Modelling non-linear variability in the frequency domain

Phil Uttley

Why non-linear(non-Gaussian) modelling?

- Even if the underlying variability process is linear, many of our favourite emission mechanisms are non-linear (and in an energy-dependent way).
- Observed broadband-noise variability shows lognormal flux distributions, and this is also expected from propagating mdot-fluctuation models.
- Spectral-timing coherence is often < 1, this can be due to independent processes in different bands, but can also be produced by non-linear transforms of variability between different energy bands.
- Conventional impulse response modelling of lags and power-spectra is simple convolution (i.e. a ***linear transform***), we need to extend this to the non-linear regime.
- And (thanks Kavitha!)..... all sources have non-zero bicoherence! This cannot be produced by linear Gaussian processes with linear transforms applied - we would like to be able to model the bispectrum of our broadband noise light curves.

Taylor series representation of the light curve

- Let's take a Gaussian-distributed light curve with mean 0, $x(t)$
- Now we transform it to create $y(t)$ with a non-linear function:

$$y(t) = f(x(t))$$

$$y(t) = f(0) + f'(0)x(t) + \frac{f''(0)}{2}x^2(t) + \frac{f'''(0)}{6}x^3(t) + \sum_{n=4}^{\infty} \frac{f_0^{(n')}}{n!} x^n(t)$$

In the Fourier domain this gives:

$$Y_\nu = f(0)\delta_\nu + f'(0)X_\nu + \frac{f''(0)}{2}(X_\nu * X_\nu) + \frac{f'''(0)}{6}(X_\nu * X_\nu * X_\nu) + \sum_{n=4}^{\infty} \frac{f_0^{(n')}}{n!} X_\nu^{(n*)}$$

constant-in-time components shift to $\nu = 0$

(where X_ν is the Fourier transform of $x(t)$ and $*$ denotes convolution)

Calculating the power spectrum 1

Since our light curves are noise processes there is not a continuous form for X_ν, which for a given light curve and Fourier frequency has real and imaginary parts drawn from independent Gaussians with mean 0:

$$X_\nu \sim (\mathcal{N}(0,1) + i\,\mathcal{N}(0,1))\sqrt{P_\nu/2}$$

where the power spectrum $P_\nu = \langle |X_\nu|^2 \rangle$

(expectation value of $X_\nu X_\nu^c$ where c denotes complex conjugation)

The power spectrum of $y(t)$ is:

$$\langle |Y_\nu|^2 \rangle = \left\langle \left(\sum_{n_1=0}^{\infty} \frac{f_0^{(n_1{}')}}{n_1!} X_\nu^{(n_1*)} \right) \left(\sum_{n_2=0}^{\infty} \frac{f_0^{(n_2{}')}}{n_2!} X_\nu^{c(n_2*)} \right) \right\rangle$$

This looks like a nightmare. However....

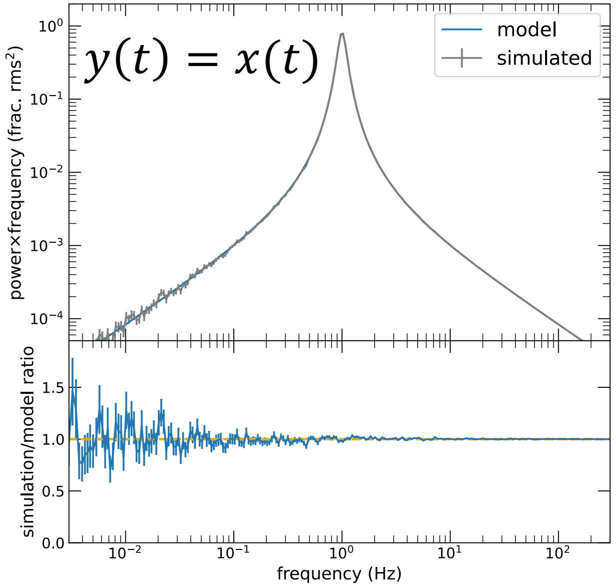
Calculating the power spectrum 2

of non-zero combinations of $\left\langle X_\nu^{(n_1*)} X_\nu^{c(n_2*)} \right\rangle$

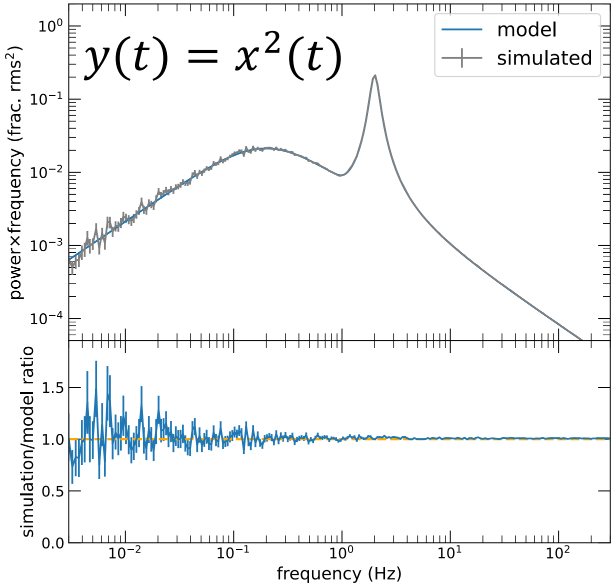
$$C_{n_1,n_2,l} = \frac{n_1!\, n_2!\, \max\big(0, n_1 - (l+1)\big)!!\, \max\big(0, n_2 - (l+1)\big)!!}{l!\,(n_1 - l)!\,(n_2 - l)!}$$

$$\left\langle \left(\frac{f_0^{(n_1\prime)}}{n_1!} X_\nu^{(n_1*)} \right) \left(\frac{f_0^{(n_2\prime)}}{n_2!} X_\nu^{c(n_2*)} \right) \right\rangle = \frac{f^{(n_1\prime)} f^{(n_2\prime)}}{n_1!\, n_2!} \sum_{\substack{l=0,\text{ even if } k \text{ even} \\ l=1,\text{ odd if } k \text{ odd}}}^{k=\min(n_1,n_2)} C_{n_1,n_2,l}\, \sigma^{(n_1+n_2-2l)} P_\nu^{(l*)}$$

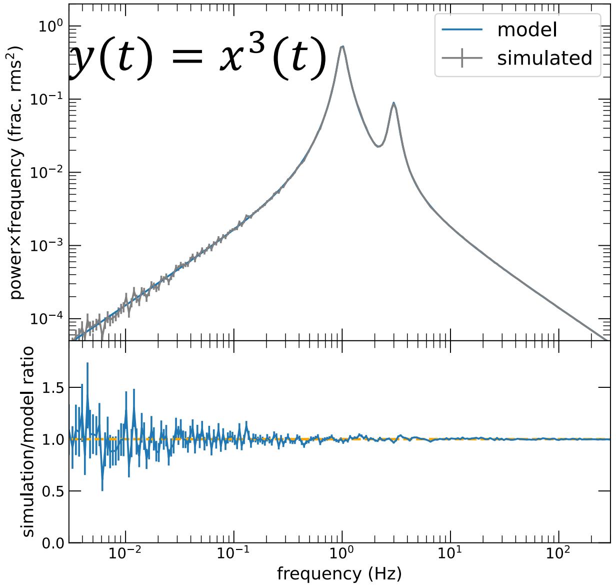
$n_1, n_2 = 1, l = 1$



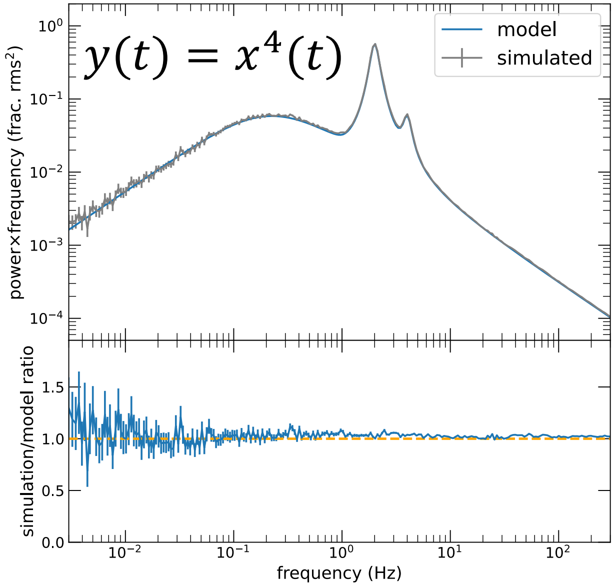
$n_1, n_2 = 2, l = 0, 2$



$n_1, n_2 = 3, l = 1, 3$



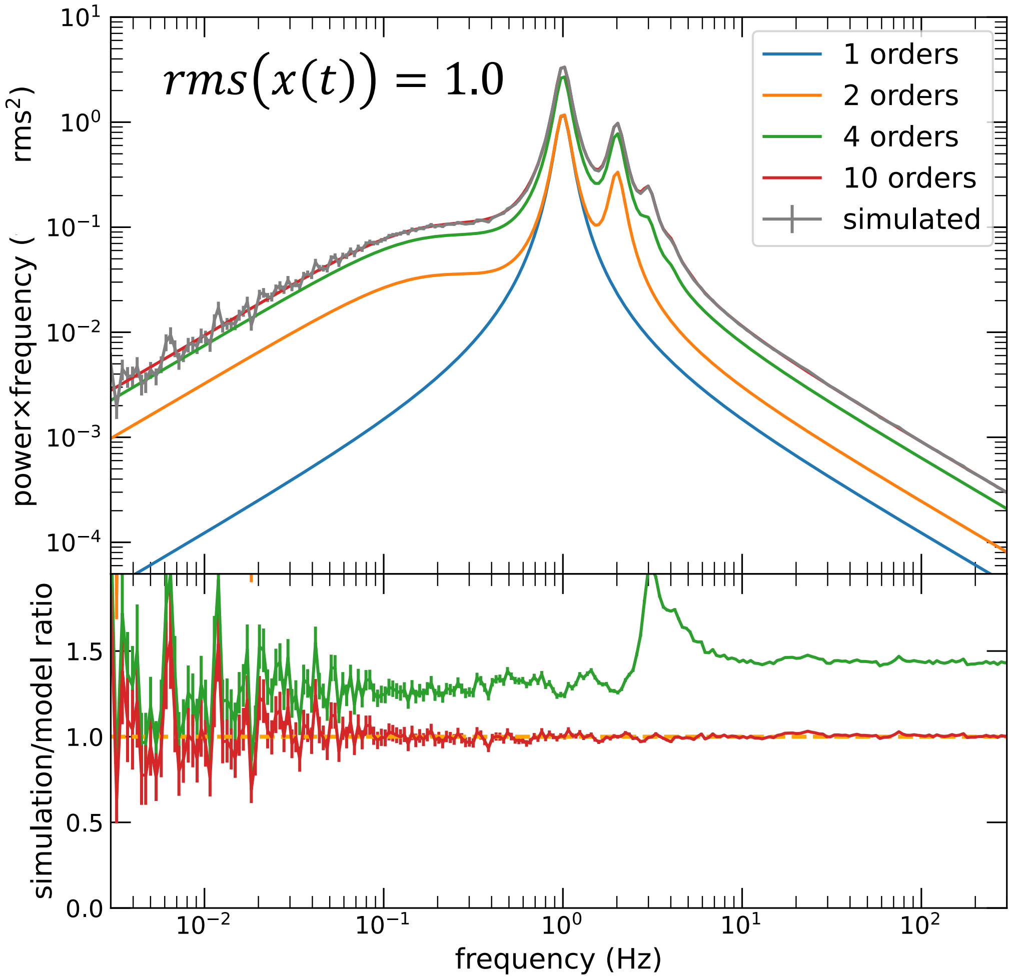
$n_1, n_2 = 4, l = 0, 2, 4$

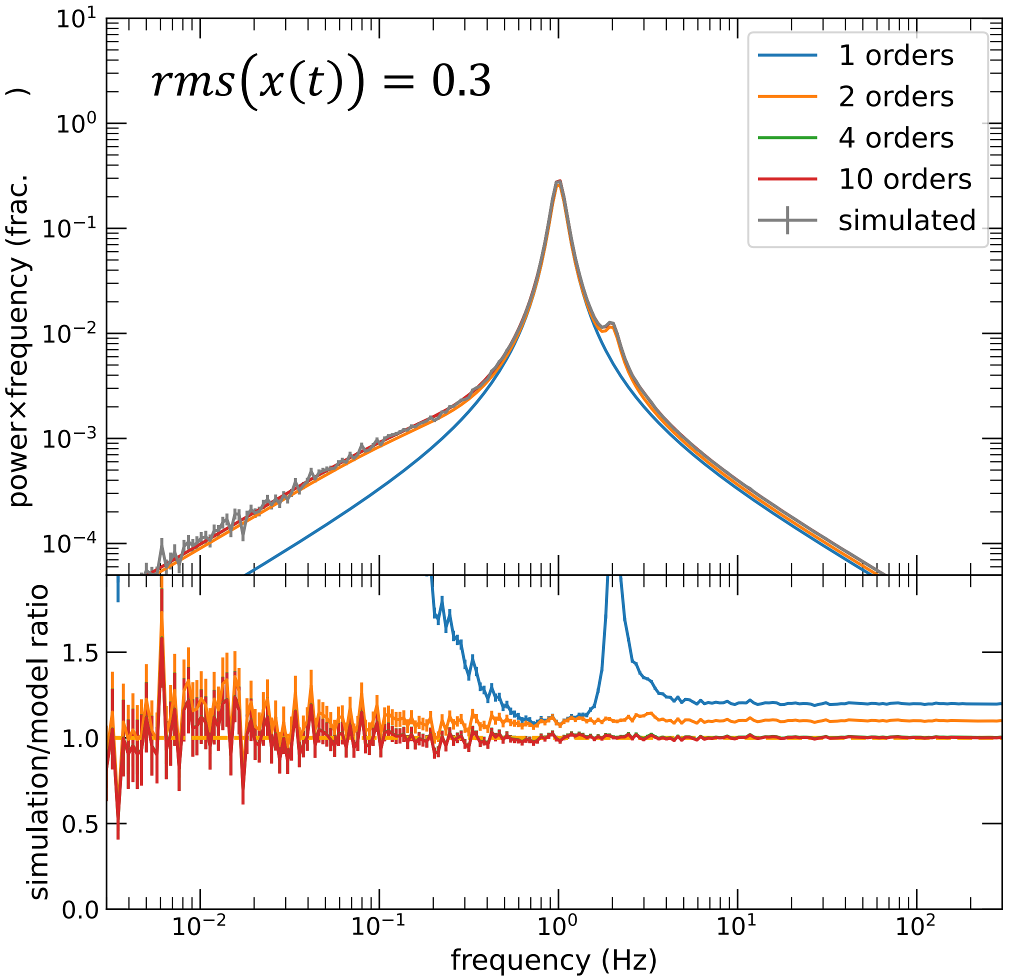


Example: exponentiated light curve

$$y(t) = \exp\big(x(t)\big) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n(t)$$

Lorentzian input power spectrum (q=5):

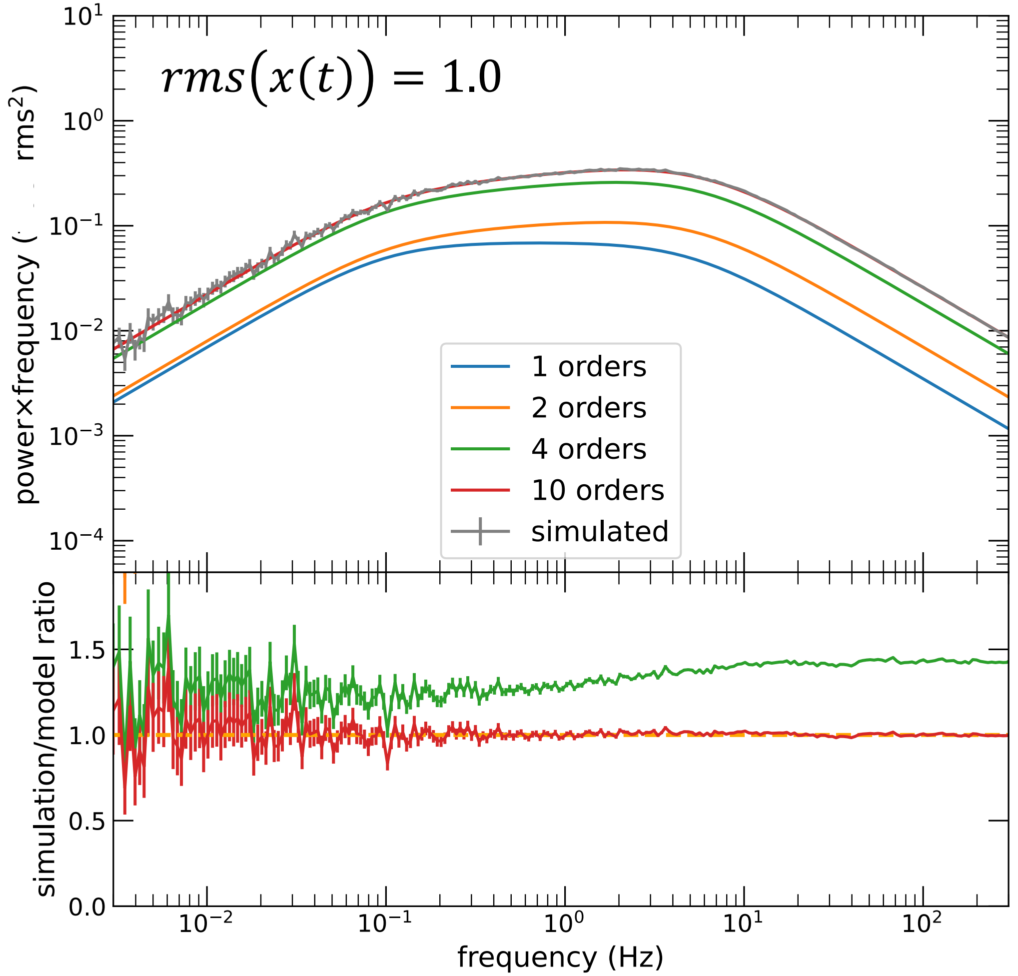


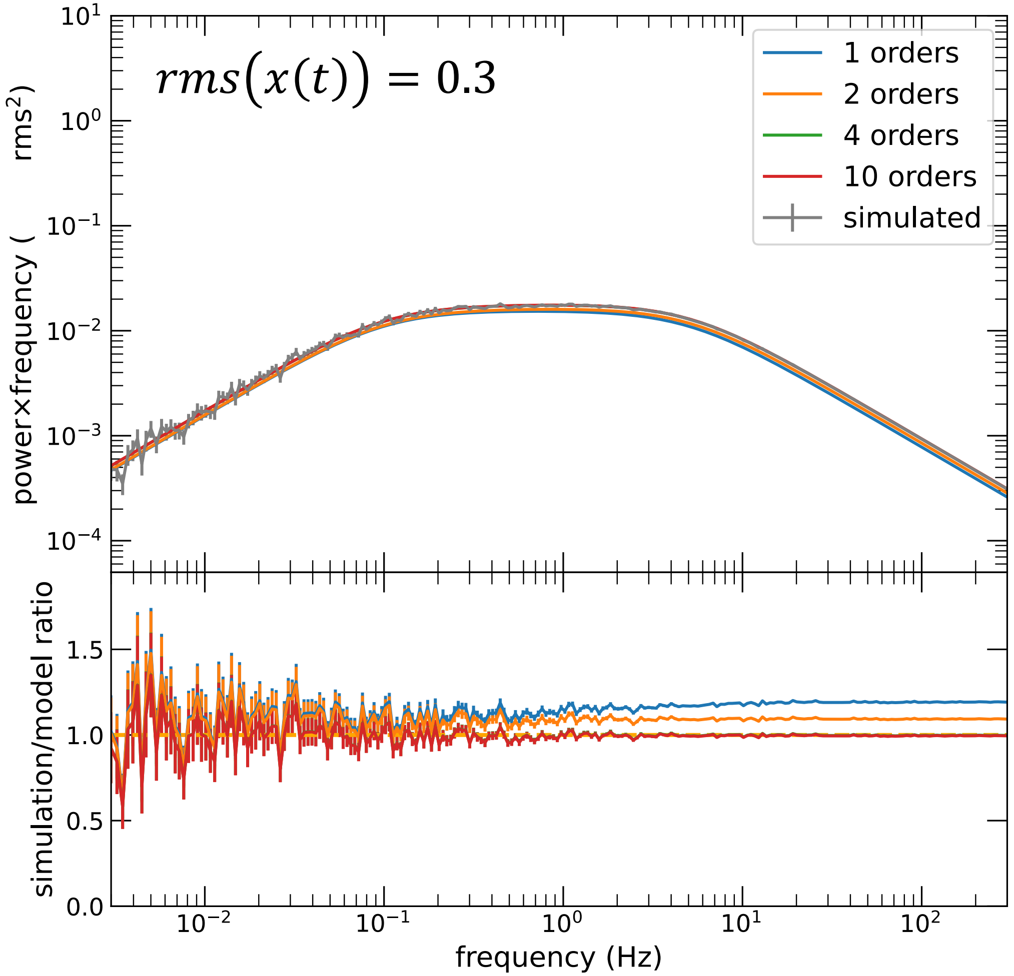


Example: exponentiated light curve

$$y(t) = \exp(x(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n(t)$$

Double-bending power-law input power spectrum:





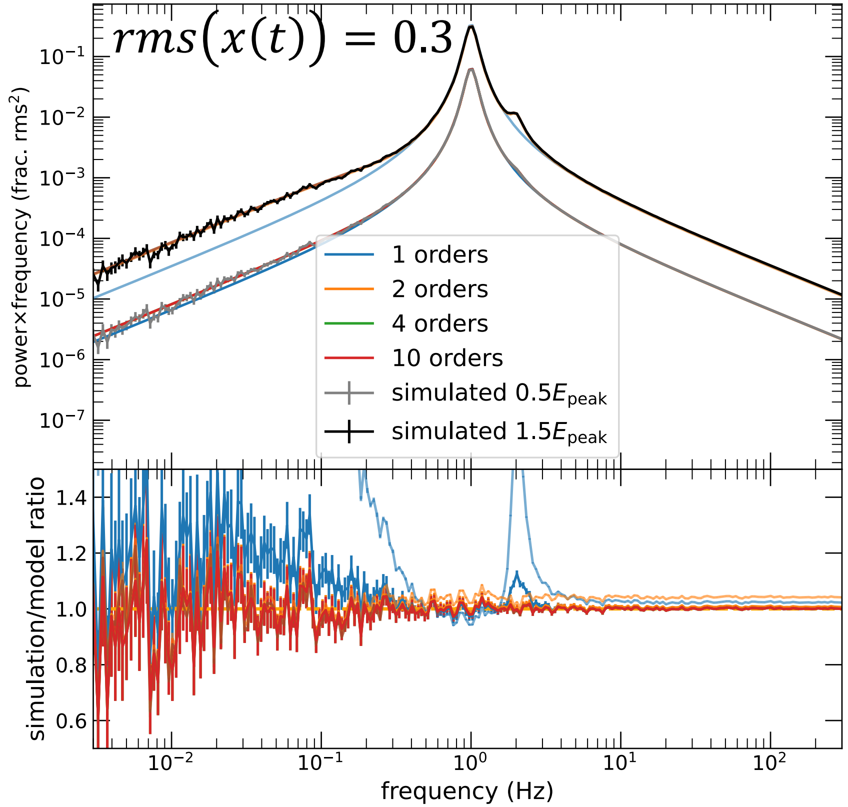
Cross-spectra and coherence (no lags yet)

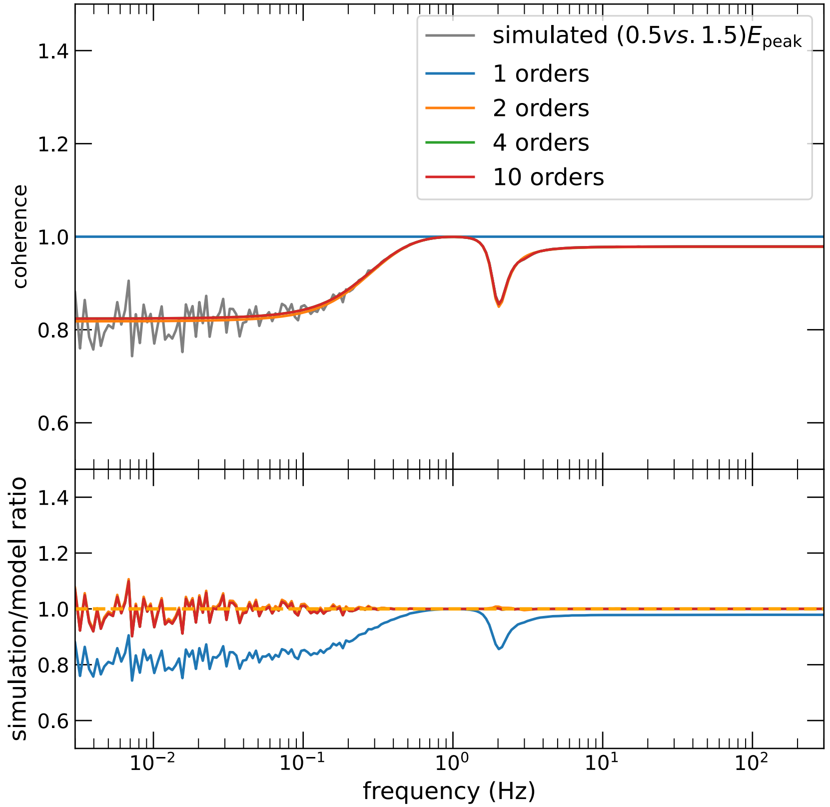
$$y(t) = f(x(t)) \qquad z(t) = g(x(t))$$

$f(x(t))$ and $g(x(t))$ can be different functions, or the same function with different parameters.

Cross-spectrum: $\langle Y_\nu Z_\nu^c \rangle = \langle C_\nu \rangle = \left\langle \left(\sum_{n_1=0}^{\infty} \frac{f_0^{(n_1\prime)}}{n_1!} X_\nu^{(n_1*)} \right) \left(\sum_{n_2=0}^{\infty} \frac{g_0^{(n_2\prime)}}{n_2!} X_\nu^{c(n_2*)} \right) \right\rangle$

Variable blackbody (constant area, kT ~ (exponentiated lightcurve)^0.25, evaluate at E=0.5x and 1.5x BB peak energy)





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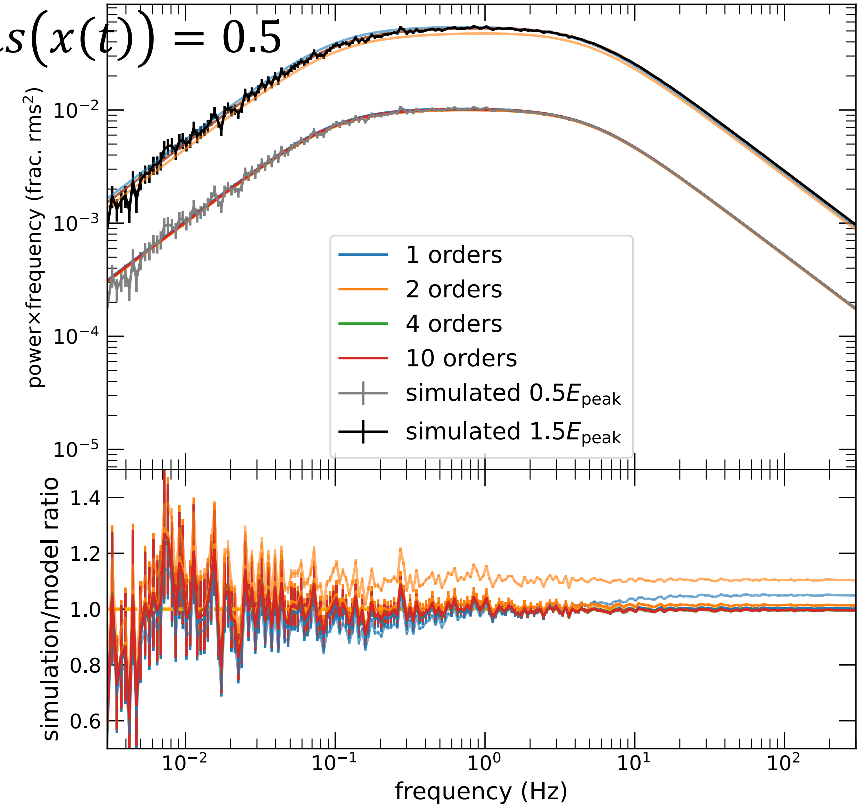
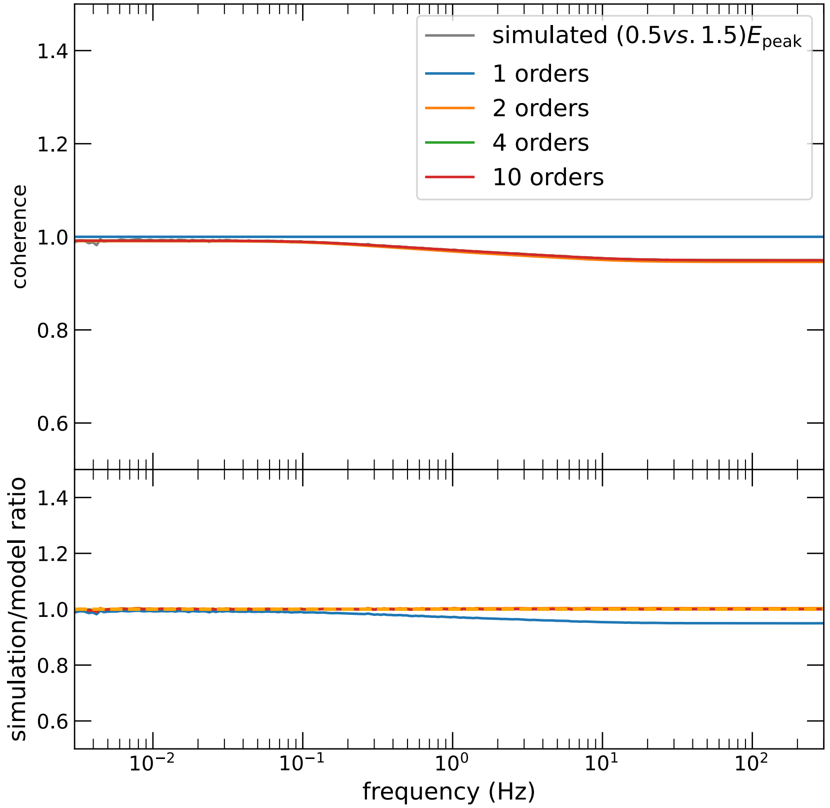
$rms(x(t)) = 0.5$





Introducing lags

Consider a single emitting region which responds to the variations with a time delay τ_i:

$$u_i(t) = \sum_{n=0}^{\infty} \frac{f_{0,i}^{(n')}}{n!} x^n(t+\tau_i) = \sum_{n=0}^{\infty} \frac{f_{0,i}^{(n')}\delta(\tau_i)}{n!} * x^n\,(t)$$

The delay can be included by convolving the original light curve with a δ-function

Now the total light curve of multiple regions with different delays:

This is an impulse response function!

$$y(t) = \sum_i u_i(t) = \sum_{n=0}^{\infty} \frac{\left(\sum_i f_{0,i}^{(n')}\delta(\tau_i)\right)}{n!} * x^n\,(t)$$

In the Fourier domain:

$$Y_\nu = \sum_{n=0}^{\infty} \frac{G_\nu^n}{n!} X_\nu^{(n*)}$$

where $G_\nu^n = \mathrm{FT}\left(\sum_i f_{0,i}^{(n')}\delta(\tau_i)\right)$, i.e. the transfer function for order *n*