_s^i.$$

- Q is an extremal point of \mathcal{M} .

Theorem 11.17. (Bjork) (The Second Fundamental Theorem) *Assume that the market is arbitrage free and consider a fixed numeraire asset S^0 . Then the market is complete if and onlt if the martingale measure Q , corresponding to the numeraire S^0 , is unique.*

Risk Neutral Valuation Formula

We have the setting of a market consisting of the assets S^0, \dots, S^N of $N + 1$ assets. We consider the numeraire S^0 being a risk free asset. We introduce a price of contingent claim X , such that the extended market consisting of the price process of X and the $N + 1$ assets is arbitrage free. Alternatively, we can, equivalently, find a replicating portfolio h such that $V_T^h = X$ with probability one.

Theorem 11.18. (Bjork) (General Pricing Equation) *The arbitrage free price process for the T -claim X is given by*

$$\Pi_t[X] = S_t^0 E^Q \left[\frac{X}{S_T^0} \middle| \mathcal{F}_t \right], \quad (11.41)$$

where Q is the (not necessarily unique) martingale measure for the a priori given market S^0, S^1, \dots, S^N , with S^0 as the numeraire.

If if we assume that the bank account takes the form

$$S_t^0 = S_0^0 e^{-\int_0^t r(s) ds},$$

where r is the short rate, then we have the familier *risk neutral valuation formula*.

Theorem 11.19. (Bjork) (Risk Neutral Valuation Formula) *Assuming the existance of a short rate, the pricing formula takes the form*

$$\Pi_t[X] = E^Q \left[e^{-\int_0^T r(s) ds} X \middle| \mathcal{F}_t \right], \quad (11.42)$$

where Q is the (not necessarily unique) martingale measure with the bank account as the numeraire.

Definition 11.20. (Bjork) *A **zero coupon bond** with **maturity date** T , also called a T -bond, is a contract which guarantees the holder one dollar to be paid on the date T . The price at time t of a bond with maturity date T is denoted by $p(t, T)$.*

Proposition 11.21. (Bjork) *The price of a zero coupon T -bond is given by*

$$p(t, T) = E^Q \left[e^{-\int_t^T r(s) ds} \middle| \mathcal{F}_t \right], \quad (11.43)$$

and in particular we have $p(T, T) = 1$ for all $T \geq 0$ (eq. 11.44).

Stochastic Discount Factors

Definition 11.22. (Bjork) *Assume the existance of a short rate r . For any fixed martingale measure Q , let the likelihood process L be defined by*

$$L_t = \frac{dQ}{dP}, \text{ on } \mathcal{F}_t. \quad (11.48)$$

The *stochastic discount factor (SDF)* process \mathbf{M} , corresponding to Q , is defined as

$$\mathbf{M}_t = e^{-\int_0^t r(s) ds} L_t \quad \left(= \frac{1}{B_t} \cdot L_t \right). \quad (11.49/50)$$

Proposition 11.23. (Bjork) *Assume absence of arbitrage. With notation as above, the following hold:*

- For any sufficiently integrable T -claim X , the arbitrage free price is given by

$$\Pi_t[X] = E^P \left[\frac{\mathbf{M}_T}{\mathbf{M}_t} X \mid \mathcal{F}_t \right]. \quad (11.51)$$

- For any arbitrage free asset price process S (derivative or underlying) the process $\mathbf{M}_t S_t$ is a (local) P -martingale.
- The P -dynamics of \mathbf{M} are given by

$$d\mathbf{M}_t = -r_t \mathbf{M}_t dt + \frac{1}{B_t} dL_t. \quad (11.53)$$

Summary

Theorem 11.24. (Bjork) (First Fundamental Theorem) The market model is free of arbitrage if and only if there exists a **martingale measure**, i.e. a measure $Q \sim P$ such that the processes

$$\frac{S_t^0}{S_t^0}, \frac{S_t^1}{S_t^0}, \dots, \frac{S_t^N}{S_t^0}$$

are (local) martingales under Q .

Proposition 11.25. (Bjork) If the numeraire S^0 is the money account, i.e.

$$S_t^0 = e^{\int_0^t r(s) ds},$$

where r is the (possibly stochastic) short rate, and if we assume that all processes are Brownian driven, then a measure $Q \sim P$ is a martingale measure if and only if all assets S^0, S^1, \dots, S^N have the short rate as their local rates of return, i.e. if the Q -dynamics are of the form

$$dS_t^i = S_t^i r_t dt + S_t^i \sigma_t^i dW_t^Q, \quad (11.54)$$

where W^Q is a (multidimensional) Q -Brownian motion.

Theorem 11.26. (Bjork) (Second Fundamental Theorem) Assuming absence of arbitrage, the market model is complete if and only if the martingale measure Q is unique.

Proposition 11.27. (Bjork)

1. In order to avoid arbitrage, X must be priced according to the formula

$$\Pi_t[X] = S_t^0 E^Q \left[\frac{X}{S_T^0} \mid \mathcal{F}_t \right], \quad (11.55)$$

where Q is a martingale measure for $[S^0, S^1, \dots, S^N]$, with S^0 as the numeraire.

2. In particular, we can choose the bank account B_t , as the numeraire. Then B has the dynamics

$$dB_t = r_t B_t dt, \quad (11.56)$$

where r is the (possibly stochastic) short rate process. In this case the pricing formula above reduces to

$$\Pi_t[X] = E^Q \left[e^{-\int_t^T r(s) ds} X \mid \mathcal{F}_t \right]. \quad (11.57)$$

3. As a special case, the price of a zero coupon T -bond is given by

$$p(t, T) = E^Q \left[e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right]. \quad (11.58)$$

4. Defining the stochastic discount factor \mathbf{M} by $\mathbf{M}_t = B_t^{-1} L_t$ we also have the pricing formula.

$$\Pi_t[X] = E^Q \left[\frac{\mathbf{M}_T}{\mathbf{M}_t} X \mid \mathcal{F}_t \right]. \quad (11.59)$$

5. Different choices of Q will generically give rise to different price processes for a fixed claim X . However, if X is attainable then all choices of Q will produce the same price process, which then is given by

$$\Pi_t[X] = V_t^h, \quad (11.60)$$

where h is the hedging portfolio. Different choices of hedging portfolios (if such exist) will produce the same price process.

6. In particular, for every replicable claim X it holds that

$$V_t^Q = E^Q \left[e^{-\int_t^T r(s) ds} X \mid \mathcal{F}_t \right]. \quad (11.61)$$

Mathematics of the martingale approach

Martingale representation theorem

Theorem 12.1. (Bjork) (Representation of Brownian Functionals) *Let W be a d dimensional Brownian motions, and let X be a random variable such that*

- $X \in \mathcal{F}_T^W$,
- $E[|X|] < \infty$.

Then there exist uniquely determined \mathcal{F}_t^W -adapted processes h^1, \dots, h^d , such that X has the representation

$$X = E[X] + \sum_{i=1}^d \int_0^T h_s^i dW_s^i. \quad (12.2)$$

Under the additional assumption

$$E[X^2] < \infty,$$

then h^1, \dots, h^d are in \mathcal{L}^2 .

Theorem 12.2. (Bjork) (The Martingale Representation Theorem) *Let W be a d dimensional Brownian motions, and assume that the filtration \mathbf{F} is defined as*

$$\mathcal{F}_t = \mathcal{F}_t^W, \quad t \in [0, T].$$

Let M be any \mathcal{F}_t -adapted martingale. Then there exist uniquely determined \mathcal{F}_t -adapted processes h^1, \dots, h^d such that M has the representation

$$M_t = M_0 + \sum_{i=1}^d \int_0^t h_s^i dW_s^i, \quad t \in [0, T]. \quad (12.9)$$

If the martingale M is square integrable, then h^1, \dots, h^d are in \mathcal{L}^2 .

Girsanov theorem

Theorem 12.3. (Bjork) (The Girsanov Theorem) *Let W be a d dimensional P -Brownian motion on $(\Omega, \mathcal{F}, P, \mathbf{F})$ and let φ be any d -dimensional adapted column vector process. Choose a fixed T and define the process L on $[0, T]$ by*

$$dL_t = \varphi_t^\top L_t dW_t, \quad (12.16)$$

$$L_0 = 1, \quad (12.17)$$

i.e.

$$L_t = \exp \left\{ \int_0^t \varphi_s^\top dW_s - \frac{1}{2} \int_0^t \|\varphi_s\|^2 ds \right\}.$$

Assume that

$$E^P[L_T] = 1, \quad (12.18)$$

and define the new probability measure Q on \mathcal{F}_T by

$$L_T = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_T. \quad (12.19)$$

Then

$$dW_t = \varphi dt + dW_t^Q, \quad (12.20)$$

where W^Q is a d dimensional Q -Brownian motion or equivalently

$$W_t^Q = W_t - \int_0^t \varphi_s ds \quad (12.21)$$

is a standard Q -Brownian motion.

We will often refer to φ as the **Girsanov kernel** of the measure transformation. Furthermore, we have written on component form above and the L dynamics will have the form

$$dL_t = L_t \sum_{i=1}^d \varphi_t^i dW_t^i,$$

and L will have the explicit form

$$L_t = \exp \left\{ \sum_{i=1}^d \int_0^t \varphi_s^i dW_s^i - \frac{1}{2} \int_0^t \sum_{i=1}^d (\varphi_s^i)^2 ds \right\}.$$

The conclusion of the Girsanov Theorem is thwn that we can write

$$dW_t^i = \varphi_t^i dt + dW_t^{Q,i},$$

for $i = 1, \dots, d$ where $W_t^{Q,1}, \dots, W_t^{Q,d}$ are independent standard Brownian motions under Q .

Definition 12.4. (Bjork) For any Brownian motion W and any kernel process φ , the **Doleans exponential** process \mathcal{E} is defined by

$$\mathcal{E}(\varphi \bullet W)_t = \exp \left\{ \int_0^t \varphi_s^\top dW_s - \frac{1}{2} \int_0^t \|\varphi_s\|^2 ds \right\}. \quad (12.24)$$

Lemma 12.5. (Bjork) (The Novikov Condition) Assume that the Girsanov kernel φ is such that

$$E^P \left[e^{\frac{1}{2} \int_0^T \|\varphi_t\|^2 dt} \right] < \infty. \quad (12.27)$$

Then L is a martingale and in particular $E^P[L_T] = 1$.

Theorem 12.6. (Bjork) (The Converse of the Girsanov Theorem) Let W be a d -dimensional standard P -Brownian motion on $(\Omega, \mathcal{F}, P, \mathbf{F})$ and assume that

$$\mathcal{F}_t = \mathcal{F}_t^W, \quad \forall t.$$

Assume that there exists a probability measure Q such that $Q \ll P$ on \mathcal{F}_T . Then there exists an adapted process φ such that the likelihood process L has the dynamics

$$\begin{aligned} dL_t &= L_t \varphi_t^\top dW_t, \\ L_0 &= 1. \end{aligned}$$

This gives us a recipe to transform dynamics of Ito processes under the measure Q as we may rewrite the dynamics of the Brownian motion. We therefore have for an Ito process X with dynamics

$$dX_t = \mu(t, X_t)X_t dt + \sigma(t, X_t)X_t dW_t,$$

may be transformed under Q as

$$\begin{aligned} dX_t &= \mu(t, X_t)X_t dt + \sigma(t, X_t)X_t dW_t \\ &= \mu(t, X_t)X_t dt + \sigma(t, X_t)X_t (\varphi_t dt + dW_t^Q) \\ &= (\mu(t, X_t) + \varphi_t)X_t dt + \sigma(t, X_t)X_t dW_t^Q. \end{aligned}$$

This may lead us into deducing that

$$\mu(t, X_t) + \sigma(t, X_t)\varphi_t = r_t \iff \varphi_t = \frac{r_t - \mu(t, X_t)}{\sigma(t, X_t)}.$$

We furthermore have the Levy characterisation of a Brownian motion.

Theorem. (Remark FinKont) (Levy Characterisation of Brownian motion) *Let X_t be an Ito process with $X_0 = 0$. Then X_t is a Brownian motion if and only if the two processes X_t and $X_t^2 - t$ are continuous martingales.*

Black-Scholes model - martingale approach

We consider the standard Black-Scholes model with a single risk free asset and risky asset with dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (13.1)$$

$$dB_t = r B_t dt. \quad (13.2)$$

We want check whether the model is arbitrage free on any time interval $[0, T]$, and find a (perhaps unique) martingale measure such that we may apply the fundamental pricing theorem 1 and 2. From Girsanov this endeavour is equivalent with searching for a (perhaps unique) Girsanov kernel φ . We therefore define as usual the likelihood process

$$dL_t = \varphi_t L_t dW_t,$$

and setting $dQ = L_T dP$ on \mathcal{F}_T , we know from Girsanov theorem that

$$dW_t = \varphi_t dt + dW_t^Q.$$

Inserting into the Black-Scholes model we have

$$dS_t = S_t(\mu + \sigma\varphi_t) dt + \sigma S_t dW_t^Q.$$

We know that for Q to be a martingale measure we know that the local rate of return under Q of S must be the short rate r i.e. we have

$$\mu + \sigma\varphi_t = r \iff \varphi_t = \frac{r - \mu}{\sigma} = -\frac{\mu - r}{\sigma}, \quad (13.3)$$

and so we see the Girsanov kernel is **constant and deterministic**. The process has the economic interpretation that the Girsanov kernel is the risk premium per unit volatility.

Lemma 13.1. (Bjork) *The Girsanov kernel φ is given by*

$$\varphi = -\lambda$$

where the market price of risk λ is defined by

$$\lambda = \frac{r - \mu}{\sigma}.$$

We therefore have determined a *martingale* and so we have the result.

Theorem 13.2. (Bjork) *The Black-Scholes model above is arbitrage free.*

We could in general have that μ, σ, r are adapted processes. If this is the case we would have to show the Novikov condition.

Pricing then of any T -claim X then is given by the risk neutral pricing formula

$$\Pi_t[X] = e^{-r(T-t)} E^Q[X | \mathcal{F}_t], \quad (13.7)$$

where the Q dynamics of S has local drift r and volatility from the Q -brownian motion W^Q .

Theorem 13.3. (Bjork) *The Black-Scholes model above is complete. This also holds for the more general model where r, μ, σ are adapted processes.*

Hedging is possible by considering a T claim with

$$E^Q \left[\frac{X}{B_T} \right] < \infty.$$

Notice the numeraire B_t as in the normalized Z -economy. Consider now the Q -martingale

$$M_t = E^Q \left[\frac{X}{B_T} \middle| \mathcal{F}_t \right], \quad (13.9)$$

and it now follows from lemma 11.15 that the model is complete if we can find a process h_t^1 such that

$$dM_t = h_t^1 dZ_t^i. \quad (13.10)$$

In order to prove existence of such a process h^1 we use the Martingale Representation Theorem 12.2, which says that there *exists* a process g_t such that

$$dM_t = g_t dW_t^Q. \quad (13.11)$$

We can now combine these two equations by the following Q dynamics

$$dZ_t^1 = Z_t^1 \sigma dW_t^Q. \quad (13.12)$$

Hence we have

$$dM_t = h_t^1 Z_t^1 \sigma dW_t^Q = g_t dW_t^Q \Rightarrow h_t^1 = \frac{g_t}{Z_t^1 \sigma}.$$

Theorem 13.4. (Bjork) *In the Black-Scholes model every T -claim X satisfying*

$$E^Q \left[\frac{X}{B_T} \right] < \infty$$

can be replicated. The replicating portfolio is given by

$$h_t^1 = \frac{g_t}{Z_t^1 \sigma}, \quad (13.13)$$

$$h_t^0 = M_t - h_t^1 Z_t^1, \quad (13.14)$$

where M is defined by the above and g is defined by above.

If the T -claim is simple that is $X = \Phi(S_T)$ we may solve a boundary value problem with Feynman-Kac to arrive at the familiar result.

Proposition 13.5. (Bjork) *In the Black-Scholes model every T -claim on the form $X = \Phi(S_T)$. Then X can be replicated by the portfolio*

$$h_t^0 = \frac{F(t, S_t) - S_t \frac{\partial F}{\partial s}(t, S_t)}{B_t}, \quad (13.15)$$

$$h_t^1 = \frac{\partial F}{\partial s}(t, S_t), \quad (13.15)$$

where F solves the **Black-Scholes equation**

$$\frac{\partial F}{\partial t}(t, s) + rs \frac{\partial F}{\partial s}(t, s) + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(t, s) - rF(t, s) = 0, \quad (13.16)$$

$$F(T, s) = \Phi(s). \quad (13.16)$$

Furthermore the value process for the replicating portfolio is given by

$$V_t = F(t, S_t).$$

Multidimensional models

We specify the general model by the assumptions below.

Assumption 14.0.1 *We assume the following:*

- There are n risky assets S^1, \dots, S^n .
- Under the objective probability measure P , the S -dynamics are given by

$$dS_t^i = \mu_t^i S_t^i dt + S_t^i \sum_{j=1}^N \sigma_t^{ij} dW_t^j, \quad (14.1)$$

for $i = 1, \dots, n$.

- The coefficients processes μ^i and σ^{ij} above are assumed to be adapted.
- We have a standard risk free asset with price process B with dynamics

$$dB_t = r_t B_t dt, \quad (14.2)$$

where the short rate process r is assumed to be an adapted stochastic process.

We can use the representation of μ^i , σ^{ij} and S^i as vectors and matrices on the form.

$$\mu = \begin{bmatrix} \mu^1 \\ \vdots \\ \mu^n \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma^{1,1} & \dots & \sigma^{1,N} \\ \vdots & \ddots & \vdots \\ \sigma^{n,1} & \dots & \sigma^{n,N} \end{bmatrix}, \quad D(S) = \begin{bmatrix} S^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & S^n \end{bmatrix}.$$

And so we have the model on compact form.

$$dS_t = D(S_t) \mu_t dt + D(S_t) \sigma_t dW_t, \quad (14.3)$$

$$dB_t = r_t B_t dt. \quad (14.4)$$

Now using Girsanov Theorem we can define the prospective likelihood process L by

$$dL_t = L_t \varphi_t^\top dW_t, \quad (14.5)$$

$$L_0 = 1, \quad (14.6)$$

where φ is an adapted N -dimensional process. Then our candidate martingale measure Q is given by $dQ = L_t dP$ on \mathcal{F}_t and the Girsanov theorem give the dynamics

$$dW_t = \varphi_t dt + dW_t^Q, \quad (14.7)$$

where W^Q is a standard Q -Brownian motion. Inserting into the P -dynamics we obtain.

$$dS_t = D(S_t) [\mu_t + \sigma_t \varphi_t] dt + D(S_t) \sigma_t dW_t^Q. \quad (14.8)$$

The from (11.54) we know that Q is a martingale measure if and only if the local rate of return (the dt -term) is the short interest rate i.e. if and only if

$$\mu_t + \sigma_t \varphi_t = \mathbf{r}_t, \quad (14.9)$$

where $\mathbf{r}_t = [r, \dots, r]^\top \in \mathbb{R}^n$. We then have that

$$\sigma_t \varphi_t = \mathbf{r}_t - \mu_t, \quad (14.11)$$

where we want to solve for φ . Thus we may write a condition for the absence of arbitrage in linear algebra terms.

Proposition 14.1. (Bjork) *A necessary condition for absence of arbitrage is that*

$$\mathbf{r}_t - \mu_t \in \text{Im}[\sigma_t]$$

with probability one for each t . A sufficient condition for absence of arbitrage is that there exists a process φ which solves (14.11) and such that L is a martingale.

Note that it is not enough for φ to solve (14.11). We also need that L is a martingale.

Definition 14.2. (Bjork) *A Girsanov kernel φ is said to be **admissible** if it generates a martingale measure, i.e. it solves (14.11) and L is a true martingale.*

Definition 14.3. (Bjork) *The model above is said to be **generically arbitrage free** if it is arbitrage free for every choice of μ .*

Proposition 14.4. (Bjork) *Disregarding integrability problems the model is generically arbitrage free if and only if, for each $t \leq T$ and P -a.s., the mapping*

$$\sigma_t : \mathbb{R}^B \rightarrow \mathbb{R}^n$$

is surjective, i.e. if and only if the volatility matrix σ_t has rank n .

Proposition 14.5. (Bjork) *Assume that the model is generically arbitrage free and that the filtration \mathbf{F} is defined by*

$$\mathcal{F}_t = \mathcal{F}_t^W. \quad (14.14)$$

Then, disregarding integrability problems, the model is complete if and only if $n = N$ and the volatility matrix σ_t is invertible P -a.s. for each $t \leq T$.

Assumption 14.3.1. (Bjork) *We assume that the model is generically free of arbitrage, i.e. that*

$$\text{Im}[\sigma_t] = \mathbb{R}^n, \quad (14.16)$$

for all t and with probability one. We also assume that the model is purely Brownian driven, i.e. that $\mathcal{F}_t = \mathcal{F}_t^W$.

Proposition 14.6. (Bjork) *Under assumption 14.3.1 the model is complete if and only if*

$$\text{Im}[\sigma_t^\top] = \mathbb{R}^N. \quad (14.23)$$

If the model is complete then, using the notation of chapter 11, the replicating portfolio $[h^0, h^S]$ is given by

$$h_t^S = g_t \sigma_t^{-1} D^{-1}(Z_t), \quad (14.24)$$

$$h_t^2 = M_t - h_t^S Z_t. \quad (14.25)$$

With M_t defined by

$$M_t = E^Q \left[\frac{\mathcal{X}}{B_T} \mid \mathcal{F}_t \right]. \quad (14.17)$$

Theorem 14.7. (Bjork) (The Second Fundamental Theorem) *Under assumptions 14.3.1 the model is complete if and only if the martingale measure is unique. This is equivalent with the statements: $\text{Ker}[\sigma_t] = \{0\}$, $\text{Im}[\sigma_t^\top] = \mathbb{R}^N$ and σ_t^{-1} exists (i.e. σ_t is invertible).*

Pricing of any T -claim \mathcal{X} is now given by the risk neutral valuation formula

$$\Pi_t[\mathcal{X}] = E^Q \left[e^{-\int_t^T r_u du} \mathcal{X} \mid \mathcal{F}_t \right], \quad (14.27)$$

where Q is some choice of martingale measure. Alternatively we can write the price as

$$\Pi_t[\mathcal{X}] = E^Q \left[\frac{\mathbf{M}_T}{\mathbf{M}_t} \mathcal{X} \mid \mathcal{F}_t \right], \quad (14.29)$$

where \mathbf{M} is the stochastic discount factor, defined by

$$\mathbf{M}_t = \frac{1}{B_t} L_t.$$

If we have a simple claim i.e. of the form $\mathcal{X} = \Phi(S_t)$ and if S is Markovian we have

$$e^{-r(T-t)} E^Q[\Phi(S_t) \mid \mathcal{F}_t] = e^{-r(T-t)} E^Q[\Phi(S_t) \mid S_t],$$

and then the pricing process must be of the form $\Pi_t[\Phi] = F(t, S_t)$. We then have F to be the solutions to the PDE

$$F_t(t, s) + \sum_{i=1}^n r s_i F_i(t, s) + \frac{1}{2} \text{tr} \{ \sigma^\top D(S) F_{ss} D(S) \sigma \} - r F(t, s) = 0, \quad (14.31)$$

$$F(T, s) = \Phi(s), \quad (14.31)$$

where $F_i = \frac{\partial F}{\partial s_i}$ and F_{ss} denotes the Hessian matrix. Furthermore, $\text{tr}(A)$ denotes the trace of A i.e. the sum of the diagonal. We have that the hedging portfolio has value process $V_t^h = F(t, S_t)$ with dynamics

$$dV_t^h = \sum_{i=1}^n F_i(t, S_t) dS_t^i.$$

Then we must have the solution

$$h_t^i = \frac{\partial F}{\partial s_i}(t, S_t), \quad i = 1, \dots, n, \quad (14.32)$$

$$h_t^0 = \frac{1}{B_t} \left\{ F(t, S_t) - \sum_{i=1}^n \frac{\partial F}{\partial s_i}(t, S_t) S_t^i \right\}. \quad (14.33)$$

Proposition 14.8. (Bjork) *With L -dynamics as in $dL_t = L_t \varphi_t^\top dW_t$, the \mathbf{M} -dynamics are*

$$d\mathbf{M}_t = -r_t \mathbf{M}_t dt + \mathbf{M}_t \varphi_t^\top dW_t, \quad (14.39)$$

or alternatively in terms of the market price of risk $\lambda_t = -\varphi_t$

$$d\mathbf{M}_t = -r_t \mathbf{M}_t dt - \mathbf{M}_t \lambda_t^\top dW_t. \quad (14.40)$$

Proposition 14.9. (Bjork) (The Hansen-Jagannathan Bounds) *Assume generic absence of arbitrage. Then the following holds for all assets, underlying or derivative, and for all admissible Girsanov kernels φ , and market prices of risk λ .*

$$\frac{|\mu_t^p - r_t|}{\|\sigma_t^p\|} \leq \|\varphi_t\|, \quad \frac{|\mu_t^p - r_t|}{\|\sigma_t^p\|} \leq \|\lambda_t\|. \quad (14.42)$$

Measure theory

Equivalent Probability Measures

The Radon-Nikodym Theorem

Definition A.50. (Bjork) Consider a measurable space (X, \mathcal{F}) on which there are defined two separate measures μ and ν :

- If, for all $A \in \mathcal{F}$, it holds that

$$\mu(A) = 0 \Rightarrow \nu(A) = 0, \quad (\text{A.7})$$

then ν is said to be **absolutely continuous** with respect to μ on \mathcal{F} and we write this as $\nu \ll \mu$.

- If we have both $\mu \ll \nu$ and $\nu \ll \mu$, then μ and ν said to be **equivalent** and we write $\mu \sim \nu$.
- If there exists two events, A and B such that:
 - $A \cup B = X$,
 - $A \cap B = \emptyset$,
 - $\mu(B) = 0$, and $\nu(A) = 0$,

then ν and μ are said to be mutually **singular**, and we write $\mu \perp \nu$.

Theorem A.52. (Bjork) (The Radon-Nikodym Theorem) Consider the measure space (X, \mathcal{F}, μ) , where we assume that μ is finite, i.e. that $\mu(X) < \infty$. Let ν be a measure on (X, \mathcal{F}) such that $\nu \ll \mu$ on \mathcal{F} . Then there exists a non-negative function $f : X \rightarrow \mathbb{R}$ such that:

$$f \text{ is } \mathcal{F}\text{-measurable} \quad (\text{A.9})$$

$$\int_X f(x) d\mu(x) < \infty, \quad (\text{A.10})$$

$$\nu(A) = \int_A f(x) d\mu(x), \text{ for all } A \in \mathcal{F}. \quad (\text{A.11})$$

The function f is called the **Radon-Nikodym derivative** of ν w.r.t. μ . It is uniquely determined μ -a.e. and we write

$$f(x) = \frac{d\nu(x)}{d\mu(x)}, \quad (\text{A.12})$$

or alternatively

$$d\nu(x) = f(x) d\mu(x). \quad (\text{A.13})$$

Equivalent Probability Measures

Lemma B.38. (Bjork) For two probability measures P and Q , the relation $P \sim Q$ on \mathcal{F} holds if and only if $P(A) = 1$ if and only if $Q(A) = 1$ for all $A \in \mathcal{F}$.

Proposition B.39. (Bjork) Assume that $Q \ll P$ on \mathcal{F} and that $\mathcal{G} \subseteq \mathcal{F}$. Then the Radon-Nikodym derivatives $L^{\mathcal{F}}$ and $L^{\mathcal{G}}$ are related by

$$L^{\mathcal{G}} = E^P[L^{\mathcal{F}} \mid \mathcal{G}]. \quad (\text{B.17})$$

Proposition B.41. (Bjork) (Bayes' Theorem) Assume that X is a random variable on (Ω, \mathcal{F}, P) , and let Q be another probability measure on (Ω, \mathcal{F}) the Radon-Nikodym derivative

$$L = \frac{dQ}{dP}$$

on \mathcal{F} . Assume that $X \in L^1(\Omega, \mathcal{F}, Q)$ and \mathcal{G} is a sigma-algebra with $\mathcal{G} \subseteq \mathcal{F}$. Then

$$E^Q[X \mid \mathcal{G}] = \frac{E^P[L \cdot X \mid \mathcal{G}]}{E^P[L \mid \mathcal{G}]}, \quad Q - a.s. \quad (\text{B.18})$$

Likelihood processes

Proposition C.12. (Bjork) Consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ on a compact interval $[0, T]$. Suppose L_T is some non-negative integrable random variable in \mathcal{F}_T . We can then define a new measure Q on \mathcal{F}_T by setting

$$dQ = L_T \, dP$$

on \mathcal{F}_T and if $E^P[L_T] = 1$ the measure Q will also be a probability measure. The likelihood process L , defined by

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t, \quad (\text{C.8})$$

is a (P, \mathcal{F}_t) -martingale.

Proposition C.13. (Bjork) A process M is a Q -martingale if and only if the process $L \cdot M$ is a P -martingale.

Linear Algebra

Invertible matrices

This sections study some fundamental properties of the *invertible matrix*. We start by defining what an invertible matrix is.

Definition. Let A be an $n \times m$ matrix and B be an $m \times n$ matrix. We say that A is invertible if

$$AB = I_{\min(m,n)}.$$

In general, we only consider square matrices. If the above holds we say that B is A 's inverse and we write $B = A^{-1}$.

We can consider some equivalence statements regarding invertible matrices.

Theorem. (The Invertible Matrix Theorem) Let A be a $n \times n$ matrix over a field K (\mathbb{R}^n), then the following statements are equivalent

1. There exists an $n \times n$ matrix B such that $AB = I_n = BA$.
2. There exist either a left inverse B or a right invers C i.e. $BA = I_n = AC$. In this case, $B = C$.
3. A has an inverse and is nonsingular and is nondegenerate.
4. A is row-equivalent to I_n .
5. A is column-equivalent to I_n .
6. A has n pivot positions.
7. A has full rank i.e. $\text{rank}(A) = n$ (spans K).
8. The equation $Ax = 0$ ($x \in K$) has only the trivial solution $x = 0$.
9. The equation $Ax = b$ has only one solution x .
10. The kernal of A is trivial i.e. $\ker(A) = \{0\}$.
11. The columns of A are linearly independent.
12. The columns of A span K .
13. $\text{span}(A) = K$.
14. The columns of A form a basis of K .
15. The linear transformation Ax is a bijection from K to K .
16. A has non-zero determinant i.e. $\det(A) \neq 0$.
17. A has not 0 as an eigenvalue.
18. The transpose of A is invertible.
19. A can be expressed as a finite product of elementary matrices.

We futhermore have some properties.

Proposition. (Properties) Let A be an $n \times n$ invertible matrix. Then

1. $(A^{-1})^{-1} = A$
2. $(kA)^{-1} = k^{-1}A^{-1}$ with $k \neq 0$.
3. $(Ax)^+ = x^+A^{-1}$ if A has orthonormal columns. $(\cdot)^+$ denotes the Moore-Penrose inverse and x is a vector.
4. If B is an $n \times n$ invertible matrix then $(AB)^{-1} = B^{-1}A^{-1}$.
5. $\det(A^{-1}) = (\det(A))^{-1}$

The property 2 is especially useful in some settings. Consider for instance

$$A = \begin{bmatrix} \sigma & 0 \\ \sigma & \sigma \end{bmatrix} = \sigma \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \sigma \tilde{A}.$$

Then we simply find the inverse of \tilde{A} and multiply by σ^{-1} . That is,

$$A^{-1} = \frac{1}{\sigma} \tilde{A}^{-1} = \frac{1}{\sigma} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sigma & 0 \\ -1/\sigma & 1/\sigma \end{bmatrix}.$$

We have an easy propositions regarding diagonal matrices.

Proposition. *If A is an diagonal matrix, then A is invertible. In particular,*

$$A^{-1} = \text{diag}(A_{11}^{-1}, \dots, A_{nn}^{-1}).$$

Random Variables

Introduction

Definition 1.1. (Hansen) A *real-valued random variable* X on a probability space (Ω, \mathbb{F}, P) is a measurable map $X : (\Omega, \mathbb{F}) \rightarrow (\mathbb{R}, \mathbb{B})$.

We never specify the background space (Ω, \mathbb{F}, P) however we always assume X is $\mathbb{F} - \mathbb{B}$ measurable. This assumption implies $(X \in A) \in \mathbb{F}$ for every $A \in \mathbb{B}$. We may want to show measurability for constructed variables and so it suffices to show measurability for generators for \mathbb{B} such as checking $(X \leq a) \in \mathbb{F}$ for every $a \in \mathbb{R}$.

Definition 1.2. (Hansen) The *distribution* of a real-valued random variable X , defined on a probability space (Ω, \mathbb{F}, P) , is the collection of probability values

$$P(X \in A) \quad \text{for } A \in \mathbb{B}. \quad (1.3)$$

In other words: the distribution of X is the image measure $X(P)$ on (\mathbb{R}, \mathbb{B}) .

Lemma 1.3. (Hansen) Let X and X' be two real-valued random variables on a probability space (Ω, \mathbb{F}, P) . If

$$P(X = X') = 1$$

then X and X' has the same distribution.

An often used way of summarizing the distribution is through the **distribution function** $F(x) = P(X \leq x)$ for some $x \in \mathbb{R}$.

Definition 1.4. (Hansen) A real-valued random variable X has a *discrete* distribution if there is a countable set $S \subset \mathbb{R}$ such that $P(X \in S) = 1$.

Usually S is one of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or a subset of these. We may in the discrete case define the distribution by the point probabilities $P(X = x) = p(x)$ for $x \in S$.

Definition 1.5. (Hansen) A real-valued random variable X has a distribution with *density* $f : \mathbb{R} \rightarrow [0, \infty)$ if

$$P(X \in A) = \int_A f(x) dx \quad \text{for } A \in \mathbb{B}. \quad (1.5)$$

If this is the case we will write $X(P) = f \cdot m$ or $X \sim f \cdot m$.

Definition 1.6. (Hansen) A real-valued random variable X defined on a probability space (Ω, \mathbb{F}, P) is said to have p 'th moment for some $p > 0$ if

$$E|X|^p < \infty \quad (1.12)$$

The collection of all variables that satisfies (1.12) is denoted by $\mathcal{L}^p(\Omega, \mathbb{F}, P)$.

Recall the definition of the **expectation** of X by

$$E X = \int X dP \in \mathbb{R} \cup \{-\infty, +\infty\}. \quad (1.11)$$

We recall that for any measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ that are continuous on a set $A \in \mathbb{B}$ such that $P(X \in A) = 1$ we may change variable simply by computing

$$E f(X) = \int f \circ X dP = \int f(x) dX(P)(x).$$

Lemma 1.7. (Hansen) *(Markov's inequality) Let X be a non-negative random variable. For any $c > 0$ it holds that*

$$P(X \geq c) \leq \frac{E X}{c} \left(\leq \frac{E X^n}{c^n} \text{ or } \leq \frac{E(\varphi(X))}{\varphi(c)} \right). \quad (1.14)$$

for some φ non-negative monotone increasing function.

Some other versions of Markov's inequality can be found in the form of **Chebyshev's inequality**, **Chebyshev-Cantelli's inequality** or **Jensen's inequality** respectively: Let X be a real-valued random variable in $\mathcal{L}^2(\Omega, \mathbb{F}, P)$ it holds for any $\varepsilon > 0$.

$$P(|X - E X| \geq \varepsilon) \leq \frac{V X}{\varepsilon^2} \quad (1.15)$$

$$P(X - E X \geq \varepsilon) \leq \frac{V X}{V X + \varepsilon^2} \quad (\text{prob: 1.13(c)})$$

$$\varphi(E X) \leq E(\varphi(X))$$

for some convex function φ .

Lemma 1.8. (Hansen) *Let X be a non-negative random variable. It holds that*

$$E X = \int_0^\infty P(X > t) dt. \quad (1.16)$$

where the integral on the right hand side is with respect to Lebesgue measure.

Definition 1.9. (Hansen) *The **joint distribution** of real-valued random variables X_1, \dots, X_k , defined on a probability space (Ω, \mathbb{F}, P) , is the collection of probability values*

$$P(\mathbf{X} \in A) \quad \text{for } A \in \mathbb{B}_k. \quad (1.21)$$

In other words: the joint distribution of X_1, \dots, X_k (or simply: the distribution of \mathbf{X}) is the image measure $\mathbf{X}(P)$ on $(\mathbb{R}^k, \mathbb{B}_k)$.

Definition 1.11. (Hansen) *Real-valued random variables X_1, \dots, X_k , defined on a probability space (Ω, \mathbb{F}, P) , are **jointly independent** if*

$$P(X_1 \in A_1, \dots, X_k \in A_k) = \prod_{i=1}^k P(X_i \in A_i) \quad \text{for } A_1, \dots, A_k \in \mathbb{B}. \quad (1.23)$$

In other words: the variables are independent if the joint distribution $\mathbf{X}(P)$ equals the product measure $X_1(P) \otimes \dots \otimes X_k(P)$.

Theorem 1.12. (Hansen) *Let X_1, \dots, X_k be real-valued random variables defined on a probability space (Ω, \mathbb{F}, P) . If the variables are independent and if $E|X_i| < \infty$ for $i = 1, \dots, k$, then the product $X_1 \cdot \dots \cdot X_k$ has first moment and*

$$E(X_1 \cdot \dots \cdot X_k) = \prod_{i=1}^k E X_i \quad (1.24)$$

The equality only holds for two independent variables. However the **Cauchy-Schwarz inequality** which closely resembles (1.24) holds whether or not X or Y are independent:

$$(E|XY|)^2 \leq E X^2 E Y^2 \text{ or } E|XY| \leq \sqrt{E X^2} \sqrt{E Y^2}. \quad (1.25)$$

Furthermore the theorem give rise to a measure for dependence i.e. the **covariance** between two variables X and Y

$$\text{Cov}(X, Y) = E((X - E X)(Y - E Y)) = E(XY) - (E X)(E Y) \quad (1.26)$$

with $\text{Cov}(X, Y) \neq 0$ if and only if X and Y are dependent. With $\text{Cov}(X, Y) = 0$ the test is inconclusive. However independence implies $\text{Cov}(X, Y) = 0$.

Conditional expectation

The theory of conditional expectation is well-known from courses on the bachelor. Because of this we will only summarise the most important results.

We consider a background space (Ω, \mathcal{F}, P) and a sub-sigma algebra $\mathcal{G} \subseteq \mathcal{F}$. We assume that some stochastic variable is \mathcal{F} -measurable, that is the mapping $X : (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}, \mathbb{B}, m)$ is \mathcal{F} - \mathbb{B} -measurable i.e. $\forall B \in \mathbb{B} : \{X \in B\} \in \mathcal{F}$. For some random variable Z defined on the subspace (Ω, \mathcal{G}, P) , we say that Z is the conditional expectation of X given \mathcal{G} if

$$\forall G \in \mathcal{G} : \int_G Z(\omega) dP(\omega) = \int_G X(\omega) dP(\omega).$$

This fact is summed up in the definition below.

Definition B.27. (Bjork) (Conditional expectation) Let (Ω, \mathcal{F}, P) be a probability space and X a random variable in $L^1(\Omega, \mathcal{F}, P)$ ($|X|$ is integrable). Let furthermore \mathcal{G} be a sigma-algebra such that $\mathcal{G} \subseteq \mathcal{F}$. If Z is a random variable with the properties that:

- i. Z is \mathcal{G} -measurable.
- ii. For every $G \in \mathcal{G}$ it holds that

$$\int_G Z(\omega) dP(\omega) = \int_G X(\omega) dP(\omega). \quad (\text{B.5})$$

Then we say that Z is the **conditional expectation of X given the sigma-algebra \mathcal{G}** . In that case we denote Z by the symbol $E[X | \mathcal{G}]$.

We see that from the above it always holds that X satisfies (ii). It does not, however, always hold that X is \mathcal{G} -measurable. Given this fact it is not trivial that a random variable $E[X | \mathcal{G}]$ even exists. This nontriviality is fortunatly resolved by the Radon-Nikodym theorem.

Theorem B.28. (Bjork) (Existence and uniqueness of Conditional expectation) Let (Ω, \mathcal{F}, P) , X and \mathcal{G} be given as in the definition above. Then the following holds:

- There will always exist a random variable Z satisfying conditions (i)-(ii) above.
- The variable Z is unique, i.e. if both Y and Z satisfy (i)-(ii) then $Y = Z$ P -a.s.

This result ensures that we may condition on any sigma-algebra for instance $\mathcal{G} = \sigma(Y)$ in that case we (pure notation) write

$$E[X | \sigma(Y)] = E[X | Y], \quad \sigma(Y) = \sigma(\{Y \in A, A \in \mathbb{B}\}).$$

In the above $\sigma(Y)$ is simply the smallest sigma-algebra containing all the pre-images of Y , that is the smallest sigma-algebra making Y measurable! Giving this foundation there are a few properties conditional expectation have which is rather useful (for instance the tower property).

Below we assume: Let (Ω, \mathcal{F}, P) be a probability space and X, Y be random variables in $L^1(\Omega, \mathcal{F}, P)$.

Proposition B.29. (Monotonicity/Linearity of Conditional expectation) The following holds:

$$X \leq Y \Rightarrow E[X | \mathcal{G}] \leq E[Y | \mathcal{G}], \quad P - \text{a.s.} \quad (\text{B.6})$$

$$E[\alpha X + \beta Y | \mathcal{G}] = \alpha E[X | \mathcal{G}] + \beta E[Y | \mathcal{G}], \quad \forall \alpha, \beta \in \mathbb{R}. \quad (\text{B.7})$$

Proposition B.30. (Bjork) (Tower property) Assume that it holds that $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$. Then the following hold:

$$E[E[X | \mathcal{G}] | \mathcal{H}] = E[X | \mathcal{H}], \quad (\text{B.8})$$

$$E[X] = E[E[X|\mathcal{G}]]. \quad (\text{B.9})$$

Proposition B.31. (Bjork) Assume X is \mathcal{G} and that both X, Y and XY are in L^1 (only assuming Y is \mathcal{F} -measurable), then

$$E[X|\mathcal{G}] = X, \quad P - \text{a.s.} \quad (\text{B.11})$$

$$E[XY|\mathcal{G}] = XE[Y|\mathcal{G}], \quad P - \text{a.s.} \quad (\text{B.12})$$

Proposition B.32. (Bjork) (Jensen inequality) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex (measurable) function and assume $f(X)$ is in L^1 . Then

$$f(E[X|\mathcal{G}]) \leq E[f(X)|\mathcal{G}], \quad P - \text{a.s.}$$

Proposition B.37. (Bjork) Let (Ω, \mathcal{F}, P) be a given probability space, let \mathcal{G} be a sub-sigma-algebra of \mathcal{F} , and let X be a square integrable random variable. Consider the problem of minimizing

$$E[(X - Z)^2]$$

where Z is allowed to vary over the class of all square integrable \mathcal{G} measurable random variables. The optimal solution \hat{Z} is then given by.

$$\hat{Z} = E[X|\mathcal{G}].$$

Proof.

Let $X \in L^2(\Omega, \mathcal{F}, P)$ be a random variable. Now consider an arbitrary $Z \in L^2(\Omega, \mathcal{G}, P)$. Recall that $\mathcal{G} \subset \mathcal{F}$ and so X is also in $L^2(\Omega, \mathcal{G}, P)$, as it is both square integrable and \mathcal{G} -measurable. Then

$$E[Z \cdot (X - E[X|\mathcal{G}])] = E[Z \cdot X] - E[Z \cdot E[X|\mathcal{G}]].$$

Then by using the law of total expectation and secondly that Z is \mathcal{G} -measurable we have that

$$E[Z \cdot X] = E[E[Z \cdot X|\mathcal{G}]] = E[Z \cdot E[X|\mathcal{G}]].$$

Combining the two equations gives the desired result. Obviously, we have that

$$X - Z = X - Z + E[X|\mathcal{G}] - E[X|\mathcal{G}] = (X - E[X|\mathcal{G}]) + (E[X|\mathcal{G}] - Z).$$

Then squaring the terms gives

$$(X - Z)^2 = (X - E[X|\mathcal{G}])^2 + (E[X|\mathcal{G}] - Z)^2 + 2(X - E[X|\mathcal{G}])(E[X|\mathcal{G}] - Z)$$

Taking expectation on each side and using linearity of the expectation we have that

$$E[(X - Z)^2] = E[(X - E[X|\mathcal{G}])^2] + E[(E[X|\mathcal{G}] - Z)^2] + 2E[(X - E[X|\mathcal{G}])(E[X|\mathcal{G}] - Z)].$$

We can now use that $E[X|\mathcal{G}] - Z$ is \mathcal{G} -measurable with the above result on the last term.

$$E[(X - Z)^2] = E[(X - E[X|\mathcal{G}])^2] + E[(E[X|\mathcal{G}] - Z)^2].$$

Now since X is given the term $E[(X - E[X|\mathcal{G}])^2]$ is simply a constant not depending on the choice of Z . The optimal choice of Z is then $E[X|\mathcal{G}]$ since this minimizes the second term. The statement is then proved.

Independence

Definition 3.1. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space. Two events $A, B \in \mathbb{F}$ are **independent** if

$$P(A \cap B) = P(A)P(B) \quad (3.1)$$

Definition 3.4. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space and let $\mathbb{G}, \mathbb{H} \subset \mathbb{F}$ be two classes of measurable sets. We say that \mathbb{G} and \mathbb{H} are independent, written $\mathbb{G} \perp \mathbb{H}$, if

$$P(A \cap B) = P(A)P(B) \quad \text{for all } A \in \mathbb{G}, B \in \mathbb{H}. \quad (3.2)$$

Lemma 3.5. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space and let $\mathbb{G}, \mathbb{H} \subset \mathbb{F}$ be two classes of measurable sets. Let $\mathbb{G}_1 \subset \mathbb{G}$ and $\mathbb{H}_1 \subset \mathbb{H}$ be two subclasses. If $\mathbb{G} \perp \mathbb{H}$ then it holds that $\mathbb{G}_1 \perp \mathbb{H}_1$.

Definition 3.6. (Hansen) A class \mathbb{H} of subsets of Ω is a **Dynkin class** if

1. $\Omega \in \mathbb{H}$,
2. $A, B \in \mathbb{H}, A \subset B \Rightarrow B \setminus A \in \mathbb{H}$,
3. $A_1, A_2, \dots \in \mathbb{H}, A_1 \subset A_2 \subset \dots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathbb{H}$.

Lemma 3.7. (Hansen) (Dynkin) Let $\mathbb{D} \subset \mathbb{H}_0 \subset \mathbb{H}$ be three nested classes of subsets of Ω . if

1. $\sigma(\mathbb{D}) = \mathbb{H}$,
2. $A, B \in \mathbb{D} \Rightarrow A \cap B \in \mathbb{D}$
3. \mathbb{H}_0 is a Dynkin class.

then it holds that $\mathbb{H}_0 = \mathbb{H}$.

Lemma 3.8. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space, and let $A \in \mathbb{F}$ be a fixed event. The class

$$\mathbb{H} = \{B \in \mathbb{F} \mid A \perp B\}$$

is a Dynkin class.

Theorem 3.9. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space, and let $\mathbb{G}_1, \mathbb{G}_2 \subset \mathbb{F}$ be two sigma-algebras. Let \mathbb{D}_1 and \mathbb{D}_2 be two classes such that $\sigma(\mathbb{D}_i) = \mathbb{G}_i$ for $i = 1, 2$. If both \mathbb{D}_1 and \mathbb{D}_2 are \cap -stable then it holds that

$$\mathbb{D}_1 \perp \mathbb{D}_2 \Rightarrow \mathbb{G}_1 \perp \mathbb{G}_2.$$

Definition 3.10. (Hansen) Two real-valued random variable X and Y on a background space (Ω, \mathbb{F}, P) are **independent**, written $X \perp Y$, if the corresponding sigma-algebras $\sigma(X)$ and $\sigma(Y)$ are independent.

Definition 3.15. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space, and let $\mathbb{G}_1, \dots, \mathbb{G}_n \subset \mathbb{F}$ be finitely many classes of measurable sets. We say that $\mathbb{G}_1, \dots, \mathbb{G}_n$ are **jointly independent**, written $\mathbb{G}_1 \perp \dots \perp \mathbb{G}_n$, if

$$P(A_1 \cap \dots \cap A_n) = \prod_{i=1}^n P(A_i) \quad \text{for } A_1 \in \mathbb{G}_1, \dots, A_n \in \mathbb{G}_n. \quad (3.8)$$

Lemma 3.16. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space, and let $\mathbb{G}_1, \dots, \mathbb{G}_n \subset \mathbb{F}$ be finitely many classes of measurable sets. It holds that

$$\mathbb{G}_1 \perp \dots \perp \mathbb{G}_n \Rightarrow \mathbb{G}_1 \perp \dots \perp \mathbb{G}_{n-1}$$

provided that $\Omega \in \mathbb{G}_n$.

Theorem 3.17. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space, and let $\mathbb{G}_1, \dots, \mathbb{G}_n \subset \mathbb{F}$ be sigma-algebras. Let $\mathbb{D}_1, \dots, \mathbb{D}_n$ be classes such that $\sigma(\mathbb{D}_i) = \mathbb{G}_i$ for $i = 1, \dots, n$. Suppose that for all lengths $k = 2, \dots, n$ an all choices of indices $j_1 < \dots < j_k \leq n$ it holds that

$$\mathbb{D}_{j_1} \perp \dots \perp \mathbb{D}_{j_k} \quad (3.9)$$

If all the generators \mathbb{D}_i are \cap -stable, then it holds that $\mathbb{G}_1 \perp \dots \perp \mathbb{G}_n$.

Lemma 3.18. (Hansen) (Grouping) Let (Ω, \mathbb{F}, P) be a probability space, and let $\mathbb{G}_1, \dots, \mathbb{G}_n \subset \mathbb{F}$ be sigma-algebras. It holds that

$$\mathbb{G}_1 \perp \dots \perp \mathbb{G}_n \quad \Rightarrow \quad \mathbb{G}_1 \perp \dots \perp \mathbb{G}_{n-2} \perp \sigma(\mathbb{G}_{n-1}, \mathbb{G}_n).$$

Definition 3.19. (Hansen) The real-valued random variables X_1, \dots, X_n on a background space (Ω, \mathbb{F}, P) are **jointly independent**, written $X_1 \perp \dots \perp X_n$, if the corresponding sigma-algebras $\sigma(X_1), \dots, \sigma(X_n)$ are jointly independent.

Definition 3.20. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space, and let $(\mathbb{G}_i)_{i \in I}$ be a family of classes of measurable sets. We say that the family $(\mathbb{G}_i)_{i \in I}$ is **jointly independent** if any finite subfamily is jointly independent.

Theorem 3.21. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space, and let $\mathbb{G}_1, \mathbb{G}_2, \dots \subset \mathbb{F}$ be sigma-algebras. Let $\mathbb{D}_1, \mathbb{D}_2, \dots$ be classes such that $\sigma(\mathbb{D}_n) = \mathbb{G}_n$ for all $n \in \mathbb{N}$. Suppose that for all lengths $k \in \mathbb{N}$ and all choices of indices $1 \leq j_1 < \dots < j_k$ it holds that

$$\mathbb{D}_{j_1} \perp \dots \perp \mathbb{D}_{j_k}. \quad (3.14)$$

If all the generators \mathbb{D}_n are \cap -stable, then it holds that $\mathbb{G}_1 \perp \mathbb{G}_2 \perp \dots$.

Lemma 3.22. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space, and let $\mathbb{G}_1, \mathbb{G}_2, \dots \subset \mathbb{F}$ be sigma-algebras. It holds that

$$\mathbb{G}_1 \perp \mathbb{G}_2 \perp \dots \quad \Rightarrow \quad \mathbb{G}_1 \perp \dots \perp \mathbb{G}_n \perp \sigma(\mathbb{G}_{n+1}, \mathbb{G}_{n+2}, \dots).$$

Definition 3.23. (Hansen) The real-valued random variables $(X_i)_{i \in I}$ on a background space (Ω, \mathbb{F}, P) are **jointly independent** if the corresponding sigma-algebras $(\sigma(X_i))_{i \in I}$ are jointly independent.

Definition 3.28. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space. A sigma-algebra $\mathbb{G} \subset \mathbb{F}$ satisfies a **zero-one law** if

$$P(A) \in \{0, 1\} \quad \text{for all } A \in \mathbb{G}.$$

Theorem 3.29. (Hansen) Let (Ω, \mathbb{F}, P) be a probability space and let $\mathbb{G} \subset \mathbb{F}$ be a sigma-algebra. The following three conditions are equivalent:

1. For any sigma-algebra $\mathbb{H} \subset \mathbb{F}$ it holds that $\mathbb{G} \perp \mathbb{H}$,
2. It holds that $\mathbb{G} \perp \mathbb{G}$,
3. \mathbb{G} satisfies a 0-1 law.

Definition 3.30. (Hansen) Let X_1, X_2, \dots be real-valued random variables on a background space (Ω, \mathbb{F}, P) . The **tail sigma-algebra** of the process is defined as

$$\mathbb{J}(X_1, X_2, \dots) = \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \dots).$$

Theorem 3.32. (Hansen) (Kolmogorov's zero-one law) Let X_1, X_2, \dots be real-valued random variables on a background space (Ω, \mathbb{F}, P) . If $X_1 \perp X_2 \perp \dots$ then the tail-algebra $\mathbb{J}(X_1, X_2, \dots)$ satisfies a 0-1 law.

Lemma 3.35. (Hansen) (2nd half of Borel-Cantelli) Let (Ω, \mathbb{F}, P) be a probability space, and let A_1, A_2, \dots be a sequence of \mathbb{F} -measurable sets. If $A_1 \perp A_2 \perp \dots$ then it holds that

$$\sum_{n=1}^{\infty} P(A_n) < \infty \iff P(A_n \text{ i.o.}) = 0.$$

Moment generating function

Let X be a random variable with distribution function $F(x) = P(X \leq x)$ and Y be a random variable with distribution function $G(y) = P(Y \leq y)$.

Definition. (Ex. FinKont) *The moment generating function or Laplace transform of X is*

$$\psi_X(\lambda) = E[e^{\lambda X}] = \int_{-\infty}^{\infty} e^{\lambda x} dF(x)$$

provided the expectation is finite for $|\lambda| < h$ for some $h > 0$.

The MGF uniquely determine the distribution of a random variable, due to the following result.

Theorem. (Ex. FinKont) (Uniqueness) *If $\psi_X(\lambda) = \psi_Y(\lambda)$ when $|\lambda| < h$ for some $h > 0$, then X and Y has the same distribution, that is, $F = G$.*

There is also the following result of independence for Moment generating functions.

Theorem. (Ex. FinKont) (Independence) *If*

$$E[e^{\lambda_1 X + \lambda_2 Y}] = \psi_X(\lambda_1)\psi_Y(\lambda_2)$$

for $|\lambda_i| < h$ for $i = 1, 2$ for some $h > 0$, then X and Y are independent random variables.

Standard distributions

Normal distribution

The following gives a comprehensive table of the standard some properties.

Description	Symbol	Normal distribution
Definition	\sim	$X \sim \mathcal{N}(\mu, \sigma^2)$
Parameters	$\theta \in \Theta$	$\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+$
Support	$\text{Im}(X)$	$x \in \mathbb{R}$
Density	f	$\frac{1}{\sqrt{2\pi}\sigma^2} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma^2}\right)^2}$
Distribution	F	$\frac{1}{2} \left(1 + N\left(\frac{x-\mu}{\sqrt{2}\sigma^2}\right)\right)$
Mean value	$E[X]$	μ
Variance	$\text{Var}(X)$	σ^2
MGF*	$\psi_X = E[e^{\lambda X}]$	$e^{\mu\lambda + \frac{1}{2}\lambda^2\sigma^2}$
Characteristic function	$\varphi_X(t) = E[e^{itX}]$	$e^{it\mu - \frac{1}{2}\sigma^2 t^2}$

In the table above we used the abbreviations: *MGF = Moment Generating function.

We also used the shorthand: N being the distribution of a standard normal distributed variable $\mathcal{N}(0, 1)$.

Continuous Time Stochastic Processes

Brownian Motion

Definition 4.1. (Bjork) A stochastic process W is called a **Brownian motion** or **Wiener process** if the following conditions hold

1. $W_0 = 0$.
2. The process W has independent increments, i.e. if $r < s \leq t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent random variables.
3. For $s < t$ the random variable $W_t - W_s$ has the Gaussian distribution $\mathcal{N}(0, t - s)$.
4. W has continuous trajectories i.e. $s \mapsto W(s; \omega)$ is continuous for all $\omega \in \Omega$.

As one can see from the simulated sample path on the right, the Brownian motion is rather erratic. In fact, the process varies infinitely on any interval with length greater than 0. This gives some of the characteristics of the process including that: W is continuous and W is non-differential everywhere. This erratic behaviour is summed up in the theorem.

Theorem 4.2. (Bjork) A Brownian motions trajectory $t \mapsto W_t$ is with probability one nowhere differential, and it has locally infinite total variation.

Filtration

Filtrations is widely used in stochastic processes, as they allow for the concept of knowledge/information. This is useful when considering mean-values of future states but in an increasing information setting. For this we introduce the term adapted processes.

Definition B.17. (Bjork) (Adapted process) *Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration on the probability space $(\mathcal{F}_t)_{t \geq 0}$. Furthermore, let $(X_t)_{t \geq 0}$ be a stochastic process on the same space. We say that X_t is adapted to the filtration \mathbf{F} if*

$$X_t \text{ is } \mathcal{F}_t - \text{measurable,} \quad \forall t \geq 0.$$

Obviously, we may introduce the **natural filtration** \mathcal{F}_t^X given by the trajectory of the process X_t :

$$\mathcal{F}_t^X = \sigma(\{X_s, s \leq t\}).$$

Indeed, X_t is adapted to this filtration.

Martingale

Definition. Let M_t be a stochastic process defined on a background space (Ω, \mathcal{F}, P) . Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration. If M_t is adapted to the filtration \mathcal{F}_t , $E|M_t| < \infty$ and

$$E[M_t | \mathcal{F}_s] = M_s, \quad P - \text{a.s.}$$

holds for any $t > s$ we say that M_t is a martingale (**F**-martingale). If the above has \leq or \geq we say that M_t is either a **submartingale** or **supermartingale** respectively.

Naturally, this definitions may easily be extended to discrete models and we have the trivial equality:

$$E[M_t - M_s \mid \mathcal{F}_s] = 0.$$

Martingales is useful, when proofing probalistic statements as the posses tractable properties. A useful technique often include the construction of the martingale

$$M_t = E[X \mid \mathcal{F}_t].$$

Stochastic calculus

Stochastic Integrals

We want to formulate financial markets in continuous time and the most elegant theory is obtained from processes that can be defined in terms of **stochastic differential equations** or in other words by their dynamics. We may call them **diffusion processes**, as they may be approximated by a stochastic difference equation:

$$X_{t+\Delta t} - X_t = \mu(t, X_t)\Delta t + \sigma(t, X_t)Z_t. \quad (4.1)$$

Above Z_t is a normally distributed random variable (a disturbance). In this formulation we say that S_t is driven by two forces: on one hand a locally deterministic velocity or drift $\mu(t, X_t)$ and on the other hand a Gaussian term amplified by the deterministic factor $\sigma(t, X_t)$.

Information We consider a primary process X_t and we introduce the notion of information generated by X_t in terms of the natural filtration. The idea can be summed up in the following definition.

Definition 4.3. (Bjork) The symbol $\mathcal{F}_t^X \subseteq \mathcal{F}$ denotes “the information generated by X_t on the interval $[0, t]$ ”, or alternatively “what has happened to X_t over the interval $[0, t]$ ”.

1. If, based upon observations of the trajectory $\{X_s; 0 \leq s \leq t\}$, it is possible to decide whether a given event A has occurred or not, then we write this as

$$A \in \mathcal{F}_t^X$$

or say that “ A is \mathcal{F}_t^X -measurable”.

2. If the value of a given random variable Z can be completely determined given observations of the trajectory $\{X_s; 0 \leq s \leq t\}$, then we also write

$$Z \in \mathcal{F}_t^X. \text{ (} Z \text{ is } \mathcal{F}_t^X\text{-measurable)}$$

3. If Y is a stochastic process such that we have

$$Y_t \in \mathcal{F}_t^X$$

for all $t \geq 0$ then we say that Y_t is **adapted** to the **filtration** $\{\mathcal{F}_t^X\}_{t \geq 0}$. For brevity of notation, we will sometimes write the filtration as $\{\mathcal{F}_t^X\}_{t \geq 0} = \mathbf{F}$.

Stochastic Integrals We will now formulate the theory of stochastic integrals, that is, processes written in terms of stochastic processes with stochastic integrator and/or stochastic integrant. We will consider some given standard Brownian motion W_t and another stochastic process X_t . We need some integrability condition on X_t in order to do the calculations. We therefore determine a selection of suitable stochastic processes X must be contained in.

Definition 4.4. (Bjork) Let X_t be a stochastic process, then

- i. We say that X_t belongs to the class $\mathcal{L}^2[a, b]$ if X_t is adapted to the filtration \mathcal{F}_t^X and the following holds

$$\int_a^b E[X_s^2] ds < \infty$$

- ii. We say that X_t belongs to the class \mathcal{L}^2 if $X_t \in \mathcal{L}^2[0, t]$ for all $t > 0$.

We now want to define what we mean by

$$\int_a^b X_t dW_s$$

for some process $X_t \in \mathcal{L}^2$. We see that a way to go about this problem is to start by defining the concept for a simple stochastic process X_t . By *simple* we mean a process X_t that is constant on between some deterministic points in time $a = t_0 < t_1 < \dots < t_n = b$. In that case we may define the integral as

$$\int_a^b X_s dW_s = \sum_{k=0}^{n-1} X_{t_k} [W_{t_{k+1}} - W_{t_k}]. \quad (4.8)$$

In the more general setting we may follow the following approach:

1. Approximate X with a sequence $\{X^n\}_{n \in \mathbb{N}}$ of simple processes such that the following convergence criteria hold

$$\int_a^b E[(X_s^n - X_s)^2] ds \rightarrow 0, \quad n \rightarrow \infty$$

2. For each n the integral $\int_a^b X_s^n dW_s := Z^n$ is well defined and it is possible to prove, using DCT, that a variable Z exists such that $Z^n \rightarrow Z$ that is in L^2 .
3. We now define the stochastic integral by the limit

$$\int_a^b X_s dW_s = \lim_{n \rightarrow \infty} \int_a^b X_s^n dW_s. \quad (4.9)$$

Obviously the hardest step is finding the processes X^n . This stochastic has some properties we will use.

Proposition 4.5. (Bjork) Let $X_t \in \mathcal{L}^2$, then

$$E \left[\int_a^b X_s dW_s \right] = 0. \quad (4.12)$$

$$E \left[\left(\int_a^b X_s dW_s \right)^2 \right] = \int_a^b E[X_s^2] dW_s. \quad (4.13)$$

$$\int_a^b X_s dW_s \text{ is } \mathcal{F}_b^W\text{-measurable.} \quad (4.14)$$

Martingales Definition 4.7. (Bjork) Let M_t be a stochastic process defined on a background space (Ω, \mathcal{F}, P) . Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration. If M_t is adapted to the filtration \mathcal{F}_t , $E|M_t| < \infty$ and

$$E[M_t | \mathcal{F}_s] = M_s, \quad P - \text{a.s.}$$

holds for any $t > s$ we say that M_t is a martingale (**F**-martingale). If the above has \geq or \leq we say that M_t is either a **submartingale** or **supermartingale** respectively.

Proposition 4.8. (Bjork) For any process $X_t \in \mathcal{L}^2[s, t]$ the following hold:

$$E \left[\int_s^t X_s dW_s \middle| \mathcal{F}_s^W \right] = 0$$

Corollary 4.9. (Bjork) For any process $X_t \in \mathcal{L}^2$ the process

$$M_t = \int_s^t X_s dW_s,$$

is an (\mathcal{F}_t^W) -martingale. In other words, modulo an integrability condition, **every stochastic integral is a martingale**.

Lemma 4.10. (Bjork) Let M_t be a stochastic process with stochastic differential, then M_t is a martingale if and only if the stochastic differential has the form $dM_t = X_t dW_t$ i.e. M_t as no dt -term.

Stochastic Calculus and the Ito Formula Given this brief introduction to stochastic integrals we may formulate some simple calculus revolving around Ito's formula. We consider the stochastic process X_t and we suppose that there exist a real number X_0 and adapted processes μ and σ wrt. \mathcal{F}_t^W such that for all $t \geq 0$ we have

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \quad (4.16)$$

where W_t is a standard Brownian motion. We know from earlier courses that the above may be written in terms of the dynamics (pure notation):

$$\begin{cases} dX_t = \mu_t dt + \sigma_t dW_t, \\ X_0 = X_0. \end{cases} \quad (4.17/18)$$

Here we interpret the above as X_t has boundary condition X_0 and evolves with a drift $\mu_t dt$ amplified and distorted by the drift $\sigma_t dW_t$. We say that X_t has **stochastic differential** dX_t and initial condition X_0 .

We want to understand how transformation of such an integral behaves and therefore we introduce some calculus which will tell how for instance $f(t, X_t)$ (for some $C^{1,2}$ -function) behaves. This insight is given by the important Ito's formula.

Theorem 4.11. (Bjork) (Ito's formula, one-dimensional) Assume that the process X has a stochastic differential form given by

$$dX_t = \mu_t dt + \sigma_t dW_t, \quad (4.28)$$

where μ and σ are adapted processes, and let $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a $C^{1,2}$ -function. Define the process Z by $Z_t = f(t, X_t)$. Then Z has stochastic differential given by

$$df(t, X_t) = \left(\frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right) dt + \sigma_t \frac{\partial f}{\partial x}(t, X_t) dW_t. \quad (4.29)$$

Proposition 4.12. (Bjork) (Ito's formula, one-dimensional) With assumptions as in theorem 4.11, df is given by

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX_t)^2, \quad (4.31)$$

where we use the following table

$$\begin{cases} (dt)^2 = 0, \\ dt \cdot dW_t = 0, \\ (dW_t)^2 = dt. \end{cases}$$

Lemma 4.18. (Bjork) Let $\sigma(t)$ be deterministic function of time and define the process X by

$$X_t = \int_0^t \sigma(s) dW_s. \quad (4.37)$$

Then

$$X_t \sim \mathcal{N}\left(0, \int_0^t \sigma^2(s) ds\right).$$

The multidimensional Ito Formula Consider a vector process $X = (X^1, \dots, X^n)^\top$ where each component X^i has stochastic differential

$$dX_t^i = \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j$$

where W^1, \dots, W^d is independent Brownian motions. Then we have respectively the drift vector process μ_t in n dimensions, the vector Brownian motion in d dimensions and a $n \times d$ -dimensional **diffusion matrix** σ_t given as below

$$\mu_t = \begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^n \end{bmatrix}, \quad W_t = \begin{bmatrix} W_t^1 \\ \vdots \\ W_t^d \end{bmatrix}, \quad \sigma_t = \begin{bmatrix} \sigma_t^{11} & \dots & \sigma_t^{1d} \\ \vdots & \ddots & \vdots \\ \sigma_t^{n1} & \dots & \sigma_t^{nd} \end{bmatrix}.$$

Given this we may write the dynamics of X as

$$dX_t = \mu_t dt + \sigma_t dW_t \in \mathbb{R}^n.$$

Consider now a function $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ which is a $C^{1,2}$ -mapping. We want to study the dynamics of the process

$$Z_t = f(t, X_t).$$

The dynamics is given in the multidimensional version of Ito's formula.

Theorem 4.19. (Bjork) (Ito's formula, multi-dimensional) *Let X be given as above. Then the following holds:*

- The process $f(t, X_t)$ has the stochastic differential given by

$$df(t, X_t) = \left(\frac{\partial f}{\partial t}(t, X_t) + \sum_{i=1}^n \mu_t^i \frac{\partial f}{\partial x^i}(t, X_t) + \frac{1}{2} \sum_{i,j=1}^n C_t^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}(t, X_t) \right) dt + \sum_{i=1}^n \sigma_t^i \frac{\partial f}{\partial x^i}(t, X_t) dW_t.$$

Here the row vector σ_t^i is the i 'th row of the matrix σ_t and the matrix C is defined by $C = \sigma \sigma^\top$.

- Alternatively, the differential is given by the formula

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(t, X_t) dX_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j}(t, X_t) dX_t^i dX_t^j,$$

with the formal multiplication table

$$\begin{cases} (dt)^2 = 0, \\ dt \cdot dW_t^i = 0, & i = 1, \dots, d, \\ (dW_t^i)^2 = dt, & i = 1, \dots, d, \\ dW_t^i \cdot dW_t^j = 0, & i \neq j. \end{cases}$$

Obviously, one can write the differential in Ito's formula in many other ways including a matrix-wise version using the Hessian matrix $H_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$.

Correlated Brownian motions In the previous section the d -dimensional Brownian was assumed to have independent Brownian motions. However we may instead consider a variation where we have some dependence between the Brownian motions.

This section has not been finished.

Discrete Stochastic Integrals

This section has not been finished.

Stochastic Differential Equations

We start the chapter by formalising some used objects. We consider the following objects.

- $M(n, d)$ denotes the class of $n \times d$ -matrices.
- W is a d -dimensional Brownian motion
- μ is a \mathbb{R}^n -valued function with arguments (t, X_t) with X_t being a n -dimensional stochastic process.
- σ a $M(n, d)$ -valued function with arguments as in μ .
- x_0 a \mathbb{R}^n -valued vector.

We want then to understand when the following has a solution

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t, \quad X_0 = x_0. \quad (5.1/2)$$

We call such an equation the **stochastic differential equation** or simply SDE. We know that the above is loosely notation for the integral form as

$$X_t = x_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad (5.3)$$

for all $t \geq 0$. The following proposition tells us when an solution exist to the problem above. In the below $\|\cdot\|$ is usual euclidian norm

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2}.$$

Proposition 5.1. (Bjork) *Suppose that there existis a constant K such that the following conditions are satisfied for all x, y and t .*

$$\|\mu(t, x) - \mu(t, y)\| \leq K\|x - y\|, \quad (5.6)$$

$$\|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|, \quad (5.7)$$

$$\|\mu(t, x)\| + \|\sigma(t, x)\| \leq K(1 + \|x\|). \quad (5.8)$$

Then there exists a unique solution to the SDE above. Furthermore, the solution has the properties

1. X is \mathcal{F}_t^W -adapted.
2. X has continuous trajectories.
3. X is a Markov process.
4. There exists a constant C such that

$$E[\|X_t\|^2] \leq Ce^{Ct}(1 + \|x_0\|^2). \quad (5.9)$$

In genereal the solution to an SDE is so complicated, that it in practical terms is unsolvable and may only be approximated on a finely subdividet grid as jumps. There does however exist som nontrivial cases where we may infer a analytical solution. One is the rather important **Geometric Brownian motion**.

Proposition 5.2. (Bjork) *Consider the SDE*

$$dX_t = \alpha X_t dt + \sigma X_t dW_t, \quad (5.13)$$

with $X_0 = x_0$. Then the solution is given as

$$X_t = x_0 \cdot \exp \left\{ \left(\alpha - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}. \quad (5.15)$$

The expected value of X is given as $E[X_t] = x_0 e^{\alpha t}$ (eq. 5.16).

One other generalisation that is analytically solvable is the Linear SDE.

Proposition 5.3. (Bjork) Consider the SDE

$$dX_t = (AX_t + b_t) dt + \sigma_t dW_t, \quad (5.19)$$

with $X_0 = x_0$ and $A \in M(n, n)$ and b_t being a real-valued function. Then the solution is given as

$$X_t = e^{At} x_0 + \int_0^t e^{A(t-s)} b_s ds + \int_0^t e^{A(t-s)} \sigma_s dW_s. \quad (5.20)$$

Where we define the exponential of a matrix as below

$$e^{At} = \sum_{k=0}^{\infty} A^k \frac{1}{k!} t^k.$$

In general with the SDE we have a partial differential operator \mathcal{A} called the **infinitesimal operator** of X which has some interesting analytical properties regarding X .

Definition 5.4. (Bjork) Consider the SDE

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t. \quad (5.21)$$

The partial differential operator \mathcal{A} is defined, for any function $h \in C^2(\mathbb{R}^n)$, by

$$\mathcal{A}h(t, x) = \sum_{i=1}^n \mu_i(t, x) \frac{\partial h}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^n (\sigma(t, x) \sigma(t, x)^\top)_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j}(x).$$

We see that in terms of Ito's formula the operator is included as such

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t}(t, X_t) + \mathcal{A}f(t, x) \right\} dt + [\nabla_x f](t, X_t) \sigma(t, X_t) dW_t,$$

where ∇_x is the gradient for function $h \in C^1(\mathbb{R}^n)$ as

$$\nabla_x h(x) = \left[\frac{\partial h}{\partial x_1}(x), \dots, \frac{\partial h}{\partial x_n}(x) \right].$$

Partial differential equations

Proposition 5.5. (Bjork) (Feynmann-Kac) Assume that F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t, x) + \mu(t, x) \frac{\partial F}{\partial x}(x, t) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x) = 0,$$

with boundary condition $F(T, x) = \Phi(x)$. Assume furthermore that the process

$$\sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s) \in \mathcal{L}^2$$

as per definition 4.4, where X is defined below. Then F has the representation

$$F(t, x) = E_{t,x}[\Phi(X_T)] = E[\Phi(X_T) \mid X_t = x], \quad (5.29)$$

where X satisfies the SDE

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s, \quad (5.30)$$

with boundary condition $X_t = x$.

Proposition 5.6. (Bjork) (Feynmann-Kac) Assume that F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t, x) + \mu(t, x) \frac{\partial F}{\partial x}(x, t) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) = 0, \quad (5.34)$$

with boundary condition $F(T, x) = \Phi(x)$. Assume furthermore that the process

$$e^{-rs} \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s) \in \mathcal{L}^2$$

as per definition 4.4, where X is defined below. Then F has the representation

$$F(t, x) = e^{-r(T-t)} E_{t,x}[\Phi(X_T)] = e^{-r(T-t)} E[\Phi(X_T) \mid X_t = x], \quad (5.36)$$

where X satisfies the SDE

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s, \quad (5.37)$$

with boundary condition $X_t = x$.

Proposition 5.8. (Bjork) (Feynmann-Kac) Assume that F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t, x) + \sum_{i=1}^n \mu_i(t, x) \frac{\partial F}{\partial x}(x, t) + \frac{1}{2} \sum_{i,j=1}^n C_{ij}(t, x) \frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) = 0,$$

with boundary condition $F(T, x) = \Phi(x)$ and $C_{ij} = \sigma \sigma^\top$. Assume furthermore that the process

$$e^{-rs} \sum_{i=1}^n \sigma_i(s, X_s) \frac{\partial F}{\partial x}(s, X_s) \in \mathcal{L}^2$$

as per definition 4.4, where X is defined below. Then F has the representation

$$F(t, x) = e^{-r(T-t)} E_{t,x}[\Phi(X_T)], \quad (5.39)$$

where X satisfies the SDE

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s, \quad (5.40)$$

with boundary condition $X_t = x$.

Proposition 5.9. (Bjork) Consider as given a vector process X with generator \mathcal{A} , and a function $F(t, x)$. Then, modulo some integrability condition, the following hold:

- The process $F(t, X_t)$ is a martingale relative to the filtration \mathcal{F}^X if and only if F satisfies the PDE

$$\frac{\partial F}{\partial t} + \mathcal{A}F = 0.$$

- The process $F(t, X_t)$ is a martingale relative to the filtration \mathcal{F}^X if and only if, for every (t, x) and $T \geq t$, we have

$$F(t, x) = E_{t,x}[F(T, X_T)].$$

Discrete Time Stochastic Processes

Convergence concepts

We start this chapter by referring to a sequence X_1, X_2, \dots of real-valued random variables as a **proces**. Consider the event $(X_n \rightarrow X) = \{\omega \in \Omega \mid X_m(\omega) \rightarrow X(\omega) \text{ for } n \rightarrow \infty\}$. We want to study such convergence in detail. However first we check measurability. Consider a family $(A_i)_{i \in I} \subset \Omega$ and observe that

$$\left\{ \omega \in \Omega \mid \forall i \in I : \omega \in A_i \right\} = \bigcap_{i \in I} A_i, \quad (2.1)$$

$$\left\{ \omega \in \Omega \mid \exists i \in I : \omega \in A_i \right\} = \bigcup_{i \in I} A_i. \quad (2.2)$$

From the standard N, ε definition of a convergent sequence $(x_n)_{n \in \mathbb{N}}$ we may formulate this convergens in the stochastic setting:

$$\begin{aligned} (X_n \rightarrow X) &= \left\{ \omega \in \Omega \mid \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : |X_n(\omega) - X(\omega)| < \varepsilon \right\} \\ &= \left(\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : |X_n - X| < \varepsilon \right) \\ &= \bigcap_{\varepsilon \in \mathbb{R}^+} \left(\exists N \in \mathbb{N} \forall n \geq N : |X_n - X| < \varepsilon \right) \\ &= \bigcap_{\varepsilon \in \mathbb{R}^+} \bigcup_{N=1}^{\infty} \left(\forall n \geq N : |X_n - X| < \varepsilon \right) \\ &= \bigcap_{\varepsilon \in \mathbb{R}^+} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left(|X_n - X| < \varepsilon \right) \in \mathbb{F} \quad \text{for all } \varepsilon > 0. \end{aligned}$$

Hence $(X_n \rightarrow X)$ lies in \mathbb{F} since $(|X_n - X| < \varepsilon)$ lies in \mathbb{F} since $X_n - X$ is measurable.

Lemma 2.1. (Hansen) *Let X, X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . It holds that*

$$(X_n \rightarrow X) \in \mathbb{F}.$$

Definition 2.2. (Hansen) *Let X, X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . We say that X_n converges to X almost surely, written $X_n \xrightarrow{a.s.} X$, if*

$$P(X_n \rightarrow X) = 1. \quad (2.6)$$

Lemma 2.3. (Hansen) *Let X, X', X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . If $X_n \xrightarrow{a.s.} X$ and $X_n \xrightarrow{a.s.} X'$ then $X = X'$ almost surely.*

Lemma 2.7. (Hansen) *Let X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . Then*

$$\left((X_n) \text{ is Cauchy} \right) \in \mathbb{F}.$$

Lemma 2.8. (Hansen) *Let X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . If $P\left((X_n) \text{ is Cauchy}\right) = 1$ then there exists and \mathbb{F} -measurable real-valued random variable X such that $X_n \xrightarrow{a.s.} X$.*

Theorem 2.10. (Hansen) *Let X_1, X_2, \dots and Y_1, Y_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . Assume that the X -process and the Y -process have the same distribution in the sense that (X_1, \dots, X_n) has the same distribution as (Y_1, \dots, Y_n) for all $n \in \mathbb{N}$. If $X_n \xrightarrow{a.s.} X$ for some limit variable X , there is a limit variable Y such that $Y_n \xrightarrow{a.s.} Y$.*

Definition 2.11. (Hansen) Let $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \dots$ be \mathbb{R}^k valued random variables on (Ω, \mathbb{F}, P) . We say that \mathbf{X}_n converges to \mathbf{X} **almost surely**, written $\mathbf{X}_n \xrightarrow{\text{a.s.}} \mathbf{X}$, if

$$|\mathbf{X}_n - \mathbf{X}| \xrightarrow{\text{a.s.}} 0. \quad (2.15)$$

Lemma 2.12. (Hansen) Let $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \dots$ be \mathbb{R}^k valued random variables on (Ω, \mathbb{F}, P) such that $\mathbf{X}_n \xrightarrow{\text{a.s.}} \mathbf{X}$. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$ be a measurable map. Assume that there is a set $A \in \mathbb{B}_k$ such that f is continuous on A and such that $P(\mathbf{X} \in A) = 1$. Then it holds that $f(\mathbf{X}_n) \xrightarrow{\text{a.s.}} f(\mathbf{X})$.

Definition 2.13. (Hansen) Let X, X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . We say that X_n converges to X **in probability**, written $X_n \xrightarrow{P} X$, if

$$\forall \varepsilon > 0 : \quad P(|X_n - X| \geq \varepsilon) \rightarrow 0 \quad \text{for } n \rightarrow \infty. \quad (2.17)$$

Lemma 2.14. (Hansen) Let X, X', X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . If $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} X'$ then $X = X'$ almost surely.

Lemma 2.14. (Hansen) Let X, X', X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . If $X_n \xrightarrow{\text{a.s.}} X$, then $X_n \xrightarrow{P} X$.

Definition 2.17. (Hansen) Let $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \dots$ be \mathbb{R}^k valued random variables on (Ω, \mathbb{F}, P) . We say that \mathbf{X}_n converges to \mathbf{X} **in probability**, written $\mathbf{X}_n \xrightarrow{P} \mathbf{X}$, if

$$|\mathbf{X}_n - \mathbf{X}| \xrightarrow{P} 0. \quad (2.23)$$

Lemma 2.18. (Hansen) Let $X, Y, X_1, Y_1, X_2, Y_2, \dots$ be real-valued random variables on (Ω, \mathbb{F}, P) . It holds that

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{P} \begin{pmatrix} X \\ Y \end{pmatrix} \iff X_n \xrightarrow{P} X \text{ and } Y_n \xrightarrow{P} Y. \quad (2.24)$$

Definition 2.19. (Hansen) Let X, X_1, X_2, \dots be real-valued random variables in $\mathcal{L}^p(\Omega, \mathbb{F}, P)$ for some $p \geq 1$. We say that X_n **converges** to X **in \mathcal{L}^p** , written $X_n \xrightarrow{\mathcal{L}^p} X$, if

$$\|X_n - X\|_p \rightarrow 0. \quad (2.27)$$

Where the p 'th norm is defined as the mapping $\|\cdot\|_p : \Omega \rightarrow [0, \infty)$ given by $X \mapsto (\int_{\Omega} |X|^p dP)^{1/p}$.

One might also define convergence in \mathcal{L}^p by simply saying if $X_n \xrightarrow{\mathcal{L}^p} X$ then $E\|X_n - X\|_p \rightarrow 0$.

Lemma 2.20. (Hansen) (Extended Cauchy-Schwarz inequality) Let $X, Y \in \mathcal{L}^p(\Omega, \mathbb{F}, P)$ for some $p \geq 1$. For any $a \in [0, p]$ it holds that

$$E|X|^a|Y|^{p-a} \leq \left(E|X|^p\right)^{\frac{a}{p}} \left(E|Y|^p\right)^{\frac{p-a}{p}}. \quad (2.29)$$

Theorem 2.21. (Hansen) Let X, X_1, X_2, \dots be real-valued random variables in $\mathcal{L}^p(\Omega, \mathbb{F}, P)$ for some $p \in \mathbb{N}$. If $X_n \xrightarrow{\mathcal{L}^p} X$, then it holds that $E X_n^p \rightarrow E X^p$.

Lemma 2.22. (Hansen) Let X, X_1, X_2, \dots be real-valued random variables in $\mathcal{L}^p(\Omega, \mathbb{F}, P)$ for some $p \geq 1$. If $X_n \xrightarrow{\mathcal{L}^p} X$, then $X_n \xrightarrow{P} X$.

Lemma 2.25. (Hansen) (Borel-Cantelli) Let (Ω, \mathbb{F}, P) be a probability space, and let A_1, A_2, \dots be a sequence of \mathbb{F} -measurable sets. It holds that

$$\sum_{n=1}^{\infty} P(A_n) < \infty \implies P(A_n \text{ i.o.}) = 0.$$

Let A_1, A_2, \dots be a sequence of subsets of Ω . We define

$$(A_n \text{ i.o.}) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m, \quad (A_n \text{ evt.}) = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m.$$

One might also define $Y = \sum_{n=1}^{\infty} 1_{A_n}$ and realise that $(A_n \text{ i.o.}) = (Y = \infty)$ and $(A_n \text{ evt.}) = (Y < \infty)$. Also by de Morgan's law it follows that $(A_n \text{ evt.})^c = (A_n^c \text{ i.o.})$.

Theorem 2.26. (Hansen) Let X, X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . If

$$\forall \varepsilon > 0 : \sum_{n=1}^{\infty} P(|X_n - X| \geq \varepsilon) < \infty, \quad (2.32)$$

then it holds that $X_n \xrightarrow{a.s.} X$.

Theorem 2.27. (Hansen) Let X, X_1, X_2, \dots be real-valued random variables on (Ω, \mathbb{F}, P) . If $X_n \xrightarrow{P} X$, then there is a subsequence X_{n_1}, X_{n_2}, \dots such that $X_{n_k} \xrightarrow{a.s.} X$ for $k \rightarrow \infty$.

Lemma 2.28. (Hansen) Let $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \dots$ be \mathbb{R}^k -valued random variables on (Ω, \mathbb{F}, P) such that $\mathbf{X}_n \xrightarrow{P} \mathbf{X}$. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$ be a measurable map. Assume that there is a set $A \in \mathbb{B}_k$ such that f is continuous on A and such that $P(\mathbf{X} \in A) = 1$. Then it holds that $f(\mathbf{X}_n) \xrightarrow{P} f(\mathbf{X})$.

Lemma. (Fatou's Lemma) Let (Ω, \mathbb{F}, P) be a measure space (here probability space). Let $f_n : \mathcal{X} \rightarrow [0, \infty]$, with $\mathcal{X} \in \mathbb{F}$, be a sequence of non-negative measurable functions. Assume f_n converge pointwise to $f : \mathcal{X} \rightarrow [0, \infty)$. Then

$$\int_{\mathcal{X}} \liminf_{n \rightarrow \infty} f_n \, dP \leq \liminf_{n \rightarrow \infty} \int_{\mathcal{X}} f_n \, dP.$$

Lemma. (Holder's Inequality) Let (Ω, \mathbb{F}, P) be a measure space (here probability space). Let f and g be real-valued (or complex-valued) functions defined on Ω . Assume f and g are measurable. For any $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ it holds that

$$\left(\int_{\Omega} |fg| \, dP \right)^1 \leq \left(\int_{\Omega} |f|^p \, dP \right)^{1/p} \left(\int_{\Omega} |g|^q \, dP \right)^{1/q}$$

Sums and average processes

Lemma 4.1. (Hansen) Let X_1, \dots, X_n be independent real-valued random variables with $E X_i^4 < \infty$ for all i . If $E X_i = 0$ for all i then it holds that

$$E \left(\sum_{i=1}^n X_i \right)^4 = \sum_{i=1}^n E X_i^4 + 6 \sum_{i=1}^{n-1} \sum_{j=i+1}^n E X_i^2 E X_j^2.$$

Theorem 4.2. (Hansen) (SLLN, weak form) Let X_1, X_2, \dots be a sequence of independent and identically distributed real-valued random variables. If $E X_1^4 < \infty$ it holds that

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\text{a.s.}} E X_1. \quad (4.3)$$

Theorem 4.10. (Hansen) (Etemadi's maximal inequality) Let X_1, \dots, X_n be independent real-valued random variables. Consider the cumulative sums

$$S_k = \sum_{i=1}^k X_i \quad \text{for } k = 1, \dots, n.$$

For any $\alpha > 0$ it holds that

$$P \left(\max_{j=1, \dots, n} |S_j| \geq 3\alpha \right) \leq 3 \max_{j=1, \dots, n} P(|S_j| \geq \alpha). \quad (4.11)$$

Theorem 4.11. (Hansen) (Levy's maximal inequality) Let X_1, \dots, X_n be independent real-valued random variables, each with a symmetric distribution. Consider the cumulative sums

$$S_k = \sum_{i=1}^k X_i \quad \text{for } k = 1, \dots, n.$$

For any $\alpha > 0$ it holds that

$$P \left(\max_{j=1, \dots, n} S_j \geq \alpha \right) \leq 2P(S_j \geq \alpha). \quad (4.13)$$

Corollary 4.12. (Hansen) Let X_1, \dots, X_n be independent real-valued random variables, each with a symmetric distribution. Consider the cumulative sums

$$S_k = \sum_{i=1}^k X_i \quad \text{for } k = 1, \dots, n.$$

For any $\alpha > 0$ it holds that

$$P \left(\max_{j=1, \dots, n} |S_j| \geq \alpha \right) \leq 2P(|S_j| \geq \alpha). \quad (4.14)$$

Theorem 4.13. (Hansen) (Skorokhod) Let X_1, X_2, \dots be a sequence of independent real-valued random variables, and consider the cumulative sums $S_k = \sum_{i=1}^k X_i$. Let S be a potential limit variable. It holds that

$$S_n \xrightarrow{P} S \quad \Rightarrow \quad S_n \xrightarrow{\text{a.s.}} S.$$

Corollary 4.14. (Hansen) (*Khintchine-Kolmogorov*) Let X_1, X_2, \dots be a sequence of independent real-valued random variables. Assume $E X_n^2 < \infty$ and that $E X_n = 0$ for every $n \in \mathbb{N}$. Consider the cumulative sums $S_k = \sum_{i=1}^k X_i$. If

$$\sum_{n=1}^{\infty} E X_n^2 < \infty \quad (4.18)$$

then there exist a limit variable S such that $S_n \rightarrow S$ almost surely and in \mathcal{L}^2 . The limit variable satisfies that

$$E S = 0 \quad \text{and} \quad V S = \sum_{n=1}^{\infty} V X_n.$$

Theorem 4.17. (Hansen) Let X_1, X_2, \dots be a sequence of independent real-valued random variables, and consider the cumulative sums $S_k = \sum_{i=1}^k X_i$. Let S be a potential limit variable. Assume that there is a constant $c > 0$ such that $P(|X_n| \leq c) = 1$ for all n , and assume that $E X_n = 0$ for all n . The three statements 1. $S_n \xrightarrow{P} S$, 2. $S_n \xrightarrow{\text{a.s.}} S$ 3. $S_n \xrightarrow{\mathcal{L}^2} S$ are equivalent.

Lemma 4.18. (Hansen) Let X_1, X_2, \dots be a sequence of independent real-valued random variables. Assume that there is a constant $c > 0$ such that $P(|X_n| \leq c) = 1$ for all n . If the associated random walk $S_n = \sum_{i=1}^n X_i$ satisfies that $S_n \rightarrow S$ almost surely for some limit variable then it holds that

$$1) \quad \sum_{n=1}^N E X_n \text{ converges in } \mathbb{R} \text{ for } N \rightarrow \infty, \quad (1)$$

$$2) \quad \sum_{n=1}^{\infty} V(X_n) < \infty. \quad (2)$$

Theorem 4.19. (Hansen) (*Kolmogorov's 3-series theorem*) Let X_1, X_2, \dots be a sequence of independent real-valued random variables. Consider the associated random walk $S_n = \sum_{i=1}^n X_i$. If there is a cut-off value $c > 0$ such that the capped variables $\tilde{X}_n = 1_{|X_n| \leq c} X_n$ satisfies that

$$\begin{aligned} 1) \quad & \sum_{n=1}^{\infty} P(X_n \neq \tilde{X}_n) < \infty, \\ 2) \quad & \sum_{n=1}^N E \tilde{X}_n \text{ converges in } \mathbb{R} \text{ for } N \rightarrow \infty, \\ 3) \quad & \sum_{n=1}^{\infty} V(\tilde{X}_n) < \infty, \end{aligned}$$

then there is a real-valued limit variable S such that $S_n \rightarrow S$ almost surely. Conversely, if $(S_n)_{n \in \mathbb{N}}$ is almost surely convergent, then the three series above converge for **any** cut-off value $c > 0$.

Lemma 4.20. (Hansen) Let $(x_n)_{n \in \mathbb{N}}$ be a real-valued sequence, and let c be a real number. It holds that

$$x_n \rightarrow c \text{ for } n \rightarrow \infty \quad \Rightarrow \quad \frac{1}{n} \sum_{i=1}^n x_i \rightarrow c \text{ for } n \rightarrow \infty.$$

Lemma 4.21. (Hansen) (*Kronecker*) Let $(x_n)_{n \in \mathbb{N}}$ be real-valued sequence, and let c be a real number. It holds that

$$\sum_{i=1}^n \frac{x_i}{i} \rightarrow c \quad \Rightarrow \quad \frac{1}{n} \sum_{i=1}^n x_i \rightarrow 0$$

for $n \rightarrow \infty$.

Lemma 4.23. (Hansen) Let X_1, X_2, \dots be a sequence of identically distributed real-valued random variables, and let $\tilde{X}_n = 1_{(|X_n| \leq n)} X_n$. If $E|X_1| < \infty$ it holds that

$$\sum_{n=1}^{\infty} \frac{E \tilde{X}_n^2}{n^2} < \infty.$$

Theorem 4.24. (Hansen) (SLLN, strong version) Let X_1, X_2, \dots be sequence of independent and identically distributed real-valued random variables. If $E|X_1| < \infty$ it holds that

$$\frac{1}{n} \sum_{i=1}^{\infty} X_i \xrightarrow{\text{a.s.}} E X_1. \quad (4.24)$$

Theorem 4.25. (Hansen) (SLLN, \mathcal{L}^p -version) Let X_1, X_2, \dots be sequence of independent and identically distributed real-valued random variables. If $E|X_1|^p < \infty$ for some $p \geq 1$, then it holds that

$$\frac{1}{n} \sum_{i=1}^{\infty} X_i \xrightarrow{\mathcal{L}^p} E X_1. \quad (4.26)$$

Lemma 4.26. (Hansen) Let X_1, X_2, \dots be a sequence of pairwise independent, identically distributed real-valued random variables with $E|X_1| < \infty$. Let $n_1 < n_2 < \dots$ be a sequence of natural numbers. If there are constants $C_1, C_2 > 0$ and $\alpha > 1$ such that

$$C_1 \alpha^k \leq n_k \leq C_2 \alpha^k \quad \text{for } k \rightarrow \infty \quad (4.27)$$

then it holds that

$$\frac{1}{n_k} \sum_{i=1}^{n_k} X_i \xrightarrow{\text{a.s.}} E X_1 \quad \text{for } k \rightarrow \infty.$$

Theorem 4.27. (Hansen) (Etemahdi's version) Let X_1, X_2, \dots be a sequence of pairwise independent, identically distributed real-valued random variables. If $E|X_1| < \infty$ it holds that

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\text{a.s.}} E X_1. \quad (4.30)$$

Ergodic Theory

Definition 5.3. (Hansen) Let $(\mathcal{X}, \mathbb{E})$ be a measurable space and let $T : \mathcal{X} \rightarrow \mathcal{X}$ be measurable. A probability measure μ on $(\mathcal{X}, \mathbb{E})$ is **invariant** under T if

$$\mu(T^{-1}(A)) = \mu(A) \quad \text{for all } A \in \mathbb{E} \quad (5.5)$$

In this case we call the quadruple $(\mathcal{X}, \mathbb{E}, \mu, T)$ a **measure-preserving dynamical system**.

We say that a set $A \in \mathbb{E}$ is an **invariant set** if $T^{-1}(A) = A$ i.e. the orbit of all $x \in A$ stays in A .

Definition 5.5. (Hansen) A measure-preserving dynamical system $(\mathcal{X}, \mathbb{E}, \mu, T)$ is **ergodic** if

$$T^{-1}(A) = A, \quad A \in \mathbb{E} \quad \Rightarrow \quad \mu(A) \in \{0, 1\}. \quad (5.7)$$

Definition 5.6. (Hansen) A measure-preserving dynamical system $(\mathcal{X}, \mathbb{E}, \mu, T)$ is **mixing** if

$$\mu(A \cap T^{-n}(B)) \rightarrow \mu(A)\mu(B) \quad \text{for all } A, B \in \mathbb{E}. \quad (5.8)$$

Lemma 5.7. (Hansen) *If a measure-preserving dynamical system $(\mathcal{X}, \mathbb{E}, \mu, T)$ is mixing then it is also ergodic.*

Lemma 5.8. (Hansen) *Let $(\mathcal{X}, \mathbb{E}, \mu, T)$ be a measure-preserving dynamical system. Let \mathbb{D} be an \cap -stable generator for \mathbb{E} . If*

$$\mu(A \cap T^{-n}(B)) \rightarrow \mu(A)\mu(B) \quad \text{for all } A, B \in \mathbb{D}. \quad (5.10)$$

then the system is mixing. (and ergodic)

Lemma 5.9. (Hansen) *Let $(\mathcal{X}, \mathbb{E}, \mu)$ be a probability space, and let $T : \mathcal{X} \rightarrow \mathcal{X}$ be a measure-preserving map. Let $(\mathcal{Y}, \mathbb{G})$ be another measurable space, and let $S : \mathcal{Y} \rightarrow \mathcal{Y}$ be a measurable map. Suppose there is a measurable map $\gamma : \mathcal{X} \rightarrow \mathcal{Y}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{T} & \mathcal{X} \\ \downarrow \gamma & & \downarrow \gamma \\ \mathcal{Y} & \xrightarrow{S} & \mathcal{Y} \end{array}$$

Then $(\mathcal{Y}, \mathbb{G}, \gamma(\mu), S)$ is a measure-preserving dynamical system.

Lemma 5.10. (Hansen) *Let $(\mathcal{X}, \mathbb{E}, \mu, T)$ and $(\mathcal{Y}, \mathbb{G}, \nu, S)$ be two measure-preserving dynamical systems. Suppose there is a measurable map $\gamma : \mathcal{X} \rightarrow \mathcal{Y}$ such that $\nu = \gamma(\mu)$ and such that the diagram in lemma 5.9 commutes. If $(\mathcal{X}, \mathbb{E}, \mu, T)$ is ergodic then $(\mathcal{Y}, \mathbb{G}, \nu, S)$ is also ergodic.*

Lemma 5.11. (Hansen) *(Maximal Ergodic Lemma) Let $(\mathcal{X}, \mathbb{E}, \mu, T)$ be a measure-preserving dynamical system, and let $f : \mathcal{X} \rightarrow \mathbb{R}$ be Borel measurable. If $f \in \mathcal{L}^1(\mu)$ then it holds that*

$$\int_{(M_n > 0)} f \, d\mu \geq 0 \quad (5.14)$$

where $M_n = \max\{0, S_1, S_2, \dots, S_n\}$ from the sequence

$$\left(f(x), f \circ T(x), f \circ T^2(x), f \circ T^3(x), \dots \right) \quad \text{with} \quad S_n = \sum_{i=0}^{n-1} f \circ T^i.$$

Theorem 5.12. (Hansen) *(Birkhoff's ergodic theorem) Let $(\mathcal{X}, \mathbb{E}, \mu, T)$ be an ergodic system. For $f \in \mathcal{L}^1(\mu)$ it holds that*

$$\frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i \xrightarrow{\text{a.s.}} \int f \, d\mu. \quad (5.16)$$

Theorem 5.13. (Hansen) *(Ergodic theorem, \mathcal{L}^p -version) Let $(\mathcal{X}, \mathbb{E}, \mu, T)$ be an ergodic system. If $f \in \mathcal{L}^p(\mu)$ for some $p \geq 1$ then it holds that*

$$\frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i \xrightarrow{\mathcal{L}^p} \int f \, d\mu. \quad (5.21)$$

Lemma 5.14. (Hansen) *Let $(\mathcal{X}, \mathbb{E})$ be a measurable space. The measurable finite dimensional product sets in $\mathcal{X}^{\mathbb{N}}$ form an \cap -stable generator for $\mathbb{E}^{\otimes \mathbb{N}}$.*

An element of the space $\mathcal{X}^{\mathbb{N}}$ is a countable set of coordinates x_n for $n \in \mathcal{N}$ with $x_n \in \mathcal{X}$. A **finite dimensional product set** in $\mathcal{X}^{\mathbb{N}}$ is set on the form

$$A_1 \times \dots \times A_k \times \mathcal{X} \times \mathcal{X} \times \dots$$

where $A_1, \dots, A_k \subset \mathcal{X}$. We also define the **projection sigma-algebra** $\mathbb{E}^{\otimes \mathbb{N}}$ as $\sigma\left((\hat{X}_n(\mathcal{X}^n))_{n \in \mathbb{N}}\right)$ where $\hat{X}_n(x_1, \dots, x_n) = x_n$.

Definition 5.15. (Hansen) Let X_1, X_2, \dots be a sequence of $(\mathcal{X}, \mathbb{E})$ -valued random variable, defined on a background space (Ω, \mathbb{F}, P) , and let $\mathbb{X} = (X_1, X_2, \dots)$ be their bundling. The **distribution** of the process is the image measure $\mathbb{X}(P)$ on $(\mathcal{X}^{\mathbb{N}}, \mathbb{E}^{\otimes \mathbb{N}})$.

Lemma 5.16. (Hansen) Let $\mathbb{X} = (X_1, X_2, \dots)$ and $\mathbb{Y} = (Y_1, Y_2, \dots)$ be two $(\mathcal{X}, \mathbb{E})$ -valued stochastic process, defined on a common background space. The two processes \mathbb{X} and \mathbb{Y} have the same distribution if and only if they have the same fidis. This can be checked by showing that

$$P(X_1 \in A_1, \dots, X_k \in A_k) = P(Y_1 \in A_1, \dots, Y_k \in A_k) \quad (5.25)$$

for any $k \in \mathbb{N}$ and any choice of $A_1, \dots, A_k \in \mathbb{E}$.

Definition 5.18. (Hansen) Let X_1, X_2, \dots be a sequence of $(\mathcal{X}, \mathbb{E})$ -valued random variable, defined on a background space (Ω, \mathbb{F}, P) , and let $\mathbb{X} = (X_1, X_2, \dots)$ be their bundling. Then we define: 1. The process \mathbb{X} is **stationary** if the distribution $\mathbb{X}(P)$ is an S -invariant probability on $(\mathbb{R}^{\mathbb{N}}, \mathbb{B}^{\otimes \mathbb{N}})$, 2. The process \mathbb{X} is **ergodic** if it is stationary and if the dynamical system $(\mathbb{R}^{\mathbb{N}}, \mathbb{B}^{\otimes \mathbb{N}}, \mathbb{X}(P), S)$ is ergodic. 3. The process \mathbb{X} is **mixing** if it is stationary and if the dynamical system $(\mathbb{R}^{\mathbb{N}}, \mathbb{B}^{\otimes \mathbb{N}}, \mathbb{X}(P), S)$ is mixing. with S being the **shift** map defined as $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$.

Theorem 5.20. (Hansen) (Khinchine's ergodic theorem) Let X_1, X_2, \dots be a stationary and ergodic sequence of real-valued random variables. If $E|X_1| < \infty$ it holds that

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\text{a.s.}} E X_1. \quad (5.27)$$

If $E|X_1|^p < \infty$ for some $p \geq 1$, the convergence is also in \mathcal{L}^p .

Theorem 5.21. (Hansen) (Ergodic transformation theorem) Let X_1, X_2, \dots be a sequence of real-valued random variables. For a measurable function $\phi : (\mathbb{R}^{\mathbb{N}}, \mathbb{B}^{\otimes \mathbb{N}}) \rightarrow (\mathbb{R}, \mathbb{B})$ we define new real-valued random variables

$$Y_n = \phi(X_n, X_{n+1}, \dots) = \phi \circ S^{n-1}(\mathbb{X}) \quad \text{for } n \in \mathbb{N}.$$

If X_1, X_2, \dots is stationary and ergodic then Y_1, Y_2, \dots is also stationary and ergodic.

Definition 5.23. (Hansen) Let $(X_n)_{n \in \mathbb{Z}}$ be a two-sided sequence of real-valued random variables, defined on a background space (Ω, \mathbb{F}, P) , and let \mathbb{X} be their bundling. 1. The process \mathbb{X} is **stationary** if the distribution $\mathbb{X}(P)$ is an S -invariant probability on $(\mathbb{R}^{\mathbb{Z}}, \mathbb{B}^{\otimes \mathbb{Z}})$, 2. The process \mathbb{X} is **ergodic** if it is stationary and if the dynamical system $(\mathbb{R}^{\mathbb{Z}}, \mathbb{B}^{\otimes \mathbb{Z}}, \mathbb{X}(P), S)$ is ergodic. 3. The process \mathbb{X} is **mixing** if it is stationary and if the dynamical system $(\mathbb{R}^{\mathbb{Z}}, \mathbb{B}^{\otimes \mathbb{Z}}, \mathbb{X}(P), S)$ is mixing.

Theorem 5.25. (Hansen) (Khinchine's ergodic theorem, two-sided version) Let $(X_n)_{n \in \mathbb{Z}}$ be a two-sided sequence of real-valued random variables. If the sequence is stationary and ergodic and if $E|X_1| < \infty$ then it holds that

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\text{a.s.}} E X_1. \quad (5.30)$$

If $E|X_1|^p < \infty$ for some $p \geq 1$, the convergence is also in \mathcal{L}^p .

Theorem 5.26. (Hansen) (Ergodic transformation theorem) Let $(X_n)_{n \in \mathbb{Z}}$ be a two-sided sequence of real-valued random variables. For a measurable function $\phi : (\mathbb{R}^{\mathbb{Z}}, \mathbb{B}^{\otimes \mathbb{Z}}) \rightarrow (\mathbb{R}, \mathbb{B})$ we define new real-valued random variables

$$Y_n = \phi \circ S^n(\mathbb{X}) \quad \text{for } n \in \mathbb{Z}.$$

If $(X_n)_{n \in \mathbb{Z}}$ is stationary and ergodic then $(Y_n)_{n \in \mathbb{Z}}$ is also stationary and ergodic.

Weak Convergence

Definition 6.1. (Hansen) A sequence of probability measures ν_1, ν_2, \dots on (\mathbb{R}, \mathbb{B}) is said to **converge weakly** to a limit probability measure ν if

$$\int f d\nu_n \rightarrow \int f d\nu \quad \text{for every } f \in C_b(\mathbb{R}) \quad (6.2)$$

We write $\nu_n \xrightarrow{wk} \nu$ to denote weak convergence.

Theorem 6.4. (Hansen) (Scheffe's) Let ν_1, ν_2, \dots and ν be probability measures on (\mathbb{R}, \mathbb{B}) . Assume that for some choice of basic measure μ it holds that $\nu_n = f_n \cdot \mu$ for every n and $\nu = f \cdot \mu$ for suitable density functions f_n and f . If

$$f_n(x) \rightarrow f(x) \quad \mu\text{-almost surely}$$

then it holds that $\nu_n \xrightarrow{wk} \nu$.

Lemma 6.8. (Hansen) Let μ and ν be two probability measures on (\mathbb{R}, \mathbb{B}) . If

$$\int f d\mu = \int f d\nu \quad \text{for all } f \in C_b(\mathbb{R}) \quad (6.7)$$

then it holds that $\mu = \nu$.

Theorem 6.9. (Hansen) Let ν_1, ν_2, \dots be a sequence of probability measures on (\mathbb{R}, \mathbb{B}) and let μ and ν be two extra probability measures. If

$$\nu_n \xrightarrow{wk} \mu \quad \text{and} \quad \nu_n \xrightarrow{wk} \nu$$

then $\mu = \nu$.

Definition 6.10. (Hansen) A sequence of real-valued variables X_1, X_2, \dots , defined on a common background space (Ω, \mathbb{F}, P) is said to **converge in distribution** to a limit variable X if

$$\int f(X_n) dP \rightarrow \int f(X) dP \quad \text{for every } f \in C_b(\mathbb{R}). \quad (6.9)$$

We will write $X_n \xrightarrow{\mathcal{D}} X$ to denote convergence in distribution.

Lemma 6.11. (Hansen) Let X_1, X_2, \dots and X be real-valued random variables. It holds that

$$X_n \xrightarrow{P} X \quad \Rightarrow \quad X_n \xrightarrow{\mathcal{D}} X.$$

Lemma 6.12. (Hansen) Let X_1, X_2, \dots be real-valued random variable and let $x_0 \in \mathbb{R}$. It holds that

$$X_n \xrightarrow{\mathcal{D}} x_0 \quad \Rightarrow \quad X_n \xrightarrow{P} x_0.$$

Theorem 6.13. (Hansen) Let ν, ν_1, ν_2, \dots be probability measures on (\mathbb{R}, \mathbb{B}) . Let \mathcal{H} be a class of bounded, non-negative and measurable functions with the following approximation property: For $f \in C_b(\mathbb{R})$ with $f \geq 0$ there is a sequence h_1, h_2, \dots of \mathcal{H} -functions such that $h_n \nearrow f$. If

$$\int h d\nu_n \rightarrow \int h d\nu \quad \text{for all } h \in \mathcal{H}. \quad (6.11)$$

then it holds that $\nu_n \xrightarrow{wk} \nu$.

Theorem 6.14. (Hansen) Let ν, ν_1, ν_2, \dots be probability measures on (\mathbb{R}, \mathbb{B}) . If

$$\int f d\nu_n \rightarrow \int f d\nu \quad \text{for all } f \in C_c(\mathbb{R}) \quad (6.13)$$

then it holds that $\nu_n \xrightarrow{wk} \nu$.

The class $C_c(\mathbb{R})$ is denoted as the set of all continuous real-valued functions with compact support i.e. there exist a $M > 0$ such that $f(x) = 0$ for all $|x| > M$.

Lemma 6.15. (Hansen) Let ν, ν_1, ν_2, \dots be probability measures on (\mathbb{R}, \mathbb{B}) , and let F, F_1, F_2, \dots be the corresponding distribution functions. If $\nu_n \xrightarrow{wk} \nu$ then it holds that

$$F_n(x_0) \rightarrow F(x_0),$$

whenever x_0 is a point of continuity for F .

Theorem 6.17. (Hansen) (Helly-Bray) Let ν, ν_1, ν_2, \dots be probability measures on (\mathbb{R}, \mathbb{B}) , and let F, F_1, F_2, \dots be the corresponding distribution functions. It holds that $\nu_n \xrightarrow{wk} \nu$ if and only if there is a dense subset $A \subset \mathbb{R}$ such that

$$F_n(x) \rightarrow F(x) \quad \text{for every } x \in A. \quad (6.16)$$

Theorem 6.18. (Hansen) Let ν, ν_1, ν_2, \dots be probability measures on (\mathbb{R}, \mathbb{B}) . Let F, F_1, F_2, \dots be the corresponding distribution functions, and let q, q_1, q_2, \dots be the corresponding quantile functions. If $\nu_n \xrightarrow{wk} \nu$ then it holds that

$$q_n(p) \rightarrow q(p)$$

for any $p \in (0, 1)$ such that the equation $F(x) = p$ has at most one solution.

Theorem 6.19. (Hansen) (Skorokhod's representation theorem) Let ν, ν_1, ν_2, \dots be probability measures on (\mathbb{R}, \mathbb{B}) . If $\nu_n \xrightarrow{wk} \nu$ then it is possible to find random variables X, X_1, X_2, \dots on a background space (Ω, \mathbb{F}, P) such that

$$X(P) = \nu, \quad X_1(P) = \nu_1, \quad X_2(P) = \nu_2, \dots$$

and such that $X_n \xrightarrow{a.s.} X$.

Corollary 6.20. (Hansen) Let ν, ν_1, ν_2, \dots be probability measures on (\mathbb{R}, \mathbb{B}) such that $\nu_n \xrightarrow{wk} \nu$. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded and measurable function. If there is a Borel-measurable set $C \subset \mathbb{R}$ such that h is continuous in every point of C and such that $\nu(C) = 1$, then it holds that

$$\int h d\nu_n \rightarrow \int h d\nu. \quad (6.20)$$

Corollary 6.21. (Hansen) (Portmanteau's lemma) Let ν, ν_1, ν_2, \dots be probability measures on (\mathbb{R}, \mathbb{B}) such that $\nu_n \xrightarrow{wk} \nu$. For any open set $G \subset \mathbb{R}$ it holds that

$$\liminf \nu_n(G) \geq \nu(G) \quad (6.21)$$

Definition 6.22. (Hansen) The **characteristic function** for a probability measure ν on (\mathbb{R}, \mathbb{B}) is the function $\phi : \mathbb{R} \rightarrow \mathbb{C}$ given by

$$\phi(\theta) = \int e^{ix\theta} d\nu(x). \quad \text{for } \theta \in \mathbb{R}. \quad (6.23)$$

Some useful observations include $|e^{ix\theta}| = 1$ hence $\phi(\theta) \leq 1$ for all $\theta \in \mathbb{R}$. We may also split the ϕ into an imaginary part and a real part with Euler's cartesian form

$$\phi(\theta) = \int \cos(x\theta) d\nu(x) + i \int \sin(x\theta) d\nu(x) \quad (6.24)$$

And lastly we have the implication

$$\nu_n \xrightarrow{\text{wk}} \nu \Rightarrow \phi_n(\theta) \rightarrow \phi(\theta) \quad \text{for all } \theta \in \mathbb{R}.$$

Furthermore, if $Y = \xi + \sigma X$ and $X \sim \mathcal{N}(0, 1)$ we have

$$\phi_Y(\theta) = e^{i\xi\theta} e^{-\sigma^2\theta^2/2}.$$

Theorem 6.28. (Hansen) *The characteristic function for any probability measure ν on (\mathbb{R}, \mathbb{B}) is uniformly continuous.*

Theorem 6.29. (Hansen) *Let ν be a probability measure on (\mathbb{R}, \mathbb{B}) . If*

$$\int |x|^k d\nu(x) < \infty$$

for some $k \in \mathbb{N}$, then the characteristic function ϕ is C^k and it holds that

$$\phi^{(k)}(\theta) = i^k \int x^k e^{i\theta x} d\nu(x) \quad \text{for } \theta \in \mathbb{R}. \quad (6.31)$$

Definition 6.30. (Hansen) *The **convolution** of two probability measures ν and ξ on (\mathbb{R}, \mathbb{B}) is the image measure*

$$\nu * \xi = \kappa(\nu \otimes \xi) \quad (6.33)$$

where $\kappa : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the addition map $\kappa(x, y) = x + y$.

Theorem 6.31. (Hansen) *Let ν and ξ be two probability measures on \mathbb{R} . If $\xi = f \cdot m$, then the convolution $\nu * \xi$ will have a density with respect to m . The density is given as*

$$g(x) = \int f(x - y) d\nu(y) \quad \text{for } x \in \mathbb{R}. \quad (6.35)$$

Lemma 6.31. (Hansen) *Let ν_1 and ν_2 be two probability measures on (\mathbb{R}, \mathbb{B}) with characteristic functions ϕ_1 and ϕ_2 . The convolution $\nu_1 * \nu_2$ has characteristic function γ given by*

$$\gamma(\theta) = \phi_1(\theta)\phi_2(\theta) \quad \text{for } \theta \in \mathbb{R}. \quad (6.37)$$

Theorem 6.34. (Hansen) *Let $\xi, \nu, \nu_1, \nu_2, \dots$ be probability measures on (\mathbb{R}, \mathbb{B}) . It holds that*

$$\nu_n \xrightarrow{\text{wk}} \nu \Rightarrow \nu_n * \xi \xrightarrow{\text{wk}} \nu * \xi.$$

Definition 6.35. (Hansen) *A probability measure $\nu = f \cdot m$ on (\mathbb{R}, \mathbb{B}) with density f with respect to Lebesgue measure is of **Polya class** if $f \in C_b(\mathbb{R})$ and if the Fourier transform \hat{f} is m -integrable.*

Lemma 6.39. (Hansen) *Let ν and ξ be two probability measures on \mathbb{R} . If ξ is of Polya class then the convolution $\nu * \xi$ is also of Polya class.*

Theorem 6.40. (Hansen) *(Inversion theorem) Let $\nu = f \cdot m$ be a probability measure on (\mathbb{R}, \mathbb{B}) of Polya class. It holds that*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\theta x} \hat{f}(\theta) d\theta, \quad x \in \mathbb{R}. \quad (6.39)$$

Theorem 6.41. (Hansen) Let ν_1 and ν_2 be two probability measures on (\mathbb{R}, \mathbb{B}) with characteristic functions ϕ_1 and ϕ_2 . If

$$\phi_1(\theta) = \phi_2(\theta), \quad \forall \theta \in \mathbb{R}$$

then $\nu_1 = \nu_2$.

Lemma 6.42. (Hansen) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be bounded and uniformly continuous. For every $\varepsilon > 0$ there is a probability measure ξ of Polya class with the property that

$$\left| f(x) - \int f(x+y) d\xi(y) \right| < \varepsilon, \quad \forall x \in \mathbb{R}. \quad (6.43)$$

Theorem 6.43. (Hansen) (Continuity theorem) Let ν, ν_1, ν_2, \dots be probability measures on (\mathbb{R}, \mathbb{B}) with characteristic functions $\phi, \phi_1, \phi_2, \dots$. If

$$\phi_n(\theta) \rightarrow \phi(\theta), \quad \theta \in \mathbb{R}, \quad (6.45)$$

then it holds that $\nu_n \xrightarrow{wk} \nu$.

Definition 6.44. (Hansen) A sequence of probability measures ν_1, ν_2, \dots on $(\mathbb{R}^k, \mathbb{B}_k)$ is said to **converge weakly** to a limit probability measure ν if

$$\int f(x) d\nu_n(x) \rightarrow \int f(x) d\nu(x), \quad \forall f \in C_b(\mathbb{R}^k). \quad (6.46)$$

We will write $\nu_n \xrightarrow{wk} \nu$ to denote weak convergence.

Theorem 6.45. (Hansen) (Continuity theorem) Let ν, ν_1, ν_2, \dots be probability measures on $(\mathbb{R}^k, \mathbb{B}_k)$ with characteristic functions $\phi, \phi_1, \phi_2, \dots$. If

$$\phi_n(\theta) \rightarrow \phi(\theta), \quad \theta \in \mathbb{R}^k, \quad (6.47)$$

then it holds that $\nu_n \xrightarrow{wk} \nu$.

Lemma 6.46. (Hansen) Let \mathbf{X} be an \mathbb{R}^k -valued random variable with characteristic function $\phi_{\mathbf{X}}$, and let \mathbf{Y} be an \mathbb{R}^m -valued random variable with characteristic function $\phi_{\mathbf{Y}}$. If $\mathbf{X} \perp\!\!\!\perp \mathbf{Y}$ then the bundle (\mathbf{X}, \mathbf{Y}) is an \mathbb{R}^{k+m} -valued random variable with

$$\phi_{(\mathbf{X}, \mathbf{Y})}(\theta_1, \theta_2) = \phi_{\mathbf{X}}(\theta_1)\phi_{\mathbf{Y}}(\theta_2), \quad \theta_1 \in \mathbb{R}^k, \theta_2 \in \mathbb{R}^m. \quad (6.49)$$

Theorem 6.47. (Hansen) (Continuous mapping theorem) Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ and \mathbf{X} be random variables with values in \mathbb{R}^k , and let $h : \mathbb{R}^k \rightarrow \mathbb{R}^m$ be continuous. If $\mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{X}$, then it holds that $h(\mathbf{X}_n) \xrightarrow{\mathcal{D}} h(\mathbf{X})$.

Theorem 6.48. (Hansen) (Cramer-Wold's device) Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ and \mathbf{X} be random variables with values in \mathbb{R}^k . If

$$\mathbf{v}^\top \mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{v}^\top \mathbf{X}, \quad (6.51)$$

for any fixed vector $\mathbf{v} \in \mathbb{R}^k$, then it holds that $\mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{X}$.

Lemma 6.49. (Hansen) Let $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \dots$ be random variables with values in \mathbb{R}^k , let $\mathbf{Y}_1, \mathbf{Y}_2, \dots$ be random variables in \mathbb{R}^m , and let \mathbf{y} be a vector in \mathbb{R}^m . If it holds that

$$\mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{X}, \quad \mathbf{Y}_n \xrightarrow{\mathcal{P}} \mathbf{y}$$

then the bundle $(\mathbf{X}_n, \mathbf{Y}_n)$ in \mathbb{R}^{k+m} will satisfy that

$$(\mathbf{X}_n, \mathbf{Y}_n) \xrightarrow{\mathcal{D}} (\mathbf{X}, \mathbf{y}).$$

Corollary 6.50. (Hansen) (*Slutsky's lemma*) Let $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \dots$ and $\mathbf{Y}_1, \mathbf{Y}_2, \dots$ be random variables with values in \mathbb{R}^k . It holds that

$$\mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{X}, \quad \mathbf{Y}_n \xrightarrow{\mathcal{P}} \mathbf{0} \quad \Rightarrow \quad \mathbf{X}_n + \mathbf{Y}_n \xrightarrow{\mathcal{D}} \mathbf{X}.$$

Corollary 6.51. (Hansen) Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ and \mathbf{X} be random variables with values in \mathbb{R}^k and let Y_1, Y_2, \dots be real-valued random variables. It holds that

$$\mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{X}, \quad Y_n \xrightarrow{\mathcal{P}} 1 \quad \Rightarrow \quad Y_n \mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{X}.$$

Corollary 6.52. (Hansen) Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ and \mathbf{X} be random variables with values in \mathbb{R}^k and let Y_1, Y_2, \dots be real-valued random variables. If

$$\mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{X}, \quad Y_n \xrightarrow{\mathcal{P}} 0 \quad \Rightarrow \quad Y_n \mathbf{X}_n \xrightarrow{\mathcal{P}} \mathbf{0}.$$

Definition 6.53. (Hansen) The \mathbb{R}^k -valued random variable $\mathbf{Z} = (Z_1, \dots, Z_k)$ has a **multivariate Gaussian distribution** if and only if the real-valued random variable $\sum_{j=1}^k c_j Z_j$ has a one-dimensional Gaussian distribution for every choice of $c_1, \dots, c_k \in \mathbb{R}$.

Theorem 6.54. (Hansen) Let $\mathbf{Z} = (Z_1, \dots, Z_k)$ have a multivariate Gaussian distribution with $E \mathbf{Z} = \xi$ and $V \mathbf{Z} = \Sigma$. Then the characteristic function is

$$\phi_{\mathbf{Z}}(\theta) = e^{i\theta^\top \xi} \exp\left(-\frac{1}{2}\theta^\top \Sigma \theta\right), \quad \theta \in \mathbb{R}^k. \quad (6.53)$$

Conversely, if \mathbf{Z} has characteristic function given by (6.53) for some $\xi \in \mathbb{R}^k$ and some symmetric, positive semi-definite $k \times k$ matrix Σ then \mathbf{Z} has a multivariate Gaussian distribution with $E \mathbf{Z} = \xi$ and $V \mathbf{Z} = \Sigma$.

Corollary 6.55. (Hansen) Let \mathbf{Z} be an \mathbb{R}^k -valued random variable, let $\mathbf{a} \in \mathbb{R}^m$ and let B be an $m \times k$ matrix. It holds that

$$\mathbf{Z} \sim \mathcal{N}(\xi, \Sigma) \quad \Rightarrow \quad \mathbf{a} + B\mathbf{Z} \sim \mathcal{N}(\mathbf{a} + B\xi, B\Sigma B^\top).$$

Lemma 6.56. (Hansen) Let $\mathbf{X} = (X_1, \dots, X_k)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ be random variables with values in \mathbb{R}^k respectively \mathbb{R}^m . If both \mathbf{X} and \mathbf{Y} have multivariate Gaussian distributions and if \mathbf{X} and \mathbf{Y} are independent, then the \mathbb{R}^{k+m} -valued bundle (\mathbf{X}, \mathbf{Y}) has a multivariate Gaussian distribution.

Lemma 6.58. (Hansen) Let $\mathbf{X} = (X_1, \dots, X_k)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ be random variables with values in \mathbb{R}^k respectively \mathbb{R}^m . If the \mathbb{R}^{k+m} -valued bundle (\mathbf{X}, \mathbf{Y}) has a multivariate Gaussian distribution, and if

$$\text{Cov}(X_j, Y_l) = 0, \quad \forall j \text{ and } l$$

then $\mathbf{X} \perp\!\!\!\perp \mathbf{Y}$.

Definition 6.59. (Hansen) Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be \mathbb{R}^k -valued random variables. Let $\xi \in \mathbb{R}^k$ be a vector and let Σ be a symmetric, positive semi-definite $k \times k$ matrix. We say that \mathbf{X}_n has an **asymptotic normal distribution** with parameters $(\xi, \frac{1}{n}\Sigma)$, written

$$\mathbf{X}_n \stackrel{\text{a.s.}}{\sim} \mathcal{N}\left(\xi, \frac{1}{n}\Sigma\right),$$

if it holds that

$$\sqrt{n}(\mathbf{X}_n - \xi) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma).$$

Lemma 6.60. (Hansen) Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ and \mathbf{Y} be random variables with values in \mathbb{R}^k . If it holds that $\mathbf{X}_n \stackrel{\text{a.s.}}{\sim} \mathcal{N}(\xi, \frac{1}{n}\Sigma)$ then it follows that $\mathbf{X}_n \xrightarrow{\mathcal{P}} \xi$.

Lemma 6.61. (Hansen) Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ and \mathbf{Y} be \mathbb{R}^k -valued random variables, and assume that $\sqrt{n}\mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{Y}$. Let $g : \mathbb{R}^k \rightarrow \mathbb{R}^m$ be a measurable map. Assume that $g(\mathbf{0}) = \mathbf{0}$ and that g is differentiable in $\mathbf{0}$ with derivative $Dg(\mathbf{0}) = A$. Then it holds that

$$\sqrt{n}g(\mathbf{X}_n) \xrightarrow{\mathcal{D}} A \mathbf{Y}.$$

Lemma 6.62. (Hansen) (Delta method) Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be \mathbb{R}^k -valued random variables, and let $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$ be measurable. If f is differentiable in ξ , then it holds that

$$\mathbf{X}_n \stackrel{\text{a.s.}}{\sim} \mathcal{N}\left(\xi, \frac{1}{n}\Sigma\right) \Rightarrow f(\mathbf{X}_n) \stackrel{\text{a.s.}}{\sim} \mathcal{N}\left(f(\xi), \frac{1}{n}Df(\xi)\Sigma Df(\xi)^\top\right).$$

Central Limit Theorems

Lemma 7.1. (Hansen) Let z_1, \dots, z_n and w_1, \dots, w_n be complex numbers. If $|z_i| \leq 1$ and $|w_i| \leq 1$ for all $i = 1, \dots, n$ then it holds that

$$\left| \prod_{i=1}^n z_i - \prod_{i=1}^n w_i \right| \leq \sum_{i=1}^n |z_i - w_i|. \quad (7.1)$$

Theorem 7.2. (Hansen) (Basic CLT) Let X_1, X_2, \dots be independent and identically distributed real-valued random variables. Assume that $E X_1 = 0$ and $E X_1^2 = 1$. Then it holds that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1) \quad (7.3)$$

Theorem 7.3. (Hansen) (Laplace's CLT) Let X_1, X_2, \dots be independent and identically distributed real-valued random variables with $E X_1^2 < \infty$. It holds that

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\mathcal{D}} \mathcal{N}\left(E X_1, \frac{V X_1}{n}\right) \quad (7.4)$$

Theorem 7.7. (Hansen) (Laplace's CLT, multivariate version) Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be independent and identically distributed random variables with values in \mathbb{R}^k . Assume that $E|\mathbf{X}_1|^2 < \infty$. It holds that

$$\frac{1}{n} \sum_{i=1}^n \mathbf{X}_i \xrightarrow{\mathcal{D}} \mathcal{N}\left(E \mathbf{X}_1, \frac{1}{n} V \mathbf{X}_1\right) \quad (7.7)$$

Definition 7.10. (Hansen) Let (X_{nm}) be a centralized array of real-valued random variables. a. The array satisfies the **vanishing variance condition** if

$$\max_{m=1, \dots, n} E X_{nm}^2 \rightarrow 0. \quad (7.8)$$

b. The array satisfies **Lindeberg's condition** if

$$\forall c > 0 : \sum_{m=1}^n \int_{(|X_{nm}| > c)} X_{nm}^2 dP \rightarrow 0. \quad (7.9)$$

c. The array satisfies **Lyapounov's condition** of order $\alpha > 2$ if

$$\sum_{m=1}^n E |X_{nm}|^\alpha \rightarrow 0. \quad (7.10)$$

Lemma 7.11. (Hansen) *Lyapounov's condition of order $\alpha > 2$ implies Lindeberg's condition.*

Lemma 7.12. (Hansen) *Lindeberg's condition implies the vanishing variance condition.*

Theorem 7.14. (Hansen) *(Lindeberg's CLT) Let (X_{nm}) be a centralized array of real-valued random variables with $E X_{nm}^2 < \infty$. Assume that the array satisfies that*

$$E X_{nm} = 0, \forall n, m,$$

and that

$$\sum_{m=1}^n E X_{nm}^2 = 1. \quad (7.13)$$

Assume that the array has independence within rows i.e. $X_{i1} \perp \dots \perp X_{in}$ for all $i = 1, \dots, n$ and satisfies Lindeberg's condition. Then it holds that

$$\sum_{m=1}^n X_{nm} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

Theorem 7.19. (Hansen) *(Lindeberg's CLT, multivariate version) Let (\mathbf{X}_{nm}) be a triangular array of random variables with values in \mathbb{R}^k with $E |\mathbf{X}_{nm}|^2 < \infty$ for all n, m . Assume that*

$$E \mathbf{X}_{nm} = \mathbf{0}, \forall n, m$$

and

$$\sum_{m=1}^n V \mathbf{X}_{nm} \rightarrow \Sigma$$

for a fixed $k \times k$ matrix Σ . Assume that the array has independence within rows, and assume that the associated real-valued array $(|\mathbf{X}_{nm}|)$ satisfies Lindeberg's condition. Then it holds that

$$\sum_{m=1}^n \mathbf{X}_{nm} \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0}, \Sigma)$$

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