Chapter 3: Dynamic discrete choice models: full solution approaches

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I. Introduction

Many economics problems describe the behavior of forward-looking agents making discrete choices taking into account how their decisions today affect tomorrow's outcomes. The seminal papers of Miller (1984), Wolpin (1984), Pakes (1986), and Rust (1987) show that estimating these dynamic discrete choice models is both feasible and important to answer key economic questions. Several examples include: education and career path decisions, migration, machine replacements, smoking, marriage, fertility, social interactions, patenting a product, entry/stay/exit from a product market,... In this part of the course, we model individual behavior of forward-looking individuals facing sequential random utility choice problems by means of a stochastic dynamic programming (DP) problems.

II. General Framework

A. Model primitives and decision problem

Time is discrete and indexed by $t = 0, 1, ..., T$ (with $T \leq \infty$). Every period t, each individual chooses among J mutually exclusive alternatives:

$$d_t \equiv \{j : j \in \mathcal{D} = \{1, 2, ..., J\}\}. \tag{1}$$

For each decision j, we define an indicator variable that equals one if the action is taken at time t, and zero otherwise, $d_{jt} \equiv \mathbb{1}\{d_t = j\}$, such that $\sum_{j=1}^{J} d_{jt} = 1$. Individual's payoff in period t depends on the vector of state variables, $\boldsymbol{s}_t \equiv \{\boldsymbol{x}_t, \boldsymbol{\varepsilon}_t\}$, where $\boldsymbol{x}_t$ is a vector state variables that are observed by the econometrician (which might include the time-varying and/or time-invariant variables, agent-specific and/or aggregate variables, the time index,...), and $\boldsymbol{\varepsilon}_t \equiv (\varepsilon_{1t}, ..., \varepsilon_{Jt})'$ is observed by the individual, but unobserved by the econometrician. State variables evolve following a choice-specific Markovian process:

$$\boldsymbol{s}_{t+1} \sim F(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t, d_t). \tag{2}$$

Individual's intertemporal payoff function is:

$$\mathbb{E}_t\left[\sum_{l=0}^{T-t}\beta^l U(\boldsymbol{s}_{t+l}, d_{t+l})\right], \tag{3}$$

where $U(.,.)$ is the one-period payoff, $\beta \in (0,1)$ is the discount factor, and $\mathbb{E}_t$ is an expectation taken with respect to individual's beliefs about future outcomes given information available in period t (which in general includes $\boldsymbol{s}_t$ and d_t). Given these payoffs and beliefs, we write the optimal decision at period t, as:

$$d_t^*(\boldsymbol{s}_t) = \arg\max_{d_t\in\mathcal{D}} \mathbb{E}_t\left[\sum_{l=0}^{T-t}\beta^l U(\boldsymbol{s}_{t+l}, d_{t+l})\right], \tag{4}$$

and we define $d_{jt}^* \equiv \mathbb{1}\{d_t^*(\boldsymbol{s}_t) = j\}$. The primitives of the model $\{U, F, \beta\}$ are known by the econometrician up to a parameter vector $\boldsymbol{\theta}$.

B. Baseline assumptions

There is not a "general class" of dynamic discrete choice models. Here we consider a set of assumptions that define somehow a class of models that we refer to as the Rust framework, following the terminology in Aguirregabiria and Mira (2010). These models are inspired by Rust (1987), which we use as a motivational example below. We consider the following assumptions.

Assumption 1 (additive separability) *The one-period utility function is additively separable between the observable and the unobservable components. This is:*

$$U(d_t, \boldsymbol{x}_t, \boldsymbol{\varepsilon}_t) = u(d_t, \boldsymbol{x}_t) + \varepsilon_t(d_t), \tag{5}$$

where $\varepsilon_t(d_t) \equiv \sum_{j\in\mathcal{D}} d_{jt}\varepsilon_{jt}$. We define $u_{jt}(\boldsymbol{x}_t)$ such that $u(d_t, \boldsymbol{x}_t) \equiv \sum_{j\in\mathcal{D}} d_{jt}u_{jt}(\boldsymbol{x}_t)$.

Assumption 2 (iid unobservables) *Unobserved state variables $\boldsymbol{\varepsilon}_t$ are independently and identically distributed across agents and over time given $\boldsymbol{x}_t$, with a cdf $F_\varepsilon(\boldsymbol{\varepsilon}_t)$ which has finite first moments and is continuous and twice differentiable.*

Assumption 3 (conditional independence of future x) *Conditional on current decision and observable state variables, next period observable variables do not depend on current $\boldsymbol{\varepsilon}_t$:*

$$F_x(\boldsymbol{x}_{t+1}|d_t, \boldsymbol{x}_t, \boldsymbol{\varepsilon}_t) = F_x(\boldsymbol{x}_{t+1}|d_t, \boldsymbol{x}_t). \tag{6}$$

Note that Assumptions 2 and 3 together imply conditional independence, this is:

$$F(\boldsymbol{x}_{t+1}, \boldsymbol{\varepsilon}_{t+1}|d_t, \boldsymbol{x}_t, \boldsymbol{\varepsilon}_t) = F_x(\boldsymbol{x}_{t+1}|d_t, \boldsymbol{x}_t)F_\varepsilon(\boldsymbol{\varepsilon}_{t+1}). \tag{7}$$

Assumption 4 (conditional logit) *Unobservables* $\{\varepsilon_{jt} : j \in \mathcal{D}\}$ *are independent across alternatives and type-I extreme value distributed.*

Later on in the chapter, we introduce some examples of papers estimating models that depart from these assumptions, and we discuss the implications of doing so.

C. *Value functions, conditional choice probabilities, and log-likelihood*

Let $V_t(\boldsymbol{x}_t)$ denote the ex-ante value function in period t, this is, the discounted sum of expected payoffs just before $\boldsymbol{\varepsilon}_t$ is revealed conditional on behaving according to the optimal decision rule:

$$V_t(\boldsymbol{x}_t) \equiv \mathbb{E}_{t-1}\left[\sum_{l=0}^{T-t}\sum_{j\in\mathcal{D}}\beta^l d^*_{jt+l}(u_{jt+l}(\boldsymbol{x}_{t+l}) + \varepsilon_{jt+l})\middle|\boldsymbol{x}_t\right] \tag{8}$$

(note the role of subscripting of $\mathbb{E}_{t-1}$ and the conditioning on $\boldsymbol{x}_t$ to define the expectation just before $\boldsymbol{\varepsilon}_t$ is revealed). Appealing to Bellman's optimality principle, and given conditional independence we can write:

$$\begin{aligned} V_t(\boldsymbol{x}_t) &= \mathbb{E}_{t-1}\left[\sum_{j\in\mathcal{D}} d^*_{jt}\left(u_{jt}(\boldsymbol{x}_t) + \varepsilon_{jt} + \beta\int V_{t+1}(\boldsymbol{x}_{t+1})dF_x(\boldsymbol{x}_{t+1}|\boldsymbol{x}_t, d^*_t)\right)\middle|\boldsymbol{x}_t\right] \\ &= \sum_{j\in\mathcal{D}}\int d^*_{jt}\left(u_{jt}(\boldsymbol{x}_t) + \varepsilon_{jt} + \beta\int V_{t+1}(\boldsymbol{x}_{t+1})dF(\boldsymbol{x}_{t+1}|\boldsymbol{x}_t, d^*_t)\right)dF_\varepsilon(\boldsymbol{\varepsilon}_t), \end{aligned} \tag{9}$$

where both integrals are multiple-dimensional. This expression is sometimes called the *Emax*, as it is the expectation of the solution of an optimization problem.

Define the conditional value function $v_{jt}(\boldsymbol{x}_t)$ as the payoff of option j without ε_{jt}:

$$v_{jt}(\boldsymbol{x}_t) \equiv u_{jt}(\boldsymbol{x}_t) + \beta\int V_{t+1}(\boldsymbol{x}_{t+1})dF_x(\boldsymbol{x}_{t+1}|\boldsymbol{x}_t, j). \tag{10}$$

The individual chooses j in period t if and only if:

$$v_{jt}(\boldsymbol{x}_t) + \varepsilon_{jt} \geq v_{kt}(\boldsymbol{x}_t) + \varepsilon_{kt} \quad \forall k \neq j, \tag{11}$$

exactly as in the random utility model described in chapter 3 of the Microeconometrics course. In this case, given assumption 4, the *conditional choice probabilities* (CCP) $p_{jt}(\boldsymbol{x}_t)$ are described by a conditional logit:

$$p_{jt}(\boldsymbol{x}_t) \equiv \mathbb{E}[d^*_{jt}|\boldsymbol{x}_t] = \frac{e^{v_{jt}(\boldsymbol{x}_t)}}{\sum_{h\in\mathcal{D}} e^{v_{ht}(\boldsymbol{x}_t)}}. \tag{12}$$

To compute the CCPs, we need the conditional value functions, $v_{jt}(\boldsymbol{x})$. In general, they do not have a closed form solution. We need to recover them solving the model for every parameter evaluation, or through a CCP estimator, suggested by Hotz and Miller (1993), discussed in Chapter 2. The type-I extreme value assumption is very convenient because it allows to write the Emax as a function of $v_{jt}(\boldsymbol{x}_t)$:

$$V_{t+1}(x) = \ln \sum_{j \in \mathcal{D}} \exp\{v_{jt+1}(x)\} + \gamma, \tag{13}$$

where γ is the Euler's constant.[1] Plugging this expression into Equation (10) yields:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \int \ln \left(\sum_{h \in \mathcal{D}} \exp\{v_{ht+1}(x)\} \right) dF_x(\boldsymbol{x}_{t+1}|\boldsymbol{x}_t, j) + \beta\gamma, \tag{14}$$

which describes a natural backwards induction (finite horizon) or fixed point (infinite horizon) algorithm to find $\{v_{jt}(\boldsymbol{x}_t)\}_{j \in \mathcal{D}}^{t=1,\dots,T}$. If we had another distribution (e.g. normal distribution), the multiple-dimensional integral embedded in the Emax should be computed numerically.

In order to write the likelihood function for our data (or the corresponding GMM estimation problem), we need to recover the conditional value function. A full information maximum likelihood estimator for a sample that includes decisions and state variables for $i = 1, \dots, N$ individuals observed over T_i periods, $\{d_{it}, \boldsymbol{x}_{it}\}_{i=1,\dots,N}^{t=1,2,\dots,T_i}$ is provided by the following likelihood function:

$$\mathcal{L}_{\mathrm{N}}(\boldsymbol{\theta}) = \sum_{\mathrm{i}=1}^{\mathrm{N}} \ln \Pr(d_{i1}, d_{i2}, \dots, d_{iT_i} \ , \ \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \dots, \boldsymbol{x}_{iT_i}; \boldsymbol{\theta}) \equiv \sum_{\mathrm{i}=1}^{\mathrm{N}} \ell_i(\boldsymbol{\theta}). \tag{15}$$

Given the previous assumptions, this probability can be factorized as:

$$\ell_i(\theta) = \sum_{t=1}^{T_i} \ln \Pr(d_{it}|\boldsymbol{x}_{it}; \boldsymbol{\theta}) + \sum_{t=2}^{T_i} \ln \Pr(\boldsymbol{x}_{it}|\boldsymbol{x}_{it-1}, d_{it-1}; \boldsymbol{\theta}) + \ln \Pr(\boldsymbol{x}_{i1}; \boldsymbol{\theta}). \tag{16}$$

The second term is $F_x(x_{it}|x_{it-1}, d_{it-1})$, whose parameters need to be estimated. The third term is the initial condition, which under current assumptions can be dismissed. The first term is the sum of CCPs, given by Equation (12). The conditional independence assumption is crucial for this factorization, because it establishes that $\boldsymbol{x}_t$ is a sufficient statistic for the previous sequence of decisions, and, hence, we can remove the dependence of CCPs on $d_{t-1}, \dots, d_{1t}$.

III. Motivational Example: Rust's Engine Replacement Model

Rust (1987) presents a discrete choice model of optimal engine replacement. The paper describes the behavior of Harold Zurcher, superintendent of maintenance

[1] As the Euler's constant appears additively in the utility associated to each of the alternatives, it is irrelevant for the maximization of utility, and, hence, we omit it hereinafter.

at the Madison Metropolitan Bus Company (Madison, Wisconsin). The objective of the paper is to test whether the behavior of Harold Zurcher coincides with that described in a regenerative optimal stopping model. The interest of the paper is not in the Harold *per se*, but as a general test of this way of modeling replacement of investment. It is one of the first dynamic discrete choice structural models ever estimated, and introduced a nested fixed point algorithm for estimation.

Every month t, Harold has to decide whether to replace the engine (of each bus i) or not. Hence, the decision variable is binary:

$$d_t = \begin{cases} 1 & \text{if replaces} \\ 0 & \text{if keeps.} \end{cases} \tag{17}$$

Harold solves an infinite horizon problem ($T = \infty$). Replacing implies a replacement cost, but a lower maintenance cost, and not replacing saves the cost of replacement at the expense of a larger maintenance cost. Hence, the utilities are:

$$U(d_t, x_t, \boldsymbol{\varepsilon}_t) = \begin{cases} -\theta_R - \theta_M 0 + \varepsilon_{1t} & \text{if } d_t = 1 \\ -\theta_M x_t + \varepsilon_{0t} & \text{if } d_t = 0, \end{cases} \tag{18}$$

where θ_R is the fixed replacement cost (the price of the new engine net of scrap value of the old one), and θ_M is the operative cost of an engine with mileage x_t. The other state variables (unobserved by the econometrician) $\boldsymbol{\varepsilon}_t = (\varepsilon_{0t}, \varepsilon_{1t})'$ satisfy the baseline assumptions described above. The interpretation of these variables is that of unobserved shocks that affect the maintenance cost of the old and new engine respectively other that mileage (a component failure that needs to be repaired vs a satisfactory report by the driver, and shortage of new engines to replace the old one or all bays in the shop are occupied vs there are engines and bays available,...).

The support of x is discrete, with a choice-specific transition matrix (x_{t+1} takes a value of zero with probability one if the engine is replaced, and one of the possible values X, with probabilities given by $F^0_{x_{t+1},x_t}$:

$$F^0_{x_{t+1},x_t} = \begin{pmatrix} \varphi_0 & \varphi_1 & 1-\varphi_0-\varphi_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \varphi_0 & \varphi_1 & 1-\varphi_0-\varphi_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \varphi_0 & \varphi_1 & 1-\varphi_0-\varphi_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \varphi_0 & \varphi_1 & 1-\varphi_0-\varphi_1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \varphi_0 & 1-\varphi_0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

Given infinite horizon, Equation (14) describes $v_j(x_t)$ as a fixed point of the system of equations given by:

$$v_j(x_t) = u_j(x_t) + \beta \sum_{x \in X} \ln \left(\sum_{h \in \mathcal{D}} \exp\{v_h(x)\} \right) F^j_{x,x_t} + \beta\gamma. \tag{20}$$

provided that conditional value functions are stationary, $v_{jt}(x) = v_j(x)\ \forall t$. This is a system of J equations and J unknowns for each $x \in X$.

IV. Estimation using full solution techniques: Rust's nested fixed point algorithm

Given the factorization in Equation (16), it is convenient to divide the parameter vector in two subsets of parameters: $\boldsymbol{\theta} = (\boldsymbol{\theta}'_U, \boldsymbol{\theta}'_x)'$, where $\boldsymbol{\theta}_x$ includes the parameters determining the law of motion of $\boldsymbol{x}_t$ (in our current example, $F^j_{l,h}$ for $j = 1, ..., J$ and $l, h = 1, ..., M$), and $\boldsymbol{\theta}_U$ includes the remaining parameters of the model. This division is convenient because it allows a two-step estimation:

$$\hat{\boldsymbol{\theta}}_x = \arg\max_{\boldsymbol{\theta}_x} \sum_{\mathrm{i}=1}^{\mathrm{N}} \sum_{t=2}^{T_i} \ln \Pr(x_{it} | x_{it-1}, d_{it-1}; \boldsymbol{\theta}_x), \tag{21}$$

and then:

$$\hat{\boldsymbol{\theta}}_U = \arg\max_{\boldsymbol{\theta}_U} \sum_{\mathrm{i}=1}^{\mathrm{N}} \sum_{t=1}^{T_i} \ln \Pr(d_{it} | x_{it}; \boldsymbol{\theta}_U, \hat{\boldsymbol{\theta}}_x). \tag{22}$$

We may add a third step in which we do a single iteration for the full likelihood optimization (Newton-Raphson or BHHH, discussed below) using $(\hat{\boldsymbol{\theta}}'_U, \hat{\boldsymbol{\theta}}'_x)'$ as starting values. This third step provides an estimator that is asymptotically equivalent to the full information ML estimator (hence consistent and efficient).

On top of separate estimation of the two sets of parameters, Rust provides an iterative algorithm that can be applied both to the partial and to the full likelihood maximization. The nested algorithm has an inner loop that solves the dynamic programming problem for every evaluation of $\boldsymbol{\theta}$ with the fixed point problem described above, and an outer algorithm that is a BHHH optimization routine (Berndt, Hall, Hall, Hausman, 1974) that iterates over $\hat{\boldsymbol{\theta}}$ (or $\hat{\boldsymbol{\theta}}_U$) to maximize the log-likelihood of the sample.

The BHHH optimization routine is similar to Newton-Raphson, except that it avoids computing the Hessian, which in this case is computationally demanding. In particular, the algorithm updates parameter guesses as follows:

$$\boldsymbol{\theta}^{m+1} = \boldsymbol{\theta}^m - \left(\sum_{\mathrm{i}=1}^{\mathrm{N}} \sum_{t=1}^{T_i} \frac{\partial \ell_{it}(\boldsymbol{\theta}^m)}{\partial \boldsymbol{\theta}} \frac{\partial \ell_{it}(\boldsymbol{\theta}^m)}{\partial \boldsymbol{\theta}'} \right)^{-1} \left(\sum_{\mathrm{i}=1}^{\mathrm{N}} \sum_{t=1}^{T_i} \frac{\partial \ell_{it}(\boldsymbol{\theta}^m)}{\partial \boldsymbol{\theta}'} \right). \tag{23}$$

Put differently, the routine uses the information matrix equality to approximate the Hessian (it is an approximation because the information matrix equality does not hold for every possible value of $\boldsymbol{\theta}$).

As noted above, we compute, in a first stage, the transition probabilities for the observable state vector are estimated. In the Rust example, these parameters are: φ_0, φ_1, and φ_2, introduced in Equation (19). Using data from Rust (1987), we obtain the following estimates:

Parameter	Group 1, 2, 3	Group 4	Group 1, 2, 3, 4
φ_0	0.29	0.40	0.33
	(0.01)	(0.01)	(0.01)
φ_1	0.70	0.59	0.66
	(0.01)	(0.01)	(0.01)
φ_2	0.01	0.01	0.01
	(0.00)	(0.00)	(0.00)

Courtesy of José García-Louzao, Sergi Marin Arànega, Alex Tagliabracci, and Alessandro Ruggieri, who replicated Rust's paper for the replication exercise in the Microeconometrics IDEA PhD course in Fall 2014.

To estimate the cost parameters, θ_R and θ_M, we fix $\beta = 0.99$. The results obtained are:

Method	Parameter	Group 1, 2, 3	Group 4	Group 1, 2, 3, 4
NFXP	θ_R	11.87	10.12	9.75
		(1.95)	(1.36)	(0.89)
	θ_M	5.02	1.18	1.37
		(1.40)	(0.28)	(0.24)

Courtesy of José García-Louzao, Sergi Marin Arànega, Alex Tagliabracci, and Alessandro Ruggieri, who replicated Rust's paper for the replication exercise in the Microeconometrics IDEA PhD course in Fall 2014.

V. Extensions: Unobserved Heterogeneity and Equilibrium

A. *Unobserved permanent heterogeneity*

The first assumption that we relax with respect to our baseline framework is the IID assumption. Assuming that unobservable characteristics are not correlated over time might be restrictive in many contexts. For instance, in a human capital model, there are some unobserved characteristics (ability) that are innate (and hence permanent) to individuals. In this section we allow for persistence of unobservables in the form of permanent unobserved heterogeneity. We use the paper by Keane and Wolpin (1997) as a motivational example.

Keane and Wolpin model the career decisions of young U.S. men. They propose a dynamic model of human capital formation (education and experience accumulation). Dynamics in this model are important because human capital investment decisions are forward-looking in its nature. The (simplified version of the) setup of the model is as follows. Individuals decide every year (from age 16 upon retirement) whether to stay home ($d_t = 0$), work in blue collar ($d_t = 1$), white collar ($d_t = 2$) or military sectors ($d_t = 3$), or attend school ($d_t = 4$). Observable state variables, $\boldsymbol{z}_t \equiv (e_t, \boldsymbol{x}_t')'$, include years of education e_t and sector-specific work experience $\boldsymbol{x}_t \equiv (x_{1t}, x_{2t}, x_{3t})'$. There are two vectors of unobserved variables: $\boldsymbol{\varepsilon}_t$, which is an i.i.d. idiosyncratic shock with $\boldsymbol{\varepsilon}_t \sim \mathcal{N}(\mathbf{0}, \Sigma)$, and $\boldsymbol{\omega}$, which is an individual-specific time-invariant component with a discrete support. Utility is given by:

$$U(d_t, \boldsymbol{z}_t, \boldsymbol{\omega}, \boldsymbol{\varepsilon}_t) = \begin{cases} \omega_0 + \varepsilon_{0t} & \text{if } d_t = 0 \\ r_j \exp\{\omega_j + \theta_{1j} e_t + \theta_{2j} x_{jt} + \theta_{3j} x_{jt}^2 + \varepsilon_{jt}\} & \text{if } d_t = 1, 2, 3 \\ \omega_4 + \theta_4 \, \mathbb{1}\{e_t \geq 12\} + \theta_5 \, \mathbb{1}\{e_t \geq 16\} + \varepsilon_{4t} & \text{if } d_t = 4. \end{cases} \tag{24}$$

The opportunity cost of investing in human capital through school attainment is the value of forgone earnings and work experience, or the forgone utility of staying home; working also has an investment value since it increases occupation-specific skills and, hence, future earnings. Career paths are additionally determined by comparative advantage embedded in endowments at age 16, both because of the initial stock of years of education, and because (as discussed below), the distribution of unobserved heterogeneity is conditional on initial education.

This setup departs from the Rust model in many ways. First, the transition probabilities for observed state variables are deterministic: if the individual goes to school, her education increases by one year and her experience remains unchanged; if she works in occupation j, her experience in that occupation increases by one year and her education and experience in other occupations stay constant; and if she stays at home, none of the observable state variables change. Second, the assumptions on the unobservable state variables are relaxed. We do not have additive separability anymore, as the unobservable state variable enters multiplicative in wage equations. The type-I extreme value assumption is changed by normally distributed errors correlated across alternatives. And, third, the conditional independence assumption is relaxed by the introduction of a permanent component of the unobservable state variables, $\boldsymbol{\omega}$, that introduces persistence in unobservable state variables.

The latter is the most important departure from Rust's framework. Because of the persistence in unobservables, $\boldsymbol{z}_t$ is no longer a sufficient statistic for d_t because lagged decisions $d_{t-1}, ..., d_1$ contain information about $\boldsymbol{\omega}$. However, conditional on $\boldsymbol{\omega}$, the transitory components ε_{it} still satisfy the CI assumption. Therefore, the likelihood conditional on unobserved heterogeneity can still be factorized as above.

For estimation, we optimize the integrated likelihood, in the same spirit as in the duration analysis or random parameters logit: we write the likelihood conditional on $\boldsymbol{\omega}$, and then, we integrate over the distribution of $\boldsymbol{\omega}$. We assume a discrete support for $\boldsymbol{\omega}$ given by $\Omega \equiv \{\boldsymbol{\omega}^k : k = 1, 2, ..., K\}$. Hence, the integral over the distribution of $\boldsymbol{\omega}$ is a sum:

$$\mathcal{L}_{\mathrm{N}}(\boldsymbol{\theta}) = \sum_{\mathrm{i=1}}^{\mathrm{N}} \ln \left\{ \sum_{k=1}^{K} \Pr(d_{i1}, d_{i2}, ..., d_{iT_i}\ ,\ \boldsymbol{z}_{i1}, \boldsymbol{z}_{i2}, ..., \boldsymbol{z}_{iT_i} | \boldsymbol{\omega}^k; \boldsymbol{\theta}) \pi_{k|\boldsymbol{z}_{i1}} \right\}, \qquad (25)$$

where $\pi_{k|\boldsymbol{z}_{i1}} \equiv \Pr(\boldsymbol{\omega}_i = \boldsymbol{\omega}^k | \boldsymbol{z}_{i1})$ is the probability that individual i is of type k (i.e. the probability that $\boldsymbol{\omega}_i$ is equal to $\boldsymbol{\omega}^k$) conditional on the initial vector of state variables. For instance, if we think of $\boldsymbol{\omega}$ as ability, we might think that the probability of being of a "high ability" type is larger for individuals that at age 16 completed 10 years of education than for those who dropped out. The points of support and the type probabilities, $\{\boldsymbol{\omega}^k, \pi_{k|\boldsymbol{z}_1} : k = 1, 2, ..., K; \boldsymbol{z}_1 \in Z_1\}$, are parameters to be estimated.

Note that the computation burden is substantially larger than in the equivalent situation without unobserved heterogeneity. First, because we need to solve the DP problem for each type of individual in every evaluation. Second, the partial/two-step likelihood approach used before is not usable anymore because the history of decisions by an individual carries information about its unobserved heterogeneity beyond the current state vector ($\boldsymbol{z}_t$).

Also note that the unobserved heterogeneity poses an initial conditions problem, as the likelihood is expressed conditional on $\boldsymbol{z}_{i1}$, which is correlated with permanent unobserved components. This problem is avoided if the DP problem is finite horizon and either we observe (left-)complete histories or a single value of $\boldsymbol{z}$ at $t = 1$ shared by all individuals.

B. Estimation of competitive equilibrium models

Lee and Wolpin (2006) extended the Keane and Wolpin framework to allow the price of skills, r_j from wage equations in (24), to be endogenously determined in equilibrium. Llull (2018) uses a similar approach, as we discuss in the application below. These papers have clear connections with general equilibrium heterogeneous agents models in macroeconomics. In this framework, r_j is, therefore, an equilibrium object, as opposed to a parameter to estimate as before. Hence, r_j is now r_{jt}. The model is expanded with an aggregate production function that determines the labor demand. Individual decisions (and hence Emaxes and CCPs) are a function of $\{r_{jt} : j = 1, ..., J\}$, which themselves are functions of individual decisions through market clearing conditions.

The endogeneization of skill prices complicates the situation in many aspects. First, the state space is augmented with skill prices, and with relevant information to predict their evolution. Second, finding prices that clear the market entails a fixed point algorithm, that increases the computational burden. Third, individuals need to forecast future skill prices. If there is aggregate uncertainty in the economy, individuals would need the entire distribution of state variables in the economy to make the best forecast of future skill prices. As this is unfeasible, we have to use an approximation to rational expectations finding quasi-sufficient statistics to predict future skill prices. In the case of Lee and Wolpin, they use an approximation which is a VAR in the differences in skill prices and in the aggregate shock. The coefficients of this rule are unknown implicit functions of the fundamental parameters of the model. Their estimation entails an additional fixed point algorithm in which a set of coefficients is used to solve the DP problem, then the U.S. economy is simulated, and expectation rules are then re-estimated with simulated aggregate data; this estimate serves as input for the next iteration, until reaching convergence. And fourth, identification of equilibrium prices through a likelihood estimation requires a representative sample for the whole population. Because of unobserved heterogeneity, we need left-complete working histories for individuals. Longitudinal data of this kind is not easy to find. Additionally, the problem is non-stationary (e.g. the evolution of the different aggregate exogenous variables, that is taken from the data, and the aggregate shock, that is an AR(1) in differences). As a result, identification of equilibrium skill prices requires left-complete histories of a sample that is representative of all cohorts alive during the estimation period, because they are exposed to different sequences of aggregate shocks, which are not observed. Because such a dataset does not exist, this model is estimated with a combination of different data sources, using a Simulated Minimum Distance estimator.

C. Using randomized experimental data to validate structural models

One of the crucial aspects for the credibility of the estimation of structural models is the validation. In particular, it is important to show that the model fits the most relevant features of the data (either in-sample or —ideally— out of sample). Todd and Wolpin (2006) propose a very nice validation strategy. The goal of their paper is to estimate a model of child schooling and fertility in order to evaluate alternative subsidies to school attendance in rural Mexico.

A very ambitious assistance program called PROGRESA was introduced in Mexico (and later exported to many Latin American and Asian countries) to foster children's (mainly girls') school attendance. At the beginning of the implementation, the program was randomly implemented in a subset of villages, and data was gathered in treated and control villages to evaluate the effectiveness of the program. The analysis of these experimental data has lead to many papers. However, in such "experimental studies", the analysis focuses always on the same treatment (namely the given subsidy). The goal of Todd and Wolpin (2006) is to use a structural model to be able to evaluate alternative subsidies other than the ones that were actually given, and to predict the long-run effects of the subsidy.

The empirical strategy consists of estimating their model using only the data from the control group (individuals from villages that did not receive the subsidy). Then, once the model is estimated, they simulate the counterfactual behavior of individuals had them received the subsidy, and compare it with the behavior of individuals in the treatment group. If randomization was successful, treatment and control households should belong to the same population, and the structural behavioral model should be invariant across both groups.

Two important assumptions are made in this exercise. First, in order to identify the impact of the subsidy (for which there is no variation in the control sample) the authors rely on variation in child wages combined with some functional form assumptions: they assume that a 1$ subsidy is equivalent to a 1$ change in the wage of the child. And second, they assume that households in control villages did not anticipate the subsidy (which was eventually extended to them later on).

VI. Application: Llull (2018)

See the paper.