Discrete Optimization

Constraint Programming: Part I

Goal of the Lecture

- Basic introduction to constraint programming

Constraint Programming

- Computational paradigm
  - use constraints to reduce the set of values that each variable can take
  - remove values that cannot appear in any solution

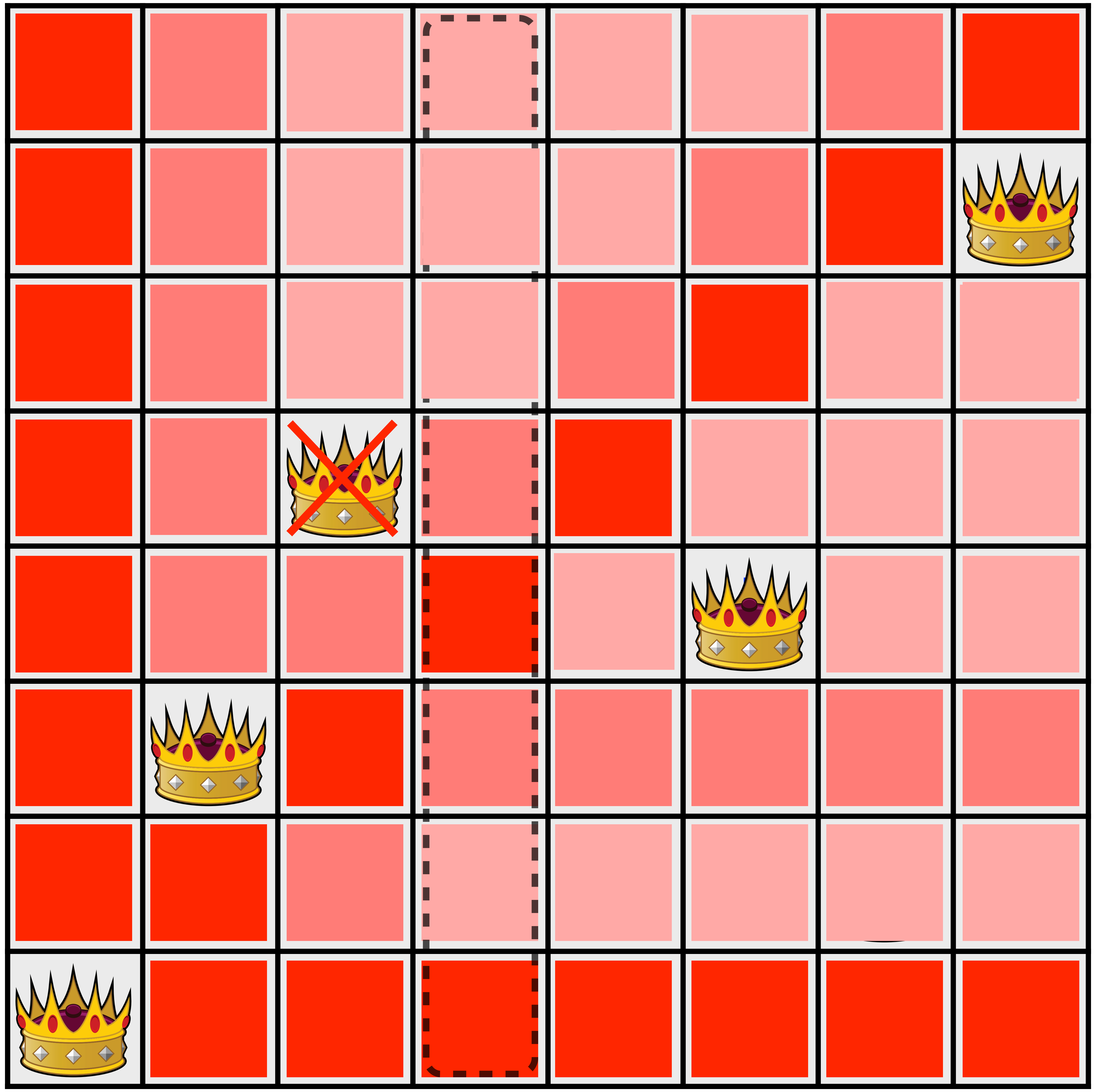
- Modeling methodology
  - convey the structure of the problem as explicitly as possible
  - express substructures of the problem
  - give solvers as much information as possible

The 8-Queens Problem

- Specification
  - place 8 queens on a chessboard so that none of them attach each other
  - two queens attack each other if they are on the same column, row, or diagonal

The 8-Queens Problem

Failure!

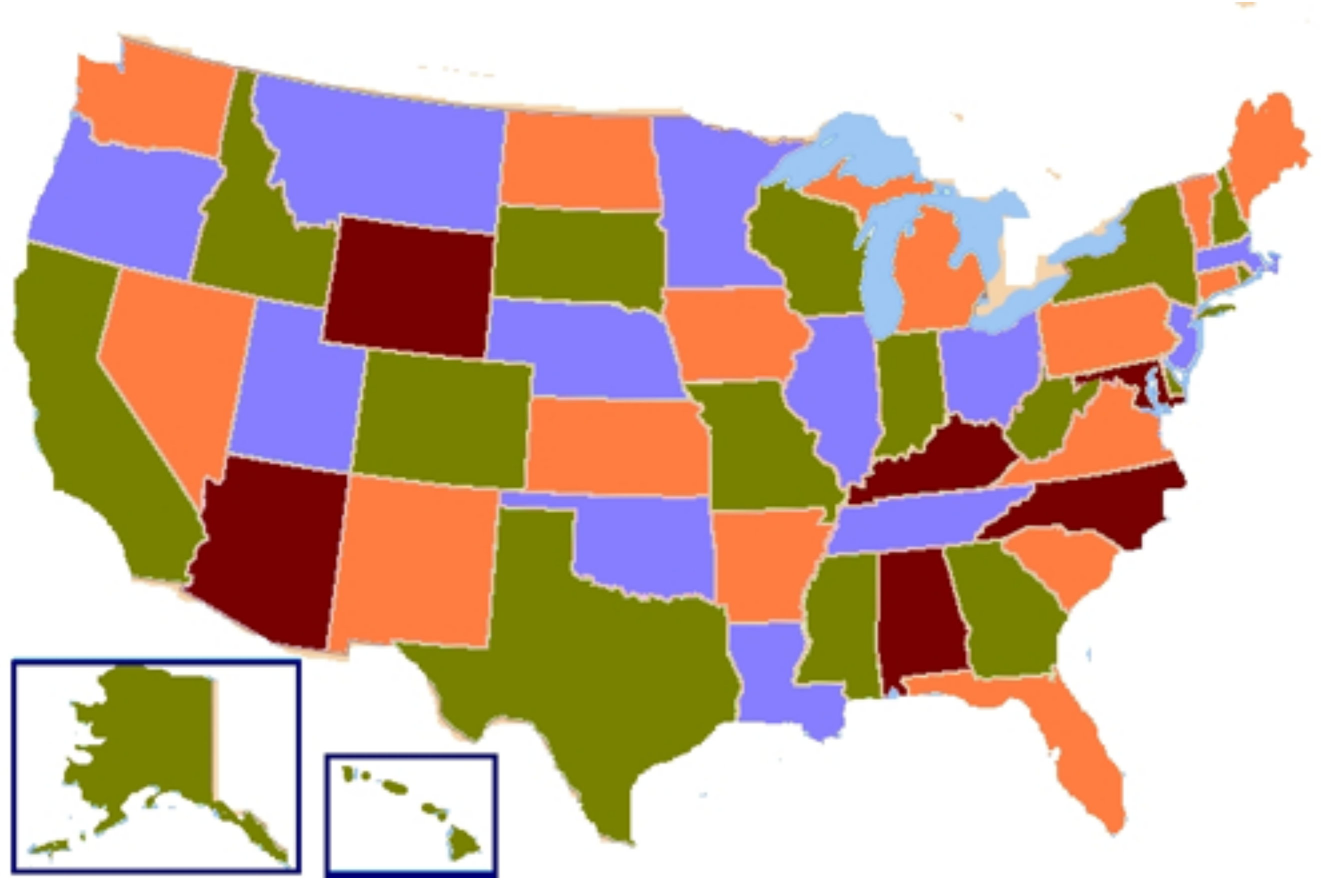


Constraint Programming

- Computational paradigm
  - use constraints to reduce the set of values that each variable can take
  - make a choice when no more deduction can be performed
- What is a choice?
  - there are many choices!
  - for the moment, assume a choice assigns a value to a variable
- Choices can be wrong
  - in optimization, they are often wrong :-(
  - the solver then backtracks,
    i.e., it tries another value

Coloring a Map

- Specification
  - color a map so that no two adjacent territories receive the same color
- The 4 Color Theorem
  - Every map can be colored with just 4 colors
  - Proven by Kenneth Appel and Wolfgang Haken
  - First major theorem proven with a computer



Coloring a Map

- How to color this map with constraint programming?
  - choose the decision variables
  - express the constraints in terms of the decision variables
- What are the decision variables?
  - the color given to each country
- What are the domains of the decision variables?
  - the domain is the set of values that a variable can take
  - four different colors
- How do you express the constraints?
  - specify that two adjacent countries cannot be given the same color

Coloring a Map

```
enum Countries = { Belgium, Denmark, France, Germany,
                   Netherlands, Luxembourg };
enum Colors = { black, yellow, red, blue };
var{Colors} color[Countries];

solve {
  color[Belgium] ≠ color[France];
  color[Belgium] ≠ color[Germany];
  color[Belgium] ≠ color[Netherlands];
  color[Belgium] ≠ color[Luxembourg];
  color[Denmark] ≠ color[Germany];
  color[France] ≠ color[Germany];
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}
```
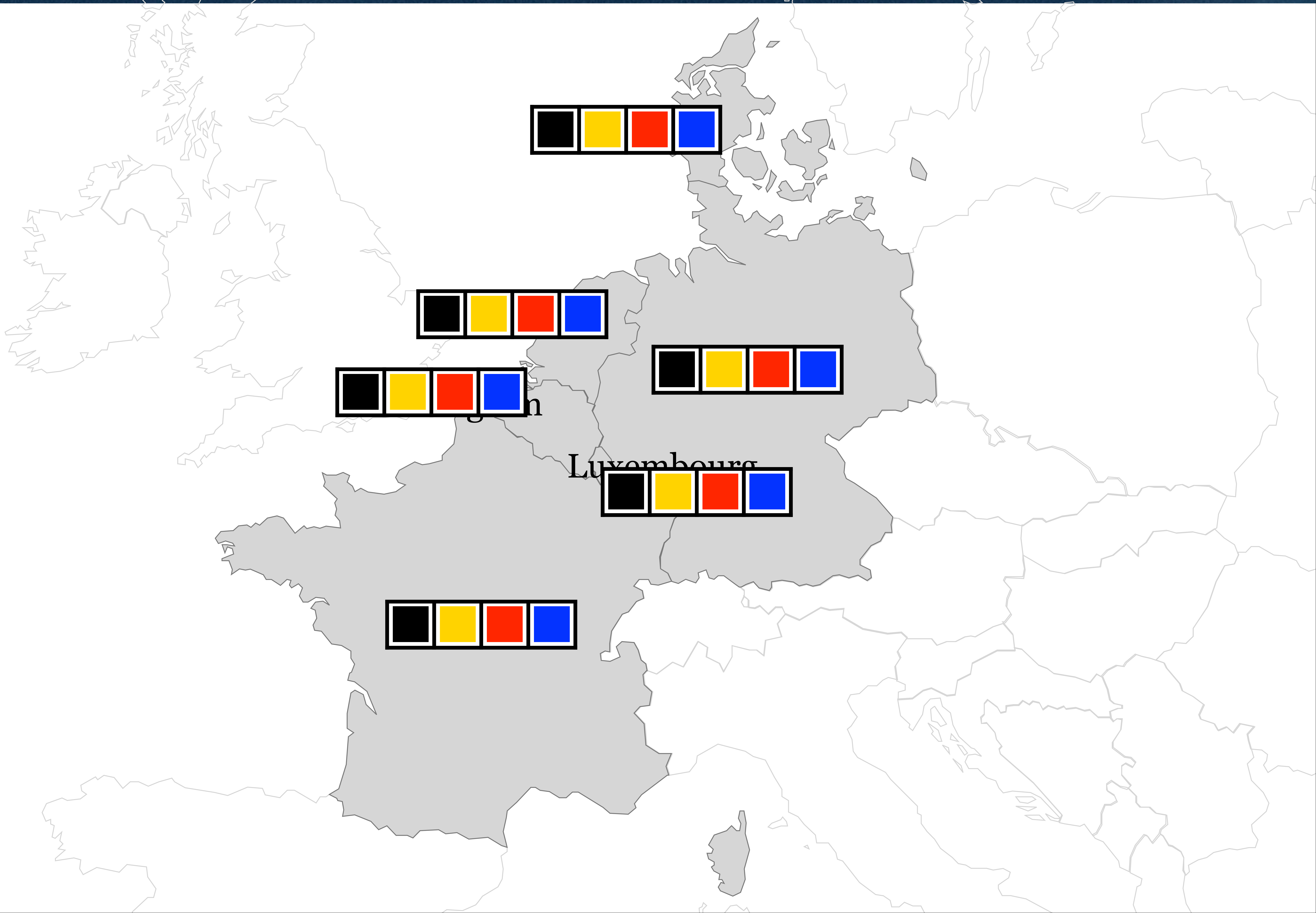

Coloring a Map



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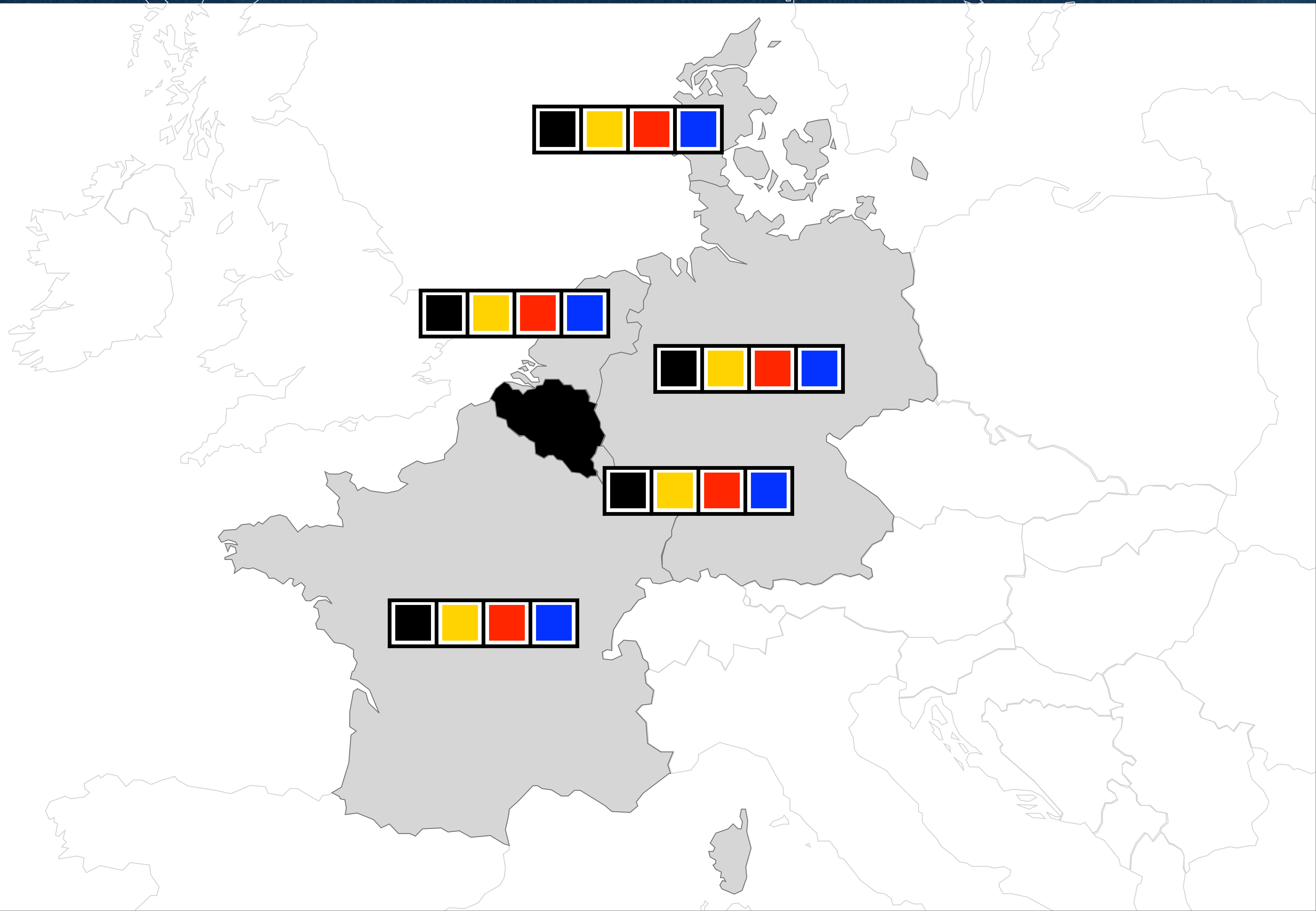

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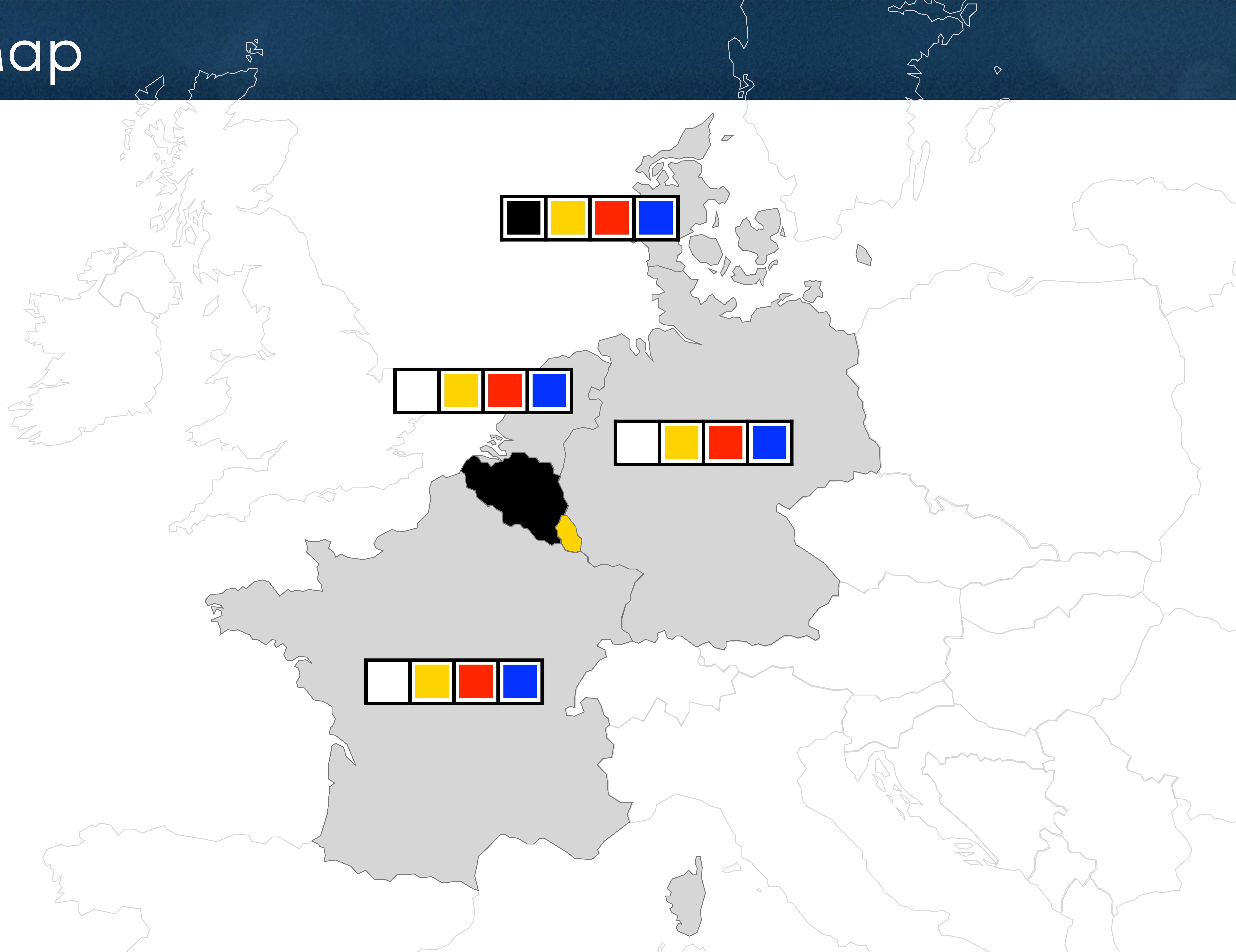

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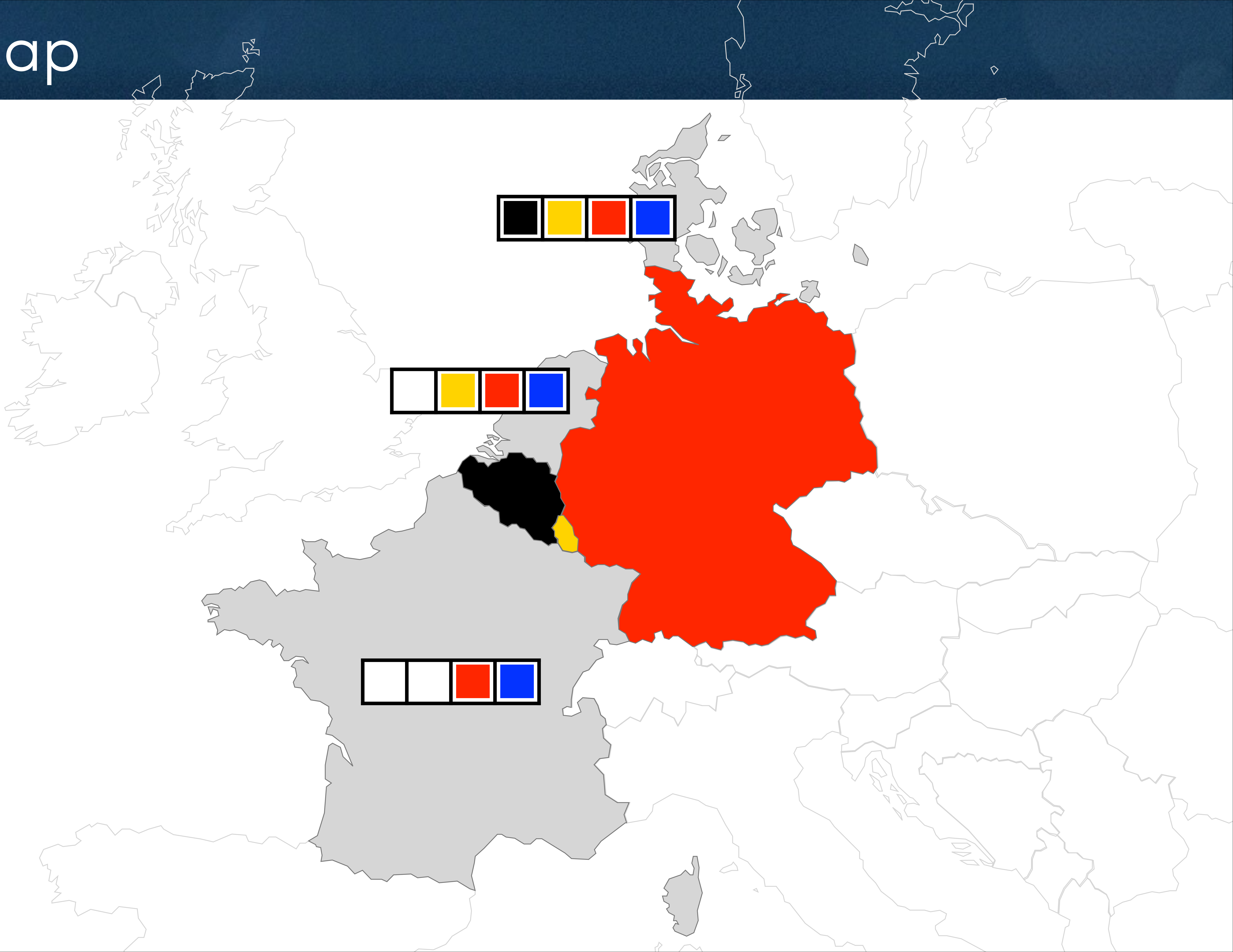

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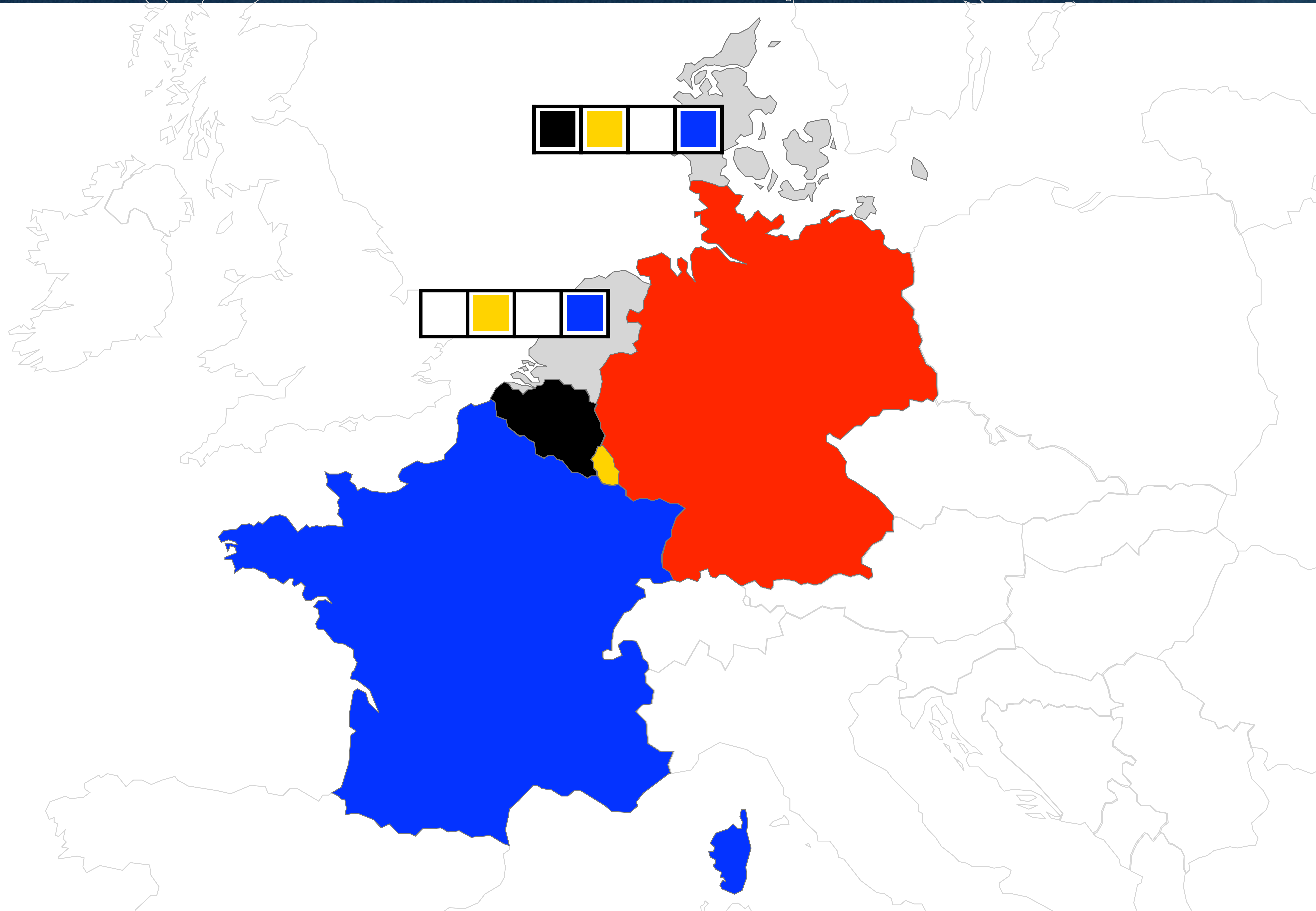

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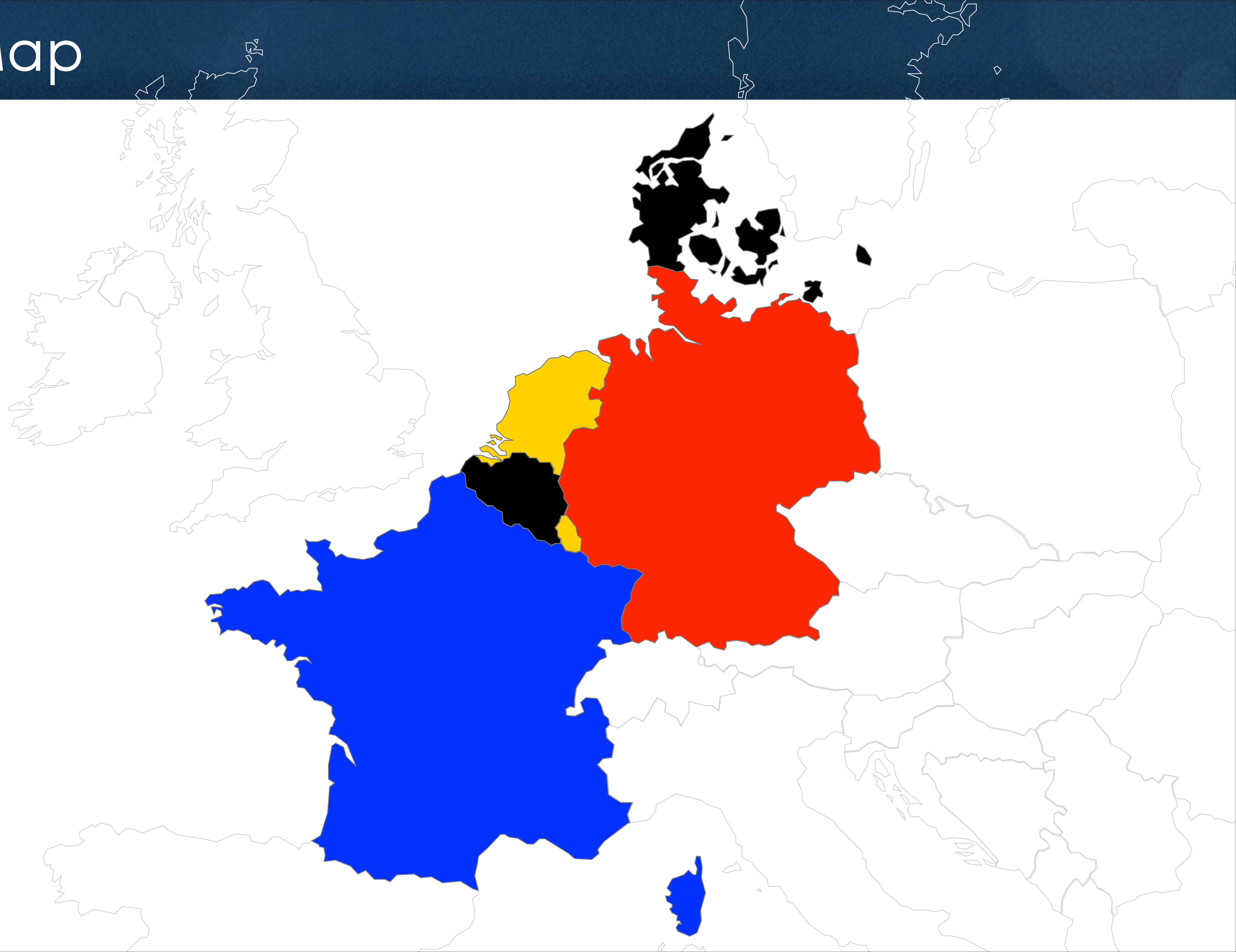

Constraint Programming

- Computational paradigm
  - use constraints to reduce the set of values that each variable can take
  - make a choice when no more deduction can be performed

- What does this mean for the coloring problem?

Coloring a Map

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  color[France] ≠ color[Luxembourg];
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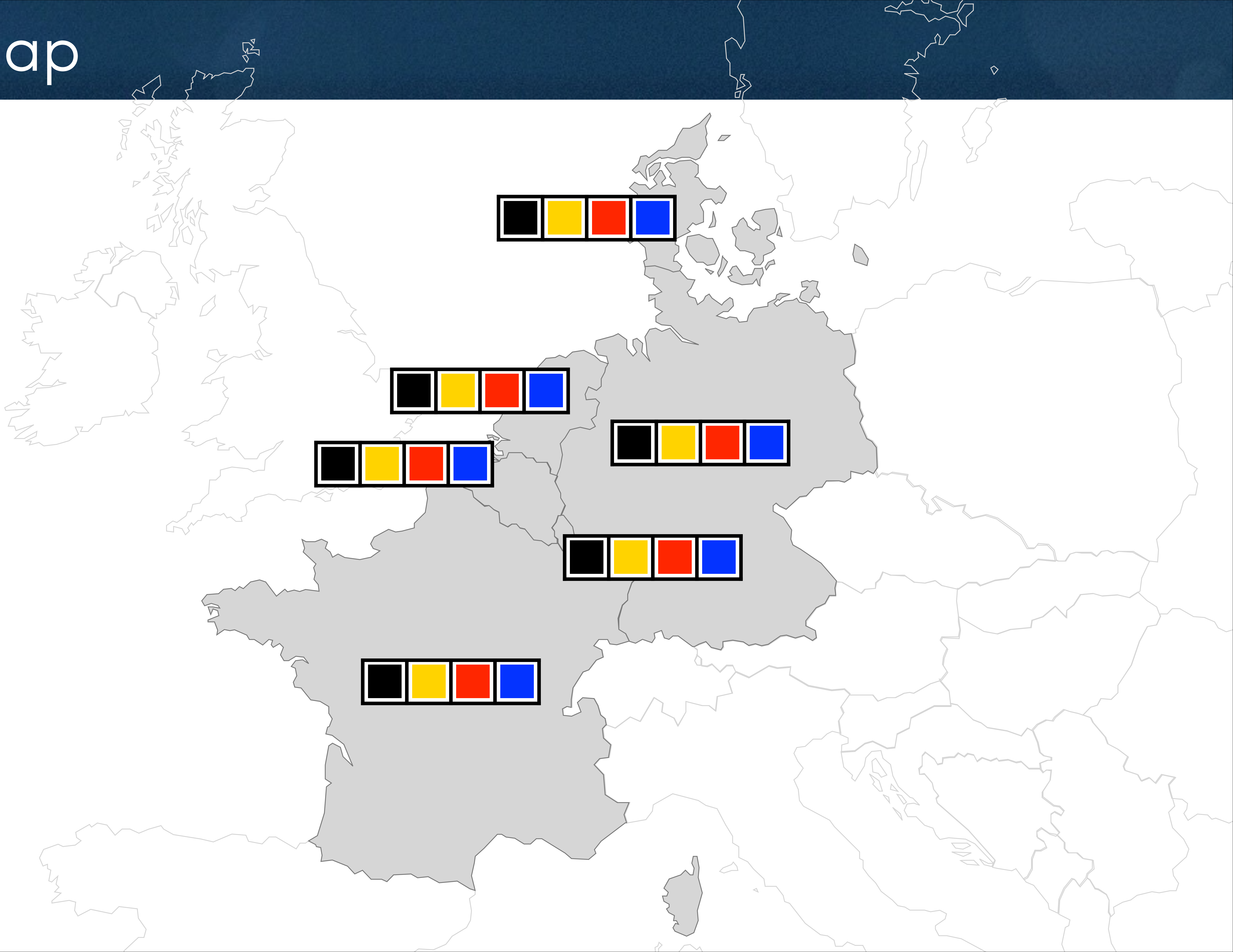
  color[Germany] ≠ color[Luxembourg];
}
```

Constraint Programming

- Computational paradigm
  - use constraints to reduce the set of values that each variable can take
  - make a choice when no more deduction can be performed

- What does this mean for the coloring problem?
  - no value can be removed initially, so the system must make a choice

Coloring a Map



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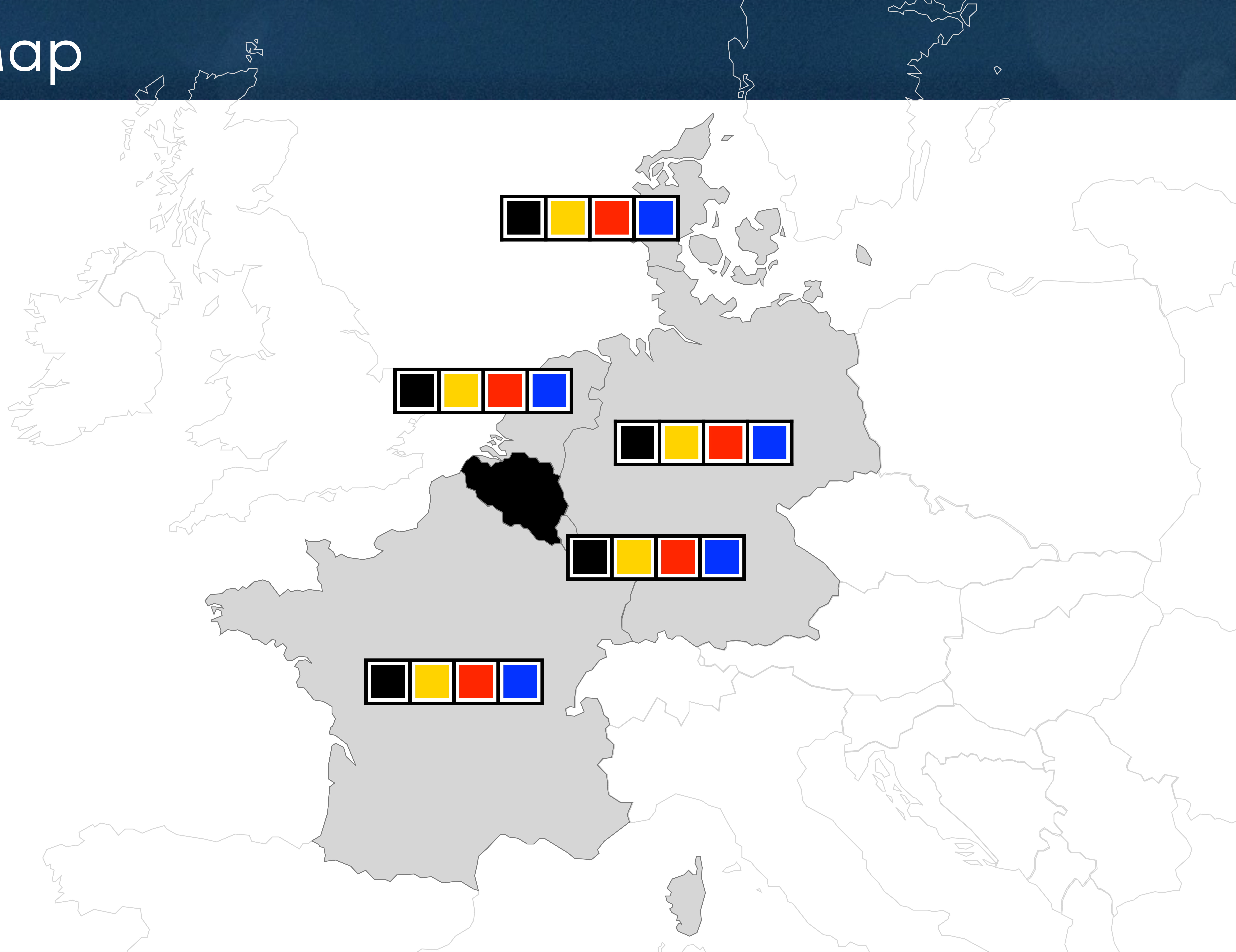

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  color[Germany] ≠ color[Luxembourg];
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Coloring a Map

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enum Colors = { black, yellow, red, blue };
var{Colors} color[Countries];

solve {
  black ≠ color[France];
  black ≠ color[Germany];
  black ≠ color[Netherlands];
  black ≠ color[Luxembourg];
  color[Denmark] ≠ color[Germany];
  color[France] ≠ color[Germany];
  color[France] ≠ color[Luxembourg];
  color[Germany] ≠ color[Netherlands];
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}
```
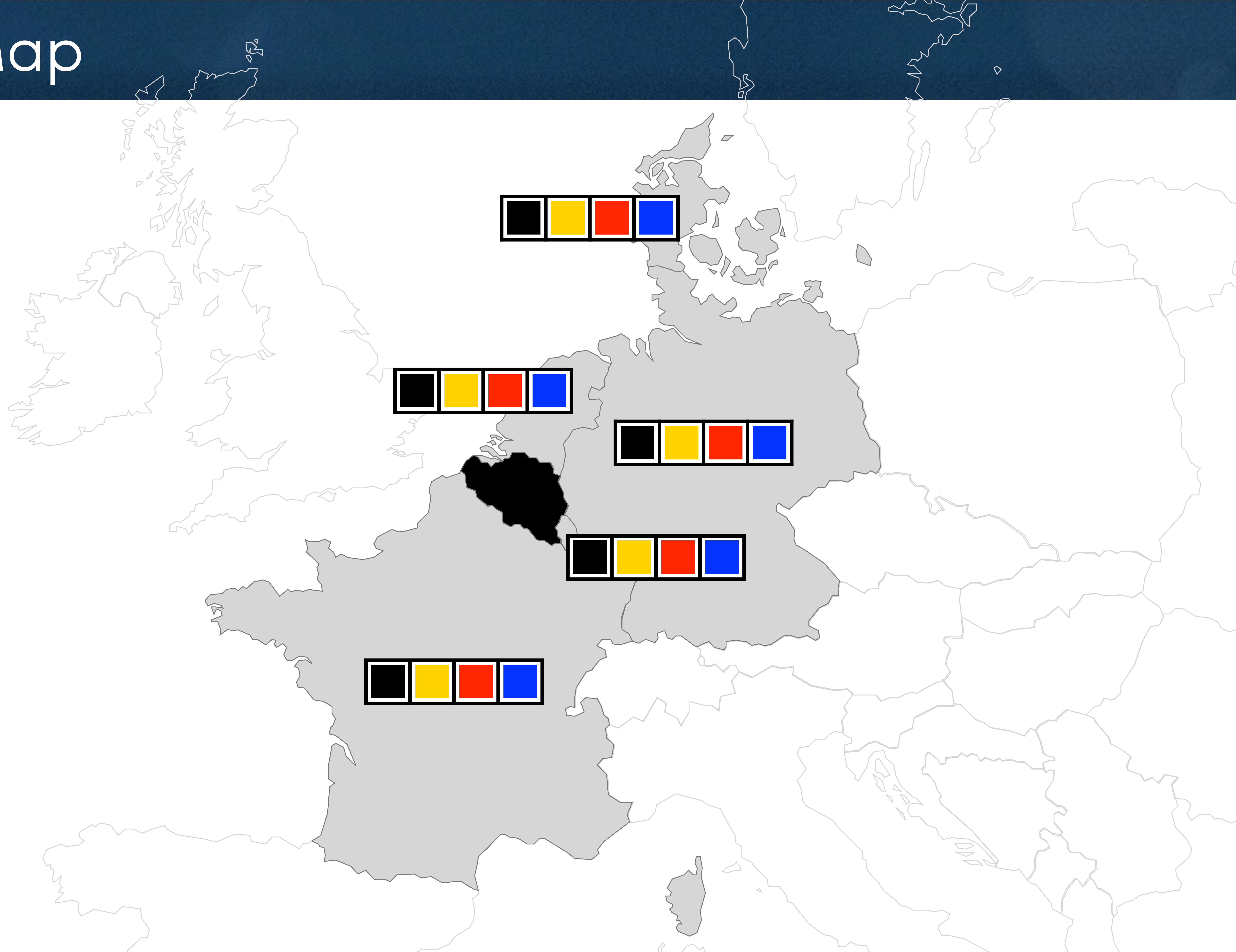

Coloring a Map



Computational Paradigm

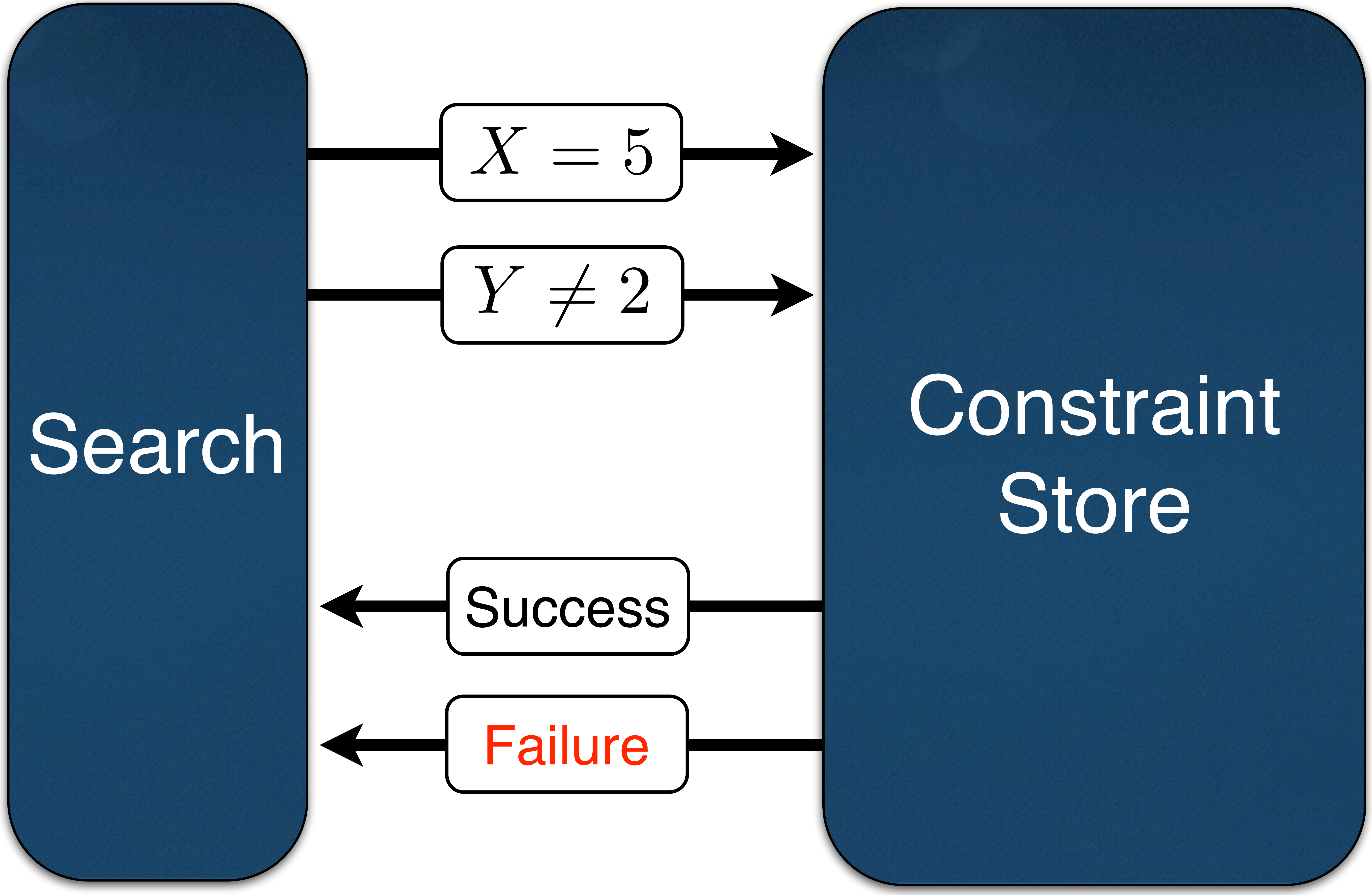
- Branch and prune
  - pruning
    - reduce the search space as much as possible
  - branching
    - decompose the problem into subproblems and explore the subproblems
- Pruning
    - use constraints to remove, from the variable domains, values that cannot belong to any solution
- Branching
    - e.g., try all the possible values of a variable until a solution is found or it can be proven that no solution exists

Computational Paradigm

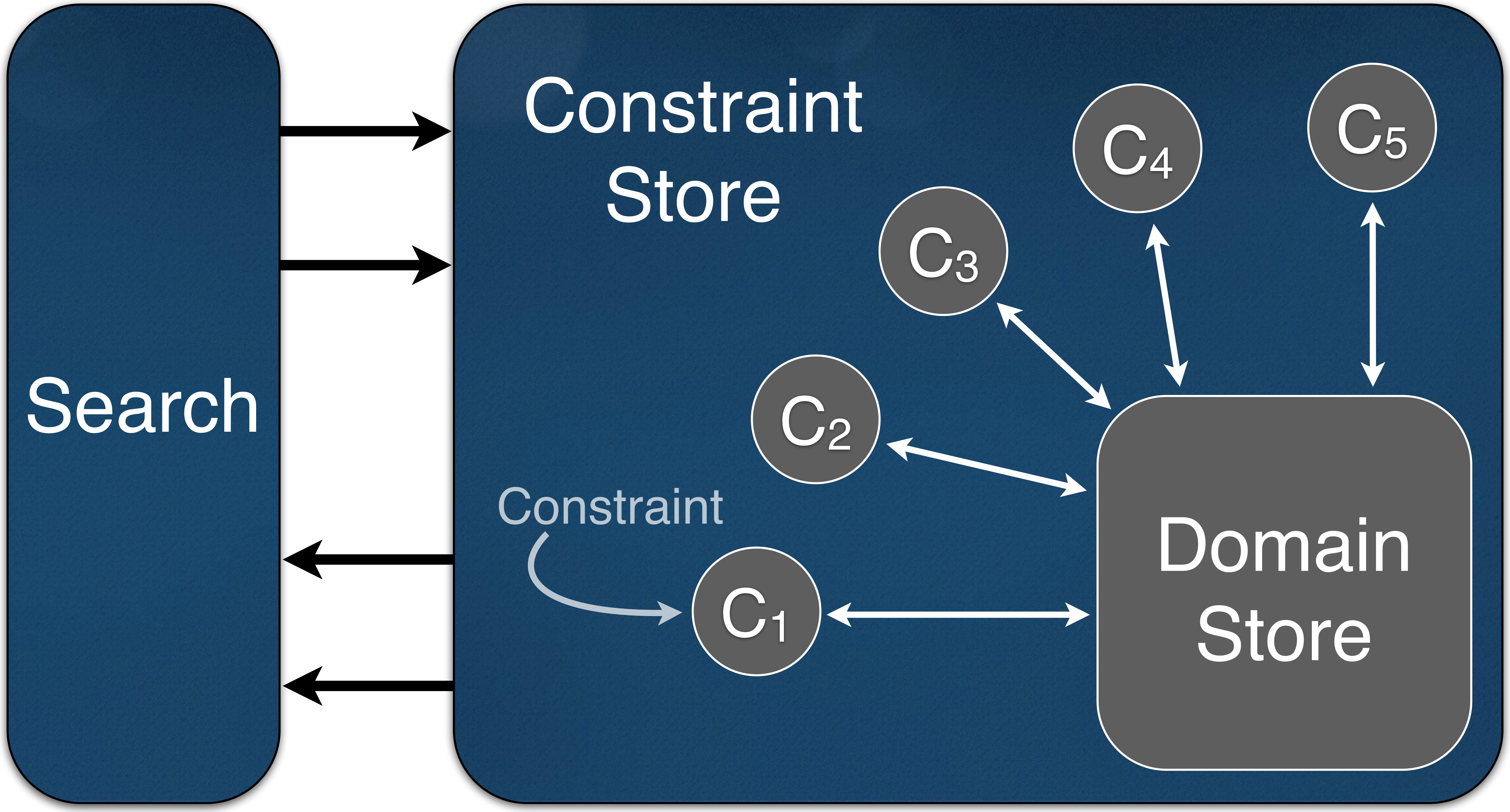
- Complete method, not a heuristic
  - given enough time, it will find a solution to a satisfaction problem
  - given enough time, it will find an optimal solution to an optimization problem

- Focus on feasibility
  - how to use constraints to prune the search space by eliminating values that cannot belong to any solution

Computational Paradigm



Computational Paradigm



Computational Paradigm

- What does a constraint do?
  - feasibility checking
  - pruning
- Feasibility checking
  - a constraint checks if it can be satisfied given the values in the domains of its variables
- Pruning
  - if satisfiable, a constraint determines which values in the domains cannot be part of any solution

Computational Paradigm

- The propagation engine
  - this is the core of any constraint-programming solver
  - a simple (fixpoint) algorithm

```
propagate()
{
  repeat
    select a constraint c;
    if c is infeasible given the domain store then
      return failure;
    else
      apply the pruning algorithm associated with c;
  until no constraint can remove any value from the
  domain of its variables;
  return success;
}
```

Back to the 8-Queens Problem

- What are the decision variables?
  - many possible modelings
    - this is what makes optimization problems interesting :-)
- Here is one modeling
  - associate a decision variable with each column
    - the variable denotes the row of the queens placed in this column
    - no two queens can be placed on the same column so this is valid
- What are the constraints?
  - the queens cannot be placed on the same,
    - row
    - upward diagonal
    - downward diagonal

A Constraint Program for the 8-Queens

- Constraints
  - the queens cannot be placed on the same,
    - row
    - upward diagonal
    - downward diagonal

```
range R = 1..8;
var{int} row[R] in R;
solve {
   forall(i in R,j in R: i < j) {
      row[i] ≠  row[j];
      row[i] ≠  row[j] + (j - i);
      row[i] ≠  row[j] - (j - i);
   }
}
```

A Constraint Program for the 8-Queens

- A simple model for the 8-queens problem

```
range R = 1..8;
var{int} row[R] in R;
solve {
   forall(i in R,j in R: i < j) {
      row[i] ≠  row[j];
      row[i] ≠  row[j] + (j - i);
      row[i] ≠  row[j] - (j - i);
   }
}
```

- What happens when the queen in column 1 is assigned the value 1

```
row[1] ≠  row[2];
...
row[1] ≠  row[8];

row[1] ≠  row[2] + 1;
...
row[1] ≠  row[8] + 7;

row[1] ≠  row[2] - 1;
...
row[1] ≠  row[8] - 7;
```

Computational Paradigm

- What does a constraint do?
  - feasibility checking
  - pruning
- Feasibility checking
  - a constraint checks if it can be satisfied given the values in the domains of its variables
- Pruning
  - if satisfiable, a constraint determines which values in the domains cannot be part of any solution

Computational Paradigm

- Consider two variables X, Y
  - X can take the values 0,1,2
  - Y can take the values 1,2,3

$$\boxed{X} \in \{0,\ 1,\ 2\}$$

$$\boxed{Y} \in \{1,\ 2,\ 3\}$$

- The domain of X is the set of values it can take

$$D(X) = \{0,\ 1,\ 2\}$$

- A short hand for ranges of integers

$$[1..5] = \{1,\ 2,\ 3,\ 4,\ 5\}$$

Computational Paradigm

- Consider constraint, X ≠ Y
- Feasibility checking

$$X \in \{0,\ 1,\ 2\}$$

$$Y \in \{1,\ 2,\ 3\}$$

$$|D(X) \cup D(Y)| \geq 2$$

$$|\{0,\ 1,\ 2\} \cup \{1,\ 2,\ 3\}| \geq 2$$

$$|\{0,\ 1,\ 2,\ 3\}| \geq 2$$

$$4 \geq 2$$

Computational Paradigm

- Consider constraint, X ≠ Y
- Pruning

X $\in \{0, 1, 2\}$

Y $\in \{1, 2, 3\}$

$$D(X) = \{1\}$$
$$\Rightarrow D(Y) \setminus \{1\}$$

$$D(Y) = \{2\}$$
$$\Rightarrow D(X) \setminus \{2\}$$

Until Next Time

Citations

Map_of_USA_four_colours.svg (http://commons.wikimedia.org/wiki/File:Map_of_United_States_vivid_colors_shown.png) of the modification : Derfel73) Dbenbenn derivative work: Tomwsulcer (Map_of_USA_four_colours.svg) [CC-BY-SA-3.0 (http://creativecommons.org/licenses/by-sa/3.0)], via Wikimedia Commons