

Theory

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Random variables

Conditional expectation

Proposition. *(Bjork, B.37.) Let (Ω, \mathcal{F}, P) be a given probability space, let \mathcal{G} be a sub-sigma-algebra of \mathcal{F} , and let X be a square integrable random variable. Consider the problem of minimizing

$$E[(X - Z)^2]$$

where Z is allowed to vary over the class of all square integrable \mathcal{G} measurable random variables. The optimal solution \hat{Z} is then given by.

$$\hat{Z} = E[X|\mathcal{G}].$$

Proof.

Let $X \in L^2(\Omega, \mathcal{F}, P)$ be a random variable. Now consider an arbitrary $Z \in L^2(\Omega, \mathcal{G}, P)$. Recall that $\mathcal{G} \subset \mathcal{F}$ and so X is also in $Z \in L^2(\Omega, \mathcal{G}, P)$, as it is both square integrable and \mathcal{G} -measurable. Then

$$E[Z \cdot (X - E[X|\mathcal{G}])] = E[Z \cdot X] - E[Z \cdot E[X|\mathcal{G}]].$$

Then by using the law of total expectation and secondly that Z is \mathcal{G} -measurable we have that

$$E[Z \cdot X] = E[E[Z \cdot X|\mathcal{G}]] = E[Z \cdot E[X|\mathcal{G}]].$$

Combining the two equations gives the desired result. Obviously, we have that

$$X - Z = X - Z + E[X|\mathcal{G}] - E[X|\mathcal{G}] = (X - E[X|\mathcal{G}]) + (E[X|\mathcal{G}] - Z).$$

Then squaring the terms gives

$$(X - Z)^2 = (X - E[X|\mathcal{G}])^2 + (E[X|\mathcal{G}] - Z)^2 + 2(X - E[X|\mathcal{G}])(E[X|\mathcal{G}] - Z)$$

Taking expectation on each side and using linearity of the expectation we have that

$$E[(X - Z)^2] = E[(X - E[X|\mathcal{G}])^2] + E[(E[X|\mathcal{G}] - Z)^2] + 2E[(X - E[X|\mathcal{G}])(E[X|\mathcal{G}] - Z)].$$

We can now use that $E[X|\mathcal{G}] - Z$ is \mathcal{G} -measurable with the above result on the last term.

$$E[(X - Z)^2] = E[(X - E[X|\mathcal{G}])^2] + E[(E[X|\mathcal{G}] - Z)^2].$$

Now since X is given the term $E[(X - E[X|\mathcal{G}])^2]$ is simply a constant not depending on the choice of Z . The optimal choice of Z is then $E[X|\mathcal{G}]$ since this minimizes the second term. The statement is then proved.

Moment generating function

Let X be a random variable with distribution function $F(x) = P(X \leq x)$ and Y be a random variable with distribution function $G(y) = P(Y \leq y)$.

Definition. The moment generating function or Laplace transform of X is

$$\psi_X(\lambda) = E[e^{\lambda X}] = \int_{-\infty}^{\infty} e^{\lambda x} dF(x)$$

provided the expectation is finite for $|\lambda| < h$ for some $h > 0$.

The MGF uniquely determine the distribution of a random variable, due to the following result.

Theorem. (*Uniqueness*) If $\psi_X(\lambda) = \psi_Y(\lambda)$ when $|\lambda| < h$ for some $h > 0$, then X and Y has the same distribution, that is, $F = G$.

There is also the following result of independence for Moment generating functions.

Theorem. (*Independence*) If

$$E[e^{\lambda_1 X + \lambda_2 Y}] = \psi_X(\lambda_1)\psi_Y(\lambda_2)$$

for $|\lambda_i| < h$ for $i = 1, 2$ for some $h > 0$, then X and Y are independent random variables.

Stochastic processes

Brownian Motion

Definition. (*Bjork, def. 4.1*) A stochastic process W is called a **Brownian motion** or **Wiener process** if the following conditions hold

1. $W_0 = 0$.
2. The process W has independent increments, i.e. if $r < s \leq t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent random variables.
3. For $s < t$ the random variable $W_t - W_s$ has the Gaussian distribution $\mathcal{N}(0, t - s)$.
4. W has continuous trajectories i.e. $s \mapsto W(s; \omega)$ is continuous for all $\omega \in \Omega$.

#Example of trajectory for BM

```
set.seed(1)
t <- 0:1000
N <- rnorm(
  n = length(t)-1, #initial value = 0
  mean = 0, #increments mean = 0
  sd = sqrt(t[2:length(t)] - t[1:(length(t)-1)]) #increment sd = sqrt(t-s)
)
W <- c(0, cumsum(N))
```

Martingale

Definition. Let M_t be a stochastic process defined on a background space (Ω, \mathcal{F}, P) . Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration. If M_t is adapted to the filtration \mathcal{F}_t , $E|M_t| < \infty$ and

$$E[M_t | \mathcal{F}_s] = M_s$$

holds for any $t > s$ we say that M_t is a martingale.

