

# Discrete Optimization

The Knapsack Problem:  
Branch and Bound

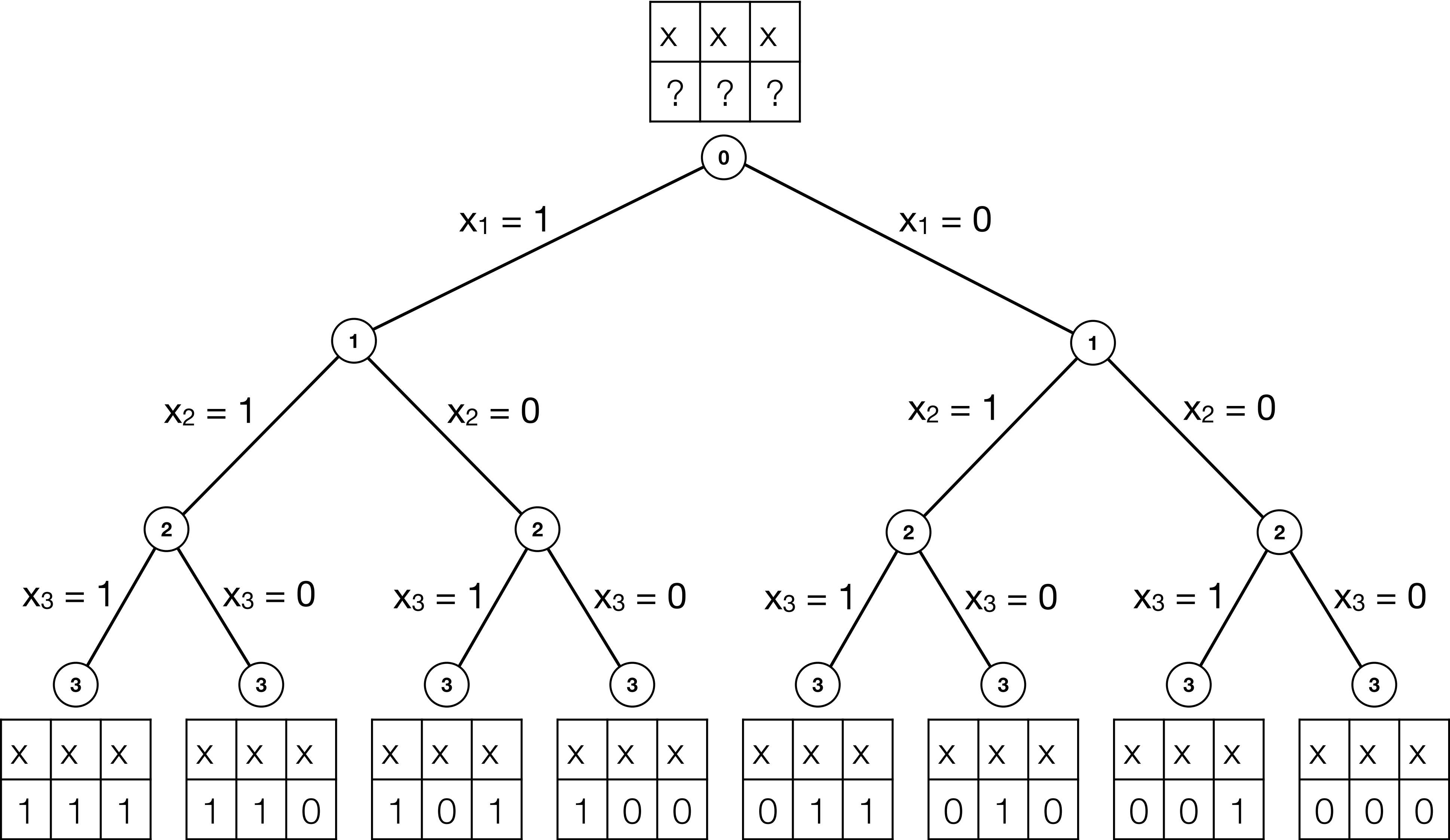
# Goals of the Lecture

- ▶ Introduce branch and bound
- ▶ The value of relaxation

# One-Dimensional Knapsack

$$\begin{array}{ll}\text{maximize} & 45x_1 + 48x_2 + 35x_3 \\ \text{subject to} & \\ & 5x_1 + 8x_2 + 3x_3 \leq 10 \\ & x_i \in \{0, 1\} \quad (i \in 1..3)\end{array}$$

# Exhaustive Search



# Branch and Bound

- ▶ Iterative two steps
  - branching
  - bounding
- ▶ Branching
  - split the problem into a number of subproblems
    - like in exhaustive search
- ▶ Bounding
  - find an ***optimistic estimate*** of the best solution to the subproblem
    - maximization: upper bound
    - minimization: lower bound



# Branch and Bound

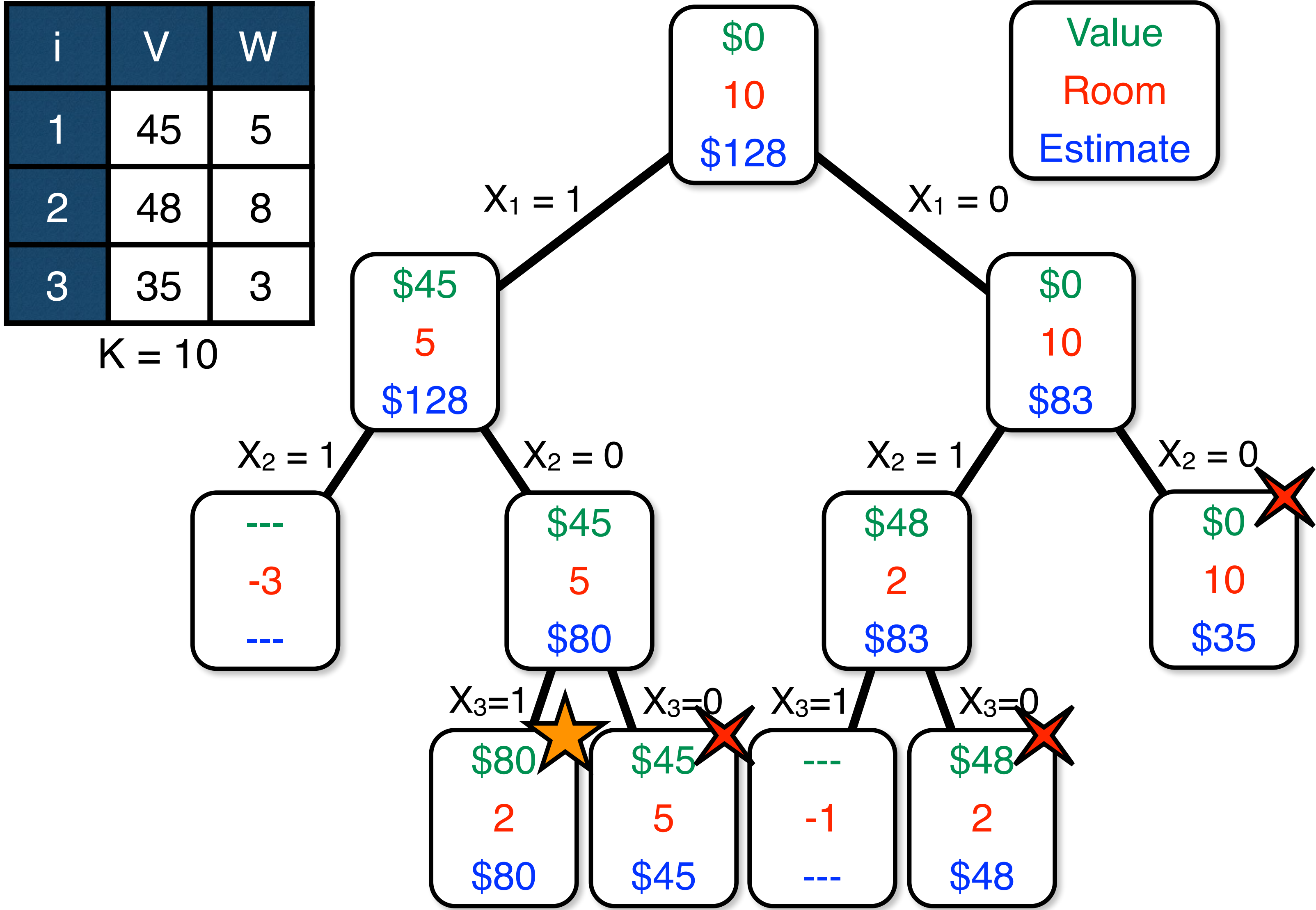
- ▶ How to find this optimistic estimate?
  - Relaxation!
- ▶ Optimization is the art of relaxation

# A Knapsack Model

$$\begin{array}{ll}\text{maximize} & 45x_1 + 48x_2 + 35x_3 \\ \text{subject to} & \\ & 5x_1 + 8x_2 + 3x_3 \leq 10 \\ & x_i \in \{0, 1\} \quad (i \in 1..3)\end{array}$$

- What can we relax?
  - we can relax the capacity constraint

# Depth-First Branch and Bound





# A Knapsack Model

$$\begin{array}{ll}\text{maximize} & 45x_1 + 48x_2 + 35x_3 \\ \text{subject to} & \\ & 5x_1 + 8x_2 + 3x_3 \leq 10 \\ & x_i \in \{0, 1\} \quad (i \in 1..3)\end{array}$$

- Can we relax something else?

# A Knapsack Model

- ▶ What if the items are bars of Belgian chocolate?
  - In that case, we could actually take a fraction of the bar!

Callebaut  
subject



- ▶ This is
  - we will
  - we rel



# A Knapsack Model

- Can we solve a knapsack when we can take parts of the items?
  - order the items by decreasing value of  $V_i/W_i$
  - “most value per kilo”

$$\begin{array}{ll}\text{maximize} & 45x_1 + 48x_2 + 35x_3 \\ \text{subject to} & \\ & 5x_1 + 8x_2 + 3x_3 \leq 10 \\ & 0 \leq x_i \leq 1 \quad (i \in 1..3)\end{array}$$

# A Knapsack Model

- ▶ How to solve the relaxation now?
  - select the items while the capacity is not exhausted
  - select a fraction of the last item

maximize  $45x_1 + 48x_2 + 35x_3$

subject to

$$5x_1 + 8x_2 + 3x_3 \leq 10$$

$$0 \leq x_i \leq 1 \quad (i \in 1..3)$$

- ▶ In this example,
  - $V_1/W_1 = 9$ ,  $V_2/W_2 = 6$ ,  $V_3/W_3 = 11.7$
  - select items 3 and 1
  - select 1/4 of item 2
  - estimation: 92



# A Knapsack Model

► Why is correct?

$$\text{let } x_i = \frac{y_i}{v_i}$$

$$\text{maximize } \sum_{i \in 1..j} y_i$$

subject to

$$\sum_{i \in 1..j} \frac{w_i}{v_i} y_i \leq K$$

$$0 \leq y_i \leq 1 \quad (i \in 1..j)$$



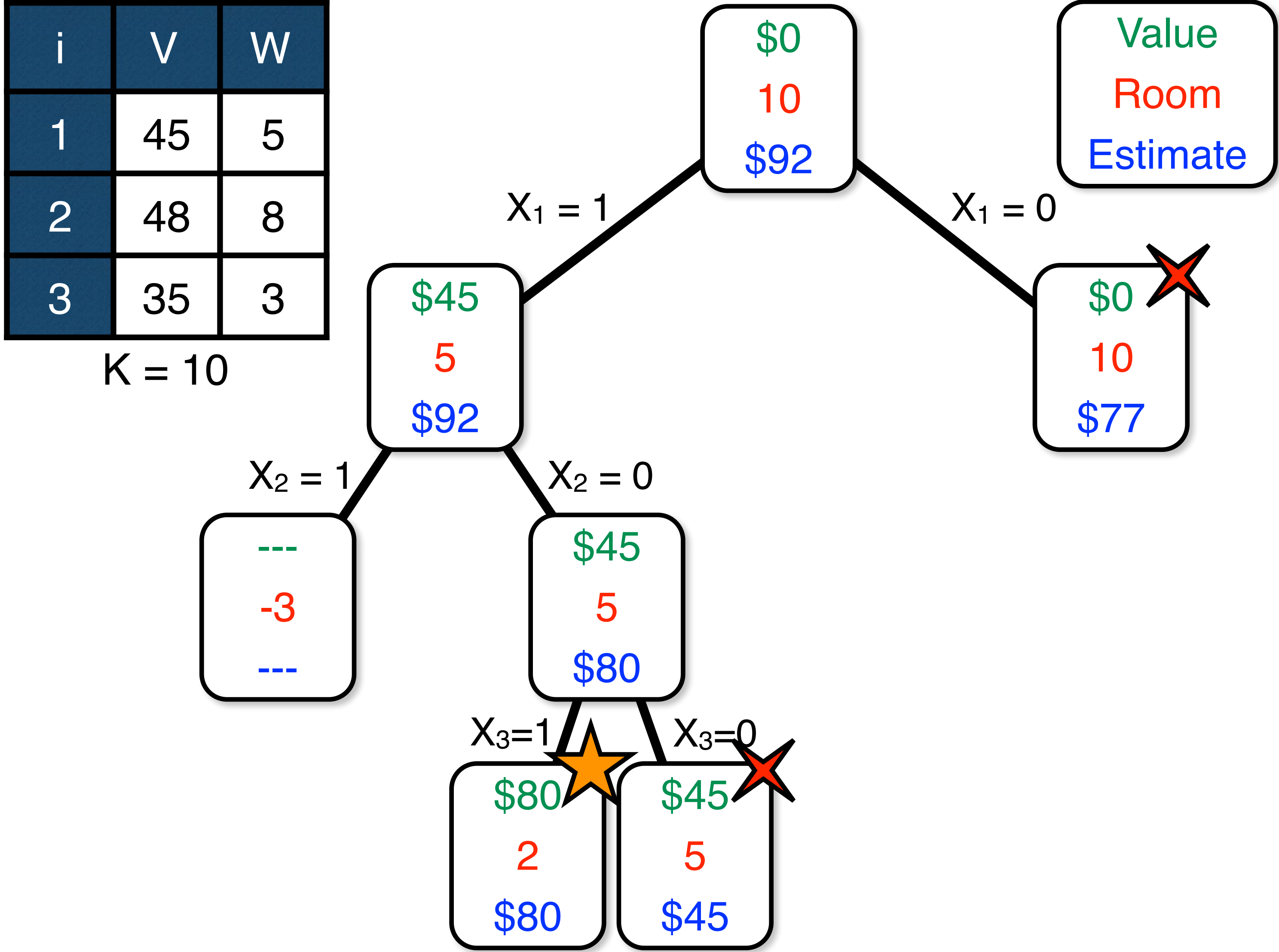
# A Knapsack Model

- Of course, if the items are archeological artifacts...





# Depth-First Branch and Bound



# Until Next Time



# Citations

Stone Foundation Tablet with Inscription of Gudea - 41221 ([http://commons.wikimedia.org/wiki/File:Sumerian \\_- \\_Stone\\_Foundation\\_Tablet\\_with\\_Inscription\\_of\\_Gudea\\_-\\_Walters\\_41221\\_-\\_View\\_A.jpg](http://commons.wikimedia.org/wiki/File:Sumerian_-_Stone_Foundation_Tablet_with_Inscription_of_Gudea_-_Walters_41221_-_View_A.jpg)). Artist Unknown. Walters Art Museum [Public domain, CC-BY-SA-3.0 (<http://creativecommons.org/licenses/by-sa/3.0> )], via Wikimedia Commons

Stone Foundation Tablet with Inscription of Gudea - 41220 ([http://commons.wikimedia.org/wiki/File:Sumerian \\_- \\_Stone\\_Foundation\\_Tablet\\_with\\_an\\_Inscription\\_of\\_Gudea\\_-\\_Walters\\_41220\\_-\\_View\\_A.jpg](http://commons.wikimedia.org/wiki/File:Sumerian_-_Stone_Foundation_Tablet_with_an_Inscription_of_Gudea_-_Walters_41220_-_View_A.jpg)). Artist Unknown. Walters Art Museum [Public domain, CC-BY-SA-3.0 (<http://creativecommons.org/licenses/by-sa/3.0>) via Wikimedia Commons

Buddha at the Moment of Victory ([http://commons.wikimedia.org/wiki/File:Thai \\_- \\_Buddha\\_at\\_the\\_Moment\\_of\\_Victory\\_-\\_Walters\\_542775.jpg](http://commons.wikimedia.org/wiki/File:Thai_-_Buddha_at_the_Moment_of_Victory_-_Walters_542775.jpg) ). Artist Unknown. Walters Art Museum [Public domain, CC-BY-SA-3.0 (<http://creativecommons.org/licenses/by-sa/3.0> )], via Wikimedia Commons

Hoxne Hoard two gold bracelets ([http://commons.wikimedia.org/wiki/File:Hoxne\\_Hoard\\_two\\_gold\\_bracelets\\_side.JPG](http://commons.wikimedia.org/wiki/File:Hoxne_Hoard_two_gold_bracelets_side.JPG)) by Fæ (<http://commons.wikimedia.org/wiki/User:F%C3%A6>) [CC-BY-SA-3.0 (<http://creativecommons.org/licenses/by-sa/3.0>)], via Wikimedia Commons

Ring with the engraved portrait of Ptolemy VI Philometor ([http://commons.wikimedia.org/wiki/File:Ring\\_with\\_engraved\\_portrait\\_of\\_Ptolemy\\_VI\\_Philometor\\_\(3rd %E2%80%93\\_2nd\\_century\\_BCE\)\\_-\\_2009.jpg](http://commons.wikimedia.org/wiki/File:Ring_with_engraved_portrait_of_Ptolemy_VI_Philometor_(3rd_%E2%80%93_2nd_century_BCE)_-_2009.jpg)<[http://commons.wikimedia.org/wiki/File:Ring\\_with\\_engraved\\_portrait\\_of\\_Ptolemy\\_VI\\_Philometor\\_\(3rd-2nd\\_century\\_BCE\)\\_-\\_2009.jpg](http://commons.wikimedia.org/wiki/File:Ring_with_engraved_portrait_of_Ptolemy_VI_Philometor_(3rd-2nd_century_BCE)_-_2009.jpg)>) By Unknown. (Photographed by PHGCOM in 2009.) [Public domain], via Wikimedia Commons

Calice du sacre Tau ([http://commons.wikimedia.org/wiki/File:Calice\\_du\\_sacre\\_Tau.jpg](http://commons.wikimedia.org/wiki/File:Calice_du_sacre_Tau.jpg)) By Vassil (<http://commons.wikimedia.org/wiki/User:Vassil>) (Own work) [Public domain], via Wikimedia Commons

# Citations

the mask of agamemnon (<http://www.flickr.com/photos/rosemania/5705122218/>) by Xuan Che (<http://www.flickr.com/people/rosemania/>) CC BY-2.0 (<http://creativecommons.org/licenses/by/2.0/deed.en>)

Terracotta Warrior (<http://www.flickr.com/photos/59627558@N00/4677378806/>) by fixermark (<http://www.flickr.com/photos/59627558@N00/>) CC BY 2.0 (<http://creativecommons.org/licenses/by/2.0/deed.en>)

Mmmmm Chocolate [274/366] (<http://www.flickr.com/photos/sackton/8041853745>) by Tim Sackton (<http://www.flickr.com/photos/sackton/>) CC BY-SA 2.0 (<http://creativecommons.org/licenses/by-sa/2.0/deed.en>)