What about Internet?

Christos Papadimitriou (STOC 2001)

“The internet is unique among all the computer systems in that it is build, operated and used by multitude of diverse economic interests, in varing relationships of collaboration and competition with each other. This suggest that the mathematical tools and insights most appropriate for understanding the Internet may come from the fusion of algorithmic ideas with concepts and techniques from Mathematical Economics and Game Theory.”

http://www.cs.berkeley.edu/~christos/games/cs294.html

What is Game Theory?

Game theory is a branch of applied mathematics and economics that studies situations where players choose different actions in an attempt to maximize their returns.

The essential feature, however, is that it provides a formal modelling approach to social situations in which decision makers interact with other minds.

Game theory extends the simpler optimization approach developed in neoclassical economics.

Where to use game theory?

Game theory studies decisions made in an environment in which players interact.

game theory studies choice of optimal behavior when personal costs and benefits depend upon the choices of all participants.

What for?

Game theory looks for states of equilibrium sometimes calles solutions

and analyzes interpretations/properties of such states

Basic Reference

- Osborne. An Introduction to Game Theory, Oxford University Press, 2004
- Nisan et al. Eds. Algorithmic game theory, Cambridge University Press, 2007

Game Theory for CS?

- Framework to analyze equilibrium states of protocols used by rational agents.
  Price of anarchy/stability.
- Tool to design protocols for internet with guarantees.
  Mechanism design.
- New concepts to analyze/justify behavior of on-line algorithms
  Give guarantees of stability to dynamic network algorithms.
- Source of new computational problems to study.
  Algorithmic game theory

Games

- Non-cooperative games
  - strategic games
  - extensive games
  - repeated games
  - Bayesian games
- Cooperative games
  - simple games
  - weighted games
  - ...

Strategic game

A strategic game Γ (with ordinal preferences) consists of:

- A finite set $N = \{1, \ldots, n\}$ of players.
- For each player $i \in N$, a nonempty set of actions A_i.
- Each player chooses his action once. Players choose actions simultaneously. No player is informed, when he chooses his action a_i, of the actions chosen by others. $A = A_1 \times \ldots \times A_n$ set of all strategy profiles $s = (a_1, \ldots, a_i, \ldots, a_n)$
- For each player $i \in N$, a preference relation (a complete, transitive, reflexive binary relation) $\preceq_i$ over the set $A = A_1 \times \cdots \times A_n$.

It is frequent to specify the players' preferences by giving utility functions $u_i(a_1, \ldots a_n)$. Also called pay-off functions.

Example: Prisoner's Dilemma

The story

- Two suspects in a major crime are held in separate cells.
- Evidence to convict each of them of a minor crime.
- No evidence to convict either of them of a major crime unless one of them finks.

The penalties

- If both stay quiet, be convicted for a minor offense (one year prison).
- If only one finks, he will be freed (and used as a witness) and the other will be convicted for a major offense (four years in prison).
- If both fink, each one will be convicted for a major offense with a reward for coperation (three years each).

The Prisoner's Dilemma models a situation in which

- there is a gain from cooperation,
- but each player has an incentive to free ride.

Game representation

- Players $N = \{\text{Suspect 1}, \text{Suspect 2}\}$.
- Actions $A_1 = A_2 = \{\text{Quiet}, \text{Fink}\}$.
- Action profiles
  $A = A_1 \times A_2 = \{(\text{Quiet}, \text{Quiet}), (\text{Quiet}, \text{Fink}), (\text{Fink}, \text{Quiet}), (\text{Fink}, \text{Fink})\}$
- Preferences
  - ⋆ Player 1

    0 days   1 day   3 days   4 days

    $$(\text{Fink}, \text{Quiet}) \precsim_1 (\text{Quiet}, \text{Quiet}) \precsim_1 (\text{Fink}, \text{Fink}) \precsim_1 (\text{Quiet}, \text{Fink})$$

  - ⋆ Player 2

    0 days   1 day   3 days   4 days

    $$(\text{Quiet}, \text{Fink}), \precsim_2 (\text{Quiet}, \text{Quiet}) \precsim_2 (\text{Fink}, \text{Fink}) \precsim_2 (\text{Fink}, \text{Quiet})$$

- Utilities
  $u_1(\text{Fink}, \text{Quiet}) = 3, u_1(\text{Quiet}, \text{Quiet}) = 2, u_1(\text{Fink}, \text{Fink}) = 1, u_1(\text{Quiet}, \text{Fink}) = 0$
  $u_2(\text{Quiet}, \text{Fink}) = 3, u_2(\text{Quiet}, \text{Quiet}) = 2, u_2(\text{Fink}, \text{Fink}) = 1, u_2(\text{Fink}, \text{Quiet}) = 0$

We can represent pay-offs in a compact way on a bi-matrix.

		Suspect 2	
		Quiet	Fink
Suspect 1	Quiet	2,2	0,3
	Fink	3,0	1,1

$u_1(\text{Fink}, \text{Quiet})$ $\quad$ $u_2(\text{Fink}, \text{Quiet})$

utilities

s1 \ s2	Quiet	Fink
Quiet	1,1	4,0
Fink	0,4	3,3

cost: years in prison

$u_1(\text{Quiet}, \text{Quiet}) = 2$

$u_2(\text{Quiet}, \text{Quiet}) = 2$

$u_1(\text{Fink}, \text{Quiet}) = 3$

$u_2(\text{Fink}, \text{Quiet}) = 0$

$u_1(\text{Quiet}, \text{Fink}) = 0$

$u_2(\text{Quiet}, \text{Fink}) = 3$

$u_1(\text{Fink}, \text{Fink}) = u_2(\text{Fink}, \text{Fink}) = 1$

$c_1(\text{Quiet}, \text{Quiet}) = 1$

$c_2(\text{Quiet}, \text{Quiet}) = 1$

$c_1(\text{Fink}, \text{Quiet}) = 0$

$c_2(\text{Fink}, \text{Quiet}) = 4$

$c_1(\text{Quiet}, \text{Fink}) = 4$

$c_2(\text{Quiet}, \text{Fink}) = 0$

$c_1(\text{Fink}, \text{Fink}) = c_2(\text{Fink}, \text{Fink}) = 3$

Example: Matching Pennies

- Two people choose, simultaneously, whether to show the head or tail of a coin.
- If they show same side, person 2 pays person 1, otherwise person 1 pays person 2.
- Payoff are equal to the amounts of money involved.

		Person 2	
		Head	Tail
Person 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

This is an example of a zero-sum game

Strategies: Notation

A strategy of player $i \in N$ in a strategic game Γ is an action $a_i \in A_i$.

A strategy profile $s = (s_1, \ldots, s_n)$ consists of a strategy for each player.

For each $s = (s_1, \ldots s_n)$ and $s'_i \in A_i$ we denote by

$$(s_{-i}, s'_i) = (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$$

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

is not a strategy profile but can be seen as a strategy for the other players.

Best response

Let Γ be a strategic game defined trhough pay-off functions

The set of best responses for player i to s_{-i} is

$$BR(s_{-i}) = \{a_i \in A_i \mid u_i(s_{-i}, a_i) = \max_{a'_i \in A_i} u_i(s_{-i}, a'_i)\}$$

Those are the actions that give maximum pay-off provided the other players do not change their strategies.

solution concepts

- Pure Nash equilibria
- (Mixed) Nash equilibria
- Dominant strategies
- Strong Nash equilibria
- Correlated equilibria $\vdots$

Pure Nash equilibrium

A pure Nash equilibrium is a strategy profile $a^* = (a_1^*, \ldots, a_n^*)$ such that

no player i can do better choosing an action different from a_i^*,

given that every other player j adheres to a_j^*:

> for every player i and for every action $a_i \in A_i$ it holds
> $u_i(a_{-i}^*, a_i^*) \geqslant u_i(a_{-i}^*, a_i)$.
>
> Equivalently, for every player i and for every action $a_i \in A_i$ it holds
> $a_i^* \in BR(a_{-i}^*)$.

Pure Nash Equilibrium

- Is a strategy profile in which all players are happy.
- Identified with a fixed point of an iterative process of computing a best response.
- However, the game is played only once!
- GT deals with the existence and analysis of equilibria assuming rational behavior.

  players try to maximize their benefit
- GT does not provide algorithmic tools for computing such equilibrium if one exists.

Pure Nash equilibria, examples

	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

	Stag	Hare
Stag	2,2	0,1
Hare	1,0	1,1

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Pure Nash equilibria, examples

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	Stag	Hare
Stag	2,2	0,1
Hare	1,0	1,1

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- Prisoner's Dilemma, (Fink, Fink).
- Bach or Stravinsky, (Bach, Bach), (Stravinsky, Stravinsky).
- Stag Hunt, (Stag, Stag), (~~Hunt~~ Hare, ~~Hunt~~ Hare).
- Matching Pennies, none.

Pure Nash equilibrium

- First notion of equilibria for non-cooperative games.
- There are strategic games with no pure Nash equilibrium.
- There are games with more than one pure Nash equilibrium.
- How to compute a Nash equilibrium if there is one?

Mixed strategies

Until now players are selecting as strategy an action.

A mixed strategy for player i is a distribution (lottery) σ_i on the set of actions A_i.

$\sigma_i : A_i \to [0,1]$ $\quad \sum_{a \in A_i} \sigma_i(a) = 1$

The utility function for player i is the expected utility under the joint distribution $\sigma = (\sigma_1, \dots, \sigma_n)$ assuming independence.

$$U_i(\sigma) = \sum_{(a_1,\dots,a_n) \in A} \sigma_1(a_1) \cdots \sigma_n(a_n) u_i(a_1 \dots, a_n)$$

Mixed Nash equilibrium

A mixed Nash equilibrium is a profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ such that no player i can get better utility choosing a distribution different from σ_i^*, given that every other player j adheres to σ_j^*.

Theorem (Nash): Every strategic game has a mixed Nash equilibrium.

From a computational point of view, mixed strategies present an additional representation problem.

In CS we can store only rational numbers. It is known

- For two player game there are always a mixed Nash equilibrium with rational probabilities.
- There are three player games without rational mixed Nash equilibrium.

[Schoenebeck and Vadhan: eccc, 2005]

Checking for a Nash equilibrium

Given a distribution σ_i on A_i define the support of σ_i to be the set

$$\{a_i \mid \sigma_i(a_i) \neq 0\}$$

$$\{a \in A_i \mid \sigma_i(a) \neq 0\}$$

A mixed strategy profile σ is a Nash equilibria iff

for any player i any action in the support of σ_i is a best response to σ_{-i}

σ és NE mixte de Γ $\Leftrightarrow$

$\forall i \quad \forall a_i \in \text{support}(\sigma_i)$

$u_i(\sigma_1 \ldots \sigma_n) = u_i(\sigma_{-i}, \sigma_i^*)$

$\sigma_i^*(a_i) = 1$

$\sigma_i^*(a_j) = 0 \quad \forall a_j \in A_i \; ; \; a_j \neq a_i$

Computational problems related to Nash equilibrium

Basic problems

Is Nash (Isn)

Given a game Γ and a strategy profile a, decide whether a is a Nash equilibrium of Γ.

Strategic Pure Nash (spn)

Given a strategic game Γ, decide whether Γ has a Pure Nash equilibrium.

How to represent a game?

- We are interested in fixing the representation of a game as an input to a program.
- It is natural to consider different levels of succinctness.
- Some components have to be represented by a TM, for example a strategy in an extensive game, the pay-off functions and so on.
- Those machines will work by a limited number of timesteps.
- All the TMs appearing in the description of games are deterministic. We use the following convention: there is a pre-fixed interpretation of the contents of the output tape of a TMso that, both when the machine stops or when the machine is stopped, it always computes a value.

The first condition implies we have only to consider rational valued functions

We gave a correct game definition from its description

$\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$

Strategic games

Strategic games in implicit form. A game is a tuple $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$. This game has n players. For each player i, their set of actions is $A_i = \Sigma^m$ and $\langle M, 1^t \rangle$ is the description of the pay-off functions.

Strategic games in general form. A game is a tuple $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$. It has n players, for each player i, their set of actions A_i is given by listing all its elements. The description of their pay-off functions is given by $\langle M, 1^t \rangle$.

Strategic games in explicit form. A game is a tuple $\Gamma = \langle 1^n, A_1, \ldots, A_m, T \rangle$. It has n players, and for each player i, their set of actions A_i is given explicitly. T is a table with an entry for each strategy profile a and a player i. In this case $u_i(a) = T(a, i)$.

Complexity results

Strategic games

representation	Exist PNE?
implicit	Σ_2^p-complete
general	NP-complete
explicit	AC ^{0}
general with fixed number of players	P-complete

Strategic - general form

Theorem: The existence of PNE problem for strategic games in general form is NP-complete.

Membership

- Guess a strategy profile $a^* \in A_1 \times \ldots \times A_n$
- Verify the correctness of the guess by checking for every player and every action whether the replacement provides a better payoff.

Hardness

- We reduce the SAT to the PNE
  $F \to \Gamma(F) = \langle 1^n, \{0,1\} \ldots \{0,1\}, M^F, 1^{(n+|F|)^{O(1)}} \rangle$
- M^F is a TMthat on input (a, i), evaluates F on assignment a and afterwards it implements the utility function of the i-th player.

$$u_1(a) = \begin{cases} 5 & \text{if } F(a) = 1, \\ 4 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 1, \\ 3 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 1, \\ 2 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 0, \\ 1 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 0, \end{cases} \quad 4 \quad u_2(a) = \begin{cases} 5 & \text{if } F(a) = 1, \\ 4 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 0, \\ 3 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 1, \\ 2 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 1, \\ 1 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 0. \end{cases}$$

And, for any $j > 2$

$$u_j(a) = \begin{cases} 5 & \text{if } F(a) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Strategic - implicit

$\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$

$N = \{1, \dots, n\}$ $A_i := \Sigma^m$ $|A_i| = 2^m$

Let $L \subseteq \Sigma^*$ a language.

$L \in \Sigma_2^p$ if and only if there is a polynomially decidable relation R, such that

$$L = \{x \mid \exists z |z| \leqslant p(|x|) \forall y |y| \leqslant p(|x|) \langle x, y, z \rangle \in R\}.$$

Theorem: The existence of PNE for strategic games in implicit form is Σ_2^p-complete.

Membership

Follows from definitions

Hardness

Follows from a reduction from a restricted version of the Quantified Boolean Formula, the Q2SAT problem, which is Σ_2^p-complete.

Q2SAT

Given $\Phi = \exists \alpha_1, \ldots, \alpha_{n_1} \forall \beta_1, \ldots \beta_{n_2} F$ where F is a Boolean formula over the boolean variables $\alpha_1, \ldots, \alpha_{n_1}, \beta_1, \ldots, \beta_{n_2}$, decide whether Φ is valid.

For each Φ we define a game $\Gamma(\Phi)$ as follows. There are four players:

- Player 1, the *existential player*, assigns truth values to the boolean variables $\alpha_1, \ldots, \alpha_{n_1}$. Their set of actions is $A_1 = \{0, 1\}^{n_1}$ and $a_1 = (\alpha_1, \ldots \alpha_{n_1}) \in A_1$.
- Player 2, the *universal player*, assigns truth values to the boolean variables $\beta_1, \ldots, \beta_{n_2}$ and then their set of actions is $A_2 = \{0, 1\}^{n_2}$ and $a_2 = (\beta_1, \ldots, \beta_{n_2}) \in A_2$.
- Players 3 and 4 avoid entering into a Nash equilibrium when the actions played by players 1 and 2 do not satisfy F. Their set of actions are $A_3 = A_4 = \{0, 1\}$.

Let us denote by $F(a_1, a_2)$ the truth value of F under the assignment given by a_1 and a_2.

$$u_1(a_1, a_2, a_3, a_4) = \begin{cases} 1 & \text{if } F(a_1, a_2) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_2(a_1, a_2, a_3, a_4) = \begin{cases} 1 & \text{if } F(a_1, a_2) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_3(a_1, a_2, a_3, a_4) = \begin{cases} 5 & \text{if } F(a_1, a_2) = 1, \\ 4 & \text{if } F(a_1, a_2) = 0 \wedge a_3 = 0 \wedge a_4 = 1, \\ 3 & \text{if } F(a_1, a_2) = 0 \wedge a_3 = 1 \wedge a_4 = 1, \\ 2 & \text{if } F(a_1, a_2) = 0 \wedge a_3 = 1 \wedge a_4 = 0, \\ 1 & \text{if } F(a_1, a_2) = 0 \wedge a_3 = 0 \wedge a_4 = 0. \end{cases}$$

$$u_4(a_1, a_2, a_3, a_4) = \begin{cases} 5 & \text{if } F(a_1, a_2) = 1, \\ 3 & \text{if } F(a_1, a_2) = 0 \wedge a_3 = 0 \wedge a_4 = 1, \\ 2 & \text{if } F(a_1, a_2) = 0 \wedge a_3 = 1 \wedge a_4 = 1, \\ 1 & \text{if } F(a_1, a_2) = 0 \wedge a_3 = 1 \wedge a_4 = 0, \\ 4 & \text{if } F(a_1, a_2) = 0 \wedge a_3 = 0 \wedge a_4 = 0. \end{cases}$$

What about the complexity of computing a mixed Nash equilibrium?

Here there is a difference as a Nash equilibrium always exists.

For three players they might not be computable as solutions might not be rationals!

for 2-players?

The problem is PPAD complete

[Chen and Deng, 2005]

PPAD = Polynomial Parity Arguments on Directed graphs