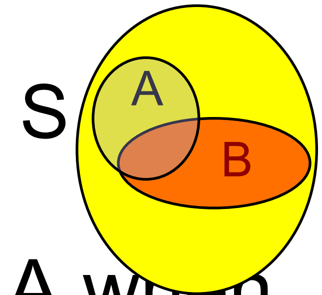
Conditional probability *(refresher)*

- Given:

1. A probability space
2. two events A and B,



- p(A|B) is the conditional probability of A when B has happened.

  e.g., p(1|red)

- The law (axiom) of conditional probability:
- $p(A|B) = p(AB) / p(B)$
- $p(AB) = p(B)p(A|B) = p(A)p(B|A)$

Independent events *(refresher)*

- B is independent of A if,

$$p(B|A)=p(B)$$

- That is $p(AB) = p(A)p(B|A) = p(A)p(B)$

- This means A is independent of B also.

$$p(A|B)=p(A)$$

Example: outcome of two rolls of the dice

Example

- X: {0, 1} - 1 means "it rains", 0 means it does not
- Y: {0, 1} - 1 means "gets cold", 0 means it does not
- We are given: p(X=0, Y=0) = 1/3
  p(X=0, Y=1) = 1/6
  p(X=1,Y=0) =1/12
  p(X=1,Y=1) = 5/12
- Find probability of (cold, given that it rains)
- p(Y=1| X=1) = $\frac{p(X=1,Y=1)}{p(X=1)} = \frac{5/12}{1/2} = \frac{5}{6}$
- Find probability of (cold, given that it does not rain)
- p(Y=1| X=0) = $\frac{p(X=0,Y=1)}{p(X=0)} = \frac{1/6}{1/2} = \frac{1}{3}$

Example

- p(1|red)
  = p(1,red)/p(red)
  =(1/6)/(2/3)= 1/4

- p(3|blue)=
  =p(3, blue)/p(blue)
  =(1/12)/(1/3)= ¼

- p(blue|3)
  =p(3,blue)/p(3)= (1/12)/ (1/12) =1

	Red	Blue
1	1/6	0
2	0	1/12
3	0	1/12
4	0	1/6
5	1/6	0
6	1/3	0

Joint distribution of XY

- Two variables X and Y with a joint distribution:

P(XY)	Y=0	Y=1
X=0	1/3	1/6
X=1	5/12	1/12

- $p(X=1,Y=1)=1/12$ $p(X=0,Y=0)= 1/3$
- $p(X=1,Y=0)=5/12$ $p(X=0,Y=1)= 1/12$

$$\Sigma_{a,b \in \{0,1\}} p(X=a,Y=b) =1$$

- Marginal distributions:

$$\Sigma_{a \in \{0,1\}} p(X = a)=1$$
$$\Sigma_{b \in \{0,1\}} p(Y = b)=1$$

Joint Entropy

- The joint entropy $H(X,Y)$ of a pair of random variables (X, Y) with joint distribution $p(X, Y)$ is defined as

$$H(X, Y) = -\Sigma_{x \in X} \Sigma_{y \in Y} p(x,y) \log p(x, y)$$

- $H(X, Y) = -E(\log p(x, y))$

  is the expected value of $-\log p(x, y)$

Joint Entropy of XY

- Two variables X and Y with a joint distribution:

P(XY)	Y=0	Y=1
X=0	2/6	1/6
X=1	1/6	2/6

- $p(X=1,Y=1) = p(X=0,Y=0) = 2/6$
- $p(X=1,Y=0) = p(X=0,Y=1) = 1/6$

- $H(X, Y) = -\sum_{x \in X} \sum_{y \in Y} p(x,y) \log p(x, y) = 1.918$ bits

Entropy of each variable

- Two variables X and Y with a joint distribution:

P(XY)	Y=0	Y=1
X=0	2/6	1/6
X=1	1/6	2/6

- Find H(X) and H(Y) using marginal distributions:
- p(Y=0)=p(Y=1)=1/2 → H(Y) = 1 bit
- p(X=0)=p(X=1)=1/2 → H(X) = 1 bit

→H(X) +H(Y)= 2 bits

→Observe: $H(X,Y) \leq H(X)+H(Y)$