W203 Homework 4, Fall 2018 Tuesday 4pm

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1a) How much do you get paid if the coin comes up heads 3 times?

There are 8 possible outcomes: 1 outcome with 3 heads, 3 outcomes with 2 heads, 3 outcomes with 1 head, 1 outcome with no heads. Let W be the winnings from the game.

$$\begin{aligned}
E(W) &= 2 * P(W = 2) + 4 * P(W = 4) + w * P(W = w) \\
6 &= 2 * \frac{3}{8} + 4 * \frac{3}{8} + w * \frac{1}{8} \\
w &= 30
\end{aligned}$$

1b) Write down a complete expression for the cumulative probability function for your winnings from the game.

$$F(w) = \begin{cases} 1/8 & 0 \le w < 2 \\ 1/2 & 2 \le w < 4 \\ 7/8 & 4 \le w < 30 \\ 1 & 30 \le w \end{cases}$$

2a) Write down a complete expression for the cumulative probability function of L.

$$f(l) = \begin{cases} 0 & l \le 0 \\ l/2 & 0 < l \le 2 \\ 0 & 2 < l \end{cases}$$

$$\begin{aligned}
F(l) &= \int_{-\infty}^{l} f(y)dy \\
&= \int_{0}^{l} \frac{y}{2} dy \\
&= \frac{1}{4} y^2 \Big|_0^l \\
&= \frac{l^2}{4}
\end{aligned}$$

$$F(l) = \begin{cases} 0 & l \le 0 \\ l^2/4 & 0 < l \le 2 \\ 1 & 2 < l \end{cases}$$

2b) Compute the expected length of the pasta, E(L).

$$\begin{aligned}
E(L) &= \int_{-\infty}^{\infty} l * f(l)dl \\
&= \int_{0}^{2} l * \frac{l^2}{4} dl \\
&= \int_{0}^{2} \frac{l^3}{4} dl \\
&= l^4 \Big|_0^2 \\
&= 2^4 \\
&= 16
\end{aligned}$$

3a) Compute the expected payout from the contract, E(X)=E(g(T))

$$\begin{aligned}
E(X) &= E(g(T)) \\
&= \int_{-\infty}^{\infty} f(t) * g(t) dt \\
&= \int_{0}^{1} 1 * 100(1-t)^{1/2} dt \\
&= 100 \int_{0}^{1} (1-t)^{1/2} dt \\
&= 100(\frac{2}{3})(1-t)^{3/2}(-1)\Big|_0^1 \\
&= 100(\frac{2}{3})(0+1) \\
&= 66\frac{2}{3}
\end{aligned}$$

3b) Compute E(X) another way

$$\begin{aligned}
g(X \leq x) \rightarrow x &\geq 100(1-t)^{1/2} \\
t &\geq 1 - (\frac{x}{100})^2
\end{aligned}$$

$$\begin{aligned}
P(X \leq x) &= (\frac{x}{100})^2 \\
pdf &= d/dx(cdf) = x/5000
\end{aligned}$$

$$\begin{aligned}
E(X) &= \int_{0}^{100} x * \frac{x}{5000} dx \\
&= \frac{x^3}{15000}\Big|_0^{100} \\
&= \frac{100^3}{15000} \\
&= 66\frac{2}{3}
\end{aligned}$$

4a) Write down an expression for E(Y) and use properties of expectation to simplify it as much as you can.

$$\begin{aligned}
Y = (X-t)^2 &= X^2 - 2Xt + t^2 \\
f(y) &= 2x - 2t + t^2
\end{aligned}$$

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} x(2x - 2t + t^2) dx \\
&= \int_{-\infty}^{\infty} (2x^2 - 2tx + xt^2) dx
\end{aligned}$$

4b) Take a partial derivative with respect to t, compute the value of t that minimizes E(Y).

$$(2x^2 - 2tx + xt^2)dt = 2x^2 - 2x + 2xt$$

E(Y) is minimized when t = 0

4c) What is the value of E(Y) for this choice of t?

$$E(Y) = 2X^2$$