Games with pure equilibria

Spring 2024

Best response dynamics

Consider a strategic game $\Gamma = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ with $N = \{1, \dots, n\}$

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- PNE are defined as the fix point among mutually best responses.
- It seems natural to consider variants of the process of local changes to try to get a PNE.
- Consider the algorithm:
  - choose $s \in A_1 \times \cdots \times A_n$
  - while s is not a NE do

    choose $i \in \{1, \ldots, n\}$ such that $s_i \notin BR(s_{-i})$

    Set s_i to be an action in $BR(s_{-i})$
- The process looks similar to local search algorithms. Is there any difference?

Best response graph

- The Nash dynamics or Best Response graph $G = (V, E)$ of Γ:
  - $V = A_1 \times \cdots \times A_n$
  - $(s, s') \in E$ iff $s' = (s_{-i}, s_i')$ for $i \in N$, $s_i \notin BR(s_{-i})$ and $s_i' \in BR(s_{-i})$.
- Performing local search on the best response graph
  - Does it produce a PNE?
  - If so, how much time?
  - Let's look to some examples.

Prisonner's dilemma

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Quiet	2,2	0,3
Fink	3,0	1,1

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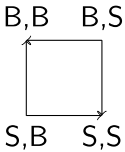

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Other games

What about Congestion Games?

- In those games we cannot get the best response graph in polynomial time.
- However we can perform a local improvement step in polynomial time.
- Although, even assuring convergence, it might take exponential time to reach a NE.

Best response graph: Properties

- A NE is a sink, a node with out-degree 0, in the best response graph.
- The existence of a cycle in the best response graph does not rule out the existence of a PNE.
- If the best response graph is acyclic, the game has a PNE.

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- If the best response graph is acyclic, the game has a PNE.
- If best response dynamics converges to a PNE, maybe with a lot of time.

Potential games

(Monderer and Shapley 96)

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- A function $\Phi : S \to \mathbb{R}$ is an exact potential function for Γ if

$$\forall\, i \in N, \forall\, s \in S, \forall\, s'_i \in A_i : \; u_i(s) - u_i(s_{-i}, s'_i) = \Phi(s) - \Phi(s_{-i}, s'_i)$$

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- Γ is a potential game if it admits a potential function.

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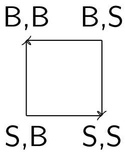

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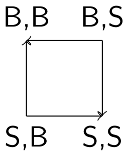

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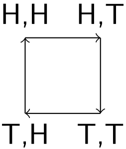

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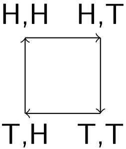

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The property on Φ cannot hold along a cycle in the best response graph.

Potential games

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- If G is acyclic, a topological sort of the graph provides a potential function for Γ.

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Proof.

Let G be the best response graph of Γ.

- If G is acyclic, a topological sort of the graph provides a potential function for Γ.
- The existence of a potential function Φ and the fact that, for each pair of connected strategy profiles in G, at least one player improves, implies the non existence of cycles in G.

$\square$

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As the best response graph is acyclic it must have a sink. $\square$

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We have a way to show that a game has a PNE by showing that it is a potential game.

1. Best response dynamics
2. Potential games
3. Congestion games
4. References

Congestion games

A congestion game is defined by

- A finite set E of m resources
- A finite set N of n players
- A delay function d: $E \times \mathbb{N} \to \mathbb{Z}$
- For each player $i \in N$:
  - A set of actions $A_i \in 2^E$
  - A cost function c_i:

$$c_i(a_1, \dots, a_n) = \left(\sum_{e \in a_i} d(e, f(a_1, \dots, a_n, e)) \right)$$

being $f(a_1, \dots, a_n, e) = |\{i \mid e \in a_i\}|$.

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A singleton congestion game has $A_i = \{\{r\} \mid r \in E\}$.

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- The cutting machine on the first line takes 1 hour to process a single order, 2 hours to process 2 and 4 hours to process 3. The cutting machine on the second line takes 4, 5 and 9 hours respectively.
- The packing machine on the first line takes 2 additional hours to pack a single order, 3 hours to pack 2 and 7 hours to pack 3. The packing machine on the second line takes instead 0, 2 and 9 hours respectively.

An example of a congestion game

- We have 4 resources C_1, C_2, P_1, P_2 and 3 players $N = \{1, 2, 3\}$
- $A_i = \{\{C_1, P_1\}, \{C_2, P_2\}\}$, $i = 1, 2, 3$
- Delay functions are defined by the processing times.

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Does this game have a PNE?

Rosenthal's theorem

Theorem (Rosenthal 73)

Every congestion game is a potential game,

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- For a strategy profile $s = (a_1, \ldots, a_n)$, define

$$\Phi(s) = \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e, k)$$

where $r(s) = \cup_{i \in N} a_i$.

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Let us show that Φ is a potential function.

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$$\Phi(s) - \Phi(s') = \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e, k) - \sum_{e' \in r(s')} \sum_{k=1}^{f(s',e')} d(e', k)$$

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$$c_i(s) - c_i(s_{-i}, s_i') = \left(\sum_{e \in a_i} d(e, f(s,e))\right) - \left(\sum_{e' \in a_i'} d(e, f(s',e'))\right)$$

$$= \sum_{e \in a_i, e \notin a_i'} d(e, f(s,e)) - \sum_{e \notin a_i, e \in a_i'} d(e, f(s',e'))$$

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$$\begin{aligned}
\Phi(s) - \Phi(s') &= \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e,k) - \sum_{e \in r(s')} \sum_{k=1}^{f(s',e)} d(e,k) \\
&= \sum_{e \in a_i, e \notin a'_i} [\sum_{k=1}^{f(s',e)+1} d(e,k) - \sum_{k=1}^{f(s',e)} d(e,k)] \\
&\quad + \sum_{e \notin a_i, e \in a'_i} [\sum_{k=1}^{f(s,e)} d(e,k) - \sum_{k=1}^{f(s,e)+1} d(e,k)]
\end{aligned}$$

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$$= \sum_{e \in a_i, e \notin a_i'} [\sum_{k=1}^{f(s',e)+1} d(e,k) - \sum_{k=1}^{f(s',e)} d(e,k)]$$

$$+ \sum_{e \notin a_i, e \in a_i'} [\sum_{k=1}^{f(s,e)} d(e,k) - \sum_{k=1}^{f(s,e)+1} d(e,k)]$$

$$= \sum_{e \in a_i, e \notin a_i'} d(e, f(s',e)+1) - \sum_{e \notin a_i, e \in a_i'} d(e, f(s,e)+1)$$

$$= \sum_{e \in a_i, e \notin a_i'} d(e, f(s,e)) - \sum_{e \notin a_i, e \in a_i'} d(e, f(s',e))$$

$$= c_i(s) - c_i(s_{-i}, s_i')$$

Network congestion games

- A network congestion game is a congestion game defined by a directed graph G and a collection of pairs of vertices (s_i, t_i).
  - The set of resources are the arcs in G.
  - The actions, for player i, are the $s_i \to t_i$ paths on G.
- A network congestion game is symmetric when $s_i = s$ and $t_i = t$, for $i \in N$.

An example of a network congestion game

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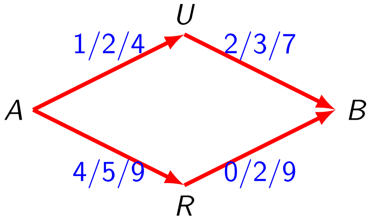

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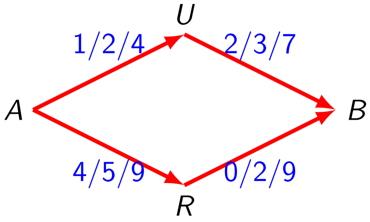

- Player's objective: going from $s = A$ to $t = B$ as fast as possible.

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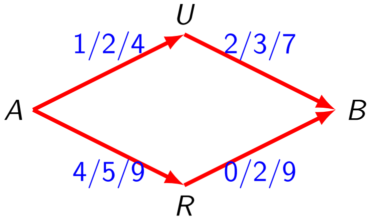

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- Strategy profiles: paths from A to B.
- A NE?

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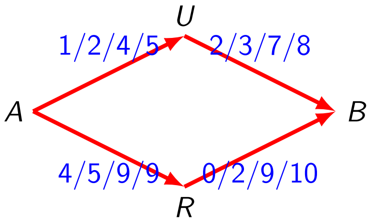

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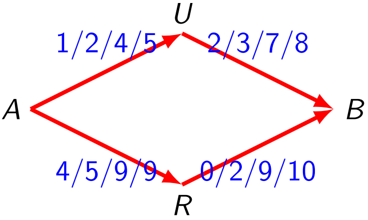

- Player's objective: send w_i units from $s = A$ to $t = B$ as fast as possible.
- Strategy profiles: paths from A to B.
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Results on convergence time

Theorem (Fabrikant, Papadimitriou, Talwar (STOC 04))

There exist network congestion games with an initial strategy profile from which all better response sequences have exponential length.

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There is a polynomial time algorithm for finding a PNE in symmetric network congestion games.

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Theorem (Ieong, McGrew, Nudelman, Shoham, Sun (AAAI 05))

In singleton congestion games all best response sequences have length at most $n^2 m$.

Complexity classification?

Optimization problem

An optimization problem is defined by a structure $\Pi = (\mathsf{I}, \mathsf{sol}, \mathsf{m}, \mathsf{goal})$, where

- I is the input set to Π;
- $\mathsf{sol}(x)$ is the set of feasible solutions for an input x.
- m is an integer measure defined over pairs (x, y), $x \in I$ and $y \in \mathsf{sol}(x)$.
- goal is the optimization criterium MAX or MIN.

An optimization problem is a functional problem whose goal, given an instance x, is to find an optimum solution

$$y = \mathsf{goal}\{(m(x, y') \mid y' \in \mathsf{sol}(x)\}.$$

Example: Given a graph and two vertices, obtain a path joining them with minimum length.

PLS

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(Johnson, Papadimitriou, Yannakakis, FOCS 85)
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- A local optimum is a solution such that all its neighbors have equal or worse cost.

(Johnson, Papadimitriou, Yannakakis, FOCS 85)
A local search problem belongs to PLS (Polynomial Local Search) if polynomial time algorithms exist for

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A local search problem belongs to PLS (Polynomial Local Search) if polynomial time algorithms exist for

- finding initial feasible solution $s \in \mathrm{sol}(x)$,
- computing the objective measure $m(x, y)$,
- checking whether a solution is a local optimum and if not finding a better solution in the neighborhood.

PLS reductions

A PLS reduction from $(\Pi_1, \mathcal{N}_1)$ to $(\Pi_2, \mathcal{N}_2)$ is

- a polynomial time computable function $f : \mathrm{I}_{\Pi_1} \to \mathrm{I}_{\Pi_2}$ and
- a polynomial time computable function $g : \mathrm{sol}_{\Pi_2}(f(x)) \to \mathrm{sol}_{\Pi_1}(x)$, for $x \in \mathrm{I}_{\Pi_1}$ such that
- if $s_2 \in \mathrm{sol}_{\Pi_2}(f(x))$ locally optimal then $g(s_2)$ is locally optimal.

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So that,

- If a local opt of Π_2 is "easy" to find then a local opt of Π_1 is easy to find.
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A PLS problem $(\Pi, \mathcal{N})$ is PLS-complete if every problem in PLS is PLS-reducible to $(\Pi, \mathcal{N})$.

PLS complete problems

- MAX-SAT (maximum satisfiability) problem
  - Given a Boolean formula in conjunctive normal form with a positive integer weight for each clause.
  - A solution is an assignment of the value 0 or 1 to all variables.
  - Its weight, to be maximized, is the sum of the weights of all satisfied clauses.
  - As neighborhood consider the Flip-neighborhood, where two assignments are neighbors if one can be obtained from the other by fliipping the value of a single variable.

PLS complete problems

- MaxCut problem.
  - Given a graph $G = (V, E)$ with non-negative edge weights.
  - A feasible solution is a partition of V into two sets A and B.
  - The objective is to maximize the weight of the edges between the two sets A and B.
  - In the Flip-neighborhood two solutions are neighbors if one can be obtained from the other by moving a single vertex from one set to the other.

PLS completeness

Theorem

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- The problem belongs to PLS taking as neighborhood the Nash dynamics because the Rosenthal's potential function can be evaluated in polynomial time.
- We provide a reduction from MaxCut under the Flip-neigborhood.

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- For each edge e, we add resources e^a and e^b, with delay 0 if used by only one player and delay w_e if used by more players.

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- For each edge e, we add resources e^a and e^b, with delay 0 if used by only one player and delay w_e if used by more players.
- Players correspond to nodes V, and the action set of $v \in V$, $A_v = \{e^a, e^b \mid e \in E \text{ incident to } v\}$

PLS completeness

- Each solution (A, B) of MaxCut corresponds to strategy profile $s = (s_1, \ldots, s_n)$ where for each $v \in V$, $s_v = \{e^a | e = (u, v) \wedge v \in A\} \cup \{e^b | e = (u, v) \wedge v \in B\}$

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- Local optima of the MaxCut instance coincide with the Nash equilibria of the congestion game.

1. Best response dynamics
2. Potential games
3. Congestion games
4. References

Reference

B. Vöcking, Congestion Games: Optimization in Competition