Unit 5 Homework Key

w203 Instructors

Q1

The covariance matrix looks like the following:

$$\begin{bmatrix} cov(W,W) & cov(W,V) \\ cov(V,W) & cov(V,V) \end{bmatrix}$$

We compute each term individually.

Since W has standard deviation 4,

$cov(W,W) = var(W) = 4^2 = 16$

$cov(W,V) = cov(W, 0.5W + U) = cov(W, 0.5W) + cov(W,U)$

The second term is zero since we know U is independent of W. Therefore,

$cov(W,V) = 0.5cov(W,W) = 0.5var(W) = 0.5 \cdot 16 = 8$

$cov(V,V) = var(V) = var(0.5W + U) = var(0.5W) + var(U) + 2cov(U,W)$

As before, the last term is zero. We are given that U has a standard normal distribution so its variance is 1. Therefore,

$cov(V,V) = 0.5^2 var(W) + var(U) = 0.5^2 16 + 1 = 5$

Put together, our covariance matrix looks like,

$$\begin{bmatrix} 16 & 8 \\ 8 & 5 \end{bmatrix}$$

Q2

You have a ruler of length 1 and you choose a place to break it using a uniform probability distribution. Let random variable X represent the length of the left piece of the ruler. X is distributed uniformly in $[0,1]$. You take the left piece of the ruler and once again choose a place to break it using a uniform probability distribution. Let random variable Y be the length of the left piece from the second break.

(a) Find the conditional expectation of Y given X, $E(Y|X)$.

The hardest thing about this question is probably just understanding that the problem is telling you what the conditional distribution of Y is, conditional on X. Notice that the second break is chosen using a uniform distribution. That distribution must be over the interval from 0 up to X, the length of the left piece from the first break. This reminds us that the first break has already occured, effectively fixing X, which may now be treated as a constant. This effectively means that we've conditioned on X. So the problem tells us that $f_{Y|X}$ is uniform on $[0, X]$.

Since the expectation of a uniform distribution is the midpoint of its interval, $E(Y|X) = X/2$.

(b) Find the unconditional expectation of Y.

We apply the law of iterated expecations:

$$E(Y) = E(E(Y|X)) = E(X/2) = E(X)/2$$

Again, the expectation of a uniform random variable is the midpoint of its interval. Therefore,

$$E(Y) = (1/2)/2 = 1/4$$

(c) Compute $E(XY)$.

We use the law of iterated expectations again:

$$E(XY) = E(E(XY|X))$$

At this point, we can pull the X out from inside the inner expectation, since we've conditioned on X, so we're just treating X as a constant. (To see this another way, note that the inner expecation is integrated over Y, so X can be pulled out in front of the integral)

$$E(XY) = E(XE(Y|X)) = E(X(X/2)) = \frac{1}{2}E(X^2)$$

Expanding the expecation,

$$E(XY) = \frac{1}{2}\int_{-\infty}^{\infty} f_X(x)x^2dx = \frac{1}{2}\int_0^1 (1)x^2dx = \frac{1}{6}x^3\Big|_0^1 = 1/6$$

(d) Using the previous results, compute $cov(X,Y)$.

We use the following basic relationship.

$$cov(X,Y) = E(XY) - E(X)E(Y)$$

We have already calculated each term on the right hand side.

$$cov(X,Y) = 1/6 - (1/2)(1/4) = 1/6 - 1/8 = 1/24$$

Q3

It's important to begin your solution by defining your random variables clearly.

Let M_1, M_2, M_3, M_4, M_5 represent the waiting times on the mornings of day 1,2,3,4,5 respectively. Similarly, Let N_1, N_2, N_3, N_4, N_5 represent the waiting times on the corresponding evenings (nights).

One common mistake is to use a single random variable, say, M, to represent all morning wait times. This effectively forces all morning wait times to be perfectly correlated, instead of independent, which will affect the computation of variance.

(a) The total wait time can be written compactly as,

$$\sum_{i=1}^{5}(M_i + N_i)$$

The expectation of a sum is the sum of expectations:

$$E\bigg(\sum_{i=1}^{5}(M_i + N_i)\bigg) = \sum_{i=1}^{5}\big(E(M_i) + E(N_i)\big)$$

Finally, the expectation of a uniform random variable is just the midpoint of its interval. Substituting in, we have

$$\sum_{i=1}^{5}(2.5+5) = 37.5$$

(b) Because we know all M_i and N_i are independent, the variance of the sum is the sum of variances:

$$var\left(\sum_{i=1}^{5}(M_i+N_i)\right) = \sum_{i=1}^{5}\big(var(M_i)+var(N_i)\big)$$

It is crucial to understand that this is only true when the random variables are independent, so all the terms that look like $cov(M_i, M_j)$ where $i \neq j$ equal zero. In particular, if you make the mistake of using one random variable M to represent all morning wait times, you would get the wrong answer because these terms would suddenly be positive.

Finally, the variance of a uniform distribution over $[a, b]$ is $\frac{1}{12}(b-a)^2$ (There's a derivation in the Devore text). Substituting in, we get

$$\sum_{i=1}^{5}\left(\frac{25}{12}+\frac{100}{12}\right) = \frac{625}{12}$$

(c) To deal with the minus signs, it may help to think about $A - B$ as $A + (-B)$. Remember also that $E(-B) = E(-1 \cdot B) = -E(B)$.

$$E\left(\sum_{i=1}^{5}(N_i-M_i)\right) = E\left(\sum_{i=1}^{5}(N_i+(-M_i))\right)$$

$$= \sum_{i=1}^{5}\big(E(N_i)+E(-M_i)\big) = \sum_{i=1}^{5}\big(5-2.5\big) = 12.5$$

(d) Using the same trick,

$$var\left(\sum_{i=1}^{5}(N_i-M_i)\right) = var\left(\sum_{i=1}^{5}(N_i+(-M_i)\right)$$

$$= \sum_{i=1}^{5}\big(var(N_i)+var(-N_i)\big) = \sum_{i=1}^{5}\big(var(N_i)+(-1)^2var(N_i)\big)$$

$$= \sum_{i=1}^{5}\left(\frac{25}{12}+\frac{100}{12}\right) = \frac{625}{12}$$

Notice that the answers to parts (b) and (d) are the same.

Q4

$Corr(X, Y) = Corr(X, aX + b) = \frac{Cov(X,aX+b)}{\sqrt{Var(X)Var(aX+b)}}$

The numerator of this fraction can be simplified as follows.

$Cov(X, aX + b) = Cov(X, aX) + Cov(X, b) = aCov(X, X) + 0 = aVar(X)$

There are other ways we could have done that. Some students may find it easier to first write down the covariance in terms of expectation. That leads to the following simplification.

$Cov(X, aX + b) = E[X(aX + b)] - E[X]E[aX + b] = aE[X^2] + bE[X] - aE[X]^2 - bE[X]$

$= a\big(E[X^2] - E[X]^2\big) = a\ Var(X)$

We know that $var(aX + b) = a^2 var(X)$ so the denominator can be written as

$\sqrt{Var(X)Var(aX + b)} = \sqrt{a^2 Var(X)Var(X)} = |a|\ Var(X)$

So, we have

$$Corr(X, aX + b) = \frac{a\ Var(x)}{|a|\ Var(x)} = \frac{a}{|a|}$$

Which means:

$$Corr(X, aX + b) = \begin{cases} -1 & a < 0 \\ \text{not defined} & a = 0 \\ +1 & a > 0 \end{cases}$$

Q5

The Poisson random variable M can be understood as the number of points that occurs for a Poisson process with rate parameter α in a unit of time. It has the the probability mass function given in the problem,

$$f_M(m) = \frac{\alpha^m}{m!} e^{-\alpha}$$

The cumulative distribution can be found by summing up the above equation for different values of m, but there is another way of expressing it that will be more useful for this problem:

$$F_M(m) = \frac{\Gamma(m + 1, \alpha)}{m!}$$

Here, Γ is the upper incomplete gamma function, a special function that is normally defined in terms of an integral,

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$$

a. It is easiest to start with the conditional probability function,

$$f_{N|M}(n|m) = 1/(m + 1) \quad \forall 0 \le n \le m$$

For the joint distribution,

$$f_{N,M}(n, m) = f_{N|M}(n|m) f_M(m) = \frac{1}{m + 1} \frac{\alpha^m}{m!} e^{-\alpha} = \frac{1}{\alpha} \frac{\alpha^{m+1}}{(m + 1)!} e^{-\alpha} = \frac{1}{\alpha} f_M(m + 1)$$

b. The marginal probability is found by taking the sum:

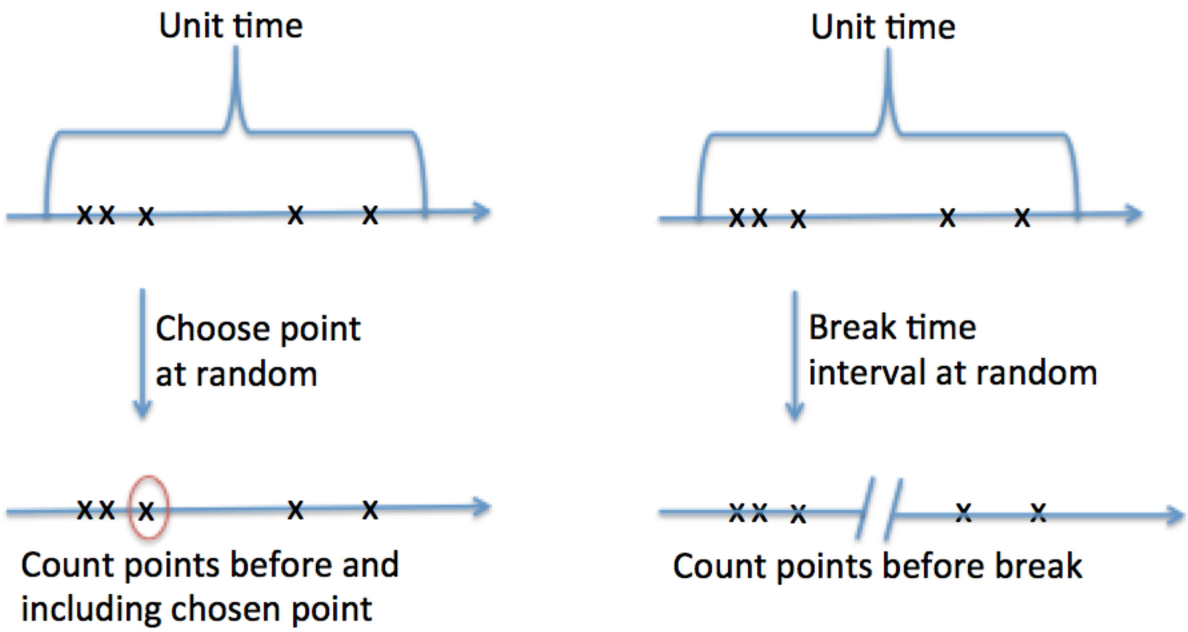


Figure 1: Descriptions of Random Variable N.

$$f_N(n) = \sum_{m=n} f_{N|M}(n|m) f_M(m) \tag{1}$$
$$= \sum_{m=n} \frac{1}{\alpha} f_M(m+1) \tag{2}$$
$$= \frac{1}{\alpha}(1 - F_M(n)) \tag{3}$$

c. One way to understand random variable N is to imagine running the Poisson process for a unit of time, then labeling the points that occur (1,2,. . . ,M). Next, select one of these events (or none of them) with equal probability. N is the number of the randomly selected points (or 0 if no event is selected).

There is a more meaningful interpretation of N. Instead of randomly selecting one of the points that occured during the unit of time, it is equivalent to first break the unit of time into two pieces at a random point, then simply count the number of points that occured before the break point.

Let $A \sim Uniform[0, 1]$ be the time until we break the unit of time. Conditional on A, the number of points that occur before the break is itself a Poisson variable, with rate parameter $A\alpha$. It therefore has a conditional distribution,

$$f_{N|Z}(n|a) = \frac{(a\alpha)^n}{n!} e^{-a\alpha}$$

We can therefore integrate to find the marginal distribution of N,

$$
\begin{aligned}
f_N(n) &= \int_0^1 f_{N|Z}(n|a)da && (4)\\
&= \int_0^1 \frac{(a\alpha)^n}{n!}e^{-a\alpha}da && (5)\\
&= -\frac{\Gamma(n+1, a\alpha)}{n!\alpha}\Big|_0^1 && (6)\\
&= -\frac{\Gamma(n+1,\alpha) - \Gamma(n+1,0)}{n!\alpha} && (7)
\end{aligned}
$$

$\Gamma(s,0) = \Gamma(s)$ is known simply as the gamma function and it equals $s!$ when s is an integer. We can therefore simplify the above as,

$$
\begin{aligned}
f_N(n) &= -\frac{\Gamma(n+1,\alpha) - n!}{n!\alpha} && (8)\\
&= \frac{1}{\alpha} - \frac{\Gamma(n+1,\alpha)}{n!\alpha} && (9)
\end{aligned}
$$

Finally, notive that the quantity on the right is almost exactly the same as the cumulative distribution for the Poisson random variable, which we provided above. Substituting in, we have,

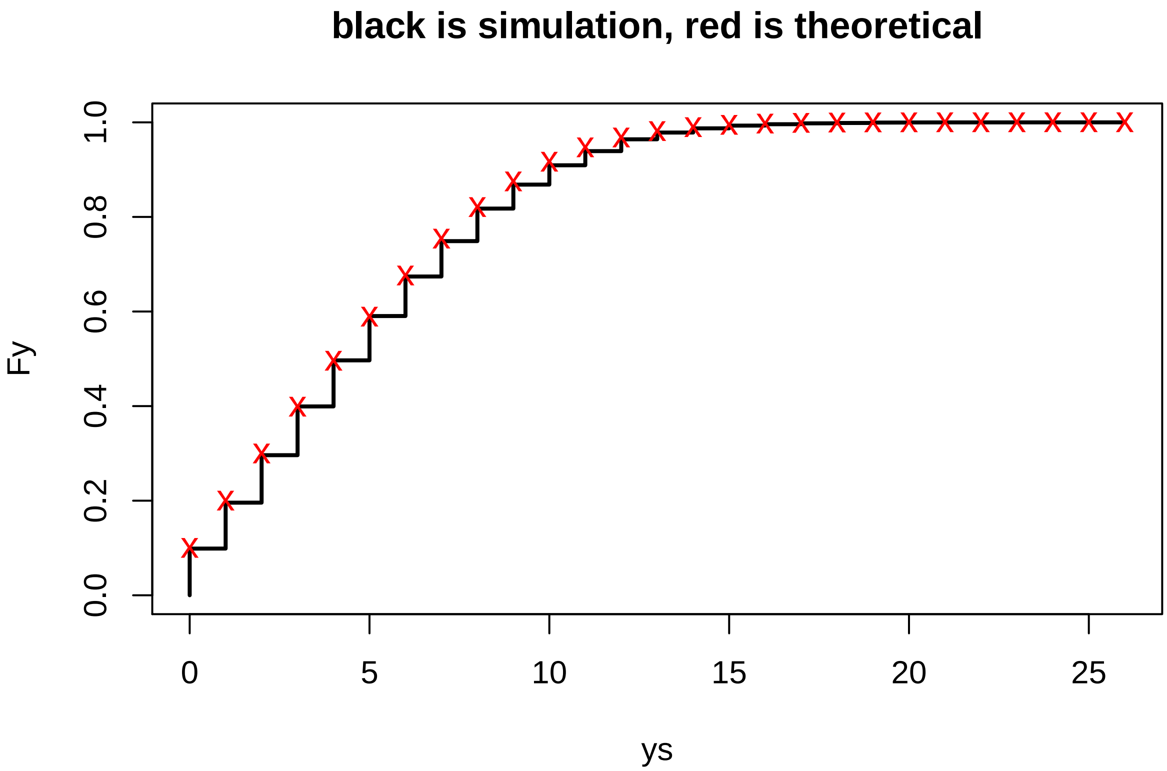
$$
f_N(n) = \frac{1}{\alpha}(1 - F_M(n))
$$

Notice that this is exactly the same as the quantity we computed above, confirming the intuition that breaking the unit time interval is equivalent to randomly selecting a Poisson point.

simulating the results

```
alpha = 10
N = 1e4
x = rpois(N, lambda =alpha)
y = sapply(x+1, sample, size= 1)-1
ys = sort(y)
Fy = (1:N)/N  # simulation
basey = 0:max(y) #basis for theoretical
th.p = 1/alpha * (1-ppois(basey,lambda = alpha) ) # density
th.c = cumsum(th.p)  # cumulative (theoretical)

plot(Fy~ys, ty='l',lwd=2, main='black is simulation, red is theoretical')
points(basey,th.c,col='red',pch='x',cex=1.2)
```


black is simulation, red is theoretical
Fy
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
ys