

# FALTINGS ELLIPTIC CURVES IN TWISTED $\mathbb{Q}$ -ISOGENY CLASSES

*Supplementary Material*

Enrique González-Jiménez and Joan-C. Lario

[arXiv: 2503.XXXXX](#)

## Tables

<b>Introduction</b>	<b>3</b>
<b>1 Type <math>L_2(2)</math></b>	<b>5</b>
1.1 Settings . . . . .	5
1.2 Kodaira symbols, minimal models, and Pal values . . . . .	6
1.3 Statement . . . . .	10
<b>2 Type <math>L_2(3)</math></b>	<b>11</b>
2.1 Settings . . . . .	11
2.2 Kodaira symbols, minimal models, and Pal values . . . . .	12
2.3 Statement . . . . .	17
<b>3 Type <math>L_2(5)</math></b>	<b>18</b>
3.1 Settings . . . . .	18
3.2 Kodaira symbols, minimal models, and Pal values . . . . .	19
3.3 Statement . . . . .	22
<b>4 Type <math>L_2(7)</math></b>	<b>23</b>
4.1 Settings . . . . .	23
4.2 Kodaira symbols, minimal models, and Pal values . . . . .	24
4.3 Statement . . . . .	29
<b>5 Type <math>L_2(11)</math></b>	<b>30</b>
5.1 Settings . . . . .	30
5.2 Kodaira symbols, minimal models, and Pal values . . . . .	31
5.3 Statement . . . . .	32
<b>6 Type <math>L_2(13)</math></b>	<b>33</b>
6.1 Settings . . . . .	33
6.2 Kodaira symbols, minimal models, and Pal values . . . . .	34
6.3 Statement . . . . .	40

<b>7</b>	<b>Type <math>L_2(17)</math></b>	<b>41</b>
7.1	Settings . . . . .	41
7.2	Kodaira symbols, minimal models, and Pal values . . . . .	42
7.3	Statement . . . . .	42
<b>8</b>	<b>Type <math>L_2(19)</math></b>	<b>44</b>
8.1	Settings . . . . .	44
8.2	Kodaira symbols, minimal models, and Pal values . . . . .	45
8.3	Statement . . . . .	45
<b>9</b>	<b>Type <math>L_2(37)</math></b>	<b>46</b>
9.1	Settings . . . . .	46
9.2	Kodaira symbols, minimal models, and Pal values . . . . .	47
9.3	Statement . . . . .	47
<b>10</b>	<b>Type <math>L_2(43)</math></b>	<b>48</b>
10.1	Settings . . . . .	48
10.2	Kodaira symbols, minimal models, and Pal values . . . . .	49
10.3	Statement . . . . .	49
<b>11</b>	<b>Type <math>L_2(67)</math></b>	<b>50</b>
11.1	Settings . . . . .	50
11.2	Kodaira symbols, minimal models, and Pal values . . . . .	51
11.3	Statement . . . . .	51
<b>12</b>	<b>Type <math>L_2(163)</math></b>	<b>52</b>
12.1	Settings . . . . .	52
12.2	Kodaira symbols, minimal models, and Pal values . . . . .	53
12.3	Statement . . . . .	53
<b>13</b>	<b>Type <math>L_3(9)</math></b>	<b>54</b>
13.1	Setting . . . . .	54
13.2	Kodaira symbols, minimal models, and Pal values . . . . .	56
13.3	Statement . . . . .	58
<b>14</b>	<b>Type <math>L_3(25)</math></b>	<b>59</b>
14.1	Settings . . . . .	59
14.2	Kodaira symbols, minimal models, and Pal values . . . . .	61
14.3	Statement . . . . .	65
<b>15</b>	<b>Type <math>L_4</math></b>	<b>66</b>
15.1	Settings . . . . .	66
15.2	Kodaira symbols, minimal models, and Pal values . . . . .	67
15.3	Statement . . . . .	67
<b>16</b>	<b>Type <math>R_4(6)</math></b>	<b>68</b>
16.1	Settings . . . . .	68
16.2	Kodaira symbols, minimal models, and Pal values . . . . .	70
16.3	Statement . . . . .	73

<b>17 Type <math>R_4(10)</math></b>	<b>75</b>
17.1 Settings . . . . .	75
17.2 Kodaira symbols, minimal models, and Pal values . . . . .	77
17.3 Statement . . . . .	83
<b>18 Type <math>R_4(14)</math></b>	<b>85</b>
18.1 Settings . . . . .	85
18.2 Kodaira symbols, minimal models, and Pal values . . . . .	86
18.3 Statement . . . . .	86
<b>19 Type <math>R_4(15)</math></b>	<b>88</b>
19.1 Settings . . . . .	88
19.2 Kodaira symbols, minimal models, and Pal values . . . . .	89
19.3 Statement . . . . .	90
<b>20 Type <math>R_4(21)</math></b>	<b>91</b>
20.1 Settings . . . . .	91
20.2 Kodaira symbols, minimal models, and Pal values . . . . .	92
20.3 Statement . . . . .	93
<b>21 Type <math>R_6</math></b>	<b>94</b>
21.1 Settings . . . . .	94
21.2 Kodaira symbols, minimal models, and Pal values . . . . .	97
21.3 Statement . . . . .	101
<b>22 Type <math>T_4</math></b>	<b>103</b>
22.1 Settings . . . . .	103
22.2 Kodaira symbols, minimal models, and Pal values . . . . .	105
22.3 Statement . . . . .	109
<b>23 Type <math>T_6</math></b>	<b>113</b>
23.1 Setting . . . . .	113
23.2 Kodaira symbols, minimal models, and Pal values . . . . .	115
23.3 Statement . . . . .	118
<b>24 Type <math>T_8</math></b>	<b>120</b>
24.1 Settings . . . . .	120
24.2 Kodaira symbols, minimal models, and Pal values . . . . .	124
24.3 Statement . . . . .	128
<b>25 Type <math>S_8</math></b>	<b>130</b>
25.1 Settings . . . . .	130
25.2 Kodaira symbols, minimal models, and Pal values . . . . .	133
25.3 Statement . . . . .	138



# Introduction

This document contains the tables providing supplementary material for the article *Faltings elliptic curves in twisted  $\mathbb{Q}$ -isogeny classes*, written by the same authors and submitted to XXX. For a detailed explanation of the terminology, notations, and contextual framework, we recommend consulting YYY.

The  $\mathbb{Q}$ -isogeny classes of elliptic curves correspond to the non-cuspidal rational points of the modular curves  $X_0(N)$ . For values of  $N$  such that  $X_0(N)$  has genus zero, these  $\mathbb{Q}$ -isogeny classes can be parameterized in terms of the rational values of a Hauptmodul  $t = t(\tau)$ , which generates the function field of modular functions on  $\Gamma_0(N)$ . In the remaining cases, where  $X_0(N)$  has genus  $\geq 1$ , the non-cuspidal rational points are finite in number.

Each  $\mathbb{Q}$ -isogeny class can be represented as a graph whose vertices correspond to the elliptic curves within the class, while the edges represent rational prime-degree isogenies among them. There are 26 possible types of labeled  $\mathbb{Q}$ -isogeny graphs:

$$\begin{aligned} &L_1, \\ &L_2(p) \text{ for } p \text{ in } \{2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, 163\}, \\ &L_3(9), L_3(25), L_4, T_4, T_6, T_8, \\ &R_4(N) \text{ for } N \text{ in } \{6, 10, 14, 15, 21\}, \\ &R_6, S_8. \end{aligned}$$

The subscript denotes the number of vertices in the graph, the letter indicates the shape of the isogeny graph ( $L$  for line,  $T$  for tetrahedron,  $R$  for rectangular, and  $S$  for special), and the level in parentheses refers to the maximal isogeny degree of a path in the graph (if no level is specified, the isogenies have degrees 2 or 3).

The following sections provide details for each type of nontrivial  $\mathbb{Q}$ -isogeny class. The first subsection, titled Settings, describes the isogeny graph, a Hauptmodul of  $X_0(N)$  in the genus zero cases, or the corresponding rational points otherwise. We then list the  $j$ -invariants along with signatures  $(c_4, c_6, \Delta)$  of the elliptic curves within the given  $\mathbb{Q}$ -isogeny class.

The choice of signatures is not completely arbitrary and satisfies the following two conditions: (1) the associated isogenies are normalized (achieved using Velús formulas), and (2) the zeros of the Hauptmodul coincide with the zeros of the discriminant (in genus zero cases). Furthermore, we describe the action of the automorphisms of  $X_0(N)$  that preserve the isogeny graph.

For each type, the second subsection presents tables that display:

- The  $p$ -adic valuations of signatures  $\text{sig}_p(\mathcal{E})$  of a minimal model  $\mathcal{E}$  of  $E$ ,
- The Weierstrass change  $u_p(E)$  that yields a  $p$ -minimal model of  $E$ ,
- The Kodaira symbol  $K_p(E)$  of  $E$  at the prime  $p$ ,
- The Pal value  $u_p(\mathcal{E}^d)$  that yields a  $p$ -minimal model of the quadratic twist  $\mathcal{E}^d$  of  $\mathcal{E}$ .

Finally, in each last subsection, we state a proposition describing the Faltings elliptic curve in the twisted isogeny classes in terms of  $p$ -valuations of the rational value of the Hauptmodul (in genus zero cases) and the square-free integers  $d$ . We also include the probability of a given vertex in the graph being the Faltings curve. The final table for each type compiles the global information derived from the local data in the previous tables, providing the necessary information to establish the main result in the article.



# 1 Type $L_2(2)$

## 1.1 Settings

### Graph

The isogeny graphs of type  $L_2(2)$  are given by two 2-isogenous elliptic curves:

$$E_1 \xrightarrow{2} E_2.$$

### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(2)$  parametrize isogeny graphs of type  $L_2(2)$ . The curve  $X_0(2)$  has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 2^{12} \left( \frac{\eta(2\tau)}{\eta(\tau)} \right)^{24}.$$

### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$j(E_1) = j(\tau) = \frac{(t+16)^3}{t}$$

$$j(E_2) = j(2\tau) = \frac{(t+256)^3}{t^2}.$$

### Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_2)$  in such a way that the isogeny graph is normalized. Their signatures are:

$L_2(2)$	
$c_4(E_1)$	$(t+16)(t+64)$
$c_6(E_1)$	$(t-8)(t+64)^2$
$\Delta(E_1)$	$t(t+64)^3$
$c_4(E_2)$	$(t+64)(t+256)$
$c_6(E_2)$	$(t-512)(t+64)^2$
$\Delta(E_2)$	$t^2(t+64)^3$

### Automorphisms

The subgroup of  $\text{Aut } X_0(2)$  that fixes the set of vertices of the graph is generated by the Fricke involution of  $X_0(2)$ , given by  $W_2(t) = 2^{12}/t$ . With regard to the action of the Fricke involution on the isogeny graph, it can be displayed as follows:

$$W_2(E_1 \xrightarrow{2} E_2) = E_2^{-2t} \xrightarrow{2} E_1^{-2t}.$$

## 1.2 Kodaira symbols, minimal models, and Pal values

Table 1:  $L_2(2)$  data for  $p \neq 2, 3$

$L_2(2)$	$p \neq 2, 3$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
$v_p(t) = 0$ $v_p(t + 64) = 4m$	$E_1$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
	$E_2$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
$v_p(t) = 0$ $v_p(t + 64) = 4m + 1$	$E_1$	$(1, 2m + 2, 3)$	$p^m$	III	1	1
	$E_2$	$(1, 2m + 2, 3)$	$p^m$	III	1	1
$v_p(t) = 0$ $v_p(t + 64) = 4m + 2$	$E_1$	$(2, 2m + 4, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_2$	$(2, 2m + 4, 6)$	$p^m$	$I_0^*$	$p$	1
$v_p(t) = 0$ $v_p(t + 64) = 4m + 3$	$E_1$	$(3, 2m + 6, 9)$	$p^m$	III*	$p$	1
	$E_2$	$(3, 2m + 6, 9)$	$p^m$	III*	$p$	1
$-m = v_p(t) < 0$ $m$ odd	$E_1$	$(2, 3, 2m + 6)$	$p^{-(m+1)/2}$	$I_{2m}^*$	$p$	1
	$E_2$	$(2, 3, m + 6)$	$p^{-(m+1)/2}$	$I_m^*$	$p$	1
$-m = v_p(t) < 0$ $m$ even	$E_1$	$(0, 0, 2m)$	$p^{-m/2}$	$I_{2m}$	1	1
	$E_2$	$(0, 0, m)$	$p^{-m/2}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	



Table 2:  $L_2(2)$  data for  $p = 3$ 

$L_2(2)$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
$v_p(t) = 0$ $v_p(t + 64) = 4m$	$E_1$	$(1, 2m + 2, 0)$	$3^m$	$I_0$	1	1
	$E_2$	$(1, 2m + 2, 0)$	$3^m$	$I_0$	1	1
$v_3(t) = 0$ $v_3(t + 64) = 4m + 1$	$E_1$	$(\geq 2, \geq 3, 3)$	$3^m$	III	1	1
	$E_2$	$(\geq 2, \geq 3, 3)$	$3^m$	III	1	1
$v_3(t) = 0$ $v_3(t + 64) = 4m + 2$	$E_1$	$(3, \geq 6, 6)$	$3^m$	$I_0^*$	3	1
	$E_2$	$(3, \geq 6, 6)$	$3^m$	$I_0^*$	3	1
$v_3(t) = 0$ $v_3(t + 64) = 4m + 3$	$E_1$	$(4, 2m + 8, 9)$	$3^m$	III*	3	1
	$E_2$	$(4, 2m + 8, 9)$	$3^m$	III*	3	1
$v_3(t) = -m < 0$ $m$ odd	$E_1$	$(2, 3, 2m + 6)$	$3^{-(m+1)/2}$	$I_{2m}^*$	3	1
	$E_2$	$(2, 3, m + 6)$	$3^{-(m+1)/2}$	$I_m^*$	3	1
$v_p(t) = -m < 0$ $m$ even	$E_1$	$(0, 0, 2m)$	$3^{-m/2}$	$I_{2m}$	1	1
	$E_2$	$(0, 0, m)$	$3^{-m/2}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	

Table 3:  $L_2(2)$  data for  $p=2$ 

$L_2(2)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 11$	$E_1$	$(6, 9, m + 6)$	2	$I_{m-4}^*$	1	$2^*$ or $4^*$	1
	$E_2$	$(6, 9, 2m - 6)$	$2^2$	$I_{2m-16}^*$	1	$2^*$ or $4^*$	1
$v_2(t) = 11$	$E_1$	$(6, 9, 17)$	2	$I_7^*$	1	2	1
	$E_2$	$(6, 9, 16)$	$2^2$	$I_6^*$	1	2	1
$v_2(t) = 10$	$E_1$	$(6, 9, 16)$	2	$I_6^*$	1	2	1
	$E_2$	$(6, 9, 14)$	$2^2$	$I_4^*$	1	2	1
$v_2(t) = 9$	$E_1$	$(6, 9, 15)$	2	$I_5^*$	1	2	1
	$E_2$	$(6, \geq 9, 12)$	$2^2$	$I_2^*$	1	2	1
$v_2(t) = 8$	$E_1$	$(6, 9, 14)$	2	$I_4^*$	1	2	1
	$E_2$	$(\geq 7, 8, 10)$	$2^2$	$I_0^*$	1	2	1
$v_2(t) = 7$	$E_1$	$(6, 9, 13)$	2	$I_2^*$	1	2	1
	$E_2$	$(5, 7, 8)$	$2^2$	III	1	1	1
$v_2(t) = 6$ $v_2(t + 64) = 4m$ $(t + 64)/2^{4m} \equiv 1 (4)$	$E_1$	$(4, 2m + 3, 6)$	$2^m$	II	1	1	1
	$E_2$	$(6, 2m + 6, 12)$	$2^m$	$I_2^*$	1	2	1
$v_2(t) = 6$ $v_2(t + 64) = 4m$ $(t + 64)/2^{4m} \equiv 3 (4)$	$E_1$	$(4, 2m + 3, 6)$	$2^m$	III	1	1	1
	$E_2$	$(6, 2m + 6, 12)$	$2^m$	$I_3^*$	1	2	1
$v_2(t) = 6$ $v_2(t + 64) = 4m + 1$	$E_1$	$(5, 2m + 5, 9)$	$2^m$	III	1	1	1
	$E_2$	$(7, 2m + 8, 15)$	$2^m$	III*	1	2	1
$v_2(t) = 6$ $v_2(t + 64) = 4m + 2$ $(t + 64)/2^{4m+2} \equiv 1 (4)$	$E_1$	$(6, 2m + 7, 12)$	$2^m$	$I_3^*$	1	2	1
	$E_2$	$(4, 2m + 4, 6)$	$2^{m+1}$	III	1	1	1
$v_2(t) = 6$ $v_2(t + 64) = 4m + 2$ $(t + 64)/2^{4m+2} \equiv 3 (4)$	$E_1$	$(6, 2m + 7, 12)$	$2^m$	$I_2^*$	1	2	1
	$E_2$	$(4, 2m + 4, 6)$	$2^{m+1}$	II	1	1	1
$v_2(t) = 6$ $v_2(t + 64) = 4m + 3$	$E_1$	$(7, 2m + 9, 15)$	$2^m$	III*	1	2	1
	$E_2$	$(5, 2m + 6, 9)$	$2^{m+1}$	III	1	1	1
$v_2(t) = 5$	$E_1$	$(5, 7, 8)$	2	III	1	1	1
	$E_2$	$(6, 9, 13)$	2	$I_2^*$	1	2	1
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

Table 3:  $L_2(2)$  data for  $p=2$  (Continued)

$L_2(2)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = 4$ $t/2^4 \equiv 1(4)$	$E_1$	$(5, 5, 4)$	2	III	1	1	1
	$E_2$	$(4, 6, 8)$	2	$I_1^*$	1	1	1
$v_2(t) = 4$ $t/2^4 \equiv 3(4)$	$E_1$	$(\geq 6, 5, 4)$	2	IV	1	1	1
	$E_2$	$(4, 6, 8)$	2	$IV^*$	1	1	1
$v_2(t) = 3$	$E_1$	$(6, \geq 9, 12)$	1	$I_2^*$	1	2	1
	$E_2$	$(6, 9, 15)$	1	$I_5^*$	1	2	1
$v_2(t) = 2$ $t/2^2 \equiv 1(4)$	$E_1$	$(4, 6, 8)$	1	$I_0^*$	1	1	1
	$E_2$	$(4, 6, 10)$	1	$I_2^*$	1	1	1
$v_2(t) = 2$ $t/2^2 \equiv 3(4)$	$E_1$	$(4, 6, 8)$	1	$I_1^*$	1	1	1
	$E_2$	$(4, 6, 10)$	1	$III^*$	1	1	1
$v_2(t) = 1$	$E_1$	$(6, 9, 16)$	$2^{-1}$	$I_6^*$	1	2	1
	$E_2$	$(6, 9, 17)$	$2^{-1}$	$I_7^*$	1	2	1
$v_2(t) = 0$ $t \equiv 1(4)$	$E_1$	$(4, 6, 12)$	$2^{-1}$	$I_4^*$	1	1	2
	$E_2$	$(4, 6, 12)$	$2^{-1}$	$I_4^*$	1	1	2
$v_2(t) = 0$ $t \equiv 3(4)$	$E_1$	$(0, 0, 0)$	1	$I_0$	1	$2^{-1}$	$2^{-1}$
	$E_2$	$(0, 0, 0)$	1	$I_0$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = -(2m+1) < 0$	$E_1$	$(6, 9, 4m+20)$	$2^{-(m+2)}$	$I_{4m+10}^*$	1	$4^*$ or $2^*$	1
	$E_2$	$(6, 9, 2m+19)$	$2^{-(m+2)}$	$I_{2m+9}^*$	1	$4^*$ or $2^*$	1
$v_2(t) = -2m < 0$ $2^{2m}t \equiv 3(4)$	$E_1$	$(0, 0, 4m)$	$2^{-m}$	$I_{4m}$	1	$2^{-1}$	$2^{-1}$
	$E_2$	$(0, 0, 2m)$	$2^{-m}$	$I_{2m}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = -2m < 0$ $2^{2m}t \equiv 1(4)$	$E_1$	$(4, 6, 12+4m)$	$2^{-(m+1)}$	$I_{4m+4}^*$	1	1	2
	$E_2$	$(4, 6, 12+2m)$	$2^{-(m+1)}$	$I_{2m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

**Remark ( $2^*$  or  $4^*$ ):** The value  $u_2(\mathcal{E}^d)$  is given by

$$u_2(\mathcal{E}^d) = \begin{cases} 2 & \text{if } d \equiv -2(8) \\ 4 & \text{if } d \equiv 2(8). \end{cases}$$

**Remark ( $4^*$  or  $2^*$ ):** The value  $u_2(\mathcal{E}^d)$  is given by

$$u_2(\mathcal{E}^d) = \begin{cases} 4 & \text{if } d \equiv -2(8) \\ 2 & \text{if } d \equiv 2(8). \end{cases}$$

### 1.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$	$d$
$v_2(t) \geq 8$	$(1 : 2)$	$(1 : 1)$	
$v_2(t) = 7$	$(1 : 2)$	$(1 : 1)$	$d \not\equiv 0 (2)$
$v_2(t) = 6$ $v_2(t + 64) \equiv 2, 3 (4)$		$(2 : 1)$	$d \equiv 0 (2)$
$v_2(t) = 6$ $v_2(t + 64) \equiv 0, 1 (4)$	$(1 : 1)$	$(1 : 1)$	$d \not\equiv 0 (2)$
$v_2(t) = 5$		$(1 : 2)$	$d \equiv 0 (2)$
$v_2(t) \leq 4$	$(1 : 1)$	$(1 : 1)$	

The contents of this table are the ingredients to prove the following result:

**Proposition 1.** *Let  $E_1 \xrightarrow{2} E_2$  be a  $\mathbf{Q}$ -isogeny graph of type  $L_2(2)$  corresponding to a given  $t$  in  $\mathbf{Q}$ ,  $t \neq 0, -64$ , with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{2} E_2^d$  is given by:*

$L_2(2)$	twisted isogeny graph	$d$	Prob
$v_2(t) \geq 8$	$E_1^d \leftarrow \textcircled{E_2^d}$		1
$v_2(t) = 7$	$E_1^d \leftarrow \textcircled{E_2^d}$	$d \not\equiv 0 (2)$	2/3
$v_2(t) = 6$ $v_2(t + 64) \equiv 2, 3 (4)$	$\textcircled{E_1^d} \rightarrow E_2^d$	$d \equiv 0 (2)$	1/3
$v_2(t) = 6$ $v_2(t + 64) \equiv 0, 1 (4)$	$\textcircled{E_1^d} \rightarrow E_2^d$	$d \not\equiv 0 (2)$	2/3
$v_2(t) = 5$	$E_1^d \leftarrow \textcircled{E_2^d}$	$d \equiv 0 (2)$	1/3
$v_2(t) \leq 4$	$\textcircled{E_1^d} \rightarrow E_2^d$		1

The column Prob gives the probability of the circled twisted curve to be the Faltings curve.

## 2 Type $L_2(3)$

### 2.1 Settings

#### Graph

The isogeny graphs of type  $L_2(3)$  are given by two 3-isogenous elliptic curves:

$$E_1 \xrightarrow{3} E_3.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(3)$  parametrize isogeny graphs of type  $L_2(3)$ . The curve  $X_0(3)$  has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 3^6 \left( \frac{\eta(3\tau)}{\eta(\tau)} \right)^{12}.$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$j(E_1) = j(\tau) = \frac{(t+3)^3(t+27)}{t}$$

$$j(E_3) = j(3\tau) = \frac{(t+27)(t+243)^3}{t^3}.$$

#### Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_3)$  in such a way that the isogeny graph is normalized. Their signatures are:

$L_2(3)$	
$c_4(E_1)$	$(t+3)(t+27)$
$c_6(E_1)$	$(t+27)(t^2+18t-27)$
$\Delta(E_1)$	$t(t+27)^2$
$c_4(E_3)$	$(t+27)(t+243)$
$c_6(E_3)$	$(t+27)(t^2-486t-19683)$
$\Delta(E_3)$	$t^3(t+27)^2$

#### Automorphisms

The subgroup of  $\text{Aut } X_0(3)$  that fixes the set of vertices of the graph is generated by the Fricke involution of  $X_0(3)$ , given by  $W_3(t) = 3^6/t$ . With regard to the action of the Fricke involution on the isogeny graph, it can be displayed as follows:

$$W_3(E_1 \xrightarrow{3} E_3) = E_3^{-t} \xrightarrow{3} E_1^{-t}.$$

## 2.2 Kodaira symbols, minimal models, and Pal values

Table 4:  $L_2(3)$  data for  $p \neq 2, 3$

$L_2(3)$	$p \neq 2, 3$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
$v_p(t) = 0$ $v_p(t + 27) = 6m$	$E_1$	$(2m, 0, 0)$	$p^m$	$I_0$	1	1
	$E_3$	$(2m, 0, 0)$	$p^m$	$I_0$	1	1
$v_p(t) = 0$ $v_p(t + 27) = 6m + 1$	$E_1$	$(1 + 2m, 1, 2)$	$p^m$	$II$	1	1
	$E_3$	$(1 + 2m, 1, 2)$	$p^m$	$II$	1	1
$v_p(t) = 0$ $v_p(t + 27) = 6m + 2$	$E_1$	$(2 + 2m, 2, 4)$	$p^m$	$IV$	1	1
	$E_3$	$(2 + 2m, 2, 4)$	$p^m$	$IV$	1	1
$v_p(t) = 0$ $v_p(t + 27) = 6m + 3$	$E_1$	$(3 + 2m, 3, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_3$	$(3 + 2m, 3, 6)$	$p^m$	$I_0^*$	$p$	1
$v_p(t) = 0$ $v_p(t + 27) = 6m + 4$	$E_1$	$(4 + 2m, 4, 8)$	$p^m$	$IV^*$	$p$	1
	$E_3$	$(4 + 2m, 4, 8)$	$p^m$	$IV^*$	$p$	1
$v_p(t) = 0$ $v_p(t + 27) = 6m + 5$	$E_1$	$(5 + 2m, 5, 10)$	$p^m$	$II^*$	$p$	1
	$E_3$	$(5 + 2m, 5, 10)$	$p^m$	$II^*$	$p$	1
$v_p(t) = -m < 0$ $m$ even	$E_1$	$(0, 0, 3m)$	$p^{-m/2}$	$I_{3m}$	1	1
	$E_3$	$(0, 0, m)$	$p^{-m/2}$	$I_m$	1	1
$v_p(t) = -m < 0$ $m$ odd	$E_1$	$(2, 3, 3m + 6)$	$p^{-(m+1)/2}$	$I_{3m}^*$	$p$	1
	$E_3$	$(2, 3, m + 6)$	$p^{-(m+1)/2}$	$I_m^*$	$p$	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 5:  $L_2(3)$  data for  $p = 3$ 

$L_2(3)$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = m \geq 6$	$E_1$	$(0, 0, m - 6)$	3	$I_{m-6}$	1	1
	$E_3$	$(0, 0, 3(m - 6))$	$3^2$	$I_{3(m-6)}$	1	1
$v_3(t) = 5$	$E_1$	$(4, 6, 11)$	1	$\text{II}^*$	3	1
	$E_3$	$(\geq 4, 6, 9)$	3	$\text{IV}^*$	3	1
$v_3(t) = 4$	$E_1$	$(4, 6, 10)$	1	$\text{IV}^*$	3	1
	$E_3$	$(3, 5, 6)$	3	$\text{IV}$	1	1
$v_3(t) = 3$ $v_3(t + 27) = 6m + 3$ $(t + 27)/3^{6m+3} \equiv 4, 5 (9)$	$E_1$	$(2m + 4, 6, 9)$	$3^m$	$\text{III}^*$	3	1
	$E_3$	$(2m + 2, 3, 3)$	$3^{m+1}$	$\text{III}$	1	1
$v_3(t) = 3$ $v_3(t + 27) = 6m + 3$ $(t + 27)/3^{6m+3} \not\equiv 4, 5 (9)$	$E_1$	$(2m + 4, 6, 9)$	$3^m$	$\text{IV}^*$	3	1
	$E_3$	$(2m + 2, 3, 3)$	$3^{m+1}$	$\text{II}$	1	1
$v_3(t) = 3$ $v_3(t + 27) = 6m + 4$	$E_1$	$(2m + 5, 7, 11)$	$3^m$	$\text{IV}^*$	3	1
	$E_3$	$(2m + 3, 4, 5)$	$3^{m+1}$	$\text{II}$	1	1
$v_3(t) = 3$ $v_3(t + 27) = 6m + 5$	$E_1$	$(2m + 6, 8, 13)$	$3^m$	$\text{II}^*$	3	1
	$E_3$	$(2m + 4, 5, 7)$	$3^{m+1}$	$\text{IV}$	1	1
$v_3(t) = 3$ $v_3(t + 27) = 6m + 6$ $(t + 27)/3^{6m+6} \equiv 4, 5 (9)$	$E_1$	$(2m + 3, 3, 3)$	$3^{m+1}$	$\text{II}$	1	1
	$E_3$	$(2m + 5, 6, 9)$	$3^{m+1}$	$\text{III}^*$	3	1
$v_3(t) = 3$ $v_3(t + 27) = 6m + 6$ $(t + 27)/3^{6m+6} \not\equiv 4, 5 (9)$	$E_1$	$(2m + 3, 3, 3)$	$3^{m+1}$	$\text{II}$	1	1
	$E_3$	$(2m + 5, 6, 9)$	$3^{m+1}$	$\text{IV}^*$	3	1
$v_3(t) = 3$ $v_3(t + 27) = 6m + 7$	$E_1$	$(2m + 4, 4, 5)$	$3^{m+1}$	$\text{II}$	1	1
	$E_3$	$(2m + 6, 7, 11)$	$3^{m+1}$	$\text{IV}^*$	3	1
$v_3(t) = 3$ $v_3(t + 27) = 6m + 8$	$E_1$	$(2m + 5, 5, 7)$	$3^{m+1}$	$\text{IV}$	1	1
	$E_3$	$(2m + 7, 8, 13)$	$3^{m+1}$	$\text{II}^*$	3	1
$v_3(t) = 2$	$E_1$	$(3, 5, 6)$	1	$\text{IV}$	1	1
	$E_3$	$(4, 6, 10)$	1	$\text{IV}^*$	3	1
$v_3(t) = 1$	$E_1$	$(\geq 2, 3, 3)$	1	$\text{II}$	1	1
	$E_3$	$(2, 3, 5)$	1	$\text{IV}$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	

Table 5:  $L_2(3)$  data for  $p = 3$  (Continued)

$L_2(3)$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = 0$	$E_1$	$(0, 0, 0)$	1	$I_0$	1	1
	$E_3$	$(0, 0, 0)$	1	$I_0$	1	1
$-m = v_3(t) < 0$ $m$ even	$E_1$	$(0, 0, 3m)$	$3^{-m/2}$	$I_{3m}$	1	1
	$E_3$	$(0, 0, m)$	$3^{-m/2}$	$I_m$	1	1
$-m = v_3(t) < 0$ $m$ odd	$E_1$	$(2, 3, 3m + 6)$	$3^{-(m+1)/2}$	$I_{3m}^*$	3	1
	$E_3$	$(2, 3, m + 6)$	$3^{-(m+1)/2}$	$I_m^*$	3	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	



Table 6:  $L_2(3)$  data for  $p=2$ 

$L_2(3)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m \geq 2$	$E_1$	$(0, 0, m)$	1	$I_m$	1	$2^{-1}$	$2^{-1}$
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 1$	$E_1$	$(4, 6, 13)$	$2^{-1}$	$I_5^*$	1	1	2
	$E_3$	$(4, 6, 15)$	$2^{-1}$	$I_7^*$	1	1	2
$v_2(t) = 0$ $v_2(t + 27) = 1$	$E_1$	$(6, 9, 14)$	$2^{-1}$	$I_4^*$	1	2	1
	$E_3$	$(6, 9, 14)$	$2^{-1}$	$I_4^*$	1	2	1
$v_2(t) = 0$ $v_2(t + 27) = 6m > 1$ $(t + 27)/2^{6m} \equiv 1 (4)$	$E_1$	$(2m + 7, 9, 12)$	$2^{m-1}$	$II^*$	1	2	2
	$E_3$	$(2m + 7, 9, 12)$	$2^{m-1}$	$II^*$	1	2	2
$v_2(t) = 0$ $v_2(t + 27) = 6m > 1$ $(t + 27)/2^{6m} \equiv 3 (4)$	$E_1$	$(2m + 3, 3, 0)$	$2^m$	$I_0$	1	1	$2^{-1}$
	$E_3$	$(2m + 3, 3, 0)$	$2^m$	$I_0$	1	1	$2^{-1}$
$v_2(t) = 0$ $v_2(t + 27) = 6m + 1 > 1$	$E_1$	$(2m + 8, 10, 14)$	$2^{m-1}$	$II^*$	1	2	1
	$E_3$	$(2m + 8, 10, 14)$	$2^{m-1}$	$II^*$	1	2	1
$v_2(t) = 0$ $v_2(t + 27) = 6m + 2$ $(t + 27)/2^{6m+2} \equiv 1 (4)$	$E_1$	$(\geq 4, 5, 4)$	$2^m$	$II$	1	1	1
	$E_3$	$(\geq 4, 5, 4)$	$2^m$	$II$	1	1	1
$v_2(t) = 0$ $v_2(t + 27) = 6m + 2$ $(t + 27)/2^{6m+2} \equiv 3 (4)$	$E_1$	$(\geq 4, 5, 4)$	$2^m$	$IV$	1	1	1
	$E_3$	$(\geq 4, 5, 4)$	$2^m$	$IV$	1	1	1
$v_2(t) = 0$ $v_2(t + 27) = 6m + 3$	$E_1$	$(2m + 6, 6, 6)$	$2^m$	$II$	1	$1^*$ or $2^*$	1
	$E_3$	$(2m + 6, 6, 6)$	$2^m$	$II$	1	$1^*$ or $2^*$	1
$v_2(t) = 0$ $v_2(t + 27) = 6m + 4$ $(t + 27)/2^{6m+4} \equiv 1 (4)$	$E_1$	$(2m + 7, 7, 8)$	$2^m$	$I_0^*$	1	1	1
	$E_3$	$(2m + 7, 7, 8)$	$2^m$	$I_0^*$	1	1	1
$v_2(t) = 0$ $v_2(t + 27) = 6m + 4$ $(t + 27)/2^{6m+4} \equiv 3 (4)$	$E_1$	$(2m + 7, 7, 8)$	$2^m$	$IV^*$	1	1	1
	$E_3$	$(2m + 7, 7, 8)$	$2^m$	$IV^*$	1	1	1
$v_2(t) = 0$ $v_2(t + 27) = 6m + 5$	$E_1$	$(2m + 8, 8, 10)$	$2^m$	$I_0^*$	1	2	1
	$E_3$	$(2m + 8, 8, 10)$	$2^m$	$I_0^*$	1	2	1
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

Table 6:  $L_2(3)$  data for  $p=2$  (Continued)

$L_2(3)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = -m < 0$ $m$ odd	$E_1$	$(6, 9, 3m + 18)$	$2^{-(m+3)/2}$	$I_{3m+8}^*$	1	2	1
	$E_3$	$(6, 9, m + 18)$	$2^{-(m+3)/2}$	$I_{m+8}^*$	1	2	1
$v_2(t) = -m < 0$ $m$ even $2^m t \equiv 1 (4)$	$E_1$	$(4, 6, 3m + 12)$	$2^{-(m+2)/2}$	$I_{3m+4}^*$	1	1	2
	$E_3$	$(4, 6, m + 12)$	$2^{-(m+2)/2}$	$I_{m+4}^*$	1	1	2
$v_2(t) = -m < 0$ $m$ even $2^m t \equiv 3 (4)$	$E_1$	$(0, 0, 3m)$	$2^{-m/2}$	$I_{3m}$	1	$2^{-1}$	$2^{-1}$
	$E_3$	$(0, 0, m)$	$2^{-m/2}$	$I_m$	1	$2^{-1}$	$2^{-1}$
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

**Remark (1\* or 2\*):** If  $v_2(t) = 0$ ,  $v_2(t + 27) = 6m + 3$ , and  $d \equiv 2 (4)$ , then the value  $u_2(\mathcal{E}^d)$  is given by

$$u_2(\mathcal{E}^d) = \begin{cases} 1 & \text{if } d \equiv 2 (8) \\ 2 & \text{if } d \equiv -2 (8). \end{cases}$$

### 2.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$	$d$
$v_3(t) \geq 5$	$(1 : 3)$	$(1 : 1)$	
$v_3(t) = 4$	$(1 : 3)$	$(1 : 1)$	$d \not\equiv 0 (3)$
$v_3(t) = 3$ $v_3(t + 27) \equiv 3, 4, 5 (6)$		$(3 : 1)$	$d \equiv 0 (3)$
$v_3(t) = 3$ $v_3(t + 27) \equiv 0, 1, 2 (6)$	$(1 : 1)$	$(1 : 1)$	$d \not\equiv 0 (3)$
$v_3(t) = 2$		$(1 : 3)$	$d \equiv 0 (3)$
$v_3(t) \leq 1$	$(1 : 1)$	$(1 : 1)$	

This table is the ingredient to prove the following result:

**Proposition 2.** *Let  $E_1 \xrightarrow{3} E_3$  be a  $\mathbf{Q}$ -isogeny graph of type  $L_2(3)$  corresponding to a given  $t$  in  $\mathbf{Q}^*$ ,  $t \neq -27$ , with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{3} E_3^d$  is given by:*

$L_2(3)$	twisted isogeny graph	$d$	Prob
$v_3(t) \geq 5$	$E_1^d \leftarrow \textcircled{E_3^d}$		1
$v_3(t) = 4$	$E_1^d \leftarrow \textcircled{E_3^d}$ $\textcircled{E_1^d} \rightarrow E_3^d$	$d \not\equiv 0 (3)$	3/4
$v_3(t) = 3$ $v_3(t + 27) \equiv 3, 4, 5 (6)$		$d \equiv 0 (3)$	1/4
$v_3(t) = 3$ $v_3(t + 27) \equiv 0, 1, 2 (6)$	$\textcircled{E_1^d} \rightarrow E_3^d$ $E_1^d \leftarrow \textcircled{E_3^d}$	$d \not\equiv 0 (3)$	3/4
$v_3(t) = 2$		$d \equiv 0 (3)$	1/4
$v_3(t) \leq 1$	$\textcircled{E_1^d} \rightarrow E_3^d$		1

The column Prob gives the probability of the circled twisted curve to be the Faltings curve.

### 3 Type $L_2(5)$

#### 3.1 Settings

##### Graph

The isogeny graphs of type  $L_2(5)$  are given by two 5-isogenous elliptic curves:

$$E_1 \xrightarrow{5} E_5.$$

##### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(5)$  parametrize isogeny graphs of type  $L_2(5)$ . The curve  $X_0(5)$  has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 5^3 \left( \frac{\eta(5\tau)}{\eta(\tau)} \right)^6.$$

##### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$j(E_1) = j(\tau) = \frac{(t^2 + 10t + 5)^3}{t}$$

$$j(E_5) = j(5\tau) = \frac{(t^2 + 250t + 3125)^3}{t^5}.$$

##### Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_5)$  in such a way that the isogeny graph is normalized. Their signatures are:

$L_2(5)$	
$c_4(E_1)$	$(t^2 + 10t + 5)(t^2 + 22t + 125)$
$c_6(E_1)$	$(t^2 + 4t - 1)(t^2 + 22t + 125)^2$
$\Delta(E_1)$	$t(t^2 + 22t + 125)^3$
$c_4(E_5)$	$(t^2 + 22t + 125)(t^2 + 250t + 3125)$
$c_6(E_5)$	$(t^2 - 500t - 15625)(t^2 + 22t + 125)^2$
$\Delta(E_5)$	$t^5(t^2 + 22t + 125)^3$

##### Automorphisms

The subgroup of  $\text{Aut } X_0(5)$  that fixes the set of vertices of the graph is generated by the Fricke involution of  $X_0(5)$ , given by  $W_5(t) = 5^3/t$ . With regard to the action of the Fricke involution on the isogeny graph, it can be displayed as follows:

$$W_5(E_1 \xrightarrow{5} E_5) = E_5^{-1} \xrightarrow{5} E_1^{-1}.$$

### 3.2 Kodaira symbols, minimal models, and Pal values

Table 7:  $L_2(5)$  data for  $p \neq 2, 5$

$L_2(5)$	$p \neq 2, 5$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_5$	$(0, 0, 5m)$	1	$I_{5m}$	1	1
$v_p(t) = 0$ $v_p(t^2 + 22t + 125) = 4m > 0$	$E_1$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
	$E_5$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
$v_p(t) = 0$ $v_p(t^2 + 22t + 125) = 4m + 1 > 0$	$E_1$	$(1, 2m + 2, 3)$	$p^m$	$III$	1	1
	$E_5$	$(1, 2m + 2, 3)$	$p^m$	$III$	1	1
$v_p(t) = 0$ $v_p(t^2 + 22t + 125) = 4m + 2 > 0$	$E_1$	$(2, 2m + 4, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_5$	$(2, 2m + 4, 6)$	$p^m$	$I_0^*$	$p$	1
$v_p(t) = 0$ $v_p(t^2 + 22t + 125) = 4m + 3 > 0$	$E_1$	$(3, 2m + 6, 9)$	$p^m$	$III^*$	$p$	1
	$E_5$	$(3, 2m + 6, 9)$	$p^m$	$III^*$	$p$	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 5m)$	$p^{-m}$	$I_{5m}$	1	1
	$E_5$	$(0, 0, m)$	$p^{-m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 8:  $L_2(5)$  data for  $p = 5$ 

$L_2(5)$	$p = 5$					
$t$	$E$	$\text{sig}_5(\mathcal{E})$	$u_5(E)$	$K_5(E)$	$u_5(\mathcal{E}^d)$	
$v_5(t) = m > 3$	$E_1$	$(0, 0, m - 3)$	5	$I_{m-3}$	1	1
	$E_5$	$(0, 0, 5(m - 3))$	$5^2$	$I_{5(m-3)}$	1	1
$v_5(t) = 3$ $v_5(t^2 + 22t + 125) = 4m + 3 > 0$	$E_1$	$(0, \geq 0, 0)$	$5^{m+1}$	$I_0$	1	1
	$E_5$	$(0, \geq 0, 0)$	$5^{m+2}$	$I_0$	1	1
$v_5(t) = 3$ $v_5(t^2 + 22t + 125) = 4m + 4 > 0$	$E_1$	$(1, \geq 2, 3)$	$5^{m+1}$	III	1	1
	$E_5$	$(1, \geq 2, 3)$	$5^{m+2}$	III	1	1
$v_5(t) = 3$ $v_5(t^2 + 22t + 125) = 4m + 5 > 0$	$E_1$	$(2, \geq 4, 6)$	$5^{m+1}$	$I_0^*$	5	1
	$E_5$	$(2, \geq 4, 6)$	$5^{m+2}$	$I_0^*$	5	1
$v_5(t) = 3$ $v_5(t^2 + 22t + 125) = 4m + 6 > 0$	$E_1$	$(3, \geq 6, 9)$	$5^{m+1}$	III*	5	1
	$E_5$	$(3, \geq 6, 9)$	$5^{m+2}$	III*	5	1
$v_5(t) = 2$	$E_1$	$(3, 4, 8)$	1	IV*	5	1
	$E_5$	$(2, 2, 4)$	5	IV	1	1
$v_5(t) = 1$	$E_1$	$(2, 2, 4)$	1	IV	1	1
	$E_5$	$(3, 4, 8)$	1	IV*	5	1
$v_5(t) = 0$ $t \not\equiv 3 \pmod{5}$	$E_1$	$(0, 0, 0)$	1	$I_0$	1	1
	$E_5$	$(0, 0, 0)$	1	$I_0$	1	1
$v_5(t) = 0$ $v_5(t^2 + 22t + 125) = 4m$	$E_1$	$(0, \geq 0, 0)$	$5^m$	$I_0$	1	1
	$E_5$	$(0, \geq 0, 0)$	$5^m$	$I_0$	1	1
$v_5(t) = 0$ $v_5(t^2 + 22t + 125) = 4m + 1$	$E_1$	$(1, \geq 2, 3)$	$5^m$	III	1	1
	$E_5$	$(1, \geq 2, 3)$	$5^m$	III	1	1
$v_5(t) = 0$ $v_5(t^2 + 22t + 125) = 4m + 2$	$E_1$	$(2, \geq 4, 6)$	$5^m$	$I_0^*$	5	1
	$E_5$	$(2, \geq 4, 6)$	$5^m$	$I_0^*$	5	1
$v_5(t) = 0$ $v_5(t^2 + 22t + 125) = 4m + 3$	$E_1$	$(3, \geq 6, 9)$	$5^m$	III*	5	1
	$E_5$	$(3, \geq 6, 9)$	$5^m$	III*	5	1
$v_5(t) = -m < 0$	$E_1$	$(0, 0, 5m)$	$5^{-m}$	$I_{5m}$	1	1
	$E_5$	$(0, 0, m)$	$5^{-m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{5}$	

The polynomial  $t^2 + 22t + 125$  factors in  $\mathbb{Q}_5[t]$  as  $(t - \alpha_1)(t - \alpha_2)$  with:

$$\begin{aligned}\alpha_1 &= 3 + 4 \cdot 5^2 + 2 \cdot 5^3 + 5^4 + 4 \cdot 5^5 + 2 \cdot 5^7 + 5^8 + 5^9 + 5^{12} + 3 \cdot 5^{13} + 3 \cdot 5^{14} + 5^{15} + O(5^{17}) \\ \alpha_2 &= 2 \cdot 5^3 + 3 \cdot 5^4 + 4 \cdot 5^6 + 2 \cdot 5^7 + 3 \cdot 5^8 + 3 \cdot 5^9 + 4 \cdot 5^{10} + 4 \cdot 5^{11} + 3 \cdot 5^{12} + 5^{13} + 5^{14} + O(5^{15})\end{aligned}$$

Table 9:  $L_2(5)$  data for  $p=2$ 

$L_2(5)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m \geq 1$	$E_1$	$(0, 0, m)$	1	$I_m$	1	$2^{-1}$	$2^{-1}$
	$E_5$	$(0, 0, 5m)$	1	$I_{5m}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 0$ $t \equiv 1 \pmod{4}$	$E_1$	$(6, 6, 6)$	1	II	1	$1^*$ or $2^*$	1
	$E_5$	$(6, 6, 6)$	1	II	1	$1^*$ or $2^*$	1
$v_2(t) = 0$ $t \equiv 3 \pmod{4}$	$E_1$	$(5, 8, 9)$	1	III	1	1	1
	$E_5$	$(5, 8, 9)$	1	III	1	1	1
$v_2(t) = -m < 0$	$E_1$	$(4, 6, 5m + 12)$	$2^{-(m+1)}$	$I_{5m+4}^*$	1	1	2
	$E_5$	$(4, 6, m + 12)$	$2^{-(m+1)}$	$I_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

**Remark ( $1^*$  or  $2^*$ ):** The value  $u_2(\mathcal{E}^d)$  is given by:

- if  $t \equiv 1 \pmod{8}$ , then  $u_2(\mathcal{E}^d) = \begin{cases} 1 & \text{if } d \equiv -2 \pmod{8} \\ 2 & \text{if } d \equiv 2 \pmod{8}; \end{cases}$
- if  $t \equiv 5 \pmod{8}$ , then  $u_2(\mathcal{E}^d) = \begin{cases} 2 & \text{if } d \equiv -2 \pmod{8} \\ 1 & \text{if } d \equiv 2 \pmod{8}. \end{cases}$

### 3.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$	$d$
$v_5(t) \geq 3$	$(1 : 5)$	$(1 : 1)$	
$v_5(t) = 2$	$(1 : 5)$	$(1 : 1)$	$d \not\equiv 0 (5)$
		$(5 : 1)$	$d \equiv 0 (5)$
$v_5(t) = 1$	$(1 : 1)$	$(1 : 1)$	$d \not\equiv 0 (5)$
		$(1 : 5)$	$d \equiv 0 (5)$
$v_5(t) \leq 0$	$(1 : 1)$	$(1 : 1)$	

**Proposition 3.** Let  $E_1 \xrightarrow{5} E_5$  be a  $\mathbf{Q}$ -isogeny graph of type  $L_2(5)$  corresponding to a given  $t$  in  $\mathbf{Q}^*$  with signatrues as aboves. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{5} E_5^d$  is given by:

$L_2(5)$	twisted isogeny graph	$d$	Prob
$v_5(t) \geq 3$	$E_1^d \leftarrow \textcircled{E_5^d}$		1
$v_5(t) = 2$	$E_1^d \leftarrow \textcircled{E_5^d}$	$d \not\equiv 0 (5)$	5/6
	$\textcircled{E_1^d} \rightarrow E_5^d$	$d \equiv 0 (5)$	1/6
$v_5(t) = 1$	$\textcircled{E_1^d} \rightarrow E_5^d$	$d \not\equiv 0 (5)$	5/6
	$E_1^d \leftarrow \textcircled{E_5^d}$	$d \equiv 0 (5)$	1/6
$v_5(t) \leq 0$	$\textcircled{E_1^d} \rightarrow E_5^d$		1

The column Prob gives the probability of the circled twisted curve to be the Faltings curve.



## 4 Type $L_2(7)$

### 4.1 Settings

#### Graph

The isogeny graphs of type  $L_2(7)$  are given by two 7-isogenous elliptic curves:

$$E_1 \xrightarrow{7} E_7.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(7)$  parametrize isogeny graphs of type  $L_2(7)$ . The curve  $X_0(7)$  has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 7^2 \left( \frac{\eta(7\tau)}{\eta(\tau)} \right)^4.$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$j(E_1) = j(\tau) = \frac{(t^2 + 5t + 1)^3 (t^2 + 13t + 49)}{t}$$

$$j(E_7) = j(7\tau) = \frac{(t^2 + 13t + 49) (t^2 + 245t + 2401)^3}{t^7}.$$

#### Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_7)$  in such a way that the isogeny graph is normalized. Their signatures are:

$L_2(7)$	
$c_4(E_1)$	$(t^2 + 5t + 1)(t^2 + 13t + 49)$
$c_6(E_1)$	$(t^2 + 13t + 49)(t^4 + 14t^3 + 63t^2 + 70t - 7)$
$\Delta(E_1)$	$t(t^2 + 13t + 49)^2$
$c_4(E_7)$	$(t^2 + 13t + 49)(t^2 + 245t + 2401)$
$c_6(E_7)$	$(t^2 + 13t + 49)(t^4 - 490t^3 - 21609t^2 - 235298t - 823543)$
$\Delta(E_7)$	$t^7(t^2 + 13t + 49)^2$

#### Automorphisms

The subgroup of  $\text{Aut } X_0(7)$  that fixes the set of vertices of the graph is generated by the Fricke involution of  $X_0(7)$ , given by  $W_7(t) = 7^2/t$ . With regard to the action of the Fricke involution on the isogeny graph, it can be displayed as follows:

$$W_7(E_1 \xrightarrow{7} E_7) = E_7^{-7} \xrightarrow{7} E_1^{-7}.$$

## 4.2 Kodaira symbols, minimal models, and Pal values

Table 10:  $L_2(7)$  data for  $p \neq 2, 3, 7$

$L_2(7)$	$p \neq 2, 3, 7$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_7$	$(0, 0, 7m)$	1	$I_{7m}$	1	1
$v_p(t) = 0$ $v_p(t^2 + 13t + 49) = 6m > 0$	$E_1$	$(2m, 0, 0)$	$p^m$	$I_0$	1	1
	$E_7$	$(2m, 0, 0)$	$p^m$	$I_0$	1	1
$v_p(t) = 0$ $v_p(t^2 + 13t + 49) = 6m + 1 > 0$	$E_1$	$(2m + 1, 1, 2)$	$p^m$	$II$	1	1
	$E_7$	$(2m + 1, 1, 2)$	$p^m$	$II$	1	1
$v_p(t) = 0$ $v_p(t^2 + 13t + 49) = 6m + 2 > 0$	$E_1$	$(2m + 2, 2, 4)$	$p^m$	$IV$	1	1
	$E_7$	$(2m + 2, 2, 4)$	$p^m$	$IV$	1	1
$v_p(t) = 0$ $v_p(t^2 + 13t + 49) = 6m + 3 > 0$	$E_1$	$(2m + 3, 3, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_7$	$(2m + 3, 3, 6)$	$p^m$	$I_0^*$	$p$	1
$v_p(t) = 0$ $v_p(t^2 + 13t + 49) = 6m + 4 > 0$	$E_1$	$(2m + 4, 4, 8)$	$p^m$	$IV^*$	$p$	1
	$E_7$	$(2m + 4, 4, 8)$	$p^m$	$IV^*$	$p$	1
$v_p(t) = 0$ $v_p(t^2 + 13t + 49) = 6m + 5 > 0$	$E_1$	$(2m + 5, 5, 10)$	$p^m$	$II^*$	$p$	1
	$E_7$	$(2m + 5, 5, 10)$	$p^m$	$II^*$	$p$	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 7m)$	$p^{-m}$	$I_{7m}$	1	1
	$E_7$	$(0, 0, m)$	$p^{-m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 11:  $L_2(7)$  data for  $p = 3$ 

$L_2(7)$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_7$	$(0, 0, 7m)$	1	$I_{7m}$	1	1
$v_3(t) = 0$ $t \equiv 1, 4 (9)$	$E_1$	$(2, 3, 4)$	1	$II$	1	1
	$E_7$	$(2, 3, 4)$	1	$II$	1	1
$v_3(t) = 0$ $t \equiv 16, 25 (27)$	$E_1$	$(3, 5, 6)$	1	$IV$	1	1
	$E_7$	$(3, 5, 6)$	1	$IV$	1	1
$v_3(t) = 0$ $t \equiv 7 (27)$	$E_1$	$(3, \geq 6, 6)$	1	$I_0^*$	3	1
	$E_7$	$(3, \geq 6, 6)$	1	$I_0^*$	3	1
$v_3(t) = -m < 0$	$E_1$	$(0, 0, 7m)$	$3^{-m}$	$I_{7m}$	1	1
	$E_7$	$(0, 0, m)$	$3^{-m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	

Table 12:  $L_2(7)$  data for  $p = 7$ 

$L_2(7)$	$p = 7$					
$t$	$E$	$\text{sig}_7(\mathcal{E})$	$u_7(E)$	$K_7(E)$	$u_7(\mathcal{E}^d)$	
$v_7(t) = m \geq 3$	$E_1$	$(2, 3, m+4)$	1	$I_{m-2}^*$	7	1
	$E_7$	$(2, 3, 7m-8)$	7	$I_{7m-14}^*$	7	1
$v_7(t) = 2$ $v_7(t^2 + 13t + 49) = 6m \geq 2$	$E_1$	$(\geq 1, 1, 2)$	$7^m$	II	1	1
	$E_7$	$(\geq 1, 1, 2)$	$7^{m+1}$	II	1	1
$v_7(t) = 2$ $v_7(t^2 + 13t + 49) = 6m + 1 \geq 2$	$E_1$	$(\geq 3, 2, 4)$	$7^m$	IV	1	1
	$E_7$	$(\geq 3, 2, 4)$	$7^{m+1}$	IV	1	1
$v_7(t) = 2$ $v_7(t^2 + 13t + 49) = 6m + 2$	$E_1$	$(\geq 2, 3, 6)$	$7^m$	$I_0^*$	7	1
	$E_7$	$(\geq 2, 3, 6)$	$7^{m+1}$	$I_0^*$	7	1
$v_7(t) = 2$ $v_7(t^2 + 13t + 49) = 6m + 3$	$E_1$	$(\geq 3, 4, 8)$	$7^m$	IV*	7	1
	$E_7$	$(\geq 3, 4, 8)$	$7^{m+1}$	IV*	7	1
$v_7(t) = 2$ $v_7(t^2 + 13t + 49) = 6m + 4$	$E_1$	$(\geq 4, 5, 10)$	$7^m$	II*	7	1
	$E_7$	$(\geq 4, 5, 10)$	$7^{m+1}$	II*	7	1
$v_7(t) = 2$ $v_7(t^2 + 13t + 49) = 6m + 5$	$E_1$	$(\geq 1, 0, 0)$	$7^{m+1}$	$I_0$	1	1
	$E_7$	$(\geq 1, 0, 0)$	$7^{m+2}$	$I_0$	1	1
$v_7(t) = 1$	$E_1$	$(1, 2, 3)$	1	III	1	1
	$E_7$	$(3, 5, 9)$	1	III*	7	1
$v_7(t) = 0$ $v_7(t^2 + 13t + 49) = 6m \geq 0$	$E_1$	$(\geq 1, 0, 0)$	$7^m$	$I_0$	1	1
	$E_7$	$(\geq 1, 0, 0)$	$7^m$	$I_0$	1	1
$v_7(t) = 0$ $v_7(t^2 + 13t + 49) = 6m + 1$	$E_1$	$(\geq 2, 1, 2)$	$7^m$	II	1	1
	$E_7$	$(\geq 2, 1, 2)$	$7^m$	II	1	1
$v_7(t) = 0$ $v_7(t^2 + 13t + 49) = 6m + 2$	$E_1$	$(\geq 3, 2, 4)$	$7^m$	IV	1	1
	$E_7$	$(\geq 3, 2, 4)$	$7^m$	IV	1	1
$v_7(t) = 0$ $v_7(t^2 + 13t + 49) = 6m + 3$	$E_1$	$(\geq 3, 3, 6)$	$7^m$	$I_0^*$	7	1
	$E_7$	$(\geq 3, 3, 6)$	$7^m$	$I_0^*$	7	1
$v_7(t) = 0$ $v_7(t^2 + 13t + 49) = 6m + 4$	$E_1$	$(\geq 4, 4, 8)$	$7^m$	IV*	7	1
	$E_7$	$(\geq 4, 4, 8)$	$7^m$	IV*	7	1
$v_7(t) = 0$ $v_7(t^2 + 13t + 49) = 6m + 5$	$E_1$	$(\geq 5, 5, 10)$	$7^m$	II*	7	1
	$E_7$	$(\geq 5, 5, 10)$	$7^m$	II*	7	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{7}$	

Table 12:  $L_2(7)$  data for  $p = 7$  (Continued)

$L_2(7)$	$p = 7$					
$t$	$E$	$\text{sig}_7(\mathcal{E})$	$u_7(E)$	$K_7(E)$	$u_7(\mathcal{E}^d)$	
$v_7(t) = -m < 0$	$E_1$	$(0, 0, 7m)$	$7^{-m}$	$I_{7m}$	1	1
	$E_7$	$(0, 0, m)$	$7^{-m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
				$d \pmod{7}$		

The polynomial  $t^2 + 13t + 49$  factors over  $\mathbb{Q}_7[t]$  as  $(t - \alpha_1)(t - \alpha_2)$  with:

$$\alpha_1 = 1 + 5 \cdot 7 + 5 \cdot 7^2 + 4 \cdot 7^3 + 7^4 + 6 \cdot 7^5 + 4 \cdot 7^6 + 7^7 + 6 \cdot 7^8 + 3 \cdot 7^9 + 6 \cdot 7^{10} + 6 \cdot 7^{11} + O(7^{12})$$

$$\alpha_2 = 7^2 + 2 \cdot 7^3 + 5 \cdot 7^4 + 2 \cdot 7^6 + 5 \cdot 7^7 + 3 \cdot 7^9 + 4 \cdot 7^{12} + 5 \cdot 7^{13} + 2 \cdot 7^{14} + O(7^{15})$$

Table 13:  $L_2(7)$  data for  $p=2$ 

$L_2(7)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m \geq 2$	$E_1$	$(4, 6, m + 12)$	$2^{-1}$	$I_{m+4}^*$	1	1	2
	$E_7$	$(4, 6, 7m + 12)$	$2^{-1}$	$I_{7m+4}^*$	1	1	2
$v_2(t) = 1$	$E_1$	$(0, 0, 1)$	1	$I_1$	1	$2^{-1}$	$2^{-1}$
	$E_7$	$(0, 0, 7)$	1	$I_7$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 0$ $t \equiv 1 \pmod{4}$	$E_1$	$(0, 0, 0)$	1	$I_0$	1	$2^{-1}$	$2^{-1}$
	$E_7$	$(0, 0, 0)$	1	$I_0$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 0$ $t \equiv 3 \pmod{4}$	$E_1$	$(4, 6, 12)$	$2^{-1}$	$I_4^*$	1	1	2
	$E_7$	$(4, 6, 12)$	$2^{-1}$	$I_4^*$	1	1	2
$v_2(t) = -1$	$E_1$	$(0, 0, 7)$	$2^{-1}$	$I_7$	1	$2^{-1}$	$2^{-1}$
	$E_7$	$(0, 0, 1)$	$2^{-1}$	$I_1$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = -m \leq -2$	$E_1$	$(4, 6, 7m + 12)$	$2^{-(m+1)}$	$I_{7m+4}^*$	1	1	2
	$E_7$	$(4, 6, m + 12)$	$2^{-(m+1)}$	$I_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

### 4.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$	$d$
$v_7(t) \geq 2$	$(1 : 7)$	$(1 : 1)$	
$v_7(t) = 1$	$(1 : 1)$	$(1 : 1)$	$d \not\equiv 0 (7)$
		$(1 : 7)$	$d \equiv 0 (7)$
$v_7(t) \leq 0$	$(1 : 1)$	$(1 : 1)$	

The contents of this table are the ingredients to prove the following result:

**Proposition 4.** *Let  $E_1 \xrightarrow{7} E_7$  be a  $\mathbf{Q}$ -isogeny graph of type  $L_2(7)$  corresponding to a given  $t$  in  $\mathbf{Q}^*$  with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{7} E_7^d$  is given by:*

$L_2(7)$	twisted isogeny graph	$d$	Prob
$v_7(t) \geq 2$	$E_1^d \leftarrow \textcircled{E_7^d}$		1
$v_7(t) = 1$	$\textcircled{E_1^d} \rightarrow E_7^d$	$d \not\equiv 0 (7)$	7/8
	$E_1^d \leftarrow \textcircled{E_7^d}$	$d \equiv 0 (7)$	1/8
$v_7(t) \leq 0$	$\textcircled{E_1^d} \rightarrow E_7^d$		1

The column Prob gives the probability of the circled twisted curve to be the Faltings curve.

## 5 Type $L_2(11)$

### 5.1 Settings

#### Graph

The isogeny graphs of type  $L_2(11)$  are given by two 11-isogenous elliptic curves:

$$E_1 \xrightarrow{11} E_{11}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(11)$  parametrize isogeny graphs of type  $L_2(11)$ . The modular curve  $X_0(11)$  is an elliptic curve of rank 0 over the rationals. More precisely, we can choose the Weierstrass model  $y^2 + y = x^3 - x^2 - 10x - 20$  for  $X_0(11)$ . The  $j$ -forgetful map  $j: X_0(11) \rightarrow X_0(1)$  is given by

$$j = \frac{P(x) + yQ(x)}{(-17x^2 - xy - 243x - 105y + 859)^3}$$

with

$$\begin{aligned} P(x) &= x^8 + 160170x^7 + 22013817x^6 - 1234891244x^5 + 18403682346x^4 - 145947253957x^3 \\ &\quad + 1422949497947x^2 + 5880426893238x + 7325611514413 \\ Q(x) &= -692x^6 - 12510792x^5 + 815793738x^4 - 17947463042x^3 + 112966993208x^2 \\ &\quad + 491634446704x - 468196759663. \end{aligned}$$

#### $j$ -invariants

One has

$$\begin{aligned} X_0(11)(\mathbb{Q}) &= \{ P_\infty = (0 : 1 : 0), P_0 = (16 : -61 : 1), \\ &\quad P_b = (5 : 5 : 1) = W_{11}(P_b), \\ &\quad P_a = (16 : 60 : 1), W_{11}(P_a) = (5 : -6 : 1) \} \end{aligned}$$

where  $W_{11}$  denotes the Fricke involution. Hence,

$$j(P_\infty) = \infty, \quad j(P_0) = \infty, \quad j(P_b) = -2^{15}, \quad j(P_a) = -11^2, \quad j(W_{11}(P_a)) = -11 \cdot 131^3.$$

Besides the cusps  $P_\infty = (\infty)$ ,  $P_0 = (0)$ , the CM point  $P_b$  corresponds to  $\tau_b = \frac{1}{2} + \frac{\sqrt{-11}}{2 \cdot 11}$ , and two non-cuspidal non-CM points  $P_a$  and  $W_{11}(P_a)$  correspond to  $\tau_a = 0.5 + 0.09227...i$  and  $11\tau_a$ . We have

$$j(\tau_b) = j(11\tau_b) = -2^{15}, \quad j(\tau_a) = -11^2, \quad j(11\tau_a) = -11 \cdot 131^3.$$



## Signatures

We choose minimal Weierstrass equations and the isogeny is normalized.

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_{1_a}$	$y^2 + xy + y = x^3 + x^2 - 30x - 76$	$-11 \cdot 131^3$	<b>121a2</b>
$E_{11_a}$	$y^2 + xy + y = x^3 + x^2 - 305x + 7888$	$-11^2$	<b>121a1</b>

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_{1_b}$	$y^2 + y = x^3 - x^2 - 7x + 10$	$-2^{15}$	<b>121.b2</b>
$E_{11_b}$	$y^2 + y = x^3 - x^2 - 887x - 10143$	$-2^{15}$	<b>121.b1</b>

Their signatures are:

$E$	$E_{1_a}$	$E_{11_a}$	$E$	$E_{1_b}$	$E_{11_b}$
$c_4(E)$	$11 \cdot 131$	$11^4$	$c_4(E)$	$2^5 \cdot 11$	$2^5 \cdot 11^3$
$c_6(E)$	$11 \cdot 4973$	$-11^5 \cdot 43$	$c_6(E)$	$-2^3 \cdot 7 \cdot 11^2$	$2^3 \cdot 7 \cdot 11^5$
$\Delta(E)$	$-11^2$	$-11^{10}$	$\Delta(E)$	$-11^3$	$-11^9$

One checks that the Faltings curve (circled) in the graph is

$$\boxed{\begin{array}{c} \textcircled{E_{1_a}} \longrightarrow E_{11_a} \end{array}} \quad \boxed{\begin{array}{c} \textcircled{E_{1_b}} \longrightarrow E_{11_b} \end{array}}$$

Note that any  $\mathbb{Q}$ -isogeny class of elliptic curves of type  $L_2(11)$  is obtained by quadratic twist from these two graphs.

## 5.2 Kodaira symbols, minimal models, and Pal values

The only prime of bad reduction for the above elliptic curves is  $p = 11$ .

$p = 11$				
$E$	$\text{sig}_{11}(\mathcal{E})$	$K_{11}(E)$	$u_{11}(\mathcal{E}^d)$	
$E_{1_a}$	$(1, 1, 2)$	$\text{II}$	1	1
$E_{11_a}$	$(4, 5, 10)$	$\text{II}^*$	11	1
$E_{1_b}$	$(1, 2, 3)$	$\text{III}$	1	1
$E_{11_b}$	$(3, 5, 9)$	$\text{III}^*$	11	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{11}$	

### 5.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1)$	$d \not\equiv 0 (11)$
$(1 : 11)$	$d \equiv 0 (11)$

This table is the ingredient to prove the following result:

**Proposition 5.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_{1_k}^d \xrightarrow{11} E_{11_k}^d$ , for  $k \in \{a, b\}$ , is given by:*

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$\bigcirc E_{1_k}^d \longrightarrow E_{11_k}^d$	$d \not\equiv 0 (11)$	$11/12$
$E_{1_k}^d \longrightarrow \bigcirc E_{11_k}^d$	$d \equiv 0 (11)$	$1/12$

*The column Prob gives the probability of the circled curve to be the Faltings curve.*

## 6 Type $L_2(13)$

### 6.1 Settings

#### Graph

The isogeny graphs of type  $L_2(13)$  are given by two 13-isogenous elliptic curves:

$$E_1 \xrightarrow{13} E_{13}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(13)$  parametrize isogeny graphs of type  $L_2(13)$ . The curve  $X_0(13)$  has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 13 \left( \frac{\eta(13\tau)}{\eta(\tau)} \right)^2.$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$\begin{aligned} j(E_1) = j(\tau) &= \frac{(t^2 + 5t + 13)(t^4 + 7t^3 + 20t^2 + 19t + 1)^3}{t} \\ j(E_{13}) = j(13\tau) &= \frac{(t^2 + 5t + 13)(t^4 + 247t^3 + 3380t^2 + 15379t + 28561)^3}{t^{13}}. \end{aligned}$$

#### Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_{13})$  in such a way that the isogeny graph is normalized. Their signatures are:

$L_2(13)$	
$c_4(E_1)$	$(t^2 + 5t + 13)(t^2 + 6t + 13)(t^4 + 7t^3 + 20t^2 + 19t + 1)$
$c_6(E_1)$	$(t^2 + 5t + 13)(t^2 + 6t + 13)^2(t^6 + 10t^5 + 46t^4 + 108t^3 + 122t^2 + 38t - 1)$
$\Delta(E_1)$	$t(t^2 + 5t + 13)^2(t^2 + 6t + 13)^3$
$c_4(E_{13})$	$(t^2 + 5t + 13)(t^2 + 6t + 13)(t^4 + 247t^3 + 3380t^2 + 15379t + 28561)$
$c_6(E_{13})$	$(t^2 + 5t + 13)(t^2 + 6t + 13)^2(t^6 - 494t^5 - 20618t^4 - 237276t^3 - 1313806t^2 - 3712930t - 4826809)$
$\Delta(E_{13})$	$t^{13}(t^2 + 5t + 13)^2(t^2 + 6t + 13)^3$

#### Automorphism

The Fricke involution of  $X_0(13)$  is given by  $W_{13}(t) = 13/t$ . With regard to the action of the Fricke involution on the isogeny graph, it can be described as:

$$W_{13}(E_1 \xrightarrow{13} E_{13}) = E_{13}^{-13} \xrightarrow{13} E_1^{-13}.$$

## 6.2 Kodaira symbols, minimal models, and Pal values

Table 14:  $L_2(13)$  data for  $p \neq 2, 3, 13$

$L_2(13)$	$p \neq 2, 3, 13$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{13}$	$(0, 0, 13m)$	1	$I_{13m}$	1	1
$v_p(t) = 0$ $v_p(t^2 + 5t + 13) = 6m$	$E_1$	$(2m, 0, 0)$	$p^m$	$I_0$	1	1
	$E_{13}$	$(2m, 0, 0)$	$p^m$	$I_0$	1	1
$v_p(t) = 0$ $v_p(t^2 + 5t + 13) = 6m + 1$	$E_1$	$(2m + 1, 1, 2)$	$p^m$	$II$	1	1
	$E_{13}$	$(2m + 1, 1, 2)$	$p^m$	$II$	1	1
$v_p(t) = 0$ $v_p(t^2 + 5t + 13) = 6m + 2$	$E_1$	$(2m + 2, 2, 4)$	$p^m$	$IV$	1	1
	$E_{13}$	$(2m + 2, 2, 4)$	$p^m$	$IV$	1	1
$v_p(t) = 0$ $v_p(t^2 + 5t + 13) = 6m + 3$	$E_1$	$(2m + 3, 3, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_{13}$	$(2m + 3, 3, 6)$	$p^m$	$I_0^*$	$p$	1
$v_{13}(t) = 0$ $v_p(t^2 + 5t + 13) = 6m + 4$	$E_1$	$(2m + 4, 4, 8)$	$p^m$	$IV^*$	$p$	1
	$E_{13}$	$(2m + 4, 4, 8)$	$p^m$	$IV^*$	$p$	1
$v_p(t) = 0$ $v_p(t^2 + 5t + 13) = 6m + 5$	$E_1$	$(2m + 5, 5, 10)$	$p^m$	$II^*$	$p$	1
	$E_{13}$	$(2m + 5, 5, 10)$	$p^m$	$II^*$	$p$	1
$v_p(t) = 0$ $v_p(t^2 + 6t + 13) = 4m$	$E_1$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
	$E_{13}$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
$v_p(t) = 0$ $v_p(t^2 + 6t + 13) = 4m + 1$	$E_1$	$(1, 2m + 2, 3)$	$p^m$	$III$	1	1
	$E_{13}$	$(1, 2m + 2, 3)$	$p^m$	$III$	1	1
$v_p(t) = 0$ $v_p(t^2 + 6t + 13) = 4m + 2$	$E_1$	$(2, 2m + 4, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_{13}$	$(2, 2m + 4, 6)$	$p^m$	$I_0^*$	$p$	1
$v_p(t) = 0$ $v_p(t^2 + 6t + 13) = 4m + 3$	$E_1$	$(3, 2m + 6, 9)$	$p^m$	$III^*$	$p$	1
	$E_{13}$	$(3, 2m + 6, 9)$	$p^m$	$III^*$	$p$	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 13m)$	$p^{-2m}$	$I_{13m}$	1	1
	$E_{13}$	$(0, 0, m)$	$p^{-2m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 15:  $L_2(13)$  data for  $p = 3$ 

$L_2(13)$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{13}$	$(0, 0, 13m)$	1	$I_{13m}$	1	1
$v_3(t) = 0$ $t \equiv 1 \pmod{3}$	$E_1$	$(1, \geq 3, 0)$	1	$I_0$	1	1
	$E_{13}$	$(1, \geq 3, 0)$	1	$I_0$	1	1
$v_3(t) = 0$ $t \equiv 5, 8 \pmod{9}$	$E_1$	$(2, 3, 4)$	1	$II$	1	1
	$E_{13}$	$(2, 3, 4)$	1	$II$	1	1
$v_3(t) = 0$ $t \equiv 2, 20 \pmod{27}$	$E_1$	$(3, 5, 6)$	1	$IV$	1	1
	$E_{13}$	$(3, 5, 6)$	1	$IV$	1	1
$v_3(t) = 0$ $t \equiv 11 \pmod{27}$	$E_1$	$(3, \geq 6, 6)$	1	$I_0^*$	3	1
	$E_{13}$	$(3, \geq 6, 6)$	1	$I_0^*$	3	1
$v_3(t) = -m < 0$	$E_1$	$(0, 0, 13m)$	$3^{-2m}$	$I_{13m}$	1	1
	$E_{13}$	$(0, 0, m)$	$3^{-2m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	

Table 16:  $L_2(13)$  data for  $p = 13$ 

$L_2(13)$	$p = 13$					
$t$	$E$	$\text{sig}_{13}(\mathcal{E})$	$u_{13}(E)$	$K_{13}(E)$	$u_{13}(\mathcal{E}^d)$	
$v_{13}(t) = m \geq 2$	$E_1$	$(2, 3, m + 5)$	1	$I_{m-1}^*$	13	1
	$E_{13}$	$(2, 3, 13m - 7)$	13	$I_{13(m-1)}^*$	13	1
$v_{13}(t) = 1$ $t/13 \not\equiv 2, 5 (13)$	$E_1$	$(2, 3, 6)$	1	$I_0^*$	13	1
	$E_{13}$	$(2, 3, 6)$	13	$I_0^*$	13	1
$v_{13}(t) = 1$ $t/13 \equiv 2 (13)$ $v_{13}(t^2 + 6t + 13) = 4m$	$E_1$	$(1, \geq 3, 3)$	$13^m$	III	1	1
	$E_{13}$	$(1, \geq 3, 3)$	$13^{m+1}$	III	1	1
$v_{13}(t) = 1$ $t/13 \equiv 2 (13)$ $v_{13}(t^2 + 6t + 13) = 4m + 1$	$E_1$	$(2, \geq 3, 6)$	$13^m$	$I_0^*$	13	1
	$E_{13}$	$(2, \geq 4, 6)$	$13^{m+1}$	$I_0^*$	13	1
$v_{13}(t) = 1$ $t/13 \equiv 2 (13)$ $v_{13}(t^2 + 6t + 13) = 4m + 2$	$E_1$	$(3, \geq 5, 9)$	$13^m$	III*	13	1
	$E_{13}$	$(3, \geq 6, 9)$	$13^{m+1}$	III*	13	1
$v_{13}(t) = 1$ $t/13 \equiv 2 (13)$ $v_{13}(t^2 + 6t + 13) = 4m + 3$	$E_1$	$(0, \geq 1, 0)$	$13^{m+1}$	$I_0$	1	1
	$E_{13}$	$(0, \geq 2, 0)$	$13^{m+2}$	$I_0$	1	1
$v_{13}(t) = 1$ $t/13 \equiv 5 (13)$ $v_{13}(t^2 + 5t + 13) = 6m$	$E_1$	$(\geq 2, 2, 4)$	$13^m$	IV	1	1
	$E_{13}$	$(\geq 2, 2, 4)$	$13^{m+1}$	IV	1	1
$v_{13}(t) = 1$ $t/13 \equiv 5 (13)$ $v_{13}(t^2 + 5t + 13) = 6m + 1$	$E_1$	$(\geq 2, 3, 6)$	$13^m$	$I_0^*$	13	1
	$E_{13}$	$(\geq 3, 3, 6)$	$13^{m+1}$	$I_0^*$	13	1
$v_{13}(t) = 1$ $t/13 \equiv 5 (13)$ $v_{13}(t^2 + 5t + 13) = 6m + 2$	$E_1$	$(\geq 3, 4, 8)$	$13^m$	IV*	13	1
	$E_{13}$	$(\geq 4, 4, 8)$	$13^{m+1}$	IV*	13	1
$v_{13}(t) = 1$ $t/13 \equiv 5 (13)$ $v_{13}(t^2 + 5t + 13) = 6m + 3$	$E_1$	$(\geq 4, 5, 10)$	$13^m$	II*	13	1
	$E_{13}$	$(\geq 5, 5, 10)$	$13^{m+1}$	II*	13	1
$v_{13}(t) = 1$ $t/13 \equiv 5 (13)$ $v_{13}(t^2 + 5t + 13) = 6m + 4$	$E_1$	$(\geq 0, 0, 0)$	$13^{m+1}$	$I_0$	1	1
	$E_{13}$	$(\geq 2, 0, 0)$	$13^{m+2}$	$I_0$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{13}$	

Continued on next page

Table 16:  $L_2(13)$  data for  $p = 13$  (Continued)

$L_2(13)$	$p = 13$					
$t$	$E$	$\text{sig}_{13}(\mathcal{E})$	$u_{13}(E)$	$K_{13}(E)$	$u_{13}(\mathcal{E}^d)$	
$v_{13}(t) = 1$ $t/13 \equiv 5(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 5$	$E_1$	$(\geq 2, 1, 2)$	$13^{m+1}$	II	1	1
	$E_{13}$	$(\geq 3, 1, 2)$	$13^{m+2}$	II	1	1
$v_{13}(t) = 0$ $t \not\equiv 7, 8(13)$	$E_1$	$(0, 0, 0)$	1	$I_0$	1	1
	$E_1$	$(0, 0, 0)$	1	$I_0$	1	1
$v_{13}(t) = 0$ $t \equiv 7(13)$ $v_{13}(t^2 + 6t + 13) = 4m$	$E_1$	$(0, \geq 1, 0)$	$13^m$	$I_0$	1	1
	$E_{13}$	$(0, \geq 0, 0)$	$13^m$	$I_0$	1	1
$v_{13}(t) = 0$ $t \equiv 7(13)$ $v_{13}(t^2 + 6t + 13) = 4m + 1$	$E_1$	$(1, \geq 3, 3)$	$13^m$	III	1	1
	$E_{13}$	$(1, \geq 2, 3)$	$13^m$	III	1	1
$v_{13}(t) = 0$ $t \equiv 7(13)$ $v_{13}(t^2 + 6t + 13) = 4m + 2$	$E_1$	$(2, \geq 5, 6)$	$13^m$	$I_0^*$	13	1
	$E_{13}$	$(2, \geq 4, 6)$	$13^m$	$I_0^*$	13	1
$v_{13}(t) = 0$ $t \equiv 7(13)$ $v_{13}(t^2 + 6t + 13) = 4m + 3$	$E_1$	$(3, \geq 7, 9)$	$13^m$	III*	13	1
	$E_{13}$	$(3, \geq 6, 9)$	$13^m$	III*	13	1
$v_{13}(t) = 0$ $t/13 \equiv 8(13)$ $v_{13}(t^2 + 5t + 13) = 6m$	$E_1$	$(\geq 1, 0, 0)$	$13^m$	$I_0$	1	1
	$E_{13}$	$(\geq 0, 0, 0)$	$13^m$	$I_0$	1	1
$v_{13}(t) = 0$ $t/13 \equiv 8(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 1$	$E_1$	$(\geq 2, 1, 2)$	$13^m$	II	1	1
	$E_{13}$	$(\geq 1, 1, 2)$	$13^m$	II	1	1
$v_{13}(t) = 0$ $t \equiv 8(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 2$	$E_1$	$(\geq 3, 2, 4)$	$13^m$	IV	1	1
	$E_{13}$	$(\geq 2, 2, 4)$	$13^m$	IV	1	1
$v_{13}(t) = 0$ $t \equiv 8(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 3$	$E_1$	$(\geq 4, 3, 6)$	$13^m$	$I_0^*$	13	1
	$E_1$	$(\geq 3, 3, 6)$	$13^m$	$I_0^*$	13	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{13}$	

Continued on next page

Table 16:  $L_2(13)$  data for  $p = 13$  (Continued)

$L_2(13)$	$p = 13$					
$t$	$E$	$\text{sig}_{13}(\mathcal{E})$	$u_{13}(E)$	$K_{13}(E)$	$u_{13}(\mathcal{E}^d)$	
$v_{13}(t) = 0$ $t \equiv 8 \pmod{13}$ $v_{13}(t^2 + 5t + 13) = 6m + 4$	$E_1$	$(\geq 5, 4, 8)$	$13^m$	$\text{IV}^*$	13	1
	$E_{13}$	$(\geq 4, 4, 8)$	$13^m$	$\text{IV}^*$	13	1
$v_{13}(t) = 0$ $t \equiv 8 \pmod{13}$ $v_{13}(t^2 + 5t + 13) = 6m + 5$	$E_1$	$(\geq 6, 5, 10)$	$13^m$	$\text{II}^*$	13	1
	$E_{13}$	$(\geq 5, 5, 10)$	$13^m$	$\text{II}^*$	13	1
$v_{13}(t) = -m < 0$	$E_1$	$(0, 0, 13m)$	$13^{-2m}$	$\text{I}_{13m}$	1	1
	$E_{13}$	$(0, 0, m)$	$13^{-2m}$	$\text{I}_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{13}$	



Table 17:  $L_2(13)$  data for  $p=2$ 

$L_2(13)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m \geq 2$	$E_1$	$(0, 0, m)$	1	$I_m$	1	$2^{-1}$	$2^{-1}$
	$E_{13}$	$(0, 0, 13m)$	1	$I_{13m}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 1$	$E_1$	$(4, 6, 13)$	$2^{-1}$	$I_5^*$	1	1	2
	$E_{13}$	$(4, 6, 25)$	$2^{-1}$	$I_{17}^*$	1	1	2
$v_2(t) = 0$ $t \equiv 1 (4)$	$E_1$	$(6, 6, 6)$	1	II	1	$1^*$ or $2^*$	1
	$E_{13}$	$(6, 6, 6)$	1	II	1	$1^*$ or $2^*$	1
$v_2(t) = 0$ $t \equiv 3 (4)$	$E_1$	$(5, 8, 9)$	1	III	1	1	1
	$E_{13}$	$(5, 8, 9)$	1	III	1	1	1
$v_2(t) = -1$	$E_1$	$(0, 0, 13)$	$2^{-2}$	$I_{13}$	1	$2^{-1}$	$2^{-1}$
	$E_{13}$	$(0, 0, 1)$	$2^{-2}$	$I_1$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = -m \leq -2$	$E_1$	$(4, 6, 13m + 12)$	$2^{-(2m+1)}$	$I_{13m+4}^*$	1	1	2
	$E_{13}$	$(4, 6, m + 12)$	$2^{-(2m+1)}$	$I_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

**Remark ( $1^*$  or  $2^*$ ):** If  $t \equiv 1 (4)$  and  $d \equiv 2 (4)$ , then the value  $u_2(\mathcal{E}^d)$  is given by:

- if  $t \equiv 1 (8)$ , then  $u_2(\mathcal{E}^d) = \begin{cases} 1 & \text{if } d \equiv 2 (8) \\ 2 & \text{if } d \equiv -2 (8); \end{cases}$
- if  $t \equiv 5 (8)$ , then  $u_2(\mathcal{E}^d) = \begin{cases} 2 & \text{if } d \equiv 2 (8) \\ 1 & \text{if } d \equiv -2 (8). \end{cases}$

### 6.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$
$v_{13}(t) > 0$	$(1 : 13)$	$(1 : 1)$
$v_{13}(t) \leq 0$	$(1 : 1)$	$(1 : 1)$

The contents of this table are the ingredients to prove the following result:

**Proposition 6.** *Let  $E_1 \xrightarrow{13} E_{13}$  be a  $\mathbf{Q}$ -isogeny graph of type  $L_2(13)$  corresponding to a given  $t$  in  $\mathbf{Q}^*$  with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{13} E_{13}^d$  is given by:*

$L_2(13)$	<i>twisted isogeny graph</i>	<i>Prob</i>
$v_{13}(t) > 0$	$E_1^d \leftarrow \textcircled{E_{13}^d}$	1
$v_{13}(t) \leq 0$	$\textcircled{E_1^d} \rightarrow E_{13}^d$	1

*The column Prob gives the probability of the circled twisted curve to be the Faltings curve.*

## 7 Type $L_2(17)$

### 7.1 Settings

#### Graph

The isogeny graphs of type  $L_2(17)$  are given by two 17-isogenous elliptic curves:

$$E_1 \xrightarrow{17} E_{17}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(17)$  parametrize isogeny graphs of type  $L_2(17)$ . The modular curve  $X_0(17)$  is an elliptic curve of rank 0 over the rationals. Its rational points are: two rational cusps and two non-cuspidal non-CM points  $\tau = 0.5 + 1.41899355973190...i$  and  $\tau' = 0.5 + 0.0834702093959941...i \in \mathbb{H}$ .

#### $j$ -invariants

The corresponding  $j$ -invariants of  $\tau$  and  $\tau'$  are:

$$j(\tau) = \frac{-17 \cdot 373^3}{2^{17}}, \quad j(\tau') = j(17\tau) = \frac{-17^2 \cdot 101^3}{2}.$$

#### Signatures

We choose minimal Weierstrass equations and the isogeny graph is normalized.

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_1$	$y^2 + xy = x^3 + x^2 - 660x - 7600$	$\frac{-17 \cdot 373^3}{2^{17}}$	14450.o2
$E_{17}$	$y^2 + xy = x^3 + x^2 - 878710x + 316677750$	$\frac{-17^2 \cdot 101^3}{2}$	14450.o1

Their signatures are:

$E$	$E_1$	$E_{17}$
$c_4(E)$	$5 \cdot 17 \cdot 373$	$5 \cdot 17^4 \cdot 101$
$c_6(E)$	$5^2 \cdot 17 \cdot 14891$	$-5^2 \cdot 17^5 \cdot 7717$
$\Delta(E)$	$-2^{17} \cdot 5^3 \cdot 17^2$	$-2 \cdot 5^3 \cdot 17^{10}$

One checks that the Faltings curve (circled) in the graph is

$$\boxed{\textcircled{E_1} \longrightarrow E_{17}}$$

Note that any  $\mathbb{Q}$ -isogeny class of elliptic curves of type  $L_2(17)$  is obtained by quadratic twist from

$$E_1 \xrightarrow{17} E_{17}.$$

## 7.2 Kodaira symbols, minimal models, and Pal values

There are three bad reduction primes involved in the conductors of these elliptic curves:  $p = 2, 5$ , and  $17$ .

$p = 2$					
$E$	$\text{sig}_2(\mathcal{E})$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$E_1$	$(0, 0, 17)$	$I_{17}$	1	$1/2$	$1/2$
$E_{17}$	$(0, 0, 1)$	$I_1$	1	$1/2$	$1/2$
			$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
			$d \pmod{4}$		

$p = 5$			
$E$	$\text{sig}_5(\mathcal{E})$	$K_5(E)$	$u_5(\mathcal{E}^d)$
$E_1$	$(1, 2, 3)$	III	1
$E_{17}$	$(1, 2, 3)$	III	1

$p = 17$				
$E$	$\text{sig}_{17}(\mathcal{E})$	$K_{17}(E)$	$u_{17}(\mathcal{E}^d)$	
$E_1$	$(1, 1, 2)$	II	1	1
$E_{17}$	$(4, 5, 10)$	II*	17	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{17}$	

## 7.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1)$	$d \not\equiv 0 \pmod{17}$
$(1 : 17)$	$d \equiv 0 \pmod{17}$

This table is the ingredient to prove the following result:

**Proposition 7.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{17} E_{17}^d$  is given by:*

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$\textcircled{E_1^d} \longrightarrow E_{17}^d$	$d \not\equiv 0 \pmod{17}$	$17/18$
$E_1^d \longrightarrow \textcircled{E_{17}^d}$	$d \equiv 0 \pmod{17}$	$1/18$

*The column Prob gives the probability of the circled twisted curve to be the Faltings curve.*

## 8 Type $L_2(19)$

### 8.1 Settings

#### Graph

The isogeny graphs of type  $L_2(19)$  are given by two 19-isogenous elliptic curves:

$$E_1 \xrightarrow{19} E_{19}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(19)$  parametrize isogeny graphs of type  $L_2(19)$ . The modular curve  $X_0(19)$  is an elliptic curve of rank 0 over the rationals. Its rational points are: two cusps and one CM point that corresponds to  $\tau = \frac{1}{2} + \frac{\sqrt{-19}}{2 \cdot 19} \in \mathbb{H}$ .

#### $j$ -invariants

The  $j$ -invariant is:

$$j(\tau) = j(19\tau) = -2^{15} \cdot 3^3.$$

#### Signatures

We can choose minimal Weierstrass equations and the isogeny graph is normalized.

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_1$	$y^2 + y = x^3 - 38x + 90$	$-2^{15} \cdot 3^3$	<b>361.a2</b>
$E_{19}$	$y^2 + y = x^3 - 13718x - 619025$	$-2^{15} \cdot 3^3$	<b>361.a1</b>

Their signatures are:

$E$	$E_1$	$E_{19}$
$c_4(E)$	$2^5 \cdot 3 \cdot 19$	$2^5 \cdot 3 \cdot 19^3$
$c_6(E)$	$-2^3 \cdot 3^3 \cdot 19^2$	$2^3 \cdot 3^3 \cdot 19^5$
$\Delta(E)$	$-19^3$	$-19^9$

One checks that the Faltings curve (circled) in the graph is

$$\boxed{\textcircled{E_1} \longrightarrow E_{19}}$$

The  $\mathbb{Q}$ -isogeny classes of elliptic curves of type  $L_2(19)$  are obtained by quadratic twist:

$$E_1^d \xrightarrow{19} E_{19}^d.$$

## 8.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime for these elliptic curves; that is  $p = 19$ .

$p = 19$				
$E$	$\text{sig}_{19}(\mathcal{E})$	$K_{19}(E)$	$u_{19}(\mathcal{E}^d)$	
$E_1$	$(1, 2, 3)$	III	1	1
$E_{19}$	$(3, 5, 9)$	III*	19	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{19}$	

## 8.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1)$	$d \not\equiv 0 \pmod{19}$
$(1 : 19)$	$d \equiv 0 \pmod{19}$

This table is the ingredient to prove the following result:

**Proposition 8.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{19} E_{19}^d$  is given by:*

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$\textcircled{E_1^d} \longrightarrow E_{19}^d$	$d \not\equiv 0 \pmod{19}$	19/20
$E_1^d \longrightarrow \textcircled{E_{19}^d}$	$d \equiv 0 \pmod{19}$	1/20

*The column Prob gives the probability of the circled twisted curve to be the Faltings curve.*

## 9 Type $L_2(37)$

### 9.1 Settings

#### Graph

The isogeny graphs of type  $L_2(37)$  are given by two 37-isogenous elliptic curves:

$$E_1 \xrightarrow{37} E_{37}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(37)$  parametrize isogeny graphs of type  $L_2(37)$ . The modular curve  $X_0(37)$  has genus 2. Its rational points are: two cusps and two non-CM points corresponding to  $\tau = 0.5 + 0.170470198193806...i$  and  $\tau' = 0.5 + 6.30739733317082...i \in \mathbb{H}$ .

#### $j$ -invariants

The corresponding  $j$ -invariants of  $\tau$  and  $\tau'$  are:

$$j(\tau) = -7 \cdot 11^3, \quad j(\tau') = j(37\tau) = -7 \cdot 137^3 \cdot 2083^3.$$

#### Signatures

We choose minimal Weierstrass equations and get the normalized isogeny.

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_1$	$y^2 + xy + y = x^3 + x^2 - 8x + 6$	$-7 \cdot 11^3$	1225.b2
$E_{37}$	$y^2 + xy + y = x^3 + x^2 - 208083x - 36621194$	$-7 \cdot 137^3 \cdot 2083^3$	1225.b1

Their signatures are:

$E$	$E_1$	$E_{37}$
$c_4(E)$	$5 \cdot 7 \cdot 11$	$5 \cdot 7 \cdot 137 \cdot 2083$
$c_6(E)$	$-5^2 \cdot 7 \cdot 47$	$5^2 \cdot 7 \cdot 11 \cdot 1433 \cdot 11443$
$\Delta(E)$	$-5^3 \cdot 7^2$	$-5^3 \cdot 7^2$

We have that the Faltings curve (circled) in the graph is

$$\boxed{\textcircled{E_1} \longrightarrow E_{37}}$$

Note that any  $\mathbb{Q}$ -isogeny class of elliptic curves of type  $L_2(37)$  can be obtained by quadratic twist:

$$E_1^d \xrightarrow{37} E_{37}^d.$$



## 9.2 Kodaira symbols, minimal models, and Pal values

There are two primes of bad reduction for these elliptic curves:  $p = 5$  and  $7$ .

$p = 5$			
$E$	$\text{sig}_5(\mathcal{E})$	$K_5(E)$	$u_5(\mathcal{E}^d)$
$E_1$	$(1, 2, 3)$	III	1
$E_{37}$	$(1, 2, 3)$	III	1

$p = 7$			
$E$	$\text{sig}_7(\mathcal{E})$	$K_7(E)$	$u_7(\mathcal{E}^d)$
$E_1$	$(1, 1, 2)$	II	1
$E_{37}$	$(1, 1, 2)$	II	1

## 9.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$
$(1 : 1)$

This table is the ingredient to prove the following result:

**Proposition 9.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{37} E_{37}^d$  is given by:*

<i>twisted isogeny graph</i>	<i>Prob</i>
$\bigcirc E_1^d \longrightarrow E_{37}^d$	1

*The column Prob gives the probability of the circled twisted curve to be the Faltings curve.*

## 10 Type $L_2(43)$

### 10.1 Settings

#### Graph

The isogeny graphs of type  $L_2(43)$  are given by two 43-isogenous elliptic curves:

$$E_1 \xrightarrow{43} E_{43}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(43)$  parametrize isogeny graphs of type  $L_2(43)$ . The modular curve  $X_0(43)$  has genus 3. Its rational points are: two cusps and one CM point given by  $\tau = \frac{1}{2} + \frac{\sqrt{-43}}{2 \cdot 43} \in \mathbb{H}$ .

#### $j$ -invariants

The corresponding  $j$ -invariant is:

$$j(\tau) = j(43\tau) = -2^{18} \cdot 3^3 \cdot 5^3.$$

#### Signatures

We can choose minimal Weierstrass equations and get the normalized isogeny graph.

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_1$	$y^2 + y = x^3 - 860x + 9707$	$-2^{18} \cdot 3^3 \cdot 5^3$	1849.b2
$E_{43}$	$y^2 + y = x^3 - 1590140x - 771794326$	$-2^{18} \cdot 3^3 \cdot 5^3$	1849.b1

Their signatures are:

$E$	$E_1$	$E_{43}$
$c_4(E)$	$2^6 \cdot 3 \cdot 5 \cdot 43$	$2^6 \cdot 3 \cdot 5 \cdot 43^3$
$c_6(E)$	$-2^3 \cdot 3^4 \cdot 7 \cdot 43^2$	$2^3 \cdot 3^4 \cdot 7 \cdot 43^5$
$\Delta(E)$	$-43^3$	$-43^9$

One checks that the Faltings curve (circled) in the graph is

$$\boxed{\textcircled{E_1} \longrightarrow E_{43}}$$

Note that any  $\mathbb{Q}$ -isogeny class of type  $L_2(43)$  is obtained by quadratic twists:

$$E_1^d \xrightarrow{43} E_{43}^d.$$

## 10.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime for these elliptic curves; that is,  $p = 43$ .

$p = 43$				
$E$	$\text{sig}_{43}(\mathcal{E})$	$K_{43}(E)$	$u_{43}(\mathcal{E}^d)$	
$E_1$	$(1, 2, 3)$	III	1	1
$E_{43}$	$(3, 5, 9)$	III*	43	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{43}$	

## 10.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1)$	$d \not\equiv 0 \pmod{43}$
$(1 : 43)$	$d \equiv 0 \pmod{43}$

This table is the ingredient to prove the following result:

**Proposition 10.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{43} E_{43}^d$  is given by:*

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$\textcircled{E_1^d} \longrightarrow E_{43}^d$	$d \not\equiv 0 \pmod{43}$	43/44
$E_1^d \longrightarrow \textcircled{E_{43}^d}$	$d \equiv 0 \pmod{43}$	1/44

*The column Prob gives the probability of the circled twisted curve to be the Faltings curve.*

## 11 Type $L_2(67)$

### 11.1 Settings

#### Graph

The isogeny graphs of type  $L_2(67)$  are given by two 67-isogenous elliptic curves:

$$E_1 \xrightarrow{67} E_{67}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(67)$  parametrize isogeny graphs of type  $L_2(67)$ . The modular curve  $X_0(67)$  has genus 5. Its rational points are: two cusps and one CM point associated with  $\tau = \frac{1}{2} + \frac{\sqrt{-67}}{2 \cdot 67} \in \mathbb{H}$ .

#### $j$ -invariant

The corresponding  $j$ -invariant is:

$$j(\tau) = j(67\tau) = -2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3.$$

#### Signatures

We choose minimal Weierstrass equations and the normalized isogeny graph is .

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_1$	$y^2 + y = x^3 - 7370x + 243528$	$-2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3$	4489.b2
$E_{67}$	$y^2 + y = x^3 - 33083930x - 73244287055$	$-2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3$	4489.b1

Their signatures are:

$E$	$E_1$	$E_{67}$
$c_4(E)$	$2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 67$	$2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 67^3$
$c_6(E)$	$-2^3 \cdot 3^3 \cdot 7 \cdot 31 \cdot 67^2$	$2^3 \cdot 3^3 \cdot 7 \cdot 31 \cdot 67^5$
$\Delta(E)$	$-67^3$	$-67^9$

One checks that the Faltings curve (circled) in the graph is

$$\boxed{\textcircled{E_1} \longrightarrow E_{67}}$$

Any other  $\mathbb{Q}$ -isogeny class of type  $L_2(67)$  is obtained by quadratic twist:

$$E_1^d \xrightarrow{67} E_{67}^d.$$

## 11.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime for these elliptic curves; that is,  $p = 67$ .

$p = 67$				
$E$	$\text{sig}_{67}(\mathcal{E})$	$K_{67}(E)$	$u_{67}(\mathcal{E}^d)$	
$E_1$	$(1, 2, 3)$	III	1	1
$E_{67}$	$(3, 5, 9)$	III*	67	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{67}$	

## 11.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1)$	$d \not\equiv 0 \pmod{67}$
$(1 : 67)$	$d \equiv 0 \pmod{67}$

This table is the ingredient to prove the following result:

**Proposition 11.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{67} E_{67}^d$  is given by:*

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$\textcircled{E_1^d} \longrightarrow E_{67}^d$	$d \not\equiv 0 \pmod{67}$	67/68
$E_1^d \longrightarrow \textcircled{E_{67}^d}$	$d \equiv 0 \pmod{67}$	1/68

*The column Prob gives the probability of the circled twisted curve to be the Faltings curve.*

## 12 Type $L_2(163)$

### 12.1 Settings

#### Graph

The isogeny graphs of type  $L_2(163)$  are given by two 163-isogenous elliptic curves:

$$E_1 \xrightarrow{163} E_{163}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(163)$  parametrize isogeny graphs of type  $L_2(163)$ . The modular curve  $X_0(163)$  has genus 13. Its rational points are: the two cusps and one CM point corresponding to  $\tau = \frac{1}{2} + \frac{\sqrt{-163}}{2 \cdot 163} \in \mathbb{H}$ .

#### $j$ -invariants

The corresponding  $j$ -invariant of is:

$$j(\tau) = j(163\tau) = -2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3.$$

#### Signatures

We choose minimal Weierstrass equations and the isogeny graph is normalized.

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_1$	$y^2 + y = x^3 - 2174420x + 1234136692$	$-2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$	26569.a2
$E_{163}$	$y^2 + y = x^3 - 57772164980x - 5344733777551611$	$-2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$	26569.a1

Their signatures are:

$E$	$E_1$	$E_{163}$
$c_4(E)$	$2^6 \cdot 3 \cdot 5 \cdot 23 \cdot 29 \cdot 163$	$2^6 \cdot 3 \cdot 5 \cdot 23 \cdot 29 \cdot 163^3$
$c_6(E)$	$-2^3 \cdot 3^3 \cdot 7 \cdot 11 \cdot 19 \cdot 127 \cdot 163^2$	$2^3 \cdot 3^3 \cdot 7 \cdot 11 \cdot 19 \cdot 127 \cdot 163^5$
$\Delta(E)$	$-163^3$	$-163^9$

One checks that the Faltings curve (circled) in the graph is

$$\boxed{\textcircled{E_1} \longrightarrow E_{163}}$$

Any other  $\mathbb{Q}$ -isogeny class of type  $L_2(163)$  is obtained by quadratic twist

$$E_1^d \xrightarrow{163} E_{163}^d.$$

## 12.2 Kodaira symbols, minimal models, and Pal values

The only bad reduction prime for these elliptic curves is  $p = 163$ .

$p = 163$				
$E$	$\text{sig}_{163}(\mathcal{E})$	$K_{163}(E)$	$u_{163}(\mathcal{E}^d)$	
$E_1$	$(1, 2, 3)$	III	1	1
$E_{163}$	$(3, 5, 9)$	III*	163	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{163}$	

## 12.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1)$	$d \not\equiv 0 \pmod{163}$
$(1 : 163)$	$d \equiv 0 \pmod{163}$

This table is the main ingredient to prove the following result:

**Proposition 12.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{163} E_{163}^d$  is given by:*

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$\textcircled{E_1^d} \longrightarrow E_{163}^d$	$d \not\equiv 0 \pmod{163}$	163/164
$E_1^d \longleftarrow \textcircled{E_{163}^d}$	$d \equiv 0 \pmod{163}$	1/164

The column *Prob* gives the probability of the circled twisted curve to be the Faltings curve.

## 13 Type $L_3(9)$

### 13.1 Setting

#### Graph

The isogeny graphs of type  $L_3(9)$  are given by three isogenous elliptic curves:

$$E_1 \xrightarrow{3} E_3 \xrightarrow{3} E_9.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(9)$  parametrize isogeny graphs of type  $L_3(9)$ . The curve  $X_0(9)$  has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 3^3 \left( \frac{\eta(9\tau)}{\eta(\tau)} \right)^3.$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$\begin{aligned} j(E_1) = j(\tau) &= \frac{(t+3)^3(t^3+9t^2+27t+3)^3}{t(t^2+9t+27)} \\ j(E_3) = j(3\tau) &= \frac{(t+3)^3(t+9)^3}{t^3(t^2+9t+27)^3} \\ j(E_9) = j(9\tau) &= \frac{(t+9)^3(t^3+243t^2+2187t+6561)^3}{t^9(t^2+9t+27)}. \end{aligned}$$

**Signatures** We can (and do) choose Weierstrass equations for  $(E_1, E_3, E_9)$  in such a way that the isogeny graph is normalized. Their signatures are:

$L_3(9)$	
$c_4(E_1)$	$(t+3)(t^3+9t^2+27t+3)$
$c_6(E_1)$	$t^6+18t^5+135t^4+504t^3+891t^2+486t-27$
$\Delta(E_1)$	$t(t^2+9t+27)$
$c_4(E_3)$	$(t+3)(t+9)(t^2+27)$
$c_6(E_3)$	$(t^2-27)(t^4+18t^3+162t^2+486t+729)$
$\Delta(E_3)$	$t^3(t^2+9t+27)^3$
$c_4(E_9)$	$(t+9)(t^3+243t^2+2187t+6561)$
$c_6(E_9)$	$t^6-486t^5-24057t^4-367416t^3-2657205t^2-9565938t-14348907$
$\Delta(E_9)$	$t^9(t^2+9t+27)$



## Automorphisms

The subgroup of  $\text{Aut } X_0(9)$  that fixes the set of vertices of the graph is generated by the Fricke involution of  $X_0(9)$ , given by  $W_9(t) = 3^3/t$ . The involution  $W_9$  acts on the isogeny graphs of type  $L_3(9)$  as:

$$W_9(E_1 \xrightarrow{3} E_3 \xrightarrow{3} E_9) = E_9^{-3} \xrightarrow{3} E_3^{-3} \xrightarrow{3} E_1^{-3}.$$

### 13.2 Kodaira symbols, minimal models, and Pal values

Table 18:  $L_3(9)$  data for  $p \neq 2, 3$

$L_3(9)$	$p \neq 2, 3$				
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$
$v_p(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1
	$E_9$	$(0, 0, 9m)$	1	$I_{9m}$	1
$v_p(t) = 0$ $m = v_p(t^2 + 9t + 27) \geq 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1
	$E_9$	$(0, 0, m)$	1	$I_m$	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 9m)$	$p^{-m}$	$I_{9m}$	1
	$E_3$	$(0, 0, 3m)$	$p^{-m}$	$I_{3m}$	1
	$E_9$	$(0, 0, m)$	$p^{-m}$	$I_m$	1

Table 19:  $L_3(9)$  data for  $p=3$

$L_3(9)$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = m \geq 3$	$E_1$	$(2, 3, m + 3)$	1	$I_{m-3}^*$	3	1
	$E_3$	$(2, 3, 3m - 3)$	3	$I_{3(m-3)}^*$	3	1
	$E_9$	$(2, 3, 9m - 21)$	$3^2$	$I_{9(m-3)}^*$	3	1
$v_3(t) = 2$	$E_1$	$(2, 3, 5)$	1	IV	1	1
	$E_3$	$(\geq 2, 3, 3)$	3	II	1	1
	$E_9$	$(\geq 4, 6, 9)$	3	IV*	3	1
$v_3(t) = 1$	$E_1$	$(\geq 2, 3, 3)$	1	II	1	1
	$E_3$	$(\geq 4, 6, 9)$	1	IV*	3	1
	$E_9$	$(4, 6, 11)$	1	II*	3	1
$v_3(t) = -m \leq 0$	$E_1$	$(0, 0, 9m)$	$3^{-m}$	$I_{9m}$	1	1
	$E_3$	$(0, 0, 3m)$	$3^{-m}$	$I_{3m}$	1	1
	$E_9$	$(0, 0, m)$	$3^{-m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	

Table 20:  $L_3(9)$  data for  $p=2$ 

$L_3(9)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 0$	$E_1$	$(4, 6, m + 12)$	$2^{-1}$	$I_{m+4}^*$	1	1	2
	$E_3$	$(4, 6, 3m + 12)$	$2^{-1}$	$I_{3m+4}^*$	1	1	2
	$E_9$	$(4, 6, 9m + 12)$	$2^{-1}$	$I_{9m+4}^*$	1	1	2
$v_2(t) = 0$	$E_1$	$(\geq 8, 9, 12)$	$2^{-1}$	$\Pi^*$	1	2	2
	$E_3$	$(\geq 8, 9, 12)$	$2^{-1}$	$\Pi^*$	1	2	2
	$E_9$	$(\geq 8, 9, 12)$	$2^{-1}$	$\Pi^*$	1	2	2
$v_2(t) = -m < 0$	$E_1$	$(4, 6, 9m + 12)$	$2^{-m-1}$	$I_{9m+4}^*$	1	1	2
	$E_3$	$(4, 6, 3m + 12)$	$2^{-m-1}$	$I_{3m+4}^*$	1	1	2
	$E_9$	$(4, 6, m + 12)$	$2^{-m-1}$	$I_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

### 13.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$	$d$
$v_3(t) \leq 0$	$(1 : 1 : 1)$	$(1 : 1 : 1)$	
$v_3(t) = 1$	$(1 : 1 : 1)$	$(1 : 1 : 1)$	$d \not\equiv 0 (3)$
		$(1 : 3 : 3)$	$d \equiv 0 (3)$
$v_3(t) = 2$	$(1 : 3 : 3)$	$(1 : 1 : 1)$	$d \not\equiv 0 (3)$
		$(1 : 1 : 3)$	$d \equiv 0 (3)$
$v_3(t) \geq 3$	$(1 : 3 : 3^2)$	$(1 : 1 : 1)$	

The contents of the above table are the ingredients to prove the following result:

**Proposition 13.** *Let  $E_1 \xrightarrow{3} E_3 \xrightarrow{3} E_9$  be a  $\mathbf{Q}$ -isogeny graph of type  $L_3(9)$  corresponding to a given  $t$  in  $\mathbf{Q}^*$  with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{3} E_3^d \xrightarrow{3} E_9^d$  is given by:*

$L_3(9)$	twisted isogeny graph	$d$	Prob
$v_3(t) \leq 0$	$\textcircled{E_1^d} \longrightarrow E_3^d \longrightarrow E_9^d$		1
$v_3(t) = 1$	$\textcircled{E_1^d} \longrightarrow E_3^d \longrightarrow E_9^d$	$d \not\equiv 0 (3)$	3/4
	$E_1^d \longleftarrow \textcircled{E_3^d} \longrightarrow E_9^d$	$d \equiv 0 (3)$	1/4
$v_3(t) = 2$	$E_1^d \longleftarrow \textcircled{E_3^d} \longrightarrow E_9^d$	$d \not\equiv 0 (3)$	3/4
	$E_1^d \longleftarrow E_3^d \longleftarrow \textcircled{E_9^d}$	$d \equiv 0 (3)$	1/4
$v_3(t) \geq 3$	$E_1^d \longleftarrow E_3^d \longleftarrow \textcircled{E_9^d}$		1

The column Prob gives the probability of the circled twisted curve to be the Faltings curve.

## 14 Type $L_3(25)$

### 14.1 Settings

#### Graph

The isogeny graphs of type  $L_3(25)$  are given by three isogenous elliptic curves:

$$E_1 \xrightarrow{5} E_5 \xrightarrow{5} E_{25}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(25)$  parametrize isogeny graphs of type  $L_3(25)$ . The curve  $X_0(25)$  has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 5 \left( \frac{\eta(25\tau)}{\eta(\tau)} \right).$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$j(E_1) = j(\tau) = \frac{(t^{10} + 10t^9 + 55t^8 + 200t^7 + 525t^6 + 1010t^5 + 1425t^4 + 1400t^3 + 875t^2 + 250t + 5)^3}{t(t^4 + 5t^3 + 15t^2 + 25t + 25)}$$

$$j(E_5) = j(5\tau) = \frac{(t^2 + 5t + 5)^3 (t^4 + 5t^2 + 25)^3 (t^4 + 5t^3 + 20t^2 + 25t + 25)^3}{t^5 (t^4 + 5t^3 + 15t^2 + 25t + 25)^5}$$

$$j(E_{25}) = j(25\tau) = \frac{(t^{10} + 250t^9 + 4375t^8 + 35000t^7 + 178125t^6 + 631250t^5 + 1640625t^4 + 3125000t^3 + 4296875t^2 + 3906250t + 1953125)^3}{t^{25} (t^4 + 5t^3 + 15t^2 + 25t + 25)}.$$

#### Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_5, E_{25})$  in such a way that the isogeny graph is normalized. Their signatures are:

$L_3(25)$	
$c_4(E_1)$	$(t^2 + 2t + 5)(t^{10} + 10t^9 + 55t^8 + 200t^7 + 525t^6 + 1010t^5 + 1425t^4 + 1400t^3 + 875t^2 + 250t + 5)$
$c_6(E_1)$	$(t^2 + 2t + 5)^2(t^4 + 4t^3 + 9t^2 + 10t + 5)(t^{10} + 10t^9 + 55t^8 + 200t^7 + 525t^6 + 1004t^5 + 1310t^4 + 1395t^3 + 725t^2 + 100t - 1)$
$\Delta(E_1)$	$t(t^2 + 2t + 5)^3(t^4 + 5t^3 + 15t^2 + 25t + 25)$
$c_4(E_5)$	$(t^2 + 2t + 5)(t^2 + 5t + 5)(t^4 + 5t^2 + 25)(t^4 + 5t^3 + 20t^2 + 25t + 25)$
$c_6(E_5)$	$(t^2 - 5)(t^2 + 2t + 5)^2(t^4 + 15t^2 + 25)(t^4 + 4t^3 + 9t^2 + 10t + 5)(t^4 + 10t^3 + 45t^2 + 100t + 125)$
$\Delta(E_5)$	$t^5(t^2 + 2t + 5)^3(t^4 + 5t^3 + 15t^2 + 25t + 25)^5$
$c_4(E_{25})$	$(t^2 + 2t + 5)(t^{10} + 250t^9 + 4375t^8 + 35000t^7 + 178125t^6 + 631250t^5 + 1640625t^4 + 3125000t^3 + 4296875t^2 + 3906250t + 1953125)$
$c_6(E_{25})$	$(t^2 + 2t + 5)^2(t^4 + 10t^3 + 45t^2 + 100t + 125)(t^{10} - 500t^9 - 18125t^8 - 163750t^7 - 871875t^6 - 3137500t^5 - 8203125t^4 - 15625000t^3 - 21484375t^2 - 19531250t - 9765625)$
$\Delta(E_{25})$	$t^{25}(t^2 + 2t + 5)^3(t^4 + 5t^3 + 15t^2 + 25t + 25)$

## Automorphisms

The subgroup of  $\text{Aut } X_0(25)$  that fixes the set of vertices of the graph is generated by the Fricke involution of  $X_0(25)$ , given by  $W_{25}(t) = 5/t$ . The involution  $W_{25}$  acts on the isogeny graphs of type  $L_3(25)$  as:

$$W_{25}(E_1 \xrightarrow{5} E_5 \xrightarrow{5} E_{25}) = E_{25}^{-5} \xrightarrow{5} E_5^{-5} \xrightarrow{5} E_1^{-5}.$$

## 14.2 Kodaira symbols, minimal models, and Pal values

Table 21:  $L_3(25)$  data for  $p \neq 2, 5$

$L_3(25)$	$p \neq 2, 3, 5$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_5$	$(0, 0, 5m)$	1	$I_{5m}$	1	1
	$E_{25}$	$(0, 0, 25m)$	1	$I_{25m}$	1	1
$v_p(t) = 0$ $v_p(t^2 + 2t + 5) = 4m$	$E_1$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
	$E_5$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
	$E_{25}$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
$v_p(t) = 0$ $v_p(t^2 + 2t + 5) = 4m + 1$	$E_1$	$(1, 2 + 2m, 3)$	$p^m$	III	1	1
	$E_5$	$(1, 2 + 2m, 3)$	$p^m$	III	1	1
	$E_{25}$	$(1, 2 + 2m, 3)$	$p^m$	III	1	1
$v_p(t) = 0$ $v_p(t^2 + 2t + 5) = 4m + 2$	$E_1$	$(2, 4 + 2m, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_5$	$(2, 4 + 2m, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_{25}$	$(2, 4 + 2m, 6)$	$p^m$	$I_0^*$	$p$	1
$v_p(t) = 0$ $v_p(t^2 + 2t + 5) = 4m + 3$	$E_1$	$(3, 6 + 2m, 9)$	$p^m$	$\text{III}^*$	$p$	1
	$E_5$	$(3, 6 + 2m, 9)$	$p^m$	$\text{III}^*$	$p$	1
	$E_{25}$	$(3, 6 + 2m, 9)$	$p^m$	$\text{III}^*$	$p$	1
$v_p(t) = 0$ $m = v_p(t^4 + 5t^3 + 15t^2 + 25t + 25) \geq 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_5$	$(0, 0, 5m)$	1	$I_{5m}$	1	1
	$E_{25}$	$(0, 0, m)$	1	$I_m$	1	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 25m)$	$p^{-3m}$	$I_{25m}$	1	1
	$E_5$	$(0, 0, 5m)$	$p^{-3m}$	$I_{5m}$	1	1
	$E_{25}$	$(0, 0, m)$	$p^{-3m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 22:  $L_3(25)$  data for  $p = 5$ 

$L_3(25)$	$p = 5$					
$t$	$E$	$\text{sig}_5(\mathcal{E})$	$u_5(E)$	$K_5(E)$	$u_5(\mathcal{E}^d)$	
$v_5(t) = m > 1$	$E_1$	$(2, 3, m + 5)$	1	$I_{m-1}^*$	5	1
	$E_1$	$(2, 3, 5m + 1)$	5	$I_{5(m-1)}^*$	5	1
	$E_1$	$(2, 3, 25m - 19)$	$5^2$	$I_{25(m-1)}^*$	5	1
$v_5(t) = 1$ $v_5(t^2 + 2t + 5) = 4m$	$E_1$	$(1, 1 + 2m, 3)$	$5^m$	III	1	1
	$E_5$	$(1, \geq 1 + 2m, 3)$	$5^{m+1}$	III	1	1
	$E_{25}$	$(1, \geq 1 + 2m, 3)$	$5^{m+2}$	III	1	1
$v_5(t) = 1$ $v_5(t^2 + 2t + 5) = 4m + 1$	$E_1$	$(2, 3 + 2m, 6)$	$5^m$	$I_0^*$	5	1
	$E_5$	$(2, \geq 3 + 2m, 6)$	$5^{m+1}$	$I_0^*$	5	1
	$E_{25}$	$(2, \geq 3 + 2m, 6)$	$5^{m+2}$	$I_0^*$	5	1
$v_5(t) = 1$ $v_5(t^2 + 2t + 5) = 4m + 2$	$E_1$	$(3, 5 + 2m, 9)$	$5^m$	III*	5	1
	$E_5$	$(3, \geq 5 + 2m, 9)$	$5^{m+1}$	III*	5	1
	$E_{25}$	$(3, \geq 5 + 2m, 9)$	$5^{m+2}$	III*	5	1
$v_5(t) = 1$ $m = v_5(t^2 + 2t + 5) = 4m + 3$	$E_1$	$(0, 1 + 2m, 0)$	$5^{m+1}$	$I_0$	1	1
	$E_5$	$(0, \geq 1 + 2m, 0)$	$5^{m+2}$	$I_0$	1	1
	$E_{25}$	$(0, \geq 1 + 2m, 0)$	$5^{m+3}$	$I_0$	1	1
$v_5(t) = 0$ $v_5(t^2 + 2t + 5) = 4m$	$E_1$	$(0, \geq 1, 0)$	$5^m$	$I_0$	1	1
	$E_5$	$(0, \geq 1, 0)$	$5^m$	$I_0$	1	1
	$E_{25}$	$(0, \geq 1, 0)$	$5^m$	$I_0$	1	1
$v_5(t) = 0$ $v_5(t^2 + 2t + 5) = 4m + 1$	$E_1$	$(1, \geq 2, 3)$	$5^m$	III	1	1
	$E_5$	$(1, \geq 2, 3)$	$5^m$	III	1	1
	$E_{25}$	$(1, \geq 2, 3)$	$5^m$	III	1	1
$v_5(t) = 0$ $v_5(t^2 + 2t + 5) = 4m + 2$	$E_1$	$(2, \geq 4, 6)$	$5^m$	$I_0^*$	5	1
	$E_5$	$(2, \geq 4, 6)$	$5^m$	$I_0^*$	5	1
	$E_{25}$	$(2, \geq 4, 6)$	$5^m$	$I_0^*$	5	1
$v_5(t) = 0$ $v_5(t^2 + 2t + 5) = 4m + 3$	$E_1$	$(3, \geq 6, 9)$	$5^m$	III*	5	1
	$E_5$	$(3, \geq 6, 9)$	$5^m$	III*	5	1
	$E_{25}$	$(3, \geq 6, 9)$	$5^m$	III*	5	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{5}$	



Table 22:  $L_3(25)$  data for  $p = 5$  (Continued)

$L_3(25)$	$p = 5$					
$t$	$E$	$\text{sig}_5(\mathcal{E})$	$u_5(E)$	$K_5(E)$	$u_5(\mathcal{E}^d)$	
$v_5(t) = -m < 0$	$E_1$	$(0, 0, 25m)$	$5^{-3m}$	$I_{25m}$	1	1
	$E_5$	$(0, 0, 5m)$	$5^{-3m}$	$I_{5m}$	1	1
	$E_{25}$	$(0, 0, m)$	$5^{-3m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{5}$	

Table 23:  $L_3(25)$  data for  $p=2$ 

$L_3(25)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	$2^{-1}$	$2^{-1}$
	$E_5$	$(0, 0, 5m)$	1	$I_{5m}$	1	$2^{-1}$	$2^{-1}$
	$E_{25}$	$(0, 0, 25m)$	1	$I_{25m}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 0$ $t \equiv 1 \pmod{4}$	$E_1$	$(5, 8, 9)$	1	III	1	1	1
	$E_5$	$(5, 8, 9)$	1	III	1	1	1
	$E_{25}$	$(5, 8, 9)$	1	III	1	1	1
$v_2(t) = 0$ $t \equiv 3 \pmod{4}$	$E_1$	$(6, 6, 6)$	1	II	1	$1^*$ or $2^*$	1
	$E_5$	$(6, 6, 6)$	1	II	1	$1^*$ or $2^*$	1
	$E_{25}$	$(6, 6, 6)$	1	II	1	$1^*$ or $2^*$	1
$v_2(t) = -m < 0$	$E_1$	$(4, 6, 25m + 12)$	$2^{-3m-1}$	$I_{25m+4}^*$	1	1	2
	$E_5$	$(4, 6, 5m + 12)$	$2^{-3m-1}$	$I_{5m+4}^*$	1	1	2
	$E_{25}$	$(4, 6, m + 12)$	$2^{-3m-1}$	$I_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

**Remark ( $1^*$  or  $2^*$ ):** If  $t \equiv 3 \pmod{4}$  and  $d \equiv 2 \pmod{4}$ , then  $u_2(\mathcal{E}^d)$  is given by

- if  $t \equiv 3 \pmod{8}$ , then  $u_2(d) = \begin{cases} 1 & \text{if } d \equiv -2 \pmod{8} \\ 2 & \text{if } d \equiv 2 \pmod{8}; \end{cases}$
- if  $t \equiv 7 \pmod{8}$ , then  $u_2(d) = \begin{cases} 2 & \text{if } d \equiv -2 \pmod{8} \\ 1 & \text{if } d \equiv 2 \pmod{8}. \end{cases}$

### 14.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$
$v_5(t) \geq 1$	$(1 : 5 : 5^2)$	$(1 : 1 : 1)$
$v_5(t) \leq 0$	$(1 : 1 : 1)$	$(1 : 1 : 1)$

The contents of the above table are the ingredients to prove the following result:

**Proposition 14.** *Let  $E_1 \xrightarrow{5} E_5 \xrightarrow{5} E_{25}$  be a  $\mathbf{Q}$ -isogeny graph of type  $L_3(25)$  corresponding to a given  $t$  in  $\mathbf{Q}^*$  with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph  $E_1^d \xrightarrow{5} E_5^d \xrightarrow{5} E_{25}^d$  is given by:*

$L_3(25)$	twisted isogeny graph	Prob
$v_5(t) \geq 1$	$E_1^d \longleftarrow E_5^d \longleftarrow \textcircled{E_{25}^d}$	1
$v_5(t) \leq 0$	$\textcircled{E_1^d} \longrightarrow E_5^d \longrightarrow E_{25}^d$	1

The column Prob gives the probability of the circled twisted curve to be the Faltings curve.

## 15 Type $L_4$

### 15.1 Settings

#### Graph

The isogeny graphs of type  $L_4$  are given by four 3-isogenous elliptic curves:

$$E_1 \xrightarrow{3} E_3 \xrightarrow{3} E_9 \xrightarrow{3} E_{27}.$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(27)$  parametrize isogeny graphs of type  $L_4$ . The modular curve  $X_0(27)$  is elliptic of rank 0 over the rationals. Its rational points are: two cusps and one CM point  $\tau = \frac{1}{2} + 3\frac{\sqrt{3}}{2}i \in \mathbb{H}$ .

#### $j$ -invariants

The corresponding  $j$ -invariant is:

$$j(\tau) = j(27\tau) = -2^{15} \cdot 3 \cdot 5^3.$$

#### Signatures

We choose minimal Weierstrass equations and the isogeny graph is normalized:

$E$	Minimal Weierstrass equation	$j(E)$	LMFDB
$E_1$	$y^2 + y = x^3 - 30x + 63$	$-2^{15} \cdot 3 \cdot 5^3$	27.a2
$E_3$	$y^2 + y = x^3$	0	27.a4
$E_9$	$y^2 + y = x^3 - 7$	0	27.a3
$E_{27}$	$y^2 + y = x^3 - 270x - 1708$	$-2^{15} \cdot 3 \cdot 5^3$	27.a1

Their signatures are:

$E$	$E_1$	$E_3$	$E_9$	$E_{27}$
$c_4(E)$	$2^5 \cdot 3^2 \cdot 5$	0	0	$2^5 \cdot 3^4 \cdot 5$
$c_6(E)$	$-2^3 \cdot 3^3 \cdot 11 \cdot 23$	$-2^3 \cdot 3^3$	$2^3 \cdot 3^6$	$2^3 \cdot 3^6 \cdot 11 \cdot 23$
$\Delta(E)$	$-3^5$	$-3^3$	$-3^9$	$-3^{11}$

One checks that the Faltings curve (circled) in the graph is

$$E_1 \longleftarrow \bigcirc E_3 \longrightarrow E_9 \longrightarrow E_{27}.$$

Note that any  $\mathbb{Q}$ -isogeny class of type  $L_4$  can be obtained by quadratic twist from

$$E_1 \xrightarrow{3} E_3 \xrightarrow{3} E_9 \xrightarrow{3} E_{27}.$$

Note that  $E_1$ ,  $E_3$ ,  $E_9$ , and  $E_{27}$  have complex multiplication and  $E_9 = E_3^{-3}$  and  $E_{27} = E_1^{-3}$ .

## 15.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime for these elliptic curves at  $p = 3$ .

$p = 3$				
$E$	$\text{sig}_3(\mathcal{E})$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$E_1$	$(2, 3, 5)$	IV	1	1
$E_3$	$(0, 3, 3)$	II	1	1
$E_9$	$(0, 6, 9)$	IV*	3	1
$E_{27}$	$(4, 6, 11)$	II*	3	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{3}$	

## 15.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1 : 1 : 1)$	$d \not\equiv 0 \pmod{3}$
$(1 : 1 : 3 : 3)$	$d \equiv 0 \pmod{3}$

This table is the ingredient to prove the following result:

**Proposition 15.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph*

$$E_1^d \xrightarrow{3} E_3^d \xrightarrow{3} E_9^d \xrightarrow{3} E_{27}^d$$

is given by:

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$E_1 \leftarrow \textcircled{E_3} \rightarrow E_9 \rightarrow E_{27}$	$d \not\equiv 0 \pmod{3}$	3/4
$E_1 \leftarrow E_3 \leftarrow \textcircled{E_9} \rightarrow E_{27}$	$d \equiv 0 \pmod{3}$	1/4

The column *Prob* gives the probability of the circled twisted curve to be the Faltings curve.

## 16 Type $R_4(6)$

### 16.1 Settings

#### Graph

The isogeny graphs of type  $R_4(6)$  are given by four isogenous elliptic curves:

$$\begin{array}{ccc} E_1 & \xrightarrow{3} & E_3 \\ \left| 2 \right. & & \left. 2 \right| \\ E_2 & \xrightarrow{3} & E_6 . \end{array}$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(6)$  parametrize isogeny graphs of type  $R_4(6)$ . The curve  $X_0(6)$  has genus 0 and a hauptmodul for this curve is:

$$t = 2^3 3^2 \frac{\eta(2\tau)\eta(6\tau)^5}{\eta(\tau)^5\eta(3\tau)} .$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one has

$$\begin{aligned} j(E_1) = j(\tau) &= \frac{(t+6)^3(t^3+18t^2+84t+24)^3}{t(t+8)^3(t+9)^2} \\ j(E_2) = j(2\tau) &= \frac{(t+12)^3(t^3+12t^2+48t+192)^3}{t^2(t+8)^6(t+9)} \\ j(E_3) = j(3\tau) &= \frac{(t+6)^3(t^3+18t^2+324t+1944)^3}{t^3(t+8)(t+9)^6} \\ j(E_6) = j(6\tau) &= \frac{(t+12)^3(t^3+252t^2+3888t+15552)^3}{t^6(t+8)^2(t+9)^3} . \end{aligned}$$

## Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_2, E_3, E_6)$  in such a way that the isogeny graph is normalized. Their signatures are:

$R_4(6)$	
$c_4(E_1)$	$(t+6)(t^3+18t^2+84t+24)$
$c_6(E_1)$	$(t^2+12t+24)(t^4+24t^3+192t^2+504t-72)$
$\Delta(E_1)$	$t(t+8)^3(t+9)^2$
$c_4(E_2)$	$(t+12)(t^3+12t^2+48t+192)$
$c_6(E_2)$	$(t^2+12t+24)(t^4+24t^3+192t^2-4608)$
$\Delta(E_2)$	$t^2(t+8)^6(t+9)$
$c_4(E_3)$	$(t+6)(t^3+18t^2+324t+1944)$
$c_6(E_3)$	$(t^2+36t+216)(t^4-216t^2-1944t-5832)$
$\Delta(E_3)$	$t^3(t+8)(t+9)^6$
$c_4(E_6)$	$(t+12)(t^3+252t^2+3888t+15552)$
$c_6(E_6)$	$(t^2+36t+216)(t^4-504t^3-13824t^2-124416t-373248)$
$\Delta(E_6)$	$t^6(t+8)^2(t+9)^3$

## Automorphisms

The subgroup of  $\text{Aut } X_0(6)$  that fixes the set of vertices of the graph is generated by the Fricke involutions of  $X_0(6)$ , given by

$$W_2(t) = -8(t+9)/(t+8), \quad W_3(t) = -9(t+8)/(t+9), \quad W_6(t) = 72/t.$$

With regard to the action of the Fricke involutions on the isogeny graph, it can be displayed as follows:

$$\begin{array}{ccc}
 E_1 & \xrightarrow{3} & E_3 \\
 \left|_2 \right. & & \left|_2 \right. \\
 E_2 & \xrightarrow{3} & E_6
 \end{array} : \text{Id} \qquad
 \begin{array}{ccc}
 E_2 & \xrightarrow{3} & E_6 \\
 \left|_2 \right. & & \left|_2 \right. \\
 E_1 & \xrightarrow{3} & E_3
 \end{array} : W_2$$

$$\begin{array}{ccc}
 E_3^{-3} & \xrightarrow{3} & E_1^{-3} \\
 \left|_2 \right. & & \left|_2 \right. \\
 E_6^{-3} & \xrightarrow{3} & E_2^{-3}
 \end{array} : W_3 \qquad
 \begin{array}{ccc}
 E_6^{-3} & \xrightarrow{3} & E_2^{-3} \\
 \left|_2 \right. & & \left|_2 \right. \\
 E_3^{-3} & \xrightarrow{3} & E_1^{-3}
 \end{array} : W_6$$

## 16.2 Kodaira symbols, minimal models, and Pal values

Table 24:  $R_4(6)$  data for  $p \neq 2, 3$

$R_4(6)$	$p \neq 2, 3$				
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$
$v_p(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1
	$E_6$	$(0, 0, 6m)$	1	$I_{6m}$	1
$v_p(t) = 0$ $m = v_p(t + 9)$	$E_1$	$(0, 0, 2m)$	1	$I_{2m}$	1
	$E_2$	$(0, 0, m)$	1	$I_m$	1
	$E_3$	$(0, 0, 6m)$	1	$I_{6m}$	1
	$E_6$	$(0, 0, 3m)$	1	$I_{3m}$	1
$v_p(t) = 0$ $m = v_p(t + 8)$	$E_1$	$(0, 0, 3m)$	1	$I_{3m}$	1
	$E_2$	$(0, 0, 6m)$	1	$I_{6m}$	1
	$E_3$	$(0, 0, m)$	1	$I_m$	1
	$E_6$	$(0, 0, 2m)$	1	$I_{2m}$	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 6m)$	$p^{-m}$	$I_{6m}$	1
	$E_2$	$(0, 0, 3m)$	$p^{-m}$	$I_{3m}$	1
	$E_3$	$(0, 0, 2m)$	$p^{-m}$	$I_{2m}$	1
	$E_6$	$(0, 0, m)$	$p^{-m}$	$I_m$	1



Table 25:  $R_4(6)$  data for  $p=3$ 

$R_4(6)$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = m > 2$	$E_1$	$(2, 3, m+4)$	1	$I_{m-2}^*$	3	1
	$E_2$	$(2, 3, 2m+2)$	1	$I_{2(m-2)}^*$	3	1
	$E_3$	$(2, 3, 3m)$	3	$I_{3(m-2)}^*$	3	1
	$E_6$	$(2, 3, 6m-6)$	3	$I_{6(m-2)}^*$	3	1
$v_3(t) = 2$ $m = v_3(t+9)$	$E_1$	$(2, 3, 2m+2)$	1	$I_{2(m-2)}^*$	3	1
	$E_2$	$(2, 3, m+4)$	1	$I_{m-2}^*$	3	1
	$E_3$	$(2, 3, 6m-6)$	3	$I_{6(m-2)}^*$	3	1
	$E_6$	$(2, 3, 3m)$	3	$I_{3(m-2)}^*$	3	1
$v_3(t) = 1$	$E_1$	$(\geq 2, 3, 3)$	1	III	1	1
	$E_2$	$(\geq 2, 3, 3)$	1	III	1	1
	$E_3$	$(\geq 4, 6, 9)$	1	III*	3	1
	$E_6$	$(\geq 4, 6, 9)$	1	III*	3	1
$v_3(t) = 0$ $m = v_3(t+8)$	$E_1$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_2$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_3$	$(0, 0, m)$	1	$I_m$	1	1
	$E_6$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
$v_3(t) = -m < 0$	$E_1$	$(0, 0, 6m)$	$3^{-m}$	$I_{6m}$	1	1
	$E_2$	$(0, 0, 3m)$	$3^{-m}$	$I_{3m}$	1	1
	$E_3$	$(0, 0, 2m)$	$3^{-m}$	$I_{2m}$	1	1
	$E_6$	$(0, 0, m)$	$3^{-m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	

Table 26:  $R_4(6)$  data for  $p=2$ 

$R_4(6)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 3$	$E_1$	$(4, 6, m + 9)$	1	$I_{m+1}^*$	1	1	2
	$E_2$	$(4, 6, 2m + 6)$	2	$I_{2m-2}^*$	1	1	2
	$E_3$	$(4, 6, 3m + 3)$	1	$I_{3m-5}^*$	1	1	2
	$E_6$	$(4, 6, 6m - 6)$	2	$I_{6m-14}^*$	1	1	2
$v_2(t) = 3$ $m = v_2(t + 8)$	$E_1$	$(4, 6, 3 + 3m)$	1	$I_{3m-5}^*$	1	1	2
	$E_2$	$(4, 6, 6m - 6)$	2	$I_{6m-14}^*$	1	1	2
	$E_3$	$(4, 6, m + 9)$	1	$I_{m+1}^*$	1	1	2
	$E_6$	$(4, 6, 2m + 6)$	2	$I_{2m-2}^*$	1	1	2
$v_2(t) = 2$	$E_1$	$(4, 6, 8)$	1	$I_0^*$	1	1	1
	$E_2$	$(\geq 4, 5, 4)$	2	$\Pi$	1	1	1
	$E_3$	$(4, 6, 8)$	1	$I_0^*$	1	1	1
	$E_6$	$(\geq 4, 5, 4)$	2	$\Pi$	1	1	1
$v_2(t) = 1$	$E_1$	$(\geq 4, 5, 4)$	1	$\Pi$	1	1	1
	$E_2$	$(4, 6, 8)$	1	$I_0^*$	1	1	1
	$E_3$	$(\geq 4, 5, 4)$	1	$\Pi$	1	1	1
	$E_6$	$(4, 6, 8)$	1	$I_0^*$	1	1	1
$v_2(t) = 0$ $m = v_2(t + 9)$	$E_1$	$(4, 6, 2m + 12)$	$2^{-1}$	$I_{2m+4}^*$	1	1	2
	$E_2$	$(4, 6, m + 12)$	$2^{-1}$	$I_{m+4}^*$	1	1	2
	$E_3$	$(4, 6, 6m + 12)$	$2^{-1}$	$I_{6m+4}^*$	1	1	2
	$E_6$	$(4, 6, 3m + 12)$	$2^{-1}$	$I_{3m+4}^*$	1	1	2
$v_2(t) = -m < 0$	$E_1$	$(4, 6, 6m + 12)$	$2^{-m-1}$	$I_{6m+4}^*$	1	1	2
	$E_2$	$(4, 6, 3m + 12)$	$2^{-m-1}$	$I_{3m+4}^*$	1	1	2
	$E_3$	$(4, 6, 2m + 12)$	$2^{-m-1}$	$I_{2m+4}^*$	1	1	2
	$E_6$	$(4, 6, m + 12)$	$2^{-m-1}$	$I_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

### 16.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u}_p = [u_p(E)]$  and  $\mathbf{u}_p(d) = [u_p(\mathcal{E}^d)]$ :

$t$	$[u_2(E)]$	$[u_2(\mathcal{E}^d)]$
$v_2(t) \geq 2$	$(1 : 2 : 1 : 2)$	$(1 : 1 : 1 : 1)$
$v_2(t) \leq 1$	$(1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1)$

$t$	$[u_3(E)]$	$[u_3(\mathcal{E}^d)]$	$d$
$v_3(t) \geq 2$	$(1 : 1 : 3 : 3)$	$(1 : 1 : 1 : 1)$	
$v_3(t) = 1$	$(1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1)$	$d \not\equiv 0 (3)$
		$(1 : 1 : 3 : 3)$	$d \equiv 0 (3)$
$v_3(t) \leq 0$	$(1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1)$	

The contents of these tables are the ingredients to prove the following result:

**Proposition 16.** *Let*

$$\begin{array}{ccc} E_1 & \xrightarrow{3} & E_3 \\ | 2 & & | 2 \\ E_2 & \xrightarrow{3} & E_6 \end{array}$$

*be a  $\mathbf{Q}$ -isogeny graph of type  $R_4(6)$  corresponding to a given  $t$  in  $\mathbf{Q} \setminus \{0, -8, -9\}$  with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted graph*

$$\begin{array}{ccc} E_1^d & \xrightarrow{3} & E_3^d \\ | 2 & & | 2 \\ E_2^d & \xrightarrow{3} & E_6^d \end{array}$$

*is given by:*

Table 27: Faltings curves in  $R_4(6)$

$R_4(6)$		twisted isogeny graph	$d$	Prob
$v_2(t) \leq 1$	$v_3(t) \leq 0$	$\begin{array}{ccc} \textcircled{E_1^d} & \longrightarrow & E_3^d \\ \downarrow & & \downarrow \\ E_2^d & \longrightarrow & E_6^d \end{array}$		1
$v_2(t) \leq 1$	$v_3(t) = 1$	$\begin{array}{ccc} \textcircled{E_1^d} & \longrightarrow & E_3^d \\ \downarrow & & \downarrow \\ E_2^d & \longrightarrow & E_6^d \end{array}$	$d \equiv 0(3)$	1/4
		$\begin{array}{ccc} E_1^d & \longleftarrow & \textcircled{E_3^d} \\ \downarrow & & \downarrow \\ E_2^d & \longleftarrow & E_6^d \end{array}$	$d \not\equiv 0(3)$	3/4
$v_2(t) \leq 1$	$v_3(t) \geq 2$	$\begin{array}{ccc} E_1^d & \longleftarrow & \textcircled{E_3^d} \\ \downarrow & & \downarrow \\ E_2^d & \longleftarrow & E_6^d \end{array}$		1
$v_2(t) \geq 2$	$v_3(t) \leq 0$	$\begin{array}{ccc} E_1^d & \longrightarrow & E_3^d \\ \uparrow & & \uparrow \\ \textcircled{E_2^d} & \longrightarrow & E_6^d \end{array}$		1
$v_2(t) \geq 2$	$v_3(t) = 1$	$\begin{array}{ccc} E_1^d & \longrightarrow & E_3^d \\ \uparrow & & \uparrow \\ \textcircled{E_2^d} & \longrightarrow & E_6^d \end{array}$	$d \equiv 0(3)$	1/4
		$\begin{array}{ccc} E_1^d & \longleftarrow & E_3^d \\ \uparrow & & \uparrow \\ E_2^d & \longleftarrow & \textcircled{E_6^d} \end{array}$	$d \not\equiv 0(3)$	3/4
$v_2(t) \geq 2$	$v_3(t) \geq 2$	$\begin{array}{ccc} E_1^d & \longleftarrow & E_3^d \\ \uparrow & & \uparrow \\ E_2^d & \longleftarrow & \textcircled{E_6^d} \end{array}$		1

The column Prob gives the probability of the circled twisted curve to be the Faltings curve.

## 17 Type $R_4(10)$

### 17.1 Settings

#### Graph

The isogeny graphs of type  $R_4(10)$  are given by four isogenous elliptic curves:

$$\begin{array}{ccc} E_1 & \xrightarrow{5} & E_5 \\ 2 \downarrow & & \downarrow 2 \\ E_2 & \xrightarrow{5} & E_{10} . \end{array}$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(10)$  parametrize isogeny graphs of type  $R_4(10)$ . The curve  $X_0(10)$  has genus 0 and a hauptmodul for this curve is:

$$t = 4 + 2^2 5 \frac{\eta(2\tau)\eta(10\tau)^3}{\eta(\tau)^3\eta(5\tau)} .$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one has

$$\begin{aligned} j(E_1) = j(\tau) &= \frac{(t^6 - 4t^5 + 16t + 16)^3}{(t-4)t^5(t+1)^2} \\ j(E_2) = j(2\tau) &= \frac{(t^6 - 4t^5 + 256t + 256)^3}{(t-4)^2 t^{10}(t+1)} \\ j(E_5) = j(5\tau) &= \frac{(t^6 - 4t^5 + 240t^4 - 480t^3 + 1440t^2 - 944t + 16)^3}{(t-4)^5 t(t+1)^{10}} \\ j(E_{10}) = j(10\tau) &= \frac{(t^6 + 236t^5 + 1440t^4 + 1920t^3 + 3840t^2 + 256t + 256)^3}{(t-4)^{10} t^2(t+1)^5} . \end{aligned}$$

## Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_2, E_5, E_{10})$  in such a way that the isogeny graph is normalized. Their signatures are:

$R_4(10)$	
$c_4(E_1)$	$(t^2 + 4)(t^6 - 4t^5 + 16t + 16)$
$c_6(E_1)$	$(t^2 - 2t - 4)(t^2 - 2t + 2)(t^2 + 4)^2(t^4 - 2t^3 - 6t^2 - 8t - 4)$
$\Delta(E_1)$	$(t - 4)t^5(t + 1)^2(t^2 + 4)^3$
$c_4(E_2)$	$(t^2 + 4)(t^6 - 4t^5 + 256t + 256)$
$c_6(E_2)$	$(t^2 - 2t - 4)(t^2 + 4)^2(t^2 + 4t + 8)(t^4 - 8t^3 + 24t^2 - 32t - 64)$
$\Delta(E_2)$	$(t - 4)^2t^{10}(t + 1)(t^2 + 4)^3$
$c_4(E_5)$	$(t^2 + 4)(t^6 - 4t^5 + 240t^4 - 480t^3 + 1440t^2 - 944t + 16)$
$c_6(E_5)$	$(t^2 - 2t + 2)(t^2 + 4)^2(t^2 + 22t - 4)(t^4 - 26t^3 + 66t^2 - 536t - 4)$
$\Delta(E_5)$	$(t - 4)^5t(t + 1)^{10}(t^2 + 4)^3$
$c_4(E_{10})$	$(t^2 + 4)(t^6 + 236t^5 + 1440t^4 + 1920t^3 + 3840t^2 + 256t + 256)$
$c_6(E_{10})$	$(t^2 + 4)^2(t^2 + 4t + 8)(t^2 + 22t - 4)(t^4 - 536t^3 - 264t^2 - 416t - 64)$
$\Delta(E_{10})$	$(t - 4)^{10}t^2(t + 1)^5(t^2 + 4)^3$

## Automorphisms

The subgroup of  $\text{Aut } X_0(10)$  that fixes the set of vertices of the graph is generated by the Fricke involutions of  $X_0(10)$ , given by

$$W_{10}(t) = 4(t + 1)/(t - 4), \quad W_5(t) = (-t + 4)/(t + 1), \quad W_2(t) = -4/t.$$

With regard to the action of the Fricke involutions on the isogeny graph, it can be displayed as follows:

$$\begin{array}{ccc}
E_1 & \xrightarrow{5} & E_5 \\
\left| \begin{array}{c} 2 \\ 2 \end{array} \right. & & \left| \begin{array}{c} 2 \\ 2 \end{array} \right. : \text{Id} \\
E_2 & \xrightarrow{5} & E_{10}
\end{array}
\qquad
\begin{array}{ccc}
E_2^2 & \xrightarrow{5} & E_{10}^2 \\
\left| \begin{array}{c} 2 \\ 2 \end{array} \right. & & \left| \begin{array}{c} 2 \\ 2 \end{array} \right. : W_2 \\
E_1^2 & \xrightarrow{5} & E_5^2
\end{array}$$
  

$$\begin{array}{ccc}
E_5^{-5} & \xrightarrow{5} & E_1^{-5} \\
\left| \begin{array}{c} 2 \\ 2 \end{array} \right. & & \left| \begin{array}{c} 2 \\ 2 \end{array} \right. : W_5 \\
E_{10}^{-5} & \xrightarrow{5} & E_2^{-5}
\end{array}
\qquad
\begin{array}{ccc}
E_{10}^{-10} & \xrightarrow{5} & E_2^{-10} \\
\left| \begin{array}{c} 2 \\ 2 \end{array} \right. & & \left| \begin{array}{c} 2 \\ 2 \end{array} \right. : W_{10} \\
E_5^{-10} & \xrightarrow{5} & E_1^{-10}
\end{array}$$

where the arrows correspond to the dual isogenies of the initial isogeny graph.

## 17.2 Kodaira symbols, minimal models, and Pal values

Table 28:  $R_4(10)$  data for  $p \neq 2, 5$

$R_4(10)$	$p \neq 2, 5$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, 5m)$	1	$I_{5m}$	1	1
	$E_2$	$(0, 0, 10m)$	1	$I_{10m}$	1	1
	$E_5$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{10}$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
$v_p(t) = 0$ $m = v_p(t+1) > 0$	$E_1$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_2$	$(0, 0, m)$	1	$I_m$	1	1
	$E_5$	$(0, 0, 10m)$	1	$I_{10m}$	1	1
	$E_{10}$	$(0, 0, 5m)$	1	$I_{5m}$	1	1
$v_p(t) = 0$ $m = v_p(t-4) > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_5$	$(0, 0, 5m)$	1	$I_{5m}$	1	1
	$E_{10}$	$(0, 0, 10m)$	1	$I_{10m}$	1	1
$v_p(t) = 0$ $v_p(t^2 + 4) = 4m$	$E_1$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
	$E_2$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
	$E_5$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
	$E_{10}$	$(0, 2m, 0)$	$p^m$	$I_0$	1	1
$v_p(t) = 0$ $v_p(t^2 + 4) = 4m + 1$	$E_1$	$(1, 2 + 2m, 3)$	$p^m$	III	1	1
	$E_2$	$(1, 2 + 2m, 3)$	$p^m$	III	1	1
	$E_5$	$(1, 2 + 2m, 3)$	$p^m$	III	1	1
	$E_{10}$	$(1, 2 + 2m, 3)$	$p^m$	III	1	1
$v_p(t) = 0$ $v_p(t^2 + 4) = 4m + 2$	$E_1$	$(2, 4 + 2m, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_2$	$(2, 4 + 2m, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_5$	$(2, 4 + 2m, 6)$	$p^m$	$I_0^*$	$p$	1
	$E_{10}$	$(2, 4 + 2m, 6)$	$p^m$	$I_0^*$	$p$	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 28:  $R_4(10)$  data for  $p \neq 2, 5$  (Continued)

$R_4(10)$	$p \neq 2, 5$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = 0$ $v_p(t^2 + 4) = 4m + 3$	$E_1$	$(3, 6 + 2m, 9)$	$p^m$	$\text{III}^*$	$p$	1
	$E_2$	$(3, 6 + 2m, 9)$	$p^m$	$\text{III}^*$	$p$	1
	$E_5$	$(3, 6 + 2m, 9)$	$p^m$	$\text{III}^*$	$p$	1
	$E_{10}$	$(3, 6 + 2m, 9)$	$p^m$	$\text{III}^*$	$p$	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 10m)$	$p^{-2m}$	$\text{I}_{10m}$	1	1
	$E_2$	$(0, 0, 5m)$	$p^{-2m}$	$\text{I}_{5m}$	1	1
	$E_5$	$(0, 0, 2m)$	$p^{-2m}$	$\text{I}_{2m}$	1	1
	$E_{10}$	$(0, 0, m)$	$p^{-2m}$	$\text{I}_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	



Table 29:  $R_4(10)$  data for  $p = 5$ 

$R_4(10)$	$p = 5$					
$t$	$E$	$\text{sig}_5(\mathcal{E})$	$u_5(E)$	$K_5(E)$	$u_5(\mathcal{E}^d)$	
$v_5(t) = m > 0$	$E_1$	$(0, 0, 5m)$	1	$I_{5m}$	1	1
	$E_2$	$(0, 0, 10m)$	1	$I_{10m}$	1	1
	$E_5$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{10}$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
$v_5(t) = 0$ $t \equiv 1 \pmod{5}$ $v_5(t^2 + 4) = 4m$	$E_1$	$(0, 1 + 2m, 0)$	$5^m$	$I_0$	1	1
	$E_2$	$(0, 1 + 2m, 0)$	$5^m$	$I_0$	1	1
	$E_5$	$(0, 2m, 0)$	$5^m$	$I_0$	1	1
	$E_{10}$	$(0, 2m, 0)$	$5^m$	$I_0$	1	1
$v_5(t) = 0$ $t \equiv 1 \pmod{5}$ $v_5(t^2 + 4) = 4m + 1$	$E_1$	$(1, 3 + 2m, 3)$	$5^m$	III	1	1
	$E_2$	$(1, 3 + 2m, 3)$	$5^m$	III	1	1
	$E_5$	$(1, 2 + 2m, 3)$	$5^m$	III	1	1
	$E_{10}$	$(1, 2 + 2m, 3)$	$5^m$	III	1	1
$v_5(t) = 0$ $t \equiv 1 \pmod{5}$ $v_5(t^2 + 4) = 4m + 2$	$E_1$	$(2, 5 + 2m, 6)$	$5^m$	$I_0^*$	5	1
	$E_2$	$(2, 5 + 2m, 6)$	$5^m$	$I_0^*$	5	1
	$E_5$	$(2, 4 + 2m, 6)$	$5^m$	$I_0^*$	5	1
	$E_{10}$	$(2, 4 + 2m, 6)$	$5^m$	$I_0^*$	5	1
$v_5(t) = 0$ $t \equiv 1 \pmod{5}$ $v_5(t^2 + 4) = 4m + 3$	$E_1$	$(3, 7 + 2m, 9)$	$5^m$	III*	5	1
	$E_2$	$(3, 7 + 2m, 9)$	$5^m$	III*	5	1
	$E_5$	$(3, 6 + 2m, 9)$	$5^m$	III*	5	1
	$E_{10}$	$(3, 6 + 2m, 9)$	$5^m$	III*	5	1
$v_5(t) = 0$ $t \equiv 14 \pmod{25}$ $v_5(t^2 + 4) = 4m$	$E_1$	$(1, \geq 2, 3)$	$5^m$	III	1	1
	$E_2$	$(1, \geq 3, 3)$	$5^m$	III	1	1
	$E_5$	$(1, \geq 2, 3)$	$5^{m+1}$	III	1	1
	$E_{10}$	$(1, \geq 2, 3)$	$5^{m+1}$	III	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{5}$	

Table 29:  $R_4(10)$  data for  $p = 5$  (Continued)

$R_4(10)$	$p = 5$					
$t$	$E$	$\text{sig}_5(\mathcal{E})$	$u_5(E)$	$K_5(E)$	$u_5(\mathcal{E}^d)$	
$v_5(t) = 0$ $t \equiv 14 (25)$ $v_5(t^2 + 4) = 4m + 1$	$E_1$	$(2, \geq 3, 6)$	$5^m$	$I_0^*$	5	1
	$E_2$	$(2, \geq 3, 6)$	$5^m$	$I_0^*$	5	1
	$E_5$	$(2, \geq 3, 6)$	$5^{m+1}$	$I_0^*$	5	1
	$E_{10}$	$(2, \geq 3, 6)$	$5^{m+1}$	$I_0^*$	5	1
$v_5(t) = 0$ $t \equiv 14 (25)$ $v_5(t^2 + 4) = 4m + 2$	$E_1$	$(3, \geq 5, 9)$	$5^m$	$\text{III}^*$	5	1
	$E_2$	$(3, \geq 5, 9)$	$5^m$	$\text{III}^*$	5	1
	$E_5$	$(3, \geq 5, 9)$	$5^{m+1}$	$\text{III}^*$	5	1
	$E_{10}$	$(3, \geq 5, 9)$	$5^{m+1}$	$\text{III}^*$	5	1
$v_5(t) = 0$ $t \equiv 14 (25)$ $v_5(t^2 + 4) = 4m + 3$	$E_1$	$(0, \geq 1, 0)$	$5^{m+1}$	$I_0$	1	1
	$E_2$	$(0, \geq 1, 0)$	$5^{m+1}$	$I_0$	1	1
	$E_5$	$(0, \geq 1, 0)$	$5^{m+2}$	$I_0$	1	1
	$E_{10}$	$(0, \geq 1, 0)$	$5^{m+2}$	$I_0$	1	1
$v_5(t) = 0$ $t \equiv 4 (25)$ $v_5(t - 4) = m$	$E_1$	$(2, 3, m + 5)$	1	$I_{m-1}^*$	5	1
	$E_2$	$(2, 3, 2m + 4)$	1	$I_{2m-2}^*$	5	1
	$E_5$	$(2, 3, 5m + 1)$	5	$I_{5m-5}^*$	5	1
	$E_{10}$	$(2, 3, 10m - 4)$	5	$I_{10m-10}^*$	5	1
$v_5(t) = 0$ $t \equiv 9, 19 (25)$	$E_1$	$(2, \geq 3, 6)$	1	$I_0^*$	5	1
	$E_2$	$(2, \geq 3, 6)$	1	$I_0^*$	5	1
	$E_5$	$(2, \geq 3, 6)$	5	$I_0^*$	5	1
	$E_{10}$	$(2, \geq 3, 6)$	5	$I_0^*$	5	1
$v_5(t) = 0$ $t \equiv 24 (25)$ $v_5(t + 1) = m$	$E_1$	$(2, 3, 2m + 4)$	1	$I_{2m-2}^*$	5	1
	$E_2$	$(2, 3, m + 5)$	1	$I_{m-1}^*$	5	1
	$E_5$	$(2, 3, 10m - 4)$	5	$I_{10m-10}^*$	5	1
	$E_{10}$	$(2, 3, 5m + 1)$	5	$I_{5m-5}^*$	5	1
$v_5(t) = 0$ $t \equiv 2, 3 (5)$	$E_1$	$(0, \geq 0, 0)$	1	$I_0$	1	1
	$E_2$	$(0, \geq 0, 0)$	1	$I_0$	1	1
	$E_5$	$(0, \geq 0, 0)$	1	$I_0$	1	1
	$E_{10}$	$(0, \geq 0, 0)$	1	$I_0$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{5}$	

Table 29:  $R_4(10)$  data for  $p = 5$  (Continued)

$R_4(10)$	$p = 5$					
$t$	$E$	$\text{sig}_5(\mathcal{E})$	$u_5(E)$	$K_5(E)$	$u_5(\mathcal{E}^d)$	
$v_5(t) = -m < 0$	$E_1$	$(0, 0, 10m)$	$5^{-2m}$	$I_{10m}$	1	1
	$E_2$	$(0, 0, 5m)$	$5^{-2m}$	$I_{5m}$	1	1
	$E_5$	$(0, 0, 2m)$	$5^{-2m}$	$I_{2m}$	1	1
	$E_{10}$	$(0, 0, m)$	$5^{-2m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{5}$	

The polynomial  $t^2 + 4$  factors over  $\mathbb{Q}_5[t]$  as  $(t - \alpha_1)(t - \alpha_2)$  with

$$\begin{aligned}\alpha_1 &= 1 + 2 \cdot 5 + 2 \cdot 5^3 + 3 \cdot 5^4 + 4 \cdot 5^6 + 2 \cdot 5^7 + 3 \cdot 5^8 + 3 \cdot 5^9 + 4 \cdot 5^{10} + 4 \cdot 5^{11} + 3 \cdot 5^{12} + O(5^{13}), \\ \alpha_2 &= 4 + 2 \cdot 5 + 4 \cdot 5^2 + 2 \cdot 5^3 + 5^4 + 4 \cdot 5^5 + 2 \cdot 5^7 + 5^8 + 5^9 + 5^{12} + 3 \cdot 5^{13} + 3 \cdot 5^{14} + O(5^{15}).\end{aligned}$$

Table 30:  $R_4(10)$  data for  $p=2$ 

$R_4(10)$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 2$	$E_1$	$(6, 9, 5m + 8)$	1	$I_{5m-2}^*$	1	$2^*$ or $4^*$	1
	$E_2$	$(6, 9, 10m - 2)$	2	$I_{10m-12}^*$	1	$2^*$ or $4^*$	1
	$E_5$	$(6, 9, m + 16)$	1	$I_{m+6}^*$	1	$2^*$ or $4^*$	1
	$E_{10}$	$(6, 9, 2m + 14)$	2	$I_{2m+4}^*$	1	$2^*$ or $4^*$	1
$v_2(t) = 2$ $m = v_2(t - 4)$	$E_1$	$(6, 9, m + 16)$	1	$I_{m+4}^*$	1	$4^*$ or $2^*$	1
	$E_2$	$(6, 9, 2m + 14)$	2	$I_{2m+4}^*$	1	$4^*$ or $2^*$	1
	$E_5$	$(6, 9, 5m + 8)$	1	$I_{5m-2}^*$	1	$4^*$ or $2^*$	1
	$E_{10}$	$(6, 9, 10m - 2)$	2	$I_{10m-12}^*$	1	$4^*$ or $2^*$	1
$v_2(t) = 1$	$E_1$	$(7, 11, 15)$	1	$\text{III}^*$	1	2	1
	$E_2$	$(5, 8, 9)$	2	$\text{III}$	1	1	1
	$E_5$	$(7, 11, 15)$	1	$\text{III}$	1	2	1
	$E_{10}$	$(5, 8, 9)$	2	$\text{III}$	1	1	1
$v_2(t) = 0$ $m = v_2(t + 1)$	$E_1$	$(0, 0, 2m)$	1	$I_{2m}$	1	$2^{-1}$	$2^{-1}$
	$E_2$	$(0, 0, m)$	1	$I_m$	1	$2^{-1}$	$2^{-1}$
	$E_5$	$(0, 0, 10m)$	1	$I_{10m}$	1	$2^{-1}$	$2^{-1}$
	$E_{10}$	$(0, 0, 5m)$	1	$I_{5m}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = -m < 0$	$E_1$	$(4, 6, 10m + 12)$	$2^{-2m-1}$	$I_{10m+4}^*$	1	1	2
	$E_2$	$(4, 6, 5m + 12)$	$2^{-2m-1}$	$I_{5m+4}^*$	1	1	2
	$E_5$	$(4, 6, 2m + 12)$	$2^{-2m-1}$	$I_{2m+4}^*$	1	1	2
	$E_{10}$	$(4, 6, m + 12)$	$2^{-2m-1}$	$I_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

**Remark ( $2^*$  or  $4^*$ ):** If  $v_2(t) > 2$  and  $d \equiv 2(4)$ , then the value  $u_2(\mathcal{E}^d)$  is given by

$$u_2(d) = \begin{cases} 2 & \text{if } d \equiv 2(8) \\ 4 & \text{if } d \equiv -2(8). \end{cases}$$

**Remark ( $4^*$  or  $2^*$ ):** If  $v_2(t) = 2$  and  $d \equiv 2(4)$ , then the value  $u_2(\mathcal{E}^d)$  is given by

$$u_2(d) = \begin{cases} 4 & \text{if } d \equiv 2(8) \\ 2 & \text{if } d \equiv -2(8). \end{cases}$$

### 17.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u}_p = [u_p(E)]$  and  $\mathbf{u}_p(d) = [u_p(\mathcal{E}^d)]$ :

$t$	$[u_2(E)]$	$[u_2(\mathcal{E}^d)]$	$d$
$v_2(t) > 1$	$(1 : 2 : 1 : 2)$	$(1 : 1 : 1 : 1)$	
$v_2(t) = 1$	$(1 : 2 : 1 : 2)$	$(1 : 1 : 1 : 1)$	$d \not\equiv 0(2)$
		$(2 : 1 : 2 : 1)$	$d \equiv 0(2)$
$v_2(t) \leq 0$	$(1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1)$	

$t$	$[u_5(E)]$	$[u_5(\mathcal{E}^d)]$
$v_5(t) \neq 0$	$(1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1)$
$v_5(t) = 0$ $t \not\equiv 4(5)$		
$v_5(t) = 0$ $t \equiv 4(5)$	$(1 : 1 : 5 : 5)$	$(1 : 1 : 1 : 1)$

The contents of these tables are the ingredients to prove the following result:

**Proposition 17.** *Let*

$$\begin{array}{ccc} E_1 & \xrightarrow{5} & E_5 \\ \left| \begin{array}{c} 2 \\ 2 \end{array} \right. & & \left| \begin{array}{c} 2 \\ 2 \end{array} \right. \\ E_2 & \xrightarrow{5} & E_{10} \end{array}$$

be a  $\mathbf{Q}$ -isogeny graph of type  $R_4(10)$  corresponding to a given  $t$  in  $\mathbf{Q} \setminus \{-1, \pm 4\}$  with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted graph

$$\begin{array}{ccc} E_1^d & \xrightarrow{5} & E_5^d \\ \left| \begin{array}{c} 2 \\ 2 \end{array} \right. & & \left| \begin{array}{c} 2 \\ 2 \end{array} \right. \\ E_2^d & \xrightarrow{5} & E_{10}^d \end{array}$$

is given by:

Table 31: Faltings curves in  $R_4(10)$

$R_4(10)$		twisted isogeny graph	$d$	Prob
$v_2(t) \leq 0$	$v_5(t) \neq 0$	$\begin{array}{ccc} \textcircled{E_1^d} & \longrightarrow & E_5^d \\ \downarrow & & \downarrow \\ E_2^d & \longrightarrow & E_{10}^d \end{array}$		1
	$v_5(t) = 0$ $t \not\equiv 4(5)$			

Continued on next page

Table 31: Faltings curves in  $R_4(10)$  (Continued)

$R_4(10)$		twisted isogeny graph	$d$	Prob
$v_2(t) = 1$	$v_5(t) \neq 0$	$\begin{array}{ccc} \textcircled{E_1^d} & \longrightarrow & E_5^d \\ \downarrow & & \downarrow \\ E_2^d & \longrightarrow & E_{10}^d \end{array}$	$d \equiv 0 \pmod{2}$	1/3
	$v_5(t) = 0$ $t \not\equiv 4 \pmod{5}$	$\begin{array}{ccc} E_1^d & \longleftarrow & E_5^d \\ \downarrow & & \downarrow \\ \textcircled{E_2^d} & \longleftarrow & E_{10}^d \end{array}$	$d \not\equiv 0 \pmod{2}$	2/3
$v_2(t) > 1$	$v_5(t) \neq 0$	$\begin{array}{ccc} E_1^d & \longleftarrow & E_5^d \\ \downarrow & & \downarrow \\ \textcircled{E_2^d} & \longleftarrow & E_{10}^d \end{array}$		1
	$v_5(t) = 0$ $t \not\equiv 4 \pmod{5}$			
	$v_5(t) = 0$ $t \equiv 4 \pmod{5}$	$\begin{array}{ccc} E_1^d & \longrightarrow & E_5^d \\ \uparrow & & \uparrow \\ E_2^d & \longrightarrow & \textcircled{E_{10}^d} \end{array}$		1
$v_2(t) = 1$	$v_5(t) = 0$ $t \equiv 4 \pmod{5}$	$\begin{array}{ccc} E_1^d & \longrightarrow & \textcircled{E_5^d} \\ \uparrow & & \uparrow \\ E_2^d & \longrightarrow & E_{10}^d \end{array}$	$d \equiv 0 \pmod{2}$	1/3
		$\begin{array}{ccc} E_1^d & \longleftarrow & E_5^d \\ \uparrow & & \uparrow \\ E_2^d & \longleftarrow & \textcircled{E_{10}^d} \end{array}$	$d \not\equiv 0 \pmod{2}$	2/3
$v_2(t) \leq 0$	$v_5(t) = 0$ $t \equiv 4 \pmod{5}$	$\begin{array}{ccc} E_1^d & \longleftarrow & \textcircled{E_5^d} \\ \uparrow & & \uparrow \\ E_2^d & \longleftarrow & E_{10}^d \end{array}$		1

The column Prob gives the probability of the circled twisted curve to be the Faltings curve.

## 18 Type $R_4(14)$

### 18.1 Settings

#### Graph

The isogeny graphs of type  $R_4(14)$  are given by four isogenous elliptic curves:

$$\begin{array}{ccc} E_1 & \xrightarrow{7} & E_7 \\ \downarrow 2 & & \downarrow 2 \\ E_2 & \xrightarrow{7} & E_{14}. \end{array}$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(14)$  parametrize isogeny graphs of type  $R_4(14)$ . The modular curve  $X_0(14)$  is elliptic of rank 0 over the rationals. Its rational points are: the four cusps and two CM points given by  $\tau = \frac{1}{2} + \frac{\sqrt{-7}}{2 \cdot 7}$  and  $2\tau$ .

#### $j$ -invariants

The corresponding  $j$ -invariants are:

$$j(\tau) = j(7\tau) = -3^3 \cdot 5^3, \quad j(14\tau) = j(2\tau) = 3^3 \cdot 5^3 \cdot 17^3.$$

#### Signatures

We choose minimal Weierstrass equations and the normalized isogeny graph.

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_1$	$y^2 + xy = x^3 - x^2 - 2x - 1$	$-3^5 \cdot 5^3$	49.a4
$E_2$	$y^2 + xy = x^3 - x^2 - 37x - 78$	$3^5 \cdot 5^3 \cdot 17^3$	49.a3
$E_7$	$y^2 + xy = x^3 - x^2 - 107x + 552$	$-3^5 \cdot 5^3$	49.a2
$E_{14}$	$y^2 + xy = x^3 - x^2 - 1822x + 30393$	$3^5 \cdot 5^3 \cdot 17^3$	49.a1

Their signatures are:

$E$	$E_1$	$E_2$	$E_7$	$E_{14}$
$c_4(E)$	$3 \cdot 5 \cdot 7$	$3 \cdot 5 \cdot 7 \cdot 17$	$3 \cdot 5 \cdot 7^3$	$3 \cdot 5 \cdot 7^3 \cdot 17$
$c_6(E)$	$3^3 \cdot 7^2$	$3^4 \cdot 7^2 \cdot 19$	$-3^3 \cdot 7^5$	$-3^4 \cdot 7^5 \cdot 19$
$\Delta(E)$	$-7^3$	$7^3$	$-7^9$	$7^3$

One checks that the Faltings curve (circled) in the graph is

$$\begin{array}{ccc} \textcircled{E_1} & \longrightarrow & E_7 \\ \downarrow & & \downarrow \\ E_2 & \longrightarrow & E_{14} \end{array}$$

Note that any  $\mathbb{Q}$ -isogeny class of type  $R_4(14)$  can be obtained by quadratic twist from

$$\begin{array}{ccc} E_1 & \xrightarrow{7} & E_7 \\ \left| 2 \right. & & \left| 2 \right. \\ E_2 & \xrightarrow{7} & E_{14}. \end{array}$$

Observe also that  $E_7 = E_1^{-7}$  and  $E_{14} = E_2^{-7}$ .

## 18.2 Kodaira symbols, minimal models, and Pal values

There is only one prime of bad reduction for these elliptic curves; that is,  $p = 7$ .

$p = 7$				
$E$	$\text{sig}_7(\mathcal{E})$	$K_7(E)$	$u_7(\mathcal{E}^d)$	
$E_1$	$(1, 2, 3)$	III	1	1
$E_2$	$(1, 2, 3)$	III	1	1
$E_7$	$(3, 5, 9)$	III*	7	1
$E_{14}$	$(3, 5, 9)$	III*	7	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{7}$	

## 18.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1 : 1 : 1)$	$d \not\equiv 0 \pmod{7}$
$(1 : 1 : 7 : 7)$	$d \equiv 0 \pmod{7}$

This table is the ingredient to prove the following result:

**Proposition 18.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{array}{ccc} E_1^d & \xrightarrow{7} & E_7^d \\ \left| 2 \right. & & \left| 2 \right. \\ E_2^d & \xrightarrow{7} & E_{14}^d. \end{array}$$

is given by:



Table 32: Faltings curves in  $R_4(14)$

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$  \begin{array}{ccc}  \textcircled{E_1^d} & \longrightarrow & E_7^d \\  \downarrow & & \downarrow \\  E_2^d & \longrightarrow & E_{14}^d  \end{array}  $	$d \not\equiv 0 \pmod{7}$	$7/8$
$  \begin{array}{ccc}  E_1^d & \longleftarrow & \textcircled{E_7^d} \\  \downarrow & & \downarrow \\  E_2^d & \longleftarrow & E_{14}^d  \end{array}  $	$d \equiv 0 \pmod{7}$	$1/8$

The column *Prob* gives the probability of the circled twisted curve to be the Faltings curve.

## 19 Type $R_4(15)$

### 19.1 Settings

#### Graph

The isogeny graphs of type  $R_4(15)$  are given by four isogenous elliptic curves:

$$\begin{array}{ccc} E_1 & \xrightarrow{5} & E_5 \\ \downarrow 3 & & \downarrow 3 \\ E_3 & \xrightarrow{5} & E_{15}. \end{array}$$

#### Modular curve

The non-cuspidal  $\mathbb{Q}$ -rational points of the modular curve  $X_0(15)$  parametrize isogeny graphs of type  $R_4(15)$ . The modular curve  $X_0(15)$  is elliptic of rank 0 over the rationals. Its rational points are: four cusps and four non-CM points given by  $\tau$ ,  $3\tau$ ,  $5\tau$ , and  $15\tau$  for a certain  $\tau \in \mathbb{H}$ .

#### $j$ -invariants

The corresponding  $j$ -invariants are:

$$j(\tau) = \frac{-5^2}{2}, \quad j(3\tau) = \frac{-5^2 \cdot 241^3}{2^3}, \quad j(5\tau) = \frac{-5 \cdot 29^3}{2^5}, \quad j(15\tau) = \frac{5 \cdot 211^3}{2^{15}}.$$

#### Signatures

We can choose minimal Weierstrass equations the isogeny graph is normalized.

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_1$	$y^2 = x^3 - x^2 - 8x + 112$	$\frac{-5^2}{2}$	400.d3
$E_3$	$y^2 = x^3 - x^2 - 2008x + 35312$	$\frac{-5^2 \cdot 241^3}{2^3}$	400.d1
$E_5$	$y^2 = x^3 - x^2 - 1208x - 19088$	$\frac{-5 \cdot 29^3}{2^5}$	400.d2
$E_{15}$	$y^2 = x^3 - x^2 + 8792x + 140912$	$\frac{5 \cdot 211^3}{2^{15}}$	400.d4

Their signatures are:

$E$	$E_1$	$E_3$	$E_5$	$E_{15}$
$c_4(E)$	$2^4 \cdot 5^2$	$2^4 \cdot 5^2 \cdot 241$	$2^4 \cdot 5^3 \cdot 29$	$-2^4 \cdot 5^3 \cdot 211$
$c_6(E)$	$-2^6 \cdot 5^2 \cdot 59$	$-2^6 \cdot 5^2 \cdot 13 \cdot 1439$	$2^6 \cdot 5^4 \cdot 421$	$-2^6 \cdot 5^4 \cdot 13 \cdot 239$
$\Delta(E)$	$-2^{13} \cdot 5^4$	$-2^{15} \cdot 5^4$	$-2^{17} \cdot 5^8$	$-2^{27} \cdot 5^8$

One checks that the Faltings curve (circled) in the graph is

$$\begin{array}{ccc} \textcircled{E_1} & \longrightarrow & E_5 \\ \downarrow & & \downarrow \\ E_3 & \longrightarrow & E_{15} \end{array}$$

Note that any other  $\mathbb{Q}$ -isogeny class of type  $R_4(15)$  can be obtained by quadratic twist from

$$\begin{array}{ccc} E_1 & \text{---} & E_5 \\ | & & | \\ E_3 & \text{---} & E_{15}. \end{array}$$

## 19.2 Kodaira symbols, minimal models, and Pal values

There are two primes of bad reduction  $p = 2$  and  $5$  for these elliptic curves.

$p = 2$					
$E$	$\text{sig}_2(\mathcal{E})$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$E_1$	$(4, 6, 13)$	$I_5^*$	1	1	2
$E_3$	$(4, 6, 15)$	$I_7^*$	1	1	2
$E_5$	$(4, 6, 17)$	$I_9^*$	1	1	2
$E_{15}$	$(4, 6, 27)$	$I_{19}^*$	1	1	2
			$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
			$d \pmod{4}$		

$p = 5$				
$E$	$\text{sig}_5(\mathcal{E})$	$K_5(E)$	$u_5(\mathcal{E}^d)$	
$E_1$	$(2, 2, 4)$	IV	1	1
$E_3$	$(2, 2, 4)$	IV	1	1
$E_5$	$(3, 4, 8)$	IV*	5	1
$E_{15}$	$(3, 4, 8)$	IV*	5	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{5}$	

### 19.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1 : 1 : 1)$	$d \not\equiv 0 (5)$
$(1 : 1 : 5 : 5)$	$d \equiv 0 (5)$

This table is the ingredient to prove the following result:

**Proposition 19.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{array}{ccc} E_1^d & \xrightarrow{5} & E_5^d \\ \downarrow 3 & & \downarrow 3 \\ E_3^d & \xrightarrow{5} & E_{15}^d. \end{array}$$

is given by:

Table 33: Faltings curves in  $R_4(15)$

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$\begin{array}{ccc} \textcircled{E_1^d} & \longrightarrow & E_5^d \\ \downarrow & & \downarrow \\ E_3^d & \longrightarrow & E_{15}^d \end{array}$	$d \not\equiv 0 (5)$	5/6
$\begin{array}{ccc} E_1^d & \longleftarrow & \textcircled{E_5^d} \\ \downarrow & & \downarrow \\ E_3^d & \longleftarrow & E_{15}^d \end{array}$	$d \equiv 0 (5)$	1/6

The column *Prob* gives the probability of the circled twisted curve to be the Faltings curve.

## 20 Type $R_4(21)$

### 20.1 Settings

The isogeny graphs of type  $R_4(21)$  are given by four isogenous elliptic curves:

$$\begin{array}{ccc} E_1 & \xrightarrow{7} & E_5 \\ \downarrow 3 & & \downarrow 3 \\ E_3 & \xrightarrow{7} & E_{21} . \end{array}$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(21)$  parametrize isogeny graphs of type  $R_4(21)$ . The modular curve  $X_0(21)$  is elliptic of rank 0 over the rationals. Its rational points are: four cusps and four non-cuspidal non-CM points corresponding to  $\tau$ ,  $3\tau$ ,  $7\tau$ ,  $21\tau$  for certain  $\tau \in \mathbb{H}$ .

#### $j$ -invariants

The corresponding  $j$ -invariants are:

$$j(\tau) = \frac{3^3 \cdot 5^3}{2}, \quad j(3\tau) = \frac{-3^2 \cdot 5^6}{2^3}, \quad j(7\tau) = \frac{-3^3 \cdot 5^3 \cdot 383^3}{2^7}, \quad j(21\tau) = \frac{-3^2 \cdot 5^3 \cdot 101^3}{2^{21}}.$$

#### Signatures

We can choose minimal Weierstrass equations and the isogeny graph is normalized.

$E$	Minimal Weierstrass model	$j(E)$	LMFDB
$E_1$	$y^2 = x^3 + 45x + 18$	$\frac{3^3 \cdot 5^3}{2}$	1296.f4
$E_3$	$y^2 = x^3 - 675x + 7074$	$\frac{-3^2 \cdot 5^6}{2^3}$	1296.f3
$E_7$	$y^2 = x^3 - 17235x - 870894$	$\frac{-3^3 \cdot 5^3 \cdot 383^3}{2^7}$	1296.f1
$E_{21}$	$y^2 = x^3 - 13635x - 1244862$	$\frac{-3^2 \cdot 5^3 \cdot 101^3}{2^{21}}$	1296.f2

Their signatures are:

$E$	$E_1$	$E_3$	$E_7$	$E_{21}$
$c_4(E)$	$-2^4 \cdot 3^3 \cdot 5$	$2^4 \cdot 3^4 \cdot 5^2$	$2^4 \cdot 3^3 \cdot 5 \cdot 383$	$2^4 \cdot 3^4 \cdot 5 \cdot 101$
$c_6(E)$	$-2^6 \cdot 3^5$	$-2^6 \cdot 3^6 \cdot 131$	$2^6 \cdot 3^5 \cdot 48383$	$2^6 \cdot 3^6 \cdot 23053$
$\Delta(E)$	$-2^{13} \cdot 3^6$	$-2^{15} \cdot 3^{10}$	$-2^{19} \cdot 3^6$	$-2^{33} \cdot 3^{10}$

One checks that the Faltings curve (circled) in the graph is

$$\begin{array}{ccc} \textcircled{E_1} & \longrightarrow & E_7 \\ \downarrow & & \downarrow \\ E_3 & \longrightarrow & E_{21} \end{array}$$

Note that any  $\mathbb{Q}$ -isogeny class of type  $R_4(21)$  can be obtained by quadratic twist:

$$\begin{array}{ccc} E_1^d & \text{---} & E_7^d \\ | & & | \\ E_3^d & \text{---} & E_{21}^d. \end{array}$$

## 20.2 Kodaira symbols, minimal models, and Pal values

There are two bad reduction rational primes  $p = 2$  and  $3$ .

$p = 2$					
$E$	$\text{sig}_2(\mathcal{E})$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$E_1$	$(4, 6, 13)$	$I_5^*$	1	1	2
$E_3$	$(4, 6, 15)$	$I_7^*$	1	1	2
$E_7$	$(4, 6, 19)$	$I_{11}^*$	1	1	2
$E_{21}$	$(4, 6, 33)$	$I_{25}^*$	1	1	2
			$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
			$d \pmod{4}$		

$p = 3$				
$E$	$\text{sig}_3(\mathcal{E})$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$E_1$	$(3, 5, 6)$	IV	1	1
$E_3$	$(4, 6, 10)$	IV*	3	1
$E_7$	$(3, 5, 6)$	IV	1	1
$E_{21}$	$(4, 6, 10)$	IV*	3	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{3}$	

### 20.3 Statement

From the above tables one gets the (projective) vector  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

$[u(\mathcal{E}^d)]$	$d$
$(1 : 1 : 1 : 1)$	$d \not\equiv 0(3)$
$(1 : 3 : 1 : 3)$	$d \equiv 0(3)$

This table is the ingredient to prove the following result:

**Proposition 20.** *For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{array}{ccc} E_1^d & \xrightarrow{7} & E_7^d \\ \downarrow 3 & & \downarrow 3 \\ E_3^d & \xrightarrow{7} & E_{21}^d \end{array}$$

is given by:

Table 34: Faltings curves in  $R_4(21)$

<i>twisted isogeny graph</i>	<i>condition</i>	<i>Prob</i>
$\begin{array}{ccc} \textcircled{E_1^d} & \longrightarrow & E_7^d \\ \downarrow & & \downarrow \\ E_3^d & \longrightarrow & E_{21}^d \end{array}$	$d \not\equiv 0(3)$	$3/4$
$\begin{array}{ccc} E_1^d & \longrightarrow & E_7^d \\ \uparrow & & \uparrow \\ \textcircled{E_3^d} & \longrightarrow & E_{21}^d \end{array}$	$d \equiv 0(3)$	$1/4$

The column *Prob* gives the probability of the circled twisted curve to be the Faltings curve.

## 21 Type $R_6$

### 21.1 Settings

#### Graph

The isogeny graphs of type  $R_6$  are given by six isogenous elliptic curves:

$$\begin{array}{ccccc} E_1 & \xrightarrow{3} & E_3 & \xrightarrow{3} & E_9 \\ 2 \downarrow & & \downarrow 2 & & \downarrow 2 \\ E_2 & \xrightarrow{3} & E_6 & \xrightarrow{3} & E_{18} \end{array}$$

#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(18)$  parametrize isogeny graphs of type  $R_6$ . The curve  $X_0(18)$  has genus 0 and a hauptmodul for this curve is:

$$t = 2 + 2 \cdot 3 \cdot \frac{\eta(2\tau)\eta(3\tau)\eta(18\tau)^2}{\eta(\tau)^2\eta(6\tau)\eta(9\tau)}.$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$\begin{aligned} j(E_1) = j(\tau) &= \frac{(t^3 - 2)^3 (t^9 - 6t^6 - 12t^3 - 8)^3}{(t - 2)t^9(t + 1)^2(t^2 - t + 1)^2(t^2 + 2t + 4)} \\ j(E_2) = j(2\tau) &= \frac{(t^3 + 4)^3 (t^9 - 12t^6 + 48t^3 + 64)^3}{(t - 2)^2t^{18}(t + 1)(t^2 - t + 1)(t^2 + 2t + 4)^2} \\ j(E_3) = j(3\tau) &= \frac{(t^3 - 2)^3 (t^3 + 6t - 2)^3 (t^6 - 6t^4 - 4t^3 + 36t^2 + 12t + 4)^3}{(t - 2)^3t^3(t + 1)^6(t^2 - t + 1)^6(t^2 + 2t + 4)^3} \\ j(E_6) = j(6\tau) &= \frac{(t^3 + 4)^3 (t^3 + 6t^2 + 4)^3 (t^6 - 6t^5 + 36t^4 + 8t^3 - 24t^2 + 16)^3}{(t - 2)^6t^6(t + 1)^3(t^2 - t + 1)^3(t^2 + 2t + 4)^6} \\ j(E_9) = j(9\tau) &= \frac{(t^3 + 6t - 2)^3 (t^9 + 234t^7 - 6t^6 + 756t^5 - 936t^4 + 2172t^3 - 1512t^2 + 936t - 8)^3}{(t - 2)^9t(t + 1)^{18}(t^2 - t + 1)^2(t^2 + 2t + 4)} \\ j(E_{18}) = j(18\tau) &= \frac{(t^3 + 6t^2 + 4)^3 (t^9 + 234t^8 + 756t^7 + 2172t^6 + 1872t^5 + 3024t^4 + 48t^3 + 3744t^2 + 64)^3}{(t - 2)^{18}t^2(t + 1)^9(t^2 - t + 1)(t^2 + 2t + 4)^2}. \end{aligned}$$

#### Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_2, E_3, E_6, E_9, E_{18})$  in such a way that the isogeny graph is normalized. Their signatures are:



R <sub>6</sub> signatures	
$c_4(E_1)$	$(t^3 - 2) \cdot (t^9 - 6t^6 - 12t^3 - 8)$
$c_6(E_1)$	$(t^6 - 4t^3 - 8) \cdot (t^{12} - 8t^9 - 8t^3 - 8)$
$\Delta(E_1)$	$(t - 2) \cdot (t + 1)^2 \cdot t^9 \cdot (t^2 + 2t + 4) \cdot (t^2 - t + 1)^2$
$c_4(E_2)$	$(t^3 + 4) \cdot (t^9 - 12t^6 + 48t^3 + 64)$
$c_6(E_2)$	$(t^6 - 4t^3 - 8) \cdot (t^{12} - 8t^9 - 512t^3 - 512)$
$\Delta(E_2)$	$(t + 1) \cdot (t - 2)^2 \cdot t^{18} \cdot (t^2 - t + 1) \cdot (t^2 + 2t + 4)^2$
$c_4(E_3)$	$(t^3 - 2) \cdot (t^3 + 6t - 2) \cdot (t^6 - 6t^4 - 4t^3 + 36t^2 + 12t + 4)$
$c_6(E_3)$	$(t^2 + 2t - 2) \cdot (t^4 - 2t^3 - 8t - 2) \cdot (t^4 - 2t^3 + 6t^2 + 4t + 4) \cdot (t^8 + 2t^7 + 4t^6 - 16t^5 - 14t^4 + 8t^3 + 64t^2 - 16t + 4)$
$\Delta(E_3)$	$(t - 2)^3 \cdot t^3 \cdot (t + 1)^6 \cdot (t^2 + 2t + 4)^3 \cdot (t^2 - t + 1)^6$
$c_4(E_6)$	$(t^3 + 4) \cdot (t^3 + 6t^2 + 4) \cdot (t^6 - 6t^5 + 36t^4 + 8t^3 - 24t^2 + 16)$
$c_6(E_6)$	$(t^2 + 2t - 2) \cdot (t^4 - 8t^3 - 8t - 8) \cdot (t^4 - 2t^3 + 6t^2 + 4t + 4) \cdot (t^8 + 8t^7 + 64t^6 - 16t^5 - 56t^4 + 128t^3 + 64t^2 - 64t + 64)$
$\Delta(E_6)$	$(t + 1)^3 \cdot (t - 2)^6 \cdot t^6 \cdot (t^2 - t + 1)^3 \cdot (t^2 + 2t + 4)^6$
$c_4(E_9)$	$(t^3 + 6t - 2) \cdot (t^9 + 234t^7 - 6t^6 + 756t^5 - 936t^4 + 2172t^3 - 1512t^2 + 936t - 8)$
$c_6(E_9)$	$(t^6 + 24t^5 + 24t^4 + 92t^3 - 48t^2 + 96t - 8) \cdot (t^{12} - 24t^{11} + 48t^{10} - 680t^9 + 792t^8 - 3312t^7 + 4704t^6 - 10656t^5 + 13968t^4 - 14792t^3 + 7968t^2 - 2112t - 8)$
$\Delta(E_9)$	$t \cdot (t - 2)^9 \cdot (t + 1)^{18} \cdot (t^2 + 2t + 4) \cdot (t^2 - t + 1)^2$
$c_4(E_{18})$	$(t^3 + 6t^2 + 4) \cdot (t^9 + 234t^8 + 756t^7 + 2172t^6 + 1872t^5 + 3024t^4 + 48t^3 + 3744t^2 + 64)$
$c_6(E_{18})$	$(t^6 + 24t^5 + 24t^4 + 92t^3 - 48t^2 + 96t - 8) \cdot (t^{12} - 528t^{11} - 3984t^{10} - 14792t^9 - 27936t^8 - 42624t^7 - 37632t^6 - 52992t^5 - 25344t^4 - 43520t^3 - 6144t^2 - 6144t - 512)$
$\Delta(E_{18})$	$t^2 \cdot (t + 1)^9 \cdot (t - 2)^{18} \cdot (t^2 - t + 1) \cdot (t^2 + 2t + 4)^2$

## Automorphisms

The subgroup of  $\text{Aut } X_0(18)$  that fixes the set of vertices of the graph is isomorphic to the Klein group of order 4.

automorphism	permutation	order
$\text{id}(t) = t$	$()$	1
$\sigma(t) = 2(t+1)/(t-2)$	$(j_1 j_{18})(j_2 j_9)(j_3 j_6)$	2
$\tau(t) = -2/t$	$(j_1, j_2)(j_3 j_6)(j_9 j_{18})$	2
$\sigma\tau(t) = -(t-2)/(t+1)$	$(j_1 j_9)(j_2 j_{18})(j_3)(j_6)$	2

Automorphism action on the graph	
id	$()$
$\sigma$	$(E_1 E_{18})^{\otimes -3}(E_2 E_9)^{\otimes -3}(E_3 E_6)^{\otimes -3}$
$\tau$	$(E_1 E_2)(E_3 E_6)(E_9 E_{18})$
$\sigma\tau$	$(E_1 E_9)^{\otimes -3}(E_2 E_{18})^{\otimes -3}(E_3)^{\otimes -3}(E_6)^{\otimes -3}$

Then notation  $(E_i E_j)^{\otimes d}$  stands for the edge isogeny corresponding to the  $d$ -quadratic twisted elliptic curves  $E_i^d$  and  $E_j^d$  from the original  $\mathbb{Q}$ -isogeny graph.

## 21.2 Kodaira symbols, minimal models, and Pal values

Table 35:  $R_6$  data for  $p \neq 2, 3$

$R_6$	$p \neq 2, 3$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, 9m)$	1	$I_{9m}$	1	1
	$E_2$	$(0, 0, 18m)$	1	$I_{18m}$	1	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_6$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_9$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{18}$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
$v_p(t) = 0$ $v_p(t-2) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_6$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_9$	$(0, 0, 9m)$	1	$I_{9m}$	1	1
	$E_{18}$	$(0, 0, 18m)$	1	$I_{18m}$	1	1
$v_p(t) = 0$ $v_p(t+1) = m > 0$	$E_1$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_2$	$(0, 0, m)$	1	$I_m$	1	1
	$E_3$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_6$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_9$	$(0, 0, 18m)$	1	$I_{18m}$	1	1
	$E_{18}$	$(0, 0, 9m)$	1	$I_{9m}$	1	1
$v_p(t) = 0$ $v_p(t^2 + 2t + 4) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_6$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_9$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{18}$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 35:  $R_6$  data for  $p \neq 2, 3$  (Continued)

$R_6$	$p \neq 2, 3$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = 0$ $v_p(t^2 - t + 1) = m > 0$	$E_1$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_2$	$(0, 0, m)$	1	$I_m$	1	1
	$E_3$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_6$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_9$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{18}$	$(0, 0, m)$	1	$I_m$	1	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 18m)$	$p^{-3m}$	$I_{18m}$	1	1
	$E_2$	$(0, 0, 9m)$	$p^{-3m}$	$I_{9m}$	1	1
	$E_3$	$(0, 0, 6m)$	$p^{-3m}$	$I_{6m}$	1	1
	$E_6$	$(0, 0, 3m)$	$p^{-3m}$	$I_{3m}$	1	1
	$E_9$	$(0, 0, 2m)$	$p^{-3m}$	$I_{2m}$	1	1
	$E_{18}$	$(0, 0, m)$	$p^{-3m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 36:  $R_6$  data for  $p = 3$ 

$R_6$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = m > 0$	$E_1$	$(0, 0, 9m)$	1	$I_{9m}$	1	1
	$E_2$	$(0, 0, 18m)$	1	$I_{18m}$	1	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_6$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_9$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{18}$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
$v_3(t) = 0$ $t \equiv 2, 5 (8)$ $v_3(t - 2) = m$	$E_1$	$(2, 3, m + 5)$	1	$I_{m-1}^*$	3	1
	$E_2$	$(2, 3, 2m + 4)$	1	$I_{2m-2}^*$	3	1
	$E_3$	$(2, 3, 3m + 3)$	3	$I_{3m-3}^*$	3	1
	$E_6$	$(2, 3, 6m)$	3	$I_{6m-6}^*$	3	1
	$E_9$	$(2, 3, 9m - 3)$	$3^2$	$I_{9m-9}^*$	3	1
	$E_{18}$	$(2, 3, 18m - 12)$	$3^2$	$I_{18m-18}^*$	3	1
$v_3(t) = 0$ $t \equiv 8 (9)$ $v_3(t + 1) = m$	$E_1$	$(2, 3, 2m + 4)$	1	$I_{2m-2}^*$	3	1
	$E_2$	$(2, 3, m + 5)$	1	$I_{m-1}^*$	3	1
	$E_3$	$(2, 3, 6m)$	3	$I_{6m-6}^*$	3	1
	$E_6$	$(2, 3, 3m + 3)$	3	$I_{3m-3}^*$	3	1
	$E_9$	$(2, 3, 18m - 12)$	$3^2$	$I_{18m-18}^*$	3	1
	$E_{18}$	$(2, 3, 9m - 3)$	$3^2$	$I_{9m-9}^*$	3	1
$v_3(t) = -m < 0$	$E_1$	$(0, 0, 18m)$	$3^{-3m}$	$I_{18m}$	1	1
	$E_2$	$(0, 0, 9m)$	$3^{-3m}$	$I_{9m}$	1	1
	$E_3$	$(0, 0, 6m)$	$3^{-3m}$	$I_{6m}$	1	1
	$E_6$	$(0, 0, 3m)$	$3^{-3m}$	$I_{3m}$	1	1
	$E_9$	$(0, 0, 2m)$	$3^{-3m}$	$I_{2m}$	1	1
	$E_{18}$	$(0, 0, m)$	$3^{-3m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	

Table 37:  $R_6$  data for  $p=2$ 

$R_6$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 1$	$E_1$	$(4, 6, 9m + 3)$	1	$I_{9m-5}^*$	1	1	2
	$E_2$	$(4, 6, 18m - 6)$	2	$I_{18m-14}^*$	1	1	2
	$E_3$	$(4, 6, 3m + 9)$	1	$I_{3m+1}^*$	1	1	2
	$E_6$	$(4, 6, 6m + 6)$	2	$I_{6m-2}^*$	1	1	2
	$E_9$	$(4, 6, m + 11)$	1	$I_{m+3}^*$	1	1	2
	$E_{18}$	$(4, 6, 2m + 10)$	2	$I_{2m+2}^*$	1	1	2
$v_2(t) = 1$ $v_2(t-2) = m$	$E_1$	$(4, 6, m + 11)$	1	$I_{m+3}^*$	1	1	2
	$E_2$	$(4, 6, 2m + 10)$	2	$I_{2m+2}^*$	1	1	2
	$E_3$	$(4, 6, 3m + 9)$	1	$I_{3m+1}^*$	1	1	2
	$E_6$	$(4, 6, 6m + 6)$	2	$I_{6m-2}^*$	1	1	2
	$E_9$	$(4, 6, 9m + 3)$	1	$I_{9m-5}^*$	1	1	2
	$E_{18}$	$(4, 6, 18m - 6)$	2	$I_{18m-14}^*$	1	1	2
$v_2(t) = 0$ $v_2(t+1) = m$	$E_1$	$(4, 6, 2m + 12)$	$2^{-1}$	$I_{2m+4}^*$	1	1	2
	$E_2$	$(4, 6, m + 12)$	$2^{-1}$	$I_{m+4}^*$	1	1	2
	$E_3$	$(4, 6, 6m + 12)$	$2^{-1}$	$I_{6m+4}^*$	1	1	2
	$E_6$	$(4, 6, 3m + 12)$	$2^{-1}$	$I_{3m+4}^*$	1	1	2
	$E_9$	$(4, 6, 18m + 12)$	$2^{-1}$	$I_{18m+4}^*$	1	1	2
	$E_{18}$	$(4, 6, 9m + 12)$	$2^{-1}$	$I_{9m+4}^*$	1	1	2
$v_2(t) = -m < 0$	$E_1$	$(4, 6, 18m + 12)$	$2^{-3m-1}$	$I_{18m+4}^*$	1	1	2
	$E_2$	$(4, 6, 9m + 12)$	$2^{-3m-1}$	$I_{9m+4}^*$	1	1	2
	$E_3$	$(4, 6, 6m + 12)$	$2^{-3m-1}$	$I_{6m+4}^*$	1	1	2
	$E_6$	$(4, 6, 3m + 12)$	$2^{-3m-1}$	$I_{3m+4}^*$	1	1	2
	$E_9$	$(4, 6, 2m + 12)$	$2^{-3m-1}$	$I_{2m+4}^*$	1	1	2
	$E_{18}$	$(4, 6, m + 12)$	$2^{-3m-1}$	$I_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

### 21.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u}_p = [u_p(E)]$  and  $\mathbf{u}_p(d) = [u_p(\mathcal{E}^d)]$ :

$t$	$[u_2(E)]$	$[u_2(\mathcal{E}^d)]$
$v_2(t) > 0$	$(1 : 2 : 1 : 2 : 1 : 2)$	$(1 : 1 : 1 : 1 : 1 : 1)$
$v_2(t) \leq 0$	$(1 : 1 : 1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1 : 1 : 1)$

$t$	$[u_3(E)]$	$[u_3(\mathcal{E}^d)]$
$v_3(t) = 0$	$(1 : 1 : 3 : 3 : 3^2 : 3^2)$	$(1 : 1 : 1 : 1 : 1 : 1)$
$v_3(t) \neq 0$	$(1 : 1 : 1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1 : 1 : 1)$

The contents of these tables are the ingredients to prove the following result:

**Proposition 21.** *Let*

$$\begin{array}{ccccc} E_1 & \xrightarrow{3} & E_3 & \xrightarrow{3} & E_9 \\ 2 \downarrow & & \downarrow 2 & & \downarrow 2 \\ E_2 & \xrightarrow{3} & E_6 & \xrightarrow{3} & E_{18} \end{array}$$

be a  $\mathbf{Q}$ -isogeny graph of type  $R_6$  corresponding to a given  $t$  in  $\mathbf{Q} \setminus \{0, -1, 2\}$  with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted graph

$$\begin{array}{ccccc} E_1^d & \xrightarrow{3} & E_3^d & \xrightarrow{3} & E_9^d \\ 2 \downarrow & & \downarrow 2 & & \downarrow 2 \\ E_2^d & \xrightarrow{3} & E_6^d & \xrightarrow{3} & E_{18}^d \end{array}$$

is given by:

Table 38: Faltings curves in  $R_6$

$R_6$		twisted isogeny graph	Prob
$v_2(t) > 0$	$v_3(t) \neq 0$	$\begin{array}{ccccc} E_1^d & \longrightarrow & E_3^d & \longrightarrow & E_9^d \\ \uparrow & & \uparrow & & \uparrow \\ \textcircled{E_2^d} & \longrightarrow & E_6^d & \longrightarrow & E_{18}^d \end{array}$	1
$v_2(t) > 0$	$v_3(t) = 0$	$\begin{array}{ccccc} E_1^d & \longleftarrow & E_3^d & \longleftarrow & E_9^d \\ \uparrow & & \uparrow & & \uparrow \\ E_2^d & \longleftarrow & E_6^d & \longleftarrow & \textcircled{E_{18}^d} \end{array}$	1

Continued on next page

Table 38: Faltings curves in  $R_6$  (Continued)

$R_6$		<i>twisted isogeny graph</i>	<i>Prob</i>
$v_2(t) \leq 0$	$v_3(t) \neq 0$	$  \begin{array}{ccccc}  \textcircled{E_1^d} & \longrightarrow & E_3^d & \longrightarrow & E_9^d \\  \downarrow & & \downarrow & & \downarrow \\  E_2^d & \longrightarrow & E_6^d & \longrightarrow & E_{18}^d  \end{array}  $	1
$v_2(t) \leq 0$	$v_3(t) = 0$	$  \begin{array}{ccccc}  E_1^d & \longleftarrow & E_3^d & \longleftarrow & \textcircled{E_9^d} \\  \downarrow & & \downarrow & & \downarrow \\  E_2^d & \longleftarrow & E_6^d & \longleftarrow & E_{18}^d  \end{array}  $	1

The column *Prob* gives the probability of the circled twisted curve to be the Faltings curve.

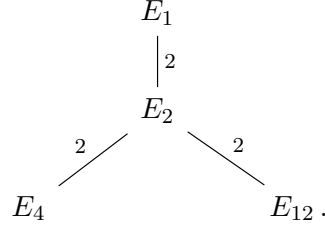


## 22 Type $T_4$

### 22.1 Settings

#### Graph

The isogeny graphs of type  $T_4$  are given by four isogenous elliptic curves:



#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(4)$  parametrize isogeny graphs of type  $T_4$ . The curve  $X_0(4)$  has genus 0 and a hauptmodul for this curve is:

$$t = 2^8 \left( \frac{\eta(4\tau)}{\eta(\tau)} \right)^8 .$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$\begin{aligned}
 j_1 = j(E_1) = j(\tau) &= \frac{(t^2 + 16t + 16)^3}{t(t + 16)} \\
 j_2 = j(E_2) = j(2\tau) &= \frac{(t^2 + 16t + 256)^3}{t^2(t + 16)^2} \\
 j_4 = j(E_4) = j(4\tau) &= \frac{(t^2 + 256t + 4096)^3}{t^4(t + 16)} \\
 j_{12} = j(E_{12}) = j(\tau + 1/2) &= -\frac{(t^2 - 224t + 256)^3}{t(t + 16)^4} .
 \end{aligned}$$

## Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_2, E_4, E_{12})$  in such a way that the isogeny graph is normalized. Their signatures are:

$c_4(E_1)$	$(t^2 + 16t + 16)$
$c_6(E_1)$	$(t + 8)(t^2 + 16t - 8)$
$\Delta(E_1)$	$t(t + 16)$
$c_4(E_2)$	$(t^2 + 16t + 256)$
$c_6(E_2)$	$(t - 16)(t + 8)(t + 32)$
$\Delta(E_2)$	$t^2(t + 16)^2$
$c_4(E_4)$	$(t^2 + 256t + 4096)$
$c_6(E_4)$	$(t + 32)(t^2 - 512t - 8192)$
$\Delta(E_4)$	$t^4(t + 16)$
$c_4(E_{12})$	$(t^2 - 224t + 256)$
$c_6(E_{12})$	$(t - 16)(t^2 + 544t + 256)$
$\Delta(E_{12})$	$-t(t + 16)^4$

## Automorphisms

The subgroup of  $\text{Aut } X_0(4)$  that fixes the set of vertices of the graph is isomorphic to the symmetric group  $\mathcal{S}_3$  with elements:

	permutation	order
$\text{id}(t) = t$	$()$	1
$\sigma(t) = -256/(t + 16)$	$(j_1 j_{12} j_4)$	3
$\sigma^2(t) = -16(t + 16)/t$	$(j_1 j_4 j_{12})$	3
$\tau(t) = 256/t$	$(j_1 j_4)$	2
$\sigma\tau(t) = -(t + 16)$	$(j_4 j_{12})$	2
$\sigma^2\tau(t) = -16t/(t + 16)$	$(j_1 j_{12})$	2

## 22.2 Kodaira symbols, minimal models, and Pal values

Table 39:  $T_4$  data for  $p \neq 2$

$T_4$	$p \neq 2$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_4$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_{12}$	$(0, 0, m)$	1	$I_m$	1	1
$v_p(t) = 0$ $v_p(t + 16) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_4$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{12}$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
$v_p(t) = -m < 0$ $m$ odd	$E_1$	$(2, 3, 4m + 6)$	$p^{-(m+1)/2}$	$I_{4m}^*$	$p$	1
	$E_2$	$(2, 3, 2m + 6)$	$p^{-(m+1)/2}$	$I_{2m}^*$	$p$	1
	$E_4$	$(2, 3, m + 6)$	$p^{-(m+1)/2}$	$I_m^*$	$p$	1
	$E_{12}$	$(2, 3, m + 6)$	$p^{-(m+1)/2}$	$I_m^*$	$p$	1
$v_p(t) = -m < 0$ $m$ even	$E_1$	$(0, 0, 4m)$	$p^{-m/2}$	$I_{4m}$	1	1
	$E_2$	$(0, 0, 2m)$	$p^{-m/2}$	$I_{2m}$	1	1
	$E_4$	$(0, 0, m)$	$p^{-m/2}$	$I_m$	1	1
	$E_{12}$	$(0, 0, m)$	$p^{-m/2}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 40:  $T_4$  data for  $p=2$ 

$T_4$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 7$	$E_1$	$(0, 0, m - 8)$	2	$I_{m-8}$	1	$2^{-1}$	$2^{-1}$
	$E_2$	$(0, 0, 2(m - 8))$	$2^2$	$I_{2(m-8)}$	1	$2^{-1}$	$2^{-1}$
	$E_4$	$(0, 0, 4(m - 8))$	$2^3$	$I_{4(m-8)}$	1	$2^{-1}$	$2^{-1}$
	$E_{12}$	$(0, 0, m - 8)$	$2^2$	$I_{m-8}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 7$	$E_1$	$(4, 6, 11)$	1	$\text{II}^*$	1	1	1
	$E_2$	$(4, 6, 10)$	2	$\text{III}^*$	1	1	1
	$E_4$	$(4, 6, 8)$	$2^2$	$I_1^*$	1	1	1
	$E_{12}$	$(4, 6, 11)$	2	$\text{II}^*$	1	1	1
$v_2(t) = 6$	$E_1$	$(4, 6, 10)$	1	$\text{III}^*$	1	1	1
	$E_2$	$(4, 6, 8)$	2	$I_1^*$	1	1	1
	$E_4$	$(5, 5, 4)$	$2^2$	$\text{III}$	1	1	1
	$E_{12}$	$(4, 6, 10)$	2	$\text{III}^*$	1	1	1
$v_2(t) = 5$	$E_1$	$(4, 6, 9)$	1	$I_0^*$	1	1	1
	$E_2$	$(4, \geq 7, 6)$	2	$\text{III}$	1	1	1
	$E_4$	$(6, \geq 10, 12)$	2	$I_3^*$	1	2	1
	$E_{12}$	$(4, 6, 9)$	2	$I_0^*$	1	1	1
$v_2(t) = 4$ $t/2^4 \equiv 1 (4)$	$E_1$	$(4, 6, 9)$	1	$I_0^*$	1	1	1
	$E_2$	$(4, \geq 7, 6)$	2	$\text{III}$	1	1	1
	$E_4$	$(4, 6, 9)$	2	$I_0^*$	1	1	1
	$E_{12}$	$(6, \geq 10, 12)$	2	$I_3^*$	1	2	1
$v_2(t) = 4$ $t/2^4 \equiv -1 (16)$ $v_2(t + 16) = m > 7$	$E_1$	$(4, 6, 4 + m)$	1	$I_{m-4}^*$	1	1	2
	$E_2$	$(4, 6, 2m - 4)$	2	$I_{2m-12}^*$	1	1	2
	$E_4$	$(4, 6, 4 + m)$	2	$I_{m-4}^*$	1	1	2
	$E_{12}$	$(4, 6, 4m - 20)$	$2^2$	$I_{4m-28}^*$	1	1	2
$v_2(t) = 4$ $t/2^4 \equiv 7 (16)$	$E_1$	$(4, 6, 11)$	1	$I_3^*$	1	1	1
	$E_2$	$(4, 6, 10)$	2	$I_2^*$	1	1	1
	$E_4$	$(4, 6, 11)$	2	$I_3^*$	1	1	1
	$E_{12}$	$(4, 6, 8)$	$2^2$	$I_0^*$	1	1	1
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

Table 40:  $T_4$  data for  $p=2$  (Continued)

$T_4$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = 4$ $t/2^4 \equiv 3 (8)$	$E_1$	$(4, 6, 10)$	1	$I_2^*$	1	1	1
	$E_2$	$(4, 6, 8)$	2	$I_0^*$	1	1	1
	$E_4$	$(4, 6, 10)$	2	$I_2^*$	1	1	1
	$E_{12}$	$(5, 5, 4)$	$2^2$	$\text{II}$	1	1	1
$v_2(t) = 3$	$E_1$	$(4, \geq 7, 6)$	1	$\text{II}$	1	1	1
	$E_2$	$(6, \geq 10, 12)$	1	$I_2^*$	1	2	1
	$E_4$	$(6, 9, 15)$	1	$I_5^*$	1	2	1
	$E_{12}$	$(6, 9, 15)$	1	$I_5^*$	1	2	1
$v_2(t) = 2$ $t/2^2 \equiv 1 (4)$	$E_1$	$(5, 5, 4)$	1	$\text{II}$	1	1	1
	$E_2$	$(4, 6, 8)$	1	$I_0^*$	1	1	1
	$E_4$	$(4, 6, 10)$	1	$I_2^*$	1	1	1
	$E_{12}$	$(4, 6, 10)$	1	$I_2^*$	1	1	1
$v_2(t) = 2$ $t/2^2 \equiv 3 (4)$	$E_1$	$(5, 5, 4)$	1	$\text{III}$	1	1	1
	$E_2$	$(4, 6, 8)$	1	$I_1^*$	1	1	1
	$E_4$	$(4, 6, 10)$	1	$\text{III}^*$	1	1	1
	$E_{12}$	$(4, 6, 10)$	1	$\text{III}^*$	1	1	1
$v_2(t) = 1$	$E_1$	$(6, 9, 14)$	$2^{-1}$	$I_4^*$	1	2	1
	$E_2$	$(6, 9, 16)$	$2^{-1}$	$I_6^*$	1	2	1
	$E_4$	$(6, 9, 17)$	$2^{-1}$	$I_7^*$	1	2	1
	$E_{12}$	$(6, 9, 17)$	$2^{-1}$	$I_7^*$	1	2	1
$v_p(t) = -2m \leq 0$ $2^{2m}t \equiv 1 (4)$	$E_1$	$(4, 6, 12 + 8m)$	$2^{-(m+1)}$	$I_{4+8m}^*$	1	1	2
	$E_2$	$(4, 6, 12 + 4m)$	$2^{-(m+1)}$	$I_{4+4m}^*$	1	1	2
	$E_4$	$(4, 6, 12 + 2m)$	$2^{-(m+1)}$	$I_{4+2m}^*$	1	1	2
	$E_{12}$	$(4, 6, 12 + 2m)$	$2^{-(m+1)}$	$I_{4+2m}^*$	1	1	2
$v_p(t) = -2m \leq 0$ $2^{2m}t \equiv 3 (4)$	$E_1$	$(0, 0, 8m)$	$2^{-m}$	$I_{8m}$	1	$2^{-1}$	$2^{-1}$
	$E_2$	$(0, 0, 4m)$	$2^{-m}$	$I_{4m}$	1	$2^{-1}$	$2^{-1}$
	$E_4$	$(0, 0, 2m)$	$2^{-m}$	$I_{2m}$	1	$2^{-1}$	$2^{-1}$
	$E_{12}$	$(0, 0, 2m)$	$2^{-m}$	$I_{2m}$	1	$2^{-1}$	$2^{-1}$
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

Table 40:  $T_4$  data for  $p=2$  (Continued)

$T_4$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_p(t) = -(2m+1) < 0$	$E_1$	$(6, 9, 8m+22)$	$2^{-(m+2)}$	$I_{8m+12}^*$	1	$2^2$	1
	$E_2$	$(6, 9, 4m+20)$	$2^{-(m+2)}$	$I_{4m+10}^*$	1	$2^2$	1
	$E_4$	$(6, 9, 2m+19)$	$2^{-(m+2)}$	$I_{2m+9}^*$	1	$2^2$	1
	$E_{12}$	$(6, 9, 2m+19)$	$2^{-(m+2)}$	$I_{2m+9}^*$	1	$2^2$	1
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

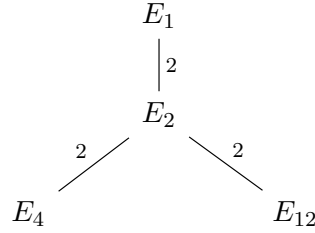
### 22.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

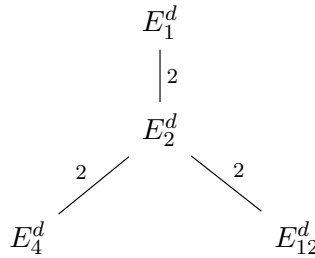
$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$	$d$
$v_2(t) \geq 6$	$(1 : 2 : 2^2 : 1)$	$(1 : 1 : 1 : 1)$	
$v_2(t) = 5$	$(1 : 2 : 2 : 2)$	$(1 : 1 : 1 : 1)$	$d \not\equiv 0 (2)$
		$(1 : 1 : 2 : 1)$	$d \equiv 0 (2)$
$v_2(t) = 4$ $t/2^4 \equiv 1 (4)$	$(1 : 2 : 2 : 2)$	$(1 : 1 : 1 : 1)$	$d \not\equiv 0 (2)$
		$(1 : 1 : 1 : 2)$	$d \equiv 0 (2)$
$v_2(t) = 4$ $t/2^4 \equiv 3 (4)$	$(1 : 2 : 2 : 2^2)$	$(1 : 1 : 1 : 1)$	
$v_2(t) = 3$	$(1 : 1 : 1 : 1)$	$(1 : 2 : 2 : 2)$	$d \not\equiv 0 (2)$
		$(1 : 1 : 1 : 1)$	$d \equiv 0 (2)$
$v_2(t) \leq 2$	$(1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1)$	

The contents of this table are the main ingredients to prove the following result:

**Proposition 22.** *Let*



be a  $\mathbf{Q}$ -isogeny graph of type  $T_4$  corresponding to a given  $t$  in  $\mathbf{Q}$ ,  $t \neq 0, -16$ , with the signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph



is given by:

Table 41: Faltings curves in  $T_4$

$T_4$	twisted isogeny graph	$d$	Prob
$v_2(t) \geq 6$			1
$v_2(t) = 5$		$d \equiv 0 (2)$	1/3
		$d \not\equiv 0 (2)$	2/3
$v_2(t) = 4$ $t/2^4 \equiv 1 (4)$		$d \equiv 0 (2)$	1/3
		$d \not\equiv 0 (2)$	2/3
$v_2(t) = 4$ $t/2^4 \equiv 3 (4)$			1
$v_2(t) = 3$		$d \equiv 0 (2)$	1/3
		$d \not\equiv 0 (2)$	2/3

Continued on next page



Table 41: Faltings curves in  $T_4$  (Continued)

$T_4$	twisted isogeny graph	$d$	Prob
$v_2(t) \leq 2$	<pre> graph TD     E1((E1^d)) --&gt; E2[E2^d]     E2 --&gt; E4[E4^d]     E2 --&gt; E12[E12^d] </pre>		1

The column *Prob* gives the probability of the circled twisted curve to be the Faltings curve.

This page illustrates the strategy followed in all the types: (i) the period lattices of  $E_i$ , (ii) their volumes, (iii) the volumes of their Néron models  $\mathcal{E}_i$ , (iv) the volumes of the Néron models of their quadratic twists  $E_i^d$ .

$$\begin{array}{c}
 \omega\langle 1, \tau \rangle \\
 | \\
 \frac{1}{2} \omega\langle 1, 2\tau \rangle \\
 \swarrow \quad \searrow \\
 \frac{1}{4} \omega\langle 1, 4\tau \rangle \quad \frac{1}{2} \omega\langle 1, \tau + 1/2 \rangle
 \end{array}$$

$$\begin{array}{c}
 V \\
 | \\
 \frac{1}{2} V \\
 \swarrow \quad \searrow \\
 \frac{1}{4} V \quad \frac{1}{4} V
 \end{array}$$

$$\begin{array}{c}
 Vu_1^2 \\
 | \\
 \frac{1}{2} Vu_2^2 \\
 \swarrow \quad \searrow \\
 \frac{1}{4} Vu_4^2 \quad \frac{1}{4} Vu_{12}^2
 \end{array}$$

$$\begin{array}{c}
 Vu_1^2 \frac{u_1(d)^2}{|d|} \\
 | \\
 \frac{1}{2} Vu_2^2 \frac{u_2(d)^2}{|d|} \\
 \swarrow \quad \searrow \\
 \frac{1}{4} Vu_4^2 \frac{u_4(d)^2}{|d|} \quad \frac{1}{4} Vu_{12}^2 \frac{u_{12}(d)^2}{|d|}
 \end{array}$$

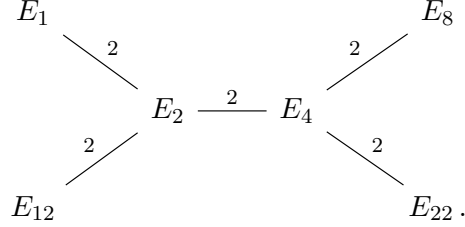
Here,  $[u(E)] = [u_1 : u_2 : u_4 : u_{12}]$  and  $[u(\mathcal{E}^d)] = [u_1(d) : u_2(d) : u_4(d) : u_{12}(d)]$ , with  $u_i = u(E_i)$  and  $u_i(d) = u(\mathcal{E}_i^d)$ .

## 23 Type $T_6$

### 23.1 Setting

#### Graph

The isogeny graphs of type  $T_6$  are given by six isogenous elliptic curves:



#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(8)$  parametrize isogeny graphs of type  $T_6$ . The curve  $X_0(8)$  has genus 0 and a hauptmodul for this curve is:

$$t = 4 + 2^5 \frac{\eta(2\tau)^2 \eta(8t)^4}{\eta(\tau)^4 \eta(4\tau)^2}.$$

#### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$\begin{aligned}
 j_1 = j(E_1) = j(\tau) &= \frac{(t^4 - 16t^2 + 16)^3}{(t-4)t^2(t+4)} \\
 j_2 = j(E_2) = j(2\tau) &= \frac{(t^4 - 16t^2 + 256)^3}{(t-4)^2 t^4 (t+4)^2} \\
 j_{12} = j(E_{12}) = j(\tau + 1/2) &= -\frac{(t^4 - 256t^2 + 4096)^3}{(t-4)t^8(t+4)} \\
 j_4 = j(E_4) = j(4\tau) &= \frac{(t^4 + 224t^2 + 256)^3}{(t-4)^4 t^2 (t+4)^4} \\
 j_8 = j(E_8) = j(8\tau) &= \frac{(t^4 + 240t^3 + 2144t^2 + 3840t + 256)^3}{(t-4)^8 t (t+4)^2} \\
 j_{22} = j(E_{22}) = j(2\tau + 1/2) &= -\frac{(t^4 - 240t^3 + 2144t^2 - 3840t + 256)^3}{(t-4)^2 t (t+4)^8}.
 \end{aligned}$$

## Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_2, E_{12}, E_4, E_8, E_{22})$  in such a way that the isogeny graph is normalized. Their signatures are:

$T_6$ signatures	
$c_4(E_1)$	$(t^4 - 16t^2 + 16)$
$c_6(E_1)$	$(t^2 - 8)(t^4 - 16t^2 - 8)$
$\Delta(E_1)$	$t^2(t - 4)(t + 4)$
$c_4(E_2)$	$(t^4 - 16t^2 + 256)$
$c_6(E_2)$	$(t^2 - 32)(t^2 - 8)(t^2 + 16)$
$\Delta(E_2)$	$t^4(t - 4)^2(t + 4)^2$
$c_4(E_{12})$	$(t^4 - 256t^2 + 4096)$
$c_6(E_{12})$	$(t^2 - 32)(t^4 + 512t^2 - 8192)$
$\Delta(E_{12})$	$-t^8(t - 4)(t + 4)$
$c_4(E_4)$	$(t^4 + 224t^2 + 256)$
$c_6(E_4)$	$(t^2 - 24t + 16)(t^2 + 16)(t^2 + 24t + 16)$
$\Delta(E_4)$	$t^2(t - 4)^4(t + 4)^4$
$c_4(E_8)$	$(t^4 + 240t^3 + 2144t^2 + 3840t + 256)$
$c_6(E_8)$	$(t^2 + 24t + 16)(t^4 - 528t^3 - 4000t^2 - 8448t + 256)$
$\Delta(E_8)$	$t(t - 4)^8(t + 4)^2$
$c_4(E_{22})$	$(t^4 - 240t^3 + 2144t^2 - 3840t + 256)$
$c_6(E_{22})$	$(t^2 - 24t + 16)(t^4 + 528t^3 - 4000t^2 + 8448t + 256)$
$\Delta(E_{22})$	$-t(t - 4)^2(t + 4)^8$

## Automorphisms

The subgroup of  $\text{Aut } X_0(8)$  that fixes the set of vertices of the graph is isomorphic to the dihedral group  $D_4$  of eight elements:

	permutation	order
$\text{id}(t) = t$	$()$	1
$\sigma(t) = -t$	$(j_8 j_{22})$	2
$\tau(t) = (4t - 16)/(t + 4)$	$(j_1 j_{22} j_{12} j_8)(j_2 j_4)$	4
$\tau^2(t) = -16/t$	$(j_{12} j_{21})(j_{22} j_8)$	2
$\tau^3(t) = (-4t - 16)/(t - 4)$	$(j_1 j_8 j_{12} j_{22})(j_2 j_4)$	4
$\sigma\tau(t) = (4t + 16)/(t - 4)$	$(j_1 j_8)(j_2 j_4)(j_{12} j_{22})$	2
$\sigma\tau^2(t) = 16/t$	$(j_1 j_{12})$	2
$\sigma\tau^3(t) = (-4t + 16)/(t + 4)$	$(j_1 j_{22})(j_2 j_4)(j_8 j_{12})$	2

## 23.2 Kodaira symbols, minimal models, and Pal values

Table 42:  $T_6$  data for  $p \neq 2$

$T_6$	$p \neq 2$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_2$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_{12}$	$(0, 0, 8m)$	1	$I_{8m}$	1	1
	$E_4$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_8$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{22}$	$(0, 0, m)$	1	$I_m$	1	1
$v_p(t) = 0$ $v_p(t - 4) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{12}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_4$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_8$	$(0, 0, 8m)$	1	$I_{8m}$	1	1
	$E_{22}$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
$v_p(t) = 0$ $v_p(t + 4) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{12}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_4$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_8$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{22}$	$(0, 0, 8m)$	1	$I_{8m}$	1	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 8m)$	$p^{-m}$	$I_{8m}$	1	1
	$E_2$	$(0, 0, 4m)$	$p^{-m}$	$I_{4m}$	1	1
	$E_{12}$	$(0, 0, 2m)$	$p^{-m}$	$I_{2m}$	1	1
	$E_4$	$(0, 0, 2m)$	$p^{-m}$	$I_{2m}$	1	1
	$E_8$	$(0, 0, m)$	$p^{-m}$	$I_m$	1	1
	$E_{22}$	$(0, 0, m)$	$p^{-m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 43:  $T_6$  data for  $p=2$ 

$T_6$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 3$	$E_1$	$(4, 6, 2m + 4)$	1	$I_{2m-4}^*$	1	1	2
	$E_2$	$(4, 6, 4m - 4)$	2	$I_{4m-12}^*$	1	1	2
	$E_{12}$	$(4, 6, 8m - 20)$	$2^2$	$I_{8m-28}^*$	1	1	2
	$E_4$	$(4, 6, 2m + 4)$	2	$I_{2m-4}^*$	1	1	2
	$E_8$	$(4, 6, m + 8)$	2	$I_m^*$	1	1	2
	$E_{22}$	$(4, 6, m + 8)$	2	$I_m^*$	1	1	2
$v_2(t) = 3$	$E_1$	$(4, 6, 10)$	1	$I_2^*$	1	1	1
	$E_2$	$(4, 6, 8)$	2	$I_0^*$	1	1	1
	$E_{12}$	$(5, 5, 4)$	$2^2$	$\Pi$	1	1	1
	$E_4$	$(4, 6, 10)$	2	$I_2^*$	1	1	1
	$E_8$	$(4, 6, 11)$	2	$I_3^*$	1	1	1
	$E_{22}$	$(4, 6, 11)$	2	$I_3^*$	1	1	1
$v_2(t) = 2$ $t \equiv 4 (32)$ $v_2(t - 4) = m + 5$	$E_1$	$(0, 0, m)$	2	$I_m$	1	$2^{-1}$	$2^{-1}$
	$E_2$	$(0, 0, 2m)$	$2^2$	$I_{2m}$	1	$2^{-1}$	$2^{-1}$
	$E_{12}$	$(0, 0, m)$	$2^2$	$I_m$	1	$2^{-1}$	$2^{-1}$
	$E_4$	$(0, 0, 4m)$	$2^3$	$I_{4m}$	1	$2^{-1}$	$2^{-1}$
	$E_8$	$(0, 0, 8m)$	$2^4$	$I_{8m}$	1	$2^{-1}$	$2^{-1}$
	$E_{22}$	$(0, 0, 2m)$	$2^3$	$I_{2m}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 2$ $t \equiv 28 (32)$ $v_2(t + 4) = m + 5$	$E_1$	$(0, 0, m)$	2	$I_m$	1	$2^{-1}$	$2^{-1}$
	$E_2$	$(0, 0, 2m)$	$2^2$	$I_{2m}$	1	$2^{-1}$	$2^{-1}$
	$E_{12}$	$(0, 0, m)$	$2^2$	$I_m$	1	$2^{-1}$	$2^{-1}$
	$E_4$	$(0, 0, 4m)$	$2^3$	$I_{4m}$	1	$2^{-1}$	$2^{-1}$
	$E_8$	$(0, 0, 2m)$	$2^3$	$I_{2m}$	1	$2^{-1}$	$2^{-1}$
	$E_{22}$	$(0, 0, 8m)$	$2^4$	$I_{8m}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 2$ $t \equiv 12 (32)$	$E_1$	$(4, 6, 11)$	1	$\Pi^*$	1	1	1
	$E_2$	$(4, 6, 10)$	2	$\text{III}^*$	1	1	1
	$E_{12}$	$(4, 6, 11)$	2	$\Pi^*$	1	1	1
	$E_4$	$(4, 6, 8)$	$2^2$	$I_1^*$	1	1	1
	$E_8$	$(4, 6, 10)$	$2^2$	$\text{III}^*$	1	1	1
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

Continued on next page

Table 43:  $T_6$  data for  $p=2$  (Continued)

$T_6$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
	$E_{22}$	$(5, 5, 4)$	$2^3$	III	1	1	1
$v_2(t) = 2$ $t \equiv 20 \pmod{32}$	$E_1$	$(4, 6, 11)$	1	$\text{II}^*$	1	1	1
	$E_2$	$(4, 6, 10)$	2	$\text{III}^*$	1	1	1
	$E_{12}$	$(4, 6, 11)$	2	$\text{II}^*$	1	1	1
	$E_4$	$(4, 6, 8)$	$2^2$	$\text{I}_1^*$	1	1	1
	$E_8$	$(5, 5, 4)$	$2^3$	III	1	1	1
	$E_{22}$	$(4, 6, 10)$	$2^2$	$\text{III}^*$	1	1	1
$v_2(t) = 1$	$E_1$	$(5, 5, 4)$	1	II	1	1	1
	$E_2$	$(4, 6, 8)$	1	$\text{I}_0^*$	1	1	1
	$E_{12}$	$(4, 6, 10)$	1	$\text{I}_2^*$	1	1	1
	$E_4$	$(4, 6, 10)$	1	$\text{I}_2^*$	1	1	1
	$E_8$	$(4, 6, 11)$	1	$\text{I}_3^*$	1	1	1
	$E_{22}$	$(4, 6, 11)$	1	$\text{I}_3^*$	1	1	1
$v_2(t) = -m \leq 0$	$E_1$	$(4, 6, 8m + 12)$	$2^{-(m+1)}$	$\text{I}_{8m+4}^*$	1	1	2
	$E_2$	$(4, 6, 4m + 12)$	$2^{-(m+1)}$	$\text{I}_{4m+4}^*$	1	1	2
	$E_{12}$	$(4, 6, 2m + 12)$	$2^{-(m+1)}$	$\text{I}_{2m+4}^*$	1	1	2
	$E_4$	$(4, 6, 2m + 12)$	$2^{-(m+1)}$	$\text{I}_{2m+4}^*$	1	1	2
	$E_8$	$(4, 6, m + 12)$	$2^{-(m+1)}$	$\text{I}_{m+4}^*$	1	1	2
	$E_{22}$	$(4, 6, m + 12)$	$2^{-(m+1)}$	$\text{I}_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

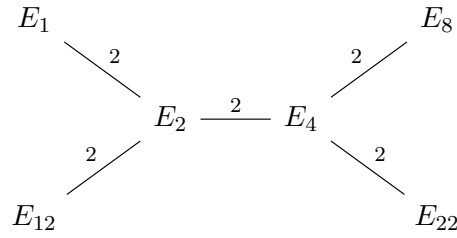
### 23.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

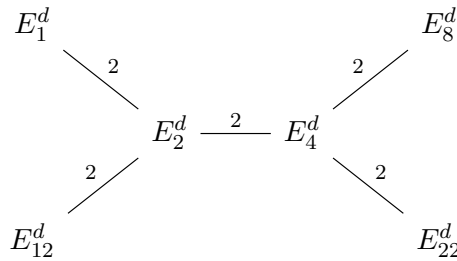
$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$
$v_2(t) \geq 3$	$(1 : 2 : 2^2 : 2 : 2 : 2)$	$(1 : 1 : 1 : 1 : 1 : 1)$
$v_2(t) = 2$ $t/2^2 \equiv 3 \pmod{4}$	$(1 : 2 : 2 : 2^2 : 2^2 : 2^3)$	$(1 : 1 : 1 : 1 : 1 : 1)$
$v_2(t) = 2$ $t/2^2 \equiv 1 \pmod{4}$	$(1 : 2 : 2 : 2^2 : 2^3 : 2^2)$	$(1 : 1 : 1 : 1 : 1 : 1)$
$v_2(t) \leq 1$	$(1 : 1 : 1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1 : 1 : 1)$

The contents of this table are the main ingredients to prove the following result:

**Proposition 23.** *Let*



be a  $\mathbf{Q}$ -isogeny graph of type  $T_6$  corresponding to a given  $t$  in  $\mathbf{Q}$ ,  $t \neq 0, \pm 4$ , with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph



is given by:



Table 44: Faltings curves in  $T_6$

$T_6$	twisted isogeny graph	Prob
$v_2(t) \geq 3$		1
$v_2(t) = 2$ $t/2^2 \equiv 3 \pmod{4}$		1
$v_2(t) = 2$ $t/2^2 \equiv 1 \pmod{4}$		1
$v_2(t) \leq 1$		1

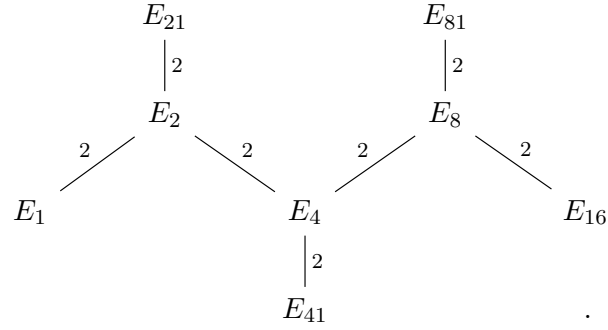
The column Prob gives the probability of the circled twisted curve to be the Faltings curve.

## 24 Type $T_8$

### 24.1 Settings

#### Graph

The isogeny graphs of type  $T_8$  are given by eight isogenous elliptic curves:



#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(16)$  parametrize isogeny graphs of type  $T_8$ . The curve  $X_0(16)$  has genus 0 and a hauptmodul for this curve is:

$$t = 2 + 2^3 \frac{\eta(2\tau)\eta(16t)^2}{\eta(\tau)^2\eta(8\tau)}.$$

### $j$ -invariants

Letting  $t = t(\tau)$ , one can write

$$j(E_1) = j(\tau) = \frac{(t^8 - 16t^4 + 16)^3}{(t-2)t^4(t+2)(t^2+4)}$$

$$j(E_2) = j(2\tau) = \frac{(t^8 - 16t^4 + 256)^3}{(t-2)^2t^8(t+2)^2(t^2+4)^2}$$

$$j(E_{21}) = j(\tau + 1/2) = -\frac{(t^8 - 256t^4 + 4096)^3}{(t-2)t^{16}(t+2)(t^2+4)}$$

$$j(E_4) = j(4\tau) = \frac{(t^4 - 4t^3 + 8t^2 + 16t + 16)^3 (t^4 + 4t^3 + 8t^2 - 16t + 16)^3}{(t-2)^4t^4(t+2)^4(t^2+4)^4}$$

$$j(E_{41}) = j(2\tau + 1/2) = -\frac{(t^4 - 16t^3 + 8t^2 + 64t + 16)^3 (t^4 + 16t^3 + 8t^2 - 64t + 16)^3}{(t-2)^2t^2(t+2)^2(t^2+4)^8}$$

$$j(E_8) = j(8\tau) = \frac{(t^8 + 240t^6 + 2144t^4 + 3840t^2 + 256)^3}{(t-2)^8t^2(t+2)^8(t^2+4)^2}$$

$$j(E_{81}) = j(4\tau + 1/2) = -\frac{(t^8 - 240t^7 + 2160t^6 - 6720t^5 + 17504t^4 - 26880t^3 + 34560t^2 - 15360t + 256)^3}{(t-2)^4t(t+2)^{16}(t^2+4)}$$

$$j(E_{16}) = j(16\tau) = \frac{(t^8 + 240t^7 + 2160t^6 + 6720t^5 + 17504t^4 + 26880t^3 + 34560t^2 + 15360t + 256)^3}{(t-2)^{16}t(t+2)^4(t^2+4)}.$$

## Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_2, E_{21}, E_4, E_{41}, E_8, E_{81}, E_{16})$  in such a way that the isogeny graph is normalized. Their signatures are:

	$T_8$
$c_4(E_1)$	$(t^8 - 16t^4 + 16)$
$c_6(E_1)$	$(t^4 - 8) \cdot (t^8 - 16t^4 - 8)$
$\Delta(E_1)$	$(t - 2) \cdot (t + 2) \cdot t^4 \cdot (t^2 + 4)$
$c_4(E_2)$	$(t^8 - 16t^4 + 256)$
$c_6(E_2)$	$(t^4 - 32) \cdot (t^4 - 8) \cdot (t^4 + 16)$
$\Delta(E_2)$	$(t - 2)^2 \cdot (t + 2)^2 \cdot t^8 \cdot (t^2 + 4)^2$
$c_4(E_{21})$	$(t^8 - 256t^4 + 4096)$
$c_6(E_{21})$	$(t^4 - 32) \cdot (t^8 + 512t^4 - 8192)$
$\Delta(E_{21})$	$(-1) \cdot (t - 2) \cdot (t + 2) \cdot t^{16} \cdot (t^2 + 4)$
$c_4(E_4)$	$(t^4 - 4t^3 + 8t^2 + 16t + 16) \cdot (t^4 + 4t^3 + 8t^2 - 16t + 16)$
$c_6(E_4)$	$(t^2 - 4t - 4) \cdot (t^2 + 4t - 4) \cdot (t^4 + 16) \cdot (t^4 + 24t^2 + 16)$
$\Delta(E_4)$	$(t - 2)^4 \cdot t^4 \cdot (t + 2)^4 \cdot (t^2 + 4)^4$
$c_4(E_{41})$	$(t^4 - 16t^3 + 8t^2 + 64t + 16) \cdot (t^4 + 16t^3 + 8t^2 - 64t + 16)$
$c_6(E_{41})$	$(t^2 - 4t - 4) \cdot (t^2 + 4t - 4) \cdot (t^8 + 528t^6 - 4000t^4 + 8448t^2 + 256)$
$\Delta(E_{41})$	$(-1) \cdot (t - 2)^2 \cdot t^2 \cdot (t + 2)^2 \cdot (t^2 + 4)^8$
$c_4(E_8)$	$(t^8 + 240t^6 + 2144t^4 + 3840t^2 + 256)$
$c_6(E_8)$	$(t^4 - 24t^3 + 24t^2 - 96t + 16) \cdot (t^4 + 24t^2 + 16) \cdot (t^4 + 24t^3 + 24t^2 + 96t + 16)$
$\Delta(E_8)$	$t^2 \cdot (t - 2)^8 \cdot (t + 2)^8 \cdot (t^2 + 4)^2$
$c_4(E_{81})$	$(t^8 - 240t^7 + 2160t^6 - 6720t^5 + 17504t^4 - 26880t^3 + 34560t^2 - 15360t + 256)$
$c_6(E_{81})$	$(t^4 - 24t^3 + 24t^2 - 96t + 16) \cdot (t^8 + 528t^7 - 3984t^6 + 14784t^5 - 31648t^4 + 59136t^3 - 63744t^2 + 33792t + 256)$
$\Delta(E_{81})$	$(-1) \cdot t \cdot (t - 2)^4 \cdot (t + 2)^{16} \cdot (t^2 + 4)$
$c_4(E_{16})$	$(t^8 + 240t^7 + 2160t^6 + 6720t^5 + 17504t^4 + 26880t^3 + 34560t^2 + 15360t + 256)$
$c_6(E_{16})$	$(t^4 + 24t^3 + 24t^2 + 96t + 16) \cdot (t^8 - 528t^7 - 3984t^6 - 14784t^5 - 31648t^4 - 59136t^3 - 63744t^2 - 33792t + 256)$
$\Delta(E_{16})$	$t \cdot (t + 2)^4 \cdot (t - 2)^{16} \cdot (t^2 + 4)$

## Automorphisms

The subgroup of  $\text{Aut } X_0(16)$  that fixes the set of vertices of the graph is isomorphic to the dihedral group of order 8:

automorphism	permutation	order
$\text{id}(t) = t$	$()$	1
$\sigma(t) = 2(t-2)/(t+2)$	$(j_1 j_{81} j_{21} j_{16})(j_2 j_8)$	4
$\sigma^2(t) = -4/t$	$(j_1 j_{21})(j_{81} j_{16})$	2
$\sigma^3(t) = -2(t+2)/(t-2)$	$(j_1 j_{16} j_{21} j_{81})(j_2 j_8)$	4
$\tau(t) = -t$	$(j_{81} j_{16})$	2
$\sigma\tau(t) = 2(t+2)/(t-2)$	$(j_1 j_{16})(j_2 j_8)(j_{21} j_{81})$	2
$\sigma^2\tau(t) = 4/t$	$(j_1 j_{21})$	2
$\sigma^3\tau(t) = -2(t-2)/(t+2)$	$(j_1, j_{81})(j_2 j_8)(j_{21} j_{16})$	2

Automorphism action on the graph	
id	$()$
$\sigma$	$(E_1 E_{81} E_{21} E_{16})^{\otimes -1} (E_2 E_8)^{\otimes -1} (E_4)^{\otimes -1} (E_{41})^{\otimes -1}$
$\sigma^2$	$(E_1 E_{21})(E_{81} E_{16})$
$\sigma^3$	$(E_1 E_{16} E_{21} E_{81})^{\otimes -1} (E_2 E_8)^{\otimes -1}$
$\tau$	$(E_{81} E_{16})$
$\sigma\tau$	$(E_1 E_{16})^{\otimes -1} (E_2 E_8)^{\otimes -1} (E_{21} E_{81})^{\otimes -1} (E_4)^{\otimes -1} (E_{41})^{\otimes -1}$
$\sigma^2\tau$	$(E_1 E_{21})$
$\sigma^3\tau$	$(E_1 E_{81})^{\otimes -1} (E_2 E_8)^{\otimes -1} (E_{21} E_{16})^{\otimes -1} (E_4)^{\otimes -1} (E_{41})^{\otimes -1}$

For the notation on the above permutations we refer to the  $R_6$ -type isogeny section.

## 24.2 Kodaira symbols, minimal models, and Pal values

Table 45:  $T_8$  data for  $p \neq 2$

$T_8$	$p \neq 2$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_2$	$(0, 0, 8m)$	1	$I_{8m}$	1	1
	$E_{21}$	$(0, 0, 16m)$	1	$I_{16m}$	1	1
	$E_4$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_{41}$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_8$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{81}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{16}$	$(0, 0, m)$	1	$I_m$	1	1
$v_p(t) = 0$ $v_p(t - 2) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{21}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_4$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_{41}$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_8$	$(0, 0, 8m)$	1	$I_{8m}$	1	1
	$E_{81}$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_{16}$	$(0, 0, 16m)$	1	$I_{16m}$	1	1
$v_p(t) = 0$ $v_p(t + 2) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{21}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_4$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_{41}$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_8$	$(0, 0, 8m)$	1	$I_{8m}$	1	1
	$E_{81}$	$(0, 0, 16m)$	1	$I_{16m}$	1	1
	$E_{16}$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 45:  $T_8$  data for  $p \neq 2$  (Continued)

$T_8$	$p \neq 2$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = 0$ $v_p(t^2 + 4) = m > 0$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{21}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_4$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_{41}$	$(0, 0, 8m)$	1	$I_{8m}$	1	1
	$E_8$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{81}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{16}$	$(0, 0, m)$	1	$I_m$	1	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 16m)$	$p^{-2m}$	$I_{16m}$	1	1
	$E_2$	$(0, 0, 8m)$	$p^{-2m}$	$I_{8m}$	1	1
	$E_{21}$	$(0, 0, 4m)$	$p^{-2m}$	$I_{4m}$	1	1
	$E_4$	$(0, 0, 4m)$	$p^{-2m}$	$I_{4m}$	1	1
	$E_{41}$	$(0, 0, 2m)$	$p^{-2m}$	$I_{2m}$	1	1
	$E_8$	$(0, 0, 2m)$	$p^{-2m}$	$I_{2m}$	1	1
	$E_{81}$	$(0, 0, m)$	$p^{-2m}$	$I_m$	1	1
	$E_{16}$	$(0, 0, m)$	$p^{-2m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 46:  $T_8$  data for  $p=2$ 

$T_8$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 1$	$E_1$	$(4, 6, 4m + 4)$	1	$I_{4m-4}^*$	1	1	2
	$E_2$	$(4, 6, 8m - 4)$	2	$I_{8m-12}^*$	1	1	2
	$E_{21}$	$(4, 6, 16m - 20)$	$2^2$	$I_{16m-28}^*$	1	1	2
	$E_4$	$(4, 6, 4m + 4)$	2	$I_{4m-4}^*$	1	1	2
	$E_{41}$	$(4, 6, 2m + 8)$	2	$I_{2m}^*$	1	1	2
	$E_8$	$(4, 6, 2m + 8)$	2	$I_{2m}^*$	1	1	2
	$E_{81}$	$(4, 6, m + 10)$	2	$I_{m+2}^*$	1	1	2
	$E_{16}$	$(4, 6, m + 10)$	2	$I_{m+2}^*$	1	1	2
$v_2(t) = 1$ $t \equiv 2(8)$ $v_2(t-2) = m$	$E_1$	$(0, 0, m - 3)$	2	$I_{m-3}$	1	$2^{-1}$	$2^{-1}$
	$E_2$	$(0, 0, 2(m - 3))$	$2^2$	$I_{2(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_{21}$	$(0, 0, m - 3)$	$2^2$	$I_{m-3}$	1	$2^{-1}$	$2^{-1}$
	$E_4$	$(0, 0, 4(m - 3))$	$2^3$	$I_{4(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_{41}$	$(0, 0, 2(m - 3))$	$2^3$	$I_{2(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_8$	$(0, 0, 8(m - 3))$	$2^4$	$I_{8(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_{81}$	$(0, 0, 4(m - 3))$	$2^4$	$I_{4(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_{16}$	$(0, 0, 16(m - 3))$	$2^5$	$I_{16(m-3)}$	1	$2^{-1}$	$2^{-1}$
$v_2(t) = 1$ $t/2 \equiv 3(4)$ $v_2(t^2 - 4) = m$ $v_2(t^2 + 4) = n$	$E_1$	$(0, 0, m - 3)$	2	$I_{m-3}$	1	$2^{-1}$	$2^{-1}$
	$E_2$	$(0, 0, 2(m - 3))$	$2^2$	$I_{2(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_{21}$	$(0, 0, m - 3)$	$2^2$	$I_{m-3}$	1	$2^{-1}$	$2^{-1}$
	$E_4$	$(0, 0, 4(m - 3))$	$2^3$	$I_{4(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_{41}$	$(0, 0, 2(m - 3))$	$2^3$	$I_{2(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_8$	$(0, 0, 8(m - 3))$	$2^4$	$I_{8(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_{81}$	$(0, 0, 16(m - 3))$	$2^5$	$I_{16(m-3)}$	1	$2^{-1}$	$2^{-1}$
	$E_{16}$	$(0, 0, 4(m - 3))$	$2^4$	$I_{4(m-3)}$	1	$2^{-1}$	$2^{-1}$
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		



Table 46:  $T_8$  data for  $p=2$  (Continued)

$T_8$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = -m \leq 0$	$E_1$	$(4, 6, 12 + 16m)$	$2^{-2m-1}$	$I_{4+16m}^*$	1	1	2
	$E_2$	$(4, 6, 12 + 8m)$	$2^{-2m-1}$	$I_{4+8m}^*$	1	1	2
	$E_{21}$	$(4, 6, 12 + 4m)$	$2^{-2m-1}$	$I_{4+4m}^*$	1	1	2
	$E_4$	$(4, 6, 12 + 4m)$	$2^{-2m-1}$	$I_{4+4m}^*$	1	1	2
	$E_{41}$	$(4, 6, 12 + 2m)$	$2^{-2m-1}$	$I_{4+2m}^*$	1	1	2
	$E_8$	$(4, 6, 12 + 2m)$	$2^{-2m-1}$	$I_{4+2m}^*$	1	1	2
	$E_{81}$	$(4, 6, 12 + m)$	$2^{-2m-1}$	$I_{4+m}^*$	1	1	2
	$E_{16}$	$(4, 6, 12 + m)$	$2^{-2m-1}$	$I_{4+m}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

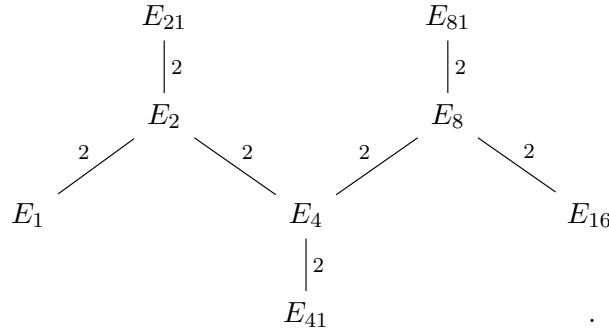
### 24.3 Statement

From the above tables one gets the (projective) vectors  $\mathbf{u} = [u(E)]$  and  $\mathbf{u}(d) = [u(\mathcal{E}^d)]$ :

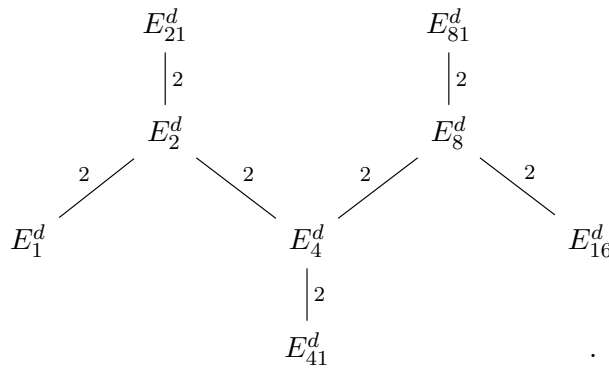
$t$	$[u(E)]$	$[u(\mathcal{E}^d)]$
$v_2(t) \geq 2$	$(1 : 2 : 2^2 : 2 : 2 : 2 : 2 : 2)$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$
$v_2(t) = 1$ $t/2 \equiv 3 \pmod{4}$	$(1 : 2 : 2 : 2^2 : 2^2 : 2^3 : 2^4 : 2^3)$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$
$v_2(t) = 1$ $t/2 \equiv 1 \pmod{4}$	$(1 : 2 : 2 : 2^2 : 2^2 : 2^3 : 2^3 : 2^4)$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$
$v_2(t) \leq 0$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$

The contents of this table are the main ingredients to prove the following result:

**Proposition 24.** *Let*



be a  $\mathbf{Q}$ -isogeny graph of type  $T_8$  corresponding to a given  $t$  in  $\mathbf{Q}$ ,  $t \neq 0, \pm 2$ , with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph



is given by:

Table 47: Faltings curves in  $T_8$

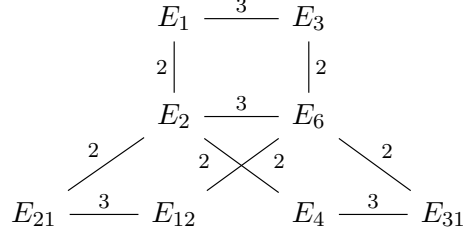
$T_8$	twisted isogeny graph	Prob
$v_2(t) \geq 2$		1
$v_2(t) = 1$ $t/2^2 \equiv 3 (4)$		1
$v_2(t) = 1$ $t/2^2 \equiv 1 (4)$		1
$v_2(t) \leq 0$		1

The column Prob gives the probability of the circled twisted curve to be the Faltings curve.

## 25 Type $S_8$

### 25.1 Settings

The isogeny graphs of type  $S_8$  are given by eight isogenous elliptic curves:



#### Modular curve

The non-cuspidal rational points of the modular curve  $X_0(12)$  parametrize isogeny graphs of type  $S_8$ . The curve  $X_0(12)$  has genus 0 and a hauptmodul for this curve is:

$$t = 3 + 2^2 3 \frac{\eta(2\tau)^2 \eta(3\tau) \eta(12\tau)^3}{\eta(\tau)^3 \eta(4\tau) \eta(6\tau)^2}.$$

#### $j$ -invariants.

Letting  $t = t(\tau)$ , one can write

$$\begin{aligned} j(E_1) &= j(\tau) = \frac{(t^2 - 3)^3 (t^6 - 9t^4 + 3t^2 - 3)^3}{(t - 3)(t - 1)^3 t^4 (t + 1)^3 (t + 3)} \\ j(E_3) &= j(3\tau) = \frac{(t^2 - 3)^3 (t^6 - 9t^4 + 243t^2 - 243)^3}{(t - 3)^3 (t - 1) t^{12} (t + 1) (t + 3)^3} \\ j(E_2) &= j(2\tau) = \frac{(t^2 + 3)^3 (t^6 - 15t^4 + 75t^2 + 3)^3}{(t - 3)^2 (t - 1)^6 t^2 (t + 1)^6 (t + 3)^2} \\ j(E_6) &= j(6\tau) = \frac{(t^2 + 3)^3 (t^6 + 225t^4 - 405t^2 + 243)^3}{(t - 3)^6 (t - 1)^2 t^6 (t + 1)^2 (t + 3)^6} \\ j(E_{21}) &= j(\tau + 1/2) = -\frac{(t^2 - 6t - 3)^3 (t^6 + 6t^5 + 27t^4 - 60t^3 - 249t^2 - 234t - 3)^3}{(t - 3)(t - 1)^{12} t (t + 1)^3 (t + 3)^4} \\ j(E_{12}) &= j(12\tau) = \frac{(t^2 + 6t - 3)^3 (t^6 + 234t^5 + 747t^4 + 540t^3 - 729t^2 - 486t - 243)^3}{(t - 3)^{12} (t - 1) t^3 (t + 1)^4 (t + 3)^3} \\ j(E_4) &= j(4\tau) = \frac{(t^2 + 6t - 3)^3 (t^6 - 6t^5 + 27t^4 + 60t^3 - 249t^2 + 234t - 3)^3}{(t - 3)^4 (t - 1)^3 t (t + 1)^{12} (t + 3)} \\ j(E_{31}) &= j(3\tau + 1/2) = -\frac{(t^2 - 6t - 3)^3 (t^6 - 234t^5 + 747t^4 - 540t^3 - 729t^2 + 486t - 243)^3}{(t - 3)^3 (t - 1)^4 t^3 (t + 1) (t + 3)^{12}}. \end{aligned}$$

## Signatures

We can (and do) choose Weierstrass equations for  $(E_1, E_3, E_2, E_6, E_{21}, E_{12}, E_4, E_{31})$  in such a way that the isogeny graph is normalized. Their signatures are:

$S_8$ signatures	
$c_4(E_1)$	$(t^2 - 3) \cdot (t^6 - 9t^4 + 3t^2 - 3)$
$c_6(E_1)$	$(t^4 - 6t^2 - 3) \cdot (t^8 - 12t^6 + 30t^4 - 36t^2 + 9)$
$\Delta(E_1)$	$(t - 3) \cdot (t + 3) \cdot (t - 1)^3 \cdot (t + 1)^3 \cdot t^4$
$c_4(E_3)$	$(t^2 - 3) \cdot (t^6 - 9t^4 + 243t^2 - 243)$
$c_6(E_3)$	$(t^4 + 18t^2 - 27) \cdot (t^8 - 36t^6 + 270t^4 - 972t^2 + 729)$
$\Delta(E_3)$	$(t - 1) \cdot (t + 1) \cdot (t - 3)^3 \cdot (t + 3)^3 \cdot t^{12}$
$c_4(E_2)$	$(t^2 + 3) \cdot (t^6 - 15t^4 + 75t^2 + 3)$
$c_6(E_2)$	$(t^4 - 6t^2 - 24t - 3) \cdot (t^4 - 6t^2 - 3) \cdot (t^4 - 6t^2 + 24t - 3)$
$\Delta(E_2)$	$(t - 3)^2 \cdot t^2 \cdot (t + 3)^2 \cdot (t - 1)^6 \cdot (t + 1)^6$
$c_4(E_6)$	$(t^2 + 3) \cdot (t^6 + 225t^4 - 405t^2 + 243)$
$c_6(E_6)$	$(t^4 - 24t^3 + 18t^2 - 27) \cdot (t^4 + 18t^2 - 27) \cdot (t^4 + 24t^3 + 18t^2 - 27)$
$\Delta(E_6)$	$(t - 1)^2 \cdot (t + 1)^2 \cdot (t - 3)^6 \cdot t^6 \cdot (t + 3)^6$
$c_4(E_{21})$	$(t^2 - 6t - 3) \cdot (t^6 + 6t^5 + 27t^4 - 60t^3 - 249t^2 - 234t - 3)$
$c_6(E_{21})$	$(t^4 - 6t^2 - 24t - 3) \cdot (t^8 - 12t^6 + 528t^5 + 30t^4 - 3168t^3 - 3996t^2 - 1584t + 9)$
$\Delta(E_{21})$	$(-1) \cdot (t - 3) \cdot t \cdot (t + 1)^3 \cdot (t + 3)^4 \cdot (t - 1)^{12}$
$c_4(E_{12})$	$(t^2 + 6t - 3) \cdot (t^6 + 234t^5 + 747t^4 + 540t^3 - 729t^2 - 486t - 243)$
$c_6(E_{12})$	$(t^4 + 24t^3 + 18t^2 - 27) \cdot (t^8 - 528t^7 - 3996t^6 - 9504t^5 + 270t^4 + 14256t^3 - 972t^2 + 729)$
$\Delta(E_{12})$	$(t - 1) \cdot t^3 \cdot (t + 3)^3 \cdot (t + 1)^4 \cdot (t - 3)^{12}$
$c_4(E_4)$	$(t^2 + 6t - 3) \cdot (t^6 - 6t^5 + 27t^4 + 60t^3 - 249t^2 + 234t - 3)$
$c_6(E_4)$	$(t^4 - 6t^2 + 24t - 3) \cdot (t^8 - 12t^6 - 528t^5 + 30t^4 + 3168t^3 - 3996t^2 + 1584t + 9)$
$\Delta(E_4)$	$t \cdot (t + 3) \cdot (t - 1)^3 \cdot (t - 3)^4 \cdot (t + 1)^{12}$
$c_4(E_{31})$	$(t^2 - 6t - 3) \cdot (t^6 - 234t^5 + 747t^4 - 540t^3 - 729t^2 + 486t - 243)$
$c_6(E_{31})$	$(t^4 - 24t^3 + 18t^2 - 27) \cdot (t^8 + 528t^7 - 3996t^6 + 9504t^5 + 270t^4 - 14256t^3 - 972t^2 + 729)$
$\Delta(E_{31})$	$(-1) \cdot (t + 1) \cdot (t - 3)^3 \cdot t^3 \cdot (t - 1)^4 \cdot (t + 3)^{12}$

## Automorphisms

The subgroup of  $\text{Aut } X_0(12)$  that fixes the set of vertices of the graph is isomorphic to the dihedral group of order 12 with elements:

automorphism	permutation	order
$\text{id}(t) = t$	$()$	1
$\sigma(t) = 3(t-1)/(t+3)$	$(j_1 j_{31} j_4 j_3 j_{21} j_{12})(j_2 j_6)$	6
$\sigma^2(t) = (t-3)/(t+1)$	$(j_1 j_4 j_{21})(j_3 j_{12} j_{31})$	3
$\sigma^3(t) = -3/t$	$(j_1 j_3)(j_2 j_6)(j_{21} j_{31})(j_{12} j_4)$	2
$\sigma^4(t) = -(t+3)/(t-1)$	$(j_1 j_{21} j_4)(j_3 j_{31} j_{12})$	3
$\sigma^5(t) = -3(t+1)/(t-3)$	$(j_1 j_{12} j_{21} j_3 j_4 j_{31})(j_2 j_6)$	6
$\tau(t) = -t$	$(j_{21} j_4)(j_{12} j_{31})$	2
$\sigma\tau(t) = 3(t+1)/(t-3)$	$(j_1 j_{31})(j_3 j_{21})(j_2 j_6)(j_{12} j_4)$	2
$\sigma^2\tau(t) = (t+3)/(t-1)$	$(j_1 j_{21} j_4)(j_3 j_{31})$	6
$\sigma^3\tau(t) = 3/t$	$(j_1 j_3)(j_2 j_6)(j_{21} j_{12})(j_4 j_{31})$	2
$\sigma^4\tau(t) = -(t-3)/(t+1)$	$(j_1 j_{21})(j_3 j_{31})$	2
$\sigma^5\tau(t) = -3(t-1)/(t+3)$	$(j_1 j_{12})(j_3 j_4)(j_2 j_6)(j_{21} j_{31})$	2

Automorphism action on the graph	
id	$()$
$\sigma$	$(E_1 E_{31} E_4 E_3 E_{21} E_{12})^{\otimes -3} (E_2 E_6)^{\otimes -3} (E_4)^{\otimes -3} (E_{41})^{\otimes -3}$
$\sigma^2$	$(E_1 E_4 E_{21})(E_3 E_{12} E_{31})$
$\sigma^3$	$(E_1 E_3)^{\otimes -3} (E_2 E_6)^{\otimes -3} (E_{21} E_{31})^{\otimes -3} (E_{12} E_4)^{\otimes -3}$
$\sigma^4$	$(E_1 E_{21} E_4)(E_3 E_{31} E_{12})$
$\sigma^5$	$(E_1 E_{12} E_{21} E_3 E_4 E_{31})^{\otimes -3} (E_2 E_6)^{\otimes -3}$
$\tau$	$(E_{21} E_4)(E_{12} E_{31})$
$\sigma\tau$	$(E_1 E_{31})^{\otimes -3} (E_3 E_{21})^{\otimes -3} (E_2 E_6)^{\otimes -3} (E_{12} E_4)^{\otimes -3}$
$\sigma^2\tau$	$(E_1 E_{21} E_4)(E_3 E_{31})$
$\sigma^3\tau$	$(E_1 E_3)^{\otimes -3} (E_2 E_6)^{\otimes -3} (E_{21} E_{12})^{\otimes -3} (E_4 E_{31})^{\otimes -3}$
$\sigma^4\tau$	$(E_1 E_{21})(E_3 E_{31})$
$\sigma^5\tau$	$(E_1 E_{12})^{\otimes -3} (E_3 E_4)^{\otimes -3} (E_2 E_6)^{\otimes -3} (E_{21} E_{31})^{\otimes -3}$

## 25.2 Kodaira symbols, minimal models, and Pal values

Table 48:  $S_8$  data for  $p \neq 2, 3$

$S$	$p \neq 2, 3$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = m > 0$	$E_1$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_3$	$(0, 0, 12m)$	1	$I_{12m}$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_6$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_{21}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{12}$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_4$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{31}$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
$v_p(t) = 0$ $v_p(t+1) = m$	$E_1$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_3$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_6$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{21}$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_{12}$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_4$	$(0, 0, 12m)$	1	$I_{12m}$	1	1
	$E_{31}$	$(0, 0, m)$	1	$I_m$	1	1
$v_p(t) = 0$ $v_p(t-1) = m$	$E_1$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_3$	$(0, 0, m)$	1	$I_m$	1	1
	$E_2$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_6$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_{21}$	$(0, 0, 12m)$	1	$I_{12m}$	1	1
	$E_{12}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_4$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_{31}$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 48:  $S_8$  data for  $p \neq 2, 3$  (Continued)

$S$	$p \neq 2, 3$					
$t$	$E$	$\text{sig}_p(\mathcal{E})$	$u_p(E)$	$K_p(E)$	$u_p(\mathcal{E}^d)$	
$v_p(t) = 0$ $v_p(t+3) = m$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_6$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_{21}$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_{12}$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_4$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{31}$	$(0, 0, 12m)$	1	$I_{12m}$	1	1
$v_p(t) = 0$ $v_p(t-3) = m$	$E_1$	$(0, 0, m)$	1	$I_m$	1	1
	$E_3$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
	$E_2$	$(0, 0, 2m)$	1	$I_{2m}$	1	1
	$E_6$	$(0, 0, 6m)$	1	$I_{6m}$	1	1
	$E_{21}$	$(0, 0, m)$	1	$I_m$	1	1
	$E_{12}$	$(0, 0, 12m)$	1	$I_{12m}$	1	1
	$E_4$	$(0, 0, 4m)$	1	$I_{4m}$	1	1
	$E_{31}$	$(0, 0, 3m)$	1	$I_{3m}$	1	1
$v_p(t) = -m < 0$	$E_1$	$(0, 0, 12m)$	$p^{-2m}$	$I_{12m}$	1	1
	$E_3$	$(0, 0, 4m)$	$p^{-2m}$	$I_{4m}$	1	1
	$E_2$	$(0, 0, 6m)$	$p^{-2m}$	$I_{6m}$	1	1
	$E_6$	$(0, 0, 2m)$	$p^{-2m}$	$I_{2m}$	1	1
	$E_{21}$	$(0, 0, 3m)$	$p^{-2m}$	$I_{3m}$	1	1
	$E_{12}$	$(0, 0, m)$	$p^{-2m}$	$I_m$	1	1
	$E_4$	$(0, 0, 3m)$	$p^{-2m}$	$I_{3m}$	1	1
	$E_{31}$	$(0, 0, m)$	$p^{-2m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	



Table 49:  $S_8$  data for  $p = 3$ 

$S$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = m > 1$	$E_1$	$(2, 3, 4m + 2)$	1	$I_{4m-4}^*$	3	1
	$E_3$	$(2, 3, 12m - 6)$	3	$I_{12m-12}^*$	3	1
	$E_2$	$(2, 3, 2m + 4)$	1	$I_{2m-2}^*$	3	1
	$E_6$	$(2, 3, 6m)$	3	$I_{6m-6}^*$	3	1
	$E_{21}$	$(2, 3, m + 5)$	1	$I_{m-1}^*$	3	1
	$E_{12}$	$(2, 3, 3m + 3)$	3	$I_{3m-3}^*$	3	1
	$E_4$	$(2, 3, m + 5)$	1	$I_{m-1}^*$	3	1
	$E_{31}$	$(2, 3, 3m + 3)$	3	$I_{3m-3}^*$	3	1
$v_3(t) = 1$ $v_3(t-3) = m$ $v_3(t+3) = n$	$E_1$	$(2, 3, m + n + 4)$	1	$I_{m+n-2}^*$	3	1
	$E_3$	$(2, 3, 3m + 3n)$	3	$I_{3m+3n-6}^*$	3	1
	$E_2$	$(2, 3, 2m + 2n + 2)$	1	$I_{2m+2n-4}^*$	3	1
	$E_6$	$(2, 3, 6m + 6n - 6)$	3	$I_{6m+6n-12}^*$	3	1
	$E_{21}$	$(2, 3, m + 4n + 1)$	1	$I_{m+4n-5}^*$	3	1
	$E_{12}$	$(2, 3, 12m + 3n - 9)$	3	$I_{12m+3n-15}^*$	3	1
	$E_4$	$(2, 3, 4m + n + 1)$	1	$I_{4m+n-5}^*$	3	1
	$E_{31}$	$(2, 3, 3m + 12n - 9)$	3	$I_{3m+12n-15}^*$	3	1
$v_3(t) = 0$ $v_3(t-1) = m$ $v_3(t+1) = n$	$E_1$	$(0, 0, 3m + 3n)$	1	$I_{3m+3n}$	1	1
	$E_3$	$(0, 0, m + n)$	1	$I_{m+n}$	1	1
	$E_2$	$(0, 0, 6m + 6n)$	1	$I_{6m+6n}$	1	1
	$E_6$	$(0, 0, 2m + 2n)$	1	$I_{2m+2n}$	1	1
	$E_{21}$	$(0, 0, 12m + 3n)$	1	$I_{12m+3n}$	1	1
	$E_{12}$	$(0, 0, m + 4n)$	1	$I_{m+4n}$	1	1
	$E_4$	$(0, 0, 3m + 12n)$	1	$I_{3m+12n}$	1	1
	$E_{31}$	$(0, 0, 4m + n)$	1	$I_{4m+n}$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	

Table 49:  $S_8$  data for  $p = 3$  (Continued)

$S$	$p = 3$					
$t$	$E$	$\text{sig}_3(\mathcal{E})$	$u_3(E)$	$K_3(E)$	$u_3(\mathcal{E}^d)$	
$v_3(t) = -m < 0$	$E_1$	$(0, 0, 12m)$	$3^{-2m}$	$I_{12m}$	1	1
	$E_3$	$(0, 0, 4m)$	$3^{-2m}$	$I_{4m}$	1	1
	$E_2$	$(0, 0, 6m)$	$3^{-2m}$	$I_{6m}$	1	1
	$E_6$	$(0, 0, 2m)$	$3^{-2m}$	$I_{2m}$	1	1
	$E_{21}$	$(0, 0, 3m)$	$3^{-2m}$	$I_{3m}$	1	1
	$E_{12}$	$(0, 0, m)$	$3^{-2m}$	$I_m$	1	1
	$E_4$	$(0, 0, 3m)$	$3^{-2m}$	$I_{3m}$	1	1
	$E_{31}$	$(0, 0, m)$	$3^{-2m}$	$I_m$	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{3}$	

Table 50:  $S_8$  data for  $p=2$ 

$S$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = m > 0$	$E_1$	$(4, 6, 4m + 12)$	$2^{-1}$	$I_{4m+4}^*$	1	1	2
	$E_3$	$(4, 6, 12m + 12)$	$2^{-1}$	$I_{12m+4}^*$	1	1	2
	$E_2$	$(4, 6, 2m + 12)$	$2^{-1}$	$I_{2m+4}^*$	1	1	2
	$E_6$	$(4, 6, 6m + 12)$	$2^{-1}$	$I_{6m+4}^*$	1	1	2
	$E_{21}$	$(4, 6, m + 12)$	$2^{-1}$	$I_{m+4}^*$	1	1	2
	$E_{12}$	$(4, 6, 3m + 12)$	$2^{-1}$	$I_{3m+4}^*$	1	1	2
	$E_4$	$(4, 6, m + 12)$	$2^{-1}$	$I_{m+4}^*$	1	1	2
	$E_{31}$	$(4, 6, 3m + 12)$	$2^{-1}$	$I_{3m+4}^*$	1	1	2
$v_2(t) = 0$ $t \equiv 3(4)$ $v_2(t-3) = m$ $v_2(t+1) = n$	$E_1$	$(4, 6, m + 3n + 4)$	1	$I_{m+3n-4}^*$	1	1	2
	$E_3$	$(4, 6, 3m + n + 4)$	1	$I_{3m+n-4}^*$	1	1	2
	$E_2$	$(4, 6, 2m + 6n - 4)$	2	$I_{2m+6n-12}^*$	1	1	2
	$E_6$	$(4, 6, 6m + 2n - 4)$	2	$I_{6m+2n-12}^*$	1	1	2
	$E_{21}$	$(4, 6, m + 3n + 4)$	2	$I_{m+3n-4}^*$	1	1	2
	$E_{12}$	$(4, 6, 12m + 4n - 20)$	4	$I_{12m+4n-28}^*$	1	1	2
	$E_4$	$(4, 6, 4m + 12n - 20)$	4	$I_{4m+12n-28}^*$	1	1	2
	$E_{31}$	$(4, 6, 3m + n + 4)$	2	$I_{3m+n-4}^*$	1	1	2
$v_2(t) = 0$ $t \equiv 1(4)$ $v_2(t-1) = m$ $v_2(t+3) = n$	$E_1$	$(4, 6, 3m + n + 4)$	1	$I_{3m+n-4}^*$	1	1	2
	$E_3$	$(4, 6, m + 3n + 4)$	1	$I_{m+3n-4}^*$	1	1	2
	$E_2$	$(4, 6, 6m + 2n - 4)$	2	$I_{6m+2n-12}^*$	1	1	2
	$E_6$	$(4, 6, 2m + 6n - 4)$	2	$I_{2m+6n-12}^*$	1	1	2
	$E_{21}$	$(4, 6, 12m + 4n - 20)$	4	$I_{12m+4n-28}^*$	1	1	2
	$E_{12}$	$(4, 6, m + 3n + 4)$	2	$I_{m+3n-4}^*$	1	1	2
	$E_4$	$(4, 6, 3m + n + 4)$	2	$I_{3m+n-4}^*$	1	1	2
	$E_{31}$	$(4, 6, 4m + 12n - 20)$	4	$I_{4m+12n-28}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

Table 50:  $S_8$  data for  $p=2$  (Continued)

$S$	$p = 2$						
$t$	$E$	$\text{sig}_2(\mathcal{E})$	$u_2(E)$	$K_2(E)$	$u_2(\mathcal{E}^d)$		
$v_2(t) = -m < 0$	$E_1$	$(4, 6, 12m + 12)$	$2^{-(2m+1)}$	$I_{12m+4}^*$	1	1	2
	$E_3$	$(4, 6, 4m + 12)$	$2^{-(2m+1)}$	$I_{4m+4}^*$	1	1	2
	$E_2$	$(4, 6, 6m + 12)$	$2^{-(2m+1)}$	$I_{6m+4}^*$	1	1	2
	$E_6$	$(4, 6, 2m + 12)$	$2^{-(2m+1)}$	$I_{2m+4}^*$	1	1	2
	$E_{21}$	$(4, 6, 3m + 12)$	$2^{-(2m+1)}$	$I_{3m+4}^*$	1	1	2
	$E_{12}$	$(4, 6, m + 12)$	$2^{-(2m+1)}$	$I_{m+4}^*$	1	1	2
	$E_4$	$(4, 6, 3m + 12)$	$2^{-(2m+1)}$	$I_{3m+4}^*$	1	1	2
	$E_{31}$	$(4, 6, m + 12)$	$2^{-(2m+1)}$	$I_{m+4}^*$	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					$d \pmod{4}$		

### 25.3 Statement

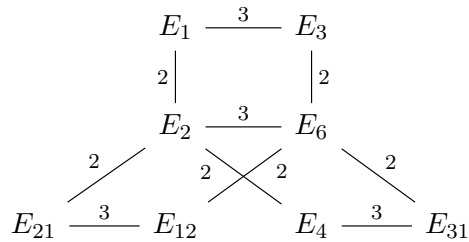
From the above tables one gets the (projective) vectors  $\mathbf{u}_p = [u_p(E)]$  and  $\mathbf{u}_p(d) = [u_p(\mathcal{E}^d)]$ :

$t$	$[u_2(E)]$	$[u_2(\mathcal{E}^d)]$
$v_2(t) \neq 0$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$
$v_2(t) = 0$ $t/2 \equiv 3 \pmod{4}$	$(1 : 1 : 2 : 2 : 2 : 2^2 : 2^2 : 2)$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$
$v_2(t) = 0$ $t/2 \equiv 1 \pmod{4}$	$(1 : 1 : 2 : 2 : 2^2 : 2 : 2 : 2^2)$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$

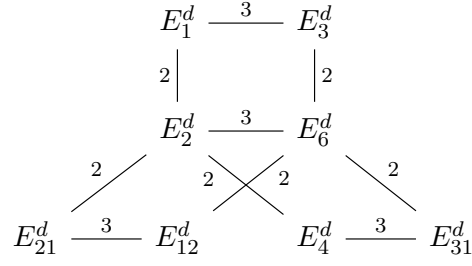
$t$	$[u_3(E)]$	$[u_3(\mathcal{E}^d)]$
$v_3(t) > 0$	$(1 : 3 : 1 : 3 : 1 : 3 : 1 : 3)$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$
$v_3(t) \leq 0$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$	$(1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)$

The contents of these tables are the main ingredients to prove the following result:

**Proposition 25.** *Let*



be a  $\mathbf{Q}$ -isogeny graph of type  $S_8$  corresponding to a given  $t$  in  $\mathbf{Q}$ ,  $t \neq 1, \pm 3$ , with signatures as above. For every square-free integer  $d$ , the Faltings curve (circled) in the twisted isogeny graph



is given by:

Table 51: Faltings curves in  $S_8$

$S$		twisted isogeny graph	Prob
$v_3(t) > 1$	$v_2(t) \neq 0$		1
$v_3(t) > 1$	$v_2(t) = 0$ $t \equiv 3(4)$		1
$v_3(t) > 1$	$v_2(t) = 0$ $t \equiv 1(4)$		1
$v_3(t) \leq 0$	$v_2(t) \neq 0$		1
$v_3(t) \leq 0$	$v_2(t) = 0$ $t \equiv 3(4)$		1

Continued on next page

Table 51: Faltings curves in  $S_8$  (Continued)

$S$		$twisted\ isogeny\ graph$	$Prob$
$v_3(t) \leq 0$	$v_2(t) = 0$ $t \equiv 1 \pmod{4}$	<pre> graph TD     E1 --&gt; E3     E3 --&gt; E6     E6 --&gt; E4     E4 --&gt; E31     E2 --&gt; E6     E2 --&gt; E12     E12 --&gt; E4     E21((E21)) --&gt; E12 </pre>	1

The column  $Prob$  gives the probability of the circled twisted curve to be the Faltings curve.