QRM Homework 1

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Problem 1

To solve the expected shortfall we will be using lemma 2.13(MFE);

$$ES_\alpha(L) = \frac{E[L; L \geq VaR_\alpha(L)]}{1-\alpha} = \frac{1}{1-\alpha}\int_{VaR_\alpha(L)}^{\infty} u \cdot g(u)du$$

Since g(u) is the density function of L. Now we shall substitute the expression of g(u) into the integral;

$$\frac{1}{1-\alpha}\int_{VaR_\alpha(L)}^{\infty} u \cdot g(u)du = \frac{1}{1-\alpha}\int_{t_\nu^{-1}(\alpha)}^{\infty} u \cdot K \cdot (1+\frac{u^2}{\nu})^{-(\nu+1)/2}du$$

Where $K = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)}$. Let $q := t_\nu^{-1}(\alpha)$. From here we get;

$$\frac{K}{1-\alpha}\int_{q}^{\infty} u \cdot (1+\frac{u^2}{\nu})^{-(\nu+1)/2}du = \frac{K}{1-\alpha}\left[\frac{\nu(1+\frac{u^2}{\nu})^{1/2-\nu/2}}{1-\nu}\right]_q^\infty$$

Since we can make the substitution $x = (1+\frac{u^2}{\nu}) \Rightarrow dx = \frac{2u}{\nu} \cdot du$ Now, since we have assumed in the problem that $\nu > 1 \Rightarrow 1/2 - \nu/2 < 0$. Hence when taking the limit $u \to \infty$, we achieve a zero. Therefore we have;

$$\begin{aligned}\frac{K}{1-\alpha}\left[\frac{\nu(1+\frac{u^2}{\nu})^{1/2-\nu/2}}{1-\nu}\right]_q^\infty =& \frac{K}{1-\alpha}\left(\frac{\nu(1+\frac{q^2}{\nu})^{1/2-\nu/2}}{\nu-1}\right) = \frac{K}{1-\alpha}\left(\frac{\nu(1+\frac{q^2}{\nu})^{-(1+\nu)/2}}{\nu-1}\cdot(1+\frac{q^2}{\nu})\right) \\ =& \frac{K}{1-\alpha}\left(\frac{\nu(1+\frac{q^2}{\nu})^{-(1+\nu)/2}}{\nu-1}\cdot(1+\frac{q^2}{\nu})\right) = \frac{K\cdot(1+\frac{q^2}{\nu})^{-(1+\nu)/2}}{1-\alpha}\left(\frac{\nu\cdot(1+\frac{q^2}{\nu})}{\nu-1}\right)\end{aligned}$$

Thus we have showed that $ES_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\left(\frac{\nu+t_\nu^{-1}(\alpha)^2}{\nu-1}\right)$

Problem 2

a)

We have $L_1, L_2 \sim \mathcal{N}(\mu, \sigma^2)$. Due to L_1 and L_2 being i.i.d we also have that $L_1 + L_2 \sim \mathcal{N}(2\mu, 2\sigma^2)$. Now using example 2.11 we have that;

$$\begin{aligned}VaR_\alpha(L_1+L_2) - (VaR_\alpha(L_1) + VaR_\alpha(L_2)) &= 2\mu + \sqrt{2\sigma^2}\cdot\Phi^{-1}(\alpha) - (2\cdot(\mu+\sigma\cdot\Phi^{-1}(\alpha))) \\ &= \sqrt{2}\cdot\Phi^{-1}(\alpha) - 2\cdot\Phi^{-1}(\alpha) = (\sqrt{2}-2)\cdot\Phi^{-1}(\alpha)\end{aligned}$$

Since $\sqrt{2} - 2 < 0$ we only achieve the desired when we choose an $\alpha < 0.5$, since $\Phi(\alpha)$ is the d.f. of a standard normal distribution, hence for all $\alpha < 0.5 \Rightarrow \Phi^{-1}(\alpha) < 0$.

b)

This example is inspired by sub-problem a) above:
We look at an $\alpha = 0.25$. We let $L_1, L_2 \sim \mathcal{N}(0,1)$ and have a dependence between them such that $\sigma_{1,2} = \text{cov}(L_1, L_2) > 0$. Then we have have that $L_1 + L_2 \sim \mathcal{N}(0+0, 1+1+2\cdot\text{cov}(L_1,L_2)) = \mathcal{N}(0, 2+2\cdot\sigma_{1,2})$ Now note that since L_1 and L_2 are both standard normal, we have;

$$VaR_\alpha(L_1+L_2) - (VaR_\alpha(L_1)+VaR_\alpha(L_2)) = (\sqrt{2}-2)\cdot\Phi^{-1}(\alpha)$$

And since $\alpha < 0.5$ we know that this means that subadditivity fails. For the expected shortfall we can use example 3.8 in HL;

$$ES_{0.25}(L_1 + L_2) = \sqrt{2 + 2\sigma_{1,2}} \frac{\phi(\Phi^{-1}(0.25))}{1 - 0.25}$$

Now we notice that the function ϕ is the density function of a standard normal, hence it is always positive; thus resulting in subadditivity holding when taking riskmeasure to be expected shortfall. It looks like VaR is fragile to some distributions where it is possible to have a negative quantile, like the standard normal.

Problem 3

a)

First thing we are doing is to transform the log-returns into losses, this is done by the transformation;

$$Y = -(\exp(X) - 1) \cdot S_0$$

Where Y is the loss, X is the log return and S_0 is the price of 1 stock at time 0(=100 in our case). To find the empirical sizes of value at risk and espected shortfall, we are using the formulas in HL notes chapter 4.1;

$$VaR_\alpha(F_n) = X_{[n(1-\alpha)]+1,n},$$
$$ES_\alpha(F) = \frac{\sum_{i=1}^{[n(1-\alpha)]+1} x_{k,n}}{[n(1-\alpha)]+1}.$$

Where $X_{i,n}$ denotes the i'th largest observation of n X's. Using the formulas we have calculated that

$$VaR_{0.99}(F_n) = 2.710064, \qquad ES_{0.99}(F) = 3.878986$$

For the confidence intervals, we also use the formulas from HL now chapter 4.2. We find the quantiles of a binomial distribution; Bonomial$(1700, 1-0.99)$. Here we get the quantiles 8, 25; meaning we have the confidence interval of value at risk at

$$\{X_{25,1700}; X_{8,1700}\} = \{2.50; 3.48\}$$

b)

Firstly, we construct our 10-day returns. This is done by starting with an investment of 100 and letting it run for 10 days by running in a for-loop:

$$S_{n+1} = S_n \cdot e^{X_{n+1}}$$

Afterwards we subtract the value of the investment after 10 days from 100 to get the loss. Then we restart the process on the next day with another investment of 100. This is done until we run out of data.
We exclude the day-one row with 0 returns because we want our investment to run through 10 instances of returns. Otherwise we would also have to include a day-one 0 return row every time we restarted the process for another 10-day run to make our 10-day returns equivalent.
By doing the above we obtain 169 independent observations of 10-day losses.
We now follow the same procedure as in problem 3a to calculate $ES_{0.99}$ and $VaR_{0.99}$ with a 95% confidence interval. We compute the quantities as $ES_{0.99} = 10.59$ and $VaR_{0.99} = 9.50$ with a 95% confidence interval $\{6.74; 11.52\}$.
In actuality the confidence interval is quite bad. We look at how good our confidence interval actually is and we see that we only achieve a 79.5% confidence interval;

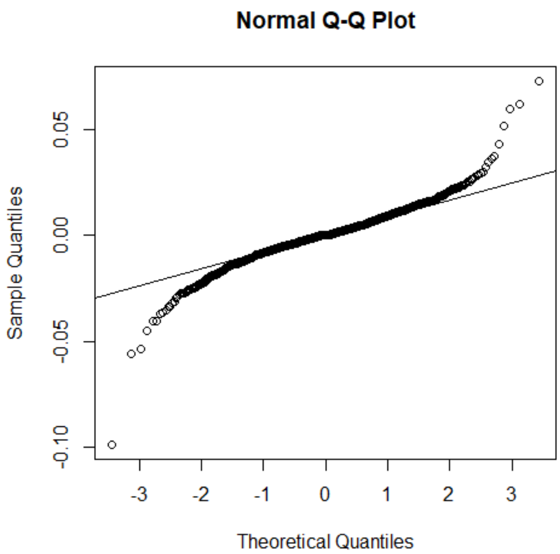
$$P(X_{6,169} < q_{0.99} < X_{1,169}) = P(Y_{0.99} < 5) \cdot P(0 < Y_{0.99}) \approx 0.794$$

Calculation taken from example 4.1 in Hult & Lindskog lecture notes.

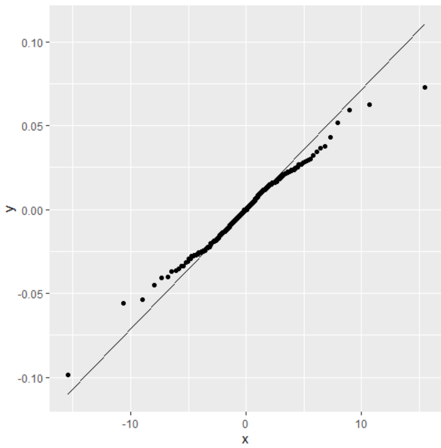
Problem 4

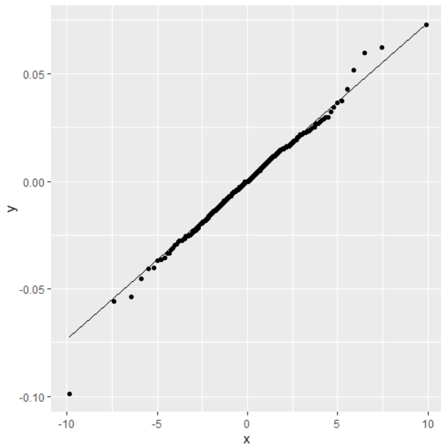
a)

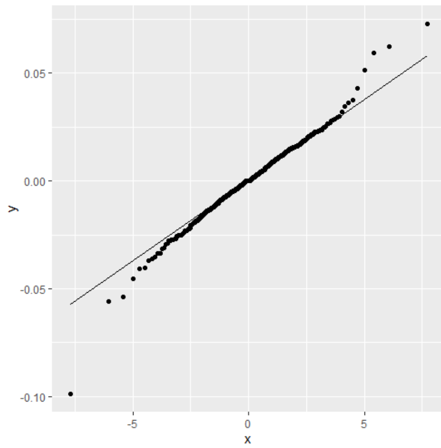
We start out with making the qq-plot where we compare to a normal distribution;



Down below one can see the qq-plots of different t-distribution. We have df=3, 4 and 5 respectively.

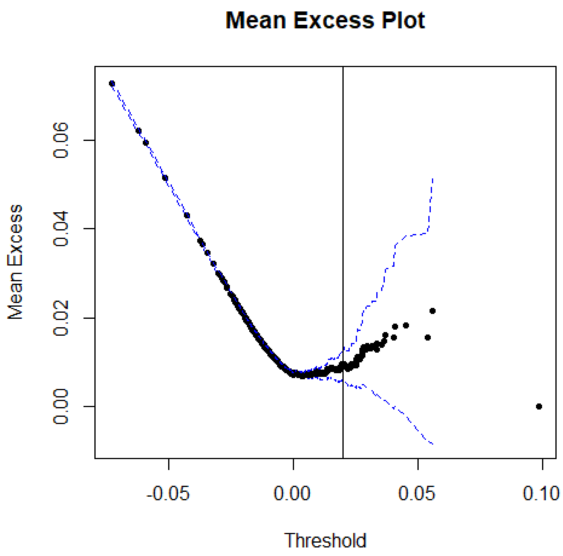


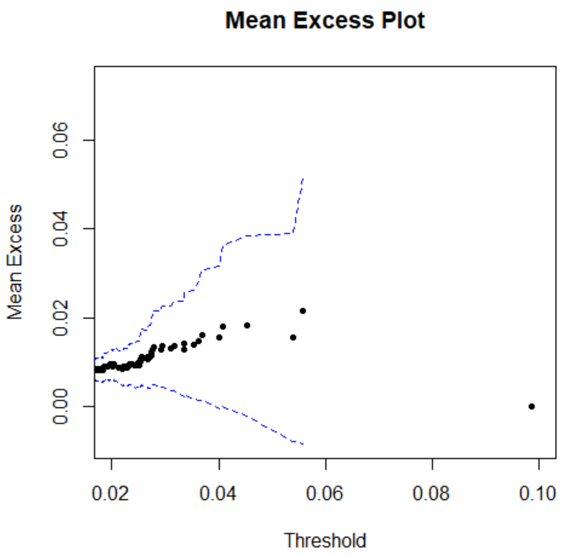




We think that the T-distribution with 4 degrees of freedom seems to fit the best, since this is the distribution that handles the tales (in both ends) the best.

b)





Above we have made the original ME-plot on the left, and zoomed in on the right image to more clearly se that we do have a tendency of some linearity from the threshold u=0.02, which means that at log-return less than 0.02 is 'above' our threshhold. After having choosen the level u=0.02 we are now left with 51 observations.

c)

We use the build in function in R (gpd.fit) to make a fitted GPD model over said observations. Using this we get the scale parameter $\beta = 0.007170337$ and the shape parameter $\gamma = 0.2191917$.

d)

The function we used in the previous question to extract estimates of β and γ uses the method of maximum likelihood. This is nice, since we know that a maximum-likelihood-estimator (MLE) has an assymptotic normal distribution. We therefore use this assymptotic result aswell as a standard 95%-confidence interval for a normally distributed object and get the intervals;

$$\beta : \{0.002824469; 0.011516205\}, \qquad \gamma : \{-0.1154214; 0.5538048\}$$

e)

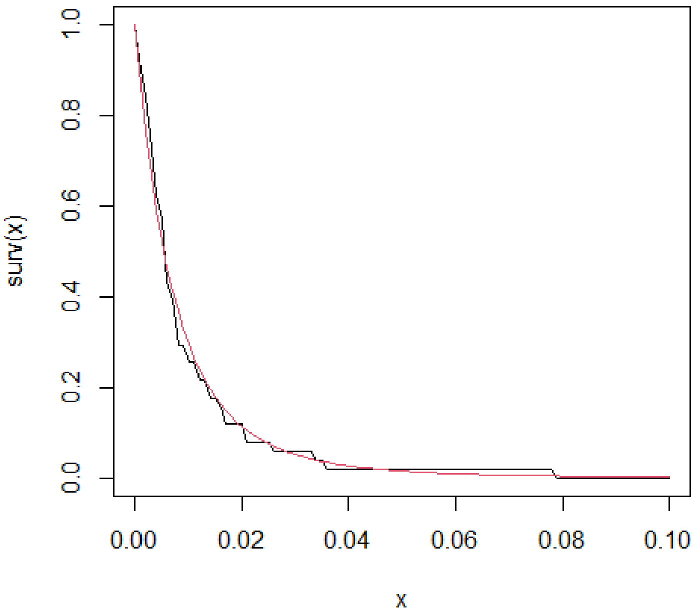
The empirical distribution function is defined as;

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i, n \leq x\}}$$

Hence if we should make an empirical counterpart of our fitted GPD it would be defined as;

$$1 - F_n(u + x) = 1 - \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i, n \leq x+u\}} = 1 - \frac{1}{n}\sum_{i=1}^{n} 1_{\{Y_i, n \leq x\}}$$

where the Y's are to be interpreted as the excess values over the threshold u. We have drawn the survival functions down below.



f)

We use the $\widehat{VaR}_{p,POT}$ as described in Hunt & Lindskog lecture notes ch. 7.4.

$$\widehat{VaR}_{p,POT} = u + \frac{\hat{\beta}}{\hat{\gamma}}\left(\left(\frac{n}{N_u}(1-p)\right)^{-\hat{\gamma}} - 1\right)$$

Plugging in the values we have and the wanted confidence we get;

$$\widehat{VaR}_{0.99,POT} = u + \frac{0.007170337}{0.2191917}\left(\left(\frac{1700}{51}(1-0.99)\right)^{-2191917} - 1\right) = 0.02890683$$

But, the excersise wanted us to compute the VaR for the one day loss! We therefore transform this log return back into a loss and get;

$$\begin{aligned}\widehat{VaR}_{0.99,POT}(\text{log-return}) &= 0.02890683 \Rightarrow \\ \widehat{VaR}_{0.99,POT}(\text{One day loss}) &= -(e^{-\widehat{VaR}_{0.99,POT}(\text{log-return})} - 1) \cdot S_n = 2.849303\end{aligned}$$

Problem 5

a)

Initially, we simulate X 10,000,000 times with the distribution described in the problem using the function *mvrnorm*. We then transform these into losses using

$$L = -\sum_{i=1}^{3} 1000 \cdot \left(e^{X_i} - 1\right)$$

We have now computed 10,000,000 different losses. We can now achieve an estimate of $VaR_{0.9999}$ and its associated confidence interval by following the steps in problem 3a:

$$VaR_{0.9999} = 215.79$$

with a confidence interval $\{214.76; 216.88\}$.

b)

We want to determine the distribution of F_ξ. We do this by MGF.

$$\begin{aligned} MGF_\xi(\alpha) &= E[e^{<\alpha,x>}] = \int e^{<\xi,x>} dF_\xi(x) = C \cdot^{<\xi,x>} e^{<\alpha,x>} dF(x) = C \cdot \int e^{<\xi+\alpha,x>} dF(x) \\ &= C \cdot e^{<\mu,\xi+\alpha>+\frac{1}{2}(\xi+\alpha)^T\Sigma(\xi+\alpha)} = C \cdot e^{<\mu,\xi>} \cdot e^{<\mu,\alpha>+\frac{1}{2}(\alpha^T\Sigma\alpha+\xi^T\Sigma\xi+\alpha^T\Sigma\xi+\xi^T\Sigma\alpha)} \\ &= \underbrace{C \cdot e^{<\mu,\xi>+\frac{1}{2}\xi^T\Sigma\xi}}_{=1} \cdot e^{<\mu,\alpha>+\frac{1}{2}(\alpha^T\Sigma\alpha+<\alpha,\Sigma\xi>+<(\xi^T\Sigma)^T,\alpha>)} \\ &= e^{<\mu,\alpha>+\frac{1}{2}<\Sigma\xi,\alpha>+\frac{1}{2}<(\xi^T\Sigma)^T,\alpha>)+\frac{1}{2}\alpha^T\Sigma\alpha} \\ &= e^{<\mu+\Sigma\xi,\alpha>+\frac{1}{2}\alpha^T\Sigma\alpha} \end{aligned}$$

Last equality follows because $\Sigma = \Sigma^T$. We can infer from this that if $C = e^{-(<\mu,\xi>+\frac{1}{2}\xi^T\Sigma\xi)}$ then $X_\xi \sim \mathcal{N}(\mu + S\xi, S)$. We can now simulate the shifted distribution. By simulating X_ξ we calculate L^ξ_{n+1} and we find $\xi = (-52, -73, -62)^T$ gives that $\{L^\xi_{n+1} \geq VaR_{0.9999}\}$ happens approximately 50% of the time.
Next we find $P(L_{n+1} \geq VaR_{0.9999})$.

$$\begin{aligned} P(L_{n+1} \geq VaR_{0.9999}) &= \int_{(L_{n+1}\geq VaR_{0.9999})} dF(x) = \int_{(L^\xi_{n+1}\geq VaR_{0.9999})} C^{-1}e^{-<\xi,x>} dF_\xi(x) \\ &= \frac{1}{N}\sum_{i=1}^{N} 1_{((L^\xi_i \geq VaR_{0.9999}))} C^{-1} e^{-<\xi,x_i>} \end{aligned}$$

We calculate this probability $1,000$ times with $N = 1,000,000$. This yields us a confidence interval of $(0.0000982; 0.0000994)$. This tells us that the estimate in problem 5a might be slightly too high since the probability is slightly lower than 0.0001.

c)

We made the function in our coding program, then tried a lot of values and found that if we took;

$$\bar{F}^{(N)}(216.41) \approx 0.0001$$

Hence our estimate of $VaR_{0.9999}$ is 216.41.