

Exam-prep (FinKont)

Joakim Bilyk

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Contents

Exam sets (2017/18-2021/22)	1
In progress	1
Exam 2017/18	1
Exam 2018/19	8
Exam 2019/20	8
Exam 2020/21	8
Exam 2021/22	8
Exam 2022/23	9

Exam sets (2017/18-2021/22)

In progress

Problem 3

Consider a two-dimensional model. The market model consist of three assets: A bank account B_t and two stocks S_1 and S_2 . The P -dynamics of B_t is

$$dB_t = rB_t dt, \quad B_0 = 1,$$

where $r \in \mathbb{R}$ is a constant interest rate. The P -dynamics of S_1 and S_2 are given by

$$\begin{aligned} dS_1(t) &= \alpha_1 S_1(t) dt + \sigma_1 S_1(t) d\bar{W}_1(t), & S_1(0) &= s_1 > 0, \\ dS_2(t) &= \alpha_2 S_2(t) dt + \sigma_2 S_2(t) d\bar{W}_2(t), & S_2(0) &= s_2 > 0, \end{aligned}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ are constants. Moreover, $\sigma_1 > 0$ is a constant and $\sigma_2(t) = \sigma_0 e^{-\gamma t}$ where $\sigma_0 > 0$ and $\gamma > 0$ are constants and $\bar{W}_1(t)$ and $\bar{W}_2(t)$ are two independent P -Brownian motions. The filtration is the one generated by the two Brownian motions, that is, $\mathcal{F}_t = \sigma(\bar{W}_1(s), \bar{W}_2(s) \mid 0 \leq s \leq t)$. Let $T > 0$ be a given and fixed (expiry) date.

- a.
 - i. Is the model arbitrage free?
 - ii. Is the model complete?

Consider the derivative that at time T pays $X = S_1(T)S_2(T)$ and let $F(t, s_1, s_2)$ be the pricing function of the derivative.

- b.
 - i. Determine the arbitrage free price of derivative X at time $t = 0$.
 - ii. Determine the equation satisfied by the pricing function $F(t, s_1, s_2)$.

Consider a new derivative that at time T pays $Y = \log(S_2(T))$.

- c. Determine the arbitrage free price of derivative Y at time $t < T$.

Solution (a).

Solution (b).

Solution (c).

Exam 2017/18

Problem 1

Let W_t denote a Brownian motion and let

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma(\{W_s \mid 0 \leq s \leq t\}).$$

Let $T > 0$ be a given and fixed time.

Let $f(t)$ be a bounded deterministic continuous function. Define the two processes

$$\begin{cases} X_t = \int_0^t f(u) dW_u, \\ M_t^{(\lambda)} = \exp \left\{ \lambda X_t - \frac{\lambda^2}{2} \int_0^t f^2(u) du \right\}, \end{cases}$$

where $\lambda \in \mathbb{R}$ is a constant.

a. Show that $M^{(\lambda)}$ is a martingale with $E[M_t^{(\lambda)}] = 1$.

Let $0 < s < t$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ be given and fixed.

b. i. Show that

$$\begin{aligned} M_s^{(\lambda_1)} &= E \left[\frac{M_s^{(\lambda_1)} M_t^{(\lambda_2)}}{M_s^{(\lambda_2)}} \middle| \mathcal{F}_s \right] \\ &= E \left[\exp \left\{ \lambda_1 X_s + \lambda_2 (X_t - X_s) - \frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du - \frac{1}{2} \lambda_2^2 \int_s^t f^2(u) du \right\} \middle| \mathcal{F}_s \right] \end{aligned}$$

ii. Show that X_s and $X_t - X_s$ are normally distributed and independent.

c. Compute the mean value of $M_T^{(\lambda)} \log(M_T^{(\lambda)})$.

Solution (a).

First, we see that since X_t is on integral form we know that

$$\begin{cases} dX_t = f(t) dW_t \\ X_0 = 0. \end{cases}$$

Hence we may represent M as $M_t^{(\lambda)} = g(t, X_t, Y_t)$ given by

$$g(t, x, y) = \exp \left\{ \lambda x - \frac{\lambda^2}{2} y \right\},$$

where $Y_t = \int_0^t f^2(u) du$ with dynamics

$$\begin{cases} dY_t = f^2(t) dt \\ Y_0 = 0. \end{cases}$$

Hence by the multidimensional Ito's formula we have the dynamics of M given by

$$\begin{aligned} dM_t^{(\lambda)} &= g_t dt + g_x dX_t + g_y dY_t + \frac{1}{2} g_{yy} (dY_t)^2 + \frac{1}{2} g_{xx} (dX_t)^2 + f_{xy} (dX_t)(dY_t) \\ &= 0 + \lambda g dX_t - \frac{\lambda^2}{2} g dY_t + 0 + \frac{1}{2} \lambda^2 g (dX_t)^2 + 0 \\ &= \lambda M_t^{(\lambda)} f(t) dW_t - \frac{1}{2} \lambda^2 M_t^{(\lambda)} f^2(t) dt + \frac{1}{2} \lambda M_t^{(\lambda)} f^2(t) dt \\ &= \lambda f(t) M_t^{(\lambda)} dW_t, \end{aligned}$$

And so we see that M is a martingale as it only has dynamics wrt. the Brownian motion W (assuming $\lambda f_t M_t^{(\lambda)} \in \mathcal{L}^2$). Furthermore we have that

$$M_0^{(\lambda)} = g(0, X_0, Y_0) = \exp \left\{ \lambda X_0 - \frac{1}{2} \lambda^2 Y_0 \right\} = e^0 = 1$$

and so we have $E[M_t^{(\lambda)}] = M_0^{(\lambda)} = 1$ as desired. \square

Solution (b).

“(i)” We have from the previous exercise

$$\begin{aligned}
& \frac{M_s^{(\lambda_1)} M_t^{(\lambda_2)}}{M_s^{(\lambda_2)}} \\
&= \exp \left\{ \lambda_1 X_s - \frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du \right\} \exp \left\{ \lambda_2 X_t - \frac{1}{2} \lambda_2^2 \int_0^t f^2(u) du \right\} \exp \left\{ \frac{1}{2} \lambda_2^2 \int_0^s f^2(u) du - \lambda_2 X_s \right\} \\
&= \exp \left\{ \lambda_1 X_s - \frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du + \lambda_2 X_t - \frac{1}{2} \lambda_2^2 \int_0^t f^2(u) du + \frac{1}{2} \lambda_2^2 \int_0^s f^2(u) du - \lambda_2 X_s \right\} \\
&= \exp \left\{ \lambda_1 X_s + \lambda_2 (X_t - X_s) - \frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du - \frac{1}{2} \lambda_2^2 \int_s^t f^2(u) du \right\}
\end{aligned}$$

and so the conclusion follows. \square

“(ii)” We have that from lemma 4.18 that

$$X_s = \int_0^s f(u) dW_u \sim \mathcal{N} \left(0, \int_0^s f^2(u) dW_u \right)$$

furthermore we have that

$$X_t - X_s = \int_s^t f(u) dW_u \sim \mathcal{N} \left(0, \int_s^t f^2(u) dW_u \right).$$

In regard to the independence claim we could check identity below

$$E[e^{t_1 X} e^{t_2 Y}] = E[e^{t_1 X}] E[e^{t_2 Y}]$$

where X, Y are independent random variables. The above identity holds if and only if X and Y are independent. From above we have that

$$M_s^{(\lambda_1)} = E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)} \mid \mathcal{F}_s] e^{-\frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du - \frac{1}{2} \lambda_2^2 \int_s^t f^2(u) du}$$

and so taking expectation we have

$$1 = E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)}] e^{-\frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du - \frac{1}{2} \lambda_2^2 \int_s^t f^2(u) du}$$

Which the gives

$$E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)}] = e^{\frac{1}{2} \lambda_1^2 \int_0^s f^2(u) du + \frac{1}{2} \lambda_2^2 \int_s^t f^2(u) du} = E[e^{\lambda_1 X_s}] E[e^{\lambda_2 (X_t - X_s)}]$$

and so the conclusion is that X_s and $X_t - X_s$ are independent. \square

Solution (c).

We recall the definition of $M_t^{(\lambda)}$ and observe that

$$\log M_t^{(\lambda)} = \lambda X_t - \frac{1}{2} \lambda^2 \int_0^t f^2(u) du.$$

Furthermore we have the dynamics of $M^{(\lambda)}$ given by the differential form

$$dM_t^{(\lambda)} = \lambda f(t) M_t^{(\lambda)} dW_t.$$

with $M_0^{(\lambda)} = 1$. Since we know that $M_t^{(\lambda)}$ is a martingale we have

$$E^P[M_T^{(\lambda)}] = E^P[M_0^{(\lambda)}] = 1,$$

and so we may define a new probability measure as

$$d\tilde{P} = M_T^{(\lambda)} dP$$

on \mathcal{F}_T . We then have a new Brownian motion \tilde{W} such that

$$dW_t = \lambda f(t) dt + d\tilde{W}_t.$$

We can then see

$$\begin{aligned} E^P[M_T^{(\lambda)} \log M_T^{(\lambda)}] &= \int M_T^{(\lambda)} \log M_T^{(\lambda)} dP = \int M_T^{(\lambda)} \log M_T^{(\lambda)} \frac{1}{M_T^{(\lambda)}} d\tilde{P} \\ &= \int \log M_T^{(\lambda)} d\tilde{P} = E^{\tilde{P}}[\log M_T^{(\lambda)}]. \end{aligned}$$

Then we can evaluate the mean value by seeing the X has representation wrt. \tilde{P} by

$$X_t = \int_0^t f(u) (\lambda f(u) du + d\tilde{W}_u) = \lambda \int_0^t f^2(u) du + \int_0^t f(u) d\tilde{W}_u.$$

Giving that

$$\begin{aligned} E^P[M_T^{(\lambda)} \log M_T^{(\lambda)}] &= E^{\tilde{P}}[\log M_T^{(\lambda)}] \\ &= E^{\tilde{P}} \left[\lambda X_T - \frac{1}{2} \lambda^2 \int_0^T f^2(u) du \right] \\ &= E^{\tilde{P}} \left[\lambda^2 \int_0^T f^2(u) du + \lambda \int_0^T f(u) d\tilde{W}_u - \frac{1}{2} \lambda^2 \int_0^T f^2(u) du \right] \\ &= \lambda E^{\tilde{P}} \left[\frac{1}{2} \lambda \int_0^T f^2(u) du + \int_0^T f(u) d\tilde{W}_u \right] \\ &= \frac{1}{2} \lambda^2 \int_0^T f^2(u) du + \lambda E^{\tilde{P}} \left[\int_0^T f(u) d\tilde{W}_u \right] \\ &= \frac{1}{2} \lambda^2 \int_0^T f^2(u) du \end{aligned}$$

Since

$$\tilde{X}_T = \int_0^T f(u) d\tilde{W}_u,$$

is a \tilde{P} -martingale. \square

Problem 2

Consider a standard Black-Scholes model, that is, a model consisting of a bank account B_t with P -dynamics given by

$$dB_t = rB_t dt, \quad B_0 = 1$$

and a stock S_t with P -dynamics given by

$$dS_t = \alpha S_t dt + \sigma S_t d\bar{W}_t, \quad S_0 = s > 0$$

where $r, \alpha \in \mathbb{R}$ and $\sigma > 0$ are constants and \bar{W}_t is a P -Brownian motion. Let $T > 0$ be a given and fixed date.

Consider the derivative that at time T pays

$$X = \max \{ \min \{ S_T, K_2 \}, K_1 \},$$

where $0 < K_1 < K_2$ are constants.

- a. Determine the arbitrage free price of derivative X at time $t < T$.

Consider a new derivative that at time T pays

$$Y = (S_T^2 - K^2)^+ - (K^2 - S_T^2)^+.$$

- b. i. Determine the arbitrage free price of derivative Y at time $t < T$.
ii. Find a hedging portfolio for derivative Y .

Let $h(t) = (h_0(t), h_1(t))$ be a portfolio where

$$h_0(t) = -e^{r(T-2t)+\sigma^2(T-t)} S^2(t)$$

is the number of units in the bank account at time t and

$$h_1(t) = 2e^{(r+\sigma^2)(T-t)} S(t)$$

is the number of shares in the stock at time t . Let $V^h(t)$ denote the associated value process.

- c. Determine whether the portfolio h is self-financing or not.
d. Compute $V^h(T)$.

Solution (a).

We see that the derivative is the bull spread given by the payout function

$$X = \begin{cases} K_2 & \text{if } S_T > K_2, \\ S_T & \text{if } K_1 \leq S_T \leq K_2, \\ K_1 & \text{if } S_T < K_1. \end{cases}$$

We know from exercise 10.3 that this can be replicated by holding K_1 bonds, one call option with strike K_1 and a short on a call with strike K_2 . The arbitrage free price of the derivative is then the value process of the mentioned portfolio i.e.

$$\Pi_t[X] = K_1 e^{-r(T-t)} + c(K_1; t, T) - c(K_2; t, T),$$

where c denotes the pricing function for a European call option (non-instructive parameters suppressed). \square

Solution (b).

(i): We start by seeing that the derivative pays out

$$Y = \begin{cases} S_T^2 - K^2 & \text{if } S_T^2 \geq K^2, \\ -(K^2 - S_T^2) & \text{if } S_T^2 < K^2. \end{cases}$$

hence the payout is $Y = S_T^2 - K^2 = \Phi(S_T)$ where $\Phi(s) = s^2 - K^2$. That is Y is in fact a simple claim. We have from the risk neutral valuation formula 7.11 that

$$\begin{aligned} \Pi_t[Y] &= e^{-r(T-t)} E_{t,s}^Q[S_T^2 - K^2] \\ &= e^{-r(T-t)} E_{t,s}^Q[S_T^2] - e^{-r(T-t)} K^2. \end{aligned}$$

Recall that under the martingale measure Q we have that S_t is a GBM hence

$$S_t = s \cdot \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) (T-t) + \sigma (W_T^Q - W_t^Q) \right\}$$

then

$$S_T^2 = s^2 \cdot \exp \left\{ 2 \left(r - \frac{1}{2} \sigma^2 \right) (T-t) + 2\sigma (W_T^Q - W_t^Q) \right\}.$$

Inserting this into the risk neutral valuation formula we get

$$\begin{aligned} \Pi_t[Y] &= e^{-r(T-t)} E_{t,s}^Q[S_T^2] - e^{-r(T-t)} K^2 \\ &= e^{-r(T-t)} s^2 e^{2(r-\frac{1}{2}\sigma^2)(T-t)} E^Q \left[\exp \left\{ 2\sigma (W_T^Q - W_t^Q) \right\} \right] - e^{-r(T-t)} K^2 \\ &= e^{-r(T-t)} s^2 e^{2(r-\frac{1}{2}\sigma^2)(T-t)} e^{\frac{1}{2}4\sigma^2(T-t)} - e^{-r(T-t)} K^2 \\ &= e^{-r(T-t)} \left(s^2 e^{(2r-\sigma^2)(T-t)+\frac{1}{2}4\sigma^2(T-t)} - K^2 \right) \\ &= e^{-r(T-t)} \left(s^2 e^{(2r+\sigma^2)(T-t)} - K^2 \right). \end{aligned}$$

The arbitrage free price of the derivative is then given above. \square

(ii): From theorem 8.5 we can determine a hedging portfolio with weightings

$$\begin{aligned} w_t^B &= \frac{\Pi_t - S_t \frac{\partial \Pi}{\partial s}}{\Pi_t} \\ &= 1 - \frac{S_t 2S_t e^{-r(T-t)} e^{(2r+\sigma^2)(T-t)}}{e^{-r(T-t)} (S_t^2 e^{(2r+\sigma^2)(T-t)} - K^2)} \\ &= 1 - \frac{2S_t^2 e^{(2r+\sigma^2)(T-t)}}{S_t^2 e^{(2r+\sigma^2)(T-t)} - K^2} \\ &= 1 - \frac{2}{1 - K^2 S_t^{-2} e^{(2r+\sigma^2)(t-T)}} \\ w_t^S &= \frac{2}{1 - K^2 S_t^{-2} e^{(2r+\sigma^2)(t-T)}}. \end{aligned}$$

In absolute terms we will hold the portfolio

$$\begin{aligned}
 h_t^S &= 2S_t e^{-r(T-t)} e^{(2r+\sigma^2)(T-t)} \\
 h_t^B &= \frac{e^{-r(T-t)} \left(s^2 e^{(2r+\sigma^2)(T-t)} - K^2 \right) - S_t h_t^S}{B_t} \\
 &= \frac{e^{-r(T-t)} \left(s^2 e^{(2r+\sigma^2)(T-t)} - K^2 \right) - S_t h_t^S}{e^{rt}} \\
 &= e^{-rT} s^2 e^{(2r+\sigma^2)(T-t)} - e^{-rT} K^2 - e^{-rt} S_t h_t^S.
 \end{aligned}$$

The portfolio above will hedge Y with probability one. \square

Solution (c).

We assume no dividends and no consumption that is $c_t = 0$ and $dD_t^i = 0$ for $i = 0, 1$. Then the portfolio is self-financing if and only if the value process has dynamics.

$$h_0(t) dB_t + h_1(t) dS_t = 0$$

This is given in lemma 6.12.

THE BELOW IS IN WORKS AND NOT CORRECT!

Now we have that the value process is given by

$$V_t^h = h_0(t)B_t + h_1(t)S_t.$$

Using the representation $V_t^h = f(h_0(t), B_t) + f(h_1(t), S_t)$ given by $f(x, y) = xy$ we have

$$dV_t^h = df(h_0(t), B_t) + df(h_1(t), S_t).$$

Using Ito's formula on each term we have

$$\begin{aligned}
 df(h_0(t), B_t) &= B_t dh_0(t) + h_0(t) dB_t + (dB_t)(dh_0(t)), \\
 df(h_1(t), S_t) &= S_t dh_1(t) + h_1(t) dS_t + (dS_t)(dh_1(t)),
 \end{aligned}$$

since of cause $f_{xx} = f_{yy} = 0$. We can the determine the dynamics of the portfolio by

$$\begin{aligned}
 dh_0(t) &= -(-2t - \sigma^2)S_t^2 e^{r(T-2t)+\sigma^2(T-t)} dt \\
 &\quad - 2S_t e^{r(T-2t)+\sigma^2(T-t)} dS_t \\
 &\quad - \frac{1}{2} 2e^{r(T-2t)+\sigma^2(T-t)} (dS_t)^2 \\
 &= (-2t - \sigma^2)h_0(t) dt + \frac{2}{S_t} h_0(t) (\mu S_t dt + \sigma S_t dW_t) + \frac{1}{S_t^2} h_0(t) \sigma^2 S_t^2 dt \\
 &= (\mu - 1)2h_0(t) dt + 2\sigma h_0(t) dW_t
 \end{aligned}$$

and

$$\begin{aligned}
dh_1(t) &= (-r - \sigma^2)2e^{(r+\sigma^2)(T-t)}S_t dt \\
&\quad + 2e^{(r+\sigma^2)(T-t)} dS_t + 0 \\
&= (-r - \sigma^2)h_1(t) dt + \frac{1}{S_t}h_1(t)(\mu S_t dt + \sigma S_t dW_t) \\
&= (-r - \sigma^2 + \mu)h_1(t) dt + h_1(t)\sigma dW_t
\end{aligned}$$

And so in total

$$\begin{aligned}
dV_t^h(t) &= df(h_0(t), B_t) + df(h_1(t), S_t) \\
&= B_t dh_0(t) + h_0(t) dB_t + (dB_t)(dh_0(t)) \\
&\quad + S_t dh_1(t) + h_1(t) dS_t + (dS_t)(dh_1(t)) \\
&= B_t ((\mu - 1)2h_0(t) dt + 2\sigma h_0(t) dW_t) + h_0(t) rB_t dt + 0 \\
&\quad + S_t ((-r - \sigma^2 + \mu)h_1(t) dt + h_1(t)\sigma dW_t) + h_1(t) (\mu S_t dt + \sigma S_t dW_t) + \sigma^2 S_t h_1(t) dt \\
&= [B_t(\mu - 1)2h_0(t) + h_0(t)rB_t + S_t(-r - \sigma^2 + \mu)h_1(t) + h_1(t)\mu S_t + \sigma^2 S_t h_1(t)] dt \\
&\quad + [B_t 2\sigma h_0(t) + S_t h_1 \sigma + h_1 \sigma S_t] dW_t \\
&= [(2\mu - 2 + r)B_t h_0(t) + (-r + 2\mu)S_t h_1(t)] dt \\
&\quad + [B_t h_0(t) + h_1 S_t] 2\sigma dW_t \\
&= V_t^h 2\mu dt + V_t^h dW_t
\end{aligned}$$

Solution (d).

We compute V_T^h easily by inserting h_0 and h_1 below

$$\begin{aligned}
V_T^h &= B_T h_0(T) + S_T h_1(T) \\
&= B_T \left(-e^{r(T-2T)+\sigma^2(T-T)} S_T^2 \right) + S_T \left(2e^{(r+\sigma^2)(T-T)} S_T \right) \\
&= -S_T^2 + 2S_T^2 = S_T^2.
\end{aligned}$$

and so h hedge the payout $\Phi(S_T) = S_T^2$. \square

Exam 2018/19

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Exam 2019/20

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Exam 2020/21

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Exam 2021/22

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Exam 2022/23

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