Homework (FinKont)

Joakim Bilyk

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Introduction

Weeks

Week 1

Material

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- Introduction (Chapter 1)
- Discrete time models (Chapter 2 and 3)

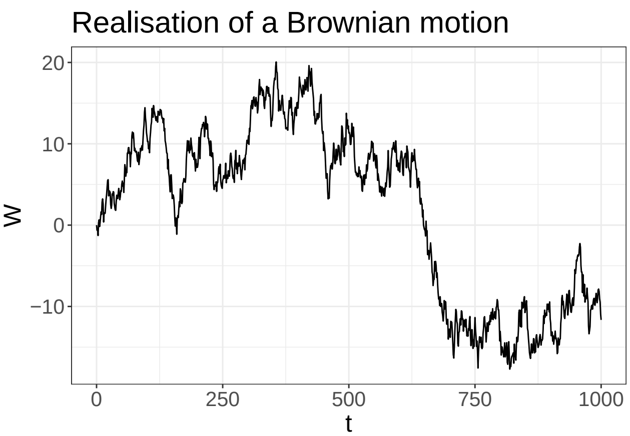
Theory

Definition 4.1. *(Brownian motion)* A stochastic process W is called a **Brownian motion** or **Wiener process** if the following conditions hold

1. $W_0 = 0$.
2. The process W has independent increments, i.e. if $r < s \leq t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent random variables.
3. For $s < t$ the random variable $W_t - W_s$ has the Gaussian distribution $\mathcal{N}(0, t-s)$.
4. W has continuous trajectories i.e. $s \mapsto W(s;\omega)$ i continuous for all $\omega \in \Omega$.

```
#Example of trajectory for BM
set.seed(1)
t <- 0:1000
N <- rnorm(
```

```
  n = length(t)-1, #initial value = 0
  mean = 0, #incements mean = 0
  sd = sqrt(t[2:length(t)] - t[1:(length(t)-1)]) #increment sd = sqrt(t-s)
)
W <- c(0,cumsum(N))
```



Definition C.1. Let M_t be a stochastic process defined on a background space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. If M_t is adapted to the filtration $\mathcal{F}_t$, $E|M_t| < \infty$ and

$$E[M_t|\mathcal{F}_s] = M_s$$

holds for any $t > s$ we say that M_t is a martingale.

Exercises

Probability exercises

Let $(W(t))_{t\geq}$ be a Brownian motion (Bjork, Definition 4.1).

Exercise 1. Show that the following processes also are Brownian motions.

i. $(-W(t))_{t\geq 0}$ (symmetry)
ii. For any $s \geq 0$, $(W(t+s) - W(s))_{t\geq 0}$ (time-homogeneity).
iii. For every $c > 0$, $(cW(t/c^2))_{t\geq 0}$ (scaling).

Solution (i).

By assumption W is a Brownian motion and so it follows that

$$-W_0 = -1 \cdot 0 = 0$$

Furthermore, for $r < s \leq t < u$ it holds that $W_u - W_t$ and $W_s - W_r$ is independent. By seperate transformations the independence property is preserved and $-(W_u - W_t)$ and $-(W_s - W_r)$ is independent. Next, for a normal distributed random variable $N \sim \mathcal{N}(\mu, \sigma^2)$ it holds, that for a scaler $c \in \mathbb{R}$ we have $cN \sim \mathcal{N}(c\mu, c^2\sigma^2)$. Then obviously;

$$-(W_t) = (-1)W_t \stackrel{d}{=} \mathcal{N}((-1)\cdot 0, (-1)^2(t-s)) \stackrel{d}{=} \mathcal{N}(0, t-s).$$

Lastly, let $\omega \in \Omega$ and consider the sample path $s \mapsto (-W_s)(\omega)$. Clearly for two continuous functions f and g it holds that $(g \circ f)$ is continuous. Then with $g(f) = -f$ and $f(t) = W_t(\omega)"/ >$ it follows that $(-W_t) = (g \circ W)(t)$ is also continuous.

Solution (ii).

Much like the previous exercise we define a new process and show the properties hold. Let $s \geq 0$ be chosen arbitrary. Now define $X_t = W(t+s) - W(s)$.

First, we let $t = 0$ and see

$$X_0 = W(0+s) - W(s) = W(s) - W(s) = 0.$$

Secondly, we have that for $r < u$:

$$X_u - X_r = W(u+s) - W(s) - (W(r+s) - W(s)) = W(u+s) - W(r+s) \sim \mathcal{N}(0, u+s-(r+s)) = \mathcal{N}(0, u-r).$$

and since for $r < u \leq k < l$ the translation $r+s < u+s \leq k+s < l+s$ still holds and $X_l - X_k = W(l+s) - W(k+s)$ and $X_u - X_r = W(u+s) - W(k+s)$ are independent. Finally since $W_t(\omega)$ is continuous in t hence the translation W_{t+s} is continouos. Adding a constant yields a function that is also continuous, hence X_t is continuous.

Solution (iii).

Let $c > 0$ be given. We show that

$$X_t = cW\left(\frac{t}{c^2}\right)$$

is a Brownian motion. We simply show the four properties. Let $t = 0$ and notice

$$X_0 = cW\left(\frac{0}{c^2}\right) = cW(0) = 0.$$

The second property follows from seperate transformation and that for $r < u \leq s < t$ we consider

$$X_u - X_r = c\left(W\left(\frac{u}{c^2}\right) - W\left(\frac{r}{c^2}\right)\right) \qquad \text{and} \qquad X_t - X_s = c\left(W\left(\frac{t}{c^2}\right) - W\left(\frac{s}{c^2}\right)\right)$$

and since $c, r, u, t, s > 0$ we have the same order for the scaled version of r, u, t, s and hence we have two independent RV scaled by c. Then by seperate transformations the variables is still independent. Next for the third property:

$$X_t - X_s = c\left(W\left(\frac{t}{c^2}\right) - W\left(\frac{s}{c^2}\right)\right) \sim \mathcal{N}\left(c \cdot 0, c^2\left(\frac{t}{c^2} - \frac{s}{c^2}\right)\right) = \mathcal{N}(0, t-s).$$

Where we use the properties of scaling a normal distributed random variable i.e. for $c > 0$ and $N \sim \mathcal{N}(\mu, \sigma^2)$ it follows that $cN \sim \mathcal{N}(c\mu, c^2\sigma^2)$. Finally, the forth property follows since $g(f) = cf$ is continuous and $h(t) = t/c^2$ is continuous, then for any continuous function $f(s)$ it follows that $(g \circ f \circ h) = g(f(h(t)))$ is continuous.

Proposition B.37. Let $(\Omega, \mathcal{F}, P)$ be a given probability space, let $\mathcal{G}$ be a sub-sigma-algebra of $\mathcal{F}$, and let X be a square integrable random variable. Consider the problem of minimizing

$$E\left[(X - Z)^2\right]$$

where Z is allowed to vary over the class of all square integrable $\mathcal{G}$ measurable random variables. The optimal solution $\hat{Z}$ is then given by.

$$\hat{Z} = E[X|\mathcal{G}].$$

Exercise 2. *(Bjork, exercise B.11.)* Prove proposition B.37 by going along the following lines.

a. Prove that the "estimation error" $X - E[X|\mathcal{G}]$ is orthogonal to $L^2(\Omega, \mathcal{G}, P)$ in the sence that for any $Z \in L^2(\Omega, \mathcal{G}, P)$ we have

$$E[Z \cdot (X - E[X|\mathcal{G}])] = 0$$

b. Now prove the proposition by writing

$$X - Z = (X - E[X|\mathcal{G}]) + (E[X|\mathcal{G}] - Z)$$

and use the result just proved.

Solution (a).

Let $X \in L^2(\Omega, \mathcal{F}, P)$ be a random variable. Now consider an arbitrary $Z \in L^2(\Omega, \mathcal{G}, P)$. Recall that $\mathcal{G} \subset \mathcal{F}$ and so X is also in $Z \in L^2(\Omega, \mathcal{G}, P)$, as it is bothe square integrable and $\mathcal{G}$-measurable. Then

$$E\left[Z \cdot (X - E[X|\mathcal{G}])\right] = E\left[Z \cdot X\right] - E\left[Z \cdot E[X|\mathcal{G}]\right].$$

Then by using the law of total expectation and secondly that Z is $\mathcal{G}$-measurable we have that

$$E\left[Z \cdot X\right] = E\left[E[Z \cdot X|\mathcal{G}]\right] = E\left[Z \cdot E[X|\mathcal{G}]\right].$$

Combining the two equations gives the desired result.

Solution (b).

Obviously, we have that

$$X - Z = X - Z + E[X|\mathcal{G}] - E[X|\mathcal{G}] = (X - E[X|\mathcal{G}]) + (E[X|\mathcal{G}] - Z).$$

Then squaring the terms gives

$$(X - Z)^2 = (X - E[X|\mathcal{G}])^2 + (E[X|\mathcal{G}] - Z)^2 + 2(X - E[X|\mathcal{G}])(E[X|\mathcal{G}] - Z)$$

Taking expectation on each side and using linearity of the expectation we have that

$$E[(X - Z)^2] = E\left[(X - E[X|\mathcal{G}])^2\right] + E\left[(E[X|\mathcal{G}] - Z)^2\right] + 2E\left[(X - E[X|\mathcal{G}])(E[X|\mathcal{G}] - Z)\right].$$

We can now use that $E[X|\mathcal{G}] - Z$ is $\mathcal{G}$-measurable with the above result on the last term.

$$E[(X - Z)^2] = E\left[(X - E[X|\mathcal{G}])^2\right] + E\left[(E[X|\mathcal{G}] - Z)^2\right].$$

Now since X is given the term $E\left[(X - E[X|\mathcal{G}])^2\right]$ is simply a constant not depending on the choice og Z. The optimal choice of Z is then $E[X|\mathcal{G}]$ since this minimizes the second term. The statement is then proved.

Exercise 3. Discuss the following theory/results of Moment generating functions (Laplace transform).

Let X be a random variable with distribution function $F(x) = P(X \leq x)$ and Y be a random variable with distribution function $G(y) = P(Y \leq y)$.

Definition. The moment generating function or Laplace transform of X is

$$\psi_X(\lambda) = E\left[e^{\lambda X}\right] = \int_{-\infty}^{\infty} e^{\lambda x} dF(x)$$

provided the expectation is finite for $|\lambda| < h$ for some $h > 0$.

The MGF uniquely determine the distribution of a random variable, due to the following result.

Theorem 1. *(Uniqueness)* If $\psi_X(\lambda) = \psi_Y(\lambda)$ when $|\lambda| < h$ for some $h > 0$, then X and Y has the same distribution, that is, $F = G$.

There is also the following result of independence for Moment generating functions.

Theorem 1. *(Independence)* If

$$E\left[e^{\lambda_1 X + \lambda_2 Y}\right] = \psi_X(\lambda_1)\psi_Y(\lambda_2)$$

for $|\lambda_i| < h$ for $i = 1, 2$ for some $h > 0$, then X and Y are independent random variables.

Example. Recall that the Moment generating function of a normal (Gaussian) distribution is given by

$$\psi_X(\lambda) = E\left[e^{\lambda X}\right] = \exp\left(\lambda\mu + \frac{\lambda^2}{2}\sigma^2\right)$$

where X is normally distributed with mean μ and variance σ^2 and $\lambda \in \mathbb{R}$ is a constant. Since a Brownian motion $W(t)$ is normally distributed with zero mean and variance t, we have that

$$E[\exp(\lambda W(t))] = \exp\left(\frac{\lambda^2}{2}t\right).$$

Discussion.

Exercise 4. *(Bjork, exercise C.8.(a-c))* Let W be a Brownian motion. Notice that for the natural filtration $\mathcal{F}_s = \sigma(W_t | t \leq s)$ $W_t - W_s$ is independent of $\mathcal{F}_s$

a. Show that W_t is a martingale.
b. Show that $W_t^2 - t$ is a martingale.
c. Show that $\exp(\lambda W_t - \frac{\lambda^2}{2}t)$ is a martingale.

Solution (a).

We show that for the natural filtration that W_t is a martingale. This include showing integrability and the martingale property. For the first we note that for a normal distributed random variable with mean 0 we have

$$E[|N|] = \int_{-\infty}^{\infty} |x| dF_N(x) = 2\int_0^{\infty} x dF_N(x)$$

since the distribution is symmetric. Substituting the distribution function $\Phi(x) = P(N \leq x)$ in we see that

$$E[|N|] = 2\int_0^{\infty} x d\Phi(x) = 2\int_0^{\infty} x\frac{1}{\sqrt{2\pi\sigma^2}}e^{-x^2/(2\sigma^2)}dx = (*)$$

by substituting $u = x^2/(2\sigma^2)$ $(x = \sqrt{2\sigma^2 u})$ we have that

$$\frac{dx}{du} = \frac{1}{2}\sqrt{2\sigma^2 u}2\sigma^2 = (\sigma^2)^{3/2}\sqrt{2}u \iff dx = (\sigma^2)^{3/2}\sqrt{2}u\ du$$

hence

$$(*) = \frac{2}{\sqrt{2\pi\sigma^2}}\int_0^{\infty}\sqrt{2\sigma^2 u}e^{-u}(\sigma^2)^{3/2}\sqrt{2}u\ du = \frac{2\sqrt{2\sigma^2}(\sigma^2)^{3/2}\sqrt{2}}{\sqrt{2\pi\sigma^2}}\int_0^{\infty}\sqrt{u}e^{-u}u\ du.$$

This then simplify to

$$(*) = \frac{(2\sigma^2)^{3/2}}{\sqrt{\pi}} \int_0^\infty u^{3/2} e^{-u}\ du = (2\sigma^2)^{1/2} \sqrt{\frac{2\sigma^2}{\pi}} \int_0^\infty u^{3/2} e^{-u}\ du = \sqrt{\frac{2\sigma^2}{\pi}} < \infty.$$

(Obviously the above is not derived correctly, but the end expression is valid, source: link) However since

$$W_t = W_t - 0 = W_t - W_0 \sim \mathcal{N}(0, t)$$

we have that $E|W_t| < \infty$ as desired.

Next, we have that

$$E[W_t|\mathcal{F}_s] = E[W_t - W_s|\mathcal{F}_s] + W_s = 0 + W_s = W_s.$$

In the above we used that $W_t - W_s$ is $\mathcal{F}_s$-measurable with mean 0. Then it follows that W_t is a martingale.

Solution (b).

Let $M_t = W_t^2 - t$. First, we observe that two measurable functions composed is still a measurable function. Hence we know that M_t is measurable wrt. the filtration since W_t is measurable and $w \mapsto w^2 + t$ is measurable. Secondly, we have that

$$E[|W_t^2 - t|] \leq E|W_t^2| + E|t| = t + t = 2t < \infty$$

where we use the triangle inequality. Thirdly, for the martingale property we have that for $t > s$:

$$E[M_t|\mathcal{F}_s] = E[W_t^2 - t|\mathcal{F}_s] = E[W_t^2 + W_s^2 - 2W_tW_s - W_s^2 + 2W_tW_s - t|\mathcal{F}_s]$$

which by linearity and independence of increments to the filtration gives

$$E[M_t|\mathcal{F}_s] = E[(W_t - W_s)^2 - W_s^2 + 2W_tW_s - t|\mathcal{F}_s] = t - s - t + E[2W_tW_s - W_s^2|\mathcal{F}_s]$$

However since W_s is measurable wrt. the filtration at time s the above is

$$E[M_t|\mathcal{F}_s] = 2W_sE[W_t|\mathcal{F}_s] - W_s^2 - s = 2W_s^2 - W_s^2 - s = W_s^2 - s = M_s.$$

Since from (a) we know that W_t is a martingale. Then we arrive at the desired result.

Solution (c).

Let $M_t = \exp\left(\lambda W_t - \frac{\lambda^2}{2}t\right)$. First, by composition of measurable functions M_t is $\mathcal{F}_t$-measurable. Secondly, we have using the MGF for a normal distributed random variable:

$$E|M_t = E\left(\exp\left(\lambda W_t - \frac{\lambda^2}{2}t\right)\right) \leq E\left(\exp\left(\lambda W_t\right)\right) = \exp\left(\frac{\lambda^2}{2}t\right) < \infty.$$

Thirdly, we consider

$$E[M_t|\mathcal{F}_s] = E\left[\left(\exp\left(\lambda W_t - \frac{\lambda^2}{2}t\right)\right)\middle|\mathcal{F}_s\right] = \exp\left(-\frac{\lambda^2}{2}t\right) E\left[\left(\exp\left(\lambda W_t\right)\right)\middle|\mathcal{F}_s\right].$$

By adding and subtracting W_s in the exponent we get

$$\begin{aligned} E[M_t|\mathcal{F}_s] &= \exp\left(-\frac{\lambda^2}{2}t\right) E\left[\left(\exp\left(\lambda(W_t - W_s) + \lambda W_s\right)\right)\middle| \mathcal{F}_s\right] \\ &= \exp\left(-\frac{\lambda^2}{2}t\right) \exp\left(\frac{\lambda^2}{2}(t-s)\right) E\left[\left(\exp\left(\lambda W_s\right)\right)\middle| \mathcal{F}_s\right]. \end{aligned}$$

Using that $E\left[\left(\exp\left(\lambda W_s\right)\right)\middle| \mathcal{F}_s\right] = \exp\left(\lambda W_s\right)$ and combining the exponents gives the desired:

$$E[M_t|\mathcal{F}_s] = \exp\left(\lambda W_s - \frac{\lambda^2}{2}s\right) = M_s.$$

Week 2

Material

MFE refers to the book by McNeil, Frey, and Embrechts, while HL refers to the notes by Hult and Lindskog posted on Absalon.

- Var-Cov method, simulation, importance sampling and bootstrap. For the Var-Cov method, see MFE Sec. 9.2. For importance sampling and bootstrap, see the supplementary reading in Absalon.
- Extreme value theory: MFE Ch. 5 or alternative reading in HL (suggested for this part).

Theory

Exercises

Week 3

Material

MFE refers to the book by McNeil, Frey, and Embrechts, while HL refers to the notes by Hult and Lindskog posted on Absalon.

- Spherical and elliptical distributions: MFE Sec. 6.3.
- Spherical and elliptical distributions, cont.; introduction to copulas.

Theory

Exercises

Week 4

Material

MFE refers to the book by McNeil, Frey, and Embrechts, while HL refers to the notes by Hult and Lindskog posted on Absalon.

- Copulas: MFE Sec. 7.1-7.5 (Sklar's theorem, Frechet bounds).
- Copulas cont. (Transformations; examples; Archimedean copulas).

Theory

Exercises

Week 5

Material

MFE refers to the book by McNeil, Frey, and Embrechts, while HL refers to the notes by Hult and Lindskog posted on Absalon.

- Copulas cont. (simulating Archimedean copulas, statistical methods, measures of dependence).
- Credit risk: the Merton model. MFE Ch. 10, particularly Sec. 10.3. [This lecture will only last two hours.]

Theory

Exercises

Week 6

Material

MFE refers to the book by McNeil, Frey, and Embrechts, while HL refers to the notes by Hult and Lindskog posted on Absalon.

- Portfolio credit risk: MFE Ch. 11, specifically Sections 11.1-11.3.
- Portfolio credit risk, cont. [This lecture will only last two hours.]

Theory

Exercises

Week 7

Material

MFE refers to the book by McNeil, Frey, and Embrechts, while HL refers to the notes by Hult and Lindskog posted on Absalon.

- Intro. to operational risk. Stochastic processes in risk management: stochastic models for operational risk; financial time series models. Connections to non-life insurance models and estimates (last lecture).

Theory

Exercises