Network Creation Game

A. Fabrikant, A. Luthra, E. Maneva,
C. H. Papadimitriou, and S. Shenker,
PODC 2003

(Part of the Slides are taken from Alex Fabrikant's presentation)

Introduction

- Introduced in [FLMPS,PODC'03]
- A **Local Connection Game** (LCG) is a game that models the *ex-novo* creation of a network
- Players are nodes that:
  - Incur a cost for the links they personally activate;
  - Benefit from having the other nodes on the network as close as possible, in terms of length of shortest paths on the created network (notice they can use all the activated edges)

[FLMPS,PODC'03]:
A. Fabrikant, A. Luthra, E. Maneva, C.H. Papadimitriou, S. Shenker, On a network creation game, PODC'03

The formal model

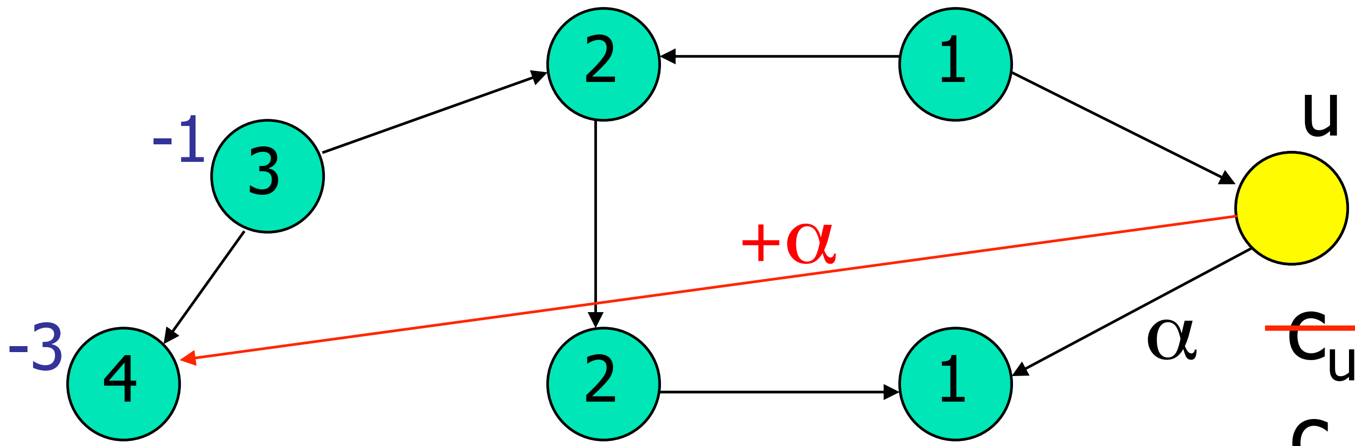
- n players: nodes $V=\{1,\ldots,n\}$ in a graph to be built
- Strategy for player u: a set of incident edges (intuitively, a player buys these edges, that will be then used bidirectionally by everybody; however, only the owner of an edge can remove it, in case he decides to change his strategy)
- Given a strategy vector $S=(s_1,\ldots, s_n)$, the constructed network will be the undirected graph $G(S)$
- player u's goal:
  - to spend as little as possible for buying edges (building cost)
  - to make the distance to other nodes as small as possible (usage cost)

The model

- Each edge has a real-value cost $\alpha \geq 0$
- $dist_{G(S)}(u,v)$: length of a shortest path (in terms of number of edges) in $G(S)$ between u and v
- n_u: number of edges bought by node u
- Player u aims to minimize its cost:

$$cost_u(S) = \alpha n_u + \sum_{v \in V} dist_{G(S)}(u,v)$$

Cost of a player: an example



$\cancel{c_u=\alpha+13}$

$c_u=2\alpha+9$

Convention: arrow *from* the node buying the link

Notice that if $\alpha<4$ this is an improving move for u

The social-choice function

- To evaluate the overall quality of a network, once again we consider the *utilitarian social cost*, i.e., the sum of all players' costs. Observe that:
  1. In $G(S)$ each term $dist_{G(S)}(u,v)$ contributes to the overall cost twice
  2. Each edge (u,v) is bough at most by one player

Social cost of a network $G(S)=(V,E)$:

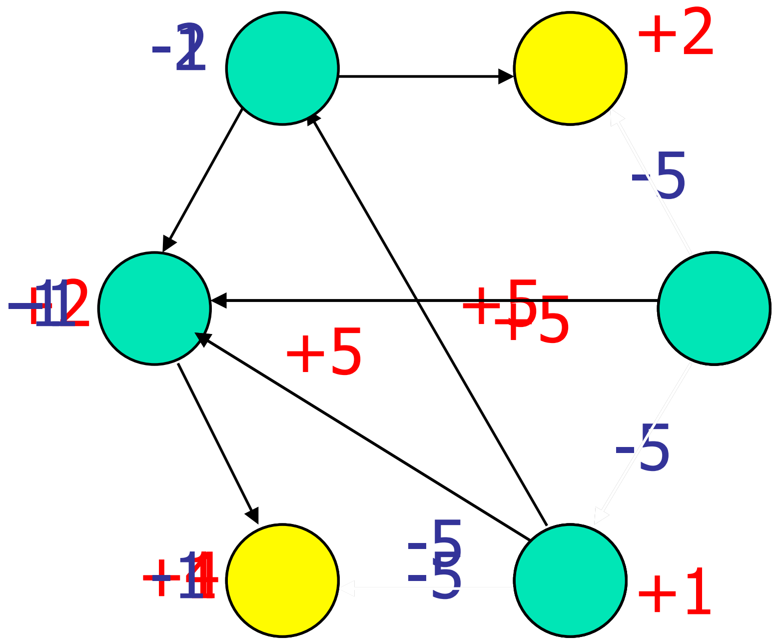
$$SC(G(S))=\alpha|E| + \Sigma_{u,v\in V}\, dist_{G(S)}(u,v)$$

Our goal

- We use Nash equilibrium (NE) as the solution concept: Given a strategy profile S, the formed network G(S)=(V,E) is stable (for the given value α) if S is a NE
- Conversely, given a graph G=(V,E), it is stable if there exists a strategy vector S such that G=G(S), and S is a NE
- Observe that any stable network must be connected, since the distance between two nodes is infinite whenever they are not connected
- A network is *optimal* or *socially efficient* if it minimizes the social cost
- We aim to characterize the efficiency loss resulting from selfishness, by bounding the Price of Stability (PoS) and the Price of Anarchy (PoA)

Stable networks: an example

- Set α=5, and consider:

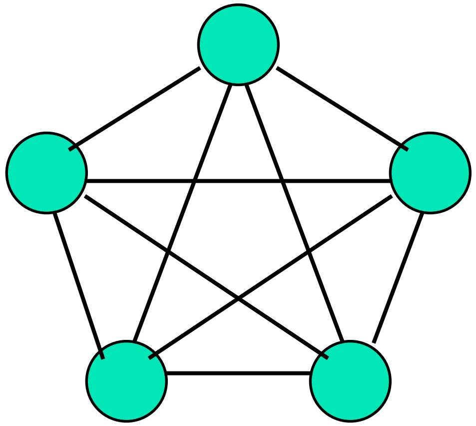


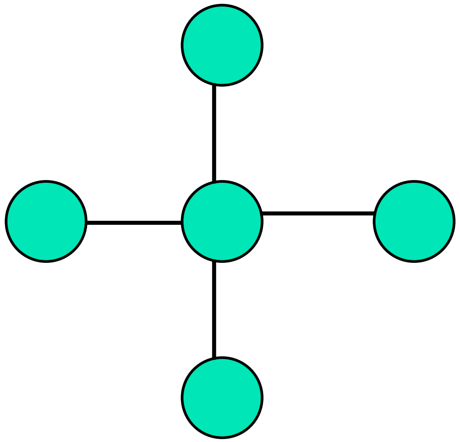
That's a stable network!

How does an optimal network look like?

Some notation

K_n: complete graph
with n nodes





A star is a tree
with height at most 1
(when rooted at its
center)

Lemma 1

Il $\alpha \le 2$ then the complete graph is an optimal solution, while if $\alpha \ge 2$ then the star is an optimal solution.

proof

Let $G=(V,E)$ be an optimal solution; $|E|=m$ and $SC(G)=OPT$

$$OPT = \alpha|E| + \Sigma_{u,v\in V}\, dist_G(u,v) \ge \alpha m + 2m \quad + \quad 2(n(n-1) -2m)$$

$$=(\alpha-2)m + 2n(n-1) \longleftarrow LB(m)$$

adjacent nodes at distance 1

non-adjacent pairs of nodes at distance ≥ 2

Notice: $LB(m)$ is equal to $SC(K_n)$ when $m=n(n-1)/2$, and to $SC(star)$ when $m=n-1$; indeed:

$$SC(K_n) = \alpha\, n(n-1)/2 + n(n-1)$$

$$SC(star) = \alpha\,(n-1) + 2(n-1) + 2(n-1)(n-2) = \alpha\,(n-1) + 2(n-1)^2$$

and it is easy to see that they correspond to $LB(n(n-1)/2)$ and to $LB(n-1)$, respectively,

Proof (continued)

$G=(V,E)$: optimal solution;
$|E|=m$ and $SC(G)=OPT$

$$LB(m)=(\alpha-2)m + 2n(n-1)$$

$$OPT \geq LB(m) \geq \begin{cases} \xrightarrow[\text{min } m]{\alpha \geq 2} LB(n-1) = SC(star) \\ \xrightarrow[\text{max } m]{\alpha \leq 2} LB(n(n-1)/2) = SC(K_n) \end{cases}$$

Are complete graphs and stars stable?

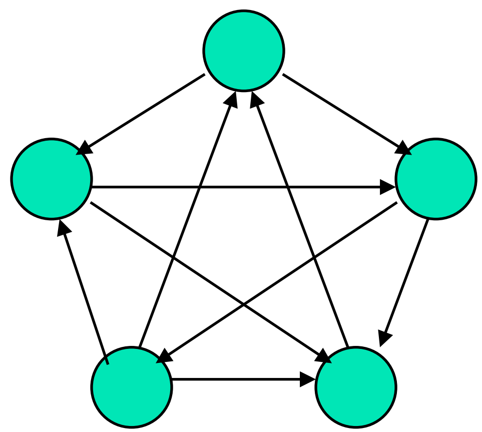
Lemma 2

Il $\alpha \leq 1$ the complete graph is stable, while if $\alpha \geq 1$ then the star is stable.

Proof:

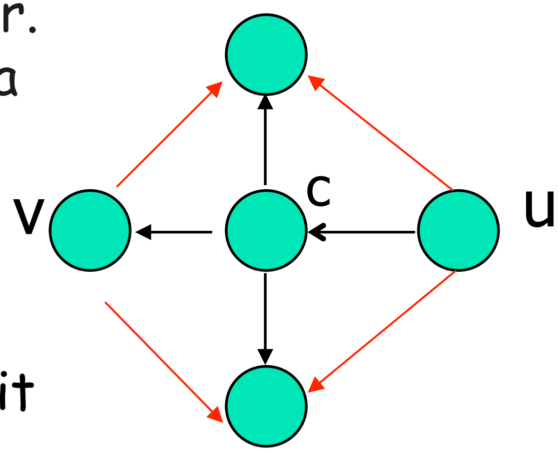
$\alpha \leq 1$

By definition, we have to find a NE S inducing a clique. Actually, any arbitrary strategy profile S inducing a clique is a NE. Indeed, if a node removes any $k \geq 1$ owned edges, it saves αk in the building cost, but it pays $k \geq \alpha k$ more in the usage cost



Proof (continued) $\alpha \geq 1$

By definition, we have to find a NE S inducing a star. Actually, any arbitrary strategy profile S inducing a star is a NE. Indeed:



Center c cannot change its strategy, otherwise its cost increase to infinity

If a leaf v not buying edges buys any $1 \leq k \leq n-2$ edges it pays αk more in the building cost, but it saves only $k \leq \alpha k$ in the usage cost

For a leaf u buying an edge, its cost is $\alpha+1+2(n-2)$ and we have two cases:

Case 1: u maintains (u,c) and buys any $1 \leq k \leq n-2$ additional edges; this case is similar to the previous one.

Case 2: u removes (u,c) and buys any $1 \leq k \leq n-2$ edges; thus, it pays αk in the building cost, and its usage cost becomes $k+2+3(n-k-2)$, and so its total cost becomes:

distance to adjacent nodes — distance to c — distance to other nodes

$$\alpha k+k+2+3n-3k-6 = \alpha+[\alpha(k-1)-2k+n]+2(n-2) \geq$$

$$\alpha+[k-1-2k+n]+2(n-2) = \alpha+[n-k-1]+2(n-2)$$

which is at least equal to the initial cost of $\alpha+1+2(n-2)$, since the quantity in square brackets is at least 1, being $1 \leq k \leq n-2$. ∎

Theorem 1

For $\alpha \le 1$ and $\alpha \ge 2$ the PoS is 1. For $1<\alpha<2$ the PoS is at most 4/3

Proof: From Lemma 1 and 2, for $\alpha \le 1$ (respectively, $\alpha \ge 2$) a complete graph (respectively, a star) is both optimal and stable, and so the claim follows.

$1<\alpha<2$ $\quad K_n$ is an optimal solution (Lemma 1), and a star T is stable (Lemma 2); then

< 0 for $1<\alpha<2$

$$\text{PoS} \le \frac{SC(T)}{SC(K_n)} = \frac{(\alpha-2)(n-1) + 2n(n-1)}{\alpha\, n(n-1)/2 + n(n-1)} < \frac{2n(n-1)}{n(n-1)/2 + n(n-1)} = 4/3$$

$> n(n-1)/2+n(n-1)$ for $1<\alpha<2$

■

What about the Price of Anarchy?

…for α<1 the complete graph is the
only stable network,
(try to prove that formally)
hence PoA=1…

…for larger value of α?

State-of-the-art

$$\alpha = 0 \quad 1 \quad 2 \quad \sqrt[3]{n/2} \quad \sqrt{n/2} \quad O(n^{1-\epsilon}) \quad 273n \quad 12n \lg n \quad \infty$$

PoA	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{\mathcal{O}(\sqrt{\log n})}$	< 5	≤ 1.5

Open Problems and Historical Results

The constant PoA conjecture: The PoA is constant for any α.

	$\alpha = 0$		1		2		$\sqrt[3]{n/2}$		$\sqrt{n/2}$		$O(n^{1-\delta})$		$4n-13$		$12n\log n$		∞
PoA		1		$\leq \frac{4}{3}$		≤ 4		≤ 6		$\Theta(1)$		$2^{O(\sqrt{\log n})}$		< 5		1.5	

The Tree conjecture: Every NE is a tree for $\alpha > n$.

Open Problems and Historical Results

The constant PoA conjecture: The PoA is constant for any α.

α	0–1	1–2	2–$\sqrt[3]{n/2}$	$\sqrt[3]{n/2}$–$\sqrt{n/2}$	$\sqrt{n/2}$–$O(n^{1-\delta})$	$O(n^{1-\delta})$–$4n-13$	$4n-13$–$12n\log n$	$12n\log n$–∞
PoA	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{O(\sqrt{\log n})}$	< 5	1.5

The Tree conjecture: Every NE is a tree for $\alpha > n$.

Two important results

Proposition (Fabrikant et al. 2003)

The price of anarchy for trees is < 5.

Lemma (Demaine et al. 2007)

The diameter plus one unit is an upper bound for the PoA.

The historical progress for $\alpha > n$

$\alpha = 0$	1	2	$\sqrt[3]{n/2}$	$\sqrt{n/2}$	$O(n^{1-\delta})$	$4n - 13$	$12n\log n$	∞
PoA	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{O(\sqrt{\log n})}$	< 5	1.5

Tree conjecture (Implying constant PoA for the corresponding range)

- Albers et al. $\alpha > 12n \log n$ (2006).
- Mihalák and Schlegel $\alpha > 273n$ (2013).
- Mamageishvilli et al. $\alpha > 65n$ (2015).
- Àlvarez and Messegué $\alpha > 17n$ (2017).
- Bilò and Lenzner $\alpha > 4n - 13$ (2018).

The historical progress for $\alpha > n$

α	0 – 1	1 – 2	2 – $\sqrt[3]{n/2}$	$\sqrt[3]{n/2}$ – $\sqrt{n/2}$	$\sqrt{n/2}$ – $O(n^{1-\delta})$	$O(n^{1-\delta})$ – $4n-13$	$4n-13$ – $12n\log n$	$12n\log n$ – ∞
PoA	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{O(\sqrt{\log n})}$	< 5	1.5

Tree conjecture (Implying constant PoA for the corresponding range)

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- Bilò and Lenzner $\alpha > 4n - 13$ (2018).

Constant PoA without relying on the tree conjecture

- Àlvarez and Messegué $\alpha > 9n$ (2017).

Our Contribution

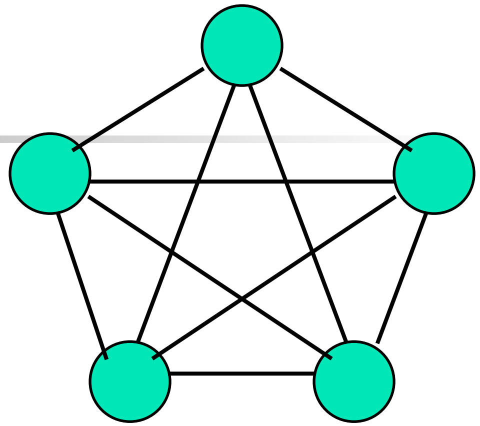
$\alpha = 0$	1	2	$\sqrt[3]{n/2}$	$\sqrt{n/2}$	$O(n^{1-\delta})$	$4n-13$	$12n\log n$	∞
PoA: 1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{O(\sqrt{\log n})}$	< 5	1.5	

For any $\epsilon > 0$ small enough constant the PoA is constant for $\alpha > n(1+\epsilon)$. (Àlvarez and Messegué 2019)

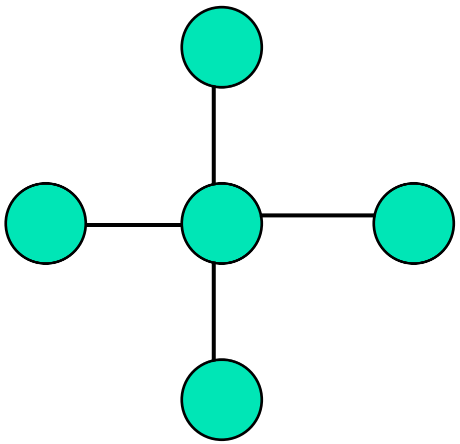
$\alpha = 0$	1	2	$\sqrt[3]{n/2}$	$\sqrt{n/2}$	$O(n^{1-\delta})$	$n(1+\epsilon)$	$4n-13$	$12n\log n$	∞
PoA: 1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{O(\sqrt{\log n})}$	$\Theta(1)$	< 5	1.5	

Some more notation

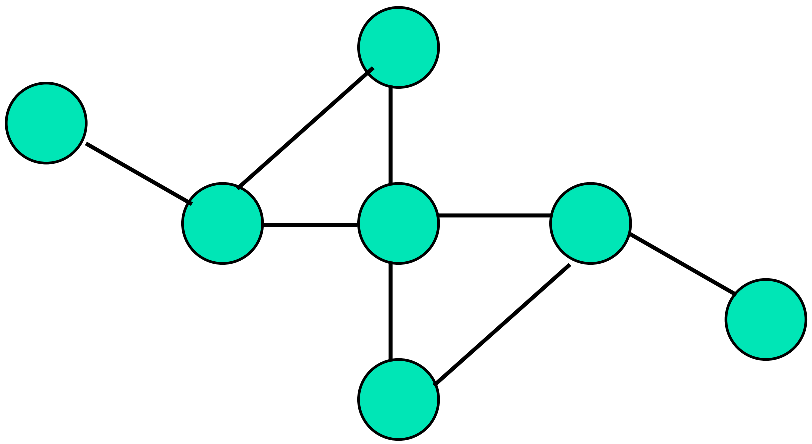
The diameter of a graph G is the maximum distance between any two nodes



diam=1



diam=2

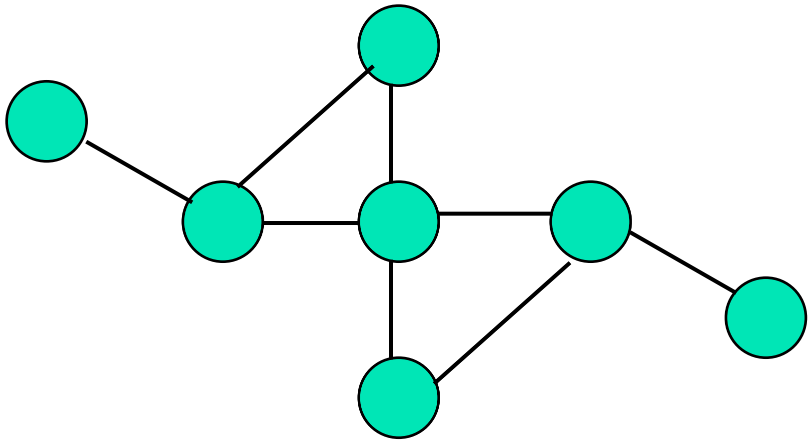


diam=4

Some more notation

An edge e is a cut edge of a graph G=(V,E) if G-e is disconnected

G-e=(V,E\{e})



A simple property:
Any graph has at most n-1 cut edges (otherwise they would induce a cycle, which is impossible…think about it)

Theorem 2

The PoA is at most $O(\sqrt{\alpha})$.

proof

It follows from the following lemmas:

Lemma 3

The diameter of any stable network is at most $2\sqrt{\alpha} + 1$.

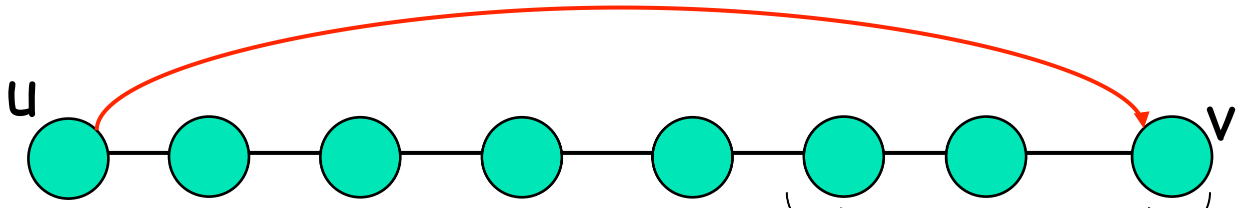
Lemma 4

The SC of any stable network with diameter d is at most $3d$ times the optimum SC.

proof of Lemma 3

G: stable network

Consider a shortest path in G between two nodes u and v



$2k \leq dist_G(u,v) \leq 2k+1$
for some k

k vertices reduce their distance from u

from $\geq 2k$ to 1 → $\geq 2k-1$

from $\geq 2k-1$ to 2 → $\geq 2k-3$

⋮

from $\geq k+1$ to k → ≥ 1

$$\sum_{i=0}^{k-1}(2i+1)=k^2$$

...since G is stable:

⇒ $\alpha \geq k^2$ ⇒ $k \leq \sqrt{\alpha}$

⇒ $dist_G(u,v) \leq 2\sqrt{\alpha} + 1$

■

Proposition 1

Let G be a network with diameter d, and let $e=(u,v)$ be a non-cut edge. Then in $G-e$, every node w increases its distance from u by at most $2d$

Proposition 2

Let G be a stable network, and let F be the set of non-cut edges bought by a node u. Then $|F| \leq (n-1)2d/\alpha$

Lemma 4

The SC of any stable network G=(V,E) with diameter d is at most 3d times the optimum SC.

proof

$$OPT \geq \alpha\,(n-1) + n(n-1)$$

$$SC(G)= \underbrace{\Sigma_{u,v} d_G(u,v)}_{\leq dn(n-1)\ \leq d\ OPT} + \alpha\,|E| \leq d\ OPT+2d\ OPT = 3d\ OPT$$

$$\alpha|E|=\alpha\underbrace{|E_{cut}|}_{\leq(n-1)} + \alpha\underbrace{|E_{non\text{-}cut}|}_{\substack{\leq n(n-1)2d/\alpha \\ \text{Prop. 2}}} \leq\alpha(n-1)+n(n-1)2d \leq 2d\ OPT$$

Theorem 2

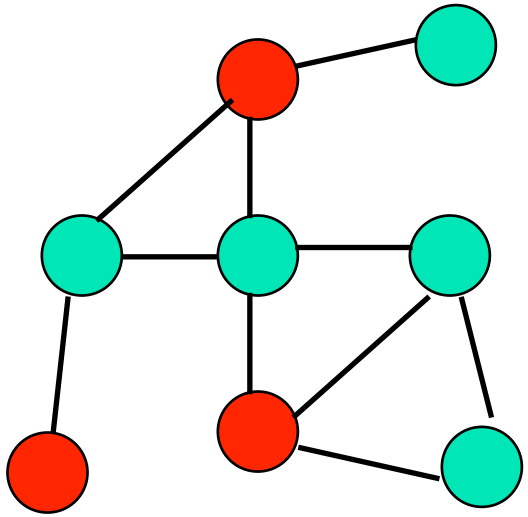
It is NP-hard, given the strategies of the other agents, to compute the best response of a given player.

proof

Reduction from dominating set problem

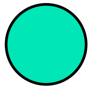
Dominating Set (DS) problem

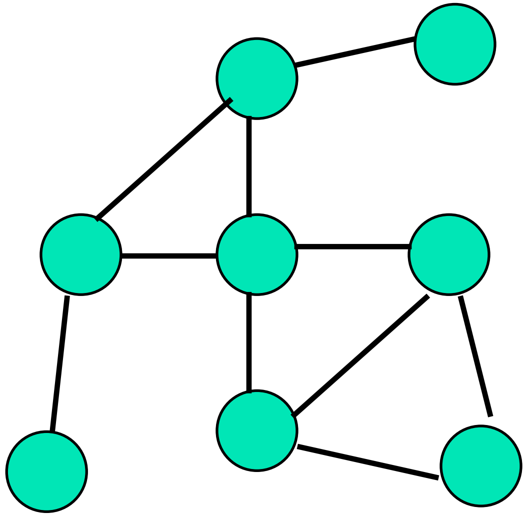
- Input:
  - a graph $G=(V,E)$
- Solution:
  - $U \subseteq V$, such that for every $v \in V-U$, there is $u \in U$ with $(u,v) \in E$
- Measure:
  - Cardinality of U



the reduction

$1<\alpha<2$

player i 

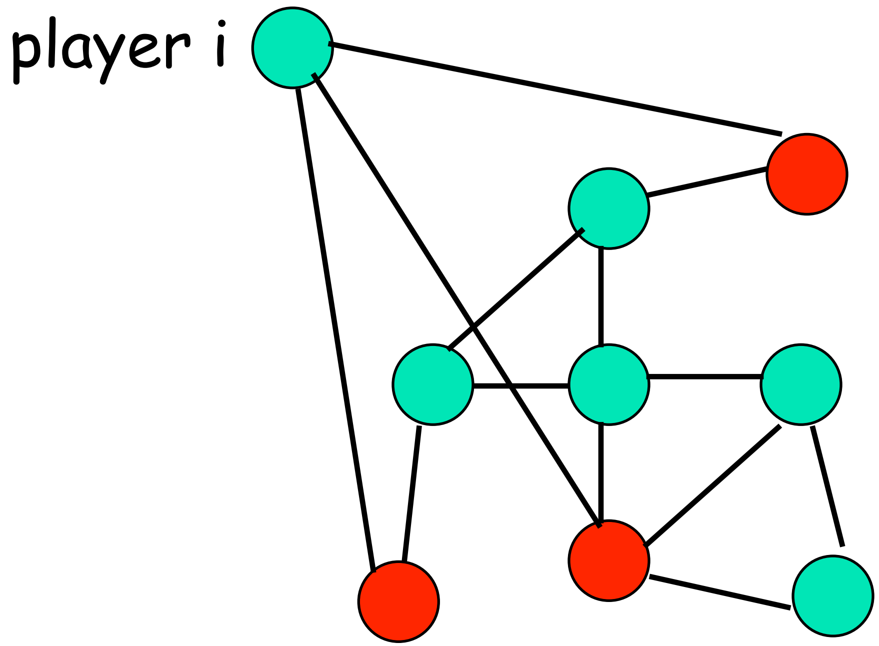


$G=(V,E) = G(S_{-i})$

Player i has a strategy yielding a cost $\leq \alpha k+2n-k$ if and only if there is a DS of size $\leq k$

the reduction

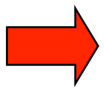
$1<\alpha<2$



$G=(V,E) = G(S_{-i})$

($\Leftarrow$)

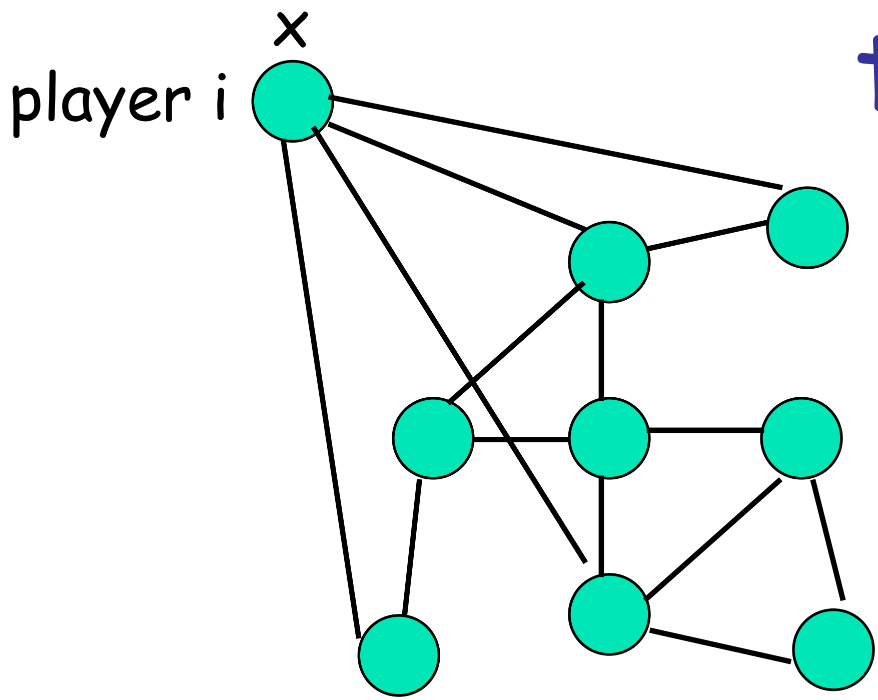
easy: given a dominating set U of size k, player i buys edges incident to the nodes in U



Cost for i is $\alpha k+2n-k$

$1<\alpha<2$

the reduction



$G=(V,E) = G(S_{-i})$

(⇒)

Let S_i be a strategy giving a cost $\leq \alpha k+2n-k$

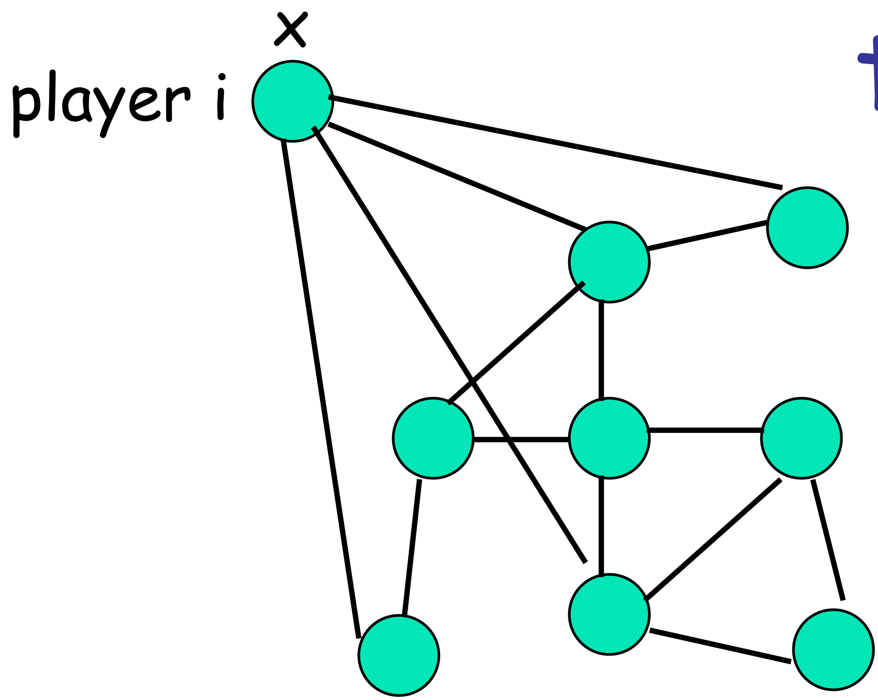
Modify S_i as follows:

repeat:

if there is a node v such with distance ≥3 from x in G(S), then add edge (x,v) to S_i (this decreases the cost)

$1<\alpha<2$

the reduction



$G=(V,E) = G(S_{-i})$

($\Rightarrow$)

Let S_i be a strategy giving a cost $\leq \alpha k+2n-k$

Modify S_i as follows:

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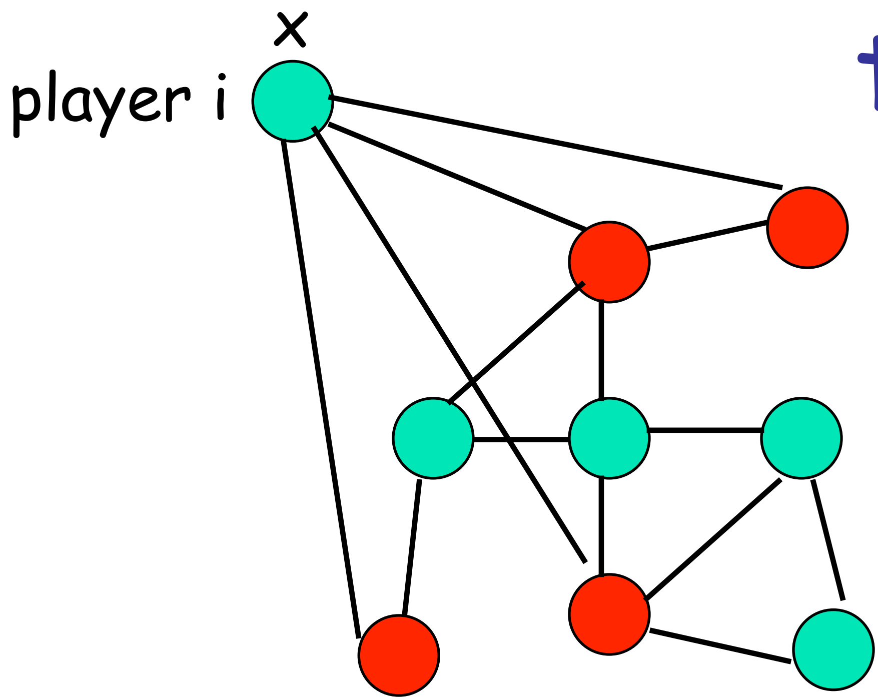
if there is a node v such with distance ≥ 3 from x in G(S), then add edge (x,v) to S_i (this decreases the cost)

Finally, every node has distance either 1 or 2 from x

$1<\alpha<2$

the reduction



$G=(V,E) = G(S_{-i})$

($\Rightarrow$)

Let S_i be a strategy giving a cost $\leq \alpha k+2n-k$

Modify S_i as follows:

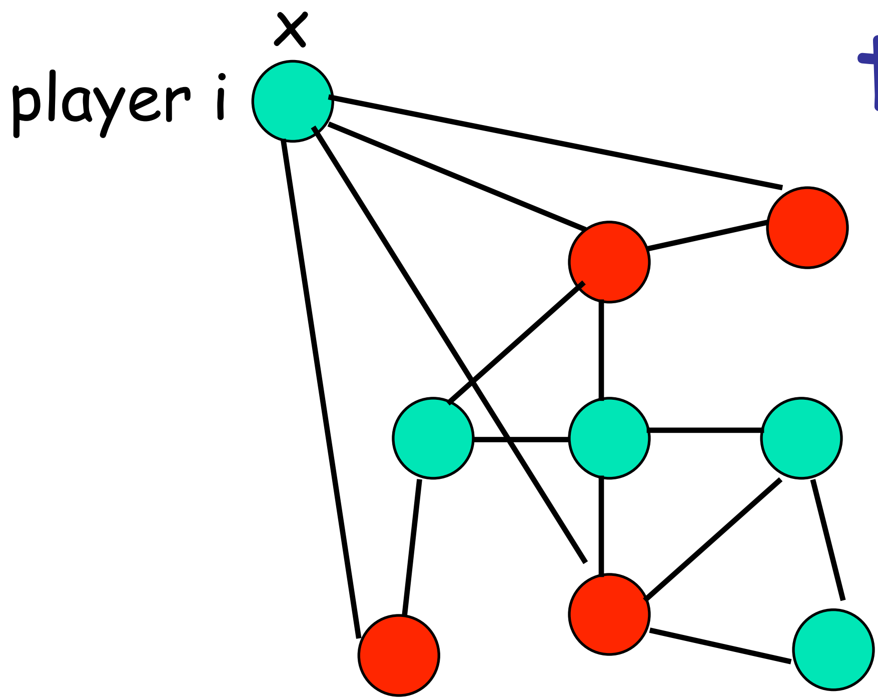
repeat:

if there is a node v such with distance ≥ 3 from x in G(S), then add edge (x,v) to S_i (this decreases the cost)

Finally, every node has distance either 1 or 2 from x

Let U be the set of nodes at distance 1 from x…

$1<\alpha<2$



the reduction

$G=(V,E) = G(S_{-i})$

$(\Rightarrow)$

…U is a dominating set of the original graph G

We have $cost_i(S)= \alpha|U|+2n-|U| \leq \alpha k+2n-k$

$\Rightarrow$ $|U| \leq k$

$\blacksquare$