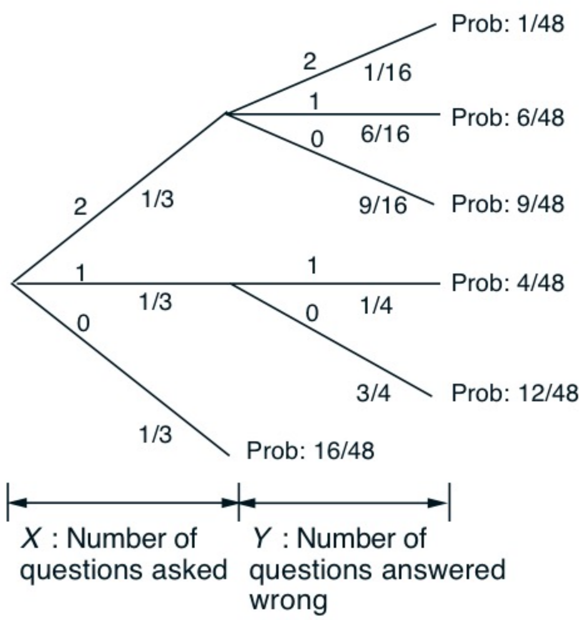
Solutions for Live Session 5 – Professorial Problem

In a live session, the number of questions that students ask is a random variable, X. X takes on three values: 0, 1, and 2, each with probability 1/3. Every time a student asks a question, Professor Paul Laskowski answers incorrectly with probability 1/4, independently of other questions. Let Y be the random variable representing the number of incorrect responses in the live session.

a. Compute the expectation of Y, conditional on X.
   Since we've conditioned on X, we can think of X as a constant.
   Let Z_i be a Bernoulli variable indicating whether question i was answered incorrectly.
   Then $E(Y|X) = E(\sum_{i=0}^{X} Z_i) = \sum_{i=0}^{X} E(Z_i) = \sum_{i=0}^{X} \frac{1}{4} = \frac{X}{4}$

$P_{Y|X=0}(0) = 1$
$P_{Y|X=1}(0) = 3/4$ $\quad$ $P_{Y|X=1}(1) = 1/4$
$P_{Y|X=2}(0) = 9/16$ $\quad$ $P_{Y|X=2}(1) = 6/16$ $\quad$ $P_{Y|X=2}(2) = 1/16$

E(Y|X=0) = 0(1) = 0 with probability 1/3
E(Y|X=1) = 0(3/4)+1(1/4) = ¼ with probability 1/3
E(Y|X=2) = 0(9/16)+1(6/16)+2(1/16) = 8/16 = ½ with probability 1/3



A quick note about why $P_{Y|X=2}(1) = 6/16$:
   Let X = number of questions asked and Y = number of incorrect answers.
   If X =2, then we have 2 possible answers, which we can denote by A1, A2.
   Each answer Ai can be either correct or incorrect so (A1, A2) must be one of the following values: (Correct, Correct), (Incorrect, Incorrect), (Correct, Incorrect), (Incorrect, Correct).
   Hence if we ask 2 questions, i.e. X =2, there are 2 ways for Y = 1 (i.e. (Incorrect, Correct) and (Correct, Incorrect) so the Pr(Y=1|X=2)=2*(1/4*3/4)= 6/16

b. Using the law of iterated expectations, compute E(Y).

E(Y) $= E(E(Y|X)) = E(\frac{X}{4}) = \frac{E(X)}{4} = \frac{1}{4}$

E(Y) $= E(E(Y|X))$

$= E(Y|X_1)P(X_1) + E(Y|X_2)P(X_2) + E(Y|X_3)P(X_3)$

$= E(Y|X=0)P(X=0) + E(Y|X=1)P(X=1) + E(Y|X=2)P(X=2)$

$= 0*1/3 + 1/4*1/3 + ½*1/3 = 3/12 = ¼$

Alternative:

$E(Y=y|X) = \frac{y}{4}$

E(Y) $= E(E(Y|X))$

$= E(Y|X_1)P(X_1) + E(Y|X_2)P(X_2) + E(Y|X_3)P(X_3)$

$= \frac{0}{4}\bullet\frac{1}{3} + \frac{1}{4}\bullet\frac{1}{3} + \frac{2}{4}\bullet\frac{1}{3}$

$= \frac{1}{4}$

c. Describe the joint probability distribution of X and Y. (You may find it easiest to use a table)

Let X and Y be the number of questions Paul is asked and the number of questions he answers wrong in a given lecture, respectively. To construct the joint PMF $p_{X,Y}(x, y)$, we need to calculate all the probabilities P(X = x, Y = y) for all combinations of values of x and y. This can be done by using a sequential description of the experiment and the multiplication rule $p_{X,Y}(x, y) = p_x(x)p_{Y|X}(y|x)$.

p(x,y)	**x=0**	**x=1**	**x=2**	
y=0	16/48	12/48	9/48	37/48
y=1	0	4/48	6/48	10/48
y=2	0	0	1/48	1/48
	1/3	1/3	1/3	

By the multiplication rule, f(x,y) = f(x) f(y|x).

As long as $y \leq x$, we can write this as f(x) f(y|x) = $\frac{1}{3}\left(\frac{1}{4}\right)^{y} \cdot \left(\frac{3}{4}\right)^{x-y}$ (x choose y).

For $y > x$, we know the probability is zero.

d. Compute the expectation of the product of X and Y, E(XY).

E(XY) = E(E(XY|X)) = E(X·E(Y|X)) = $E\left(\frac{X^2}{4}\right) = \frac{E(X^2)}{4}$ = [$\frac{0+1+4}{3}$]/4 = $\frac{5}{12}$

$$E[XY] = \sum_{i,j=0}^{2} x_i y_j \cdot p(x_i, y_j)$$

$$= \left[(0\cdot 0)\cdot\frac{16}{48}\right]+\left[(1\cdot 0)\cdot\frac{12}{48}\right]+\left[(1\cdot 1)\cdot\frac{4}{48}\right]+\left[(2\cdot 0)\cdot\frac{9}{48}\right]+\left[(2\cdot 1)\cdot\frac{6}{48}\right]+\left[(2\cdot 2)\cdot\frac{1}{48}\right]$$

$$= \frac{4}{48} + \frac{12}{48} + \frac{4}{48}$$

$$= \frac{20}{48}$$

e. Using the previous result, compute cov(X, Y).

Cov(X,Y) = E[XY] − E[X]E[Y]

Cov(X,Y) = E[XY] − E[X]E[Y] = $\frac{5}{12} - 1(\frac{1}{4}) = \frac{5}{12} - \frac{3}{12} = \frac{1}{6}$

E[X]E[Y] = $(0\cdot P[X=0] + 1\cdot P[X=1] + 2\cdot P[X=2])\cdot(0\cdot P[Y=0] + 1\cdot P[Y=1] + 2\cdot P[Y=2])$

$$= (1\cdot\frac{1}{3} + 2\cdot\frac{1}{3})\,(1\cdot\frac{10}{48} + 2\cdot\frac{1}{48})$$

$$= \frac{12}{48}$$

=> Cov(X,Y) = E[XY] − E[X]E[Y] = $\frac{20}{48} - \frac{12}{48} = \frac{8}{48}$

Question 2

Continuous random variables X and Y have a joint distribution with probability density function,

$$f(x,y) = \begin{cases} 1 & 0 < y < 1, \quad a \cdot y < x < a \cdot y + 1 \\ 0 & otherwise \end{cases}$$

where a is a constant.

a. Choose 2 example values for a and draw a graph of the region for which X and Y have positive probability density.
b. Derive the marginal distribution of Y.
c. Compute the conditional expectation of X, conditional on Y.
d. As a slight variation on the previous part, compute E(XY|Y). Note that since we're conditioning on Y, the Y inside the expectation is just a constant.
e. Derive cov(X,Y). Hint: an nice way to do this is to use the law of iterated expectations. Write down the definition of covariance, then break the expectation up into two expectations. The inner expectation should be conditional on Y, and the outer expectation should be unconditional. The results you derived above should help you finish the proof.
f. Check what cov(X,Y) equals when a = **0** . What is cov(X,Y) when a = −**1** ?
g. Choose an example value for a, then use R to simulate 100 draws from the given joint distribution and plot them.

Hints/Solutions:

$$f_Y(y) = \int_{-\infty}^{\infty} f_y(x,y)dx = \int_{x=ay+1}^{x=a} 1dx = 1$$

$E(X|Y) = ay + \frac{1}{2}$

$f_{X|Y}$ is uniform over [ay, ay+1]

For Y = constant:

$$E(XY|Y) = Y\, E(X|Y) = Y\left(aY + \frac{1}{2}\right) = aY^2 + \frac{Y}{2}$$

$$cov(X,Y) = E(XY) - E(X)E(Y) = \frac{a}{3} + \frac{1}{4} - \left(\frac{a}{2} + \frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{a}{3} + \frac{1}{4} - \frac{a}{4} - \frac{1}{4} = \frac{a}{12}$$

$$E(XY) = E(E(XY|Y)) = \mathrm{E}\left(aY^2 + \frac{Y}{2}\right) = aE(Y^2) + \frac{1}{2}E(Y)$$

$$E(Y^2) = \int_0^1 Y^2(1)dy = \frac{1}{3}y^3\Big|_0^1 = \frac{1}{3}$$

$$E(Y) = \int_0^1 y f_y(y)dy = \int_0^1 y\, dy = \frac{1}{2}y^2\Big|_0^1 = \frac{1}{2}$$

$$E(XY) = E(XY|Y) = aE(Y^2) + \frac{1}{2}E(Y) \ = \ a\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{a}{3} + \frac{1}{4}$$

$$\begin{aligned} cov(X,Y) \ &= E(XY) - E(X)E(Y) \\ &= \frac{a}{3} + \frac{1}{4} - \left(\frac{a}{2} + \frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{a}{3} + \frac{1}{4} - \frac{a}{4} - \frac{1}{4} = \frac{a}{12} \end{aligned}$$