Homework 5 (W203)

Joanna Yu (Fall 2018 Tuesday 4pm)

10/7/2018

1a) Using properties of variance and covariance, derive each element of the variance-covariance matrix for W and V.

$$
\begin{aligned}
V = 0.5W + U \implies Cov(W,V) &= E(WV) - E(W)E(V) \\
&= E(W * (0.5W + U)) - E(W)E(0.5W + U) \\
&= 0.5E(W^2) + E(WU) - 0.5(E(W))^2) - E(W)E(U) \\
&= 0.5E(W^2) - 0.5(E(W))^2 + E(WU) - E(W)E(U) \\
&= 0.5Var(W) + 0 \\
&= 0.5(4^2) \\
&= 8
\end{aligned}
$$

$$
\begin{aligned}
V = 0.5W + U \implies Var(V) &= Var(0.5W + U) \\
&= Var(0.5W) + Var(U) + 2Cov(W,U) \\
&= 0.25Var(W) + 1^2 + 0 \\
&= 0.25(4)^2 + 1 \\
&= 5
\end{aligned}
$$

$$
\begin{bmatrix} Cov(W,W) & Cov(W,V) \\ Cov(V,W) & Cov(V,V) \end{bmatrix} = \begin{bmatrix} Var(W) & 8 \\ 8 & Var(V) \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 8 & 5 \end{bmatrix}
$$

2a) Find the conditional expectation of Y given X, E(Y|X) X is uniformly distributed in [0,1] so f(X)=1, E(X)=1/2, Var(X)=1/12

Y is uniformly distributed in [0,X], so f(Y|X)=1/X, E(Y|X)=X/2

2b) Find the unconditional expectation of Y.

$$
\begin{aligned}
E(Y) &= E(E(Y|X)) \\
&= E(\frac{X}{2}) \\
&= \frac{1}{2}E(X) \\
&= \frac{1}{2} * \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}
$$

2c) Compute E(XY)

$$\begin{aligned}
E(XY) &= \int_0^1 \int_0^x xy * f(xy)dydx \\
&= \int_0^1 \int_0^x xy * \frac{1}{x}dydx \\
&= \int_0^1 \int_0^x ydydx \\
&= \int_0^1 \frac{y^2}{2}\Big|_0^x dx \\
&= \int_0^1 \frac{x^2}{2}dx \\
&= \frac{x^3}{6}\Big|_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

2d) Using the previous results, computer cov(X,Y)

$$\begin{aligned}
Cov(X,Y) &= E(XY) - E(X)E(Y) \\
&= 1/6 - (1/2)(1/4) \\
&= 1/24
\end{aligned}$$

3a) What is the total expected waiting time? Let X represent the morning wait time and Y represent the evening wait time.

$$\begin{aligned}
E(5X+5Y) &= E(5X) + E(5Y) \\
&= 5 * E(X) + 5 * E(Y) \\
&= 5 * (5 * 1/2) + 5 * (10 * 1/2) \\
&= 5(2.5) + 5(5) \\
&= 37.5
\end{aligned}$$

3b) What is the variance of your total waiting time?

$$\begin{aligned}
Var(5X+5Y) &= Var(5X) + Var(5Y) + 2Cov(5X,5Y) \\
&= Var(5X) + Var(5Y) \\
&= 25Var(X) + 25Var(Y) \\
&= 25 * (5^2 * 1/12) + 25 * (10^2 * 1/12) \\
&= 260.41
\end{aligned}$$

3c) What is the expected value of the difference between the total evening waiting time and the total morning waiting time over all 5 days?

$$\begin{aligned}
E(5Y-5X) &= E(5Y) - E(5X) \\
&= 5 * E(Y) - 5 * E(X) \\
&= 5 * (10 * 1/2) - 5 * (5 * 1/2) \\
&= 5(5) - 5(2.5) \\
&= 12.5
\end{aligned}$$

3d) What is the variance of the difference between the total evening waiting time and the total morning waiting time over all 5 days?

$$\begin{aligned}
Var(5X - 5Y) &= Var(5X) + Var(5Y) - 2Cov(5X, 5Y) \\
&= Var(5X) + Var(5Y) \\
&= 25Var(X) + 25Var(Y) \\
&= 25 * (5^2 * 1/12) + 25 * (10^2 * 1/12) \\
&= 260.41
\end{aligned}$$

4) Show that if Y = aX + b where X and Y are random variables and a != 0, corr(X,Y) = -1 or 1.

If Y = aX + b, then Var(Y) = a^2*Var(X) and SD(Y)=|a|SD(X)

$$\begin{aligned}
Corr(X, Y) &= \frac{Cov(X, Y)}{SD(X)SD(Y)} \\
&= \frac{E(XY) - E(X)E(Y)}{SD(X) * |a|SD(X)} \\
&= \frac{E(X * (aX + b)) - E(X)E(aX + b)}{|a|Var(X)} \\
&= \frac{aE(X^2) + bE(X) - E(X)(aE(X) + E(b))}{|a|Var(X)} \\
&= \frac{aE(X^2) + bE(X) - a(E(X))^2 - bE(X)}{|a|Var(X)} \\
&= \frac{aE(X^2) - a(E(X))^2}{|a|Var(X)} \\
&= \frac{a(E(X^2) - (E(X))^2)}{|a|Var(X)} \\
&= \frac{aVar(X)}{|a|Var(X)} \\
&= -1 or 1
\end{aligned}$$