Ampliació d'Algorismia: Aproximation, Parameterization and Streaming

QP 2023-2024: Sheet 2

20. **(Puzzle)**

    Consider the $n \times n$ sliding puzzle which consists of a frame with $n \times n$ with n^2 tiles. $n^2 - 1$ tiles hold numbers from 1 to $n^2 - 1$. The frame has one empty tile and this enables the others to move horizontally and vertically. The puzzle is solved if all numbers are row sorted. Show that the problem of deciding whether a sliding puzzle can be solved in $\leq k$ moves, parameterized by k, belongs to FPT.

21. **(Triangle elimination)**

    Show that the following parameterized edge deletion problem belongs to FPT.

    TRIANGLE ELIMINATION Input: A graph G and an integer $k \leq 0$.
    Parameter: k.
    Question: Does it exist a edge subset F so that $|F| \leq k$ and $G - F$ has no triangles?

22. **(Maximizing support)**

    Consider an undirected graph $G = (V, E)$. We say that a set $S \subseteq V$ supports a vertex $u \in V$ (or that u is supported by S) if u and all its neighbors in G belong to S.

    The Max Restricted Support problem is defined as follows: Given an undirected graph $G = (V, E)$ and an integer s, $1 \leq s \leq |V|$, find a subset $S \subseteq V$ with $|S| = s$ such that the number of vertices supported by S is maximum over all sets with s vertices.

    Show that the bounded version of this optimization problem under the natural parameterization belongs to XP.

23. **(Clustering)**

    El problema de l'obtenció de clústers per esborrat de vèrtexs es defineix així: donat un graf $G = (V, E)$ i un enter k, existeix un conjunt S de mida com a màxim k tal que $G[V \setminus S]$ consisteix en una col·lecció de cliques disjunts?

    Les cliques han de ser disjunts en el sentit que no comparteixen vèrtexs i/o arestes i que a més no hi ha cap aresta amb un vertex a una clique i l'altre a un altre.

    Dissenyeu un algorisme FPT quan la parametrització és $\kappa(G, k) = k$.

    Teniu en compte que un graf està format per un conjunt de cliques disjunts si i només si no conté, com a subgraf induit, cap camí de llargada 2. Recordeu que per $U \subseteq V$ el subgraph induit per U es el graph $G[U] = (U, E \cap (U \times U))$.

24. **(Edge clique cover)**

    In the EDGE CLIQUE COVER problem, we are given a graph G and a nonnegative integer k, and the goal is to decide whether the edges of G can be covered by at most k cliques.

    Consider the following reduction rules:

    - R1. Remove isolated vertices.

- R2. If there is an isolated edge (u, v) (i.e., a connected component that is just an edge), delete it and decrease k by 1. The new instance is $(G - \{u, v\}, k - 1)$.
- R3. If there is an edge (u, v) whose endpoints have exactly the same closed neighborhood including at least another point, that is, $N[u] = N[v] \neq \{u, v\}$, then delete v. The new instance is $(G - v, k)$.

(a) Show that by applying any of the above rules the obtained new instance is equivalent to the initial one.

(b) Show that if (G, k) is a reduced yes-instance, on which none of the three reduction rules can be applied, then $|V(G)| \leq 2^k$.

(c) Can we conclude that p-Edge Clique Cover belongs to FPT?

25. **(Induced matching)**

Given a graph $G = (V, E)$, an induced matching of G is a matching $F \subseteq E$, such that the edge set of the induced subgraph $G[V(F)]$ is F itself. The size of an induced matching is the number of edges in it. The Induced Matching problem is given a graph G and an integer kk, to decide whether G has an induced matching of size at least k.

Consider the following reduction rules:

- R1. Remove isolated vertices.
- R2. If there is a non isolated edge (u, v) (i.e., an edge such that u or v (or both) have degree bigger than 1), delete the vertices in $(N(u) \cup N(v)) - \{u, v\}$.

(a) Show that by applying the above rules until none of them can be applied, we get an induced matching of G.

(b) Can the previous preprocessing be used to define a kernelization for induced matching parameterized by $d + k$ where d is the maximum degree of the graph G?

26. **(Kernel for vertex cover)**

Considereu el problema de la coberta de vèrtexs (min vertex cover). Saps construir en temps polinomial una solució òptima x a la versió relaxada del problema de programació entera que el descriu. En aquesta sol·lució, $x_u \in [0, 1]$. Considereu els conjunts

$$S_1 = \{v \in V \mid x_v > \frac{1}{2}\}$$
$$S_{\frac{1}{2}} = \{v \in V \mid x_v = \frac{1}{2}\}$$
$$S_0 = \{v \in V \mid x_v < \frac{1}{2}\}$$

Demostreu que sota la parametrització natural, $(G[S_{\frac{1}{2}}], k - |S_1|)$ és un kernel. Dona una mida del kernel com a funció del paràmetre.

27. Consider the Independent set problem: Given a graph $G = (V, E)$ and an integer k, decide if there is a set $S \subseteq V$ of size k, so that no pair of vertices in S is connected by an edge. Show that the problem parameterized by treewidth belongs to FPT.

28. ***Minimizing incongruity.*** We have a social network modeling the interaction among members of a company. The social situation is modeled by an edge weighted undirected graph. The manager keeps as edge weights the *level of incongruity* among members which is a non negative number measuring the damage done during activities in which both members participated. The manager wants to split the members into two groups that will be assigned to two different activities. For doing so, the company defines the *incongruity of a group* as the sum of the level of incongruity of each pair of participants in the group. The goal of the manager is to find a splitting of all the members into two groups so that the sum of the incongruity of the two groups is minimized.

    (i) Provide a formalization of the problem.

    (ii) Show that the problem parameterized by the treewidth of the graph belongs to FPT.