Discrete Optimization

The Knapsack Problem: Modeling

Goals of the Lecture

- How to formalize an optimization task as a mathematical model

The (1-Dimensional) Knapsack Problem

- Given a set of items $\mathtt{I}$, each item $i \in \mathtt{I}$ characterized by
  - its weight w_i
  - its value v_i

  and
  - a capacity K for a knapsack

  find the subset of items in $\mathtt{I}$
  - that has maximum value
  - does not exceed the capacity K of the knapsack

Optimization Models

- How to model an optimization problem
  - choose some decision variables
    - they typically encode the result we are interested in
  - express the problem constraints in terms of these variables
    - they specify what the solutions to the problem are
  - express the objective function
    - the objective function specifies the quality of each solution
- The result is an optimization model
  - It is a declarative formulation
    - specify the “what”, not the “how”
  - There may be many ways to model an optimization problem

A Knapsack Model

- Decision variables
  - x_i denotes whether item i is selected in the solution
    - $x_i = 1$ means the item is selected
    - $x_i = 0$ means that it is not selected
- Problem constraint
  - The selected item cannot exceed the capacity of the knapsack

$$\sum_{i \in I} w_i x_i \leq K$$

- Objective function
  - Captures the total value of the selected items

$$\sum_{i \in I} v_i x_i$$

A Knapsack Model

- Putting it all together

$$
\begin{aligned}
&\text{maximize} && \sum_{i \in I} v_i \; x_i \\
&\text{subject to} && \\
& && \sum_{i \in I} w_i x_i \le K \\
& && x_i \in \{0, 1\} \quad (i \in I)
\end{aligned}
$$

Exponential Growth

- How many possible configurations?
  - (0,0,0,...,0), (0,0,0,...,1), ..., (1,1,1,...,1)
- Not all of them are feasible
  - They cannot exceed the capacity of the knapsack
- How many are they?
  - $2^{|\mathtt{I}|}$
- How much time to explore them all?
  - 1 millisecond to test a configuration
  - if $|\mathtt{I}| = 50$, it will take 1,285,273,866 centuries

Until Next Time