Calculus

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MAT03c: Mathematics 3c

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Contents

1 The Infinituple & Infinitesimal

An *infinituple* (∞) is a number which is so great in size that it can not be defined numerically.
An *infinitesimal* is a number which is *almost* 0, but not equal to 0. It can be represented in 2 ways:

$$\frac{1}{\infty}$$

$$0.000\cdots1$$

2 The Derivative

2.1 Definition of the derivative

Derivatives may be denoted as such for a function $y = f(x)$:

$$\dot{y} = f'(x)$$

OR:

$$\dot{y} = \frac{dy}{dx}f(x)$$

A derivative is the instantaneous rate of change defined by the function's values at two points separated by some infinitesimal h:

$$\frac{d}{dx} = \lim_{h\to0}\frac{f(x+h)-f(x)}{h} \tag{1}$$

Alternatively, using two points approaching eachother:

$$\frac{d}{dx} = \lim_{a\to x}\frac{f(x)-f(a)}{x-a} \tag{2}$$

Note that both these equations are constructed such that the numerator is Δy, and the denominator Δx.
However, when written in infinitesimal sizes, we replace the Δ with a d, hence why we may express derivatives as $\frac{dy}{dx}$.

2.2 Derivation Rules

It is *possible* to use the *definition of the derivative* (**2.1**) to calculate derivatives of functions. It is however cumbersome.
Instead of using that, we may utilise derivation rules, the derivation rules are different for each type of polynomial.

2.2.1 The Power Rule

The *power rule* is applicable for *power polynomials* ($f(x) = kx^n$). The power rule is as follows:

$$\frac{dy}{dx}kx^n = knx^{(n-1)} \tag{3}$$

The Power Rule

Example:

Q: Provided $f(x) = x^2$, find $f'(x)$.

$$\because \frac{dy}{dx}kx^n = knx^{(n-1)}$$
$$\frac{dy}{dx}x^2 = 2x^{(2-1)} = 2x$$

2.2.2 The Exponential Derivative

The rule for the *exponential derivative* (it has no good name) is as follows:

$$ka^{nx} = nka^{nx} \ln a \tag{4}$$

NOTE:

In polynomials containing e^x there is no $\ln e$ needed as $\ln e = 1$.

Example:

Q: Find $\frac{dy}{dx} 3 \times 2^x$

$$\because \frac{dy}{dx} ka^{nx} = nka^{nx} \ln a$$
$$\frac{dy}{dx} 3 \times 2^x = 3 \times 2^x \ln 2$$

LOOK OUT!

If the power of x is negative, there is a -1 in the place of n, meaning that the polynomial switches sign as you derivate it.

2.2.3 The logarithmic derivative

The *logarithmic derivative* is related to the *exponential derivative* (**2.2.2**).

$$\frac{dy}{dx} \log_a x = \frac{1}{x \ln a}$$

Example:

Q: Find $\frac{dy}{dx} \log_2 x$.

$$\because \frac{dy}{dx} \log_a x = \frac{1}{x \ln a}$$
$$\frac{dy}{dx} \log_2 x = \frac{1}{x \ln 2}$$

3 The Integral

The integal is less well defined intuitively, but it may be defined as the area under a graph.

3.1 Primitive Functions

A *primitive function* or an *indefinite integral* is the *antiderivative* of a function, i.e. the function is the derivative of the primitive function. They are denoted as such for the primitive function of $f(x)$:

$$F(x)$$

A primitive function is then defined such that:

$$\frac{dy}{dx}F(x) = f(x)$$

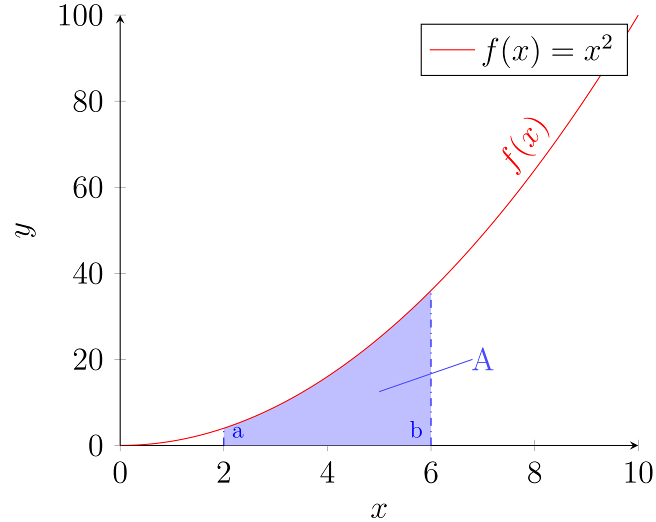
3.1.1 The Constant of Integration

All primitive functions contain some constant that is "lost in derivation", represented by C, as such:

$$f(x) = x^n$$
$$F(x) = \frac{x^n + 1}{n + 1} + C$$

3.2 Definite Integrals

A *definite integral* may be thought of as the area between two bounds of a function, a & b.



$$A = \int_a^b f(x)\ dx = F(b) - F(a) \tag{5}$$

Above equation is true for *all definite integrals*.

NOTE:

In an integral, both $F(b)$ & $F(a)$ contain a *constant of integration* (**3.1.1**). Since $F(a)$ is subtracted from $F(b)$, C may be ignored;

<u>Proof that *C* may be ignored *for definite integrals*:</u>

Definitions: Let $\phi(x)$ be a component of $F(x)$ such that:

$$F(x) = \phi(x) + C$$

$$\int_a^b f(x)\ dx = F(b) - F(a) = (\phi(b) + C) - (\phi(a) + C)$$

$$\int_a^b f(x)\ dx = \phi(b) + \cancel{C} - \phi(a) - \cancel{C} = \phi(b) - \phi(a)$$

C cancels, hence it may be ignored.

$\square$