Problem 1

a) $P(x_1, \cdots, x_n \mid \pi, r)$

$$= \prod_{i=1}^{N} \binom{x_i + r - 1}{x_i} \pi^{x_i} (1-\pi)^{r}$$

b) $l = \log (P(x_1, \cdots, x_n \mid \pi, r)$

$$= \sum_{i=1}^{N} \left[\binom{x_i + r - 1}{x_i} + x_i \log \pi + r \log(1-\pi) \right]$$

$$= \sum_{i=1}^{N} \left[\binom{x_i + r - 1}{x_i} \right] + \sum_{i=1}^{N} x_i \log \pi + nr \log(1-\pi)$$

$$\Rightarrow \frac{\partial}{\partial \pi} l = \sum_{i=1}^{N} x_i \cdot \frac{1}{\pi} + nr \cdot \frac{1}{1-\pi} \cdot (-1)$$

set $\frac{\partial}{\partial \pi} l$ to 0:

$$\sum_{i=1}^{N} x_i \frac{1}{\pi} = nr \cdot \frac{1}{1-\pi}$$

$$\sum_{i=1}^{N} x_i \cdot (1-\pi) = nr\pi$$

$$\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} x_i \cdot \pi = nr\pi$$

$$\pi \left(\sum_{i=1}^{N} x_i + nr \right) = \sum_{i=1}^{N} x_i$$

$$\Rightarrow \hat{\Pi}_{ML} = \frac{\sum_{i=1}^{N} X_i}{\sum_{i=1}^{N} X_i + nr}$$

c) $P(\pi) = beta(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \pi^{a-1} (1-\pi)^{b-1}$

$$p = P(\pi \mid x_1, \cdots, x_n) = \frac{P(x_1, \cdots x_n \mid \pi) \cdot P(\pi)}{\int_0^1 P(x_1, \cdots, x_n \mid \pi) P(\pi) d\pi}$$

$$\Rightarrow p = \frac{\prod_{i=1}^{N} \binom{x_i + r - 1}{x_i} \pi^{x_i} (1-\pi)^r \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \pi^{a-1} (1-\pi)^{b-1}}{\int_0^1 P(x_1, \cdots, x_n \mid \pi) P(\pi) d\pi}$$

$$\propto \prod_{i=1}^{N} \binom{x_i + r - 1}{x_i} \pi^{x_i} (1-\pi)^r \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \pi^{a-1} (1-\pi)^{b-1}$$

$$\Rightarrow l = \sum_{i=1}^{N} \binom{x_i + r - 1}{x_i} + \sum_{i=1}^{N} x_i \log(\pi) + nr \log(1-\pi)$$

$$+ \log\left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right) + (a-1)\log \pi + (b-1)\log(1-\pi)$$

$$\Rightarrow \frac{\partial}{\partial \pi} l = \sum_{i=1}^{N} X_i \cdot \frac{1}{\pi} + \frac{(-nr)}{1-\pi} + \frac{(a-1)}{\pi} + \frac{(-(b-1))}{1-\pi}$$

$$= \left[\sum_{i=1}^{n} x_i + (a-1)\right]\frac{1}{\pi} - (nr + b - 1)\frac{1}{1-\pi}$$

set $\frac{\partial}{\partial \pi} l = 0$:

$$\left[\sum_{i=1}^{n} x_i + (a-1)\right](1-\pi) = (nr + b - 1)\,\pi$$

$$\pi = \frac{\sum_{i=1}^{n} x_i + (a-1)}{\sum_{i=1}^{n} x_i + (a-1) + nr + b - 1}$$

$$= \frac{\sum_{i=1}^{n} x_i + a - 1}{\sum_{i=1}^{n} x_i + nr + a + b - 2}$$

d) $$P(\pi \mid x_1, \cdots, x_n) = \frac{P(x_1, \cdots, x_n \mid \pi)\, p(\pi)}{\int_0^1 p(x_1, \cdots, x_n)\, p(\pi)\, d\pi}$$

$$\propto \prod_{i=1}^{n} \binom{x_i + r - 1}{x_i} \pi^{x_i} (1-\pi)^{r} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \pi^{a-1} (1-\pi)^{b-1}$$

$$= \prod_{i=1}^{n} \binom{x_i + r - 1}{x_i} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \pi^{\sum_{i=1}^{n} x_i + a - 1} (1-\pi)^{nr + b - 1}$$

$$\propto \text{Beta}\left(\sum_{i=1}^{n} x_i + a \ , \ nr + b\right)$$

e) $P(\pi \mid x_1, \cdots, x_n) \sim \text{Beta}\left(\sum_{i=1}^{n} x_i + a, \ nr + b\right)$

$$\Rightarrow \overline{\pi}_{MAP} = \frac{\sum_{i=1}^{n} x_i + a}{\sum_{i=1}^{n} x_i + a + nr + b}$$

$$Var(\pi_{MAP}) = \frac{\left(\sum_{i=1}^{n} x_i + a\right) \cdot (nr + b)}{\left(\sum_{i=1}^{n} x_i + a + nr + b\right)^2 \left(\sum_{i=1}^{n} x_i + a + nr + b + 1\right)}$$

$\hat{\pi}_{ML}$ is a point estimator. It only depends on the given data and its distribution. $\hat{\pi}_{MAP}$ provides the distribution of π based on the proposed distribution, and adjusts according to the given data. The variance gives us a way to quantify our confidence for our estimation.