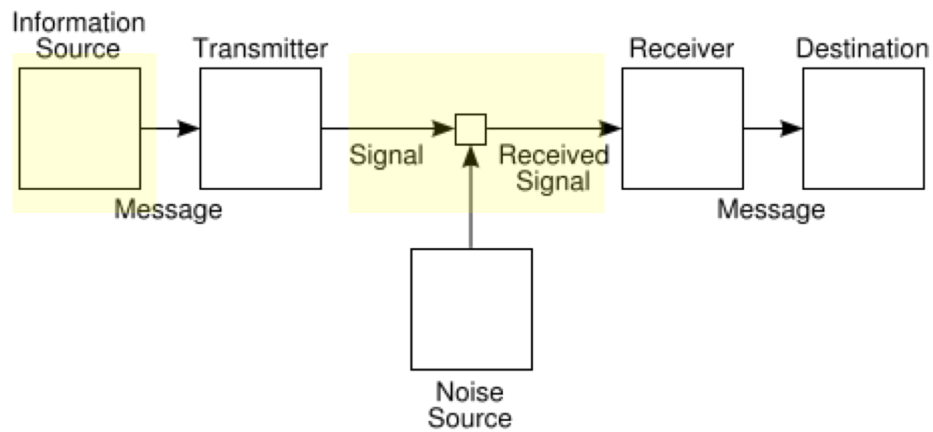
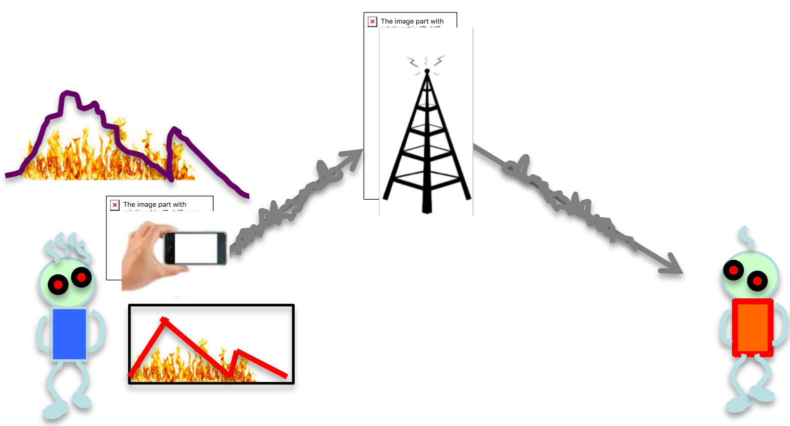
Entropy measures for two variables

- Joint Entropy
- Conditional Entropy
- Mutual Information





Motivation:

-relation between two variable

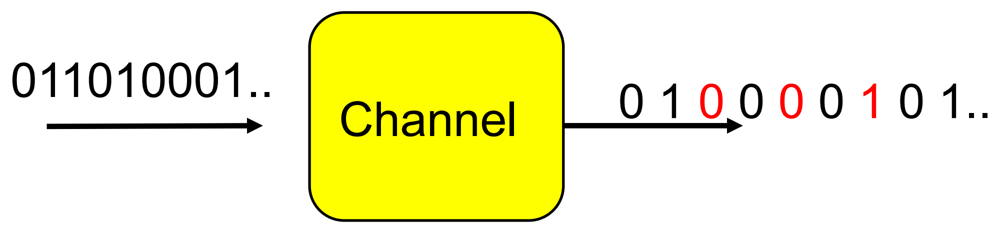
- model a channel

Plan

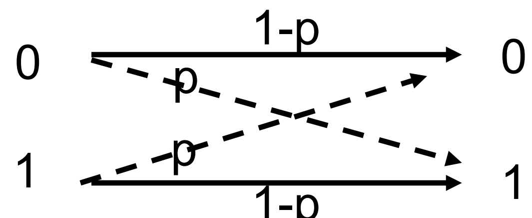
- Models with two variable
- Joint distribution
- Entropy measures
  - Joint entropy, conditional entropy, mutual information, relative entropy
- Min-entropy
- From data to distribution

Modeling a channel

- A channel is specified by,
- an input alphabet
- an output alphabet
- a transition function
  - probabilistic



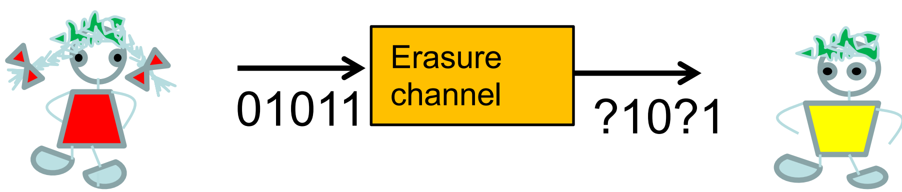
p is the bit error probability

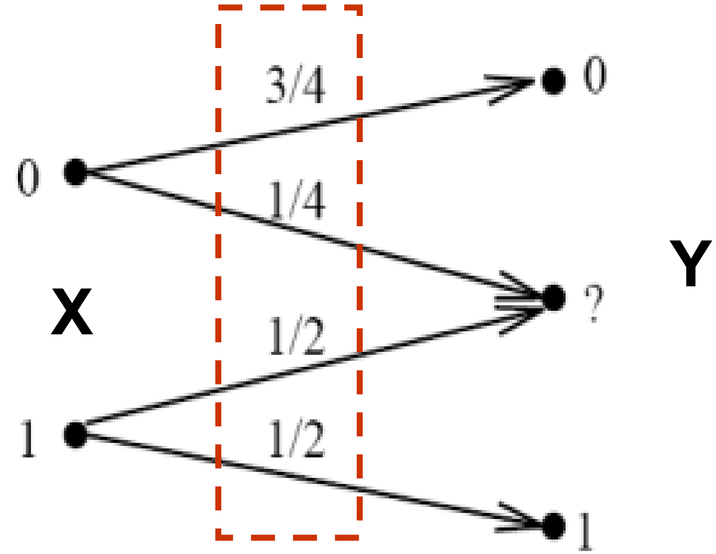


Example

An erasure channel:

- X={0, 1} is the input set
- Y= {0,1, ?} is output set





- Bob receives the string S = ?10?1

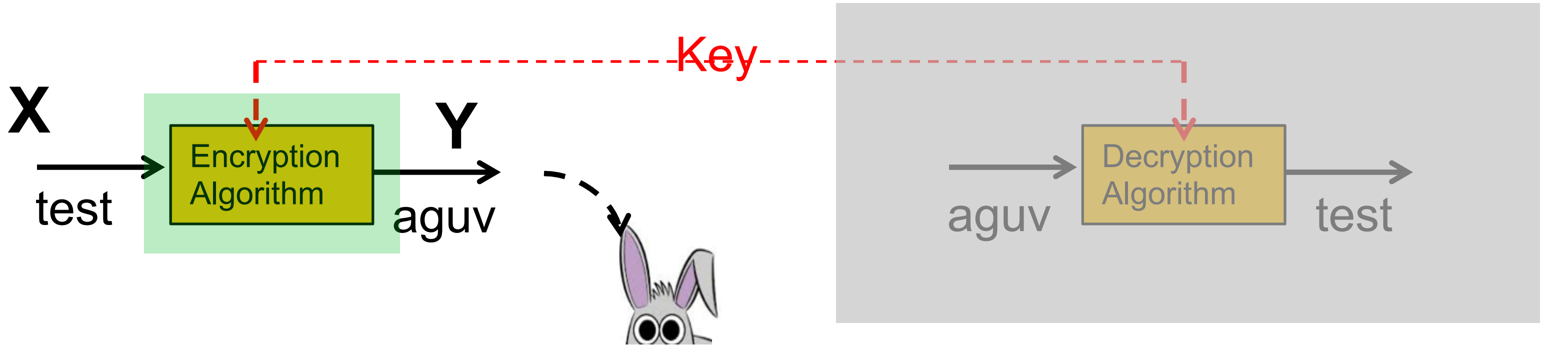
- How much information has passed through the channel?

Example

An encryption system.

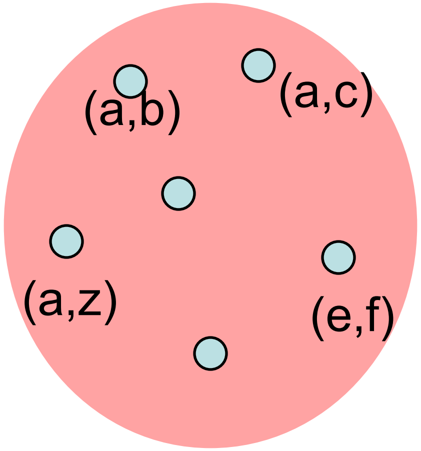
Messages: X

- Key space K
- Encryption: $y = f_k(x)$
- Decryption $x = f_k^{-1}(y)$
- How much information does eavesdropper receive after seeing the ciphertext?



Joint Distributions: *(refresher)*

Sample set is $\mathcal{X}\mathcal{Y}$: { (x,y): x in $\mathcal{X}$, y in $\mathcal{Y}$}



X, Y $\quad p(X,Y)$ $\quad p(X = a, Y = b)$

$p(X,Y)$ is called the joint distribution.

Example: How is the weather today?

- X: {0, 1} - 1 means "it rains", 0 means it does not
- Y: {0, 1} - 1 means "gets cold", 0 means it does not

- We are given: $p(X=0, Y=0) = 1/3$
  $p(X=0, Y=1) = 1/6$
  $p(X=1, Y=0) = 1/12$
  $p(X=1, Y=1) = 5/12$

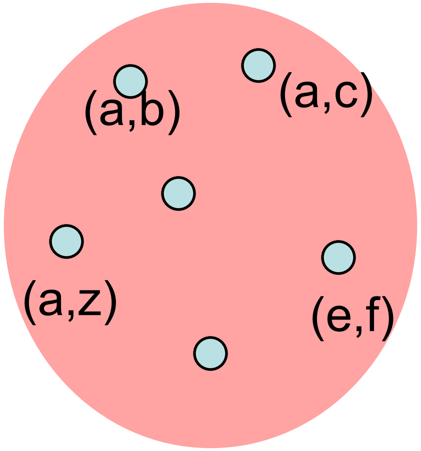
P(XY)	Y=0	Y=1
X=0	1/3	1/6
X=1	5/12	1/12

- Probability that it rains:
- $p(X=1) = p(X=1,Y=0) + p(X=1,Y=1) = 1/12 + 5/12 = 1/2$
- $p(X=0) = 1/2$

- Probability that it gets cold:
- $p(Y=1) = p(X=0,Y=1) + p(X=1,Y=1) = 7/12$
- $p(Y=0) = 5/12$

Joint Distributions: *(refresher)*

Sample set is $\mathcal{X}\mathcal{Y}$: { (x,y): x in $\mathcal{X}$, y in $\mathcal{Y}$}



X, Y $\quad p(X, Y)$ $\quad p(X = a, Y = b)$

$p(X, Y)$ is called the joint distribution.

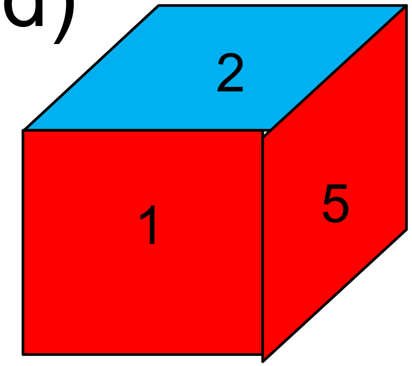
Marginal distributions $p(X)$ and $p(Y)$ can be obtained:

$$p(X = a) = \sum_i p(X = a, Y = i)$$

$$p(Y = b) = \sum_i p(X = i, Y = b)$$

Joint distribution

- (1, red)
- (2,blue)
- (3,blue)
- (4,blue)
- (5,red)
- (6,red)



	Red	Blue
1	1/6	0
2	0	1/12
3	0	1/12
4	0	1/6
5	1/6	0
6	1/3	0

Joint distribution

- p(1,red)
  = 1/6
- p(1)=
  =1/6
- p(red)
  =2/3
- p(blue) =1/3

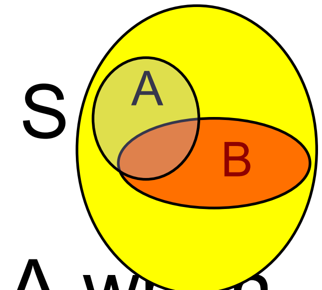
- p(even, red)=

	Red	Blue
1	1/6	0
2	0	1/12
3	0	1/12
4	0	1/6
5	1/6	0
6	1/3	0

Conditional probability *(refresher)*

- Given:

1. A probability space
2. two events A and B,



- p(A|B) is the conditional probability of A when B has happened.

  e.g., p(1|red)

- The law (axiom) of conditional probability:
- $p(A|B) = p(AB) / p(B)$
- $p(AB) = p(B)p(A|B) = p(A)p(B|A)$

Independent events *(refresher)*

- B is independent of A if,

$$p(B|A)=p(B)$$

- That is $p(AB) = p(A)p(B|A) = p(A)p(B)$

- This means A is independent of B also.

$$p(A|B)=p(A)$$

Example: outcome of two rolls of the die

Example

- X: {0, 1} - 1 means "it rains", 0 means it does not
- Y: {0, 1} - 1 means "gets cold", 0 means it does not

- We are given: p(X=0, Y=0) = 1/3
  p(X=0, Y=1) = 1/6
  p(X=1,Y=0) =1/12
  p(X=1,Y=1) = 5/12

- Find probability of (cold, given that it rains)

- p(Y=1| X=1) $=\frac{p(X=1,Y=1)}{p(X=1)} = \frac{5/12}{1/2} = \frac{5}{6}$

- Find probability of (cold, given that it does not rain)

- p(Y=1| X=0) $= \frac{p(X=0,Y=1)}{p(X=0)} = \frac{1/6}{1/2} = \frac{1}{3}$