Rapid Distortion Theory

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Rapid distortion theory (RDT) was a workhorse method of obtaining information about rapid pressure-strain for the development of the MARV Reynolds stress model for our book **Functional Reynolds Stress Modeling** (by Joan G. Moore and John Moore, Blacksburg, VA, 2006). At either high dimensionless strain rate, Sk/ε, or low turbulence Reynolds number, RDT gives results which are close to DNS - cheaply - only a matter of minutes on my Mac Mini. We used RDT to create a database of $\overline{u_i u_j}/k$ as a function of dimensionless time, *St,* for a variety of homogeneous flows to supplement available DNS and experimental data.

This folder, rdt.jgm, contains rdt.oneksets.f, a Fortran program to do rapid-distortion-theory calculations. Starting with isotropic turbulence, it gives the development of the Reynolds stresses in time for a specified velocity gradient tensor, $\partial U_i/\partial x_j$, and coordinate rotation rate vector, Ω_k. It may be run in the pure mode without dissipation, or with a (approximately) specified dimensionless strain rate, Sk/ε as a function of time. It also gives tables showing how rapid pressure-strain depends on individual components of mean strain rate and vorticity tensors, as given in Tables 9.5.1, 9.5.3, 9.5.5, and 9.5.7 of our book.

The descriptions for compiling the program and running the examples are for Unix. I use the MAC OS 10 app Applications/Utilities/Terminal which brings up a Unix shell window.

Rapid Distortion Theory Equations

Rapid distortion theory considers the spectral form of the Reynolds stresses. The analysis is valid for homogeneous turbulence where the gradients of the mean velocity are uniform and the flow is incompressible. The equation for the Fourier transform of the two-point double velocity correlation gives terms which may be integrated over wave-number space to give the terms in the Reynolds stress equations. In the limit of rapid distortion (high strain rate), the term which gives the cascade of energy, and the term which integrates to the slow pressure-strain, may be neglected, and a solvable set of equations which require no modeling, is obtained. Solutions to these equations, when integrated, give the development in time of both the Reynolds stresses and the rapid pressure-strain.

Townsend presented the equations in a form which follows the development along a streamline of a single initial wave-number component and its movement in wave-number space. The result is a set of 9 equations for each point in wave-number space, three for the location in wave-number space, K_k, and nine for the components of the spectrum tensor, Φ_{ij}. From our book, Chapter 9,

$$\frac{dK_k}{dt} = -K_m \frac{\partial U_m}{\partial x_k}, \tag{9.3.3}$$

$$\frac{d\Phi_{ij}}{dt} = \underbrace{-\left(\Phi_{mj}\frac{\partial U_i}{\partial x_m} + \Phi_{im}\frac{\partial U_j}{\partial x_m}\right)}_{\text{production}} \underbrace{-2\nu K^2\Phi_{ij}}_{\text{dissipation}}$$

$$\underbrace{-2\Omega_k\left(\varepsilon_{ikm}\Phi_{mj} + \varepsilon_{jkm}\Phi_{im}\right)}_{\text{Coriolis}} + \underbrace{2\left(\frac{\partial U_n}{\partial x_m} + \varepsilon_{nkm}\Omega_k\right)\left(\frac{K_iK_n}{K^2}\Phi_{mj} + \frac{K_jK_n}{K^2}\Phi_{im}\right)}_{\text{rapid pressure—strain}}. \tag{9.3.4}$$

Both K_k and Φ_{ij} are functions of the <u>initial</u> location in wave-number space, $K_{1,0}, K_{2,0}, K_{3,0}$, and time, t.

Since the sizes of the elemental volumes do not change with time,

$$dK_1 dK_2 dK_3 = dK_{1,0} dK_{2,0} dK_{3,0} \ , \tag{9.1.13}$$

the Reynolds stresses are obtained from

$$\overline{u_i u_j}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{ij}(K_{1,0}, K_{2,0}, K_{3,0}, t)\, dK_{1,0} dK_{2,0} dK_{3,0} \ , \tag{9.3.5}$$

and the rapid pressure-strain term integrates to give

$$\Pi_{ij}^R(t) = 2\left(\frac{\partial U_n}{\partial x_m} + \varepsilon_{nkm}\Omega_k\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{K_iK_n}{K^2}\Phi_{mj} + \frac{K_jK_n}{K^2}\Phi_{im}\right) dK_{1,0} dK_{2,0} dK_{3,0} \ ,$$

or in terms of the strain rate and absolute vorticity tensors,

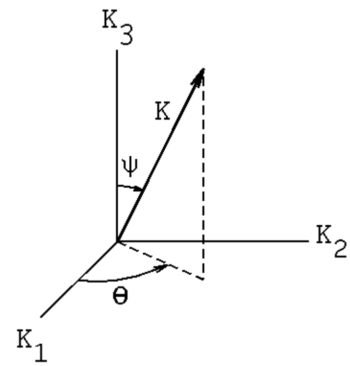
$$S_{nm} = \frac{1}{2}\left(\frac{\partial U_n}{\partial x_m} + \frac{\partial U_m}{\partial x_n}\right) \text{ and } W_{nm} = \frac{1}{2}\left(\frac{\partial U_n}{\partial x_m} - \frac{\partial U_m}{\partial x_n} + 2\varepsilon_{nkm}\Omega_k\right),$$

$$\Pi_{ij}^R(t) = 2\left(S_{nm} + W_{nm}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{K_iK_n}{K^2}\Phi_{mj} + \frac{K_jK_n}{K^2}\Phi_{im}\right) dK_{1,0} dK_{2,0} dK_{3,0} \ .$$

Program Overview

If the dissipation term in Eq. 9.3.4, is small, the development of Φ_{ij} becomes independent of the magnitude of the wave number, K, and it is only necessary to consider the equations over one initial spherical surface in wave-number space. The symmetries of Φ_{ij} further reduce the region to be gridded to a hemisphere.

In terms of spherical coordinates the initial wave numbers were defined over a grid with uniform spacing in θ (spanning 0 to π) and $\cos\psi$ (spanning 0 to 1), so that each point represented the same area on the surface. In Cartesian coordinates, the initial locations of the grid points are



$$K_{1,0} = K_0 \sin\psi\cos\theta,\ K_{2,0} = K_0 \sin\psi\sin\theta,\ K_{3,0} = \pm K_0 \cos\psi.$$

With an initial uniform thickness δK of the hemisphere, volume integrals are then simple sums of the values at the grid points.

The spectrum tensor is initialized as isotropic, $\Phi_{ij} = \delta_{ij} - \dfrac{K_i K_j}{K^2}$.

Equations 9.3.3 and 9.3.4 are then integrated in time using a fifth-order Runge-Kutta solver. Note that the results for a starting point in wave-number space are independent of what happens at all other starting points in wave-number space. So the solution procedure is fairly simple – a 1d solution in time, for each of the 2d set of grid points. Thank Townsend for this insight. It makes the program very simple compared with many RDT methods in the literature.

Compiling Rdt

Compile the RDT code with a Fortran compiler, e.g.,

```
f77 -o a.rdt rdt.oneksets.f
```

to give the program a.rdt. If you are running MAC OS 10, you may be able to skip recompiling the program. The file a.rdt in directory rdt.jgm was compiled under MAC OS 10.6.4 but using the -m32 C compiler option (with a Fortran to C converter) so it should run on either older 32-bit Macs or newer 64-bit Macs. If you attempt to read the code, note the subroutine runge was initially written in the 1960's and uses an old form of if statements. E.g.,

```
    IF (A) 10,20,30
```

means if A<0 branch to statement 10, if A=0 to statement 20, and if A>0 branch to statement 30.

Running Rdt

The program reads from standard input and writes to standard output. E.g.,

```
./a.rdt < in.rdt/in.shear.vort0 > out/out.shear.vort0
```

The directory in.rdt contains a selection of input files.

Input Parameters for Rdt

```
read (5,*) jmax,kmax,sfix
read (5,*) ((dudy(i,j),i=1,3),j=1,3)
read (5,*) (rot(i),i=1,3)
read (5,*) dtout,tmax,tolf,tolmin
if (sfix<0) then
  read (5,*) isetm, (stset(i),sset(i),i=1,isetm)
endif
```

jmax - number of grid points spanning θ from 0 to π.
kmax - number of grid points spanning cosψ from 0 to 1. (Doubled for $\pm K_3$)
sfix - the dimensionless strain rate. If set to zero, the laminar viscosity is set to zero. Otherwise after each print step, the viscosity is recalculated to give Sk/ε = sfix. If sfix is negative, sfix as a function of St is read after the other input.
dudy(i,j) - the mean velocity gradient tensor, $\partial U_i / \partial x_j$.
rot(i) - the frame rotation vector, Ω_i.
dtout - the time between print results. Initial timestep set at 0.1dtout.
tmax - the final time
tolf - the maximum fractional change in the properties per time step. Runge adjusts the time step as needed.
tolmin - the tolf check is omitted if the absolute value of the property is less than tolmin. (All the examples set tolmin=0)
isetm - number of points for sfix versus St array. Must be <=10.
stset(i) - the dimensionless times, St, at which Sk/ε is specified.
sset(i) - the values of Sk/ε at times stset(i). sfix is set by linear interpolation, with linear extrapolation if stset(i) does not cover the range 0 to S*tmax.

Sample input file: file in.rdt/in.shear.vort0

```
64 64 0.          / jmax,kmax,sfix
0. 0. 0. 10. 0. 0. 0. 0. 0. / dudy
0. 0. 5.                / rot
.005 1.2 .001 .0          / dtout,tmax,tolf,tolmin
```

Output Format

The directory out contains a few sample output files. Sample output in red.
From out/out.shear.vort0:

First line: Input parameters
jmax,kmax 64 64 sfix   .0 dUidxj   .0   .0   .0 10.0   .0   .0   .0   .0   .0 rot   .0   .0   5.0
Line 2: number of properties printed each dtout
88
Next: names of the properties printed
t beta kdk0 s
con1 con2 con3 DkDt dkdt dkdtP

```
 uu vv ww   uv  uw vw
 duu dvv dww   duv  duw dvw
 b11 b22 b33  b12 b13 b23
 dispuu dispvv dispww dispuv dispuw dispvw
 cs1111 cs1122 cs1133 cs1112 cs1113 cs1123  cw1112 cw1113 cw1123
 cs2211 cs2222 cs2233 cs2212 cs2213 cs2223  cw2212 cw2213 cw2223
 cs3311 cs3322 cs3333 cs3312 cs3313 cs3323  cw3312 cw3313 cw3323
 cs1211 cs1222 cs1233 cs1212 cs1213 cs1223  cw1212 cw1213 cw1223
 cs1311 cs1322 cs1333 cs1312 cs1313 cs1323  cw1312 cw1313 cw1323
 cs2311 cs2322 cs2333 cs2312 cs2313 cs2323  cw2312 cw2313 cw2323
```

t - time
beta - dimensionless time, St
kdk0 - turbulence kinetic energy, k, over values at t=0
s - calculated Sk/ε if sfix>0, otherwise S
coni - continuity checks, should be much less than the Reynolds stresses
DkDt - $\Delta k/\Delta t$ change in k between printouts / dtout
dkdt - dk/dt current rate of change of k with time
dkdtP - production rate (= dk/dt when the viscosity is zero)
uu to vw - the Reynolds stress tensor components, $\overline{u_i u_j}$ (arbitrary units)
duu to dvw - current rate of change of Reynolds stresses, $d\overline{u_i u_j}/dt$
b11 to b33 - anisotropy of the Reynolds stress tensor, $b_{ij} = \overline{u_i u_j}/(2k) - \delta_{ij}/3$
dispuu to dispvw - dissipation rate of Reynolds stresses, $\varepsilon_{ij}/(\nu k)$
csijnm, cwijnm - Coefficients of strain rate, S_{nm}, and absolute vorticity, W_{nm}, tensors in rapid pressure-strain,

$$\frac{\Pi_{ij}^{\mathrm{R}}}{k} = c_{sij11}S_{11} + c_{sij22}S_{22} + c_{sij33}S_{33} + c_{sij12}S_{12} + c_{sij13}S_{13} + c_{sij23}S_{23} + c_{wij12}W_{12} + c_{wij13}W_{13} + c_{wij23}W_{23} \,.$$

Next: values for each dtout

```
.00000E+00  .00000E+00  1.0000      10.000
 .11426E-13 -.79844E-05  .00000E+00  .00000E+00  .00000E+00  .00000E+00
 5461.2      5461.2      5461.5     -.37484E-13 -.17070E-14  .33222E-13
 .00000E+00  .00000E+00  .00000E+00  .00000E+00  .00000E+00  .00000E+00
-.50966E-05 -.50970E-05  .10163E-04 -.22879E-17 -.10419E-18  .20277E-17
 1.3333      1.3333      1.3334     -.65494E-17 -.12841E-17  .11719E-17
   .533  -.267  -.267   .000   .000   .000   .000   .000   .000
  -.267   .533  -.267   .000   .000   .000   .000   .000   .000
  -.267  -.267   .533   .000   .000   .000   .000   .000   .000
   .000   .000   .000   .800   .000   .000   .000   .000   .000
   .000   .000   .000   .000   .800   .000   .000   .000   .000
   .000   .000   .000   .000   .000   .800   .000   .000   .000
```

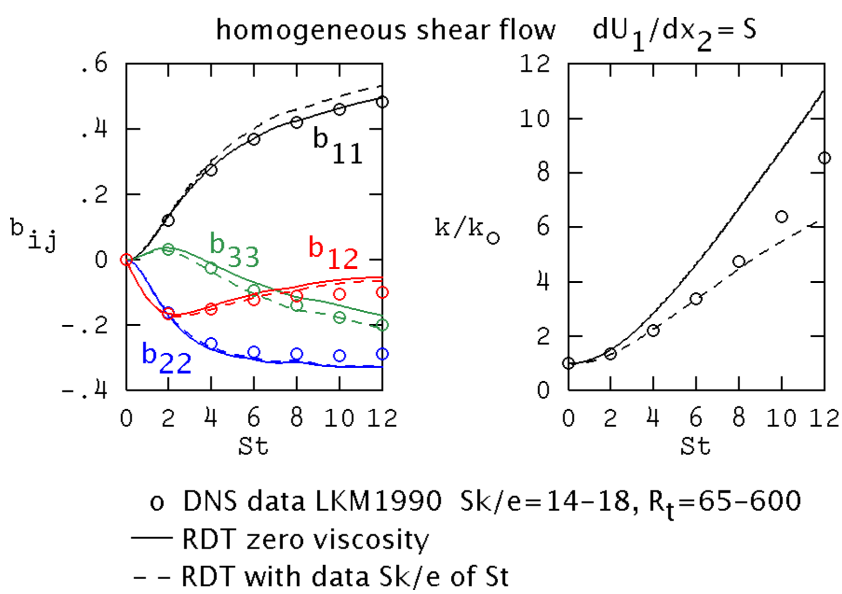
---- part omitted --- results for St=2

```
.20000      2.0000      1.4324      10.000
 -.20803E-05 -.36387E-05  .00000E+00  29587.      29761.      29761.
  4045.3      9997.4      9425.0     -2976.1     -.50626E-13 -.15077E-12
 -8345.1      35150.      32717.     -6867.1     -.18696E-12 -.22915E-12
 -.16096      .92675E-01  .68281E-01 -.12682     -.21573E-17 -.64245E-17
  .97594      1.6697      3.8703     -.34689      .38793E-17 -.15782E-16
   .186  -.108  -.078  -.142   .000   .000   .000   .000   .000
  -.108   .743  -.635  -.415   .000   .000   .000   .000   .000
  -.078  -.635   .713   .558   .000   .000   .000   .000   .000
  -.071  -.208   .279   .572   .000   .000   .000   .000   .000
   .000   .000   .000   .000   .436  -.457   .000   .000   .000
   .000   .000   .000   .000  -.457  1.350   .000   .000   .001
```

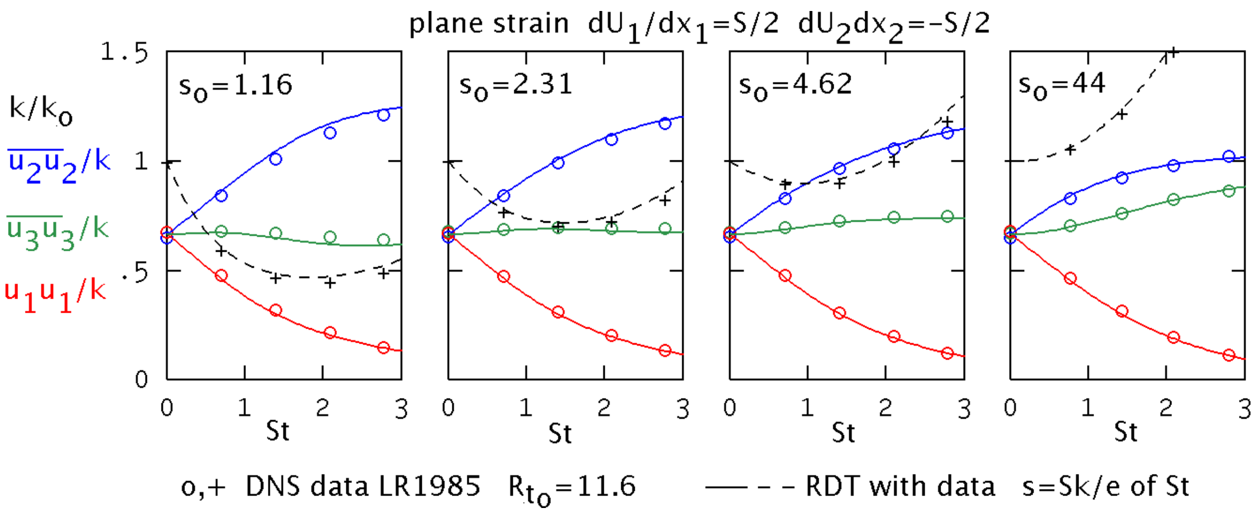
Note the above rapid pressure-strain coefficients at *St*=2 are Table 9.5.3 of our book.

Method Validation - Comparison with DNS

The Lee, Kim and Moin (1990) DNS for homogeneous shear flow is at moderately high Sk/ε, 14 to 18. This may be compared with Sk/ε ~ 3.5 in the log-law region of a turbulent boundary layer. In the near-wall region of a turbulent boundary layer, at about *y*+ of 12, the conditions, Sk/ε ~16, R_t ~150, are similar to the LKM DNS. Lee et al. noted that their results are like rapid distortion theory and (at *St*=12) like near-wall turbulence. The results below, from the current RDT code, show that the turbulence anisotropy is reasonable well represented by RDT with or without dissipation. If the DNS dissipation rate is included by setting Sk/ε from the data, the increase in turbulence is also well represented up to St=6.



Lee and Reynolds (1985) made DNS calculations for homogeneous plane strain at 4 different initial dimensionless strain rates, 1.16 to 44. This was very early days for DNS so the turbulence Reynolds number was small, R_{t_o} = 11.6, to keep the spectral grid requirements manageable. It is well known that slow pressure-strain is reduced at low R_t. Models in our book suggest that at R_t=11.6, slow pressure-strain is only 6% of its high R_t value, so that neglecting it is reasonable. The current RDT code was run for the 4 cases imposing the DNS $Sk/\varepsilon(St)$ distributions. The results below show good agreement for all the different initial dimensionless strain rates. Note that the differences in $\overline{u_i u_j}/k$ for the different cases are from the dissipation term and in particular how the movement of points in wave-number space (Eq. 9.3.3) to different magnitudes of K, selectively dissipates some components of the spectrum tensor more rapidly, and some less rapidly. It is encouraging that this good agreement can be obtained with this simple method which only considers one initial hemisphere of grid points.



Relevance

High Sk/ε is found in flows approaching bluff bodies, in the near-wall region of turbulent boundary layers and in pre-transitional boundary layers.

Standard Disclaimer

The program and information about it in the folder rdt.jgm are free software which comes with no warranties as to its usefulness or accuracy. Only the hope that you will find it as useful as I have.

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