Lévy Characterization of Brownian motion and proof of Girsanov theorem (d=1)

Theorem (Lévy Characterization of Brownian motion).
Let $X(t)$ be an Ito process with $X(0) = 0$. Then $X(t)$ is a Brownian motion if and only if the two processes $X(t)$ and $X^2(t) - t$ are (continuous) martingales.

Proof. Assume $X(t)$ is a Brownian motion then we have that $X(t)$ and $X^2(t) - t$ are martingales.

Assume $X(t)$ and $Y(t) = X^2(t) - t$ are martingales. Since $X(t)$ is an Ito process and a martingale then for an integrand $\varphi(t)$ the stochastic integral $\varphi(t)\,dX(t)$ $(= \varphi(t)\sigma(t)\,dW^P(t)))$ is a martingale by Lemma 4.10. By Ito formula, the dynamics of $Y(t)$ is

$$dY(t) = 2X(t)dX(t) + \big(dX(t)\big)^2 - dt.$$

Since $dY(t)$ and $2X(t)dX(t)$ are martingales then $\big(dX(t)\big)^2 = dt$ by Lemma 4.10. For fix $\theta \in$ we define $f(t,x) = e^{\theta x - (\theta^2/2)t}$ and apply Ito formula

$$\begin{aligned} df(t,X(t)) &= f_t(t,X(t))\,dt + f_x(t,X(t))\,dX(t) + \frac{1}{2} f_{xx}(t,X(t))\,(dX(t))^2 \\ &= -\frac{\theta^2}{2} f(t,X(t))dt + f_x(t,X(t))\,dX(t) + \frac{\theta^2}{2} f(t,X(t))\,dt \\ &= f_x(t,X(t))\,dX(t). \end{aligned}$$

Then $f(t,X(t))$ is also a martingale. For $0 \le s < t$ the martingale property imply that

$$\mathbf{E}[e^{\theta X(t) - (\theta^2/2)t} | \mathcal{F}_s] = e^{\theta X(s) - (\theta^2/2)s}$$

and multiply with $e^{-\theta X(s)}$ and $e^{(\theta^2/2)t}$ we get that

$$\mathbf{E}[e^{\theta(X(t) - X(s))} | \mathcal{F}_s] = e^{(\theta^2/2)(t-s)}.$$

By the definition of conditional expectation we get for $A \in \mathcal{F}_s$ that

$$\mathbf{E}[e^{\theta(X(t)-X(s))} 1_A] = \mathbf{E}[e^{(\theta^2/2)(t-s)} 1_A] = e^{(\theta^2/2)(t-s)} \mathbf{P}(A).$$

Thus $X(t) - X(s)$ is normally distributed with mean 0, variance $t - s$, and is independent of $\mathcal{F}_s$. Thus $X(t)$ is a Brownian motion.

Proof of Girsanov theorem (Theorem 12.3) with $d = 1$.
Using Lévy Characterization of Brownian motion to show that $W^Q(t)$ is a Brownian motion under $\mathbf{Q}$ we have to verify that $W^Q(t)$ and $W^Q(t)^2 - t$ are martingales under $\mathbf{Q}$. By Proposition C.13 this is equivalent to show that $W^Q(t)L(t)$ and $(W^Q(t)^2 - t)L(t)$ are martingales under $\mathbf{P}$. First we compute

$$\begin{aligned} d\big(W^Q(t)L(t)\big) & \\ &= L(t)\,dW^Q(t) + W^Q(t)\,dL(t) + dW^Q(t)\,dL(t) \\ &= L(t)\,(dW^Q(t) - \phi(t)\,dt) + W^Q(t)\phi(t)L(t)\,dW^P(t) + \phi(t)L(t)\,dt \\ &= L(t)(1 + W^Q(t)\phi(t))\,dW^P(t) \end{aligned}$$

and next

$$\begin{aligned} d\big((W^Q(t)^2 - t)L(t)\big) & \\ &= -L(t)\,dt + 2W^Q(t)L(t)\,dW^Q(t) + L(t)\,(dW^Q(t))^2 + (W^Q(t)^2 - t)\,dL(t) + 2W^Q(t)\,dW^Q(t)dL(t) \\ &= -L(t)\,dt + 2W^Q(t)L(t)\,(dW^p(t) - \phi(t)\,dt) + L(t)\,dt \\ &\quad + (W^Q(t)^2 - t)\phi(t)L(t)\,dW^P(t) + 2W^Q(t)\phi(t)L(t)\,dt \\ &= L(t)(2W^Q(t) + (W^Q(t)^2 - t)\phi(t))\,dW^P(t). \end{aligned}$$

We see that the two processes are martingales under $\mathbf{P}$.