Linear Model

With the given model (scalar)

$$\begin{aligned} Y_i &= X_i\beta + \epsilon_i \\ X_i &= U_i + 2V_i + \delta_i \end{aligned} \tag{1}$$

We have $i = 1, ..., n$,

$$\begin{aligned} U_i &\sim^{i.i.d.} N(0,1) \\ V_i &\sim^{i.i.d.} N(0,1) \\ (\epsilon_i, \delta_i) &\sim^{i.i.d} N(\mu, \Sigma) \end{aligned} \tag{2}$$

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma^2 & \rho \\ \rho & \sigma^2 \end{pmatrix}$$

So, we can see that X is endogenous and (U, V) are instruments. We would like to use both OLS and IVLS to estimate parameter β and σ^2.

Parameter Estimation

In Monte-Carlo simulations (with $N = 1000$ runs), we set the true parameter values to be

$$\beta = 3, \sigma^2 = 1, \rho = 3/4$$

With $n = 100$ vector $(U_i, V_i, \epsilon_i, \delta_i, X_i, Y_i)$ samples in each run, we can estimate parameters with different methods.

1. Coefficient parameter estimate

We can get estimates $\hat{\beta}_{OLS}, \hat{\beta}_{IVLS}$ using OLS and IVLS. Here, in real implementation, IVLS is carried out as two-stage LS.

```
ols <- lm(Y ~ X - 1)
beta.hat.ols[i] <- ols$coeff

iv <- lm(X ~ U + V - 1)
X.hat <- iv$fit
ivls <- lm(Y ~ X.hat - 1)
beta.hat.ivls[i] <- ivls$coeff
```

The statistics from the MC simulations are given below in the table, and histogram in Figure 1.

$(\beta = 3)$	$\hat{\beta}_{OLS}$	$\hat{\beta}_{IVLS}$
mean	3.1279	3.0037
SD	0.0378	0.0448
RMS	0.1333	0.0449

We can see that $\hat{\beta}_{IVLS}$ has smaller bias than $\hat{\beta}_{OLS}$, thus smaller root mean-squared error (RMS). However, $\hat{\beta}_{OLS}$ has smaller standard deviation (SD) than $\hat{\beta}_{IVLS}$, which can also be seen in the histogram. Since IVLS is equivalent to two-stage OLS, when getting better unbiased estimate of the coefficient parameter, the SD of the estimate is aggregated from the two stages of LS, which is expected to be larger than simple OLS.

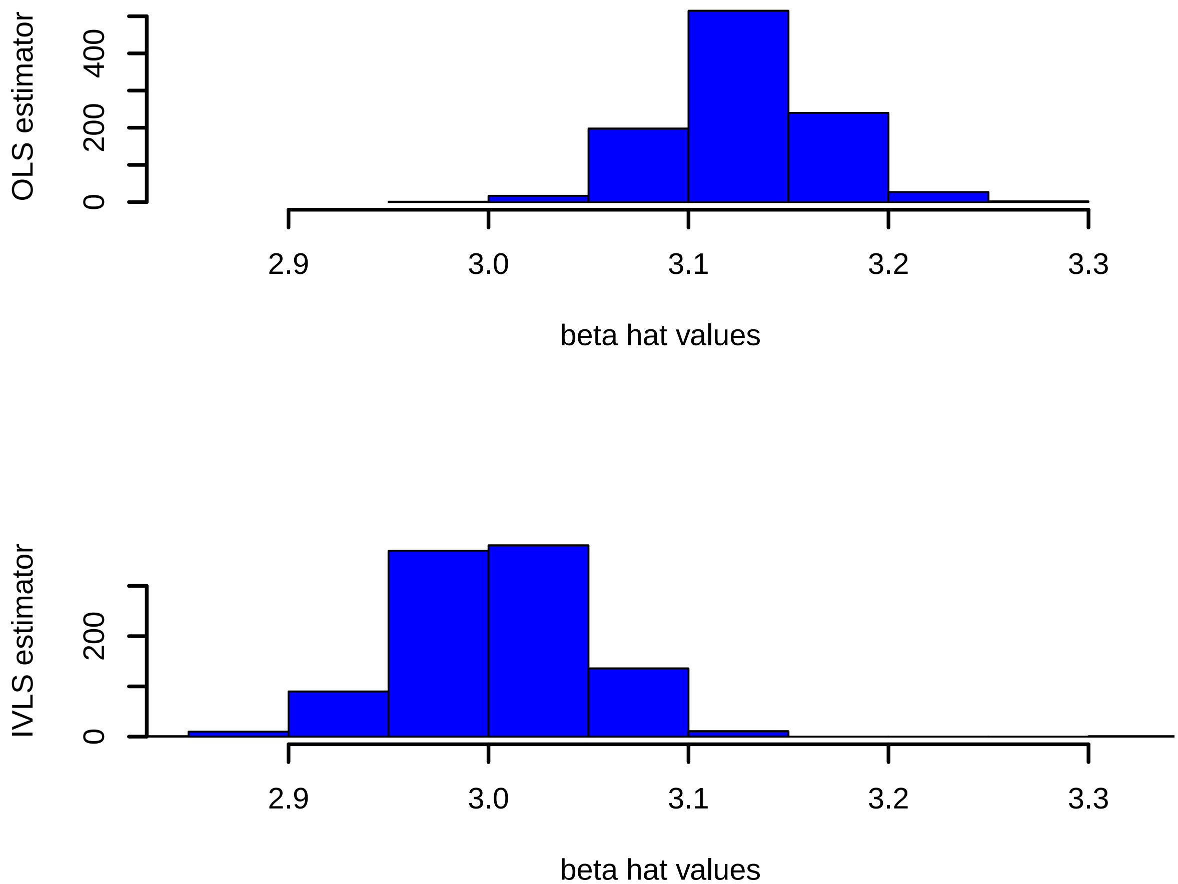

Figure 1: Histogram of OLS and IVLS coefficient parameter estimates

2. Error variance parameter estimate

We estimate the error variance σ^2 in two ways. First, we plug in $\hat{\beta}_{IVLS}$ into the original model (1) and use the residuals for error variance parameter estimation. The second method that we employ use the residuals from the transformed equation

$$(Z'Z)^{-1/2}Z'Y = (Z'Z)^{-1/2}Z'X\beta + \eta \quad (3)$$

where Z is the $n \times 2$ matrix of instruments, X is the $n \times 1$ design matrix and Y is the $n \times 1$ vector of responses.

```
SS <- sum((Y - X * ivls$coeff)^2)
sigma2.hat.ivls[i] <- SS/(n - 1)

tZ <- t(cbind(U, V))
ZZ.sqinv <- sqrt.m22(solve(tcrossprod(tZ)))
L <- ZZ.sqinv %*% tZ %*% Y
M <- ZZ.sqinv %*% tZ %*% X
trans <- lm(L ~ M - 1)
sigma2.hat.zz[i] <- sum(trans$res^2)
```

The appropriate denominator for each method and their statistics from the MC simulations are given below in the table, and histogram in Figure 2.

$(\sigma^2 = 1)$	Plug-in IVLS	Transformed OLS
denominator	$n-1$	1
mean	0.994	1.0381
SD	0.1476	1.5067

We can see that $\hat{\sigma^2}_{IVLS}$ has almost the same bias as $\hat{\sigma^2}_{trans}$, but much smaller SD. This can be seen from the histogram. The plug-in IVLS estimator keeps the normal error variance characteristics of the OLS. However, since the degree of freedom in the transformed model is only 2 (two instruments), the error variance is no longer normal, which gives much larger SD.

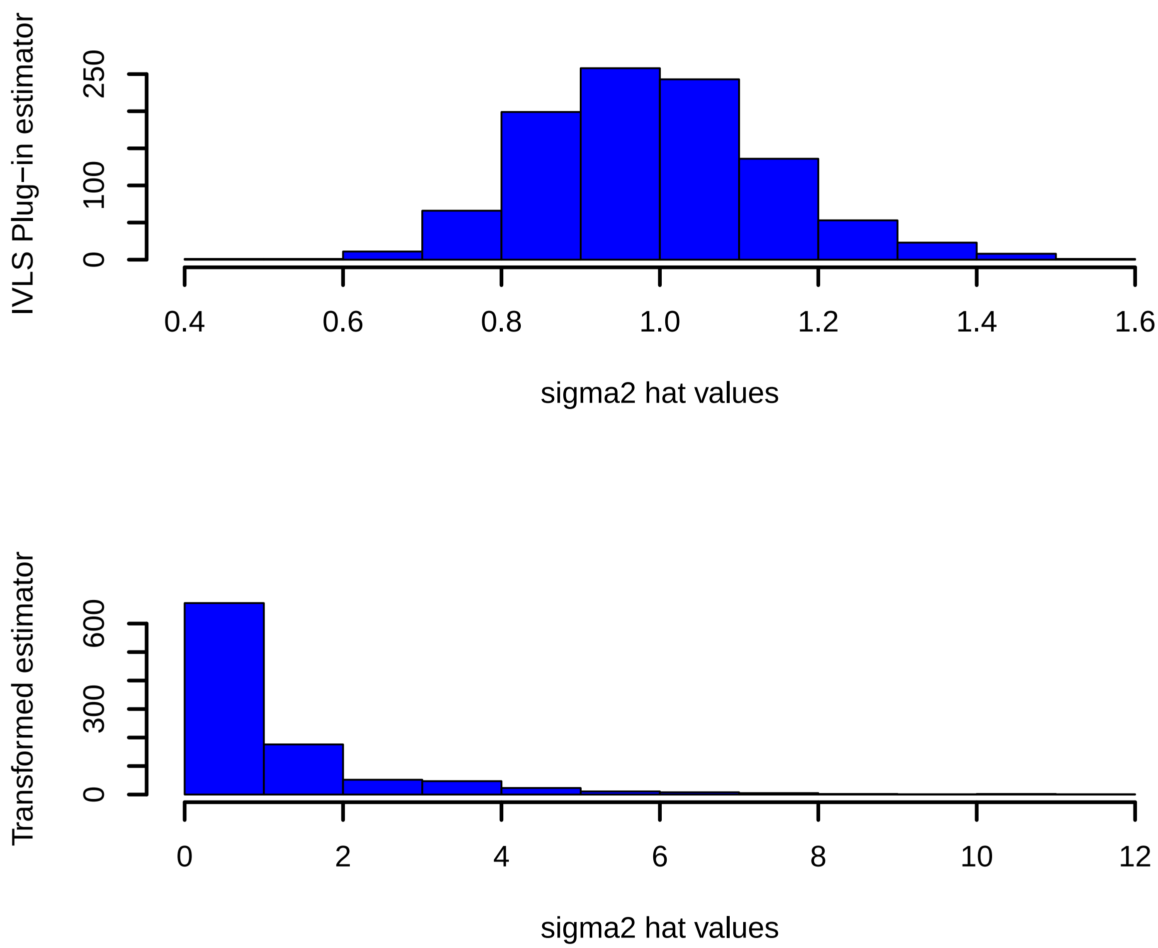

Figure 2: Histogram of IVLS and Transformed OLS error variance estimates