University of Copenhagen

Continuous time finance (FinKont)

Exam Papers

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Problem 2

Consider a standard Black-Scholes model, that is, a model consisting of a bank account $B(t)$ with P-dynamics given by

$$dB(t) = rB(t)\ dt,$$

with $B(0) = 1$ and a stock $S(t)$ with P-dynamics given by

$$dS(t) = \alpha S(t)\ dt + \sigma S(t)\ d\overline{W}(t),$$

with $S(0) = s > 0$ where $r, \alpha \in \mathbb{R}$ and $\sigma > 0$ are constants and $\overline{W}(t)$ is a P-Brownian motion. Let $T > 0$ be a given and fixed (expiry) date.

Consider the derivative that at time T pays $X = \min\Big[\max\Big[S(T), K_1\Big], K_2\Big]$ where $0 < K_1 < K_2$ are constants. Let $F(t, s)$ be the pricing function of the derivative.

a. i. Determine the equations satisfied by the pricing function $F(t, s)$.
   ii. Find a hedging portfolio for the derivative X. (Hint: Draw a picture of the payoff function).

Let $h(t) = \Big(h_0(t), h_1(t)\Big)$ be a self-financing portfolio given by

$$h_0(t) = (1-u)\frac{V^h(t)}{B(t)},\ h_1(t) = u\frac{V^h(t)}{S(t)}$$

where u is a constant and set $V^h(0) = 1$. Note that $h_0(t)$ is the number of units of the bank account at time t, and $h_1(t)$ is the number of shares in the stock at time t, and $V^h(t)$ denotes the associated value process. Consider the derivative that at time T pays $Y = \sqrt{V^h(T)}$.

b. Determine the arbitrage free price of derivative Y at time $t = 0$.

Solution (a).

Solution (b).

Problem 3

Solution (a).

Solution (b).

Solution (c).

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Solution (e).

Exam 2017/18

Problem 1

Let W_t denote a Brownian motion and let

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma(\{W_s \mid 0 \le s \le t\}).$$

Let $T > 0$ be a given and fixed time.

Let $f(t)$ be a bounded deterministic continuous function. Define the two processes

$$\begin{cases} X_t = \int_0^t f(u)\ dW_u, \\ M_t^{(\lambda)} = \exp\left\{\lambda X_t - \frac{\lambda^2}{2}\int_0^t f^2(u)\ du\right\}, \end{cases}$$

where $\lambda \in \mathbb{R}$ is a constant.

a. Show that $M^{(\lambda)}$ is a martingale with $E[M_t^{(\lambda)}] = 1$.

Let $0 < s < t$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ be given and fixed.

b. i. Show that

$$\begin{aligned} M_s^{(\lambda_1)} &= E\left[\frac{M_s^{(\lambda_1)} M_t^{(\lambda_2)}}{M_s^{(\lambda_2)}} \;\middle|\; \mathcal{F}_s\right] \\ &= E\left[\exp\left\{\lambda_1 X_s + \lambda_2(X_t - X_s) - \frac{1}{2}\lambda_1^2\int_0^s f^2(u)\ du - \frac{1}{2}\lambda_2^2\int_s^t f^2(u)\ du\right\} \;\middle|\; \mathcal{F}_s\right] \end{aligned}$$

ii. Show that X_s and $X_t - X_s$ are normally distributed and independent.

c. Compute the mean value of $M_T^{(\lambda)}\log(M_T^{(\lambda)})$.

Solution (a).

First, we see that since X_t is on integral form we know that

$$\begin{cases} dX_t = f(t)\ dW_t \\ X_0 = 0. \end{cases}$$

Hence we may represent M as $M_t^{(\lambda)} = g(t, X_t, Y_t)$ given by

$$g(t,x,y) = \exp\left\{\lambda x - \frac{\lambda^2}{2}y\right\},$$

where $Y_t = \int_0^t f^2(u)\ du$ with dynamics

$$\begin{cases} dY_t = f^2(t)\ dt \\ Y_0 = 0. \end{cases}$$

Hence by the multidimensional Ito's formula we have the dynamics of M given by

$$\begin{aligned} dM_t^{(\lambda)} &= g_t\ dt + g_x\ dX_t + g_y\ dY_t + \frac{1}{2}g_{yy}\ (dY_t)^2 + \frac{1}{2}g_{xx}\ (dX_t)^2 + f_{xy}(dX_t)(dY_t) \\ &= 0 + \lambda g\ dX_t - \frac{\lambda^2}{2}g\ dY_t + 0 + \frac{1}{2}\lambda^2 g\ (dX_t)^2 + 0 \\ &= \lambda M_t^{(\lambda)} f(t)\ dW_t - \frac{1}{2}\lambda^2 M_t^{(\lambda)} f^2(t)\ dt + \frac{1}{2}\lambda M_t^{(\lambda)} f^2(t)\ dt \\ &= \lambda f(t) M_t^{(\lambda)}\ dW_t, \end{aligned}$$

And so we see that M is a martingale as it only has dynamics wrt. the Brownian motion W (assuming $\lambda f_t M_t^{(\lambda)} \in \mathcal{L}^2$). Furthermore we have that

$$M_0^{(\lambda)} = g(0, X_0, Y_0) = \exp\left\{\lambda X_0 - \frac{1}{2}\lambda^2 Y_0\right\} = e^0 = 1$$

and so we have $E[M_t^{(\lambda)}] = M_0^{(\lambda)} = 1$ as desired. □

Solution (b).

"(i)" We have from the previous exercise

$$\begin{aligned}
&\frac{M_s^{(\lambda_1)} M_t^{(\lambda_2)}}{M_s^{(\lambda_2)}} \\
&= \exp\left\{\lambda_1 X_s - \frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du\right\} \exp\left\{\lambda_2 X_t - \frac{1}{2}\lambda_2^2 \int_0^t f^2(u)\ du\right\} \exp\left\{\frac{1}{2}\lambda_2^2 \int_0^s f^2(u)\ du - \lambda_2 X_s\right\} \\
&= \exp\left\{\lambda_1 X_s - \frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du + \lambda_2 X_t - \frac{1}{2}\lambda_2^2 \int_0^t f^2(u)\ du + \frac{1}{2}\lambda_2^2 \int_0^s f^2(u)\ du - \lambda_2 X_s\right\} \\
&= \exp\left\{\lambda_1 X_s + \lambda_2 (X_t - X_s) - \frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du - \frac{1}{2}\lambda_2^2 \int_s^t f^2(u)\ du\right\}
\end{aligned}$$

and so the conclusion follows. □

"(ii)" We have that from lemma 4.18 that

$$X_s = \int_0^s f(u)\ dW_u \sim \mathcal{N}\left(0, \int_0^s f^2(u)\ dW_u\right)$$

furthermore we have that

$$X_t - X_s = \int_s^t f(u)\ dW_u \sim \mathcal{N}\left(0, \int_s^t f^2(u)\ dW_u\right).$$

In regard to the independence claim we could check identity below

$$E[e^{t_1 X} e^{t_2 Y}] = E[e^{t_1 X}] E[e^{t_2 Y}]$$

where X, Y are independent random variables. The above identity holds if and only if X and Y are independent. From above we have that

$$M_s^{(\lambda_1)} = E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)} \mid \mathcal{F}_s] e^{-\frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du - \frac{1}{2}\lambda_2^2 \int_s^t f^2(u)\ du}$$

and so taking expectation we have

$$1 = E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)}] e^{-\frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du - \frac{1}{2}\lambda_2^2 \int_s^t f^2(u)\ du}$$

Which the gives

$$E[e^{\lambda_1 X_s} e^{\lambda_2 (X_t - X_s)}] = e^{\frac{1}{2}\lambda_1^2 \int_0^s f^2(u)\ du + \frac{1}{2}\lambda_2^2 \int_s^t f^2(u)\ du} = E[e^{\lambda_1 X_s}] E[e^{\lambda_2 (X_t - X_s)}]$$

and so the conclusion is that X_s and $X_t - X_s$ are independent. $\square$

Solution (c).

We recall the definition of $M_t^{(\lambda)}$ and observe that

$$\log M_t^{(\lambda)} = \lambda X_t - \frac{1}{2}\lambda^2 \int_0^t f^2(u)\ du.$$

Furthermore we have the dynamics of $M^{(\lambda)}$ given by the differential form

$$dM_t^{(\lambda)} = \lambda f(t) M_t^{(\lambda)}\ dW_t.$$

with $M_0^{(\lambda)} = 1$. Since we know that $M_t^{(\lambda)}$ is a martingale we have

$$E^P[M_T^{(\lambda)}] = E^P[M_0^{(\lambda)}] = 1,$$

and so we may define a new probability measure as

$$d\tilde{P} = M_T^{(\lambda)}\ dP$$

on $\mathcal{F}_T$. We then have a new Brownian motion $\tilde{W}$ such that

$$dW_t = \lambda f(t)\ dt + d\tilde{W}_t.$$

We can then see

$$\begin{aligned} E^P[M_T^{(\lambda)} \log M_T^{(\lambda)}] &= \int M_T^{(\lambda)} \log M_T^{(\lambda)}\ dP = \int M_T^{(\lambda)} \log M_T^{(\lambda)} \frac{1}{M_T^{(\lambda)}}\ d\tilde{P} \\ &= \int \log M_T^{(\lambda)}\ d\tilde{P} = E^{\tilde{P}}[\log M_T^{(\lambda)}]. \end{aligned}$$

Then we can evaluate the mean value by seeing the X has representation wrt. $\tilde{P}$ by

$$X_t = \int_0^t f(u)\ (\lambda f(u)\ du + d\tilde{W}_u) = \lambda \int_0^t f^2(u)\ du + \int_0^t f(u)\ d\tilde{W}_u.$$

Giving that

$$
\begin{aligned}
E^{P}[M_T^{(\lambda)} \log M_T^{(\lambda)}] &= E^{\tilde{P}}[\log M_T^{(\lambda)}] \\
&= E^{\tilde{P}}\left[\lambda X_T - \frac{1}{2}\lambda^2 \int_0^T f^2(u)\ du\right] \\
&= E^{\tilde{P}}\left[\lambda^2 \int_0^T f^2(u)\ du + \lambda \int_0^T f(u)\ d\tilde{W}_u - \frac{1}{2}\lambda^2 \int_0^T f^2(u)\ du\right] \\
&= \lambda E^{\tilde{P}}\left[\frac{1}{2}\lambda \int_0^T f^2(u)\ du + \int_0^T f(u)\ d\tilde{W}_u\right] \\
&= \frac{1}{2}\lambda^2 \int_0^T f^2(u)\ du + \lambda E^{\tilde{P}}\left[\int_0^T f(u)\ d\tilde{W}_u\right] \\
&= \frac{1}{2}\lambda^2 \int_0^T f^2(u)\ du
\end{aligned}
$$

Since

$$\tilde{X}_T = \int_0^T f(u)\ d\tilde{W}_u,$$

is a $\tilde{P}$-martingale. $\square$

Problem 2

Consider a standard Black-Scholes model, that is, a model consisting of a bank account B_t with P-dynamics given by

$$dB_t = rB_t\ dt,\ B_0 = 1$$

and a stock S_t with P-dynamics given by

$$dS_t = \alpha S_t\ dt + \sigma S_t\ d\overline{W}_t,\ S_0 = s > 0$$

where $r, \alpha \in \mathbb{R}$ and $\sigma > 0$ are constants and $\overline{W}_t$ is a P-Brownian motion. Let $T > 0$ be a given and fixed date.

Consider the derivative that at time T pays

$$X = \max\left\{\min\left\{S_T, K_2\right\}, K_1\right\},$$

where $0 < K_1 < K_2$ are constants.

a. Determine the arbitrage free price of derivative X at time $t < T$.

Consider a new derivative that at time T pays

$$Y = (S_T^2 - K^2)^+ - (K^2 - S_T^2)^+.$$

b. i. Determine the arbitrage free price of derivative Y at time $t < T$.
   ii. Find a hedging portfolio for derivative Y.

Let $h(t) = (h_0(t), h_1(t))$ be a portfolio where

$$h_0(t) = -e^{r(T-2t)+\sigma^2(T-t)} S^2(t)$$

is the number of units in the bank account at time t and

$$h_1(t) = 2e^{(r+\sigma^2)(T-t)} S(t)$$

is the number of shares in the stock at time t. Let $V^h(t)$ denote the associated value process.

c. Determine whether the portfolio h is self-financing or not.
d. Compute $V^h(T)$.

Solution (a).

We see that the derivative is the bull spread given by the payout function

$$X = \begin{cases} K_2 & \text{if } S_T > K_2, \\ S_T & \text{if } K_1 \le S_T \le K_2, \\ K_1 & \text{if } S_T < K_1. \end{cases}$$

We know from exercise 10.3 that this can be replicated by holding K_1 bonds, one call option with strike K_1 and a short on a call with strike K_2. The arbitrage free price of the derivative is then the value process of the mentioned portfolio i.e.

$$\Pi_t[X] = K_1 e^{-r(T-t)} + c(K_1; t, T) - c(K_2; t, T),$$

where c denotes the pricing function for a European call option (non-instructive parameters supressed). □

Solution (b).

(i): We start by seeing that the derivative pays out

$$Y = \begin{cases} S_T^2 - K^2 & \text{if } S_T^2 \ge K^2, \\ -(K^2 - S_T^2) & \text{if } S_T^2 < K^2. \end{cases}$$

hence the payout is $Y = S_T^2 - K^2 = \Phi(S_T)$ where $\Phi(s) = s^2 - K^2$. That is Y is in fact a simple claim. We have from the risk neutral valueation formula 7.11 that

$$\begin{aligned} \Pi_t[Y] &= e^{-r(T-t)} E_{t,s}^Q[S_T^2 - K^2] \\ &= e^{-r(T-t)} E_{t,s}^Q[S_T^2] - e^{-r(T-t)} K^2. \end{aligned}$$

Recall that under the martingale measure Q we have that S_t is a GBM hence

$$S_t = s \cdot \exp\left\{ \left(r - \frac{1}{2}\sigma^2 \right)(T-t) + \sigma \left(W_T^Q - W_t^Q \right) \right\}$$

then

$$S_T^2 = s^2 \cdot \exp\left\{ 2\left(r - \frac{1}{2}\sigma^2 \right)(T-t) + 2\sigma \left(W_T^Q - W_t^Q \right) \right\}.$$

Inserting this into the risk neutral valuation formula we get

$$\begin{aligned}
\Pi_t[Y] &= e^{-r(T-t)} E^Q_{t,s}[S_T^2] - e^{-r(T-t)} K^2 \\
&= e^{-r(T-t)} s^2 e^{2\left(r-\frac{1}{2}\sigma^2\right)(T-t)} E^Q\left[\exp\left\{2\sigma\left(W_T^Q - W_t^Q\right)\right\}\right] - e^{-r(T-t)}K^2 \\
&= e^{-r(T-t)} s^2 e^{2\left(r-\frac{1}{2}\sigma^2\right)(T-t)} e^{\frac{1}{2}4\sigma^2(T-t)} - e^{-r(T-t)}K^2 \\
&= e^{-r(T-t)}\left(s^2 e^{(2r-\sigma^2)(T-t)+\frac{1}{2}4\sigma^2(T-t)} - K^2\right) \\
&= e^{-r(T-t)}\left(s^2 e^{(2r+\sigma^2)(T-t)} - K^2\right).
\end{aligned}$$

The arbitrage free price of the derivative is then given above. □

(ii): From theorem 8.5 we can determine a hedging portfolio with weightings

$$\begin{aligned}
w_t^B &= \frac{\Pi_t - S_t\frac{\partial \Pi}{\partial s}}{\Pi_t} \\
&= 1 - \frac{S_t 2 S_t e^{-r(T-t)} e^{(2r+\sigma^2)(T-t)}}{e^{-r(T-t)}\left(S_t^2 e^{(2r+\sigma^2)(T-t)} - K^2\right)} \\
&= 1 - \frac{2S_t^2 e^{(2r+\sigma^2)(T-t)}}{S_t^2 e^{(2r+\sigma^2)(T-t)} - K^2} \\
&= 1 - \frac{2}{1 - K^2 S_t^{-2} e^{(2r+\sigma^2)(t-T)}} \\
w_t^S &= \frac{2}{1 - K^2 S_t^{-2} e^{(2r+\sigma^2)(t-T)}}.
\end{aligned}$$

In absolute terms we will hold the portfolio

$$\begin{aligned}
h_t^S &= 2S_t e^{-r(T-t)} e^{(2r+\sigma^2)(T-t)} \\
h_t^B &= \frac{e^{-r(T-t)}\left(s^2 e^{(2r+\sigma^2)(T-t)} - K^2\right) - S_t h_t^S}{B_t} \\
&= \frac{e^{-r(T-t)}\left(s^2 e^{(2r+\sigma^2)(T-t)} - K^2\right) - S_t h_t^S}{e^{rt}} \\
&= e^{-rT} s^2 e^{(2r+\sigma^2)(T-t)} - e^{-rT}K^2 - e^{-rt} S_t h_t^S.
\end{aligned}$$

The portfolio above will hedge Y with probability one. □

Solution (c).

We assume no dividends and no consumption that is $c_t = 0$ and $dD_t^i = 0$ for $i = 0, 1$. Then the portfolio is self-financing if and only if the value process has dynamics.

$$h_0(t)\ dB_t + h_1(t)\ dS_t = 0$$

This is given in lemma 6.12.

THE BELOW IS IN WORKS AND NOT CORRECT!

Now we have that the value process is given by

$$V_t^h = h_0(t)B_t + h_1(t)S_t.$$

Using the representation $V_t^h = f(h_0(t), B_t) + f(h_1(t), S_t)$ given by $f(x, y) = xy$ we have

$$dV_t^h = df(h_0(t), B_t) + df(h_1(t), S_t).$$

Using Ito's formula on each term we have

$$\begin{aligned} df(h_0(t), B_t) &= B_t \ dh_0(t) + h_0(t) \ dB_t + (dB_t)(dh_0(t)), \\ df(h_1(t), S_t) &= S_t \ dh_1(t) + h_1(t) \ dS_t + (dS_t)(dh_1(t)), \end{aligned}$$

since of cause $f_{xx} = f_{yy} = 0$. We can the determine the dynamics of the portfolio by

$$\begin{aligned} dh_0(t) &= -(-2t - \sigma^2)S_t^2 e^{r(T-2t)+\sigma^2(T-t)} \ dt \\ &\quad - 2S_t e^{r(T-2t)+\sigma^2(T-t)} \ dS_t \\ &\quad - \frac{1}{2} 2e^{r(T-2t)+\sigma^2(T-t)} \ (dS_t)^2 \\ &= (-2t - \sigma^2)h_0(t) \ dt + \frac{2}{S_t} h_0(t) \ (\mu S_t \ dt + \sigma S_t \ dW_t) + \frac{1}{S_t^2} h_0(t)\sigma^2 S_t^2 \ dt \\ &= (\mu - 1)2h_0(t) \ dt + 2\sigma h_0(t) \ dW_t \end{aligned}$$

and

$$\begin{aligned} dh_1(t) &= (-r - \sigma^2)2e^{(r+\sigma^2)(T-t)} S_t \ dt \\ &\quad + 2e^{(r+\sigma^2)(T-t)} \ dS_t + 0 \\ &= (-r - \sigma^2)h_1(t) \ dt + \frac{1}{S_t} h_1(t)(\mu S_t \ dt + \sigma S_t \ dW_t) \\ &= (-r - \sigma^2 + \mu)h_1(t) \ dt + h_1(t)\sigma \ dW_t \end{aligned}$$

And so in total

$$
\begin{aligned}
dV_t^h(t) &= df(h_0(t), B_t) + df(h_1(t), S_t) \\
&= B_t \; dh_0(t) + h_0(t) \; dB_t + (dB_t)(dh_0(t)) \\
&+ S_t \; dh_1(t) + h_1(t) \; dS_t + (dS_t)(dh_1(t)) \\
&= B_t \; ((\mu - 1)2h_0(t) \; dt + 2\sigma h_0(t) \; dW_t) + h_0(t) \; rB_t \; dt + 0 \\
&+ S_t \; ((-r - \sigma^2 + \mu)h_1(t) \; dt + h_1(t)\sigma \; dW_t) + h_1(t) \; (\mu S_t \; dt + \sigma S_t \; dW_t) + \sigma^2 S_t h_1(t) \; dt \\
&= \left[B_t(\mu - 1)2h_0(t) + h_0(t)rB_t + S_t(-r - \sigma^2 + \mu)h_1(t) + h_1\mu S_t + \sigma^2 S_t h_1(t)\right] \; dt \\
&+ \left[B_t 2\sigma h_0(t) + S_t h_1 \sigma + h_1 \sigma S_t\right] \; dW_t \\
&= \left[(2\mu - 2 + r)B_t h_0(t) + (-r + 2\mu)S_t h_1(t)\right] \; dt \\
&+ \left[B_t h_0(t) + h_1 S_t\right] 2\sigma \; dW_t \\
&= V_t^h 2\mu \; dt + V_t^h \; dW_t
\end{aligned}
$$

Solution (d).

We compute V_T^h easily by inserting h_0 and h_1 below

$$
\begin{aligned}
V_T^h &= B_T h_0(T) + S_T h_1(T) \\
&= B_T \left(-e^{r(T-2T)+\sigma^2(T-T)} S_T^2\right) + S_T \left(2e^{(r+\sigma^2)(T-T)} S_T\right) \\
&= -S_T^2 + 2S_T^2 = S_T^2.
\end{aligned}
$$

and so h hedge the payout $\Phi(S_T) = S_T^2$. $\square$

Problem 3

Consider a two-dimensional model. The market model consist of three assets: A bank account B_t and two stocks S_1 and S_2. The P-dynamics of B_t is

$$dB_t = rB_t \; dt, \; B_0 = 1,$$

where $r \in \mathbb{R}$ is a constant interest rate. The P-dynamics of S_1 and S_2 are given by

$$
\begin{aligned}
dS_1(t) &= \alpha_1 S_1(t) \; dt + \sigma_1 S_1(t) \; d\overline{W}_1(t), \qquad && S_1(0) = s_1 > 0, \\
dS_2(t) &= \alpha_2 S_2(t) \; dt + \sigma_2 S_2(t) \; d\overline{W}_2(t), \qquad && S_2(0) = s_2 > 0,
\end{aligned}
$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ are constants. Moreover, $\sigma_1 > 0$ is a constant and $\sigma_2(t) = \sigma_0 e^{-\gamma t}$ where $\sigma_0 > 0$ and $\gamma > 0$ are constants and $\overline{W}_1(t)$ and $\overline{W}_2(t)$ are two independent P-Brownian motions. The filtration is the one generated by the two Brownian motions, that is, $\mathcal{F}_t = \sigma(\overline{W}_1(s), \overline{W}_2(s) \mid 0 \leq s \leq t)$. Let $T > 0$ be a given and fixed (expiry) date.

a. i. Is the model arbitrage free?
   ii. Is the model complete?

Consider the derivative that at time T pays $X = S_1(T)S_2(T)$ and let $F(t, s_1, s_2)$ be the pricing function of the derivative.

b. i. Determine the arbitrage free price of derivative X at time $t = 0$.
   ii. Determine the equation satisfied by the pricing function $F(t, s_1, s_2)$.

Consider a new derivative that at time T pays $Y = \log(S_2(T))$.

c. Determine the arbitrage free price of derivative Y at time $t < T$.

Solution (a).

(i): We know that the model is arbitrage free if and only if there exist a martingale measure Q. This is equivalent with finding a likelihood process L with Radon-Nikodym derivative φ given by the solution to the equation

$$\sigma_t \varphi_t = r_t - \mu_t.$$

We see that

$$\sigma_t = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_0 e^{-\gamma t} \end{bmatrix} \Rightarrow \sigma_t^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 \\ 0 & e^{\gamma t}/\sigma_0 \end{bmatrix}.$$

Hence we trivially have **a solution** given by

$$\varphi_t = \begin{bmatrix} 1/\sigma_1 & 0 \\ 0 & e^{\gamma t}/\sigma_0 \end{bmatrix} \begin{bmatrix} r - \alpha_1 \\ r - \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{r-\alpha_1}{\sigma_1} \\ \frac{r-\alpha_2}{\sigma_0} e^{\gamma t} \end{bmatrix}.$$

Proposition 14.1 gives now that if L, given by

$$dL_t = \varphi_t^\top L_t \, dW_t, \; L_0 = 1,$$

is a martingale then the market is arbitrage free. This is true if the Novikov condition is satisfied. We have

$$E^P \left[e^{\frac{1}{2} \int_0^T \|\varphi_t\|^2 \, dt} \right] = e^{\frac{1}{2} \int_0^T \|\varphi_t\|^2 \, dt} = e^{\frac{1}{2} \int_0^T (\frac{r-\alpha_1}{\sigma_1})^2 + (\frac{r-\alpha_2}{\sigma_0} e^{\gamma t})^2 \, dt} < \infty$$

since of cause

$$\int_0^T (\frac{r-\alpha_1}{\sigma_1})^2 + (\frac{r-\alpha_2}{\sigma_0} e^{\gamma t})^2 \, dt = T(\frac{r-\alpha_1}{\sigma_1})^2 + (\frac{r-\alpha_2}{\sigma_0})^2 \int_0^T e^{2\gamma t} \, dt < \infty$$

for all $T \geq 0$. Then the Novikov condition is satisfied and L is martingale with $E[L_T] = 1$. □

(ii): The model is complete if the martingale measure is unique. This is equivalent with $Ker[\sigma_t] = \{0\}$ and since σ_t is invertible (diagonal) we have that the model is complete. □

Solution (b).

(i): We may determine the price of the derivative using the risk neutral valueation formula

$$\Pi_t[X] = E^Q \left[e^{-\int_t^T r(u) \, du} X \;\middle|\; \mathcal{F}_t \right]$$

Hence we have for $t = 0$ and $S_1(0) = s_1$ and $S_2(0) = s_2$ that

$$\Pi_0[X] = E^Q \left[e^{-\int_0^T r(u) \, du} X \;\middle|\; \mathcal{F}_0 \right] = e^{-rT} E^Q \left[S_1(T) S_2(T) \mid \mathcal{F}_0 \right],$$

Since we have that S_1 and S_2 have dynamics wrt. two independent Brownian motions we know that the price processes are independent. If we multiply by $B(T)/B(T)$ we obtain two martingale processes under the measure Q:

$$\begin{aligned}\Pi_0[X] &= e^{-rT}B(T)^2E^Q\left[\frac{S_1(T)}{B(T)}\;\middle|\;\mathcal{F}_0\right]E^Q\left[\frac{S_2(T)}{B(T)}\;\middle|\;\mathcal{F}_0\right]\\ &= e^{-rT}e^{2rT}s_1(0)s_2(0) = e^{rT}s_1(0)s_2(0),\end{aligned}$$

and so the arbitrage free price is given above. □

(ii): We have from Bjork (14.31) that Π satisfies the PDE below

$$\begin{cases} F_t + \sum_{i=1}^2 rs_iF_{s_i} + \frac{1}{2}\mathrm{tr}[\sigma_t^\top D(S)F_{ss}D(S)\sigma_t] - rF = 0\\ F(T,s_1,s_2) = \Phi(s_1,s_2)\end{cases}$$

The PDE is in detail

$$\begin{aligned}&0 + rS_1(t)S_2(t) + rS_2S_1 + \frac{1}{2}\mathrm{tr}\begin{bmatrix}S_1\sigma_1 & 0\\ 0 & S_2\sigma_0e^{-\gamma t}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}S_1\sigma_1 & 0\\ 0 & S_2\sigma_0e^{-\gamma t}\end{bmatrix} - r\Pi_t\\ &= 2rS_1(t)S_2(t) + \frac{1}{2}\mathrm{tr}\begin{bmatrix}S_1\sigma_1 & 0\\ 0 & S_2\sigma_0e^{-\gamma t}\end{bmatrix}\begin{bmatrix}0 & S_2\sigma_0e^{-\gamma t}\\ S_1\sigma_1 & 0\end{bmatrix}\\ &= 2rS_1(t)S_2(t) + \frac{1}{2}\mathrm{tr}\begin{bmatrix}0 & S_1S_2\sigma_0\sigma_1e^{-\gamma t}\\ S_1S_2\sigma_0\sigma_1e^{-\gamma t} & 0\end{bmatrix} - r\Pi_t\\ &= 2rS_1(t)S_2(t) - r\Pi_t = 0.\end{aligned}$$

or

$$F(t,s_1,s_2) = 2s_1s_2,\;\; F(T,s_1,s_2) = s_1s_2$$

this ends the question. □

Solution (c).

We have the derivative $Y = \log(S_2(T))$. By the risk neutral valuation formula we have that the arbitrage free price is given by

$$\Pi_t[Y] = E^Q\left[e^{-\int_t^T r(u)\ du}Y\;\middle|\;\mathcal{F}_t\right] = e^{-r(T-t)}E^Q\left[\log(S_2(T))\;|\;\mathcal{F}_t\right].$$

Under the measure Q the dynamics of S_2 is that of a GBM hence

$$d\log(S_2(t)) = \left(r - \frac{1}{2}\sigma_0^2e^{-2\gamma t}\right)\ dt + \sigma_0^2e^{-2\gamma t}\ dW_t^Q,$$

and so with the knowledge that $S_2(t) = s_2$ we have

$$
\begin{aligned}
\Pi_t[Y] &= e^{-r(T-t)} E^Q \left[\log(s_2) + \int_t^T \left(r - \frac{1}{2}\sigma_0^2 e^{-2\gamma s} \right) \, ds + \int_t^T \sigma_0^2 e^{-2\gamma t} \, dW_t^Q \,\middle|\, \mathcal{F}_t \right] \\
&= e^{-r(T-t)} \left[\log(s_2) + \int_t^T \left(r - \frac{1}{2}\sigma_0^2 e^{-2\gamma s} \right) \, ds \right] \\
&= e^{-r(T-t)} \left[\log(s_2) + r(T-t) - \frac{1}{2}\sigma_0^2 \int_t^T e^{-2\gamma s} \, ds \right] \\
&= e^{-r(T-t)} \left[\log(s_2) + r(T-t) + \frac{1}{4\gamma}\sigma_0 \left[e^{-2\gamma s} \right]_t^T \right] \\
&= e^{-r(T-t)} \left[\log(s_2) + r(T-t) + \frac{1}{4\gamma}\sigma_0 (e^{-2\gamma T} - e^{-2\gamma t}) \right].
\end{aligned}
$$

The arbitrage free price of the derivative is then given above. □

Exam 2018/19

Problem 1

Let $W(t)$ denote a Brownian motion and let $\mathcal{F}_t = \mathcal{F}_t^W$. Let $T > 0$ be a given and fixed time.

Consider the stochastic differential equation

$$dX(t) = \alpha \, dt + \sqrt{X(t)} \, dW(t),$$

and $X(0) = x > 0$ where $\alpha \in \mathbb{R}$.

a. i. Compute the mean value of $X(T)$.
   ii. Compute the variance of $X(T)$.

b. Find the solution of the partial differential equation

$$
\begin{aligned}
&4F_t(t,x) + 8x^2 F_{xx}(t,x) + 3xF_x(t,x) = 5F(t,x) \text{ for } t < T \text{ and } x > 0. \\
&F(T,x) = x^3.
\end{aligned}
$$

Let $\widetilde{W}(t)$ be another Brownian motion such that $W(t)$ and $\widetilde{W}(t)$ are two independent Brownian motions. Let $Y(t)$ and $Z(t)$ be two martingales given by the following dynamics

$$
\begin{aligned}
dY(t) &= W(t) \, dW(t) + \widetilde{W}(t) \, d\widetilde{W}(t), \\
dZ(t) &= \widetilde{W}(t) \, dW(t) - W(t) \, d\widetilde{W}(t).
\end{aligned}
$$

with $Y(0) = Z(0) = 0$.

c. Show that $M(t) = Y(t)Z(t)$ is a martingale.

Solution (a).

(i): We start by writing X on integral form given as

$$X(t) = x + \int_0^t \alpha \, dt + \int_0^t \sqrt{X(s)} \, dW(s).$$

Taking expectation yields.

$$E[X(t)] = E\left[x + \alpha t + \int_0^t \sqrt{X(s)}\; dW(s)\right] = x + \alpha t,$$

since

$$E\left[\int_0^t \sqrt{X(s)}\; dW(s)\right] = E\left[\int_0^0 \sqrt{X(s)}\; dW(s)\right] = 0.$$

This result follows from lemma 4.10 as the process $M_t = \int_0^t \sqrt{X(s)}\; dW(s)$ is a martingale. From this we have that $E[X(T)] = x + \alpha T$. $\square$

(ii): We have that the variance is given by

$$Var(X(t)) = E(X^2(t)) - (E(X(t))^2.$$

and so we have

$$\begin{aligned}
Var(X(t)) + E(X(t))^2 &= E\left[\left(x + t\alpha + \int_0^t \sqrt{X(s)}\; dW(s)\right)^2\right] \\
&= (x+\alpha t)^2 + E\left[\left(\int_0^t \sqrt{X(s)}\; dW(s)\right)^2\right] + 2(x+\alpha t)E\left[\int_0^t \sqrt{X(s)}\; dW(s)\right] \\
&= (x+\alpha t)^2 + E\left[\left(\int_0^t \sqrt{X(s)}\; dW(s)\right)^2\right].
\end{aligned}$$

Now by setting $Z(t) = \int_0^t \sqrt{X(s)}\; dW(s)$ we see that Z has dynamics $dZ(t) = \sqrt{X(t)}\; dW(t)$ with $Z(0) = 0$. By Ito's formula on the variable $f(t, Z(t))$ with $f(t, z) = z^2$ we have

$$\begin{aligned}
df(t, Z(t)) &= 0\; dt + 2Z(t)\; dZ(t) + \frac{1}{2}2\; (dZ(t))^2 \\
&= 2Z(t)\sqrt{X(t)}\; dW(t) + X(t)\; dt.
\end{aligned}$$

Obviously when taking expectation on $f(t, Z(t))$ we see that the integral part related to the Brownian motion is a martingale with mean 0 and then

$$E[f(t, Z(t))] = E\left[\int_0^t X(s)\; ds\right].$$

In total we have

$$Var(X(t)) = (x+\alpha t)^2 + E\left[\int_0^t X(s)\; ds\right] - (x+\alpha t)^2 = E\left[\int_0^t X(s)\; ds\right].$$

Moving the expectation inside the integral then gives

$$Var(X(t)) = \int_0^t (x + \alpha s)\; ds = xt + \frac{1}{2}\alpha t^2.$$

Inserting $t = T$ gives the desired result. □

Solution (b).

We see by dividing by 4 we have the PDE given by

$$F_t + 2x^2 F_{xx} + \frac{3}{4} x F_x = \frac{5}{4} F$$

hence by setting $r = 5/4$, $\mu = 3x/4$ and $\sigma^2 = 4x^2$ we have the boundary value problem

$$\begin{cases} F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx} - rF = 0, \\ F(T, x) = x^3. \end{cases}$$

From Feymann-Kac we know this has solution on $[0, T] \times \mathbb{R}$ given by the stochastic representation

$$F(t, x) = e^{-r(T-t)} E_{t,x}[X_T^3],$$

where X satisfies the SDE

$$dX_t = \frac{3}{4} X_t \; dt + 2X_t \; dW_t.$$

Giving that $X(t) = x$ and X is a GBM we have

$$X_T = x \cdot e^{\left(r - \frac{1}{2}2^2\right)(T-t) + 2(W_T - W_t)} = x \cdot e^{\frac{-5}{4}(T-t) + 2(W_T - W_t)}.$$

The relevant mean value is then

$$\begin{aligned} F(t, x) &= e^{-\frac{5}{4}(T-t)} E\left[x^3 \cdot e^{\frac{-15}{4}(T-t) + 6(W_T - W_t)}\right] \\ &= e^{-\frac{5}{4}(T-t)} x^3 e^{\frac{-15}{4}(T-t)} E\left[e^{6(W_T - W_t)}\right] \\ &= x^3 e^{\frac{-20}{4}(T-t)} e^{\frac{1}{2}6^2(T-t)} = x^3 e^{-5(T-t) + 18(T-t)} \\ &= x^3 e^{13(T-t)}. \end{aligned}$$

The solution is the given above. □

Solution (c).

We show that M has dynamics solely given in terms of Brownian motions. We have that $M(t) = f(t, Y(t), Z(t))$ for $f(t, y, z) = yz$ the dynamics given by Ito's formula:

$$dM(t) = 0 \; dt + Z(t) \; dY(t) + Y(t) \; dZ(t) + (dY(t))(dZ(t))$$

since the only second derivative not zero is $f_{yz} = f_{zy} = 1$. The product $(dY(t))(dZ(t))$ is computed first

$$\begin{aligned} (dY(t))(dZ(t)) &= (W(t) \; dW(t) + \widetilde{W}(t) \; d\widetilde{W}(t)) \cdot (\widetilde{W}(t) \; dW(t) - W(t) \; d\widetilde{W}(t)) \\ &= W(t)\widetilde{W}(t) \; dt - \widetilde{W}(t)W(t) \; dt = 0, \end{aligned}$$

where we use that $dW(t)d\widetilde{W}(t) = dt$ is the only non-zero term. Then we obviously have

$$\begin{aligned} dM(t) &= Z(t)\ dY(t) + Y(t)\ dZ(t) \\ &= Z(t)W(t)\ dW(t) + Z(t)\widetilde{W}(t)\ d\widetilde{W}(t) + Y(t)\widetilde{W}(t)\ dW(t) - Y(t)W(t)\ d\widetilde{W}(t). \end{aligned}$$

Giving that $M(t)$ is a martingale. (lemma 4.11)

Problem 2

Consider a standard Black-Scholes model, that is, a model consisting of a bank account $B(t)$ with P-dynamics given by

$$dB(t) = rB(t)\ dt,$$

with $B(0) = 1$ and a stock $S(t)$ with P-dynamics given by

$$dS(t) = \alpha S(t)\ dt + \sigma S(t)\ d\overline{W}(t),$$

with $S(0) = s > 0$ and where $r, \alpha \in \mathbb{R}$ and $\sigma > 0$ are constants and $\overline{W}(t)$ is a P-Brownian motion. Let $T > 0$ be a given fixed (expiry) date.

Let $h(t) = (h_0(t), h_1(t))$ be a portfolio where

$$h_0(t) = \exp\left(\frac{1}{2}\sigma\overline{W}(t) + \left(\frac{\alpha - r}{2} - \frac{1}{8}\sigma^2\right)t\right)$$

is the number of units in the bank account at time t and

$$h_1(t) = \frac{1}{s}\exp\left(-\frac{1}{2}\sigma\overline{W}(t) + \left(\frac{r - \alpha}{2} - \frac{3}{8}\sigma^2\right)t\right)$$

is the number of shares in the stock at time t. Let $V^h(t)$ denote the associated value process and let $u(t) = (u_0(t), u_1(t))$ denote the relative portfolio.

a. i. Determine whether the portfolio h is self-financing or not.
   ii. Compute $u_1(t)$.

Consider two derivatives that at time T pay $X_1 = \Phi_1(S(T))$ and $X_2 = \Phi_2(S(T))$. For $i = 1, 2$, the arbitrage free price of derivative X_i is given by $\pi_i(t) = F_i(t, S(t))$ where $F_i(t, s)$ is the pricing function of the derivative. Assume that $\pi_i(t) > 0$. The price process $\pi_i(t)$ has dynamics (under the P-measure) given by

$$d\pi_i(t) = \alpha_i(t)\pi_i(t)\ dt + \sigma_i(t)\pi_i(t)\ d\overline{W}(t).$$

b. i. Determine $\alpha_i(t)$ and $\sigma_i(t)$ for $i = 1, 2$.
   ii. Show that
$$\frac{r - \alpha_1(t)}{\sigma_1(t)} = \frac{r - \alpha_2(t)}{\sigma_2(t)}.$$

Let $C(t, s; K, T)$ denote the Black-Scholes price at time t of an European call option with strike K and expiry date T when the current price of the underlying is s. Similary, let $P(t, s; K, T)$ denote the Black-Scholes price at time t of an European put option with strike K and expiry date T when the current price of the underlying is s. Consider a new derivative that at time T pays

$$Y = \max\{C(T, S(T); K, T_1), P(T, S(T); K, T_1\}$$

where $T < T_1$ is a fixed date.

c. Determine the arbitrage free price of derivative Y at time $t < T$. (Hint: recall $\max(x, y) = (x + y)^+ y$)

Assume that the call option and the put option do not have the same strike prices, that is, a derivative that at time T pays

$$\widetilde{Y} = \max\{C(T, S(T); K_1, T_1), P(T, S(T); K_2, T_1\}$$

where the strike prices $K_1 \neq K_2$. Let $F(t, s)$ be the pricing function of the derivative.

d. Determine the equation satisfied by the pricing function $F(t, s)$.

Solution (a).

We have that h is self-financing if and only if the equation

$$dV^h(t) = h_0(t)\ dB(t) + h_1(t)\ dS(t)$$

is satisfied. And so, we start by determining the dynamics of the number of assets denoted by h_0 and h_1. From Ito's formula we can conclude that

$$\begin{aligned}
dh_0(t) &= \left(\frac{\alpha - r}{2} - \frac{1}{8}\sigma^2\right) h_0(t)\ dt + \frac{1}{2}\sigma h_0(t)\ dW(t) + \frac{1}{2}\frac{1}{2}\sigma\frac{1}{2}\sigma h_0\ (dW(t))^2 \\
&= \left(\frac{\alpha - r}{2} - \frac{1}{8}\sigma^2\right) h_0(t)\ dt + \frac{1}{2}\sigma h_0(t)\ dW(t) + \frac{1}{2^3}\sigma^2 h_0\ dt \\
&= \left(\frac{\alpha - r}{2} - \frac{1}{8}\sigma^2 + \frac{1}{8}\sigma^2\right) h_0(t)\ dt + \frac{1}{2}\sigma h_0(t)\ dW(t) \\
&= \frac{\alpha - r}{2} h_0(t)\ dt + \frac{1}{2}\sigma h_0(t)\ dW(t).
\end{aligned}$$

For the number of stocks we have

$$\begin{aligned}
dh_1(t) &= \left(\frac{r - \alpha}{2} + \frac{3}{8}\sigma^2\right) h_1(t)\ dt - \frac{1}{2}\sigma h_1(t)\ dW(t) + \frac{1}{2}\frac{1}{2}\sigma\frac{1}{2}\sigma h_1(t)\ (dW(t))^2 \\
&= \left(\frac{r - \alpha}{2} + \frac{1}{2}\sigma^2\right) h_1(t)\ dt - \frac{1}{2}\sigma h_1(t)\ dW(t).
\end{aligned}$$

We may derive the dynamics of the portfolio as

$$\begin{aligned}
dV^h(t) &= d(h_0(t)B(t) + h_1(t)S(t)) \\
&= B(t)\ dh_0(t) + h_0(t)\ dB(t) + (dh_0(t))(dB(t)) \\
&+ S(t)\ dh_1(t) + h_1(t)\ dS(t) + (dh_1(t))(dS(t))
\end{aligned}$$

and so we want that

$$(*) = B(t)\ dh_0(t) + (dh_0(t))(dB(t)) + S(t)\ dh_1(t) + (dh_1(t))(dS(t)) = 0.$$

Inserting the dynamics given and portfolio dynamics above we have

$$\begin{aligned}
(*) &= B(t)\frac{\alpha - r}{2}h_0(t)\ dt + B(t)\frac{1}{2}\sigma h_0(t)\ dW(t) + 0 \\
&+ S(t)\left(\frac{r-\alpha}{2} + \frac{1}{2}\sigma^2\right)h_1(t)\ dt - S(t)\frac{1}{2}\sigma h_1(t)\ dW(t) \\
&- \frac{1}{2}\sigma h_1(t)\sigma S(t)\ dt \\
&= \left(B(t)\frac{\alpha - r}{2}h_0(t) - \frac{\alpha - r}{2}S(t)h_1(t)\right)\ dt \\
&+ \left(\frac{1}{2}B(t)\sigma h_0(t) - \frac{1}{2}S(t)\sigma h_1(t)\right)\ dW(t)
\end{aligned}$$

We see that this is zero if $h_0(t)B(t) = h_1(t)S(t)$. First we have

$$\begin{aligned}
h_0(t)B(t) &= \exp\left(\frac{1}{2}\sigma\overline{W}(t) + \left(\frac{\alpha - r}{2} - \frac{1}{8}\sigma^2\right)t\right)B(t) \\
&= \exp\left(\frac{1}{2}\left(\alpha - r - \frac{1}{4}\sigma^2\right)t + \frac{1}{2}\sigma\overline{W}(t)\right)B(t) \\
&= \exp\left(\frac{1}{2}\left(\alpha - \frac{1}{2}\sigma^2\right)t + \frac{1}{2}\sigma\overline{W}(t)\right)\exp\left(\frac{1}{2}\left(-r + \frac{1}{4}\sigma^2\right)t\right)B(t) \\
&= (S(t))^{1/2}\exp\left(\left(-\frac{r}{2} + \frac{1}{8}\sigma^2\right)t\right)B(t),
\end{aligned}$$

and

$$\begin{aligned}
h_1(t)S(t) &= \frac{1}{s}\exp\left(-\frac{1}{2}\sigma\overline{W}(t) + \left(\frac{r-\alpha}{2} - \frac{3}{8}\sigma^2\right)t\right)s\cdot\exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma\overline{W}(t)\right) \\
&= \exp\left(\frac{1}{2}\sigma\overline{W}(t) + \left(\frac{r+\alpha}{2} - \frac{7}{8}\sigma^2\right)t\right) \\
&= \exp\left(\frac{1}{2}\sigma\overline{W}(t) + \frac{1}{2}\left(\alpha - \frac{2}{4}\sigma^2\right)t\right)\exp\left(\frac{1}{2}\left(r - \frac{5}{4}\sigma^2\right)t\right) \\
&= (S(t))^{1/2}\exp\left(\left(-\frac{5}{8}\sigma^2\right)t\right)B(t)
\end{aligned}$$

Which does not hold. **THIS EXERCISE SHOULD BE ABLE TO BE SOLVED..** $\square$

(ii): We have that

$$u_1(t) = \frac{h_1(t)S(t)}{V^h(t)}.$$

Using that S is a GBM and $B(t) = e^{rt}$ we have

$$
\begin{aligned}
u_1(t) &= \frac{h_1(t)S(t)}{h_1(t)S(t)+h_0(t)B(t)} \\
&= \frac{e^{-\frac{1}{2}\sigma\overline{W}(t)+\left(\frac{r-\alpha}{2}-\frac{3}{8}\sigma^2\right)t}e^{\left(\alpha-\frac{1}{2}\sigma^2\right)t+\sigma\overline{W}_t}}{e^{-\frac{1}{2}\sigma\overline{W}(t)+\left(\frac{r-\alpha}{2}-\frac{3}{8}\sigma^2\right)t}e^{\left(\alpha-\frac{1}{2}\sigma^2\right)t+\sigma\overline{W}_t}+e^{\frac{1}{2}\sigma\overline{W}(t)+\left(\frac{\alpha-r}{2}-\frac{1}{8}\sigma^2\right)t}e^{rt}} \\
&= \frac{se^{\left(\frac{r-\alpha}{2}-\frac{3}{8}\sigma^2+\alpha-\frac{1}{2}\sigma^2\right)t+\frac{1}{2}\sigma\overline{W}_t}}{e^{\left(\frac{r-\alpha}{2}-\frac{3}{8}\sigma^2+\alpha-\frac{1}{2}\sigma^2\right)t+\frac{1}{2}\sigma\overline{W}_t}+e^{\frac{1}{2}\sigma\overline{W}(t)+\left(\frac{\alpha-r}{2}-\frac{1}{8}\sigma^2+r\right)t}} \\
&= \frac{e^{\left(\frac{r+\alpha}{2}-\frac{7}{8}\sigma^2\right)t+\frac{1}{2}\sigma\overline{W}_t}}{e^{\left(\frac{r+\alpha}{2}-\frac{7}{8}\sigma^2\right)t+\frac{1}{2}\sigma\overline{W}_t}+e^{\frac{1}{2}\sigma\overline{W}(t)+\left(\frac{\alpha+r}{2}-\frac{1}{8}\sigma^2\right)t}} \\
&= \frac{e^{-\frac{7}{8}\sigma^2 t}}{e^{-\frac{7}{8}\sigma^2 t}+e^{-\frac{1}{8}\sigma^2 t}} = \frac{e^{-\frac{6}{8}\sigma^2 t}}{e^{-\frac{6}{8}\sigma^2 t}+1}.
\end{aligned}
$$

OBVIOUSLY had the previous exercise been done correct we would have $h_1(t)S(t) = h_0(t)B(t)$ i.e. $u_1(t) = \frac{1}{2}$.
□

Solution (b).

(i): We know that $\pi_i(t) = F_i(t, S(t))$ and so from Ito's formula we have the dynamics (we suppress the argument $(t, S(t))$ in the derivatives):

$$
\begin{aligned}
d\pi_i(t) &= \frac{\partial F_i}{\partial t}\ dt + \frac{\partial F_i}{\partial s}\ dS(t) + \frac{1}{2}\frac{\partial^2 F_i}{\partial s^2}\ (dS(t))^2 \\
&= \frac{\partial F_i}{\partial t}\ dt + \frac{\partial F_i}{\partial s}\ (\alpha S(t)\ dt + \sigma S(t)\ d\overline{W}(t)) + \frac{1}{2}\frac{\partial^2 F_i}{\partial s^2}\ \sigma^2 S(t)^2\ dt \\
&= \left(\frac{\partial F_i}{\partial t} + \frac{\partial F_i}{\partial s}\alpha S(t) + \frac{1}{2}\frac{\partial^2 F_i}{\partial s^2}\ \sigma^2 S(t)^2\right)\ dt + \frac{\partial F_i}{\partial s}\sigma S(t)\ d\overline{W}(t) \\
&= \underbrace{\frac{\frac{\partial F_i}{\partial t} + \frac{\partial F_i}{\partial s}\alpha S(t) + \frac{1}{2}\frac{\partial^2 F_i}{\partial s^2}\ \sigma^2 S(t)^2}{\pi_i(t)}}_{=\alpha_i(t)}\pi_i(t)\ dt + \underbrace{\frac{\frac{\partial F_i}{\partial s}\sigma S(t)}{\alpha_i(t)}}_{=\sigma_i(t)}\pi_i(t)\ d\overline{W}(t)
\end{aligned}
$$

as desired. □

(ii): We have

$$
\begin{aligned}
\frac{r-\alpha_i(t)}{\sigma_i(t)} &= \frac{r - \frac{\frac{\partial F_i}{\partial t}+\frac{\partial F_i}{\partial s}\alpha S(t)+\frac{1}{2}\frac{\partial^2 F_i}{\partial s^2}\ \sigma^2 S(t)^2}{\pi_i(t)}}{\frac{\frac{\partial F_i}{\partial s}\sigma S(t)}{\alpha_i(t)}} \\
&= \frac{r\pi_i(t) - \frac{\partial F_i}{\partial t} - \frac{\partial F_i}{\partial s}\alpha S(t) - \frac{1}{2}\frac{\partial^2 F_i}{\partial s^2}\ \sigma^2 S(t)^2}{\frac{\partial F_i}{\partial s}\sigma S(t)} \\
&= \frac{r\pi_i(t) - \frac{\partial F_i}{\partial t} - \frac{\partial F_i}{\partial s}rS(t) - \frac{1}{2}\frac{\partial^2 F_i}{\partial s^2}\ \sigma^2 S(t)^2 + \frac{\partial F_i}{\partial s}rS(t) - \frac{\partial F_i}{\partial s}\alpha S(t)}{\frac{\partial F_i}{\partial s}\sigma S(t)} \\
&= \frac{\frac{\partial F_i}{\partial s}rS(t) - \frac{\partial F_i}{\partial s}\alpha S(t)}{\frac{\partial F_i}{\partial s}\sigma S(t)} = \frac{r-\alpha}{\sigma},
\end{aligned}
$$

where we used the Black-Scholes equation i.e.

$$r\pi_i(t) - \frac{\partial F_i}{\partial t} - \frac{\partial F_i}{\partial s} rS(t) - \frac{1}{2}\frac{\partial^2 F_i}{\partial s^2}\ \sigma^2 S(t)^2 = 0$$

for any derivative's arbitrage free pricing process. Since i is not included in the fraction above we have the desired result. □

Solution (c).

We follow the hint and see that the payout is

$$\begin{aligned}
Y &= \max\{C(T,S(T);K,T_1), P(T,S(T);K,T_1)\} \\
&= \Big(C(T,S(T);K,T_1) - P(T,S(T);K,T_1)\Big)^+ + P(T,S(T);K,T_1) \\
&= \Big(C(T,S(T);K,T_1) - Ke^{-r(T_1-T)} - C(T,S(T);K,T_1) + S(T)\Big)^+ + P(T,S(T);K,T_1) \\
&= \Big(S(T) - Ke^{-r(T_1-T)}\Big)^+ + P(T,S(T);K,T_1) \\
&= C(T,S(T);Ke^{-r(T_1-T)},T) + P(T,S(T);K,T_1)
\end{aligned}$$

Hence we can hedge this payout with a call option with strike $Ke^{-r(T_1-T)}$ at expiry T and a put with strike K at expiry T_1, that is

$$\Pi_t[Y] = C(t,S(t);Ke^{-r(T_1-T)},T) + P(t,S(t);K,T_1)$$

as desired. □

Solution (d).

We have that the arbitrage free pricing function $F(t,s)$ has to satisfie the Black-Scholes equation 7.10 i.e.

$$\begin{aligned}
F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF = 0 \\
F(T,s) = \max\{C(T,s;K_1,T_1), P(T,s;K_2,T_1\}.
\end{aligned}$$

which may be written differently in terms of call options, stock price s and zero-coupon bonds. □

Problem 3

Consider a two-dimensional Black-Scholes model. The market model consist of three assets: A bank account $B(t)$ and two stocks $S_1(t)$ and $S_2(t)$. The P-dynamics of $B(t)$ is

$$dB(t) = rB(t)\ dt$$

with $B(0) = 1$ where $r \in \mathbb{R}$ is a constant interest rate. The P-dynamics of $S_1(t)$ and $S_2(t)$ are given by

$$\begin{aligned}
dS_1(t) &= \alpha_1 S_1(t)\ dt + \sigma S_1(t)\ d\overline{W}_1(t), \\
dS_2(t) &= \alpha_2 S_2(t)\ dt + \sigma S_2(t)\ \big(d\overline{W}_1(t) + d\overline{W}_2(t)\big),
\end{aligned}$$

with $S_1(0) = s_1 > 0$ and $S_2(0) = s_2 > 0$ where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\sigma > 0$ are constants and $\overline{W}_1(t)$ and $\overline{W}_2(t)$ are two independent P-Brownian motions. The filtration is the one generated by the two Brownian motions. Let $T > 0$ be a given and fixed (expiry) date.

a. i. Is the model arbitrage free?
   ii. Is the model complete?

b. Compute the covariance of $S_1(T)$ and $S_2(T)$. (Hint: recall $cov(X,Y) = E[XY] - E[X]E[Y]$).

Consider the derivative that at time T pays $X = S_1(T_0) + S_2(T)$ where $0 < T_0 < T$ is a fixed date.

c. Find a hedge portfolio for derivative X.

Solution (a).

(i): The model is arbitrage free if and only if a martingale measure exists. That is if the equation

$$\sigma_t \varphi_t = r - \alpha$$

has at least one solution. We have the following market on matrix form

$$dS_t = D(S_t)\alpha_t \; dt + D(S_t)\sigma_t \; dW_t$$

or written out in total

$$\begin{bmatrix} dS_1(t) \\ dS_2(t) \end{bmatrix} = \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} dt \\ dt \end{bmatrix} + \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ \sigma & \sigma \end{bmatrix} \begin{bmatrix} d\overline{W}_1(t) \\ d\overline{W}_2(t) \end{bmatrix}.$$

Hence we want to solve

$$\begin{bmatrix} \sigma & 0 \\ \sigma & \sigma \end{bmatrix} \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = \begin{bmatrix} r - \alpha_1 \\ r - \alpha_2 \end{bmatrix}.$$

This is easy if σ is invertible. We see that we in fact have that the inverse of σ is

$$\sigma_t^{-1} = \begin{bmatrix} 1/\sigma & 0 \\ -1/\sigma & 1/\sigma \end{bmatrix}$$

as we have

$$\begin{bmatrix} 1/\sigma & 0 \\ -1/\sigma & 1/\sigma \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ \sigma & \sigma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Then we clearly have the solution

$$\varphi_t = \begin{bmatrix} 1/\sigma & 0 \\ -1/\sigma & 1/\sigma \end{bmatrix} \begin{bmatrix} r - \alpha_1 \\ r - \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{r-\alpha_1}{\sigma} \\ \frac{-r+\alpha_1+r-\alpha_2}{\sigma} \end{bmatrix} = \begin{bmatrix} \frac{r-\alpha_1}{\sigma} \\ \frac{\alpha_1-\alpha_2}{\sigma} \end{bmatrix}.$$

By defining the likelihood process L_t as

$$dL_t = \varphi_t^\top L_t \; d\overline{W}_t, \; L_0 = 1,$$

we know from the Novikov condition that if the integral $E^P[e^{1/2 \int_0^T \|\varphi_t\|^2 \; dt}]$ is finite then L is a martingale with $E^P[L_T] = 1$. We see that

$$E^P\left[e^{\frac{1}{2}\int_0^T \|\varphi_t\|^2 \; dt}\right] = E^P\left[e^{\frac{1}{2}\int_0^T \left(\frac{r-\alpha_1}{\sigma}\right)^2 + \left(\frac{\alpha_1-\alpha_2}{\sigma}\right)^2 \; dt}\right] = E^P\left[e^{\frac{1}{2}T\left(\frac{r-\alpha_1}{\sigma}\right)^2 + \frac{1}{2}T\left(\frac{\alpha_1-\alpha_2}{\sigma}\right)^2}\right] < \infty.$$

Hence we have found a martingale measure defined by the likelihood process L above. We conclude that the market is arbitrage free. □

(ii): The model is complete if the martingale measure is unique. This is equivalent with $Ker[\sigma_t] = \{0\}$ and since σ_t is invertible (diagonal) we have from theorem 14.7 that the model is complete. □

Solution (b).

We have by definition:

$$cov(S_1(T), S_2(T)) = E\left[S_1(T)S_2(T)\right] - E[S_1(T)]E[S_2(T)].$$

Thus we set $Z(t) = S_1(t)S_2(t)$ and evaluate the mean value of Z. By Ito's formula on $f(s_1, s_2) = s_1 s_2$ we have

$$\begin{aligned}
dZ(t) &= df(S_1(t), S_2(t)) \\
&= S_2(t)\ dS_1(t) + S_1(t)\ dS_2(t) + \frac{1}{2}(dS_1(t))(dS_2(t)) \\
&= S_2(t)\alpha_1(t)S_1(t)\ dt + S_2(t)\sigma S_1(t)\ d\overline{W}_1(t) \\
&+ S_1(t)\alpha_2(t)S_2(t)\ dt + S_1(t)\sigma S_2(t)\ (d\overline{W}_1(t) + d\overline{W}_2(t)) \\
&+ \frac{1}{2}\sigma^2 S_1(t)S_2(t)\ d\overline{W}_1(t)(d\overline{W}_1(t) + d\overline{W}_2(t)) \\
&= (\alpha_1(t) + \alpha_2(t))S_1(t)S_2(t)\ dt + 2\sigma S_1(t)S_2(t)\ d\overline{W}_1(t) + \sigma S_1(t)S_2(t)\ d\overline{W}_2(t) \\
&+ \frac{1}{2}\sigma^2 S_1(t)S_2(t)\ dt \\
&= (\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2)Z(t)\ dt + 2\sigma Z(t)\ d\overline{W}_1(t) + \sigma Z(t)\ d\overline{W}_2(t).
\end{aligned}$$

Thus we have that the terms invovling the Brownian motions will vanish when takings expectation hence

$$\begin{aligned}
E[Z(t)] &= Z(0) + E\left[\int_0^t (\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2)Z(s)\ ds\right] \\
&= Z(0) + (\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2)\int_0^t E\left[Z(s)\right]\ ds.
\end{aligned}$$

Then we have the dynamics of $E[Z(t)]$ is given as

$$dE[Z(t)] = (\alpha_1(t) + \alpha_2(t) + \frac{1}{2}\sigma^2)E\left[Z(t)\right]\ dt.$$

We may then solve this using $E[Z(0)] = Z(0) = s_1 s_2$:

$$E[Z(T)] = Z(0)e^{(\alpha_1(t)+\alpha_2(t)+\frac{1}{2}\sigma^2)T} = s_1 s_2 e^{(\alpha_1(t)+\alpha_2(t)+\frac{1}{2}\sigma^2)T}.$$

Inserting in the formula for covariance we arrive at

$$
\begin{aligned}
cov(S_1(T), S_2(T)) &= E\left[S_1(T)S_2(T)\right] - E[S_1(T)]E[S_2(T)] \\
&= s_1 s_2 e^{(\alpha_1(t)+\alpha_2(t)+\frac{1}{2}\sigma^2)T} - E[S_1(T)]E[S_2(T)] \\
&= s_1 s_2 e^{(\alpha_1+\alpha_2+\frac{1}{2}\sigma^2)T} - s_1 e^{\alpha_1 T} s_2^{\alpha_2 T} \\
&= s_1 s_2 e^{(\alpha_1+\alpha_2)T}\left(e^{\frac{1}{2}\sigma^2 T} - 1\right).
\end{aligned}
$$

as desired. □

Solution (c).

We may look at this problem on two subintervals: $[0, T_0]$ and $(T_0, T]$. On the latter we know that the portfolio should consist of $S_1(T_0)$ zero coupon bonds with expiry T and one position in the second stock. Hence on the interval $(T_0, T]$ the hedging portfolio is

$$
h(t) = \Big(h_0(t), h_1(t), h_2(t)\Big) = \Big(e^{-r(T-T_0)}S(T_0), 0, 1\Big),\ t > T_0.
$$

Hence we on the interval $[0, T_0]$ we want to replicate the derivative $\widetilde{X} = e^{-r(T-T_0)}S(T_0)$. This is obviously easy since we should hold $e^{-r(T-T_0)}$ of the first stock. Then we have

$$
h(t) = \begin{cases} \Big(0, e^{-r(T-T_0)}, 1\Big) & \text{for } t \le T_0, \\ \Big(e^{-r(T-T_0)}S(T_0), 0, 1\Big) & \text{for } t > T_0. \end{cases}
$$

This then give a self-financing portfolio with value process

$$
V^h(t) = \begin{cases} S_1(t)e^{-r(T-T_0)} + S_2(t) & \text{for } t \le T_0, \\ e^{-r(T-t)}S(T_0) + S_2(t) & \text{for } t > T_0. \end{cases}
$$

as desired. □

Exam 2019/20

Problem 1

Let $W(t)$ denote a Brownian motion and let $\mathcal{F}_t = \mathcal{F}_t^W$. Let $T > 0$ be a given and fixed time.

Consider the two dimensional stochastic differential equation

$$
\begin{aligned}
dX(t) &= \frac{1}{2}X(t)\ dt + Y(t)\ dW(t), \\
dY(t) &= \frac{1}{2}Y(t)\ dt + X(t)\ dW(t),
\end{aligned}
$$

with $X(0) = 0$ and $Y(1) = 1$.

a. Show that $(X(t), Y(t)) = (\sinh(W(t)), \cosh(W(t)))$ solves the two-dimensional stochastic differential equation. (Hint: Recall that $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$).

b. i. Show that $M(t) = e^{-t/2}\cosh(W(t))$ is a martingale.
   ii. Find a constant z and a process $h(t)$ such that

$$
\cosh(W(T)) = z + \int_0^T h(t)\ dW(t).
$$

Let $L(t)$ be a Likelihood process and let $dQ = L(T)dP$ be a new probability measure.

c. Determine the Likelihood process $L(t)$ such that $\sinh(W(t))$ is a martingale under the probability measure Q.

Solution (a).

Assume that $X(t) = \sinh(W(t))$ and $Y(t) = \cosh(W(t))$. The relevant derivatives is then

$$\frac{d}{dw}\sinh(w) = \frac{1}{2}e^w + \frac{1}{2}e^{-w} = \cosh(w), \quad \frac{d^2}{dw^2}\sinh(w) = \frac{d}{dw}\cosh(w) = \frac{1}{2}e^w - \frac{1}{2}e^{-w} = \sinh(w).$$

That is sinus and cosinus hyperbolic are their each others derivative. Then by Ito's formula we have

$$\begin{aligned}
dX(t) &= \cosh(W(t))\ dW(t) + \frac{1}{2}\sinh(W(t))\ (dW(t))^2 \\
&= \frac{1}{2}\sinh(W(t))\ dt + \cosh(W(t))\ dW(t) \\
&= \frac{1}{2}X(t)\ dt + Y(t)\ dW(t).
\end{aligned}$$

and

$$\begin{aligned}
dY(t) &= \sinh(W(t))\ dW(t) + \frac{1}{2}\cosh(W(t))\ (dW(t))^2 \\
&= \frac{1}{2}\cosh(W(t))\ dt + \sinh(W(t))\ dW(t) \\
&= \frac{1}{2}Y(t)\ dt + X(t)\ dW(t).
\end{aligned}$$

And thus the result has been prooved. $\square$

Solution (b).

(i): Consider the function $f(z,y) = zy$. Then we have that $M(t) = f(Z(t), Y(t))$ for $Z(t) = e^{-t/2}$ hence M has dynamics given by Ito's formula:

$$\begin{aligned}
dM(t) &= df(Z(t), Y(t)) \\
&= Y(t)\ dZ(t) + Z(t)\ dY(t) + (dZ(t))(dY(t)) \\
&= Y(t)\ (-\frac{1}{2}Z(t)\ dt) + Z(t)\ (\frac{1}{2}Y(t)\ dt + X(t)\ dW(t)) + (-\frac{1}{2}Z(t)\ dt)(\frac{1}{2}Y(t)\ dt + X(t)\ dW(t)) \\
&= X(t)Z(t)\ dW(t).
\end{aligned}$$

and so we see that pr. lemma 4.11 M is a martingale. $\square$

(ii): We have from above

$$M(T) = M(0) + \int_0^T X(t)Z(t)\ dW(t) = Z(T)\cosh(W(t))$$

Hence it follows that

$$\cosh(W(T)) = \frac{M(0)}{Z(T)} + \int_0^T \frac{X(t)Z(t)}{Z(T)}\, dW(t).$$

Using that the martingale has initial value

$$M(0) = e^{-0/2}\cosh(0) = 1$$

we have

$$\cosh(W(T)) = e^{T/2} + \int_0^T \sinh(W(t))e^{(T-t)/2}\, dW(t).$$

In total we have $z = e^{T/2}$ and $h(t) = \sinh(W(t))e^{(T-t)/2}$ as desired. □

Solution (c).

We have that under the measure Q the dynamics of W is given by the Girsanov Theorem

$$dW(t) = \varphi\, dt + dW_t^Q,$$

where φ is the Girsanov kernel associated with L. Then we know that X has dynamics under the Q measure:

$$\begin{aligned} dX(t) &= \frac{1}{2}X(t)\, dt + Y(t)\, (\varphi\, dt + dW_t^Q) \\ &= \left(\frac{1}{2}X(t) + \varphi Y(t)\right)\, dt + Y(t)\, dW_t^Q \end{aligned}$$

and so we would have that X is a martingale under Q if

$$\varphi_t = -\frac{1}{2}\frac{X(t)}{Y(t)} = -\frac{1}{2}\frac{\sinh(W(t))}{\cosh(W(t))} = -\frac{1}{2}\tanh(W(t)).$$

Then we can define a Likelihood process with initial condition $L_0 = 1$ and dynamics $dL_t = \varphi_t L_t\, dW(t)$ i.e. L is the solution

$$L_t = \exp\left\{\int_0^s -\frac{1}{2}\tanh(W(s))\, dW(t) - \frac{1}{2}\int_0^t \left(-\frac{1}{2}\tanh(W(t))\right)^2\, dW(t)\right\} > 0.$$

We lastly show that the Novikov condition is satisfied i.e.

$$E^P\left[e^{\frac{1}{2}\int_0^T \|\varphi_s\|^2\, ds}\right] = E^P\left[e^{\frac{1}{8}\int_0^T \tanh^2(W(t))\, ds}\right] < \infty$$

and so L is a P-martingale and L is a Likelihood process. We thus have found a Likelihood process such that X is a martingale under the measure Q given by $dQ = L_T\, dP$. □

Exam 2020/21

Some content

Exam 2021/22

Some content

Exam 2022/23

Some content