Homework (FinKont)

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2023-01-19

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Theory

This week revolves around the theory of the Brownian motion and martingale processes. Other main topics are the binomial model and an introduction to financial derivatives. Financial derivatives is contingent on the outcome of a stochastic process at some future time $t = T$ and often is a function Φ of some assets price S_t. As such the derivative will give a stochastic payout, at time $t = T$ of the size $X_T = \Phi(S_T)$. Naturally we want to say something about the *fair* price of the derivative in the form of

$$\Pi_t(X_T) = \mathbb{E}\left[\Phi(S_T) \mid \mathcal{F}_t\right],$$

where $\mathcal{F}_t \subset \mathcal{F}$ is the available information at time t. We will by defualt intepret the times $t = 0$ as *today* and $t = T$ as *tomorrow*. This indeed require some fundamental understanding of the behaviour of the asset price S_t. This lead us over to discussing the process in center of the *Black-Scholes* model: the Brownian motion.

The Brownian motion **Definition 4.1.** *(Brownian motion)* A stochastic process W is called a **Brownian motion** or **Wiener process** if the following conditions hold

1. $W_0 = 0$.
2. The process W has independent increments, i.e. if $r < s \leq t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent random variables.
3. For $s < t$ the random variable $W_t - W_s$ has the Gaussian distribution $\mathcal{N}(0, t - s)$.
4. W has continuous trajectories i.e. $s \mapsto W(s; \omega)$ i continuous for all $\omega \in \Omega$.

As one can see from the simulated sample path on the right, the Brownian motion is rather irratic. In fact, the process varies infinitely on any interval with length greater than 0. This gives some of the characteristics of the process including that: W is continuous and W is non-differential everywhere. This irratic behaviour is summed up in the theorem.

Theorem 4.2. A Brownian motions trajectory $t \mapsto W_t$ is with probability one nowhere differential, and it has locally infinite total variation.

This may seem not that horrifying since we can observe the process at any time and conclude an increment $W_{t+\Delta t} - W_t$ for any $\Delta t > 0$ but any integral constructed with W_t as integrator becomes nonsensible. We will be studying processes on the form

Realisation of a Brownian motion

$$S_{t+\Delta t} - S_t = \mu(t, S_t)\Delta t + \sigma(t, S_t)\Delta W_t, \qquad \Delta W_t = W_{t+\Delta t} - W_t.$$

where W_t is a standard Brownian motion and $\mu(t, S_t)$ is locally deterministic (velocity), that is $\mu(t, S_t)$ is deterministic on a small time interval. One could consider the dynamics of the process S_t by studying the equation below as $\Delta t \to 0_+$

$$\frac{S_{t+\Delta t} - S_t}{\Delta t} = \mu(t, S_t) + \sigma(t, S_t)\frac{W_{t+\Delta t} - W_t}{\Delta t}.$$

The limit however is impossible to determine as W_t is non-differentiable and as such dS_t is not well-defined. From LivStok we know that the dynamics of S_t is given by letting Δt tend to 0 without dividing by it, that is
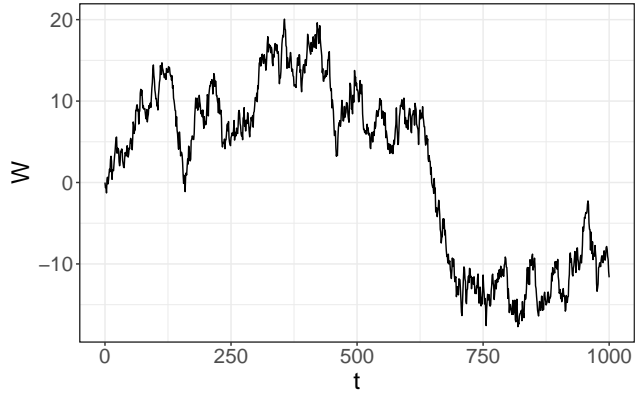
$$dS_t = \mu(t, S_t)\,dt + \sigma(t, S_t)\,dW_t.$$

Giving that S_0 is observable we could intepret the dynamics on the integral form

$$S_t = S_0 + \int_0^t \mu(s, S_s)\,ds + \int_0^t \sigma(s, S_s)\,dW_s,$$

where the above integrals is Riemann-Stieltjes integral. This is however still at dead-end, since from theorem 4.2 we know that W_t has unbounded variation on any interval. So **we cannot define S_t for each W-trajectory seperately** we will despite this define another integral (the Ito integral) that in some other sense give a global solution to this integral. To this we will be considering a L^2-definition.

Conditional expectation The theory of conditional expectation is well-known from courses on the bachelor. Because of this we will only summarise the most important results.

We consider a background space $(\Omega, \mathcal{F}, P)$ and a sub-sigma algebra $\mathcal{G} \subseteq \mathcal{F}$. We assume that some stochastic variable is $\mathcal{F}$-measurable, that is the mapping $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathbb{B}, m)$ is $\mathcal{F} - \mathbb{B}$-measurable i.e. $\forall B \in \mathbb{B} : \{X \in B\} \in \mathcal{F}$. For some random variable Z defined on the subspace $(\Omega, \mathcal{G}, P)$, we say that Z is the conditional expectation of X given $\mathcal{G}$ if

$$\forall G \in \mathcal{G} : \int_G Z(\omega) \, dP(\omega) = \int_G X(\omega) \, dP(\omega).$$

This fact is summed up in the definition below.

Definition B.27. *(Conditional expectation)* Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable in $L^1(\Omega, \mathcal{F}, P)$ ($|X|$ is integrable). Let furthermore $\mathcal{G}$ be a sigma-algebra such that $\mathcal{G} \subseteq \mathcal{F}$. If Z is a random variable with the properties that:

    i. Z is $\mathcal{G}$-measurable.
    ii. For every $G \in \mathcal{G}$ it holds that

$$\int_G Z(\omega) \, dP(\omega) = \int_G X(\omega) \, dP(\omega).$$

Then we say that Z is the *conditional expectation of X given the sigma-algebra $\mathcal{G}$*. In that case we denote Z by the symbol $E[X \mid \mathcal{G}]$.

We see that from the above it always holds that X satisfies (ii). It does not, however, always hold that X is $\mathcal{G}$-measurable. Given this fact it is not trivial that a random variable $E[X \mid \mathcal{G}]$ even exists. This nontriviality is fortunately resolved by the Radon-Nikodym theorem.

Theorem B.28. *(Existance and uniqueness of Conditional expectation)* Let $(\Omega, \mathcal{F}, P)$, X and $\mathcal{G}$ be given as in the definition above. Then the following holds:

- There will always exist a random variable Z satisfying conditions (i)-(ii) above.
- The variable Z is unique, i.e. if both Y and Z satisfy (i)-(ii) then $Y = Z$ P-a.s.

This result ensures that we may condition on any sigma-algebra for instance $\mathcal{G} = \sigma(Y)$ in that case we (pure notation) write

$$E[X \mid \sigma(Y)] = E[X \mid Y], \qquad \sigma(Y) = \sigma\left(\{Y \in A, \ A \in \mathbb{B}\}\right).$$

In the above $\sigma(Y)$ is simply the smallest sigma-algebra containing all the pre-images of Y, that is the smallest sigma-algebra making Y measurable! Giving this foundation there are a few properties conditional expectation have which is rather useful (for instance the tower property).

Below we assume: Let $(\Omega, \mathcal{F}, P)$ be a probability space and X, Y be random variables in $L^1(\Omega, \mathcal{F}, P)$.

Proposition B.29. *(Monotinicity/Linearity of Conditional expectation)* The following holds:

$$X \leq Y \ \Rightarrow \ E[X \mid \mathcal{G}] \leq E[Y \mid \mathcal{G}], \qquad P - \text{a.s.}$$
$$E[\alpha X + \beta Y \mid \mathcal{G}] = \alpha E[X \mid \mathcal{G}] + \beta E[Y \mid \mathcal{G}], \qquad \forall \alpha, \beta \in \mathbb{R}.$$

Proposition B.30. *(Tower property)* Assume that it holds that $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$. Then the following hold:

$$E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}],$$
$$E[X] = E[E[X|\mathcal{G}]].$$

Proposition B.31. Assume X is $\mathcal{G}$ and that both X, Y and XY are in L^1 (only assuming Y is $\mathcal{F}$-measurable), then

$$E[X|\mathcal{G}] = X, \qquad P - \text{a.s.}$$
$$E[XY|\mathcal{G}] = XE[Y|\mathcal{G}], \qquad P - \text{a.s.}$$

Proposition B.32. *(Jensen inequality)* Let $f : \mathbb{R} \to \mathbb{R}$ be a convex (measurable) function and assume $f(X)$ is in L^1. Then

$$f(E[X|\mathcal{G}]) \leq E[f(X)|\mathcal{G}], \qquad P - \text{a.s.}$$

Filtrations Let $(\Omega, \mathcal{F}, P)$ be a probability space. We define a filtration as an increasing familily of sub-sigma-algebras in the following definition.

Definition B.16. *(Filtration)* Let $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ be an time indexed family of sub-sigma-algebras such that $F_s \subseteq F_t$ for $s \leq t$ and $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \geq 0$. We may given this filtration define $\mathcal{F}_\infty$ as $\sigma \left(\bigcup_{t \geq 0} \mathcal{F}_t \right)$.

Filtrations is widely used in stochastic processes, as they allow for the concept of knowledge/information. This is useful when considering mean-values of future states but in an increasing information setting. For this we introduce the term adapted processes.

Definition B.17. *(Adapted process)* Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration on the probability space $(\mathcal{F}_t)_{t \geq 0}$. Furthermore, let $(X_t)_{t \geq 0}$ be a stochastic process on the same space. We say that X_t is adapted to the filtration $\mathbf{F}$ if

$$X_t \quad \text{is} \quad \mathcal{F}_t - \text{measurable}, \qquad \forall t \geq 0.$$

Obviously, we may introduce the **natural filtration** $\mathcal{F}_t^X$ given by the tragetory of the process X_t:

$$\mathcal{F}_t^X = \sigma(\{X_s, \ s \leq t\}).$$

Indeed, X_t is adapted to this filtration.

Martingales **Definition C.1.** Let M_t be a stochastic process defined on a background space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration. If M_t is adapted to the filtration $\mathcal{F}_t$, $E|M_t| < \infty$ and

$$E[M_t|\mathcal{F}_s] = M_s, \qquad P - \text{a.s.}$$

holds for any $t > s$ we say that M_t is a martingale (**F**-martingale). If the above has $\geq$ or $\leq$ we say that M_t is either a **submartingale** or **supermartingale** respectively.

Naturally, this defintions may easily be extended to discrete models and we have the trivial equality:

$$E[M_t - M_s \mid \mathcal{F}_s] = 0.$$

Martingales is useful, when proofing probalistic statements as the posses tractable properties. A useful technique often include the construction of the martingale

$$M_t = E[X \mid \mathcal{F}_t].$$

Discrete time models

One-period time models   The study of this course is the **European call** option (and *put* option). This financial derivative is an agreement between two parties where the holder of the option has the right to *"exercise"* the derivative, at a future time $t = T$. Exercising means buying an asset at a certain agreed opon price-strike K. In the case of the put-option: the holder has the right (but not obligation) to sell the asset at the strike price K. As such the derivative has the payoff

$$\text{Call option:} \quad \Phi(S_T) = (S_T - K)^+, \qquad \text{Put option:} \quad \Phi(S_T) = (K - S_T)^+.$$

Our objective is to understand when an arbitrage exist and to find the fair price of these derivative. The strategy in pricing is finding a replicating portfolio with the same payoff as the option (with probability one) and then price the derivative accordingly.

Model description   In the one-period model we consider the simplest possible market. We have two distinct times $t = 0$ (today) and $t = 1$ (tomorrow) and we may buy any portfolio as a mixture of bonds and one stock. We denote the bonds price by B_t and the stocks price by S_t and we assume the following:

$$B_0 = 1, \ B_1 = 1 + R, \qquad S_0 = s, \ S_1 = \begin{cases} s \cdot u, & \text{with probability } p_u. \\ s \cdot d, & \text{with probability } p_d. \end{cases}$$

We may introduce Z as the random variable

$$Z = u \cdot (I) + d \cdot (1 - I),$$

for an bernoulli variable I with succes probability p_u. Naturally, we assume $d \leq (1+R) \leq u$ (this is imperative to ensure no arbitrage as we will see).

Portfolios and arbirtage   We study any portfolio on the (B, S) market as a vector $h = (x, y)$ where x is the amount of bonds and y is the amount of stock held in the portfolio. Notice that we allow for shorting, that is $x < 0$ or $y < 0$. As such, we have that $h \in \mathbb{R}^2$. In this we have made some unrealistic, but attractable assumptions included in the assumptions:

- We allow short positions and fractional holding, i.e. $h \in \mathbb{R}^2$,
- We assume no spread between ask and bids,
- No transaction costs and
- A completely liquid market i.e. we may borrow and buy as much stock and bonds as wanted.

Given that we have chosen a portfolio h we may introduce the value process.

Definition 2.1. The **value process** of the porfolio $h \in \mathbb{R}^2$ is the stochastic process

$$V_t^h = x B_t + y S_t, \ t = 0, 1.$$

Given this notation we may define what an arbitrage is.

Definition 2.2. An **arbitrage** is a portfolio h with the properties: 1) $V_0^h = 0$, 2) $P(V_1^h \geq 0) = 1$ and 3) $P(V_1^h > 0) > 0$.

That is h is an deterministic money-machine where we at least never loose any money. Granted the bonds give a determinictic non-negative return, but an arbitrage does not require any money out of pocket. With the notion of an arbitrage we will show the first proposition regarding the choice of R, u, d as defined above.

Proposition 2.3. The one-period binomial model is arbitrage free if and only if the following inequality hold:

$$d \leq (1 + R) \leq u. \tag{2.1}$$

Proof.

The statement is proofed by contradiction. Assume that $d > 1 + R$ holds. Then by definition $u > d > 1 + R$. Notice that any portfolio satisfying $V_0^h = 0$ must satisfy

$$0 = xB_0 + yS_0 = x + ys \iff x = -ys$$

That is for some choice y the only arbitrage candidate is the portfolio $h = (-ys, y)$. Calculating the value at time $t = 1$ we have

$$V_1^h = -ys \cdot (1 + R) + y \cdot s \cdot Z = ys(Z - 1 - R)$$

However since $Z \geq d$ we have $Z - (1 + R) \geq 0$ and therefore an arbitrage (for $y > 0$). The other inequality $1 + R > u$ follows analog steps. Simply choose some $y < 0$ and the result follows. $\blacksquare$

From inequality (2.1) we see that since $1 + R$ is between u and d we may find a pair $q_d, q_u \geq 0$ with $q_d + q_u = 1$ such that

$$1 + R = q_u \cdot u + q_d \cdot d.$$

This yields the important risk neutral valuation formula as summed op in the following definition

Definition 2.4. A probability measure Q is called a **martingale meausre** if the following condition holds:

$$S_0 = \frac{1}{1 + R} E^Q[S_1].$$

The above measure Q is the measure $Q(Z = d) = q_d$ and $Q(Z = u) = q_u$ for the binomial model. This does in fact yield the risk neautral valuation formula:

$$\begin{aligned} S_0 &= \frac{1}{1 + R} E^Q[S_1] = \frac{1}{1 + R}(Q(Z = d) \cdot d \cdot s + Q(Z = u) \cdot u \cdot s) \\ &= s\frac{1}{1 + R}(q_d \cdot d + q_u \cdot u) = s, \end{aligned}$$

where we simply use $1 + R = q_d \cdot d + q_u \cdot u$. We call this the risk neautral valuation formula because it in some sense gives an expected discounted value of the future stock price. We end this endavour with reformulating the arbitrage proposition and determining the values of the Q-measure.

Proposition 2.5. The one-period binomial model is arbitrage free if and only if there exists a martingale measure Q.

Proposition 2.6. The one-period binomial model has martingale probabilities given by:

$$\begin{cases} q_u = \frac{(1+R)-d}{u-d}, \\ q_u = \frac{u-(1+R)}{u-d}. \end{cases}$$

Contingent Claims This chapter revolves around the financial derivative and we start by stating the definition of the financial derivative.

Definition 2.7. A **contingent claim** (financial derivative) is *any* stochastic variable X of the form $\Phi(Z)$, where Z is the stochastic varible driving the stock price process.

We may also call the function Φ the **contract function** as it states how the contract is resolved once the stochastic variable Z has been realised. Our objective is now to study, what a buyer of said contract would have to pay at any given time t. We call the fair price of X at time t: $\Pi_t[X]$. As such it is easy to see that the fair price at the time of maturity T is simply the payout X i.e. $\Pi_T[X] = X$. Our strategy is to find a replicating portfolio h and determine the price of said portfolio.

Definition 2.8. A contingent claim X can be **replicated**, or said to be **reachable** if there exist a portfolio h such that

$$V_1^h = X,$$

with probability one. In that case, we say that the portfolio h is a **hedging** portfolio or a **replicationg** portfolio. If all claims can be replicated we say that the market is **complete**.

Our pricing strategy is then to determine the value process of the replicating portfolio and then by the first pricing principle below we say that the price is imply the value of the replicating portfolio.

Pricing principle 1. If a clain X is reachable with replicating portfolio h, then the only reasonable price process for X is given by

$$\Pi_t[X] = V_t^h.$$

Notice, that this assumes that a replicating portfolio exist and even so we have a uniqueness statement to solve. We end this section by writing two important results.

Proposition 2.9. Suppose that a claim X is reachable with replicating portfolio h. Then any price at time $t \geq 0$ of the claim X other than the value process of h will lead to an arbitrage on the extended market (B, S, X).

Proposition 2.10. If the one-period binomial model is free of arbitrage, then it is also complete.

The hedging portfolio in the one-period binomial model is given by the portfolio (x, y) below

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u - d}, \qquad y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u - d}.$$

Risk Neutral Valuation We see that since the one-period model is complete we can price any contingent claim and we see that

$$
\begin{aligned}
\Pi_0[X] &= \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u - d} + s\frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u - d} \\
&= \frac{1}{1+R} \left\{ \frac{u\Phi(d) - d\Phi(u)}{u - d} + (1+R)\frac{\Phi(u) - \Phi(d)}{u - d} \right\} \\
&= \frac{1}{1+R} \left\{ \frac{(1+R) - d}{u - d}\Phi(u) + \frac{u - (1+R)}{u - d}\Phi(d) \right\} \\
&= \frac{1}{1+R} E^Q[X].
\end{aligned}
$$

i.e. the price at time $t = 0$ should simply be the expected discounted payout according to the martingale measure. This leads to the important pricing proposition:

Proposition 2.11. If the one-period binomial model is free of arbitrage, then the arbitrage free price of a contingent claim X is given by

$$\Pi_0[X] = \frac{1}{1+R}E^Q[X].$$ (2.4)

Here the martingale measure Q is uniquely determined by the relation

$$S_0 = \frac{1}{1+R}E^Q[S_1],$$ (2.5)

and the explicit expressions for q_u and q_d are given in proposition 2.6. Furthermore the claim X can be replicated using the portfolio

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u - d},$$ (2.6)

$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u - d}.$$ (2.7)

Multi-period model The one-period binomial model can easily be extended to a multi-period model, by assuming that the bond and stock pricess evolve by the processes:

$$t \geq 1: \ B_t = (1+R)B_{t-1} \qquad \text{and} \qquad B_0 = 1,$$

$$t \geq 1: \ S_t = Z_{t-1}S_{t-1} \qquad \text{and} \qquad S_0 = s,$$

where we obviously have that $B_t = (1+R)^t$ for $t \geq 0$. In the above Z_t is u with probability p_u and d with probability p_d. In this context, we need to define a portfolio in terms of a strategy.

Definition 2.13. A **portfolio strategy** is a stochastic process on $\{1, ..., T\}$

$$h = \{h_t = (x_t, y_t); \ t = 1, ..., T\}$$

such that h_t is a function of $S_0, S_1, ..., S_{t-1}$. For a given portfolio strategy h we set $h_0 = h_1$ by convention. The associated **value process** corresponding to the portfolio h is defined by

$$V_t^h = x_t(1+R) + y_tS_t.$$

Given this notation we may define what an arbitrage is, but first we introduce the notion of a self-financing portfolio. A self-financing portfolio in an intuative sense is a portfolio that is not withdrawn from or deposited into.

Definition 2.14. A portfolio strategy h is said to be **self-financing** if the following condition holds for all $t = 0, ..., T - 1$:

$$x_t(1+R) + y_tS_t = x_{t+1} + y_{t+1}S_t.$$

The above equation says that the portfolio purchased at time t and helt until $t + 1$ (x_{t+1}, y_{t+1}) can only be financed by the market value of the portfolio held from $[t - 1, t)$ i.e. (x_t, y_t). We now define an arbitrage.

Definition 2.15. An **arbitrage** is a self-financing portfolio h with the properties: 1) $V_0^h = 0$, 2) $P(V_T^h \geq 0) = 1$ and 3) $P(V_T^h > 0) > 0$.

The multiperiod binomial model has an just like the oneperiod model a result regarding when an arbitrage exists.

Lemma 2.16. If $d \leq (1 + R) \leq u$ then the multiperiod model is arbitrage-free.

As one can see, the multiperiod model is rather similar to the one period model. We wil in the following summarise equivalent statements for the multiperiod model as the ones in the oneperiod model.

Definition 2.17. The martingale probabilities q_u and q_d are defined as the probabilities for which the relation below holds.

$$s = \frac{1}{1 + R} E^Q[S_{t+1} \mid S_t].$$

Proposition 2.18. The martingale probabilities q_u and q_d are given by

$$\begin{cases} q_u = \frac{(1+R)-d}{u-d}, \\ q_u = \frac{u-(1+R)}{u-d}. \end{cases}$$

Definition 2.19. A **contingent claim** is a stochastic variable X of the form

$$X = \Phi(S_T),$$

where the **contract function Φ** is some given real valued function.

Definition 2.20. A given contingent claim X is said to be **reachable** if there exists a self-financing portfolio h such that

$$V_T^h = X,$$

with probability one. In that case we say that the portfolio h is a **hedging** portfolio or a **replicating** portfolio. If all claims can be replicated we say that the market is *(dynamically)* **complete**.

Pricing principle 2. If a claim X is reachable with replicating portfolio h, then the only reasonable price process for X os given by

$$\Pi_t[X] = V_t^h, \ t = 0, 1, ..., T.$$

Proposition 2.21. Assume X is reachable by h, then any price other than V_t^h for some $t \geq 0$ leads to an arbitrage opportunity.

Proposition 2.22. The multiperiod model is complete, i.e. every claim can be replicated by a self-financing portfolio.

Proposition 2.24. (Binomial algorithm) Consider a T-claim $X = \Phi(S_T)$. Then this claim can be replicated using af self-financing portfolio. If $V_t(k)$ denotes the value of the portfolio at the node (t, k) (k referring to k amount of up-moves for the stock), then $V_t(k)$ can be computed recursively by the scheme

$$\begin{cases} V_t(k) = \frac{1}{1+R} \{q_u V_{t+1}(k+1) + q_d V_{t+1}(k)\}, \\ \qquad V_T(k) = \Phi(su^k d^{T-k}). \end{cases}$$

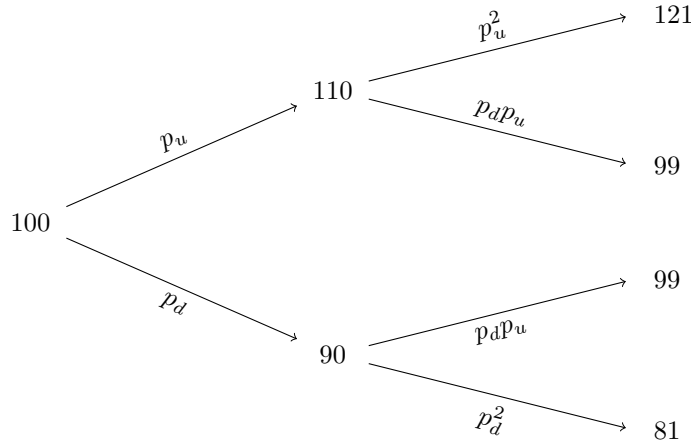where the martingale probabilities q_u and q_d are given by

$$\begin{cases} q_u = \frac{(1+R)-d}{u-d}, \\ q_u = \frac{u-(1+R)}{u-d}. \end{cases}$$

With the notation as above, the hedging portfolio is given by

$$\begin{cases} x_t(k) = \frac{1}{1+R} \cdot \frac{uV_t(k)-dV_t(k+1)}{u-d}, \\ y_t(k) = \frac{1}{S_{t-1}} \cdot \frac{V_t(k+1)-V_t(k)}{u-d}. \end{cases}$$

In particular, the arbitrage free price of the claim at $t = 0$ is given by $V_0(0)$.

Example.

Consider $R = 0.04$, $s = 100$, $u = 1.1$, $d = 0.9$, $p_u = 0.6$ and $p_d = 0.4$. We consider a model of length $T = 2$ and we want to evaluate the price of the european call option with srike $K = 90$ that is the contingent claim

$$X = (S_T - K)^+, \qquad \Phi(s) = (s - K)^+.$$

For each time t we know the replicating portfolio, if we know the payoff the following period. Therefore we start from the leaves of the tree and work towards the root. Since the strike price is $K = 90$ the end result will be the following payoffs:

$$\begin{aligned} u^2 : & \quad (121 - 90)^+ = 31 \\ ud : & \quad (99 - 90)^+ = 9 \\ du : & \quad (99 - 90)^+ = 9 \\ d^2 : & \quad (81 - 90)^+ = 0 \end{aligned}$$

Therefore by the risk neautral valuation formula with $q_u = \frac{(1+R)-d}{u-d} = 0.7$ and $q_d = \frac{u-(1+R)}{u-d} = 0.3$ we have that the cost of the replicating portfolio at time $t = 1$ is respectively

$$\begin{aligned} u : & \quad \frac{1}{1+R} \{31 \cdot q_u + 9 \cdot q_d\} \approx 23.46 \\ d : & \quad \frac{1}{1+R} \{9 \cdot q_u + 0 \cdot q_d\} \approx 6.06 \end{aligned}$$

To replicate this payoff at time $t = 1$ we can use the risk neutral valuation formula once more to find the base cost of the replicating portfolio i.e. the price of X at time $t = 0$

$$\frac{1}{1+R}\{23.46 \cdot q_u + 6.06 \cdot q_d\} \approx 17.54.$$

Working from the root to the leaves we can now calculate the hedging portfolio at time $t = 0, 1$ for each path. For time $t = 0$ we calculate

$$x = \frac{1}{1+R} \cdot \frac{u \cdot 6.06 - d \cdot 23.46}{u - d} \approx -69.46,$$
$$y = \frac{1}{s} \cdot \frac{23.46 - 6.06}{u - d} \approx 0.87$$

We see by calculations that this does indeed replicate the payoff at time $t = 1$:

$$u: \quad V_1^h = (1+R) \cdot x + 110 \cdot y \approx 23.46,$$
$$d: \quad V_1^h = (1+R) \cdot x + 90 \cdot y \approx 6.06.$$

We also see by calculation that the initial portfolio does cost the expected 17.54 as

$$x \cdot 1 + y \cdot 100 = 87 - 69.46 = 17.54.$$

Following these steps at time $t = 1$ the portfolios $(-86.54, 1)$ (for the up-scenario) and $(-38.94, 0.5)$ (for the down-scenario) would arise. Notice when calculating y one has to use the current price $S_1 = S_0 \cdot Z$ not S_0. One should also check by similar calculations as above, that these portfolios does indeed replicate the payoff of the contingent claim X. $\square$

Proposition 2.25. The arbitrage free price at $t = 0$ of a T-claim X is given by

$$\Pi_0[X] = \frac{1}{(1+R)^T} E^Q[X]$$

where Q denotes the martingale measure, or more explicitly

$$\Pi_0[X] = \frac{1}{(1+R)^T} \sum_{k=0}^{T} \binom{T}{k} q_u^k q_d^{T-k} \Phi(su^k d^{T-k}).$$

Example.

We follow an analog example as the one after proposition 2.24. Let $K = 90$ and we see that

$$\Pi_0[X]$$
$$= \frac{1}{(1+0.04)^2} \sum_{k=0}^{2} \binom{2}{k} \cdot 0.7^k \cdot 0.3^{2-k} \cdot \Phi(100 \cdot 1.1^k \cdot 0.9^{2-k})$$
$$= 0.9245562 \cdot \left(\underbrace{1 \cdot 1 \cdot 0.09 \cdot 0}_{k=0} + \underbrace{2 \cdot 0.7 \cdot 0.3 \cdot 9}_{k=1} + \underbrace{1 \cdot 0.49 \cdot 1 \cdot 31}_{k=2} \right)$$
$$= 0.9245562 \cdot (0 + 3.78 + 15.19)$$
$$= 17.53883$$

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Since we know that K must meaningfully range in $[0, 121]$ we could try to calculate the price of the contingent claim at time $t = 0$ for all integers in this interval. We see that the price range between S_0 and 0 as expected. One can also see that the price changes slope at the prices 99 and 121 as the function is linear in Φ and som realisations loose any effect on the price when the strike is higher than the outcome. $\square$

Proposition 2.26. The condition $d < (1 + R) < u$ is necessary and sufficient condition for absence of arbitrage.

Price of european call option
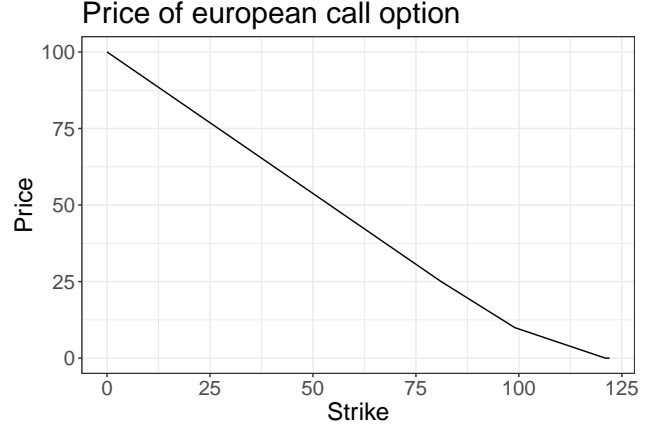
Generelised one-period model In the previous we had the simpel model where we only had one stochastic asset S and only one stochastic variable Z determining the future stock price. Now we will generelise this model by introducing N assets and introducing som stochastic behaviour to the system.

Model specification We consider the market consisting of a collection of stochastic prices assets $i = 1, ..., N$ with N-dimensional price process.

$$
S_t = \begin{bmatrix} S_t^1 \\ \vdots \\ S_t^N \end{bmatrix}
$$

We now assume that S_t is defined on a background space with finite sample space $\Omega = \{\omega_1, ..., \omega_M\}$ with associated probabilities $p_j = P(\omega_j)$, $j = 1, ..., M$. We can then for eact time $t = 1, ..., T$ define the $N \times M$ matrix D_t as such

$$
D_t = \begin{bmatrix} S_t^1(\omega_1) & \cdots & S_t^1(\omega_M) \\ \vdots & \ddots & \vdots \\ S_t^N(\omega_1) & \cdots & S_t^M(\omega_M) \end{bmatrix}.
$$

We will assume that $S_0^1 > 0$ and $S_1^1(\omega_j) > 0$, $j = 1, ..., M$.

Absence of Arbitrage We now define a **portfolio** as an N-dimensional row vector

$$
h = \begin{bmatrix} h^1, \ldots, h^N \end{bmatrix}
$$

representing the amount of assets held at time $t = 0$ and held until $t = 1$. The **value process** is then

$$
V_t^h = h \cdot S_t = \sum_{i=1}^{N} h^i S_t^i, \ t = 0, 1.
$$

For a given $\omega_j \in \Omega$ we have the realisation

$$
V_t^h = h S_t(\omega_j) = h d_j = (hD)_j.
$$

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Definition 3.1. The portfolio h is an **arbitrage portfolio** fil it satisfies the conditions: $V_0^h = 0$, $P(V_1^h \geq 0) = 1$ and $P(V_1^h > 0) > 0$.

Lemma 3.2. (Farkas' Lemma) Suppose that $d_0, d_1, ..., d_M$ are column vectors in $\mathbb{R}^N$. Then exactly one of the following problems possesses a solution.

- **Problem 1**: There exist $\lambda_1, ..., \lambda_M \geq 0$ such that $d_0 = \sum_{j=1}^M \lambda_j d_j$.
- **Problem 2**: There exist $h \in \mathbb{R}^N$ such that $h^\top d_0 < 0$ and $h^\top d_j \geq 0$ for $j = 1, ..., M$.

We now investegate this system for any possible arbitrage portfolios. However first we acknowledge that there exist a nominal price system S_t and a normalised price system Z_t. The latter we define as the nominel pricess under the numeraire S_t^1 that is

$$Z_t = \begin{bmatrix} S_t^1/S_t^1 \\ S_t^2/S_t^1 \\ \vdots \\ S_t^N/S_t^1 \end{bmatrix} = \begin{bmatrix} 1 \\ S_t^2/S_t^1 \\ \vdots \\ S_t^N/S_t^1 \end{bmatrix}.$$

The reason for introducing the normalized price system is that we can without much effort translate results in this system to the nominal system and the normalised system is easier to analize. For this, however, we need af few results.

Lemma 3.3. With notation as above, the following hold.

1. The Z_t value process i related to the S_t value process by

$$V_t^{h,Z} = hZ_t = \frac{1}{S_t^1} V_t^h.$$

2. A portfolio is an arbitrage in the S_t system if and only if there is an arbitrage in the Z_t system.
3. In the Z_t price system, the numeraie asset Z^1 has unit constant prices i.e. $Z_t^1 = 1$ for all $t \geq 0$.

One of the reason that the normalised system is attractable is that the numeraire asset is constant i.e. risk free in the normalised system. Let us formulate our first main result.

Proposition 3.4. The market is arbitrage free if and only iff there exists strictly positive real numbers $q_1, ..., q_M \geq 0$ with $q_1 + \cdots + q_M = 1$ (probability vector) such that the following vector equality holds

$$\begin{bmatrix} Z_0^1 \\ \vdots \\ Z_N^1 \end{bmatrix} = \begin{bmatrix} Z_1^1(\omega_1) \\ \vdots \\ Z_1^N(\omega_1) \end{bmatrix} q_1 + \cdots + \begin{bmatrix} Z_1^1(\omega_M) \\ \vdots \\ Z_1^N(\omega_M) \end{bmatrix} q_M. \tag{3.3}$$

Proof.

Martingale Measures **Definition 3.5.** Given the objective probability measure P on $(\Omega, \mathcal{F}, P)$, we say that another probability measure Q defined on Ω is **equivalent** to P if

$$\forall A \in \mathcal{F} : P(A) = 0 \iff Q(A) = 0,$$

or equivalently

$$\forall A \in \mathcal{F} : P(A) = 1 \iff Q(A) = 1.$$

Definition 3.7. Consider the market model above and set S^1 as the numeraire asset. We say that a probability measure Q defined on Ω is a **martingale measure** if it satisfies the following conditions:

1. Q is equivalent to P, i.e. $Q \sim P$.
2. For every $i = 1, ..., N$, the normalized asset price process

$$Z_t^i = \frac{S_t^i}{S_t^1},$$

is martingale under the measure Q.

Theorem 3.8. (First Fundamental Theorem) Given a fixed numeraire, ther market is free of arbitrage possibilities if and only if there exists a martingale measure Q.

By assuming that the numeraire asset is risk free (i.e. does not depend on ω) then by scaling we can derive the short interest rate as

$$1 + R = \frac{S_1^1}{S_0^1}.$$

With this in mind we can formulate theorem 3.8 in its more widely used form.

Theorem 3.9. (First Fundamental Theorem) Assume that there exist a risk free asset, and denote the corresponding risk free interest rate by R. Then the market is arbitrage free if and only if there exist a measure $Q \sim P$ such that

$$S_0^i = \frac{1}{1 + R} E^Q[S_1^i], \qquad \text{for all } i = 1, ..., N. \tag{3.9}$$

Martingale Pricing   Moving forward we will assume that there exist a risk free asset and we will denote it by B_t $(B_t = S_t^1/S_0^1)$.

Definition 3.10. A **contingent claim** is any random variable X, defined on the sample space Ω.

To ensure no arbitrage in the extended market containing the N assets and the contingent claim we can apply the first fundamental pricing theorem on the extended market.

Proposition 3.11. Consider a given claim X. In order to avoid arbitrage, X must then be priced according to the formula

$$\Pi_0[X] = \frac{1}{1 + R} E^Q[X], \tag{3.10}$$

where Q is a martingale measure for the underlying market $(\Pi, S^1, ..., S^N)$.

Completeness   Given that a market is arbitrage-free we may run into a uniqueness issue when determining the price of a contingent claim. If a martingale measure exist we will very much like it to be unique as this will ensure that the price from the risk neutral valuation formula is unique. To this we need the market to be complete.

Definition 3.12. Consider a contingent claim X. If there exists a portfolio h, based on the underlying assets, such that

$$V_1^h = X, \text{ with probability } 1 \tag{3.11}$$

i.e.

$$V_1^h(\omega_j) = X(\omega_j), \ j = 1, ..., M, \tag{3.12}$$

then we say that X is **replicated**, or **hedged** by h. Such a portfolio h is called a replicating, or hedging portfolio. If every contingent claim can be replicated, we say that the market is **complete**.

We can now formulate a proposition on when the market is complete in terms of the matrix D.

Proposition 3.13. The market is complete if and only if the rows of the matrix D span $\mathbb{R}^M$, i.e. if and only if D has rank M.

Now we formulate the second fundamental pricing theorem in terms of the martingale measure Q.

Proposition 3.14. (Second Fundamental Theorem) Assume that the model is arbitrage free i.e. Q exist. Then the market is unique if and only if the martingale measure is unique.

Stochastic Discount Factors Definition 3.16. The random variable L on Ω is defined by

$$L(\omega_i) = \frac{q_i}{p_i}, \qquad i = 1, ..., M.$$

Definition 3.17. Assume the absence of arbitrage, and fix a martingale measure Q. With notation as above, the **stochastic discount factor** (or "state price deflator") is the random variable Λ on Ω by

$$\mathbf{M}(\omega) = \frac{1}{1+R} \cdot L(\omega). \tag{3.19}$$

Proposition 3.18. The arbitrage free price of any claim X is given by the formula

$$\Pi_0[X] = E^P[\mathbf{M} \cdot X]$$

where $\mathbf{M}$ is a stochastic discount factor.

Exercises

Probability exercises

Let $(W(t))_{t\geq}$ be a Brownian motion (Bjork, Definition 4.1).

Exercise 1. Show that the following processes also are Brownian motions.

    i. $(-W(t))_{t\geq0}$ (symmetry)
   ii. For any $s \geq 0$, $(W(t+s) - W(s))_{t\geq0}$ (time-homogeneity).
  iii. For every $c > 0$, $(cW(t/c^2))_{t\geq0}$ (scaling).

Solution (i).

By assumption W is a Brownian motion and so it follows that

$$-W_0 = -1 \cdot 0 = 0$$

Furthermore, for $r < s \leq t < u$ it holds that $W_u - W_t$ and $W_s - W_r$ is independent. By seperate transformations the independence property is preserved and $-(W_u - W_t)$ and $-(W_s - W_r)$ is independent. Next, for a normal distributed random variable $N \sim \mathcal{N}(\mu, \sigma^2)$ it holds, that for a scaler $c \in \mathbb{R}$ we have $cN \sim \mathcal{N}(c\mu, c^2\sigma^2)$. Then obviously;

$$-(W_t) = (-1)W_t \overset{d}{=} \mathcal{N}((-1) \cdot 0, (-1)^2(t-s)) \overset{d}{=} \mathcal{N}(0, t-s).$$

Lastly, let $\omega \in \Omega$ and consider the sample path $s \mapsto (-W_s)(\omega)$. Clearly for two continuous functions f and g it holds that $(g \circ f)$ is continuous. Then with $g(f) = -f$ and $f(t) = W_t(\omega)"/ >$ it follows that $(-W_t) = (g \circ W)(t)$ is also continuous.

Solution (ii).

Much like the previous exercise we define a new process and show the properties hold. Let $s \geq 0$ be chosen arbitrary. Now define $X_t = W(t+s) - W(s)$.

First, we let $t = 0$ and see

$$X_0 = W(0+s) - W(s) = W(s) - W(s) = 0.$$

Secondly, we have that for $r < u$:

$$X_u - X_r = W(u+s) - W(s) - (W(r+s) - W(s)) = W(u+s) - W(r+s) \sim \mathcal{N}(0, u+s-(r+s)) = \mathcal{N}(0, u-r).$$

and since for $r < u \leq k < l$ the translation $r + s < u + s \leq k + s < l + s$ still holds and $X_l - X_k = W(l+s) - W(k+s)$ and $X_u - X_r = W(u+s) - W(k+s)$ are independent. Finally since $W_t(\omega)$ is continuous in t hence the translation W_{t+s} is continouos. Adding a constant yields a function that is also continuous, hence X_t is continuous.

Solution (iii).

Let $c > 0$ be given. We show that

$$X_t = cW\left(\frac{t}{c^2}\right)$$

is a Brownian motion. We simply show the four properties. Let $t = 0$ and notice

$$X_0 = cW\left(\frac{0}{c^2}\right) = cW(0) = 0.$$

The second property follows from seperate transformation and that for $r < u \leq s < t$ we consider

$$X_u - X_r = c\left(W\left(\frac{u}{c^2}\right) - W\left(\frac{r}{c^2}\right)\right) \qquad \text{and} \qquad X_t - X_s = c\left(W\left(\frac{t}{c^2}\right) - W\left(\frac{s}{c^2}\right)\right)$$

and since $c, r, u, t, s > 0$ we have the same order for the scaled version of r, u, t, s and hence we have two independent RV scaled by c. Then by seperate transformations the variables is still independent. Next for the third property:

$$X_t - X_s = c\left(W\left(\frac{t}{c^2}\right) - W\left(\frac{s}{c^2}\right)\right) \sim \mathcal{N}\left(c \cdot 0, c^2\left(\frac{t}{c^2} - \frac{s}{c^2}\right)\right) = \mathcal{N}(0, t-s).$$

Where we use the properties of scaling a normal distributed random variable i.e. for $c > 0$ and $N \sim \mathcal{N}(\mu, \sigma^2)$ it follows that $cN \sim \mathcal{N}(c\mu, c^2\sigma^2)$. Finally, the forth property follows since $g(f) = cf$ is continuous and $h(t) = t/c^2$ is continuous, then for any continuous function $f(s)$ it follows that $(g \circ f \circ h) = g(f(h(t)))$ is continuous.

Proposition B.37. Let $(\Omega, \mathcal{F}, P)$ be a given probability space, let $\mathcal{G}$ be a sub-sigma-algebra of $\mathcal{F}$, and let X be a square integrable random variable. Consider the problem of minimizing

$$E\left[(X - Z)^2\right]$$

where Z is allowed to vary over the class of all square integrable $\mathcal{G}$ measurable random variables. The optimal solution $\hat{Z}$ is then given by.

$$\hat{Z} = E[X|\mathcal{G}].$$

Exercise 2. *(Bjork, exercise B.11.)* Prove proposition B.37 by going along the following lines.

a. Prove that the "estimation error" $X - E[X|\mathcal{G}]$ is orthogonal to $L^2(\Omega, \mathcal{G}, P)$ in the sence that for any $Z \in L^2(\Omega, \mathcal{G}, P)$ we have

$$E[Z \cdot (X - E[X|\mathcal{G}])] = 0$$

b. Now prove the proposition by writing

$$X - Z = (X - E[X|\mathcal{G}]) + (E[X|\mathcal{G}] - Z)$$

and use the result just proved.

Solution (a).

Let $X \in L^2(\Omega, \mathcal{F}, P)$ be a random variable. Now consider an arbitrary $Z \in L^2(\Omega, \mathcal{G}, P)$. Recall that $\mathcal{G} \subset \mathcal{F}$ and so X is also in $Z \in L^2(\Omega, \mathcal{G}, P)$, as it is bothe square integrable and $\mathcal{G}$-measurable. Then

$$E\left[Z \cdot (X - E[X|\mathcal{G}])\right] = E\left[Z \cdot X\right] - E\left[Z \cdot E[X|\mathcal{G}]\right].$$

Then by using the law of total expectation and secondly that Z is $\mathcal{G}$-measurable we have that

$$E\left[Z \cdot X\right] = E\left[E[Z \cdot X|\mathcal{G}]\right] = E\left[Z \cdot E[X|\mathcal{G}]\right].$$

Combining the two equations gives the desired result.

Solution (b).

Obviously, we have that

$$X - Z = X - Z + E[X|\mathcal{G}] - E[X|\mathcal{G}] = (X - E[X|\mathcal{G}]) + (E[X|\mathcal{G}] - Z).$$

Then squaring the terms gives

$$(X - Z)^2 = (X - E[X|\mathcal{G}])^2 + (E[X|\mathcal{G}] - Z)^2 + 2(X - E[X|\mathcal{G}])(E[X|\mathcal{G}] - Z)$$

Taking expectation on each side and using linearity of the expectation we have that

$$E[(X - Z)^2] = E\left[(X - E[X|\mathcal{G}])^2\right] + E\left[(E[X|\mathcal{G}] - Z)^2\right] + 2E\left[(X - E[X|\mathcal{G}])(E[X|\mathcal{G}] - Z)\right].$$

We can now use that $E[X|\mathcal{G}] - Z$ is $\mathcal{G}$-measurable with the above result on the last term.

$$E[(X - Z)^2] = E\left[(X - E[X|\mathcal{G}])^2\right] + E\left[(E[X|\mathcal{G}] - Z)^2\right].$$

Now since X is given the term $E\left[(X - E[X|\mathcal{G}])^2\right]$ is simply a constant not depending on the choice og Z. The optimal choice of Z is then $E[X|\mathcal{G}]$ since this minimizes the second term. The statement is then proved.

Exercise 3. Discuss the following theory/results of Moment generating functions (Laplace transform).

Let X be a random variable with distribution function $F(x) = P(X \leq x)$ and Y be a random variable with distribution function $G(y) = P(Y \leq y)$.

Definition. The moment generating function or Laplace transform of X is

$$\psi_X(\lambda) = E\left[e^{\lambda X}\right] = \int_{-\infty}^{\infty} e^{\lambda x} dF(x)$$

provided the expectation is finite for $|\lambda| < h$ for some $h > 0$.

The MGF uniquely determine the distribution of a random variable, due to the following result.

Theorem 1. *(Uniqueness)* If $\psi_X(\lambda) = \psi_Y(\lambda)$ when $|\lambda| < h$ for some $h > 0$, then X and Y has the same distribution, that is, $F = G$.

There is also the following result of independence for Moment generating functions.

Theorem 1. *(Independence)* If

$$E\left[e^{\lambda_1 X + \lambda_2 Y}\right] = \psi_X(\lambda_1)\psi_Y(\lambda_2)$$

for $|\lambda_i| < h$ for $i = 1, 2$ for some $h > 0$, then X and Y are independent random variables.

Example. Recall that the Moment generating function of a normal (Gaussian) distribution is given by

$$\psi_X(\lambda) = E\left[e^{\lambda X}\right] = \exp\left(\lambda \mu + \frac{\lambda^2}{2}\sigma^2\right)$$

where X is normally distributed with mean μ and variance σ^2 and $\lambda \in \mathbb{R}$ is a constant. Since a Brownian motion $W(t)$ is normally distributed with zero mean and variance t, we have that

$$E[\exp(\lambda W(t))] = \exp\left(\frac{\lambda^2}{2}t\right).$$

Discussion.

Exercise 4. *(Bjork, exercise C.8.(a-c))* Let W be a Brownian motion. Notice that for the natural filtration $\mathcal{F}_s = \sigma(W_t | t \leq s)$ $W_t - W_s$ is independent of $\mathcal{F}_s$

    a. Show that W_t is a martingale.
    b. Show that $W_t^2 - t$ is a martingale.
    c. Show that $\exp(\lambda W_t - \frac{\lambda^2}{2}t)$ is a martingale.

Solution (a).

We show that for the natural filtration that W_t is a martingale. This include showing integrability and the martingale property. For the first we note that for a normal distributed random variable with mean 0 we have

$$E[|N|] = \int_{-\infty}^{\infty} |x| dF_N(x) = 2\int_{0}^{\infty} x dF_N(x)$$

since the distribution is symmetric. Substituting the distribution function $\Phi(x) = P(N \leq x)$ in we see that

$$E[|N|] = 2 \int_0^\infty x d\Phi(x) = 2 \int_0^\infty x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)} dx = (*)$$

by substituting $u = x^2/(2\sigma^2)$ $(x = \sqrt{2\sigma^2 u})$ we have that

$$\frac{dx}{du} = \frac{1}{2}\sqrt{2\sigma^2 u}2\sigma^2 = (\sigma^2)^{3/2}\sqrt{2}u \iff dx = (\sigma^2)^{3/2}\sqrt{2}u \; du$$

hence

$$(*) = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^\infty \sqrt{2\sigma^2 u}e^{-u}(\sigma^2)^{3/2}\sqrt{2}u \; du = \frac{2\sqrt{2\sigma^2}(\sigma^2)^{3/2}\sqrt{2}}{\sqrt{2\pi\sigma^2}} \int_0^\infty \sqrt{u}e^{-u}u \; du.$$

This then simplify to

$$(*) = \frac{(2\sigma^2)^{3/2}}{\sqrt{\pi}} \int_0^\infty u^{3/2}e^{-u} \; du = (2\sigma^2)^{1/2}\sqrt{\frac{2\sigma^2}{\pi}} \int_0^\infty u^{3/2}e^{-u} \; du = \sqrt{\frac{2\sigma^2}{\pi}} < \infty.$$

(Obviously the above is not derived correctly, but the end expression is valid, source: link) However since

$$W_t = W_t - 0 = W_t - W_0 \sim \mathcal{N}(0, t)$$

we have that $E|W_t| < \infty$ as desired.

Next, we have that

$$E[W_t|\mathcal{F}_s] = E[W_t - W_s|\mathcal{F}_s] + W_s = 0 + W_s = W_s.$$

In the above we used that $W_t - W_s$ is $\mathcal{F}_s$-measurable with mean 0. Then it follows that W_t is a martingale.

Solution (b).

Let $M_t = W_t^2 - t$. First, we observe that two measurable functions composed is still a measurable function. Hence we know that M_t is measurable wrt. the filtration since W_t is measurable and $w \mapsto w^2 + t$ is measurable. Secondly, we have that

$$E[|W_t^2 - t|] \leq E|W_t^2| + E|t| = t + t = 2t < \infty$$

where we use the triangle inequality. Thirdly, for the martingale property we have that for $t > s$:

$$E[M_t|\mathcal{F}_s] = E[W_t^2 - t|\mathcal{F}_s] = E[W_t^2 + W_s^2 - 2W_tW_s - W_s^2 + 2W_tW_s - t|\mathcal{F}_s]$$

which by linearity and independence of increments to the filtration gives

$$E[M_t|\mathcal{F}_s] = E[(W_t - W_s)^2 - W_s^2 + 2W_tW_s - t|\mathcal{F}_s] = t - s - t + E[2W_tW_s - W_s^2|\mathcal{F}_s]$$

However since W_s is measurable wrt. the filtration at time s the above is

$$E[M_t|\mathcal{F}_s] = 2W_sE[W_t|\mathcal{F}_s] - W_s^2 - s = 2W_s^2 - W_s^2 - s = W_s^2 - s = M_s.$$

Since from (a) we know that W_t is a martingale. Then we arrive at the desired result.

Solution (c).

Let $M_t = \exp\left(\lambda W_t - \frac{\lambda^2}{2}t\right)$. First, by composition of measurable functions M_t is $\mathcal{F}_t$-measurable. Secondly, we have using the MGF for a normal distributed random variable:

$$E|M_t = E\left(\exp\left(\lambda W_t - \frac{\lambda^2}{2}t\right)\right) \leq E\left(\exp\left(\lambda W_t\right)\right) = \exp\left(\frac{\lambda^2}{2}t\right) < \infty.$$

Thirdly, we consider

$$E[M_t|\mathcal{F}_s] = E\left[\left(\exp\left(\lambda W_t - \frac{\lambda^2}{2}t\right)\right)\middle|\mathcal{F}_s\right] = \exp\left(-\frac{\lambda^2}{2}t\right) E\left[\left(\exp\left(\lambda W_t\right)\right)|\mathcal{F}_s\right].$$

By adding and subtracting W_s in the exponent we get

$$E[M_t|\mathcal{F}_s] = \exp\left(-\frac{\lambda^2}{2}t\right) E\left[\left(\exp\left(\lambda(W_t - W_s) + \lambda W_s\right)\right)|\mathcal{F}_s\right]$$

$$= \exp\left(-\frac{\lambda^2}{2}t\right) \exp\left(\frac{\lambda^2}{2}(t-s)\right) E\left[\left(\exp\left(\lambda W_s\right)\right)|\mathcal{F}_s\right].$$

Using that $E\left[\left(\exp\left(\lambda W_s\right)\right)|\mathcal{F}_s\right] = \exp\left(\lambda W_s\right)$ and combining the exponents gives the desired:

$$E[M_t|\mathcal{F}_s] = \exp\left(\lambda W_s - \frac{\lambda^2}{2}s\right) = M_s.$$

Week 2

Material

- Stochastic integrals and Ito formula (Chapter 4 and Appendix C.2)
- Stochastic differential equations (Chapter 5.1-4)

Theory

Stochastic Integrals We want to formulate financial markets in continuous time and the most elegant theory is obtained from processes that can be defined in terms of **stochastic differential equations** or in other words by their dynamics. We may call them **diffusion processes**, as they may be approximated by a stochastic difference equation:

$$X_{t+\Delta t} - X_t = \mu(t, X_t)\Delta t + \sigma(t, X_t)Z_t. \tag{4.1}$$

Above Z_t is a normally distributed random variable (a disturbance). In this formulation we say that S_t is driven by two forces: on one hand a locally deterministic velocity or drift $\mu(t, X_t)$ and on the other hand a Gaussian term amplified by the deterministic factor $\sigma(t, X_t)$.

Information We consider a primary process X_t and we introduce the notion of information generated by X_t in terms of the natural filtration. The idea can be summed up in the following definition.

Definition 4.3. The symbol $\mathcal{F}_t^X \subseteq \mathcal{F}$ denotes "the information generated by X_t on the interval $[0, t]$", or alternatively "what has happened to X_t over ther interval $[0, t]$".

1. If, based upon observations of the trajectory $\{X_s;\ 0 \le s \le t\}$, it is possible to decide whether a given event A has occurred or not, then we write this as

$$A \in \mathcal{F}_t^X$$

   or say that "A is $\mathcal{F}_t^X$-measurable".
2. If the value of a given random variable Z can be completely determined given observations of the tragectory $\{X_s;\ 0 \le s \le t\}$, then we also write

$$Z \in \mathcal{F}_t^X.\ (Z \text{ is } \mathcal{F}_t^X\text{-measurable})$$

3. If Y is a stochastic process such that we have

$$Y_t \in \mathcal{F}_t^X$$

   for all $t \ge 0$ then we say that Y_t is **adapted** to the **filtration** $\{\mathcal{F}_t^X\}_{t \ge 0}$. For brevity of notation, we will sometimes write the filtration as $\{\mathcal{F}_t^X\}_{t \ge 0} = \mathbf{F}$.

Stochastic Integrals We will now formulate the theory of stochastic integrals, that is, processes written in terms of stochastic processes with stochastic integrator and/or stochastic integrant. We will consider some given standard Brownian motion W_t and another stochastic process X_t. We need som integrability condition on X_t in order to do the calculations. We therefore determine a selection of suitable stochastic processes X must be contained in.

Definition 4.4. Let X_t be a stochastic process, then

   i. We say that X_t belongs to the class $\mathcal{L}^2[a,b]$ if X_t is adapted to the filtration $\mathcal{F}_t^X$ and the following holds

$$\int_a^b E[X_s^2]\ ds < \infty$$

   ii. We say that X_t belongs to the class $\mathcal{L}^2$ if $X_t \in \mathcal{L}^2[0,t]$ for all $t > 0$.

We now want to define what we mean by

$$\int_a^b X_t\ dW_s$$

for some process $X_t \in \mathcal{L}^2$. We see that a way to go about this problem is to start by defining the concept for a simple stochastic process X_t. By *simple* we mean a process X_t that is constant on between some deterministic points in time $a = t_0 < t_1 < \cdots < t_n = b$. In that case we may define the integral as

$$\int_a^b X_s\ dW_s = \sum_{k=0}^{n-1} X_{t_k}[W_{t_{k+1}} - W_{t_k}].$$

In the more general setting we may follow the following approach:

1. Approximate X with a sequence $\{X^n\}_{n \in \mathbb{N}}$ of simple processes such that the following convergence criteria hold

$$\int_a^b E[(X_s^n - X_s)^2]\ ds \to 0,\ n \to \infty$$

2. For each n the integral $\int_a^b X_s^n\ dW_s := Z^n$ is well defined and it is possible to prove, using DCT, that a variable Z exists such that $Z^n \to Z$ that is in L^2.
3. We now define the stochastic integral by the limit

$$\int_a^b X_s \, dW_s = \lim_{n \to \infty} \int_a^b X_s^n \, dW_s.$$

Obviously the hardest step is finding the processes X^n. This stochastic har some properties we will use.

Proposition 4.5. Let $X_t \in \mathcal{L}^2$, then

$$E \left[\int_a^b X_s \, dW_s \right] = 0. \tag{4.11}$$

$$E \left[\left(\int_a^b X_s \, dW_s \right)^2 \right] = \int_a^b E[X_s^2] \, dW_s. \tag{4.13}$$

$$\int_a^b X_s \, dW_s \quad \text{is } \mathcal{F}_b^W\text{-measurable}. \tag{4.14}$$

Martingales   Definition 4.7. Let M_t be a stochastic process defined on a background space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration. If M_t is adapted to the filtration $\mathcal{F}_t$, $E|M_t| < \infty$ and

$$E[M_t | \mathcal{F}_s] = M_s, \qquad P - \text{a.s.}$$

holds for any $t > s$ we say that M_t is a martingale (**F**-martingale). If the above has $\geq$ or $\leq$ we say that M_t is either a **submartingale** or **supermartingale** respectively.

Proposition 4.8. For any process $X_t \in \mathcal{L}^2[s, t]$ the following hold:

$$E \left[\int_s^t X_s \, dW_s \, \middle| \, \mathcal{F}_s^W \right] = 0$$

Corollary 4.9. For any process $X_t \in \mathcal{L}^2$ the process

$$M_t = \int_s^t X_s \, dW_s,$$

is an $(\mathcal{F}_t^W)$-martingale. In other words, modulo an integrability condition, *every stochastic integral is a martingale.*

Lemma 4.10. Let M_t be a stochastic process with stochastic differential, then M_t is a martingale if and only if the stochastic differential has the form $dM_t = X_t \, dW_t$ i.e. M_t as no dt-term.

Stochastic Calculus and the Ito Formula   Given this breif introduction to stochastic integrals we may formulate som simple calculus revolving around Ito's formula. We consider the stochastic process X_t and we suppose that there exist a real number X_0 and adapted processes μ and σ wrt. $\mathcal{F}_t^W$ such that for all $t \geq 0$ we have

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s, \tag{4.16}$$

where W_t is a standard Brownian motion. We know from earlier courses that the above may be written in terms of the dynamics (pure notation):

$$\begin{cases} dX_t = \mu_t \ dt + \sigma_t \ dW_t, \\ \qquad X_0 = X_0. \end{cases}$$

Here we intepret the above as X_t has boundary condition X_0 and evolves with a drift $\mu_t \ dt$ amplified and distorted by the drift $\sigma_t \ dW_t$. We say that X_t has **stochastic differential** dX_t and initial condition X_0.

We want to understand how transformation of such an integral behaves and therefore we introduce some calculus which will tell how for instance $f(t, X_t)$ (for some $C^{1,2}$-function) behaves. This insight is given by the important Ito's formula.

Thoerem 4.11. Assume that the process X has a stochastic differential form given by

$$dX_t = \mu_t \ dt + \sigma_t \ dW_t, \tag{4.28}$$

where μ and σ are adapted processes, and let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a $C^{1,2}$-function. Define the process Z by $Z_t = f(t, X_t)$. Then Z has stochastic differential given by

$$df(t, X_t) = \left(\frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right) \ dt + \sigma_t \frac{\partial f}{\partial x}(t, X_t) \ dW_t. \tag{4.29}$$

Proposition 4.12. With assumptions as in theorem 4.11, df is given by

$$df = \frac{\partial f}{\partial t} \ dt + \frac{\partial f}{\partial x} \ dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \ (dX_t)^2, \tag{4.31}$$

where we use the following table

$$\begin{cases} (dt)^2 = 0, \\ dt \cdot dW_t = 0, \\ (dW_t)^2 = dt. \end{cases}$$

Lemma 4.18. Let $\sigma(t)$ be deterministic function of time and define the process X by

$$X_t = \int_0^t \sigma(s) \ dW_s. \tag{4.37}$$

Then

$$X_t \sim \mathcal{N} \left(0, \int_0^t \sigma^2(s) \ ds \right).$$

The multidimensional Ito Formula Consider a vector process $X = (X^1, ..., X^n)^\top$ where each component X^i has stochastic differential

$$dX_t^i = \mu_t^i \ dt + \sum_{j=1}^d \sigma_t^{ij} \ dW_t^j$$

where $W^1, ..., W^d$ is independent Brownian motions. Then we have respectively the drift vector process μ_t in n dimensions, the vector Brownian motion in d dimensions and a $n \times d$-dimensional **diffusion matrix** σ_t given as below

$$\mu_t = \begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^n \end{bmatrix}, \quad W_t = \begin{bmatrix} W_t^1 \\ \vdots \\ W_t^d \end{bmatrix}, \quad \sigma_t = \begin{bmatrix} \sigma_t^{11} & \cdots & \sigma_t^{1d} \\ \vdots & \ddots & \vdots \\ \sigma_t^{n1} & \cdots & \sigma_t^{nd} \end{bmatrix}.$$

Given this we may write the dynamics of X as

$$dX_t = \mu_t \ dt + \sigma_t \ dW_t \in \mathbb{R}^n.$$

Consider now a function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ which is a $C^{1,2}$-mapping. We want to study the dynamics of the process

$$Z_t = f(t, X_t).$$

The dynamics is given in the multidimensional version of Ito's formula.

Thoerem 4.19. Let X be given as above. Then the following holds:

- The process $f(t, X_t)$ has the stochastisc differential given by

$$df(t, X_t) = \left(\frac{\partial f}{\partial t}(t, X_t) + \sum_{i=1}^{n} \mu_t^i \frac{\partial f}{\partial x^i}(t, X_t) + \frac{1}{2} \sum_{i,j=1}^{n} C_t^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}(t, X_t) \right) dt + \sum_{i=1}^{n} \sigma_t^i \frac{\partial f}{\partial x^i}(t, X_t) \ dW_t.$$

  Here the row vector σ_t^i is the i'th row of the matrix σ_t and the matrix C is defined by $C = \sigma \sigma^\top$.
- Alternatively, the differential is given by the formula

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) \ dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(t, X_t) \ dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j}(t, X_t) \ dX_t^i dX_t^j,$$

  with the formal multiplication table

$$\begin{cases} (dt)^2 = 0, \\ dt \cdot dW_t^i = 0, & i = 1, ..., d, \\ (dW_t^i)^2 = dt, & i = 1, ..., d, \\ dW_t^i \cdot dW_t^i = 0, & i \neq j. \end{cases}$$

Obviously, one can write the differential in Ito's formula in many other ways including a matrix-wise version using the Hessian matrix $H_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$.

Correlated Brownian motions In the previous section the d-dimensional Brownian was assumed to have independent Brownian motions. However we may instead consider a variation where we have some dependence between the Brownian motions.

This section has not been finished.

Discrete Stochastic Integrals This section has not been finished.

Stochastic Differential Equations We start the chapter by formalising some used objects. We consider the following objects.

- $M(n, d)$ denotes the class of $n \times d$-matrices.
- W is a d-dimensional Brownian motion
- μ is a $\mathbb{R}^n$-valued function with arguments (t, X_t) with X_t being a n-dimensional stochastic process.
- σ a $M(n, d)$-valued function with arguments as in μ.

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- x_0 a $\mathbb{R}^n$-valued vector.

We want then to understand when the following has a solution

$$dX_t = \mu(t, X_t) \; dt + \sigma(t, X_t) \; dW_t, \quad X_0 = x_0.$$

We call such an equation the **stochastic differential equation** or simply SDE. We know that the above is loosely notation for the integral form as

$$X_t = x_0 + \int_0^t \mu(s, X_s) \; ds + \int_0^t \sigma(s, X_s) \; dW_s,$$

for all $t \geq 0$. The following proposition tells us when an solution exist to the problem above. In the below $\| \cdot \|$ is usual euclidian norm

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Proposition 5.1. Suppose that there existis a constant K such that the following conditions are satisfied for all x, y and t.

$$\|\mu(t, x) - \mu(t, y)\| \leq K \|x - y\|,$$
$$\|\sigma(t, x) - \sigma(t, y)\| \leq K \|x - y\|,$$
$$\|\mu(t, x)\| + \|\sigma(t, x)\| \leq K(1 + \|x\|).$$

Then there exists a unique solution to the SDE above. Furthermore, the solution has the properties

1. X is $\mathcal{F}_t^W$-adapted.
2. X has continuous trajectories.
3. X is a Markov process.
4. There exists a constant C such that

$$E[\|X_t\|^2] \leq Ce^{Ct}(1 + \|x_0\|^2).$$

In genereal the solution to an SDE is so complicated, that it in practical terms is unsolvable and may only be approximated on a finely subdividet grid as jumps. There does however exist som nontrivial cases where we may infer a analytical solution. One is the rather important **Geometric Brownian motion**.

Proposition 5.2. Consider the SDE

$$dX_t = \alpha X_t \; dt + \sigma X_t \; dW_t,$$

with $X_0 = x_0$. Then the solution is given as

$$X_t = x_0 \cdot \exp\left\{ \left(\alpha - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}.$$

The expected value of X is given as $E[X_t] = x_0 e^{\alpha t}$.

One other generalisation that is analytically solvable is the Linear SDE.

Proposition 5.3. Consider the SDE

$$dX_t = (AX_t + b_t)\ dt + \sigma_t\ dW_t,$$

with $X_0 = x_0$ and $A \in M(n,n)$ and b_t being a real-valued function. Then the solution is given as

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}b_s\ ds + \int_0^t e^{A(t-s)}\sigma_s\ dW_s.$$

Where we define the exponential of a matrix as below

$$e^{At} = \sum_{k=0}^{\infty} A^k \frac{1}{k!} t^k.$$

In general with the SDE we have a partial differential operator $\mathcal{A}$ called the **infinitesimal operator** of X which has some interesting analytical properties regarding X.

Definition 5.4. Consider the SDE

$$dX_t = \mu(t, X_t)\ dt + \sigma(t, X_t)\ dW_t.$$

The partial differential operator $\mathcal{A}$ is defined, for any function $h \in C^2(\mathbb{R}^n)$, by

$$\mathcal{A}h(t,x) = \sum_{i=1}^n \mu_i(t,x)\frac{\partial h}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^n (\sigma(t,x)\sigma(t,x)^\top)_{ij}\frac{\partial^2 h}{\partial x_i \partial x_j}(x).$$

We see that in terms of Ito's formula the operator is included as such

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t}(t, X_t) + \mathcal{A}f(t,x) \right\}\ dt + [\nabla_x f](t, X_t)\sigma(t, X_t)\ dW_t,$$

where ∇_x is the gradient for function $h \in C^1(\mathbb{R}^n)$ as

$$\nabla_x h(x) = \left[\frac{\partial h}{\partial x_1}(x), ..., \frac{\partial h}{\partial x_n}(x) \right].$$

Exercises

Exercise 1 *(Bjork 4.1)* Compute the stochastic differential dZ_t when

a. $Z_t = e^{\alpha t}$.
b. $Z_t = \int_0^t g_s\ dW_s$, where g is an adapted stochastic process.
c. $Z_t = e^{\alpha W_t}$.
d. $Z_t = e^{\alpha X_t}$, where X has stochastic differential $dX_t = \mu\ dt + \sigma\ dW_t$ and μ, σ is constants.
e. $Z_t = X_t^2$, where X has stochastic differential $dX_t = \alpha X_t\ dt + \sigma X_t\ dW_t$.

Solution (a).

Let $Z_t = e^{\alpha t}$, then we see that $f(t,x) = e^{\alpha t}$ and the the following relevant derivatives is

$$\frac{\partial f}{\partial t}(t,x) = \alpha e^{\alpha t}, \quad \frac{\partial f}{\partial x}(t,x) = 0, \quad \frac{\partial f}{\partial x^2}(t,x) = 0.$$

Since Z does not depend on any stochastic process, we will content with $X_t = 0$, that is $\mu_t = \sigma_t = 0$. Then by theorem 4.11 (Ito's formula) we have

$$dZ_t = \left(\alpha e^{\alpha t} + 0 + 0\right) \ dt + 0 = \alpha e^{\alpha t} \ dt,$$

as expected. □

Solution (b).

Let $Z_t = \int_0^t g_s \ dW_s$, where g is an adapted stochastic process. We see that if we set $X_t = \int_0^t g_s \ dW_s$ then

$$dX_t = 0 \ dt + g_t \ dW_t.$$

Then we have the function $f(t, x) = x$ and the relevant derivatives are:

$$\frac{\partial f}{\partial t}(t, x) = 0, \quad \frac{\partial f}{\partial x}(t, x) = 1, \quad \frac{\partial f}{\partial x^2}(t, x) = 0.$$

This then gives

$$dZ_t = \left(0 + 0 + \frac{1}{2}g_t \cdot 0\right) \ dt + g_t \cdot 1 \ dW_t = g_t \ dW_t,$$

as expected. □

Solution (c).

Let $Z_t = e^{\alpha W_t}$. Then we may set $X_t = W_t$ and we then have $\mu_t = 0$ and $\sigma_t = 1$. The function $f(t, x) = e^{\alpha x}$ and the relevant derivatives are:

$$\frac{\partial f}{\partial t}(t, x) = 0, \quad \frac{\partial f}{\partial x}(t, x) = \alpha e^{\alpha x}, \quad \frac{\partial f}{\partial x^2}(t, x) = \alpha^2 e^{\alpha x}.$$

Then the dynamics of Z_t is as follows

$$\begin{aligned}
dZ_t &= \left(0 + 0 + \frac{1}{2}1^2\alpha^2 e^{\alpha X_t}\right) \ dt + 1\alpha e^{\alpha X_t} \ dW_t \\
&= \frac{\alpha^2}{2}e^{\alpha X_t} \ dt + \alpha e^{\alpha X_t} \ dW_t \\
&= \frac{\alpha^2}{2}Z_t \ dt + \alpha Z_t \ dW_t.
\end{aligned}$$

As desired. □.

Solution (d).

Let $Z_t = e^{\alpha X_t}$, where X has stochastic differential $dX_t = \mu \ dt + \sigma \ dW_t$ and μ, σ is constants. Then we have been given the definition of X_t and we set $f(t, x) = e^{\alpha x}$. The relevant derivatives are then:

$$\frac{\partial f}{\partial t}(t, x) = 0, \quad \frac{\partial f}{\partial x}(t, x) = \alpha e^{\alpha x}, \quad \frac{\partial f}{\partial x^2}(t, x) = \alpha^2 e^{\alpha x}.$$

We may now derive the dynamics of Z_t:

$$dZ_t = \left(0 + \mu \alpha e^{\alpha X_t} + \frac{1}{2} \sigma^2 \alpha^2 e^{\alpha X_t} \right) \, dt + \sigma \alpha e^{\alpha X_t} \, dW_t$$

$$= \left(\mu + \frac{1}{2} \sigma^2 \alpha \right) \alpha e^{\alpha X_t} \, dt + \sigma \alpha e^{\alpha X_t} \, dW_t$$

$$= \left(\mu + \frac{1}{2} \sigma^2 \alpha \right) \alpha Z_t \, dt + \sigma \alpha Z_t \, dW_t.$$

As desired. $\square$.

Solution (e).

Let $Z_t = X_t^2$, where X has stochastic differential $dX_t = \alpha X_t \, dt + \sigma X_t \, dW_t$. Then we set $f(t, x) = x^2$ and the relevant derivatives are:

$$\frac{\partial f}{\partial t}(t, x) = 0, \quad \frac{\partial f}{\partial x}(t, x) = 2x, \quad \frac{\partial f}{\partial x^2}(t, x) = 2.$$

Given this we have the dynamics of Z_t as follows

$$dZ_t = \left(0 + \alpha X_t 2 X_t + \frac{1}{2}(\sigma X_t)^2 2 \right) \, dt + \sigma X_t 2 X_t \, dW_t$$

$$= \left(2\alpha + \sigma^2 \right) X_t^2 \, dt + 2\sigma X_t^2 \, dW_t$$

$$= \left(2\alpha + \sigma^2 \right) Z_t \, dt + 2\sigma Z_t \, dW_t.$$

As desired. $\square$.

Exercise 2 *(Bjork 4.2)* Compute the stochastic differential for Z when $Z_t = (X_t)^{-1}$ and X has the stochastic differential

$$dX_t = \alpha X_t \, dt + \sigma X_t \, dW_t.$$

Furthermore, by using the definition $Z = X^{-1}$ you can in fact express the right-hand side of dZ entirely in terms of Z itself (rather then in terms of X). Thus Z satisfies a stochastic differential equation. Which one?

Solution.

We see that $f(t, x) = 1/x$ and so the relevant derivatives is

$$\frac{\partial f}{\partial t}(t, x) = 0, \quad \frac{\partial f}{\partial x}(t, x) = -\frac{1}{x^2}, \quad \frac{\partial f}{\partial x^2}(t, x) = \frac{2}{x^3}.$$

Then we by Ito's formula we have

$$dZ_t = \left(0 - \alpha X_t \frac{1}{X_t^2} + \frac{1}{2} \sigma^2 X_t^2 \frac{2}{X_t^3} \right) \, dt - \sigma X_t \frac{1}{X_t^2} \, dW_t$$

$$= \left(-\alpha \frac{1}{X_t} + \sigma^2 \frac{1}{X_t} \right) \, dt - \sigma \frac{1}{X_t} \, dW_t$$

$$= (\sigma^2 - \alpha) Z_t \, dt - \sigma Z_t \, dW_t.$$

We also notice that

$$Z_t = \frac{1}{X_t} \Rightarrow dZ_t = d\left(\frac{1}{X_t}\right) = -\left(\frac{1}{X_t}\right)^2 dX_t = -Z_t^2(\alpha X_t \ dt + \sigma X_t \ dW_t)$$

Hence we may insert $X_t = Z_t^{-1}$ and optain

$$dZ_t = -Z_t^2\left(\alpha \frac{1}{Z_t} \ dt + \sigma \frac{1}{Z_t} \ dW_t\right) = -\alpha Z_t \ dt - \sigma Z_t \ dW_t.$$

Which clearly is faulty.. $\square$

Exercise 3. *(Bjork 4.3)* Let $\sigma(t)$ be a given deterministic function of time and define the process X by

$$X_t = \int_0^t \sigma(s) \ dW_s.$$

Use the technique discribed in example 4.17 in order to show that the characteristic function of X_t (for a fixed t) is given by

$$E[e^{iuX_t}] = \exp\left\{-\frac{u^2}{2}\int_0^t \sigma^2(s) \ ds\right\}, \quad u \in \mathbb{R},$$

thus showing that X_t is normally distributed with zero mean and a variance given by

$$Var[X_t] = \int_0^t \sigma^2(s) \ ds.$$

Solution.

We follow along the lines of

1. Determine the dynamics of $Z_t = e^{iuX_t}$ (for fixed u).
2. Write the integral form of Z_t.
3. Take expectation.
4. Solve ODE.

"1)" Set $f(t,x) = e^{iuX_t}$ then the relevant derivatives are

$$\frac{\partial f}{\partial t}(t,x) = 0, \quad \frac{\partial f}{\partial x}(t,x) = iue^{iuX_t} = iuZ_t, \quad \frac{\partial f}{\partial x^2}(t,x) = i^2u^2e^{iuX_t} = -u^2Z_t.$$

Recall that $dX_t = \sigma(t) \ dW_t$, then by Ito's formula we have

$$dZ_t = \left(-\sigma(t)^2\frac{1}{2}u^2Z_t\right) \ dt + \sigma(t)iuZ_t \ dW_t. \tag{*}$$

"2)" We can now write (*) on integral form as below

$$Z_t = Z_0 - \frac{u^2}{2}\int_0^t \sigma^2(s)Z_s \ ds + iu\int_0^t \sigma(s)Z_s \ dW_s,$$

where $Z_0 = e^{iuX_0} = 1$.

"3)" Taking expectation now yields

$$E[Z_t] = 1 - \frac{u^2}{2} \int_0^t \sigma^2(s) E[Z_s] \, ds + iuE\left[\int_0^t \sigma(s) Z_s \, dW_s\right] = 1 - \frac{u^2}{2} \int_0^t \sigma^2(s) E[Z_s] \, ds,$$

since any expectaion of an integral wrt. a Brownian motion is 0 (proposition 4.5).

"4)" Now we see that the t-derivative gives

$$dE[Z_t] = -\frac{u^2}{2} \sigma^2(t) E[Z_t] \, dt, \quad E[Z_0] = 1.$$

This is a ordinary differential equation with solution $y(t) = \exp\{-u^2/2 \int_0^t \sigma^2(s) \, ds\}$ (check by differentiating) hence

$$E[e^{iuX_t}] = E[Z_t] = \exp\left\{-\frac{u^2}{2} \int_0^t \sigma^2(s) \, ds\right\}.$$

We recognize this as the characteristic function of a normally distributed random variable with variance $\int_0^t \sigma^2(s) \, ds$ as desired. (X_t follows this distributions since characteristic functions determine the distribution)
□

Exercise 4 *(Bjork 4.4)* Suppose that X has the stochastic differential

$$dX_t = \alpha X_t \, dt + \sigma_t \, dW_t,$$

where α is a real number and σ_t is a integrable adapted stochastic process. Use the technique in example 4.17 in order to determine the function $m(t) = E[X_t]$.

Solution.

We follow the same steps as the previous exercise. We have been given the dynamics of X hence we may write it on integral form.

$$X_t = X_0 + \alpha \int_0^t X_s \, ds + \int_0^t \sigma(s) \, dW_s.$$

Then taking expectation now gives

$$E[X_t] = X_0 + \alpha \int_0^t E[X_s] \, ds.$$

Hence $E[X_t]$ follows from the solution to the ODE below

$$dE[X_t] = \alpha E[X_t] \Rightarrow E[X_t] = C \cdot \exp\{\alpha t\}.$$

Then obviously $C = X_0$ and we arrive at the solution $E[X_t] = X_0 e^{\alpha t}$, where X_0 is some deterministic value.
□

Exercise 5 *(Bjork 4.5)* Suppose that the process X has a stochastic differential

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

and that $\mu_t \geq 0$ with probability one for all $t \geq 0$. Show that this implies that X is a sub-martingale.

Solution.

Note that we are (strictly speaking) supposed to show adaptation and integrability, we will however only fokus on the submartingale property.

"$E[X_t|\mathcal{F}_s] \geq X_s$" Intuitively speaking, the statement is obvious since we have with probability one a positive upwards drift with Brownian distortion (i.e. martingale). Formally, we will show the statement by first writing X_t on integral form

$$X_t = x_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s.$$

And so

$$X_t - X_s = \int_s^t \mu_u \, du + \int_s^t \sigma_u \, dW_u.$$

We then have

$$
\begin{aligned}
E[X_t \mid \mathcal{F}_s] - X_s &= E[X_t - X_s \mid \mathcal{F}_s] \\
&= E\left[\int_s^t \mu_u \, du + \int_s^t \sigma_u \, dW_u \;\middle|\; \mathcal{F}_s \right] \\
&= E\left[\int_s^t \mu_u \, du \;\middle|\; \mathcal{F}_s \right] + E\left[\int_s^t \sigma_u \, dW_u \;\middle|\; \mathcal{F}_s \right] \\
&= E\left[\int_s^t \mu_u \, du \;\middle|\; \mathcal{F}_s \right] \geq 0.
\end{aligned}
$$

Then adding X_s to the above inequality yields the result. $\square$

Exercise 6 *(Bjork 4.7)* The objective of this exercise is to give an argument for the formal identity

$$dW_1(t) \cdot dW_2(t) = 0,$$

when W_1 and W_2 are independent Brownian motions. Let us therefore fix a time t, and divide the inerval $[0, t]$ into equidistant points $0 = t_0 < t_1 < \cdots < t_n = t$, where $t_i = \frac{i}{n} \cdot t$. We use the notation

$$\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1}), \quad i = 1, 2.$$

Now define Q_n by

$$Q_n = \sum_{k=1}^{n} \Delta W_1(t_k) \cdot \Delta W_2(t_k).$$

Show that $Q_n \to 0$ in L^2, i.e. show that

$$E[Q_n] = 0, Var[Q_n] \to 0.$$

Solution.

We wish to show the statement

$$E[(Q_n - 0)^2] = E[Q_n^2] \to 0,$$

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as $n \to \infty$. Recall that

$$Var[Q_n] = E[Q_n^2] - E[Q_n]^2,$$

hence if Q_n has mean 0, then showing convergence in L^2 is equivalent to showing variance going to 0. Let us start by showing the mean is 0.

We have that

$$
\begin{aligned}
Q_n &= \sum_{k=1}^{n} \Delta W_1(t_k) \cdot \Delta W_2(t_k) \\
&= \sum_{k=1}^{n} (W_1(t_k) - W_1(t_{k-1})) \cdot (W_2(t_k) - W_2(t_{k-1})) \\
&\overset{\mathcal{D}}{=} \sum_{k=1}^{n} XY,
\end{aligned}
$$

where $X, Y \sim \mathcal{N}(0, t_k - t_{k-1}) = \mathcal{N}(0, 1/n)$ and independent random variable. This is justified since the increments of the Brownian motion has mean 0 and variance equal to the increment size. Now this implies, that we need to show that $E[XY] = 0$ and that $Var[XY]$ is sufficiently small in terms of n such that it is summable. We see that

$$E[XY] = E[X]E[Y] = 0^2 = 0.$$

Here we use independence. We now know that the mean is

$$E[Q_n] = \sum_{k=1}^{n} E[XY] = 0.$$

We know from basic properties of variance that

$$
\begin{aligned}
Var(Q_n) &= \sum_{k=1}^{n} Var(XY) = \sum_{k=1}^{n} E[(XY)^2] \\
&= \sum_{k=1}^{n} \frac{1}{n^2} = \frac{1}{n^2} n \\
&= \frac{1}{n} \to 0, \ n \to \infty.
\end{aligned}
$$

And so the result follows. $\square$

Exercise 7 *(Bjork 4.8)* Let X and Y be given as the solutions to the following system of stochastic differential equations.

$$
\begin{aligned}
dX_t &= \alpha X_t \ dt - Y_t \ dW_t, & X_0 &= x_0, \\
dY_t &= \alpha Y_t \ dt + X_t \ dW_t, & Y_0 &= y_0.
\end{aligned}
$$

Note that the initial values x_0 and y_0 are deterministic constants.

a. Prove that the process R defined by $R_t = X_t^2 + Y_t^2$ is deterministic.

b. Compute $E[X_t]$.

Solution (a).

We see that

$$dR_t = d(X_t^2 + Y_t^2) = d(X_t^2) + d(Y_t^2)$$

Hence we may start by considering de dynamics of the processes X_t^2 and Y_t^2. We see that for the process $Z_t = X_t^2$ we may set $f(t,x) = x^2$ and the relevant derivatives are

$$\frac{\partial f}{\partial t}(t,x) = 0, \ \frac{\partial f}{\partial x}(t,x) = 2x, \ \frac{\partial^2 f}{\partial x^2}(t,x) = 2.$$

By Ito's formula we have

$$d(X_t^2) = \left(\alpha X_t 2X_t + Y_t^2 2\right) \ dt - Y_t 2X_t \ dW_t = 2(\alpha X_t^2 + Y_t^2) \ dt - 2X_t Y_t \ dW_t.$$

By the same concept we have

$$d(Y_t^2) = \left(\alpha Y_t 2Y_t + X_t^2 2\right) \ dt + X_t 2Y_t \ dW_t = 2(\alpha Y_t^2 + X_t^2) \ dt + 2X_t Y_t \ dW_t.$$

Combining we get the dynamics

$$\begin{aligned}
dR_t &= 2(\alpha X_t^2 + Y_t^2) \ dt - 2X_t Y_t \ dW_t \\
&\quad + 2(\alpha Y_t^2 + X_t^2) \ dt + 2X_t Y_t \ dW_t \\
&= (2\alpha + 1)(X_t^2 + Y_t^2) \ dt \\
&= (2\alpha + 1)R_t \ dt
\end{aligned}$$

Hence R_t has deterministic derivative and therefore a deterministic process. In fact, the solution to above is

$$R_t = R_0 \exp\left\{(2\alpha + 1)t\right\} = (x_0^2 + y_0^2)e^{(2\alpha+1)t},$$

which is clearly deterministic. $\square$

Solution (b).

We start by acknowledging that the differential form of X may be written on integral form:

$$X_t = x_0 + \alpha \int_0^t X_s \ ds - \int_0^t Y_s \ dW_s.$$

Taking expectation we see that

$$E[X_t] = x_0 + \int_0^t E[X_s] \ ds$$

as the last term has mean 0 according to proposition 4.5. Then the above may be written on the differential form

$$dE[X_t] = E[X_t]\, dt$$

Hence we have that

$$E[X_t] = x_0 e^t.$$

Hence X_t has mean not depending on the tragetory of the sister-process Y_t. $\square$

Week 3

Material

- Partial differential equations (Chapter 5.5)
- Self-financing portfolios (Chapter 6)
- Black-Scholes PDE (classic approach) and risk neutral valuation (Chapter 7.1-5)

Theory

Partial differential equations **Proposition 5.5. (Feynmann-Kac)** Assume that F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(x,t) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) = 0,$$

with boundary condition $F(T,x) = \Phi(x)$. Assume furthermore that the process

$$\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s) \in \mathcal{L}^2$$

as per definition 4.4, where X is defined below. Then F has the representation

$$F(t,x) = E_{t,x}[\Phi(X_T)] = E[\Phi(X_T) \mid X_t = x],$$

where X satisfies the SDE

$$dX_s = \mu(s,X_s)\, ds + \sigma(s,X_s)\, dW_s,$$

with boundary condition $X_t = x$.

Proposition 5.6. (Feynmann-Kac) Assume that F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(x,t) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0,$$

with boundary condition $F(T,x) = \Phi(x)$. Assume furthermore that the process

$$e^{-rs}\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s) \in \mathcal{L}^2$$

as per definition 4.4, where X is defined below. Then F has the representation

$$F(t,x) = e^{-r(T-t)}E_{t,x}[\Phi(X_T)] = e^{-r(T-t)}E[\Phi(X_T) \mid X_t = x],$$

where X satisfies the SDE

$$dX_s = \mu(s, X_s) \, ds + \sigma(s, X_s) \, dW_s,$$

with boundary condition $X_t = x$.

Proposition 5.8. (Feynmann-Kac) Assume that F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t, x) + \sum_{i=1}^{n} \mu_i(t, x)\frac{\partial F}{\partial x}(x, t) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t, x)\frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) = 0,$$

with boundary condition $F(T, x) = \Phi(x)$ and $C_{ij} = \sigma\sigma^\top$. Assume furthermore that the process

$$e^{-rs}\sum_{i=1}^{n} \sigma_i(s, X_s)\frac{\partial F}{\partial x}(s, X_s) \in \mathcal{L}^2$$

as per definition 4.4, where X is defined below. Then F has the representation

$$F(t, x) = e^{-r(T-t)}E_{t,x}[\Phi(X_T)],$$

where X satisfies the SDE

$$dX_s = \mu(s, X_s) \, ds + \sigma(s, X_s) \, dW_s,$$

with boundary condition $X_t = x$.

Proposition 5.9. Consider as given a vector process X with generator $\mathcal{A}$, and a function $F(t, x)$. Then, modulo some integrability condition, the following hold:

- The process $F(t, X_t)$ is a martingale relative to the filtration $\mathcal{F}^X$ if and only if F satisfies the PDE

$$\frac{\partial F}{\partial t} + \mathcal{A}F = 0.$$

- The process $F(t, X_t)$ is a martingale relative to the filtration $\mathcal{F}^X$ if and only if, for every (t, x) and $T \geq t$, we have
$$F(t, x) = E_{t,x}[F(T, X_T)].$$

Self-financing portfolios We move forward in this chapter by first defining a self-financing portfolio in discrete time and then by letting the step length tend to zero obtain the continuous time analogue.

Discrete time SF portfolio We consider N different adapted price processes $S^1, ..., S^N$. We use the following definition.

Definition 6.1. We use the following definitions.

- S_n^i is th price of asset i at time n,
- h_n^i is the number of units of asset i held during $[n, n+1)$, that is bought at time n,
- d_n^i is the dividends from asset i in the time-interval $[n-1, n)$, that is recieved at time n,
- h_n is the portfolio $(h_n^1, ..., h_n^N)$ held during $[n, n+1)$,
- c_n is the consumption i.e. withdrawel at time n (negative being deposits/saving),
- V_n is the value of the portfolio just before time n i.e. of the portfolio h_{n-1} at time n.

We are now ready to define the self-financing portfolio

Definition 6.2. A **self-financing portfolio supporting the consumption stream c** is a portfolio adhering to the **budget constraint** given as

$$h_{n+1} S_{n+1} + c_{n+1} = h_n S_{n+1} + h_n d_{n+1}.$$

The interpretation being, that we may only use funds obtained from selling the old portfolio h_n and recieved in dividends to buy the new portfolio h_{n+1} and consume the amount c_{n+1}.

Before studying the self-financing portfolio we define the operator Δ (in definition 6.3) as the increment $\Delta x_n = x_{n+1} - x_n$ of a countable sequence $(x_n)_{n \in \mathbb{N}_0}$. Notice that we define the increment forward so the increment n is the increment over the time period $[n, n+1)$ with the first increment being $[0, 1)$. Using this notation we can derive the lemma below.

Lemma 6.4. For any pair of sequences of real numbers $(x_n)_{n \in \mathbb{N}_0}$ and $(y_n)_{n \in \mathbb{N}_0}$ we have the relations

$$\Delta(xy)_n = x_n \Delta y_n + y_{n+1} \Delta x_n,$$
$$\Delta(xy)_n = y_n \Delta x_n + x_{n+1} \Delta y_n,$$
$$\Delta(xy)_n = x_n \Delta y_n + y_n \Delta x_n + \Delta x_n \Delta y_n.$$

This is also valid if the sequances are N-dimensional, where we interpret the products above as scalar products $(xy^\top)$.

Using these definitions and the lemma above we see that the dynamics of the self-financing portfolio is given below.

Proposition 6.6. The dynamics of any self-financing portfolio supporting the consumption stream c are given by

$$\Delta V_n = h_n \Delta S_n + h_n d_{n+1} - c_{n+1},$$

or, in more detail

$$\Delta V_n = \sum_{i=1}^N h_n^i (\Delta S_n^i + d_{n+1}^i) - c_{n+1}.$$

We may rewrite the dividends as accumulating dividends $D_n^i = \sum_{k=1}^n d_k^i$ and see that $d_{n+1}^i = \Delta D_n^i$ and so the above condition is equivalent with.

Proposition 6.8. The dynamics of any self-financing portfolio supporting the consumption stream c are given by

$$\Delta V_n = h_n \Delta S_n + h_n \Delta D_n - c_{n+1},$$

or, in more detail

$$\Delta V_n = \sum_{i=1}^N h_n^i (\Delta S_n^i + \Delta D_n^i) - c_{n+1}.$$

Continuous time SF portfolio Formulating the dynamics of the self-financing portfolio in continuous time is easy work given the discrete setup above. However since we now are in continuous time we will change the n with a t and cosider the behavour $V_{t+dt} - V_t$ as we let $dt \to 0$. First we formulate some basic notation.

Definition 6.9. We use the following definitions.

- S_t^i is th price of asset i at time t,
- h_t^i is the number of units of asset i held at time t,
- D_t^i is the cumulative dividend processs for asset i,
- h_t is the portfolio $(h_t^1, ..., h_t^N)$ held at time t,
- c_t is the consumption rate at time n (negative being deposits/saving),
- V_t is the value of the portfolio at time t i.e. of the portfolio h_t at time t.

Given these definitions we may define a portfolio strategy that is self-financing.

Definition 6.10. Let S be and adapted N-dimensional price process. We define the following

1. A **portfolio strategy** is any adapted N-dimensional process h.
2. The **value process** V^h corresponding to the portfolio h is given by

$$V_t^h = \sum_{i=1}^N h_t^i S_t^i.$$

3. A **consumption process** is any adapted one-dimensional process c.
4. A portfolio-consumption pair (h, c) is called **self-financing** if the value process V^h satisfies the condition

$$dV_t^h = \sum_{i=1}^N h_t^i (dS_t^i + dD_t^i) - c_t \ dt,$$

   i.e. if
$$dV_t^h = h_t \ dS_t + h_t \ dD_t - c_t \ dt.$$

5. The **gain process** G is defined by
$$G_t = S_t + D_t$$

   so we can write the self-financing condition as

$$dV_t = h_t \ dG_t - c_t \ dt.$$

6. The portfolio h is said to be **Markovian** if it is of the form

$$h_t = h(t, S_t),$$

   for some function $h : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$.

Portfolio weights **Definition 6.11.** For a given portfolio h the corresponding **relative portfolio** or **portfolio weights** w are defined by

$$w_t^i = \frac{h_t^i S_t^i}{V_t^h}, \ i = 1, ..., N,$$

so, in particular, we have $\sum_{i=1}^N w_i = 1$.

Lemma 6.12. A portfolio-consumption par (h, c) is self-financing if and only if

$$dV_t^h = V_t^h \sum_{i=1}^N w_t^i \frac{dS_t^i + dD_t^i}{S_t^i} - c_t \ dt.$$

Lemma 6.13. Consider the case with no dividends. Let c be a consumption process, and assume that there exist a scalar process Z and a vector process $q = (q^1, ..., q^N)$ such that

$$dZ_t = Z_t \sum_{i=1}^{N} q_t^i \frac{dS_t^i}{S_t^i} - c_t \ dt,$$

and $\sum_{i=1}^{N} qq^i = 1$. Now define a portfolio h by

$$h_t^i = \frac{q_t^i Z_t}{S_t^i}.$$

Then the value process V^h is given by $V^h = Z$, the pair (h, c) is self-financing, and the corresponding relative portfolio w is given by $w = q$.

Black-Scholes PDE/Risk-neutral valuation The Black-Scholes model revolves arround SDE's as seen above. In this model we have two assets a risk free asset B and a stochastic priced asset S. We therefore start by defining what we mean by a quote-on-qoute *risk free* asset.

Definition 7.1. The price process B is the price of a **risk free** asset if it has the dynamics

$$dB_t = r_t B_t \ dt,$$

where r is any $\mathcal{F}_t$ adapted process.

We see from this definition that the meaning of "risk free" is the property, that B is priced locally deterministic in the sence that r is adapted and therefore known at time t and we therefore know the yield on a short term basis. This is also why we may call r the **short interest rate**. Given the dynamics above, we know that B in fact is represented by the process

$$B_t = B_0 e^{\int_0^t r_s \ ds},$$

for some B_0 initial value. We will moving forward assume that $B_0 = 1$. The stochastic asset S has dynamics.

$$dS_t = \mu(t, S_t) \ dt + \sigma(t, S_t) \ dW_t,$$

where as usual μ and σ are deterministic functions and W_t is a standard Brownian motion. Note that the risk free asset has a similarly process with $\sigma = 0$. We may now include this in the definition of the Black-Scholes model.

Definition 7.2. The **Black-Scholes model** consists of two assets with dynamics given by

$$dB_t = r B_t \ dt,$$
$$dS_t = \mu S_t \ dt + \sigma S_t \ dW_t,$$

where r, μ, σ are deterministic constants.

Definition 7.3. A **zero coupon bond** with maturity T (henceforth "T-bond") is an asset which pays the holder the face value 1 dollar at time T. The price at time n of a T-bond is denoted by $p(n, T)$.

Definition 7.4. The (possible stochastic) discrete **short rate** r_n, for the period $[n, n+1]$, is defined as

$$p(n, n+1) = \frac{1}{1 + r_n}.$$

From this short rate we may derive the dynamics of the bank account recieving zero-coupon rates for each distinct time interval.

Definition 7.5. The dynamics of the bank account are given by

$$\Delta B_n = r_n B_n.$$

Contingent Claims and Arbitrage   Definition 7.6. A **European call option** with **exercise price** (or strike price) K and **time of maturity** (exercise date) T on the **underlying asset** S is a contract defined by the following clauses:

- The holder of the option has, at time T, the right to buy one share of the underlying stock at the price K dollars from the underwriter of the option.
- The holder of the option is in no way obliged to buy the underlying stock.
- The right to buy the underlying stock at the price K can only be exercised at the precise time T.

Obviously, we also have the **european put** option which gives the owner the right to sell an asset at price K at time T. Let os formally define a contingent claim.

Definition 7.7. Consider a financial market with vector price process S. A **contingent claim** with **date of maturity** T, also called a T-claim, is any random variable $\mathcal{X} \in \mathcal{F}_T^S$. A contingent claim $\mathcal{X}$ is called a **simple** claim if it is of the form $\mathcal{X} = \Phi(S_t)$. The function Φ is called the **contract function**.

Definition 7.8. An **arbitrage** possibility on a financial market is a self-financed portfolio h such that

$$V^h(0) = 0,$$
$$P(V_T^h \geq 0) = 1,$$
$$P(V_T^h > 0) > 0.$$

We say that the market is **arbitrage free** if there are no arbitrage possibilities.

Definition 7.9. Suppose that there exists a self-financing portfolio h, such that the value process V^h has the dynamics

$$dV_t^h = k_t V_t^h \ dt,$$

where k is an adapted process. Then it must hold that $k_t = r_t$ for all t, ore there exists an arbitrage possibility.

Theorem 7.10. (Black-Scholes equation) Assume that the market is specified by the equations

$$dB_t = rB_t \ dt,$$
$$dS_t = \mu(t, S_t)S_t \ dt + \sigma(t, S_t)S_t \ dW_t,$$

and that we want to price a contingent claim of the form $\mathcal{X} = \Phi(S_t)$. Then the only pricing function of the form $\Pi_t[\Phi(S_t)] = F(t, S_t)$ which is consistent with the absence of arbitrage in the market $[B_t, S_t, \Pi_t]$ is when F is the solution of the following boundary value problem in the domain $[0, T] \times \mathbb{R}_+$:

$$F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2(t, s)F_ss(t, s) - rF(t, s) = 0,$$
$$F(T, s) = \Phi(s).$$

Risk Neutral Valuation Theorem 7.11. (Risk Neutral Valuation) The arbitrage free price of the claim $\Phi(S_t)$ is given by $\Pi_t[\Phi] = F(t, S_t)$, where F is given by the formula

$$F(t, s) = e^{-r(T-t)} E_{t,s}^Q[\Phi(S_T)],$$

where the Q-dynamics of S are those of

$$dS_t = rS_t \ dt + S_t\sigma(t, S_t) \ dW_t^Q.$$

Property 7.12. (The Martingale Property) In the Black-Scholes model, the price process Π_t for every traded asset, be it the underlying or derivate asset, has the property the the normalized price process

$$Z_t = \frac{\Pi_t}{B_t},$$

(including S_t/B_t) is a martingale under the measure Q.

Exercises

Exercise 1 *(Bjork 5.1)* Show that the scalar SDE

$$\begin{cases} dX_t = \alpha X_t \ dt + \sigma \ dW_t, \\ \qquad X_0 = x_0, \end{cases}$$

has the solution

$$X(t) = e^{\alpha t}x_0 + \sigma \int_0^t e^{\alpha(t-s)} \ dW_s,$$

by differentiating X as defined by the equation above and showing that X so defined satisfies the SDE.

Solution.

We move forward by rewriting the solution in terms of three processes Z, Y and R as

$$X_t = \underbrace{x_0 e^{\alpha t}}_{:=Y_t} + \underbrace{\sigma e^{\alpha t}}_{:=Z_t} \underbrace{\int_0^t e^{-\alpha s} \ dW_s}_{R_t} = Y_t + Z_t \cdot R_t.$$

We furthermore see easily that the dynamics of the processes individually has dynamics

$$\begin{aligned} dY_t &= \alpha x_0 e^{\alpha t} \ dt = \alpha Y_t \ dt, & Y_0 &= x_0, \\ dZ_t &= \alpha \sigma^{\alpha t} \ dt = \alpha Z_t \ dt, & Z_0 &= \sigma, \\ dR_t &= e^{-\alpha t} \ dW_s, & R_0 &= 0. \end{aligned}$$

We then have the following function

$$f(t, y, z, r) = y + zr.$$

With the following multi-dimensional process

$$dM_t = \begin{bmatrix} \alpha Y_t \\ \alpha Z_t \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-\alpha t} \end{bmatrix} \begin{bmatrix} dW_t \\ dW_t \\ dW_t \end{bmatrix},$$

with

$$C = \sigma\sigma^\top = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-\alpha t} \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-2\alpha t} \end{bmatrix}.$$

That is $X_t = f(t, M_t)$. We can then use the multidimensional version of Ito's formula.

$$
\begin{aligned}
dX_t &= df(t, M_t) \\
&= \left(\frac{\partial f}{\partial t}(t, M_t) + \sum_{i=1}^{3} \mu_i \frac{\partial f}{\partial x^i}(t, M_t) + \frac{1}{2} \sum_{i,j=1}^{3} C_t^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}(t, M_t) \right) dt + \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(t, M_t) \sigma_t^i \ dW_t \\
&= (0 + \alpha Y_t + \alpha Z_t R_t) \ dt + Z_t e^{-\alpha t} \ dW_t \\
&= \left(\alpha x_0 e^{\alpha t} + \alpha \sigma e^{\alpha t} \int_0^t e^{-\alpha \sigma} \ dW_s \right) dt + \sigma e^{\alpha t} e^{-\alpha t} \ dW_t \\
&= \left(\alpha x_0 e^{\alpha t} + \alpha \sigma \int_0^t e^{(t-s)\alpha} \ dW_s \right) dt + \sigma \ dW_t \\
&= \alpha X_t \ dt + \sigma \ dW_t.
\end{aligned}
$$

Then this solution does in fact satisfies the differential form. We furthermore have that $X_0 = x_0$ and the desired result follows. $\square$

Exercise 2 *(Bjork 5.5)*

Solution.

Exercise 3 *(Bjork 5.6)*

Solution.

Exercise 4 *(Bjork 5.9)*

Solution.

Exercise 5 *(Bjork 5.13)*

Solution.

Exercise 6 *(Exam 2017/18, problem 1, question (a)-(b))*

Solution.

Exercise 7 *(Exam 2019/20, problem 1, question (a))*

Solution.

Exercise 8 *(Exam 2020/21, problem 1, question (a)-(b))*

Solution.

Week 4

Material

- Black-Scholes formula (Chapter 7.6, see also Remark to Black-Scholes formula)
- Completeness and hedging (Chapter 8)
- Put-call parity (Chapter 10.1)
- The Greeks (Chapter 10.2)
- Risk neutral valuation formula (Chapter 11.6)
- Equivalent probability measures (Appendix A.11, B.6 and C.3)

Theory

Exercises

Week 5

Material

- Girsanov theorem (Chapter 12, see also Levy characterization of Brownian motion and proof of Girsanov)
- Martingale representation theorem (Chapter 12)

Theory

Exercises

Week 6

Material

- Black-Scholes model, martingale approach (Chapter 13)
- Multidimensional models (Chapter 14)

Theory

Exercises

Week 7

Material

- Pricing and proof of fundamental pricing theorem I and II (Chapter 11)
- Incomplete Markets (Chapter 9)

Theory

Exercises