PONTRYAGIN-WEISS CLASSES AND A RATIONAL DECOMPOSITION OF SPACES OF HOMEOMORPHISMS

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ABSTRACT. We construct a rational homotopy pullback decomposition for variants of the classifying space of the group of homeomorphisms for a large class of manifolds. This has various applications, including a rational section of the stabilisation map $Top(d) \to Top$ of the space of homeomorphisms of \mathbb{R}^d for $d \geq 6$, or a new method to construct topological bundles and detect characteristic classes thereof. Some steps in the proofs may be of independent interest, such as the construction of a nullhomotopy of the twice-iterated stabilisation map for the space of orientation-preserving derived automorphisms of the rational E_d -operad, results on recovering boundaries of manifolds from the interior in the context of embedding calculus, or a treatment of tensor products of truncated ∞ -operads.

Contents

 Introduc 	etion	1
2. Prelimin	2. Preliminaries	
3. Twice-iterated stabilisation for $Aut(E_{d,Q})$		12
4. Boundary conditions in embedding calculus		17
5. A pullback decomposition of spaces of self-embeddings		26
6. Applications		33
Appendix A.	Tensor products of truncated operads	42
Appendix B.	Tensor products of rational pro-operads	54
References		58

1. Introduction

From a homotopy-theoretic perspective, the study of fibre bundles with fibre a compact topological d-manifold M and trivialised boundary bundle amounts to the study of the homotopy type of the classifying space $\operatorname{BHomeo}_{\partial}(M)$ of the topological group of homeomorphisms fixing the boundary. In this work we construct a homotopy pullback decomposition of variants of $\operatorname{BHomeo}_{\partial}(M)$ for a large class of manifolds. Instead of a stand-alone result, we consider this decomposition rather as a tool in the study of $\operatorname{BHomeo}_{\partial}(M)$ and thus believe that it is best appreciated through its applications. We begin by explaining three of them, and only then discuss the actual decomposition theorem.

Application I: Pontryagin–Weiss classes. The first application concerns *topological Pontryagin classes*. Recall that the ring of rational characteristic classes for stable vector bundles—the rational cohomology ring of the corresponding classifying space BO—is a polynomial ring H*(BO; Q) \cong Q[p_1, p_2, \ldots] in the rational Pontryagin classes p_k . By construction, the kth Pontryagin class p_k is trivial on vector bundles of fixed rank d as long as $k > \frac{d}{2}$, i.e. $p_k \in H^{4k}(BO; Q)$ vanishes in this range when pulled back along the stabilisation map BO(d) \to BO that classifies the underlying stable vector bundle of a vector bundle of rank d. It turns out that rational Pontryagin classes can be defined more generally for d-dimensional *Euclidean bundles* i.e. fibre bundles whose fibres are homeomorphic to \mathbb{R}^d , without linear structure [Sch70, Sie71, Kah72]. Indeed, the map BO \to BTop that classifies the underlying stable Euclidean bundle of a stable vector bundle induces an isomorphism on rational cohomology, so Pontryagin classes extend uniquely to characteristic classes $p_k \in H^{4k}(BTop; Q)$ for stable Euclidean bundles, but with this definition it is not clear whether they have the same vanishing

1

behaviour when evaluated on Euclidean bundles of fixed rank d as in the case of vector bundles, that is, whether the pullback $p_k \in \mathrm{H}^{4k}(\mathrm{BTop}(d); \mathbf{Q})$ along the stabilisation map $\mathrm{BTop}(d) \to \mathrm{BTop}$ vanishes for $k > \frac{d}{2}$. In fact, Weiss showed that this is *not* the case for many choices of pairs (d,k) in this range [Wei21, Theorem 1.1.1]. He moreover proved that for these (d,k), the class p_k is nontrivial on a d-dimensional Euclidean bundle *over a sphere*, i.e. that p_k evaluates nontrivially even on the image of the Hurewicz homomorphism $\pi_{4k}(\mathrm{BTop}(d)) \to \mathrm{H}_{4k}(\mathrm{BTop}(d))$. More recently, Galatius and Randal-Williams [GRW23, Theorem 1.1] showed via a different method that the stabilisation map $\mathrm{BTop}(d) \to \mathrm{BTop}$ is injective on rational cohomology in *all* degrees for $d \ge 6$, so in particular $p_k \ne 0 \in \mathrm{H}^{4k}(\mathrm{BTop}(d); \mathbf{Q})$ for all $k \ge 1$. Their argument however does not prove that these classes are nontrivial on a bundle over a sphere (see Remark 1.3 loc.cit.). Extending Weiss' original approach using the aforementioned pullback decomposition, we show that they are:

Theorem A. For all $d \ge 6$ and $k \ge 1$, there exists a fibre bundle over the 4k-sphere with fibres homeomorphic to \mathbb{R}^d whose kth rational Pontryagin class is nontrivial.

Theorem A has the following homotopy-theoretic reformulation:

Corollary B. For $d \ge 6$ the stabilisation map $BTop(d) \to BTop$ admits a rational section after taking loop spaces. Equivalently, it is surjective on rational homotopy groups.

Via Morlet's equivalence $\mathrm{BDiff}_{\partial}(D^d) \simeq \Omega_0^d \mathrm{Top}(d)/\mathrm{O}(d)$ in dimensions $d \neq 4$ [KS77, V.3.4], Corollary B has the following consequence for the classifying space $\mathrm{BDiff}_{\partial}(D^d)$ of the group of diffeomorphisms of a closed d-dimensional disc fixing the boundary pointwise, equipped with the smooth topology (see e.g. [KrRW21, Section 8.2.1] for a definition of the maps in the statement):

Corollary C. For $n \ge 3$, the following maps are surjective on rational homotopy groups:

$$\begin{split} \left(\Omega^{2n}(p_n-e^2)^{\tau} \times \sqcap_{k>n} \Omega^{2n} p_k^{\tau}\right) &: & \mathrm{BDiff}_{\partial}(D^{2n}) &\longrightarrow \prod_{k=n}^{\infty} K(\mathbf{Q}, 4k-2n-1) \\ \left(\Omega^{2n+1}(p_n-E)^{\tau} \times \sqcap_{k>n} \Omega^{2n+1} p_k^{\tau}\right) &: & \mathrm{BDiff}_{\partial}(D^{2n+1}) &\longrightarrow \prod_{k=n}^{\infty} K(\mathbf{Q}, 4k-2n-2). \end{split}$$

Remark.

- (i) Corollary C implies that BDiff $_{\partial}(D^d)$ has for $d \geq 6$ infinitely many nontrivial rational homotopy groups, most of which are outside of the range where $\pi_*(\mathrm{BDiff}_{\partial}(D^d)) \otimes \mathbf{Q}$ has been computed in [KuRW25, KrRW21] and many lie in different degrees than Watanabe's "graph classes" detected by configuration space integrals [Wat09, Wat22, Wat21].
- (ii) From the case of discs, one can obtain nontriviality of infinitely many rational homotopy groups for $BDiff_{\partial}(M)$ for many other smooth compact d-manifolds M, e.g. whenever M is closed, parallelisable, and admits a non-negatively curved metric [BFJ18, Lemma 2.1].
- (iii) We do not show that there is a section as in Corollary B before taking loop spaces, so our result is neither stronger nor weaker than the main result of [GRW23] in that it does not show that the stabilisation $BTop(d) \rightarrow BTop$ is surjective on rational homology groups, but it does show that it is surjective on rational homotopy groups. For applications such as Corollary C, the surjectivity on homotopy as opposed to homology groups is essential.

Application II: A rationally unconstrained Burghelea–Lashof splitting. The second application of our pullback decomposition concerns the forgetful map

$$\operatorname{Aut}_{\partial}(M)/\operatorname{Homeo}_{\partial}(M) \longrightarrow \operatorname{Aut}_{\partial}(M)/\overline{\operatorname{Homeo}_{\partial}(M)}$$
 (1)

from the classifying space for fibre homotopically trivialised fibre bundles with a compact d-manifold M as fibre and trivialised boundary bundle, to the corresponding classifying space for fibre homotopically trivialised block bundles. Burghelea and Lashof [BL82] proved that after Postnikov truncation in a range growing with d, taking loop spaces, and localising away from 2, the map (1) admits a section. Since the target of the map admits a complete description in terms of algebraic L-theory by surgery theory, this gives a convenient method to construct fibre bundles with specified properties (see e.g. [KKRW21, Fre21] for recent applications of this strategy in the smooth setting).

Using the aforementioned pullback decomposition, we show that, when looped once, the forgetful map (1) often admits a section *before* Postnikov truncation as long as one is willing to rationalise:

Theorem D. For a 2-connected smoothable compact manifold M of dimension $d \ge 13$ with 2-connected boundary, the forgetful map (1) admits a rational section after taking loop spaces.

Remark.

- (i) The hypothesis on the dimension in Theorem D can often be improved, and under additional assumptions on M the section also exists before taking loop spaces (see Theorem 6.10).
- (ii) In contrast to Burghelea–Lashof's section, the section in Theorem D without Postnikov truncation is a purely topological phenomenon and does not exist in the smooth category in general (this can for instance be deduced from [RW17, Proposition 3.1]).

Application III: Detecting characteristic classes. Our pullback decomposition can also serve as a tool to detect characteristic classes of topological manifold bundles, in particular *tautological classes*, that is, fibre integrals of characteristic classes of the vertical topological tangent bundle. To illustrate the general method, we prove the following as an example:

Theorem E. Let M be a 2-connected smoothable compact manifold of dimension $d \ge 8$ with 2-connected boundary, such that M contains $S^m \times S^{d-m}$ as a connected summand for some m.

- (i) The tautological class $\kappa_{p_ip_j} \in H^{4(i+j)-d}(BHomeo_{\partial}(M); \mathbf{Q})$ associated to a product of Pontryagin classes is nontrivial for all $i, j > \max(m, d m)/4$.
- (ii) If m and d-m are even, then the total dimension of the cohomology ring $H^*(BHomeo_{\partial}(M); \mathbb{Q})$ in degree $* \le k$ grows with k faster than any polynomial.

The pullback decomposition. We now explain the pullback decomposition on which the above applications rely. For a topological d-manifold M, taking topological derivatives yields a factorisation

$$\operatorname{Homeo}_{\partial}(M) \longrightarrow \operatorname{Aut}_{\partial}(TM) \longrightarrow \operatorname{Aut}_{\partial}(M)$$
 (2)

of the forgetful map to the topological monoid $\operatorname{Aut}_{\partial}(M)$ of homotopy equivalences of M fixed on ∂M , through the topological monoid $\operatorname{Aut}_{\partial}(TM)$ of homotopy equivalences of M fixed on ∂M covered by a bundle map of the topological tangent bundle TM of M fixed on $TM|_{\partial M}$. We show that the delooping of (2) can "often" be "partially" completed to a homotopy pullback square

$$\begin{array}{ccc} \operatorname{BHomeo}_{\partial}(M) & \longrightarrow Z_{M} \\ \downarrow & \downarrow \\ \operatorname{BAut}_{\partial}(TM) & \longrightarrow \operatorname{BAut}_{\partial}(M) \end{array}$$

for some space Z_M . "Often" in that we assume $d \ge 5$ and that M is smoothable, 2-connected, and has 2-connected boundary, and "partially" in that we replace the sequence (2) by a variant explained below, involving a *weaker boundary condition*, *tangential structures*, and certain *rationalisations*.

To introduce the *weaker boundary condition*, we assume that M has nonempty boundary, fix an embedded disc $D^{d-1} \subset \partial M$, set $1/2\partial M := \partial M \setminus \operatorname{int}(D^{d-1})$, and replace $\operatorname{Aut}_{\partial}(TM)$ in (2) by the topological monoid $\operatorname{Aut}_{\partial}(TM, 1/2\partial)$ defined in the same way as $\operatorname{Aut}_{\partial}(TM)$, except that the bundle map is only required to fix $TM|_{1/2\partial M}$ instead of $TM|_{\partial M}$. Regarding *tangential structures*, for a map $\theta \colon B \to \operatorname{BTop}(d)$ and a θ -structure $\ell_{1/2\partial}$ on $TM|_{1/2\partial M}$ (a lift of its classifying map along θ), the space of θ -structures on TM extending $\ell_{1/2\partial}$ is acted upon by $\operatorname{Homeo}_{\partial}(M)$ and $\operatorname{Aut}_{\partial}(TM, 1/2\partial)$, so by taking homotopy orbits we obtain a variant $\operatorname{BHomeo}_{\partial}^{\theta}(M; \ell_{1/2\partial}) \to \operatorname{BAut}_{\partial}^{\theta}(TM; \ell_{1/2\partial}) \to \operatorname{BAut}_{\partial}(M)$ of the sequence (2). Finally, with respect to $\operatorname{rationalisations}$, we replace the final map in the latter sequence by the initial map in its $\operatorname{Moore-Postnikov}$ 1-factorisation $\operatorname{BAut}_{\partial}^{\theta}(TM, 1/2\partial; \ell_{1/2\partial}) \to \operatorname{BAut}_{\partial}(M)^{\ell_{1/2\partial}} \to \operatorname{BAut}_{\partial}(M)$ and replace $\operatorname{BHomeo}_{\partial}^{\theta}(M; \ell_{1/2\partial})$ as well as $\operatorname{BAut}_{\partial}^{\theta}(TM; \ell_{1/2\partial})$ by their fibrewise rationalisation over $\operatorname{BAut}_{\partial}(M)^{\ell_{1/2\partial}}$, indicated by a $(-)_{fO}$ -subscript. In total, we arrive at a sequence of the form

$$BHomeo_{\partial}^{\theta}(M; \ell_{1/2\partial})_{fQ} \longrightarrow BAut_{\partial}^{\theta}(TM; \ell_{1/2\partial})_{fQ} \longrightarrow BAut_{\partial}(M)^{\ell_{1/2\partial}}$$
(3)

which is the variant of (2) that our homotopy pullback decomposition result is about:

Theorem F. Let M be a smoothable compact d-manifold M with $d \ge 5$ and $\partial M \ne \emptyset$, $\theta: B \to BSTop(d)$ an oriented tangential structure such that B is nilpotent, and $\ell_{1/2\partial}$ a θ -structure on $TM|_{1/2\partial}$. If

- (i) θ factors through the stabilisation map BSTop $(d-2) \to BSTop(d)$ after rationalisation,
- (ii) M and ∂M are both 2-connected,

then the sequence (3) can be completed to a homotopy pullback square.

Informally speaking and neglecting tangential structures, rationalisation, and boundary conditions, Theorem F says that specifying a topological fibre bundle with fibre M is equivalent to specifying a fibration with fibre M together with two pieces of data on it that are independent from each other: a "vertical tangent bundle" i.e. a d-dimensional Euclidean bundle on the total space that is equivalent to TM on each fibre, and whichever extra data on a fibration the additional space in the pullback decomposition of Theorem F classifies. Since the two pieces of data can be chosen independently, this allows one for instance to manipulate the vertical tangent bundle of a fibre bundle without changing the underlying fibration.

Remark.

- (i) We prove a stronger version of Theorem F that allows more general boundary conditions and also applies to certain spaces of self-embeddings (see Theorem 5.1).
- (ii) The proof of Theorem F in principle also describes the additional space in the pullback extension of (3), but we prefer to state the result as above to emphasise that the mere existence of such a pullback decomposition suffices for many applications, in particular for Theorems A, D, and E. Roughly speaking, the additional space is constructed out of the homotopy types of Aut_∂(M) and the limit of fibrewise rationalisations of the truncated particle embedding calculus approximations to a certain self-embedding space of M, in the sense of [KK24c] and closely related to Boavida de Brito–Weiss' theory of configuration categories [BdBW18a].

Twice-iterated stabilisation of the rationalised E_d -**operad.** The proof of Theorem F is inspired by Weiss' work on topological Pontryagin classes [Wei21] and relies on our earlier work [KK24c]. One of the crucial additional ingredients is a result on the classifying space BAut($E_{d,Q}$) of the derived automorphism group of the rationalisation of the operad E_d of little d-discs. The space of 2-ary operations of $E_{d,Q}$ is homotopy equivalent to a rational (d-1)-sphere, so acting on its top homology yields a morphism Aut($E_{d,Q}$) \rightarrow GL(Q) whose kernel we denote by SAut($E_{d,Q}$). A rational version of the additivity theorem of Dunn and Lurie yields a stabilisation map BSAut($E_{d-1,Q}$) \rightarrow BSAut($E_{d,Q}$) which we show to be nullhomotopic when iterated twice:

Theorem G. The stabilisation map $BSAut(E_{d-2,Q}) \to BSAut(E_{d,Q})$ is nullhomotopic for all $d \ge 2$. Remark.

- (i) Theorem **G** is optimal in the sense that passing to the subgroup $SAut(E_{d-2,Q}) \le Aut(E_{d-2,Q})$, rationalising, and stabilising twice instead of once are all necessary in general (see Remark 3.1).
- (ii) Theorem G is related to work of Fresse-Willwacher [FW20] and Khoroshkin-Willwacher [KW17], but our proof is independent of their results and methods (see Remark 3.9).
- (iii) The strategy of proof of Theorem G is inspired by work of Boavida de Brito and Horel [BdBH21] on a proof of formality for the E_d -operads in positive characteristic.
- (iv) As an input to the proof of Theorem G, we establish various results on tensor products of truncated ∞ -operads that may be of independent interest (see Appendix A).

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2. Preliminaries

We begin with some preliminaries on rationalisation (Section 2.1), operads (Section 2.2), and the Bousfield–Kan spectral sequence (Section 2.3).

Convention. We work in the setting of ∞ -categories as in [Lur09, Lur17] throughout. In particular, a *category* is an ∞ -category, an *operad* is an ∞ -operad, a *space* is an object in the ∞ -category of spaces \mathcal{S} , and a *pointed space* is an object in the undercategory $\mathcal{S}_{*/}$, etc.. Operations that we perform with (pointed) spaces—such as various (co)limits—are formed within these ∞ -categories. An E_1 -space is a monoid object in \mathcal{S} and an E_1 -group is a group object in \mathcal{S} . An action of an E_1 -group G on a space X is a functor $BG \to \mathcal{S}$ whose values are equivalent to X, and the (co)invariants X_G and X^G are the (co)limit of this functor.

- 2.1. **Rationalisation.** Throughout this work, by *rationalisation of spaces*, we mean Bousfield–Kan Q-completion, viewed as an endofunctor $(-)_Q \colon \mathcal{S} \to \mathcal{S}$ of the category of spaces, together with a natural transformation $\mathrm{id}_{\mathcal{S}} \to (-)_Q$. It arises by considering the adjunction between \mathcal{S} and the category $\mathrm{Mod}_Q^{\mathrm{aug}}$ of augmented of HQ-module spectra whose left adjoint is given by mapping a space X to the HQ-module $\Sigma_+^\infty X \otimes \mathrm{HQ}$ equipped with the augmentation induced by $X \to *$ and whose right adjoint maps an augmented HQ-module $E \to \mathrm{HQ}$ to the fibre $\mathrm{fib}_1(\Omega^\infty E \to \Omega^\infty \mathrm{HQ} \simeq \mathrm{Q})$. The natural transformation $\mathrm{id}_{\mathcal{S}} \to (-)_Q$ is then obtained as the limit of the coaugmented canonical resolution of the monad associated to this adjunction. A classical reference for this is [BK72, I.4.2], where the functor $(-)_Q$ is denoted $\mathrm{Q}_\infty(-)$. We call this functor simply *rationalisation*.
- 2.1.1. Fibrewise rationalisation. For a fixed space B, by fibrewise rationalisation over B, we mean the endofunctor on the overcategory $S_{/B}$ given as the composition $S_{/B} \simeq \operatorname{Fun}(B, S) \to \operatorname{Fun}(B, S) \simeq S_{/B}$ induced by postcomposition with $(-)_Q$. We denote this functor by $(-)_{fQ}$; the space B will be clear from the context. The natural transformation $\operatorname{id}_S \to (-)_Q$ of endofunctors on S induces an analogous natural transformation $\operatorname{id}_{S/B} \to (-)_{fQ}$ of endofunctors on $S_{/B}$.
- 2.1.2. Nilpotent spaces. A space X is nilpotent if for all $x \in X$ the action of $\pi_1(X, x)$ on $\pi_k(X, x)$ is nilpotent for all $k \ge 1$ in the sense of II.4.1 loc.cit.. For k = 1 this is equivalent to the group $\pi_1(X, x)$ being nilpotent for all $x \in X$. On the full subcategory of nilpotent spaces, the functor $(-)_Q$ agrees with usual rationalisation of nilpotent spaces, so the map $X \to X_Q$ induces rationalisations in the algebraic sense on all homotopy and homology groups of each path component in positive degrees, see V.3.1 loc.cit. (see V.2 loc.cit. for what it means to rationalise a nilpotent group).
- 2.1.3. Non-nilpotent spaces. If X is not nilpotent, the map $X \to X_Q$ need not induce isomorphisms on rational homotopy or homology groups, but the rational (co)homology of X_Q still contains the rational (co)homology of X as a natural summand. This because the coaugmented cosimplicial object in S arising from the monad associated to the adjunction between S and $\operatorname{Mod}_Q^{\operatorname{aug}}$ admits an extra codegeneracy after applying the left adjoint $\Sigma_+^\infty(-) \otimes \operatorname{HQ}$; see I.5.4 loc.cit.. Working over a base space B, the rational (co)homology of the total space of the fibrewise rationalisation $X_{\mathrm{fQ}} \to B$ of a map $X \to B$ similarly contains the rational (co)homology of X as a natural summand. This is because taking vertical fibrewise rationalisation of the square

$$\begin{array}{c}
X = X \\
\downarrow \qquad \downarrow \\
B \to *
\end{array}$$

gives a map $X_{fQ} \to X_Q$ from the fibrewise to the nonfibrewise rationalisation of X whose precomposition with the fibrewise rationalisation $X \to X_{fQ}$ gives the nonfibrewise rationalisation $X \to X_Q$. Since the latter has a cosection after applying $\Sigma_+^{\infty}(-) \otimes HQ$, so does the former.

2.1.4. *Inheritance properties.* The functor $(-)_Q : S \to S$ preserves coproducts, finite products, and connectivity of maps by I.7.1, I.7.2., IV.5.1 loc.cit.. In addition, as the following lemma shows, it preserves pullbacks along 1-connected maps $f : X \to Y$ that are *nilpotent* in that for all $x \in X$, the action of $\pi_1(X,x)$ on $\pi_k(\operatorname{fib}_{f(x)}(f),x)$ for every $k \ge 1$ is nilpotent in the sense of II.4.1 loc.cit..

Lemma 2.1. For a pullback square of spaces

$$\begin{array}{c} X \longrightarrow Y \\ \downarrow \qquad \downarrow \\ Z \longrightarrow W. \end{array}$$

whose bottom row is 1-connected and nilpotent, the rationalisation of the square remains a pullback. The nilpotency assumption on the bottom row is automatic if Z and W are both nilpotent.

Proof. In the case where all four spaces are connected, the claim follows from II.5.3, II.5.4 loc.cit.. To deduce the general case from this, note that since rationalisation preserves connectivity, the rationalised square has still 1-connected rows. A square with 1-connected rows is a pullback square if and only if the squares of connected spaces obtained by restriction to the components induced by points in the top-left corner are again pullback squares, so we may indeed assume that all spaces are connected. The addendum of the claim follows from II.4.5 loc.cit..

2.2. **Operads.** We adopt the conventions and notation for operads from [KK24c, Section 1.4]. As in that reference, we consider the category $\mathfrak{Op}^{\mathrm{un}}$ of unital operads (i.e. operads \mathfrak{O} with $\mathrm{Mul}_{\mathfrak{O}}(\emptyset;c) \simeq *$ for all colours c) and its truncated variants $\mathfrak{Op}^{\leq k,\mathrm{un}}$ for $1 \leq k < \infty$, fitting into a tower of categories

$$\mathfrak{O}p^{un} = \mathfrak{O}p^{\leq \infty, un} \to \cdots \to \mathfrak{O}p^{\leq 2, un} \to \mathfrak{O}p^{\leq 1, un} \simeq \mathfrak{C}at \tag{4}$$

with $\mathbb{O}\mathrm{p}^{\mathrm{un}} \simeq \lim_k \mathbb{O}\mathrm{p}^{\leq k,\mathrm{un}}$; see Section 1.4.4 loc.cit.. The composition $\mathbb{O}\mathrm{p}^{\mathrm{un}} \to \mathbb{C}\mathrm{at}$ sends an operad to its *category of colours* $\mathbb{O}^{\mathrm{col}}$. All functors in the tower $\tau \colon \mathbb{O}\mathrm{p}^{\leq j,\mathrm{un}} \to \mathbb{O}\mathrm{p}^{\leq k,\mathrm{un}}$ for $1 \leq k \leq j \leq \infty$ admit fully faithful right and left adjoints $\tau_* \colon \mathbb{O}\mathrm{p}^{\leq k,\mathrm{un}} \to \mathbb{O}\mathrm{p}^{\leq j,\mathrm{un}}$ and $\tau_! \colon \mathbb{O}\mathrm{p}^{\leq k,\mathrm{un}} \to \mathbb{O}\mathrm{p}^{\leq j,\mathrm{un}}$ (see [DL24, Thm 1.2] or [KK24c, Thm A.4] loc.cit.). We are mostly interested in the full subcategories $\mathbb{O}\mathrm{p}^{\leq k,\mathrm{red}} \subset \mathbb{O}\mathrm{p}^{\leq k,\mathrm{un}}$ of *reduced operads*, which are those (truncated) unital operads \mathbb{O} such that $\mathbb{O}^{\mathrm{col}}$ is trivial, i.e. $\mathrm{Mul}_{\mathbb{O}}(c;d) \simeq *$ for all colours c,d. Alternatively, the category $\mathbb{O}\mathrm{p}^{\leq k,\mathrm{red}}$ is the fibre fib* $(\mathbb{O}\mathrm{p}^{\leq k,\mathrm{un}} \to \mathbb{O}\mathrm{p}^{\leq 1,\mathrm{un}} \simeq \mathbb{C}\mathrm{at})$. In particular, we have $\mathbb{O}\mathrm{p}^{\leq 1,\mathrm{red}} \simeq *$. The category $\mathbb{O}\mathrm{p}^{\leq 2,\mathrm{red}}$ admits a simple description as well: taking spaces of 2-ary operations yields an equivalence $\mathbb{O}\mathrm{p}^{\leq 2,\mathrm{red}} \simeq \mathbb{S}^{\Sigma_2} = \mathrm{Fun}(\Sigma_2,\mathbb{S})$ where Σ_2 is the symmetric group in two letters (see Lemma 2.4 below).

2.2.1. Tensor products of (truncated) operads. The category of all operads Op admits a symmetric monoidal structure ⊗ which can be thought of as the ∞-categorical version of the classical Boardman-Vogt product for ordinary operads in sets [Lur17, 2.2.5.13]. By 2.1.4.6, 2.2.5.7 loc.cit., it turns Op into a presentable symmetric monoidal category in the sense of 3.4.4.1 loc.cit., so in particular the tensor products of operads preserves colimits in both variables. The unit for the monoidal structure is the trivial operad Triv (the initial object in Op) whose category of colours is the trivial category (which has a single object * with contractible endomorphism space) and all whose spaces of operations are empty except $Mul_{Triv}(*;*) \simeq *$, so in particular Triv is not unital. There is a closely related operad E_0 which only differs from Triv in that there is a unary operation $\mathrm{Mul}_{E_0}(\varnothing;*) \simeq *$. The functor $(-) \otimes E_0 \colon \mathcal{O}p \to \mathcal{O}p$ takes values in the full subcategory $\mathcal{O}p^{\mathrm{un}} \subset \mathcal{O}p$ and the resulting corestriction $(-) \otimes E_0 : \mathcal{O}p \to \mathcal{O}p^{\mathrm{un}}$ is left adjoint to the inclusion $\mathcal{O}p^{\mathrm{un}} \subset \mathcal{O}p$ (see p. 246 loc.cit.), so in particular Opun is a localisation of Op (it is also a colocalisation; see p. 246 loc.cit.). Since the localisation is given by taking tensor product with an object, it is compatible with the symmetric monoidal structure of Op in the sense of 2.2.1.7 loc.cit., so an application of 2.2.1.9 loc.cit. yields a symmetric monoidal structure on $\mathbb{O}p^{\mathrm{un}}$ such that $(-) \otimes E_0 \colon \mathbb{O}p \to \mathbb{O}p^{\mathrm{un}}$ can be lifted to a symmetric monoidal functor. As $E_0 \otimes E_0 \simeq E_0$ by 2.3.1.6 loc.cit., it follows that the tensor product of two objects in $\mathfrak{Op}^{\mathrm{un}}$ is simply given by the tensor product in Op, and that the monoidal unit in Op^{un} is E_0 instead of Triv as in Op. In Appendix A, we will show that there is an extension of this symmetric monoidal

structure to symmetric monoidal structure on the category of k-truncated operads $\mathbb{O}p^{\leq k, \mathrm{un}}$ for all k which (a) agrees for k=1 with the cartesian structure on $\mathbb{O}p^{\leq 1, \mathrm{un}} \simeq \mathbb{C}$ at, (b) is closely related to the join of spaces for k=2, and (c) for which (4) lifts to a tower of symmetric monoidal categories.

2.2.2. Dunn–Lurie additivity. Recall the operad E_d of little d-discs for $d \ge 0$ which has a single colour * and whose space of k-ary operations $E_d(k) := \operatorname{Mul}_{E_d}(\{1,\ldots,k\} \times *;*)$ is homotopy equivalent to the space of ordered configurations of k points in \mathbf{R}^d [Lur17, 5.1.0.3, 5.1.0.4, 5.1.1.3]. In particular, $E_d(k)$ is contractible for $k \le 1$, so E_d is a reduced operad. These operads are additive under taking tensor products, meaning that there are preferred equivalences $E_n \otimes E_m \simeq E_{n+m}$ [Lur17, 5.1.1.2] for $n, m \ge 0$. Moreover, there is a preferred map $t : \operatorname{BTop}(d) \to \operatorname{BAut}(E_d)$ where $\operatorname{BTop}(d)$ is the classifying space of the topological group of homeomorphisms of \mathbf{R}^d [KK24c, Section 5.1.1] which turns out to be compatible with additivity in that there exists a commutative square of spaces

$$BTop(n) \times BTop(m) \xrightarrow{(t,t)} BAut(E_n) \times BAut(E_m)$$

$$\downarrow^{\times} \qquad \qquad \downarrow^{\otimes}$$

$$BTop(n+m) \xrightarrow{t} BAut(E_{n+m})$$
(5)

whose vertical arrows are induced by taking direct products and tensor products respectively.

Lemma 2.2. There exists a commutative square as in (5).

Proof. Recall from [KK24c, Theorem 2.2] that taking colimits in \mathfrak{Op} of functors $\theta \colon X \to \mathfrak{Op}$ where X is a groupoid and θ has values in reduced operads, induces an equivalence between the cartesian unstraightening of the functor Fun $(-, \mathfrak{Op}^{\mathrm{red}}) \colon \mathcal{S}^{\mathrm{op}} \to \mathcal{C}$ at and the pullback $(-)^{\mathrm{col}} \colon \mathfrak{Op}^{\mathrm{gc}} \to \mathcal{S}$ of $(-)^{\mathrm{col}} \colon \mathfrak{Op}^{\mathrm{un}} \to \mathcal{C}$ at along $\mathcal{S} \subset \mathcal{C}$ at. Using this equivalence and viewing $\mathrm{BAut}(E_n)$ as a subcategory of \mathfrak{Op} , it suffices to construct an equivalence of operads

$$\operatorname{colim} \left(\operatorname{BTop}(n) \times \operatorname{BTop}(m) \xrightarrow{(t,t)} \operatorname{Op} \times \operatorname{Op} \xrightarrow{\otimes} \operatorname{Op} \right) \simeq \operatorname{colim} \left(\operatorname{BTop}(n) \times \operatorname{BTop}(m) \xrightarrow{\times} \operatorname{BTop}(n+m) \xrightarrow{t} \operatorname{Op} \right)$$

which upon applying $(-)^{\operatorname{col}}$ yields $\operatorname{id}_{\operatorname{BTop}(n)\times\operatorname{BTop}(m)}$. Since the tensor product preserves colimits in both variables, the left-hand side agrees with $\operatorname{colim}(\operatorname{BTop}(n) \to \operatorname{Op}) \otimes \operatorname{colim}(\operatorname{BTop}(m) \to \operatorname{Op})$. Moreover, $\operatorname{colim}(\operatorname{BTop}(n) \to \operatorname{Op})$ agrees, by definition of the map $t \colon \operatorname{BTop}(n) \to \operatorname{BAut}(E_n)$, with the operad denoted $\mathbb{E}^{\otimes}_{\operatorname{BTop}(n)}$ in [Lur17, 5.4.2.10] (see [KK24c, Section 5.1.1] where this operad is denoted E_n^t), and by a similar argument as in the proof of Lemma 2.7 loc.cit. the right-hand colimit in the asserted equivalence agrees with $\mathbb{E}^{\otimes}_{\operatorname{BTop}(n)\times\operatorname{BTop}(m)}$, so it suffices to establish an equivalence

$$\mathbb{E}_{\mathrm{BTop}(n)}^{\otimes} \otimes \mathbb{E}_{\mathrm{BTop}(m)}^{\otimes} \simeq \mathbb{E}_{\mathrm{BTop}(n) \times \mathrm{BTop}(m)}^{\otimes}$$

that yields $id_{BTop(n)\times BTop(m)}$ on categories of colours. This follows from [Lur17, 5.4.2.14].

2.2.3. *Dendroidal models*. In addition to Lurie's model for operads, we will also rely on the *dendroidal model* for operads. We now summarise the relevant points. Writing Op for the ordinary 1-category of coloured operads in sets in the classical sense, there is a fully faithful inclusion [Lur17, 2.1.1.7]

$$\ell \colon \mathsf{Op} \longrightarrow \mathfrak{Op}.$$

The category Op has a full subcategory $\Omega \subset \text{Op}$, the *dendroidal tree category*, which is spanned by operads encoded by finite rooted trees (see [HM22, Sections 1.3 and 3.2]). There is also an intermediate full subcategory $\Omega \subset \Phi \subset \text{Op}$, obtained from Ω by formally adjoining finite coproducts, which can be thought of as finite forests of finite rooted trees (see [HM24, Section 2.2.2]). The Yoneda embedding of $\mathbb{O}p$ followed by restriction along $\ell|_{\Phi}$ yields a functor $\delta_{\Phi} \colon \mathbb{O}p \to \text{PSh}(\Phi)$. By [HM24, Theorem 3.1.4] this functor is fully faithful and its essential image $\text{PSh}(\Phi)^{\text{seg.}c} \subset \text{PSh}(\Phi)$ (denoted DOp in loc.cit.) is the category of *complete dendroidal Segal spaces*, which is the full subcategory on those presheaves that satisfies three conditions: a completeness one, a dendroidal Segal one, and a forest Segal one (see (D1)–(D3) in Section 2.2.3 loc.cit.). We obtain a left-hand equivalence in

$$\delta_{\Phi} \colon \mathfrak{Op} \xrightarrow{\simeq} \mathrm{PSh}(\Phi)^{\mathrm{seg},c}, \qquad \qquad \delta_{\Omega} \colon \mathfrak{Op} \xrightarrow{\simeq} \mathrm{PSh}(\Omega)^{\mathrm{seg},c}.$$

As indicated by the right-hand equivalence, one can avoid using forests and only work with Ω : this right-hand equivalence is given similarly by the Yoneda embedding of $\mathbb{O}p$ followed by restriction along $\ell|_{\Omega}$. It differs from the right-hand equivalence by postcomposition with the restriction $\mathrm{PSh}(\Phi) \to \mathrm{PSh}(\Omega)$ along $\Omega \subset \Phi$, which restricts—by design of the forest Segal condition—to an equivalence $\mathrm{PSh}(\Phi)^{\mathrm{seg},c} \simeq \mathrm{PSh}(\Omega)^{\mathrm{seg},c}$ to the full subcategory of presheaves on Ω that only satisfy the completeness and dendroidal Segal condition (see (D1) and (D2) in loc.cit.). Note that, since ℓ is fully faithful, the compositions $\delta_{\Phi} \circ \ell|_{\Phi}$ and $\delta_{\Omega} \circ \ell|_{\Omega}$ agree with the respective Yoneda embeddings.

To model the subcategory of unital operads $\mathfrak{Op}^{\mathrm{un}}\subset \mathfrak{Op}$ and the tower (4), one considers the full subcategory $\overline{\Omega}\subset \Omega$ of closed trees (see $[\mathrm{HM22},\mathrm{p.92},97]$). Imposing a dendroidal Segal and a completeness condition yields a full subcategory $\mathrm{PSh}(\overline{\Omega})^{\mathrm{seg.}c}\subset \mathrm{PSh}(\overline{\Omega})$ which is equivalent to $\mathfrak{Op}^{\mathrm{un}}$ such that the inclusion $\mathfrak{Op}^{\mathrm{un}}\subset \mathfrak{Op}$ corresponds via the equivalence δ_Ω to the functor $\mathrm{PSh}(\overline{\Omega})^{\mathrm{seg.}c}\to \mathrm{PSh}(\Omega)^{\mathrm{seg.}c}$ induced by left Kan extension along $\overline{\Omega}\subset \Omega$ (see $[\mathrm{KK24c},\mathrm{Sections\ A.1}]$ for a summary). For the tower (4), one considers the sequence of full subcategories $\overline{\Omega}_{\leq 1}\subset \overline{\Omega}_{\leq 2}\subset \cdots \overline{\Omega}_{\leq \infty}=\overline{\Omega}$ where $\overline{\Omega}_{\leq k}\subset \overline{\Omega}$ are those closed trees whose vertices have all $\leq k$ incoming edges. Imposing the Segal and completeness condition gives full subcategories $\mathfrak{Op}^{\leq k,\mathrm{un}}\coloneqq \mathrm{PSh}(\overline{\Omega}_{\leq k})^{\mathrm{seg.}c}\subset \mathrm{PSh}(\overline{\Omega}_{\leq k})$ which are preserved by restriction along the inclusions $\overline{\Omega}_{\leq k-1}\subset \overline{\Omega}_{\leq k}$ as well as left and right Kan extension along them (see $[\mathrm{DL}24]$ and $[\mathrm{KK}24c,\mathrm{Sections\ A.2}]$). The resulting tower yields (4) and the fully faithful left and right adjoints are given by the left and right Kan extensions, respectively. Since left Kan extension preserves representables, this in particular implies that the inclusion $\ell|_{\overline{\Omega}_{\leq k}}\colon \overline{\Omega}_{\leq k}\hookrightarrow \mathfrak{Op}^{\mathrm{un}}$ factors over the left adjoint $\tau_!\colon \mathfrak{Op}^{\leq k,\mathrm{un}}\hookrightarrow \mathfrak{Op}^{\mathrm{un}}$ to truncation.

2.2.4. Rationalisation of reduced operads. In terms of the dendroidal model, the full subcategory of reduced operads $\mathfrak{O}\mathsf{p}^{\leq k,\mathrm{red}}\subset \mathfrak{O}\mathsf{p}^{\leq k,\mathrm{un}}\subset \mathrm{PSh}(\overline{\Omega}_{\leq k})$ corresponds to those presheaves on $\overline{\Omega}_{\leq k}$ that satisfy the dendroidal Segal condition and whose values at the closed n-corolla \overline{C}_n (the unique closed tree with 1 internal and n external vertices) is the terminal object for n=0,1. The completeness condition is automatic in this case and the pullback that appears in the dendroidal Segal condition becomes a product. In particular, for any endofunctor $\varphi\colon \mathbb{S}\to\mathbb{S}$ that preserves finite products, the functor $\varphi_*\colon \mathrm{PSh}(\overline{\Omega}_{\leq k})\to \mathrm{PSh}(\overline{\Omega}_{\leq k})$ given by postcomposition preserves the full subcategory of reduced operads, so it induces an endofunctor $\varphi_*\colon \mathbb{O}\mathsf{p}^{\leq k,\mathrm{red}}\to \mathbb{O}\mathsf{p}^{\leq k,\mathrm{red}}$ by restriction, compatible with truncation. Moreover, if it φ receives a natural transformation id $_{\mathbb{S}}\to\varphi$ from the identity then so does φ_* , compatible with truncation. The component $\mathbb{O}\to\varphi_*(\mathbb{O})$ of this natural transformation is on spaces of multi-operations given by the respective component of the natural transformation id $_{\mathbb{S}}\to\varphi$. Applying this to the rationalisation functor $\varphi=(-)_{\mathbb{Q}}$ from Section 2.1, this allows us to rationalise reduced k-truncated operads for any $1\leq k\leq \infty$, compatible with truncation.

2.2.5. Decomposing maps of operads. In view of the equivalence $\mathfrak{Op}^{\mathrm{un}} \simeq \lim_k \mathfrak{Op}^{\leq k,\mathrm{un}}$, the mapping space $\mathrm{Map}(\mathfrak{O}, \mathfrak{P}) := \mathrm{Map}_{\mathfrak{Op}^{\mathrm{un}}}(\mathfrak{O}, \mathfrak{P})$ for unital operads $\mathfrak{O}, \mathfrak{P} \in \mathfrak{Op}^{\mathrm{un}}$ agrees with the limit of the tower of mapping spaces

$$\cdots \longrightarrow \operatorname{Map}_{\leq k}(\mathcal{O}, \mathcal{P}) \longrightarrow \operatorname{Map}_{\leq k-1}(\mathcal{O}, \mathcal{P}) \longrightarrow \cdots \longrightarrow \operatorname{Map}_{\leq 1}(\mathcal{O}, \mathcal{P}) \simeq \ast$$

for the corresponding truncated operads; here $\operatorname{Map}_{\leq k}(\mathcal{O}, \mathcal{P}) := \operatorname{Map}_{\mathcal{O}_p^{\leq k, \operatorname{un}}}(\mathcal{O}, \mathcal{P})$. In the case where \mathcal{O} and \mathcal{P} are reduced, Göppl and Weiss [GW24] gave a convenient description of the fibres

$$L_k^f(\mathcal{O}, \mathcal{P}) := \mathrm{fib}_f \big(\mathrm{Map}_{< k}(\mathcal{O}, \mathcal{P}) \to \mathrm{Map}_{< k-1}(\mathcal{O}, \mathcal{P}) \big) \quad \text{for } f \in \mathrm{Map}_{< k-1}(\mathcal{O}, \mathcal{P}). \tag{6}$$

To state a suitable version of their result, we consider the commutative square

$$\begin{array}{c} \operatorname{PSh}(\overline{\Omega}_{\leq k}) \xrightarrow{\left(\operatorname{Latch}_{k}(-) \to (-)(k) \to \operatorname{Match}_{k}(-)\right)} \operatorname{PSh}(\Sigma_{k})^{[2]} \\ \downarrow^{\operatorname{res}} & \downarrow^{(0 \leq 2)^{*}} \\ \operatorname{PSh}(\overline{\Omega}_{\leq k-1}) \xrightarrow{\left(\operatorname{Latch}_{k}(-) \to \operatorname{Match}_{k}(-)\right)} \operatorname{PSh}(\Sigma_{k})^{[1]} \end{array}$$

of categories. Here the left vertical arrow is given by restriction. The category $\mathrm{PSh}(\Sigma_k)^{[2]}$ is the category of sequences $X \to Y \to Z$ of spaces with an action of the symmetric group Σ_k . The latter maps to the category $\mathrm{PSh}(\Sigma_k)^{[1]}$ of maps $X \to Z$ of Σ_k -spaces by composition; this is the right vertical functor. For $\emptyset \in \mathrm{PSh}(\overline{\Omega}_{\leq k-1})$, the kth latching object $\mathrm{Latch}_k(\emptyset)$ is the Σ_k -space given by left Kan extending \emptyset to $\overline{\Omega}_{\leq k}$ and then evaluating at the k-corolla $\overline{C}_k \in \overline{\Omega}_{\leq k}$ with its Σ_k -action by permuting the k leaves. The matching object $\mathrm{Match}_k(X)$ is defined analogously using right Kan extensions. The universal property of Kan extensions gives a map $\mathrm{Latch}_k(X) \to \mathrm{Match}_k(X)$ of Σ_k -spaces; this gives the bottom horizontal functor. If X is the restriction of a presheaf of $\mathrm{PSh}(\overline{\Omega}_{\leq k})$ then the universal property of Kan extensions gives a factorisation of the map of Σ_k -spaces $\mathrm{Latch}_k(X) \to \mathrm{Match}_k(X)$ through the value $\emptyset(k) \coloneqq \emptyset(\overline{C}_k)$ at the k-corolla (which corresponds to the space of k-ary operations if \emptyset is a reduced operad); this defines the top horizontal functor.

By Theorem 3.2.7 loc.cit., this square of categories induces pullbacks on mapping spaces between reduced operads. By taking vertical fibres in the pullback on mapping spaces, it follows that the fibre (6) is for $\mathbb{O}, \mathbb{P} \in \mathbb{O}p^{\leq k, \mathrm{red}}$ naturally equivalent to the space of Σ_k -equivariant dashed fillers in

$$\begin{array}{ccc} \operatorname{Latch}_{k}(\mathbb{O}) & \xrightarrow{f_{*}} & \operatorname{Latch}_{k}(\mathbb{P}) \\ & \downarrow & & \downarrow \\ \mathbb{O}(k) & ----- & \mathbb{P}(k) \\ & \downarrow & & \downarrow \\ \operatorname{Match}_{k}(\mathbb{O}) & \xrightarrow{f_{*}} & \operatorname{Match}_{k}(\mathbb{P}), \end{array}$$

so it fits into a fibre sequence of the form

$$L_k^f(\mathcal{O}, \mathcal{P}) \longrightarrow \mathrm{Map}_{\mathrm{Latch}_k(\mathcal{O})}(\mathcal{O}(k), \mathcal{P}(k))^{\Sigma_k} \longrightarrow \mathrm{Map}_{\mathrm{Latch}_k(\mathcal{O})}(\mathcal{O}(k), \mathrm{Match}_k(\mathcal{P}))^{\Sigma_k} \tag{7}$$

where $\operatorname{Map}_{\operatorname{Latch}_k(\mathcal{O})}(\mathcal{O}(k), \operatorname{Match}_k(\mathcal{P}))$ is the space of maps $\mathcal{O}(k) \to \operatorname{Match}_k(\mathcal{P})$ extending the map $\operatorname{Latch}_k(\mathcal{O}) \to \operatorname{Match}_k(\mathcal{P})$ (based at the composition of $\mathcal{O}(k) \to \operatorname{Match}_k(\mathcal{O})$ with $f_* \colon \operatorname{Match}_k(\mathcal{O}) \to \operatorname{Match}_k(\mathcal{P})$), the space $\operatorname{Map}_{\operatorname{Latch}_k(\mathcal{O})}(\mathcal{O}(k), \mathcal{P}(k))$ is the space of maps $\mathcal{O}(k) \to \mathcal{P}(k)$ extending the map $\operatorname{Latch}_k(\mathcal{O}) \to \mathcal{P}(k)$, and $(-)^{\Sigma_k}$ denotes taking Σ_k -invariants.

2.2.6. Matching and latching objects. For reduced operads $\mathbb O$, Göppl and Weiss also gave a simpler description of $\mathrm{Latch}_k(\mathbb O)$ and $\mathrm{Match}_k(\mathbb O)$: For the latching object, one considers the poset (Ψ_k,\leq) of finite rooted trees T with k leaves labeled by $\underline k=\{1,\ldots,k\}$ and no vertex of valence 2, where $T\leq T'$ if T' can be obtained from T by collapsing an interior forest. This has a subposet (Ψ_k^-,\leq) on those labeled trees with more than one interior vertex (i.e. those that are not equivalent to the k-corolla). There is an evident way of how to identify (Ψ_k,\leq) with a subcategory of the overcategory $\overline{\Omega}_{/\overline{C}_k}$ (the labelings of the leaves describe maps to \overline{C}_k using the identification of the leaves of \overline{C}_k with k), so we get a map $\mathrm{colim}_{T\in\Psi_k^-}\mathbb O(T)\to \mathbb O(\overline{C}_k)=\mathbb O(k)$ which the argument in $[\mathrm{GW}24,\mathrm{Example}\ 3.2.6]$ shows to be naturally equivalent to $\mathrm{Latch}_k(\mathbb O)\to \mathbb O$. Note that Ψ_k^- is empty for $k\leq 2$, so $\mathrm{Match}_k(\mathbb O)=\emptyset$ for $k\leq 2$. For the matching object, one uses that any $S\subseteq \underline k$ defines a map $\overline{C}_{|S|}\to \overline{C}_k$ in $\overline{\Omega}$, so there is a $\underline k$ -cubical diagram $\underline k\supseteq S\mapsto \mathbb O(\overline{C}_{|S|})$. By the argument of $[\mathrm{Wei}21,\mathrm{Lemma}\ 3.4.7]$, the resulting map $\mathbb O(k)=\mathbb O(\overline{C}_k)\to \overline{\lim}_{S\subseteq k}\mathbb O(\overline{C}_{|S|})$ is equivalent to the map $\mathbb O(k)\to \mathrm{Match}_k(\mathbb O)$.

Example 2.3. Using this, Göppl and Weiss [GW24, Examples 2.1.6, 3.2.6] showed that for the little d-discs operad $\mathcal{O} = E_d$, the map $\operatorname{Latch}_k(\mathcal{O}) \to \mathcal{O}(k)$ is equivalent to the boundary inclusion of an explicit compact manifold of dimension (k-1)d-1 whose boundary is empty if and only if $k \leq 2$.

We conclude this section by justifying the aforementioned description of the category $Op^{\leq 2, \text{red}}$.

Lemma 2.4. The functor $(-)(2): \mathfrak{Op}^{\leq 2, \text{red}} \to \mathbb{S}^{\Sigma_2}$ that takes 2-ary operations is an equivalence.

Proof. Using $\mathfrak{Op}^{\leq 1,\mathrm{red}} \simeq *$, $\mathrm{Latch}_2(\mathfrak{O}) = \emptyset$, and $\mathrm{Match}_2(\mathfrak{O}) = *$, the description of the fibres of the map on mapping spaces induced by $\mathfrak{Op}^{\leq 2,\mathrm{red}} \to \mathfrak{Op}^{\leq 1,\mathrm{red}}$ from Section 2.2.5 shows that the functor in the claim is fully faithful. Essential surjectivity is equivalent to the functor $(-)(2): \mathfrak{Op}^{\mathrm{red}} \to \mathfrak{S}^{\Sigma_2}$ on nontruncated operads being essentially surjective, i.e. that any Σ_2 -space X arises as the space of

2-ary operations of a reduced operad. Modelling X by a Kan-complex with a strict Σ_2 -action, there is a 1-coloured operad in Kan complexes in the classical sense whose *n*-ary operations are given by $\prod_{J\subseteq n, |J|=2} X$. Its operadic nerve (see [Lur17, 2.1.1.27]) yields an operad as required.

2.3. The Bousfield-Kan spectral sequence of a tower. Given a tower of pointed spaces

$$\cdots \to X_s \to X_{s-1} \to \cdots \to X_1 \to X_0 \longrightarrow X_{-1} \simeq *,$$

the homotopy groups of the limit $X := \lim_s X_s$ can be studied by a spectral sequence known as the Bousfield-Kan spectral sequence [BK72, IX.4], which is natural in maps of towers of pointed spaces. Its entries are defined for $t \ge s \ge 0$, starting on the first page $E_1^{s,t}$. The differentials are of the form $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$, so decrease t-s by 1. The first page is given by

$$E_1^{s,t} = \pi_{t-s}(F_s)$$
 where $F_s := \text{fib}(X_s \to X_{s-1})$.

Note that these entries are not always abelian groups: they are possibly non-abelian groups for t-s=1, and only pointed sets for t-s=0. In general, the spectral sequence is an extended spectral sequence in the sense of IX.4.2 loc.cit., which means that:

- (i) E_r^{s,t} is a pointed set for t − s = 0, a group for t − s ≥ 1, and an abelian group for t − s ≥ 2.
 (ii) There is an action of the group E_r^{s-r,s-r+1} on the pointed set E_r^{s,s}.
 (iii) For t − s ≥ 2 the differential d_r: E_r^{s,t} → E_r^{s+r,t+r-1} is a group homomorphism which lands in the centre of the target for t-s=2. For t-s=1, the differential is given by acting on the basepoint using the action of $E_r^{s-r,s-r+1}$ on $E_r^{s,s}$.
- (iv) We have $E_r^{s,t} = \ker(d_r)/\operatorname{im}(d_r)$ for $t s \ge 1$, and $E_{r+1}^{s,s} \subset E_r^{s,s}/(\operatorname{action of } E_r^{s-r,s-r+1})$.

General convergence properties of this spectral sequence are discussed in IX.5 loc.cit.. We will only need a special case which we describe now. Note that there are only finitely many differentials into each entry since $E_r^{s,t}$ is only defined for $t \ge s \ge 0$, so we have $E_{r+1}^{s,t} \subset E_r^{s,t}$ for fixed $0 \le s \le t$ and all $r \gg 0$. If for a fixed $i \ge 1$ we have $E_r^{s,s+i} = E_{r+1}^{s,s+i}$ for each $s \ge 0$ and $r \ge N(s)$ for some $N(s) \ge 0$, then one says that the spectral sequence Mittag-Leffler converges in degree i (see IX.5.5 loc.cit.). In this case, the tower of surjections of groups

$$\cdots \twoheadrightarrow \pi_i(X)_1 \twoheadrightarrow \pi_i(X)_0 \twoheadrightarrow \pi_i(X)_{-1} = * \text{ given by } \pi_i(X)_s := \operatorname{im}(\pi_i(X) \to \pi_i(X_s))$$
 (8)

satisfies

$$\pi_i(X) \cong \lim_s \pi_i(X)_s$$
 and $E^{s,s+i}_\infty \cong \ker \left(\pi_i(X)_s \twoheadrightarrow \pi_i(X)_{s-1}\right)$ for $s \ge 0$

where $E_{\infty}^{s,s+i} := E_r^{s,s+i}$ for $r \ge N(s)$ (see IX.5.3 loc.cit.).

Remark 2.5. In practice, for fixed $i \ge 1$ the entries $E_1^{s,s+i}$ and $E_1^{s,s+i-1}$ are often zero for $s \gg 0$, and hence the same holds for $E_r^{s,s+i}$ and $E_r^{s,s+i-1}$ for all $1 \le r \le \infty$. This in particular implies Mittag-Leffler convergence in degree i and that the tower of surjections (8) is eventually constant. In particular, by setting $\pi_i(X)^s := \ker(\pi_i(X) \twoheadrightarrow \pi_i(X)_s)$, this property implies that $\pi_i(X)$ has a finite normal series $\{e\} = \pi_i(X)^N \triangleleft \cdots \triangleleft \pi_i(X)^{-1} = \pi_i(X)$ with quotients $\pi_i(X)^{s-1}/\pi_i(X)^s \cong E_\infty^{s,s+i}$.

We now discuss three examples of this spectral sequence that will play a role in this work.

Example 2.6. Given an E_1 -group G acting on a pointed space X, the Postnikov tower $(\cdots \to X_{\leq 1} \to X_{\leq 1})$ $X_{\leq 0} \to X_{\leq -1} \simeq *)$ of X gives a limit decomposition of the fixed points $X^G \simeq \lim_k (X_{\leq k})^G$. In this case $F_s \simeq K(\pi_s(X), s)^G$, so the E_1 -page of the associated spectral sequence satisfies

$$E_1^{s,t} \cong H^{2s-t}(BG; \pi_s(X))$$
 for $t \ge s \ge 0$.

These cohomology groups (and pointed sets) are interpreted to be 0 for 2s < t. In the case s = 0, 1where the coefficients need not be abelian groups, the possibly nontrivial entries are $H^1(BG; \pi_1(X))$, $H^0(BG; \pi_1(X))$, and $H^0(BG; \pi_0(X))$, which are defined as the first nonabelian cohomology of $\pi_0(G)$ acting on $\pi_1(X)$, the fixed point group of this action, and the fixed point set of the action on the pointed set $\pi_0(X)$ respectively. There are two situations in which we are going to apply this spectral sequence: in the first BG is finite dimensional and in the second G is discrete and finite and $\pi_s(X)$

admits for $s \ge 1$ a G-invariant finite normal series whose consecutive quotients are Q-vector space. In both cases $H^k(BG; \pi_s(X))$ vanishes for large enough k and s (by an induction on the length of the normal series in the second case), so the spectral sequences in particular converge in all degrees $i \ge 1$ in the strong sense of Remark 2.5.

Example 2.7. Given a sequence of maps $A \to X \to Y$ between nonempty spaces, we consider the space $\operatorname{Map}_A(X,Y)$ of maps $X \to Y$ that extend the composition $A \to Y$, i.e. the fibre at $A \to Y$ of the map $\operatorname{Map}(X,Y) \to \operatorname{Map}(A,Y)$ induced by precomposition with $A \to X$. This is pointed by the given map $X \to Y$. Applying $\operatorname{Map}_A(X,-)$ to the Postnikov tower $(\cdots \to Y_{\leq 1} \to Y_{\leq 0} \to Y_{\leq -1} \simeq *)$ of Y provides a limit decomposition $\operatorname{Map}_A(X,Y) \simeq \lim \left(\cdots \to \operatorname{Map}_A(X,Y_{\leq 0}) \to \operatorname{Map}_A(X,Y_{\leq -1}) \simeq * \right)$ as pointed spaces. For simplicity, we assume that Y is 1-connected. In this case, by obstruction theory, the E_1 -page of the Bousfield–Kan spectral sequence is given by

$$E_1^{s,t} = \pi_{t-s} \big(\operatorname{fib}(\operatorname{Map}_A(X, Y_{\leq s}) \to \operatorname{Map}_A(X, Y_{\leq s-1})) \big) \cong \operatorname{H}^{2s-t}(X/A; \pi_s(Y)) \quad \text{for } t \geq s \geq 0.$$

Here X/A is the cofibre of $A \to X$ and the groups $H^{2s-t}(X/A; \pi_s(Y))$ are interpreted to be 0 for $s \le 1$ or 2s - t < 0. This identification can be seen by using the fibre sequence $Y_{\le s} \to Y_{\le s-1} \to K(\pi_s(Y), s+1)$ involving the sth k-invariant. In particular, if X/A is homologically finite-dimensional, then the spectral sequence converges in all degrees $i \ge 1$ in the strong sense of Remark 2.5.

Example 2.8. Given a map $A \to X$ to a 1-connected space X, we consider the E_1 -group $\operatorname{Aut}_A(X)$ of automorphisms of X under A, i.e. the fibre of the map $\operatorname{Aut}(X) \to \operatorname{Map}(A,X)$ at the given map $A \to X$. Postnikov truncating X gives a limit decomposition as E_1 -groups $\operatorname{Aut}_A(X) \simeq \lim_k \operatorname{Aut}_A(X_{\leq k})$ which deloops to a limit decomposition $\operatorname{BAut}_A(X) \simeq (\lim_k \operatorname{BAut}_A(X_{\leq k}))_{\mathrm{id}}$ where $(-)_{\mathrm{id}}$ denotes the basepoint component. To identify the E_1 -page of the associated Bousfield–Kan spectral sequence, it is convenient to pass to the subgroup $\operatorname{SAut}_A(X) \subset \operatorname{Aut}_A(X)$ of those homotopy automorphisms that induce the identity on all homotopy groups of X, which has a similar limit decomposition as pointed spaces $\operatorname{BSAut}_A(X) \simeq \lim(\cdots \to \operatorname{BSAut}_A(X_{\leq 0}) \to \operatorname{BSAut}_A(X_{\leq -1}) \simeq *)_{\mathrm{id}}$. The E_1 -page of the Bousfield–Kan spectral sequence of this tower satisfies

$$E_1^{s,t} \cong \begin{cases} \text{pointed subset of H}^{s+1}(X_{\leq s-1}/A; \pi_s X) & \text{for } t-s=0, \text{ as pointed sets} \\ H^{2s-t+1}(X_{\leq s-1}/A; \pi_s X) & \text{for } t-s \geq 1, \text{ as groups.} \end{cases}$$
(9)

This identification becomes clearest when being deduced from a more general statement: fixing a space A and an integer $s \geq 2$, we write $\mathcal{S}_{A/}^{\leq s,1\text{-ctd},\simeq} \subset \mathcal{S}_{A/}^{\simeq}$ for the full subcategory of the core of the category of spaces under A on those maps $A \to X$ for which X is s-truncated and 1-connected. Writing $\mathcal{A}b^{\simeq}$ for the core of the category of abelian groups, there is a fibre sequence $\operatorname{Map}_A(Z,K(M,s+1)) \to \mathcal{S}_{A/}^{\leq s,1\text{-ctd},\simeq} \to \mathcal{S}_{A/}^{\leq s-1,1\text{-ctd},\simeq} \times \mathcal{A}b^{\simeq}$ in which the fibre is taken over $(A \to Z,M) \in \mathcal{S}_{A/}^{\leq s-1,1\text{-ctd}} \times \mathcal{A}b$. Here $\operatorname{Map}_A(Z,K(M,s+1))$ is the space of maps $Z \to K(M,s+1)$ that agree with the constant map on A, the first map is given by taking fibres, and the second map sends $A \to Y$ to the composition $A \to Y \to Y_{\leq s-1}$ in the first argument and to $\pi_s(Y)$ in the second (see [BDG04, Remark 3.9] for a non- ∞ -categorical reference in the case A = * and [Pst23, Section 2] for an ∞ -categorical reference in the case $A = \emptyset$; the argument for general A is similar). Restricting components and taking fibres at $(\pi_i(X))_{i\geq 1}$ of the functors to $Ab^{\times s}$ given by taking homotopy groups, we obtain a fibre sequence $\operatorname{Map}_A(X_{\leq s-1}, K(\pi_s(X), s+1))_{X_{\leq s}} \to \operatorname{BSAut}_A(X_{\leq s}) \to \operatorname{BSAut}_A(X_{\leq s-1})$ where $\operatorname{Map}_A(X_{\leq s-1}, K(\pi_s(X), s+1))_{X_{\leq s}} \subset \operatorname{Map}_A(X_{\leq s-1}, K(\pi_s(X), s+1))$ is the collection of those components of maps $f: X_{\leq s-1} \to K(\pi_s(X), s+1)$ that are trivialised on A and are such that the fibre inclusion fib $(f) \to X_{\leq s-1}$ is equivalent, as a map under A, to the truncation $X_{\leq s} \to X_{\leq s-1}$. Taking homotopy groups gives the identification (9). Now note that $E_1^{s,s+k} = H^{s-k+1}(X_{\leq s-1}/A; \pi_s(X))$ injects for $k \ge 1$ into $H^{s-k+1}(X/A; \pi_s(X))$ since $X \to X_{\le s-1}$ is s-connected. Consequently, if X/A is homologically finite-dimensional, these groups vanish for $s \gg 0$, so the spectral sequence converges in degrees $i \ge 2$ in the strong sense of Remark 2.5. In dimension i = 1, the situation is more subtle since the pointed sets $E_1^{s,s} \subset H^{s+1}(X_{\leq s-1}/A; \pi_s X)$ could be nontrivial for infinitely many s. However, one still gets that $\pi_0(\operatorname{SAut}_A(X_{\leq s})) = \pi_1(\operatorname{BSAut}_A(X_{\leq s}))$ admits for each fixed finite $s \geq 1$ a finite

normal series with quotients isomorphic to $H^s(X_{\leq s-1}/A;\pi_s(X)) \leq H^s(X/A;\pi_s(X))$ for varying s. Often this is good enough, since $\pi_0(\operatorname{SAut}_A(X)) \to \pi_0(\operatorname{SAut}_A(X_{\leq s}))$ is injective for $s \gg 0$ by obstruction theory if X/A is homologically finite-dimensional.

3. Twice-iterated stabilisation for $Aut(E_{d,O})$

As mentioned in the final part of the introduction, a crucial ingredient for the proof of our main results is that the twice-iterated stabilisation map for the group of orientation preserving automorphisms of the rationalised E_d -operad is nullhomotopic. Let us now be more precise about this. Recall that the space of k-ary operations of the little d-discs operad E_d is equivalent to the ordered configuration space $F_k(\mathbf{R}^d)$ (see Section 2.2.2). The latter is contractible for $k \leq 1$, so E_d is a reduced operad and we may apply the rationalisation procedure described in Section 2.2.4 to define $E_{d,Q} := (E_d)_Q$. As $F_2(\mathbf{R}^d)$ is equivalent to a (d-1)-sphere, the space of binary operations in $E_{d,Q}$ is equivalent to a rational (d-1)-sphere, so acting on its top homology gives a morphism

$$\chi : \operatorname{Aut}(E_{d,Q}) \longrightarrow \operatorname{Aut}(\widetilde{\operatorname{H}}_{d-1}(E_{d,Q}(2))) \cong \operatorname{GL}(Q),$$

whose kernel we denote as $SAut(E_{d,Q}) \leq Aut(E_{d,Q})$. We will see shortly that there is a stabilisation E_1 -map $Aut(E_{d-1,Q}) \to Aut(E_{d,Q})$ which is compatible with the map χ , so it restricts to an E_1 -map $SAut(E_{d-1,Q}) \to SAut(E_{d,Q})$. The purpose of this section is to prove Theorem G, which says that its twice-iterate $SAut(E_{d-2,Q}) \to SAut(E_{d,Q})$ is nullhomotopic as an E_1 -map.

Remark 3.1. Theorem **G** is optimal in the following three senses.

- (i) The restriction to the kernel $SAut(E_{d,Q}) \leq Aut(E_{d,Q})$ is necessary since the morphism $\chi \colon Aut(E_{d,Q}) \to GL(Q)$ is compatible with stabilisation and surjective (see Section 3.1).
- (ii) The *single* E_1 -stabilisation map $SAut(E_{d-1,Q}) \to SAut(E_{d,Q})$ is often *not* null: for any odd $d = 2n + 1 \ge 3$ the composition (involving the evident action of O(d) on E_d)

$$\mathsf{BSO}(2n) \to \mathsf{BSAut}(E_{2n,\mathbb{Q}}) \to \mathsf{BSAut}(E_{2n+1,\mathbb{Q}}) \to \mathsf{BSAut}(E_{2n+1,\mathbb{Q}}(2)) \simeq \mathsf{BSAut}(S^{2n})_{\mathbb{Q}}$$

pulls back the Hopf class $E \in H^{4n}(BSAut(S^{2n}); \mathbb{Q})$ to the (nontrivial) square of the Euler class $p_n = e^2 \in H^{4n}(BSO(2n); \mathbb{Q})$ (see e.g. [Sul05, p. 95]), so the second map is not null.

- (iii) There is a version $SAut(E_{d-2}) \to SAut(E_d)$ of the double stabilisation map between the unrationalised E_d -operads which is compatible with the version for the rationalised operads. The nonrationalised version can be seen to often be *not* nullhomotopic, by precomposing it with the map $BSO(d-2) \to BSAut(E_{d-2})$ and postcomposing it with the composition $BSAut(E_d) \to BSAut(S^{d-1}) \to BSAut_*(S^d)$ induced by restriction to arity 2 and unreduced suspension. The resulting map $BSO(d-2) \to BSAut_*(S^d)$ induces the appropriate unstable J-homomorphism on homotopy groups, so it is often nontrivial.
- 3.1. Strategy of proof. The first step to proving Theorem G is to construct a dashed map in

making this a commutative diagram of E_1 -groups for $n, m \ge 1$. Here the maps labeled $(-)_Q$ are induced by the rationalisation functor from Section 2.2.4, the leftmost vertical map is induced by the tensor product of operads from Section 2.2.1 and the additivity equivalence $E_n \otimes E_m \simeq E_{n+m}$ from Section 2.2.2, and the right vertical map by $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$, so is given by multiplication in the group $GL(\mathbb{Q}) \cong \mathbb{Q}^{\times}$. The construction of the dashed filler is somewhat subtle, since we do not know whether one has $E_{n,\mathbb{Q}} \otimes E_{m,\mathbb{Q}} \simeq E_{n+m,\mathbb{Q}}$, so the dashed filler will *not* just be given by taking tensor products in \mathbb{O} p, but will instead use a tensor product in a certain category of *rational pro-operads*.

Remark 3.2. A similar diagram was constructed by Boavida de Brito–Horel when replacing the leftmost vertical map with a certain tensor product for configuration categories from [BdBW18b] (see [BdBH21, 1.5, 5.5., 6.3, 7.2]; by the discussion in Appendix B.5 below our definition of $\operatorname{Aut}(E_{d,Q})$ agrees with theirs). Their construction inspired ours, and it seems likely that the resulting diagrams are in fact equivalent. Showing this would involve proving that the tensor product for configuration categories from [BdBW18b] is compatible with the one for operads from [Lur17]. We learned from Boavida de Brito that there is a natural comparison map between them, but it has not yet been shown to be an equivalence. An attempt in this direction was made in [Col19].

Once (10) is established, we can specialise the dashed map to n=d-2 and m=2 and restrict to SAut-subgroups to obtain an E_1 -map $(-)\otimes_{\mathbb{Q}}(-)$: SAut $(E_{d-2,\mathbb{Q}})\times \mathrm{SAut}(E_{2,\mathbb{Q}})\to \mathrm{SAut}(E_{d,\mathbb{Q}})$ which is $\mathrm{Aut}(E_{2,\mathbb{Q}})$ -equivariant with respect to the action on the source by acting trivially on SAut $(E_{d-2,\mathbb{Q}})$ and by conjugation on SAut $(E_{2,\mathbb{Q}})$, and on the target by stabilisation followed by conjugation. Restricting the source to SAut $(E_{d-2,\mathbb{Q}})\times \{\mathrm{id}\}$ and taking invariants, it follows that the stabilisation map $(-)\otimes_{\mathbb{Q}}\mathrm{id}_{E_{2,\mathbb{Q}}}\colon \mathrm{SAut}(E_{d-2,\mathbb{Q}})\to \mathrm{SAut}(E_{d,\mathbb{Q}})$ factors through the $\mathrm{Aut}(E_{2,\mathbb{Q}})$ -invariants $\mathrm{SAut}(E_{d,\mathbb{Q}})^{\mathrm{Aut}(E_{2,\mathbb{Q}})}$, and thus also through the invariants $\mathrm{SAut}(E_{d,\mathbb{Q}})^G$ of any E_1 -group G that acts on $\mathrm{SAut}(E_{d,\mathbb{Q}})$ through $\mathrm{Aut}(E_{2,\mathbb{Q}})$. In order to prove Theorem G, it will thus suffice to find such a G for which the invariants $\mathrm{SAut}(E_{d,\mathbb{Q}})^G$ are trivial. Our choice of G relies the following result going back to Drinfel'd (see [BdBH21, Theorem 7.1] and [Dri90]), which can also be used to give a proof of rational formality of the E_d -operads (see [Pet14, BdBH21]).

Theorem 3.3 (Drinfel'd). The map $\chi: Aut(E_{2,\mathbb{Q}}) \to GL(\mathbb{Q})$ is surjective.

Fixing an automorphism $\Lambda \in \operatorname{Aut}(E_{2,\mathbb{Q}})$, we get a morphism $\langle \Lambda \rangle \to \operatorname{Aut}(E_{2,\mathbb{Q}})$ from the free E_1 -group generated by Λ . By the above discussion, Theorem G would follow if we could choose Λ so that the invariants $\operatorname{SAut}(E_{d,\mathbb{Q}})^{\langle \Lambda \rangle}$ are trivial. We will show that this is indeed possible:

Proposition 3.4. For $d \geq 3$ and $\Lambda \in Aut(E_{2,\mathbb{Q}})$ with $\chi(\Lambda) \neq \pm 1$, we have $SAut(E_{d,\mathbb{Q}})^{\langle \Lambda \rangle} \simeq *$.

To summarise the discussion, in order to prove Theorem G, it suffices to (a) construct the dashed filler in (10), and (b) prove Proposition 3.4. Part (a) will be taken care of in Appendix B.5, so the remainder of this section deals with (b), the proof of Proposition 3.4.

3.2. **Cyclotomic actions.** The key ingredient to the proof of Proposition 3.4 is to establish a certain qualitative property of the action of $\operatorname{Aut}(E_{2,\mathbb{Q}})$ on the homotopy groups of the automorphism spaces $\operatorname{SAut}_{\leq k}(E_{d,\mathbb{Q}})$ of the truncations of $E_{d,\mathbb{Q}}$. For brevity, we write

$$GT := \pi_0(Aut(E_{2,\mathbf{O}})).$$

This notation is motivated by a result of Fresse [Fre17, Theorem A] that says that $\pi_0(\operatorname{Aut}(E_{2,Q}))$ is isomorphic to Drinfel'ds pro-unipotent rational version of the Grothendieck–Teichmüller group, though we will not rely on this. The qualitative property of the GT-action on the homotopy groups of $\operatorname{SAut}_{\leq k}(E_{d,Q})$ we prove is that they are $\operatorname{gr-cyclotomic}$ of nontrivial weight in the following sense:

Definition 3.5. Let *G* be a group with a GT-action.

- (i) The GT-action is *cyclotomic* if it factors through the morphism $\chi \colon GT \to GL(Q)$.
- (ii) A cyclotomic action is of weight n for an integer $n \in \mathbb{Z}$ if G is abelian, torsion-free, and divisible (equivalently: a **Q**-vector space) and the action is given by $\varphi(g) = \chi(\varphi)^n \cdot g$ for all $\varphi \in G$ T and $g \in G$. It is of nontrivial weight if it is of weight n for some $n \neq 0$.
- (iii) The action is gr-cyclotomic if there exists a GT-invariant finite normal series $1 = G_0 \triangleleft \cdots \triangleleft G_k = G$ such that the action on the subquotients G_i/G_{i-1} is cyclotomic. Such a gr-cyclotomic action is of nontrivial weight if the action on the subquotients is of nontrivial weight (the weights may be different for the different subquotients).

Being gr-cyclotomic of nontrivial weight enjoys various inheritance properties:

Lemma 3.6.

- (i) Being gr-cyclotomic of nontrivial weight is preserved under taking GT-invariant subgroups, quotients by GT-invariant normal subgroups, and GT-invariant extensions.
- (ii) Let Q be a group and G a group with $(Q \times GT)$ -action. If G is gr-cyclotomic of nontrivial weight then so is $H^k(BQ;G)$ for $0 \le k \le 1$. If G is abelian, then this also holds for $k \ge 2$.

Proof. We prove the part of (i) regarding subgroups; the other properties are similar. Fix a group G with GT-action that is gr-cyclotomic of nontrivial weight, i.e. there is a GT-invariant finite normal series $1 = G_0 \triangleleft \cdots \triangleleft G_k = G$ such that G_i/G_{i-1} is cyclotomic of weight $n_i \neq 0$. Given a GT-invariant subgroup $H \leq G$ we consider the normal series $1 = H_0 \triangleleft \ldots \triangleleft H_k = H$ with $H_i \coloneqq H \cap G_i$, and claim that the GT-invariant subgroup $H_i/H_{i-1} \leq G_i/G_{i-1}$ is cyclotomic of nontrivial weight. That H_i/H_{i-1} is cyclotomic, abelian, torsion free, and that the action is given by $\varphi(x) = \chi(\varphi)^{n_i} \cdot x$ follows from the corresponding properties for G_i/G_{i-1} , so it remains to show that any $x \in H_i/H_{i-1}$ is divisible by any m > 0. For this use Theorem 3.3 to find an $\varphi \in GT$ with $\chi(\varphi) = m^{-n_i/|n_i|}$, then for $x \in H_i/H_{i-1}$, we have by GT-invariance and the fact that G_i/G_{i-1} has weight n_i that $\varphi(x) = \chi(\varphi)^{n_i} \cdot x = m^{-|n_i|} \cdot x \in H_i/H_{i-1}$, so x is divisible by $m^{|n_i|}$ and thus also by m.

Many of the GT-actions related to $E_{d,\mathbb{Q}}$ turn out to be gr-cyclotomic of nontrivial weight:

Lemma 3.7. Fix $d \ge 3$ and $k \ge 2$.

- (i) The groups $H_*(E_{d,Q}(k))$ are trivial except possibly in degrees *=n(d-1) for $0 \le n \le k-1$ in which case the GT-action is cyclotomic of weight n.
- (ii) The groups $\pi_*(E_{d,Q}(k))$ and $\pi_*(\operatorname{Match}_k(E_{d,Q}))$ are trivial except possibly in degrees *=n(d-2)+1 for $n\geq 1$ in which case the GT-action is cyclotomic of weight n.
- (iii) The group $H_{(k-1)d-1}(E_{d,O}(k), Latch_k(E_{d,O}))$ is cyclotomic of weight (k-1).
- (iv) The groups $H^*(E_{d,Q}(k), Latch_k(E_{d,Q}))$ are trivial except possibly in degrees * = (k-1)d n(d-1) 1 for $0 \le n \le k-1$ in which case the GT-action is cyclotomic of weight n-k+1.
- (v) For $k \ge 3$ and all $j \ge i \ge 0$, the GT-action on the groups

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H^i(E_{d,Q}(k), Latch_k(E_{d,Q}); \pi_j(E_{d,Q}(k))) and H^i(E_{d,Q}(k), Latch_k(E_{d,Q}); \pi_j(Match_k(E_{d,Q}))) is cyclotomic of nontrivial weight. For k = 2, the same holds except in degrees i = j = d - 1.
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Proof. Throughout the proof, we will make use of two facts: (a) the map $\operatorname{Latch}_k(E_d) \to E_d(k)$ is for each k equivalent to the boundary inclusion of a compact ((k-1)d-1)-dimensional manifold (see Example 2.3), and (2) the space $E_d(k)$ is 1-connected for all k. The reason for the latter is that $E_d(k)$ is equivalent to the space $\operatorname{Emb}(\underline{k}, \mathbf{R}^d)$ of ordered configurations of k points in \mathbf{R}^d which is 1-connected by transversality since we assumed $d \geq 3$. In particular, the spaces $E_d(k)$ are nilpotent, so their rationalisations $E_{d,Q}(k)$ are well-behaved in the sense of Section 2.1.2.

For (i), we use that the operad $H_*(E_{d,\mathbb{Q}}) = H_*(E_d;\mathbb{Q})$ in graded vector spaces is the d-Poisson operad (see e.g. [Sin13, Theorem 6.3, Section 5]) which is generated under operadic compositions in arity 2. This reduces the claim to showing that the arity 2 part is concentrated in degree 0 and d-1 and acted upon by GT cyclotomically with weight 0 in degree 0 and weight 1 in degree d-1. But $H_*(E_{d,\mathbb{Q}}(2)) \cong H_*(S^{d-1};\mathbb{Q})$, so this holds by definition of the morphism χ .

The first part of (ii) follows from (i) together with the fact that the homotopy groups $\pi_*(E_{d,Q}(k)) = \pi_*(E_d(k)) \otimes \mathbb{Q}$ are generated under taking Whitehead brackets by $\pi_{d-1}(E_d(k)) \otimes \mathbb{Q} \cong H_{d-1}(E_d(k); \mathbb{Q})$ which follows e.g. from [CG02, Theorem 2.3] together with the Milnor-Moore theorem. The second part follows from the first part by showing that the map $E_{d,Q}(k) \to \mathrm{Match}_k(E_{d,Q})$ is surjective on homotopy groups. From the discussion in Section 2.2.6, it follows that the map $E_{d,Q}(k) \to \mathrm{Match}_k(E_{d,Q})$ is equivalent to the map $\mathrm{Emb}(\underline{k}, \mathbb{R}^d)_{\mathbb{Q}} \to \lim_{S \subseteq \underline{k}} (\mathrm{Emb}(S, \mathbb{R}^d)_{\mathbb{Q}})$ induced by forgetting points, so it suffices to show that the fibre inclusion tohofib $S \subseteq \underline{k}$ (Emb $(S, \mathbb{R}^d)_{\mathbb{Q}}$) $\to \mathrm{Emb}(\underline{k}, \mathbb{R}^d)_{\mathbb{Q}}$ is injective on homotopy groups. Adding new points to the configuration near ∞ shows that the cubical diagram $\underline{k} \supseteq S \to \mathrm{Emb}(S, \mathbb{R}^d)_{\mathbb{Q}}$ is split up to homotopy in the sense of [KuRW25, Definition 5.24], so the claim follows from Lemma 5.25 loc.cit..

To show (iii), we distinguish two cases. If k=2, then $\operatorname{Latch}_k(E_{d,\mathbb{Q}})=\emptyset$ (see Section 2.2.6), so the group in question is $H_{d-1}(E_d(2);\mathbb{Q})$ on which GT acts indeed cyclotomically of weight

1=k-1. If $k\geq 3$, then ${\rm Latch}_k(E_d)$ is nonempty (see Example 2.3), so ${\rm H}_{(k-1)d-1}(E_d(k);{\bf Q})$ vanishes by Poincaré duality and hence the connecting map ${\rm H}_{(k-1)d-1}(E_{d,{\bf Q}}(k),{\rm Latch}_k(E_{d,{\bf Q}}))\to {\rm H}_{(k-1)d-2}({\rm Latch}_k(E_{d,{\bf Q}}))$ is injective, so it suffices to prove that ${\rm H}_{(k-1)d-2}({\rm Latch}_k(E_{d,{\bf Q}}))$ is cyclotomic of weight (k-1). For this we use the colimit decomposition ${\rm colim}_{T\in\Psi_k^-}E_{d,{\bf Q}}(T)\simeq {\rm Latch}_k(E_{d,{\bf Q}})$ from Section 2.2.6. Filtering the bar-construction model for colimits by skeleta, we get a spectral sequence of GT-modules

$$E_{p,q}^{1} = \bigoplus_{T_0 \le \dots \le T_p} H_q(E_{d,Q}(T_0)) \implies H_{p+q}(\operatorname{Latch}_k(E_{d,Q}))$$
(11)

where the sum runs through non-degenerate chains of length p in the poset Ψ_k^- . Note that any such chain has length $\leq (k-3)$ since rooted trees with k leaves that have no interior vertex of valence 2 and have at least one interior vertex differ from each other by collapsing at most k-3 edges, so $E_{p,q}^1=0$ for p>k-3. By the dendroidal Segal property we have $E_{d,\mathbb{Q}}(T_0)\simeq \sqcap_{i=1}^l E_{d,\mathbb{Q}}(k_i)$ with $\sum_{i=1}^l (k_i-1)=k-1$, and as $E_{d,\mathbb{Q}}(n)$ has homology concentrated in degrees in degrees $*\leq (n-1)(d-1)$, the homology of $E_{d,\mathbb{Q}}(T_0)$ is concentrated in degrees $*\leq (k-1)(d-1)$, so $E_{p,q}^1=0$ for q>(k-1)(d-1). As (k-3)+(k-1)(d-1) agrees with (k-1)d-2, the only entry that can contribute to $H_{(k-1)d-2}(\operatorname{Latch}_k(E_{d,\mathbb{Q}}))$ is $E_{k-3,(k-1)(d-1)}^1$. Since the action of GT on the latter is cyclotomic of weight k-1 by (i), Item (iii) follows.

For (iv), we first compare the spectral sequence (11) with the corresponding spectral sequence for E_d to see that the map $\operatorname{Latch}_k(E_d) \to \operatorname{Latch}_k(E_{d,Q})$ is a Q-homology isomorphism, from which it follows that the pair $(E_{d,Q}(k), \operatorname{Latch}_k(E_{d,Q}))$ has rational Poincaré duality with relative fundamental class in degree (k-1)d-1. Capping with this class thus gives an isomorphism $\operatorname{H}^*(E_{d,Q}(k), \operatorname{Latch}_k(E_{d,Q})) \cong \operatorname{H}_{(k-1)d-1-i}(E_{d,Q}(k))$ which together with (i) implies the first part of (iv). The second part follows by observing that since the right-hand side of the Poincaré duality isomorphism is cyclotomic of weight n by (i) and the fundamental class has weight (k-1) by (iii), so the left-hand side is cyclotomic of weight n-(k-1).

For (v) we note that the universal coefficient theorem provides a GT-equivariant isomorphism

$$\mathsf{H}^i\big(E_{d,\mathbb{Q}}(k),\mathsf{Latch}_k(E_{d,\mathbb{Q}});\pi_j(E_{d,\mathbb{Q}}(k))\big)\cong \mathsf{H}^i\big(E_{d,\mathbb{Q}}(k),\mathsf{Latch}_k(E_{d,\mathbb{Q}});\mathbb{Q}\big)\otimes \pi_j(E_{d,\mathbb{Q}}(k))$$

and similarly for $\mathrm{H}^i\big(E_{d,\mathbb{Q}}(k),\mathrm{Latch}_k(E_{d,\mathbb{Q}});\pi_j(\mathrm{Match}_k(E_{d,\mathbb{Q}}))\big)$; here the tensor products are acted upon diagonally. By combining (ii) and (iv) it thus follows that the groups in (v) are 0 unless (i,j)=((k-1)d-n(d-1)-1,m(d-2)+1) for some $n\geq 0$ and $m\geq 1$ and in this case the GT-action is cyclotomic of weight w:=n-k+1+m. It remains to show that $w\neq 0$. As $0\leq j-i=w(d-1)-m-k+3$, the only case in which the weight could be 0 is when $m+k\leq 3$. As $m\geq 1$ this does not happen for $k\geq 3$ and it happens for k=2 exactly when j-i=0 and m=1. For m=1 and k=2 we have n=0 if k=0, and thus k=1 are claimed.

Proposition 3.8. For $d \ge 3$, the GT-action on the groups $\pi_i(\operatorname{SAut}_{\le k}(E_{d,Q}))$ is gr-cyclotomic of nontrivial weight for all $i \ge 0$ and $2 \le k < \infty$.

Proof. During the proof, we will repeatedly use the fact that any automorphism in $SAut_{\leq k}(E_{d,Q}) = \ker(\chi_{\leq k} : Aut_{\leq k}(E_{d,Q}) \to GL(Q))$ for $k \geq 2$ acts as the identity on all homotopy and homology groups of $E_{d,Q}(\ell)$ for all $\ell \leq k$. This follows from the proof of Lemma 3.7 (i) and (ii).

By the long exact sequence in homotopy groups and Lemma 3.6, it suffices to show that GT acts on the homotopy groups of $\mathrm{SAut}_{\leq 2}(E_{d,\mathbb{Q}})$ and of $L_k^\times(E_{d,\mathbb{Q}}) \coloneqq \mathrm{fib}_{\mathrm{id}}(\mathrm{Aut}_{\leq k}(E_{d,\mathbb{Q}}) \to \mathrm{Aut}_{\leq k-1}(E_{d,\mathbb{Q}}))$ for $k \geq 3$ gr-cyclotomically of nontrivial weight.

We begin with $L_k^{\times}(E_{d,Q})$. Taking group-like components of the fibre sequence (7) in the case $\mathcal{P} = \mathcal{O} = E_{d,Q}$ and looping once gives a fibre sequence of E_1 -groups with GT-action

$$\Omega \left(\operatorname{Map}_{\operatorname{Latch}_k(E_{d,\mathbb{Q}})} \left(E_{d,\mathbb{Q}}(k), \operatorname{Match}_k(E_{d,\mathbb{Q}}) \right)^{\Sigma_k} \right) \to L_k^{\operatorname{id}}(E_{d,\mathbb{Q}})^\times \\ \to \operatorname{Aut}_{\operatorname{Latch}_k(E_{d,\mathbb{Q}})}(E_{d,\mathbb{Q}}(k))^{\Sigma_k},$$

Since any automorphism in $\operatorname{Aut}_{\leq k}(E_{d,Q})$ that lifts to the fibre $L_k^{\operatorname{id}}(E_{d,Q})^{\times}$ lies in the kernel of $\chi_{\leq k}$, the second map lands in the components $\operatorname{SAut}_{\operatorname{Latch}_k(E_{d,Q})}(E_{d,Q}(k))^{\Sigma_k} \leq \operatorname{Aut}_{\operatorname{Latch}_k(E_{d,Q})}(E_{d,Q}(k))^{\Sigma_k}$. By the long exact sequence in homotopy groups, it thus suffices to show that GT acts on

$$\pi_i \left(\operatorname{Map}_{\operatorname{Latch}_k(E_{d,Q})} (E_{d,Q}(k), \operatorname{Match}_k(E_{d,Q}))^{\Sigma_k} \right) \text{ and } \pi_i \left(\operatorname{BSAut}_{\operatorname{Latch}_k(E_{d,Q})} (E_{d,Q}(k))^{\Sigma_k} \right) \text{ for } i \geq 1 \quad (12)$$

gr-cyclotomically of nontrivial weight. We first show this for the homotopy groups of the corresponding spaces without the Σ_k -superscripts. For the first group, we apply the Bousfield–Kan spectral sequence as in Example 2.7. We have seen in the proof of Lemma 3.7 (ii) that $E_{d,O}(k) \rightarrow$ $Match_k(E_{d,Q})$ is surjective on homotopy groups, so $Match_k(E_{d,Q})$ is 1-connected. Moreover, since Latch_k $(E_{d,Q}) \to E_{d,Q}(k)$ is Q-homology equivalent to the boundary inclusion of a compact manifold (see the proof of Lemma 3.7 (iv)), the homology of the cofibre vanishes in large degrees, so by the discussion in Example 2.7 and Remark 2.5, the group $\pi_i(\mathrm{Map}_{\mathrm{Latch}_k(E_{d,Q})}(E_{d,Q}(k),\mathrm{Match}_k(E_{d,Q})))$ admits for $i \ge 1$ a GT-invariant normal series whose quotients are subquotients of the groups $H^{s-i}(E_{d,O}(k)/\text{Latch}_k(E_{d,O}); \pi_s(\text{Match}_k(E_{d,O})))$, so the claim follows from the fact that GT acts grcyclotomically with nontrivial weight on these groups for $k \geq 3$ by Lemma 3.7 (v). Similarly, to deal with the second group in (12) (without the Σ_k -superscript) we apply the Bousfield–Kan spectral sequence as in Example 2.8. Arguing as before, this shows the claim as a consequence of the fact that GT acts for $k \ge 3$ on $H^{s-i+1}(E_{d,O}(k)/\text{Latch}_k(E_{d,O}); \pi_s(E_{d,O}(k)))$ gr-cyclotomically with nontrivial weight by Lemma 3.7 (v). Finally, to show that GT acts gr-cyclotomically with nontrivial weights on the groups (12) with the Σ_k -superscript, we apply the Bousfield–Kan spectral sequence as in Example 2.6 to see that these groups admit finite normal series whose quotients are subquotients of Σ_k -cohomology groups with coefficients in the groups without the Σ_k -superscripts which are gr-cyclotomic of nontrivial weight, so the claim follows from Lemma 3.6.

It remains to show that the GT-action on the homotopy groups of $SAut_{\leq 2}(E_{d,Q})$ is gr-cyclotomic of nontrivial weight. Since $Match_2(E_{d,Q})$ and $Aut_{\leq 1}(E_{d,Q})$ are contractible and $Latch_2(E_{d,Q})$ empty (see Section 2.2.6), we have $SAut_{\leq 2}(E_{d,Q}) \cong SAut(E_{d,Q}(2))^{\Sigma_2}$ by (7). By the same argument as in the final step above, it suffices to show that the GT-action on $\pi_i(SAut(E_{d,Q}(2)))$ is cyclotomic of nontrivial weight. As $E_{d,Q}(2)$ is the rationalisation of a (d-1)-sphere, the morphism $\chi \colon \pi_0(Aut(E_{d,Q}(2))) \to GL(Q)$ is an isomorphism, so $\pi_i(SAut(E_{d,Q}(2)))$ vanishes for i=0. For $i \geq 1$, we have $\pi_i(SAut(E_{d,Q}(2))) = \pi_i(Map(E_{d,Q}(2), E_{d,Q}(2)))$, so applying the Bousfield–Kan spectral sequence as in Example 2.7, the claim follows from the final part of Lemma 3.7 (v).

Proof of Theorem G and Proposition 3.4. We already explained in Section 3.1 how Proposition 3.4 implies Theorem G. To show the former, note that since $\lim_k \operatorname{SAut}_{\leq k}(E_{d,Q}) \cong \operatorname{SAut}(E_{d,Q})$ and taking invariants commutes with limits, it suffices to show $\operatorname{SAut}_{\leq k}(E_{d,Q})^{\langle \Lambda \rangle} \cong *$ for $2 \leq k < \infty$. We have $\operatorname{SAut}_{\leq k}(E_{d,Q})^{\langle \Lambda \rangle} \cong \Omega(\operatorname{BSAut}_{\leq k}(E_{d,Q})^{\langle \Lambda \rangle})$, so it suffices to show that $\pi_i(\operatorname{BSAut}_{\leq k}(E_{d,Q})^{\langle \Lambda \rangle})$ vanishes for $i \geq 1$. From the Bousfield–Kan spectral sequence as in Example 2.6, we see that it suffices to show that $\operatorname{H}^*(\operatorname{B}(\langle \Lambda \rangle); \pi_s(\operatorname{BSAut}_{\leq k}(E_{d,Q})))$ vanishes for all s. As $\pi_s(\operatorname{BSAut}_{\leq k}(E_{d,Q}))$ is gr-cyclotomic of nontrivial weight, it admits a GT-invariant finite normal series whose quotients that are cyclotomic of nontrivial weight, so an induction over the length of the series shows that it suffices to show that $\operatorname{H}^*(\operatorname{B}(\langle \Lambda \rangle); A)$ is trivial for any cyclotomic GT-module A of fixed weight $n \neq 0$. Since $\operatorname{B}(\langle \Lambda \rangle) \cong \operatorname{BZ}$, the only potentially nontrivial cohomology groups occur for *=0,1 where they are given by the (co)invariants of $\langle \Lambda \rangle$ acting on A. But by assumption, A is a rational vector space and Λ acts by multiplication with $\chi(\Lambda)^n \neq 1$, so the (co)invariants vanish.

Remark 3.9.

- (i) The proof of Theorem G can likely be shortened by relying on the description of $\mathrm{BAut}(E_{d,Q})$ in terms of a graph complex [FW20] to show that the GT-action on the homotopy groups of $\mathrm{SAut}(E_{d,Q})$ is cyclotomic of nontrivial weight (the weight should correspond to the loop order in the graph complex). Such an argument was suspected by Willwacher. We decided against pursuing this route when we discovered the more self-contained argument given above.
- (ii) Theorem **G** in particular implies that the map $BSO(d-2) \to BSAut(E_{d,Q})$ induced by the SO(d)-action on $BSAut(E_{d,Q})$ is nullhomotopic, because it factors over $BAut(E_{d-2,Q}) \to BAut(E_{d,Q})$ as a result of (5) and (10). This corollary of Theorem **G** ought to be closely related to work of Khoroshkin–Willwacher [KW17] which suggests that the map $BSO(d) \to BSAut(E_{d,Q})$ factors for even d=2n through the Euler class $e:BSO(2n) \to K(2n,Q)$ and for odd d=2n+1 through the Pontryagin class $p_n:BSO(2n+1) \to K(4n,Q)$.

3.2.1. Alternative nontriviality of $S_{\partial}^{\mathcal{D}isc}(D^d)$. We briefly mention an application of Theorem G that is independent of the remainder of this work. There is an equivalence (see [KK24c, Remark 5.39])

$$\Omega^{d+1} \operatorname{Aut}(E_d) / \operatorname{Top}(d) \simeq S_{a}^{\mathcal{D}isc}(D^d) \quad \text{for } d \neq 4$$
 (13)

between the (d+1)st loop space of the fibre $\operatorname{Aut}(E_d)/\operatorname{Top}(d)$ of the map $t\colon\operatorname{BTop}(d)\to\operatorname{BAut}(E_d)$ from Section 2.2.2 and the $\operatorname{\mathcal{D}isc}$ -structure space $S^{\operatorname{\mathcal{D}isc}}_{\partial}(D^d)$ in the sense of [KK24a]. In Theorems E and 8.13 loc.cit. we proved that $\operatorname{Aut}(E_d)/\operatorname{Top}(d)$ and $S^{\operatorname{\mathcal{D}isc}}_{\partial}(D^d)$ are nontrivial for most values of d:

Theorem 3.10. We have
$$\operatorname{Aut}(E_d)/\operatorname{Top}(d) \neq * \text{ for } d \geq 3, \text{ and } S_{\partial}^{\mathcal{D}\mathrm{isc}}(D^{d'}) \neq * \text{ for } d' = 3 \text{ and } d' \geq 5.$$

The proof in [KK24a] relied on a computation of the homotopy groups of $Aut(E_{d,Q})$ in terms of a graph complex due to Fresse–Turchin–Willwacher (see Theorem 8.3 loc.cit.). Using Theorem G, we can offer (for most values of d and d') a simpler proof which is independent of their work:

Alternative proof of Theorem 3.10 for $d \ge 6$ and $d' \ge 8$. Consider the commutative diagram

whose left and right squares are induced by (5) and (10) respectively. Assume there is a class in $\pi_i(\mathrm{BSTop}(d-2))_{\mathbb{Q}}$ that maps nontrivially to $\pi_i(\mathrm{BSTop}(d))_{\mathbb{Q}}$. Since the rightmost column is nullhomotopic by Theorem G, the class is trivial in $\pi_i(\mathrm{BSAut}(E_{d,\mathbb{Q}}))_{\mathbb{Q}}$. The argument in $[\mathrm{KK24a}$, Section 8.3.1] shows that the class is either zero in $\pi_i(\mathrm{BSAut}(E_d))_{\mathbb{Q}}$ and thus $\pi_i(\mathrm{Aut}(E_{d,\mathbb{Q}})/\mathrm{Top}(d))_{\mathbb{Q}} \neq 0$, or $\pi_*(\mathrm{Aut}(E_{d,\mathbb{Q}})/\mathrm{Top}(d))$ is uncountable for *=i-1 or *=i-2. Thus, as soon as there is an element that is not in the kernel of $\pi_i(\mathrm{BSTop}(d-2))_{\mathbb{Q}} \to \pi_i(\mathrm{BSTop}(d))_{\mathbb{Q}}$, then $\pi_*(\mathrm{Aut}(E_d)/\mathrm{Top}(d))$ is nontrivial in at least one of the degrees *=i,i-1,i-2. For $d\geq 6$, the composition $\mathrm{BSO}(4)\to \mathrm{BSTop}(d-2)\to \mathrm{BSTop}(d)\to \mathrm{BTop}$ is nontrivial on rational homotopy groups, so there is indeed such an element and the claim regarding $\mathrm{Aut}(E_d)/\mathrm{Top}(d)$ follows. To apply the same strategy to show that $S_{\partial}^{\mathcal{D}\mathrm{isc}}(D^{d'}) \cong \Omega^{d'+1}\mathrm{Aut}(E_{d'})/\mathrm{Top}(d')$ is nontrivial, we need a class that is not in the kernel $\pi_i(\mathrm{BSTop}(d'-2))_{\mathbb{Q}} \to \pi_i(\mathrm{BSTop}(d'))_{\mathbb{Q}}$ for some $i-2\geq d'+1$. Such classes are known to exist as long as $d'-2\geq 6$ (see [KK24a, Theorem 8.10]), so $S_{\partial}^{\mathcal{D}\mathrm{isc}}(D^{d'}) \not\cong * \text{for } d'\geq 8$.

4. Boundary conditions in embedding calculus

In this section, as a further ingredient to the proof of our main result Theorem **F**, we discuss how the homotopy type of the boundary of a manifold M can often be recovered in a natural way from the interior of M in the context of Goodwillie–Weiss' embedding calculus and variants of it. We first fix some notation. Fix a compact d-dimensional manifold triad $(M, \partial^v M, \partial^h M)$, by which we mean a compact topological d-manifold M together with a decomposition $\partial M = \partial^v M \cup \partial^h M$ of its boundary into two codimension zero submanifolds that intersect in their common boundary $\partial^{vh} M := \partial(\partial^v M) = \partial^v M \cap \partial^h M = \partial(\partial^h M)$ (the v stands for v-ertical, the h for h-orizontal; this choice of terminology will become clear later). We consider the commutative square of E_1 -groups

- Homeo $_{\partial^v}(M)$ is the space of homeomorphisms of M that fix a neighbourhood of $\partial^v M$ pointwise, viewed as an E_1 -group by composition,
- Emb $_{\partial^{v}}(M)^{\times}$ is space of topological self-embeddings of M that are the identity in a neighbourhood of $\partial^{v}M$ and are invertible up to isotopy fixed on $\partial^{v}M$, viewed as an E_{1} -group by composition (these are the group-like elements of the E_{1} -space Emb $_{\partial^{v}}(M)$ of all self-embeddings),

• Aut $_{\partial^v}(M, \partial^h)$ is the E_1 -group of homotopy self-equivalences of M that "fix $\partial^v M$ pointwise and $\partial^h M$ setwise", i.e. the automorphism group of the square induced by the inclusions

$$\begin{array}{ccc}
\partial^{vh} M \to \partial^{h} M \\
\downarrow & \downarrow \\
\partial^{v} M \longrightarrow M
\end{array} \tag{15}$$

considered as an object in the undercategory $(S^{[1]})_{(\partial^{vh}M \to \partial^{v}M)/}$ where S is the category of spaces,

- Aut $_{\partial^v}(M)$ is the E_1 -group of self-equivalences of M that only "fix $\partial^v M$ pointwise", i.e. the automorphism group of $(\partial^v M \to M) \in \mathbb{S}_{\partial^v M/}$, and
- the solid maps in (14) are the evident forgetful maps. For the left vertical map, this uses that homeomorphisms of a manifold always preserve the boundary setwise, by invariance of domain.

We will explain below that there exists a dashed filler in (14), which informally says that a self-embedding that fixes $\partial^v M$ pointwise also preserve $\partial^h M$ setwise "in a homotopical sense" even though it does not need to actually preserve it in a point-set sense, unlike for homeomorphism. The main point of this section will then be to show that after replacing $\partial^h M$ and $\partial^{vh} M$ with certain approximations resulting from variants of Goodwillie–Weiss' embedding calculus [Wei99], the dashed map in (14) admits a further factorisation

$$\operatorname{Emb}_{\partial^v}(M)^{\times} \longrightarrow T_{\infty} \operatorname{Emb}_{\partial^v}(M)^{\times} \longrightarrow T_{\infty} \operatorname{Emb}_{\partial^v}^p(M)^{\times} \xrightarrow{----} \operatorname{Aut}_{\partial^v}(M, \partial^h),$$

through the composition of the topological embedding calculus approximation followed by the map to particle embedding calculus, both in the sense of [KK24c] (we omit the t-superscripts from the notation in loc.cit. since we will only consider topological embeddings and topological embedding calculus in this work). If (i) the triad $(M, \partial^v M, \partial^h M)$ is smoothable, (ii) $d \geq 5$, (iii) the inclusion $\partial^h M \subset M$ is an equivalence on tangential 2-types (i.e. it induces an equivalence on fundamental groupoids and for all components the Stiefel–Whitney class $w_2 \colon \pi_2(M) \to \mathbb{Z}/2$ is nontrivial if and only if $w_2 \colon \pi_2(\partial^h M) \to \mathbb{Z}/2$ is), and (iv) the same holds for the inclusion $\partial^{vh} M \subset \partial^v M$, then we will see that the approximations to $\partial^h M$ and $\partial^{vh} M$ recover $\partial^h M$ and $\partial^{vh} M$, which will lead to a proof of:

Theorem 4.1. Let $(M, \partial^{\upsilon}M, \partial^h M)$ be a compact smoothable d-manifold triad with $d \geq 5$. If $\partial^h M \subset M$ and $\partial^{\upsilon h}M \subset \partial^{\upsilon}M$ are equivalences on tangential 2-types, then there exists a dashed filler in the diagram

$$\begin{array}{c} \operatorname{Homeo}_{\partial^v}(M) \to \operatorname{Emb}_{\partial^v}(M)^\times \to T_\infty \operatorname{Emb}_{\partial^v}(M)^\times \to T_\infty \operatorname{Emb}_{\partial^v}^p(M)^\times \\ \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ \operatorname{Aut}_{\partial^v}(M, \partial^h) \xrightarrow{\longleftarrow} \operatorname{Aut}_{\partial^v}(M). \end{array}$$

of E_1 -groups.

Remark 4.2.

- (i) In the smooth setting and under slightly more restrictive assumptions on the triad, a variant of this result was proved by Weiss as the main result of [Wei20]. The *statement* of Theorem 4.1 is strongly inspired by Weiss' work, but the *proofs* are considerably different. The main results of [TW17] can be recovered and slightly extended by a similar approach. One advantage of our construction is that it has better naturality properties, and we believe that (extensions of) this construction will see further applications in the future (see also Remark 4.10).
- (ii) Under the hypothesis of Theorem 4.1 on $\partial^h M \subset M$, the map $\operatorname{Emb}_{\partial^v}(M) \to T_\infty \operatorname{Emb}_{\partial^v}(M)$ is an equivalence by convergence of topological embedding calculus [KK24c, Theorem 6.4], so (14) already gives a dashed filler with domain $T_\infty \operatorname{Emb}_{\partial^v}(M)^\times$. The nontrivial part of Theorem 4.1 is that it can be extended to $T_\infty \operatorname{Emb}_{\partial^v}^p(M)^\times$ under the additional assumption on $\partial^{vh} M \subset \partial^v M$.

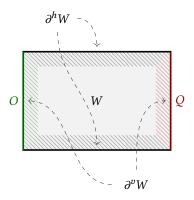


Figure 1. A bordism $W: O \rightarrow Q$ of manifolds with boundary. A collar is dashed.

4.1. **Embeddings.** We begin by constructing the dashed filler in (14). To do so, instead of considering compact manifold triads $(M, \partial^v M, \partial^h M)$ as above, we work in the slightly more general setting of compact bordisms $W: O \leadsto Q$ of manifolds with boundary, i.e. compact manifold triads $(W, \partial^v W, \partial^h W)$ with an ordered decomposition $\partial^v W = O \sqcup Q$ of the vertical boundary as a disjoint union of submanifolds; see Figure 1. We view manifold triads as bordisms of the form $\partial^v M \leadsto \emptyset$. It will be convenient to fix a closed collar $\partial W \times I \subset W$ of $\partial W = \partial W \times \{0\}$ in W and replace Homeo $_{\partial^v}(W)$ by the equivalent E_1 -group of homeomorphisms ϕ with $\phi|_{\partial^v W \times I} = \operatorname{id}$ and $\phi|_{\partial^h W \times I} = \phi|_{\partial^h W} \times \operatorname{id}_I$. The strategy for obtaining the filler in (14) is to replace the square (15) of inclusions Homeo $_{\partial^v}(M)$ -equivariantly by an equivalent square of Homeo $_{\partial^v}(M)$ -spaces whose Homeo $_{\partial^v}(M)$ -action visibly factors over $\operatorname{Emb}_{\partial^v}(M)^\times$, in a way that is compatible with the a priori action of $\operatorname{Emb}_{\partial^v}(M)^\times$ on the bottom row (which factors through $\operatorname{Aut}_{\partial^v}(M)$). In doing so, we will repeatedly use the Kister–Mazur theorem [Kis64] which says that the inclusion $\operatorname{Emb}(\mathbb{R}^d) \subset \operatorname{Top}(d) = \operatorname{Homeo}(\mathbb{R}^d)$ is an equivalence, so in particular the E_1 -space $\operatorname{Emb}(\mathbb{R}^d)$ of self-embeddings of \mathbb{R}^d under composition is an E_1 -group.

4.1.1. Recovering horizontal boundaries from embeddings. The first observation leading to the dashed filler in (14) is that the $\operatorname{Emb}(W)^{\times}$ -equivariant homotopy type of W can, as a consequence of the Kister–Mazur theorem, be recovered as the orbits $\operatorname{Emb}(\mathbb{R}^d,W)_{\operatorname{Emb}(\mathbb{R}^d)}$ of the space of topological embeddings $\mathbb{R}^d \hookrightarrow W$ with respect to the $\operatorname{Emb}(\mathbb{R}^d)$ -action by precomposition (cf. [KK24c, Proof of Proposition 5.10]). Here $\operatorname{Emb}(W)$ acts on the orbits by postcomposition. More precisely, we have equivalences $\operatorname{Emb}(\mathbb{R}^d,W)_{\operatorname{Emb}(\mathbb{R}^d)} \cong \operatorname{Map}_{\mathcal{S}}(\mathbb{R}^d,W)_{\operatorname{Aut}_{\mathcal{S}}}(\mathbb{R}^d) \cong W$ where the first equivalence is induced by the forgetful map and the second by the inclusion $W \to \operatorname{Map}_{\mathcal{S}}(\mathbb{R}^d,W)$ as the constant maps. The second observation is that, as consequence of the isotopy extension theorem, there is a homotopy fibre sequence of the form

$$\operatorname{Emb}(\mathbb{R}^d, \partial^h W \times I) \longrightarrow \operatorname{Emb}_{\partial^v}(W \sqcup \mathbb{R}^d, W) \longrightarrow \operatorname{Emb}_{\partial^v}(W),$$

where the left-hand map is induced by taking disjoint union with W, followed by postcomposition with an embedding $\partial^h W \times I \sqcup W \hookrightarrow W$ which "pushes in the collar". The right-hand map is given by restriction, and the fibre is taken at id_W . This is equivariant with respect to the Homeo $_{\partial^v}(W)$ -action induced by conjugation. Combining these observations, we see that the Homeo $_{\partial^v}(W)$ -equivariant homotopy type of $\partial^h W \subset W$ can be described as the following map induced by restriction

$$\partial_{E}^{h}W := \operatorname{fib}_{\operatorname{id}_{W}}\left(\operatorname{Emb}_{\partial^{v}}(W \sqcup \mathbf{R}^{d}, W) \to \operatorname{Emb}_{\partial^{v}}(W)\right)_{\operatorname{Emb}(\mathbf{R}^{d})} \qquad \qquad \downarrow \qquad (16)$$

$$EW := \operatorname{Emb}_{\partial^{v}}(\mathbf{R}^{d}, W)_{\operatorname{Emb}(\mathbf{R}^{d})}.$$

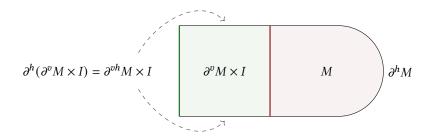


FIGURE 2. The bordism $M^c : \partial^v M \to \emptyset$ is obtained by composing the left-hand bordism $\partial^v M \times I : \partial^v M \to \partial^v M$ with the right-hand bordism $M : \partial^v M \to \emptyset$.

Note that, written like this, the action on $\partial^h W \subset W$ in the arrow category $\mathcal{S}^{[1]}$ of the ∞ -category of spaces visibly factors through $\operatorname{Homeo}_{\partial^v}(W) \to \operatorname{Emb}_{\partial^v}(W)^\times$, in a way that is compatible with the a priori action of $\operatorname{Emb}_{\partial^v}(W)^\times$ on ${}_E W \simeq W$. Moreover, this construction is natural with respect to gluing bordisms: if W arises a composition $W = U \cup V$ of two bordisms $U \colon O \leadsto P$ and $V \colon P \leadsto Q$, then we have two equivalent squares in \mathcal{S}

$$\begin{pmatrix}
\partial_E^h U \to \partial_E^h W \\
\downarrow & \downarrow \\
E U \to E W
\end{pmatrix} \simeq \begin{pmatrix}
\partial^h U \to \partial^h W \\
\downarrow & \downarrow \\
U \longrightarrow W
\end{pmatrix}$$
(17)

where the left square induced by extension of embeddings along $U \subset W$ and the right square by inclusion. The equivalence between these two squares is equivariant with respect to the E_1 -group $\operatorname{Homeo}_{\partial^v}(U) \times \operatorname{Homeo}_{\partial^v}(V) \to \operatorname{Homeo}_{\partial^v}(W)$ in $S^{[1]}$, where the action on the left column is through the projection to $\operatorname{Homeo}_{\partial^v}(U)$. It is then clear that the action on the left square factors through the forgetful map from $\operatorname{Homeo}_{\partial^v}(U) \times \operatorname{Homeo}_{\partial^v}(V) \to \operatorname{Homeo}_{\partial^v}(W)$ to $\operatorname{Emb}_{\partial^v}(U)^{\times} \times \operatorname{Emb}_{\partial^v}(V)^{\times} \to \operatorname{Emb}_{\partial^v}(W)^{\times}$, compatibly with the a priori action of the latter on the bottom row.

4.1.2. Recovering vh-boundaries from embeddings. To construct the filler in (14) from this, we consider the situation of compact triads $(M, \partial^v M, \partial^h M)$, so W = M, $O = \partial^v M$, $Q = \emptyset$. We replace M up to homeomorphism relative to $\partial^v M$ with the composition $M^c = \partial^v M \times I \cup_{\partial^v} M$ of the trivial bordism $\partial^v M \times I : \partial^v M \longrightarrow \partial^v M$ with the nullbordism $M : \partial^v M \longrightarrow \emptyset$. Since $\partial^h (\partial^v M \times I) = \partial^{vh} M \times I \cong \partial^{vh} M$ (see Figure 2), the equivalence (17) specialises to

$$\begin{pmatrix} \partial_E^{vh} M := \partial_E^h (\partial^v M \times I) \longrightarrow \partial_E^h M \\ \downarrow & \downarrow & \downarrow \\ \partial_E^v M := {}_E (\partial^v M \times I) \longrightarrow {}_E M \end{pmatrix} \simeq \begin{pmatrix} \partial^{vh} M \longrightarrow \partial^h M \\ \downarrow & \downarrow \\ \partial^v M \longrightarrow M \end{pmatrix}$$
(18)

of squares. Moreover, the action by $\operatorname{Homeo}_{\partial^v}(M^c) \simeq \operatorname{Homeo}_{\partial^v}(M)$ on the right column of the right-square extends to an $\operatorname{Homeo}_{\partial^v}(M^c)$ -action on the square considered as an object in the undercategory $\mathcal{S}^{[1]}_{/(\partial^{vh}M \to \partial^v M)}$, since the restriction $\{\operatorname{id}\} \times \operatorname{Homeo}_{\partial^v}(M) \to \operatorname{Homeo}_{\partial^v}(M^c)$ of the gluing morphism $\operatorname{Homeo}_{\partial^v \times \{0,1\}}(\partial^v M \times I) \times \operatorname{Homeo}_{\partial^v}(M) \to \operatorname{Homeo}_{\partial^v}(M^c)$ is an equivalence. The same holds compatibly for the $\operatorname{Emb}_{\partial^v}(M)^\times$ -action on the left square, in a way that is compatible with the a priori action on the bottom row, so the equivariant equivalence (18) yields the filler in (14).

4.2. A generalisation to double categories. We now explain an abstraction of the previous constructions to the situation of a symmetric monoidal double category $\mathcal M$ satisfying two conditions. We refer to [KK24c, Section 1.1] for a recollection on double categories and an explanation of the notation and terminology we use. To state the conditions on $\mathcal M$ that we impose, note that the unit

of the symmetric monoidal structure yields an object $\emptyset \in \mathcal{M}_{[0]}$ in the category of objects of \mathcal{M} , and an object $\emptyset_m \in \mathcal{M}_{\emptyset,\emptyset}$ in the mapping category from \emptyset to itself. The two conditions on \mathcal{M} are:

- (i) the mapping category $\mathcal{M}_{A,B}$ admits an initial object $\mathbb{I}_{A,B}$ for all $A, B \in \mathcal{M}_{[0]}$, and
- (ii) the object $\emptyset_m \in \mathcal{M}_{\emptyset,\emptyset}$ is an initial object of $\mathcal{M}_{\emptyset,\emptyset}$, so $\emptyset_m \simeq \mathbb{I}_{\emptyset,\emptyset}$.

In particular, for objects $A, B \in \mathcal{M}_{[0]}$, we have functors

$$(-)_{\varnothing} \coloneqq \left(\mathcal{M}_{A,B} \xrightarrow{(-) \cup \mathbb{I}_{B,\varnothing}} \mathcal{M}_{A,\varnothing}\right) \ _{\varnothing}(-)_{\varnothing} \coloneqq \left(\mathcal{M}_{A,B} \xrightarrow{\mathbb{I}_{\varnothing,A} \cup (-) \cup \mathbb{I}_{B,\varnothing}} \mathcal{M}_{A,\varnothing}\right) \ _{\varnothing}(-) \coloneqq \left(\mathcal{M}_{A,B} \xrightarrow{\mathbb{I}_{\varnothing,A} \cup (-)} \mathcal{M}_{\varnothing,B}\right)$$

where \cup denotes the composition functors in \mathfrak{M} (the notation is motivated by Example 4.4 below). To explain the analogue of (16), we fix an object $R \in \mathfrak{M}_{\varnothing,\varnothing}$ that will play the role of \mathbf{R}^d , and objects $O, Q \in \mathfrak{M}_{[0]}$ as well as $W \in \mathfrak{M}_{O,Q}$ that will play the role of the same-named manifolds and the bordism between them. Abbreviating $\operatorname{Aut}(R) := \operatorname{Aut}_{\mathfrak{M}_{\varnothing,\varnothing}}(R,R)$, the analogue of (16) is

$$\partial_{\mathcal{M}}^{h}W := \operatorname{fib}_{\operatorname{id}_{W}} \left(\operatorname{Map}_{\mathcal{M}_{O,Q}}(W \sqcup R, W) \to \operatorname{Map}_{\mathcal{M}_{O,Q}}(W, W) \right)_{\operatorname{Aut}(R)}$$

$$\downarrow \qquad \qquad \downarrow \qquad$$

where \sqcup denotes the symmetric monoidal structure. The horizontal map we take fibres of is induced by precomposition with $\mathrm{id}_W \sqcup (\varnothing_m \to R)$ and the vertical map is induced by the composition of the inclusion of the fibre inclusion into $\mathrm{Map}_{M \cap \Omega}(W \sqcup R, W)$ followed by

the inclusion of the fibre inclusion into
$$\operatorname{Map}_{\mathcal{M}_{O,Q}}(W \sqcup R, W)$$
 followed by
$$\operatorname{Map}_{\mathcal{M}_{O,Q}}(W \sqcup R, W) \xrightarrow{\varnothing^{(-)_{\varnothing}}} \operatorname{Map}_{\mathcal{M}_{\varnothing,\varnothing}}({}_{\varnothing}W_{\varnothing} \sqcup R, {}_{\varnothing}W_{\varnothing}) \xrightarrow{((\varnothing_m \to {}_{\varnothing}W_{\varnothing}) \sqcup \operatorname{id}_R)^*} \operatorname{Map}_{\mathcal{M}_{\varnothing,\varnothing}}(R, {}_{\varnothing}W_{\varnothing}) \quad (20)$$

where the superscript $(-)^*$ indicates precomposition. Conjugation induces an $\operatorname{Aut}_{\mathcal{M}_{O,Q}}(W)$ -action on the map (19). If the bordism W arises as a composition $W = U \cup V$ of $U \in \mathcal{M}_{O,P}$ and $V \in \mathcal{M}_{P,Q}$, then there is a commutative square of the same form as the left square in (17), with the E-subscripts replaced by \mathcal{M} -subscripts, together with an action of the gluing morphism $\operatorname{Aut}_{\mathcal{M}_{O,P}}(U) \times \operatorname{Aut}_{\mathcal{M}_{P,Q}}(V) \to \operatorname{Aut}_{\mathcal{M}_{O,Q}}(W)$ considered as a E_1 -group in $\mathcal{S}^{[1]}$. This square is induced by the following commutative diagram whose unlabeled arrows are instances of the composition (20)

$$\begin{split} \operatorname{Map}_{\mathcal{M}_{O,P}}(U,U) &\xleftarrow{(\operatorname{id}_{U} \sqcup (\varnothing_{m} \to R))^{*}} \operatorname{Map}_{\mathcal{M}_{P,Q}}(U \sqcup R,U) &\longrightarrow \operatorname{Map}_{\mathcal{M}_{\varnothing,\varnothing}}(R,{_{\varnothing}U_{\varnothing}}) \\ &\downarrow^{(-) \cup V} & \downarrow^{(-) \cup V} & \downarrow^{({_{\varnothing}U} \cup (\mathbb{I}_{P,\varnothing} \to V_{\varnothing}))_{*}} \operatorname{Map}_{\mathcal{M}_{\varnothing,\varnothing}}(W,W) & \xleftarrow{(\operatorname{id}_{W} \sqcup (\varnothing_{m} \to R))^{*}} \operatorname{Map}_{\mathcal{M}_{\varnothing,\varnothing}}({_{\varnothing}W_{\varnothing}} \sqcup R,{_{\varnothing}W_{\varnothing}}) &\longrightarrow \operatorname{Map}_{\mathcal{M}_{\varnothing,\varnothing}}(R,{_{\varnothing}W_{\varnothing}}). \end{split}$$

By design, this construction is natural in functors $\varphi \colon \mathcal{M} \to \mathcal{N}$ of symmetric monoidal double categories that preserve the initial objects $\mathbb{I}_{A,B}$ of the mapping categories. In particular, there is a map of squares from the analogue of the left square in (17) for a composition $W = U \cup V$ in \mathcal{M} to that for the composition $\varphi(W) = \varphi(U) \cup \varphi(V)$ in \mathcal{N} resulting from applying φ , and this map of squares comes with an action of the square of E_1 -groups

Example 4.3. The analogue in the context of double categories of the special case discussed in Section 4.1.2 is the following: given objects $\partial^v M \in \mathcal{M}_{[0]}$ and $M \in \mathcal{M}_{\partial^v M, \emptyset}$, we can view M as a composition $M^c := \partial^v M \times I \cup M \simeq M$ where $\partial^v M \times I \in \mathcal{M}_{\partial^v M, \partial^v M}$ now stands for the unit with respect to the composition \cup in \mathcal{M} . By the same argument as in Section 4.1.2 and setting

$$\left(\partial_{\mathcal{M}}^{vh}M \to \partial_{\mathcal{M}}^{v}M\right) \coloneqq \left(\partial_{\mathcal{M}}^{h}(\partial^{v}M \times I) \to _{\mathcal{M}}(\partial^{v}M \times I)\right),$$

the $\operatorname{Aut}_{\mathcal{M}_{\partial^v M,\varnothing}}(M)$ -action on the right-column of the square resulting from the construction, extends to an action in the overcategory $\mathcal{S}^{[1]}_{/(\partial^{oh}_{\mathcal{M}}M \to \partial^v_{\mathcal{M}}M)}$ since $\operatorname{Aut}_{\mathcal{M}_{\partial^v M,\varnothing}}(M) \simeq \operatorname{Aut}_{\mathcal{M}_{\partial^v M,\varnothing}}(M^c)$.

Example 4.4. An example of a symmetric monoidal double category \mathfrak{M} satisfying (i) and (ii) is the topological noncompact bordism double category $\operatorname{ncBord}^t(d)$ introduced in [KK24c, Section 5.3]. Its category of objects $\operatorname{ncBord}^t(d)_{[0]}$ has as objects (d-1)-dimensional (potentially noncompact) topological manifolds without boundary and mapping spaces are embeddings between them. Its mapping categories have as objects bordisms and mapping spaces are spaces of embeddings between bordisms fixing the boundary, the composition functors are induced by gluing bordisms, the composition in the mapping categories is given by by composing embeddings, and the symmetric monoidal structure is given by taking disjoint unions. The unit \emptyset is given by the empty manifold, the object \emptyset_m by the empty bordism, the initial objects in condition (i) are given by bordisms $(A \times [0,1) \sqcup (-1,0] \times B) : A \leadsto B$, and (ii) is satisfied since $\emptyset = \emptyset \times [0,1) \sqcup (-1,0] \times \emptyset$.

Given a compact bordism of manifolds with boundary $W: O \to Q$ as in Section 4.1, we may cut out the horizontal part of the boundary to obtain a bordism $W^{\circ} := (W \setminus \partial^h W) := \operatorname{int}(O) \to \operatorname{int}(Q)$ in $\operatorname{nc}\mathcal{B}\operatorname{ord}^t(d)$. Note that the left-hand square in (17) only depends on W° and its decomposition $W^{\circ} = U^{\circ} \cup_{\operatorname{int}(P)} V^{\circ}$ since U, V, and W are compatibly isotopy equivalent (relative to the interior of their vertical boundary) to the respective versions with a $(-)^{\circ}$ subscript. Moreover, going through the construction, one sees that the constructions from this section in the case $\mathcal{M} = \operatorname{nc}\mathcal{B}\operatorname{ord}^t(d)$ and $R = \mathbb{R}^d := \emptyset \to \emptyset$ specialise to those in Sections 4.1.1 and 4.1.2.

Example 4.5. A second example of a symmetric monoidal double category that satisfies (i) and (ii) is the cospan double category $Cosp(\mathcal{C})$ of a category \mathcal{C} admitting finite colimits: its category of objects is \mathcal{C} , the mapping category between objects A and B is the overcategory $\mathcal{C}_{A \sqcup B}$, the composition functor are given by taking pushouts, and the symmetric monoidal structure by taking coproducts (see [KK24c, Section 1.7.1] and [KK24a, Section 2.10] for more on $Cosp(\mathcal{C})$). The initial object $\mathbb{I}_{A,B}$ of $Cosp(\mathcal{C})_{A,B} \simeq \mathcal{C}_{A \sqcup B}$ is $id_{A \sqcup B}$. In the case $\mathcal{M} = Cosp(\mathcal{C})$, we have

$$\operatorname{Map}_{\mathcal{M}_{O,Q}}(W \sqcup R,W) = \operatorname{Map}_{\mathcal{C}_{/O \sqcup Q}}(W \sqcup R,W) \simeq \operatorname{Map}_{\mathcal{C}}(R,W) \times \operatorname{Map}_{\mathcal{C}_{/O \sqcup Q}}(W,W),$$

so it follows that the map $\partial^h_{\operatorname{Cosp}(\mathfrak{C})}(W) \to \operatorname{Cosp}(\mathfrak{C})W = \operatorname{Map}_{\mathfrak{C}}(R,W)_{\operatorname{Aut}_{\mathfrak{C}}(R)}$ is always an equivalence. If $\mathfrak{C} = \mathbb{S}_{/B}$ is the category of spaces over a connected space B and we choose for $R = (* \to B)$ the inclusion of a point, then $\operatorname{Aut}(R) \simeq \Omega B$ and for any space $W \to B$ over B, we get

$$\partial^h_{\operatorname{Cosp}(\mathcal{C})}(W) \simeq {}_{\operatorname{Cosp}(\mathcal{C})}W = \operatorname{Map}_{\mathcal{C}}(R,W)_{\operatorname{Aut}_{\mathcal{C}}(R)} \simeq \operatorname{fib}(W \to B)_{\Omega B} \simeq W.$$

Example 4.6. Generalising Example 4.5, one may consider any symmetric monoidal category $\mathbb C$ whose tensor product is compatible with geometric realisations in the sense of [KK24c, Section 1.6.4] and which is *unital*, i.e. the monoidal unit \emptyset is an initial object. Its Morita category ALG($\mathbb C$) in the sense of Section 1.7 loc.cit. is then a symmetric monoidal double category which satisfies (i) and (ii), since the free (A,B)-bimodule $F_{A,B}(\emptyset) \in \mathrm{BMod}_{A,B}(\mathbb C) = \mathrm{ALG}(\mathbb C)_{A,B}$ on the monoidal unit \emptyset for associative algebras $A,B \in \mathrm{Ass}(\mathbb C) = \mathrm{ALG}(\mathbb C)_{[1]}$ is an initial object, using that \emptyset is initial in $\mathbb C$. Given another symmetric monoidal category $\mathbb D$ satisfying the above conditions, and a symmetric monoidal functor $\varphi \colon \mathbb C \to \mathbb D$ compatible with geometric realisations in the sense of Section 1.6.4 loc.cit. the induced functor $\varphi \colon \mathrm{ALG}(\mathbb C) \to \mathrm{ALG}(\mathbb D)$ preserves the initial objects in the mapping categories, since $\varphi(F_{A,B}(\emptyset)) \cong F_{\varphi(A),\varphi(B)}(\emptyset)$; see Lemma 1.10 loc.cit. for an explanation.

4.3. **Embedding calculus.** In [KK24c, Section 5.3.2], we introduced a version of Goodwillie–Weiss embedding calculus for spaces of topological embeddings. It arose as the special case of a construction for any unital operad \mathbb{O} —namely the case when $\mathbb{O}=E_d^t$ is the BTop(d)-framed E_d -operad in the sense of Section 5.1.1 loc.cit.—and it took the form of a tower of symmetric monoidal double categories under the bordism double category nc \mathbb{B} ord $^t(d)$ from Example 4.4

$$\operatorname{ncBord}^{t}(d) \xrightarrow{E} r \operatorname{Mod}_{\infty}^{\operatorname{un}}(E_{d}^{t}) \to \cdots \to r \operatorname{Mod}_{2}^{\operatorname{un}}(E_{d}^{t}) \to r \operatorname{Mod}_{1}^{\operatorname{un}}(E_{d}^{t}) \simeq \operatorname{Cosp}(\mathbb{S}_{/\operatorname{BTop}(d)})$$
 (21)

which yields the topological embedding calculus tower upon taking mapping spaces in the mapping categories. The double categories in this sequence all satisfy conditions (i) and (ii) by Examples 4.4 and 4.6 since $r \operatorname{Mod}_k^{\operatorname{un}}(E_d^t)$ is, by definition, the Morita category of a unital symmetric monoidal category, namely the category $\operatorname{PSh}^{\operatorname{un}}(\operatorname{Disc}_{d < k}^t)$ of unital space-valued presheaves on a symmetric

monoidal category \mathbb{D} isc $_{d,\leq k}^t$ of topological d-manifolds homeomorphic to $\sqcup^n \mathbf{R}^d$ for some $n\leq k$ and topological embeddings between those, equipped with a Day convolution symmetric monoidal structure induced by disjoint union. Here unital means that the value at the presheaf at $\emptyset\in \mathbb{D}$ isc $_{d,\leq k}$ is contractible. Moreover, all functors in (21) preserve initial objects in mapping categories: for all but the first functor this was explained in Example 4.6 since the functors are by definition induced by symmetric monoidal functors compatible with geometric realisations, and for the first functor it follows from the argument in the proof of Lemma 5.7 loc.cit.. As a result, combining the discussion in Example 4.4 with the naturality of the construction in Section 4.2 and adopting the notation regarding towers of Section 1.2 loc.cit., given a compact d-dimensional bordism $W: O \leadsto Q$, we obtain a tower $(\partial_{\leq \bullet}^h W \to \leq_{\bullet} W) \in \operatorname{Tow}(\mathbb{S}^{[1]}_{|\partial^h W \to W})$ in $\mathbb{S}^{[1]}$ under $\partial^h W \to W$

$$\partial^{h}W \simeq \partial_{E}^{h}W \longrightarrow \partial_{\leq \infty}^{h}W \longrightarrow \cdots \longrightarrow \partial_{\leq 2}^{h}W \longrightarrow \partial_{\leq 1}^{h}W$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \simeq$$

$$W \simeq {}_{E}W \xrightarrow{\simeq} {}_{\leq \infty}W \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} {}_{\leq 2}W \xrightarrow{\simeq} {}_{\leq 1}W$$

$$(22)$$

where the column involving the $\leq k$ -subscripts is the instance of (19) for the image of $W^{\circ} \in \operatorname{ncBord}^t(d)_{\operatorname{int}(O),\operatorname{int}(Q)}$ in $r\operatorname{Mod}_k^{\operatorname{un}}(E_d^t)$. The tower (22) is acted upon by the tower $\operatorname{Emb}_{\partial^v}(W)^{\times} \to T_{\bullet}\operatorname{Emb}_{\partial^v}(W)^{\times}$ of E_1 -groups under $\operatorname{Emb}_{\partial^v}(W)^{\times}$ consisting of the automorphism groups of the images of W° in $r\operatorname{Mod}_{\bullet}^{\operatorname{un}}(E_d^t)$. The fact that the rightmost vertical arrow in (22) is an equivalence was explained in Example 4.5 since $r\operatorname{Mod}_1^{\operatorname{un}}(E_d^t) \simeq \operatorname{Cosp}(S_{/\operatorname{BTop}(d)})$. Regarding the horizontal maps, we can deduce the following lemma from convergence of embedding calculus:

Lemma 4.7. Fix a compact d-dimensional bordism $W: O \rightarrow Q$ of manifolds with boundary.

- (i) The map $W \to {}_{\leq k} W$ is an equivalence for all $1 \leq k \leq \infty$.
- (ii) If $d \ge 5$, the bordism W is smoothable, and $\partial^h W \subset W$ is 2-connected, then the map $\partial^h W \to \partial^h_{\le k} W$ is (k-d+3)-connected for $1 \le k \le \infty$. In particular, it is an equivalence if $k = \infty$.
- (iii) For the final conclusion in (ii), the 2-connectivity assumption can be weakened to only assuming that the inclusion $\partial^h W \subset W$ is an equivalence on tangential 2-types.

Proof. Adopting the notation from [KK24c, Section 5.3.2], the map $\text{Emb}(S \times \mathbb{R}^d, N) \to T_k \text{Emb}(S \times \mathbb{R}^d, N)$ is an equivalence for any finite set S or cardinality $\leq k$, and all topological d-manifolds N, e.g. by combining Theorem 5.3 (iii) loc.cit. with the Yoneda lemma. This implies that the map

$$W \simeq {}_{E}W = \operatorname{Emb}(\mathbf{R}^{d}, W)_{\operatorname{Emb}(\mathbf{R}^{d})^{\times}} \longrightarrow T_{k}\operatorname{Emb}(\mathbf{R}^{d}, W)_{T_{k}\operatorname{Emb}(\mathbf{R}^{d})^{\times}} = {}_{\leq k}W$$

from Part (i) is an equivalence. Under the hypothesis of Part (ii), the maps $\operatorname{Emb}_{\partial^{o}}(W \sqcup \mathbf{R}^{d}, W) \to T_{k}\operatorname{Emb}_{\partial^{o}}(W \sqcup \mathbf{R}^{d}, W)$ and $\operatorname{Emb}_{\partial^{o}}(W, W) \to T_{k}\operatorname{Emb}_{\partial^{o}}(W, W)$ are (k - d + 4)-connected by Theorem 6.3 and Remark 6.11 loc.cit., so the map between fibres

$$\operatorname{fib_{idw}}(\operatorname{Emb}_{\partial^{v}}(W \sqcup \mathbf{R}^{d}, W) \to \operatorname{Emb}_{\partial^{v}}(W, W)) \to \operatorname{fib_{idw}}(T_{k}\operatorname{Emb}_{\partial^{v}}(W \sqcup \mathbf{R}^{d}, W) \to T_{k}\operatorname{Emb}_{\partial^{v}}(W, W))$$

is (k-d+3)-connected. After taking orbits (which preserves connectivity) with respect to the action by $\operatorname{Emb}(\mathbf{R}^d)^\times \simeq T_k \operatorname{Emb}(\mathbf{R}^d)^\times$, this is precisely the map $\partial^h W \simeq \partial^h_E W \to \partial^h_{\leq k} W$ in the claim, so Part (ii) follows. The final Part (iii) follows from the same argument using Theorem 6.3 loc.cit..

4.4. **Particle embedding calculus.** As explained in [KK24c, Section 5.7], there is a map of towers of symmetric monoidal double categories $r\mathcal{M}\mathrm{od}^{\mathrm{un}}_{\bullet}(E^t_d) \to r\mathcal{M}\mathrm{od}^{\mathrm{un}}_{\bullet}(E^p_{d,\leq \bullet})$ such that the composition $\mathrm{nc}\mathcal{B}\mathrm{ord}^t(d) \to r\mathcal{M}\mathrm{od}^{\mathrm{un}}_{\bullet}(E^t_d) \to r\mathcal{M}\mathrm{od}^{\mathrm{un}}_{\bullet}(E^p_{d,\leq \bullet})$ yields the particle embedding calculus tower in the sense of loc.cit. upon taking mapping spaces in mapping categories. This induces a map of towers in $\mathcal{S}^{[1]}_{\partial^h W \to W}$ from $(\partial^h_{\leq \bullet} W \to {}_{\leq \bullet} W)$ in (22) to the analogous tower $(\partial^h_{\leq \bullet, p} W \to {}_{\leq \bullet, p} W)$ obtained by replacing $r\mathcal{M}\mathrm{od}^{\mathrm{un}}_{\bullet}(E^t_d)$ with $r\mathcal{M}\mathrm{od}^{\mathrm{un}}_{\bullet}(E^p_{d,\leq \bullet})$. This map turns out to consist of equivalences:

Lemma 4.8. The maps $\partial_{\leq k}^h W \to \partial_{\leq k,p}^h W$ and $d \leq k W \to d \leq k,p W$ are equivalences for all $1 \leq k \leq \infty$.

Proof. Recall from [KK24c, Section 5.7] that there is a pullback square

$$\begin{split} r \mathcal{M} \mathrm{od}_k^{\mathrm{un}}(E_d^t) & \longrightarrow \mathrm{Cosp}(\mathbb{S}_{/\mathrm{BTop}(d)}) \\ \downarrow & \downarrow \\ r \mathcal{M} \mathrm{od}_k^{\mathrm{un}}(E_{d,\leq k}^p) & \to \mathrm{Cosp}(\mathbb{S}_{/\mathrm{BAut}_{\leq k}(E_d)}) \end{split}$$

of symmetric monoidal double categories whose right vertical row is induced by postcomposition with the composition of the map $t \colon \mathrm{BTop}(d) \to \mathrm{BAut}(E_d)$ from Section 2.2.2 with the map $\mathrm{BAut}(E_d) \to \mathrm{BAut}_{\leq k}(E_d)$ induced by truncation (see Section 2.2). This induces pullback squares

so it suffices to show that the maps ① and ② are equivalences. By the discussion in Example 4.5, the map ① is equivalent to ② which is in turn equivalent to id_W, so it is in particular an equivalence. \Box

Lemma 4.8 implies that the action of the tower of E_1 -groups $T_{\bullet}\mathrm{Emb}_{\partial^v}(W)^{\times}$ on $(\partial_{\leq \bullet}^h W \to_{\leq \bullet} W)$ factors through to the map of towers of E_1 -groups $T_{\bullet}\mathrm{Emb}_{\partial^v}(W)^{\times} \to T_{\bullet}\mathrm{Emb}_{\partial^v}^p(W)^{\times}$ induced by the map of towers of symmetric monoidal double categories $r \mathcal{M} \mathrm{od}_{\bullet}^{\mathrm{un}}(E_d^t) \to r \mathcal{M} \mathrm{od}_{\bullet}^{\mathrm{un}}(E_{d,\leq \bullet}^p)$.

4.5. **Proof of Theorem 4.1.** Applying the above discussion to the case of triads $M: \partial^{v} M \rightsquigarrow \emptyset$ from Section 4.1.2 and Example 4.3, we obtain map of towers of squares under (18)

$$\frac{\partial^{vh}M}{\partial^{h}M} \xrightarrow{\simeq} \frac{\partial^{vh}M}{\partial^{s}} \xrightarrow{\simeq} \frac{\partial^{vh}M}{\partial^{vh}M} \xrightarrow{\simeq}$$

acted upon by $\operatorname{Emb}_{\partial^v}(M)^\times \to T_{\bullet}\operatorname{Emb}_{\partial^v}(M)^\times \to T_{\bullet}\operatorname{Emb}_{\partial^v}^p(M)^\times$, compatibly with the a priori action of $T_{\bullet}\operatorname{Emb}_{\partial^v}^p(M)^\times$ on $\partial^v M \to M$ induced by the composition of symmetric monoidal double categories $r\mathcal{M}\operatorname{od}_{\bullet}^{\mathrm{un}}(E_{d,\leq \bullet}^p) \to \operatorname{Cosp}(\mathbb{S}_{/\operatorname{BAut}_{\leq \bullet}(E_d)}) \to \operatorname{Cosp}(\mathbb{S})$. The arrows labeled by \cong are equivalences as a result of Lemmas 4.8, and 4.7 (i) in the cases $W = \partial^v M$ or W = M. Moreover, by Lemma 4.7 (iii), if the hypothesis of Theorem 4.1 is satisfied, the two unlabeled horizontal maps are for $\bullet = \infty$ also equivalences, so we obtain the asserted filler in Theorem 4.1.

4.5.1. Truncated variant of Theorem 4.1. One might think that Theorem 4.1 ought to, under suitable hypothesis, be extendable to a factorisation of maps of towers when replacing $T_{\infty} \text{Emb}_{\partial^{v}}(M)^{\times}$, $T_{\infty} \text{Emb}_{\partial^{v}}^{p}(M)^{\times}$, and $\text{Aut}_{\partial^{v}}(M, \partial^{h})$ by towers $T_{\bullet} \text{Emb}_{\partial^{v}}(M)^{\times}$, $T_{\bullet} \text{Emb}_{\partial^{v}}^{p}(M)^{\times}$, and $\text{Aut}_{\partial^{v}}(M, \partial^{h}_{\leq \bullet})$. Note, however, that the final tower $\text{Aut}_{\partial^{v}}(M, \partial^{h}_{\leq \bullet})$ is not even defined: the existence of the tower of squares given as the middle diagonal face of (23) does not yield a tower on automorphism spaces. However, there is the following workaround to this issue: Given $k \in \mathbb{N} \cup \{\infty\}$, we may consider the endofunctor $\rho_k \colon \mathbb{S}^{[1]} \to \mathbb{S}^{[1]}$ that sends a map $X \to Y$ to the first map $X \to \rho_k X$ in the Moore-Postnikov k-factorisation of $X \to Y$ (the unique factorisation into a k-connected map followed by a k-coconnected map). This defines a localisation of the category $\mathbb{S}^{[1]}$ [Lur09, 5.2.8.16,5.2.8.19], so we obtain a localisation $\mathbb{S}^{[1] \times [1]} = \text{Fun}([1], \mathbb{S}^{[1]}) \to \text{Fun}([1], \mathbb{S}^{[1]}) = \mathbb{S}^{[1] \times [1]}$ of the category of squares $\mathbb{S}^{[1] \times [1]}$, given by postcomposition with ρ_k . This satisfies $\rho_n \circ \rho_k \simeq \rho_{\min(n,k)}$. In particular, for any monotonically increasing function $c \colon \mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}$, we obtain a tower of E_1 -groups $\text{Aut}_{\partial^{v}}(M, \partial^{h}) \to \text{Aut}_{\partial^{v}}(M, \rho_{\alpha(\bullet)} \partial^{h})$ under $\text{Aut}_{\partial^{v}}(M, \partial^{h})$ whose kth stage is the space of automorphisms of the leftmost diagonal square in (23) when applying ρ_k to the two vertical

maps, considered as an object in ${\mathbb S^{[1]}}_{/(\rho_k\partial^{vh}M)\to\partial^v M)}$. Using this, a minor extension of the proof of Theorem 4.1 in which the role of Lemma 4.7 (iii) is replaced by Lemma 4.7 (ii) shows:

Theorem 4.9. Let $(M, \partial^v M, \partial^h M)$ be a compact smoothable d-dimensional manifold triad with $d \ge 5$ and $c \colon \mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}$ monotonically increasing function. If

- (i) $c(k) \le k d + 2$ for all k and
- (ii) the inclusions $\partial^h M \subset M$ and $\partial^{vh} M \subset \partial^v M$ are 2-connected,

then there exists a dashed filler in the diagram

$$\operatorname{Homeo}_{\partial^{v}}(M) \longrightarrow \operatorname{Emb}_{\partial^{v}}(M)^{\times} \to T_{\bullet} \operatorname{Emb}_{\partial^{v}}(M)^{\times} \to T_{\bullet} \operatorname{Emb}_{\partial^{v}}^{p}(M)^{\times}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{Aut}_{\partial^{v}}(M, \rho_{c(\bullet)}\partial^{h}) \xrightarrow{\longleftarrow} \operatorname{Aut}_{\partial^{v}}(M).$$

of towers of E_1 -groups.

- 4.6. **Digression:** T_{∞} -**boundaries.** We conclude this section with some remarks on the embedding calculus approximation $\partial^h M \to \partial^h_{\leq \infty} M$ to the "moving" part of the boundary of a compact d-dimensional manifold triad. Lemma 4.7 (iii) implies that this map is an equivalence if M is smoothable, $d \geq 5$, and the inclusion $\partial^h M \subset M$ is a tangential 2-type equivalence, but it would be interesting to determine $\partial^h_{\leq \infty} M$ when these assumptions are not satisfied. Here are some observations regarding this, for simplicity phrased in the case $\partial^h M = \partial M$ where we write $\partial^h_{<\infty} M := \partial_{\leq \infty} M$:
- (i) For $d \le 2$, we have $\partial M \simeq \partial_{<\infty} M$ as a consequence of the main result of [KK24b].
- (ii) Comparing the definition of $\partial M \simeq \partial_E M$ with that of $\partial_{\leq \infty} M$ and using that $T_{\infty} \text{Emb}(-, M)$ satisfies descent for complete Weiss ∞ -covers [KK24c, Lemma 5.9], it follows that the map $\partial M \to \partial_{\leq \infty} M$ is equivalent to the map

$$\partial M \longrightarrow \lim_{D \subset \text{int}(M)} M \backslash D$$
 (24)

induced by inclusion, where the (homotopy) limit runs is taken over the opposite of the poset of submanifolds $D \subset \text{int}(M)$ homeomorphic to $\sqcup^k \mathbb{R}^d$ for some $k \geq 0$, ordered by inclusion.

(iii) There are examples for which (24) is not an equivalence: Fix a homology (d-1)-sphere Σ with nontrivial fundamental group and $d \geq 3$ which bounds a contractible manifold W_{Σ} (e.g. Σ could be the Poincaré homology sphere) and consider the commutative square

$$\Sigma \longrightarrow \lim_{D \subset \operatorname{int}(W_{\Sigma})} W_{\Sigma} \backslash D$$

$$\downarrow \qquad \qquad \downarrow$$

$$\Sigma^{+} \longrightarrow \lim_{D \subset \operatorname{int}(W_{\Sigma})} (W_{\Sigma} \backslash D)^{+}$$

whose vertical arrows are induced by taking Quillen's plus-construction. Since $d \ge 3$, the inclusion $W_{\Sigma} \setminus D \to W_{\Sigma}$ is 2-connected by transversality, so $W_{\Sigma} \setminus D$ is simply connected which implies that the right vertical map is an equivalence. In particular, as $\Sigma^+ \simeq S^{d-1}$, the upper horizontal map is trivial on fundamental groups, so it cannot be an equivalence.

- (iv) There are also examples for which (24) is an equivalence but that are not covered by Lemma 4.7 (iii), for instance any nonempty closed manifold M. To see this, it suffices to prove that $\partial_{\leq \infty} M$ is empty if M is closed. Assume for a contradiction that it is not, so there is an element of $T_{\infty} \text{Emb}(M \sqcup \mathbf{R}^d, M)$ that hits the identity component of $T_{\infty} \text{Emb}(M, M)$. By isotopy extension for embedding calculus [KK24c, Section 5.4.2], we have a fibre sequence $T_{\infty} \text{Emb}(M, M \backslash \mathbf{R}^d) \to T_{\infty} \text{Emb}(M \sqcup \mathbf{R}^d, M) \to T_{\infty} \text{Emb}(\mathbf{R}^d, M)$, so we conclude that there is an element of $T_{\infty} \text{Emb}(M, M \backslash \mathbf{R}^d)$ that maps to the identity component of $T_{\infty} \text{Emb}(M, M)$. By considering the underlying maps of spaces, this shows that the identity on M factors up to homotopy over $M \backslash \mathbf{R}^d \subset M$. This is impossible if M is closed (consider the top \mathbf{F}_2 -homology).
- (v) The map $\partial M \to \partial_{\leq \infty} M$ is a retract after applying double suspension $\Sigma^2_+(-)$ if M has no closed components and $d \geq 3$. The argument for this uses the *double M-suspension* of a map $X \to M$, defined as the pushout $\Sigma^2_M X := M \times S^1 \cup_{X \times S^1} X \times D^2$. For the inclusion $Y \subset M$ of

a reasonable subspace of the interior (e.g. a submanifold inclusion), the map $\Sigma_M^2(M \setminus Y) \to (M \times D^2) \setminus (Y \times \{0\})$ is an equivalence. Using this, one checks that the composition $\Sigma_M^2 \partial M \to \Sigma_M^2 \lim_{D \subset \operatorname{int}(M)} M \setminus D \to \lim_{D \subset \operatorname{int}(M)} \Sigma_M^2(M \setminus D)$ agrees with the map $\partial M \times D^2 \to \partial_{\leq \infty}^h(M \times D^2)$, which is an equivalence as a result of Lemma 4.7 since $\partial M \times D^2 \to M \times D^2$ is 2-connected. This implies that $\Sigma_M^2 \partial M \to \Sigma_M^2 \partial_{\leq \infty} M$ admits a cosection compatible with the inclusions of $M \times S^1$. Collapsing $M \times S^1$ then gives the statement. We suspect that $\partial M \to \partial_{\leq \infty} M$ is an equivalence after applying $\Sigma_+^2(-)$, not just a retract, and it also seems possible that suspending once suffices.

Remark 4.10 (Boundaries and complements of \mathcal{D} isc-presheaves). The construction of the embedding calculus approximation $\partial^h M \to \partial^h_{\leq k} M$ to the moving part of the manifold can be generalised in various directions. We briefly outline two of them, for simplicity in the case $\partial^h M = \partial M$ and $k = \infty$:

(i) Given a presheaf $X \in \mathrm{PSh}(\mathrm{Disc}_d^t)$, its underlying homotopy type is the orbit space $X(\mathbf{R}^d)_{\mathrm{Emb}(\mathbf{R}^d)^{\times}}$. With this terminology, $\partial_{\leq \infty} M$ arises as the underlying homotopy type of a presheaf ∂E_M , namely $\mathrm{Disc}_d^t \ni D \mapsto \mathrm{fib}_{\mathrm{id}} (T_\infty \mathrm{Emb}(M \sqcup D, M) \to T_\infty \mathrm{Emb}(M, M))$. The latter arises as a fibre

$$\partial E_M \simeq \operatorname{fib}_{\operatorname{id}}(\underline{\operatorname{Map}}(E_M, E_M) \to \operatorname{Map}(E_M, E_M)) \in \operatorname{PSh}(\operatorname{Disc}_d^t)$$

where $\underline{\mathrm{Map}}(-,-)$ is the internal mapping object in $\mathrm{PSh}(\mathbb{D}\mathrm{isc}_d^t)$ and the map to the constant presheaf on the mapping space in $\mathrm{PSh}(\mathbb{D}\mathrm{isc}_d^t)$ from E_M to itself is induced by the fact that $\emptyset \in \mathbb{D}\mathrm{isc}_d^t$ is initial. The latter formula also makes sense when E_M is replaced by an arbitrary presheaf X on $\mathbb{D}\mathrm{isc}_d^t$, not necessarily induced by a manifold. The maps $\mathrm{Emb}(-,-) \to T_\infty \mathrm{Emb}(-,-)$ induce to a map of presheaves $E_{\partial M \times I} \to \partial E_M$ which yields the map $\partial M \to \partial_{\leq \infty} M$ considered above on underlying homotopy types. Moreover, the internal composition in $\underline{\mathrm{Map}}(-,-)$ equips ∂E_M (more generally: ∂X) with an associative algebra structure such that $E_{\partial M \times I} \to \partial E_M$ is an algebra map through which the $E_{\partial M \times I}$ -module structure on E_M given by "stacking" factors (more generally: any presheaf X is a module over ∂X). Much of the discussion in this chapter on $\partial_{\leq \infty} M$ can be generalised to the level of presheaves, e.g. the conditions we discussed under which $\partial M \to \partial_{\leq \infty} M$ is an equivalence in fact imply that the map of presheaves $E_{\partial M \times I} \to \partial E_M$ is an equivalence. One can also define particle embedding calculus versions of this "presheaf boundary". There are further variants of the construction, e.g. when part of the boundary is fixed.

(ii) More generally, given a map of $\mathbb{D}\mathrm{isc}_d^t$ -presheaves $\varphi\colon X\to Y$, one can define a "complement" $Y\setminus^{\varphi}X:=\mathrm{fib}_{\varphi}(\underline{\mathrm{Map}}(X,Y)\to\mathrm{Map}(X,Y))\in\mathrm{PSh}(\mathbb{D}\mathrm{isc}_d^t)$ which generalises ∂X in that $\partial X\simeq X\setminus^{\mathrm{id}}X$. This complement admits a module structure over ∂X , and is the analogue on the level of presheaves of taking the complement of a codimension 0-embedding of manifolds. In fact, given a codimension 0 embedding $e\colon M\hookrightarrow\mathrm{int}(N)$ between compact manifolds, there is a map of presheaves $E_{M\setminus e(\mathrm{int}(N))}\to E_N\setminus^{E_e}E_M$ which is a module map with respect to the algebra map $E_{\partial M\times I}\to\partial E_M$ and an equivalence in favourable circumstances.

These "boundaries" and "complements" of presheaves, as well as extensions of these constructions, allow one to mimic various constructions for manifolds on the level of presheaves. We expect this to find further applications in future work.

5. A PULLBACK DECOMPOSITION OF SPACES OF SELF-EMBEDDINGS

This section serves to prove a version of the pullback decomposition as in the introduction for certain spaces of self-embeddings instead of homeomorphisms. Theorem F will be a special case.

5.1. **The pullback decomposition.** For a compact d-dimensional topological manifold triad $(M, \partial^v M, \partial^h M)$, we saw in (14) that the forgetful map $\operatorname{Homeo}_{\partial^v}(M) \to \operatorname{Aut}_{\partial^v}(M, \partial^h)$ factors through the group-like components $\operatorname{Emb}_{\partial^v}(M)^\times$ of the space of self-embeddings of M that fix a neighbourhood of $\partial^v M$. By taking derivatives, the latter maps to the E_1 -group of self-equivalences $\operatorname{Aut}_{\partial^v}(TM)$ of the topological tangent bundle that fix $TM|_{\partial^v}$. More precisely, the latter is defined as the automorphisms of a tangent classifier $M \to \operatorname{BTop}(d)$ in the under-/overcategory $(\delta_{/\operatorname{BTop}(d)})_{\partial^v M/}$. This

induces a factorisation

$$\operatorname{Emb}_{\partial^{v}}(M)^{\times} \longrightarrow \operatorname{Aut}_{\partial^{v}}(TM, \partial^{h}) \longrightarrow \operatorname{Aut}_{\partial^{v}}(M, \partial^{h})$$
 (25)

through the pullback $\operatorname{Aut}_{\partial^{v}}(TM,\partial^{h}) := \operatorname{Aut}_{\partial^{v}}(TM) \times_{\operatorname{Aut}_{\partial^{h}}(M)} \operatorname{Aut}_{\partial^{v}}(M,\partial^{h})$ which can be thought as the space of self-equivalences of TM as an Euclidean bundle that fix $TM|_{\partial^{v}M}$ pointwise and $TM|_{\partial^{h}M}$ setwise. The general version of Theorem F will show that the delooping of this sequence can often completed to a pullback square once we add appropriate tangential structures and rationalise. We will now make these modifications precise.

5.1.1. Tangential structures and rationalisations. A tangential structure is a map $\theta \colon B \to \operatorname{BTop}(d)$, and a θ -structure ℓ^v on $TM|_{\partial^v M}$ is a lift of the restriction $TM|_{\partial^v M} \colon \partial^v M \to \operatorname{BTop}(d)$ along θ . Fixing such B and ℓ^v , the first two E_1 -groups in (25) act on the space of θ -structures on TM that agree with ℓ^v on $TM|_{\partial^v M}$, that is, the space of lifts $\ell \colon M \to B$ of $TM \colon M \to \operatorname{BTop}(d)$ along θ that extend $\ell^v \colon TM|_{\partial^v M} \to B$. Taking orbits, we obtain a sequence of the form

$$\operatorname{BEmb}_{\partial^{v}}^{\theta}(M; \ell^{v})^{\times} \longrightarrow \operatorname{BAut}_{\partial^{v}}^{\theta}(TM, \partial^{h}; \ell^{v}) \longrightarrow \operatorname{BAut}_{\partial^{v}}(M, \partial^{h}), \tag{26}$$

which will now further modify by replacing the second map by the initial map in its Moore–Postnikov 1-factorisation $\mathrm{BAut}_{\partial^v}^\theta(TM,\partial^h;\ell^v)\to\mathrm{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}\to\mathrm{BAut}_{\partial^v}(M,\partial^h)$. This modification is mild in that the homotopy type of $\mathrm{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}$ is given by

$$\mathrm{BAut}_{\partial^{v}}(M, \partial^{h})^{\ell^{v}} \simeq \bigsqcup_{\lceil \ell \rceil \in \theta \text{-st}} \mathrm{BAut}_{\partial^{v}}^{\ell}(M, \partial^{h}),$$

where θ -st is the set of components of $\operatorname{BAut}_{\partial^v}^\theta(TM, \partial^h; \ell^v)$, which is the set of orbits of the action of $\pi_0(\operatorname{Aut}_{\partial^v}(TM, \partial^h))$ on the path components of the space of θ -structures that agree with ℓ^v on $TM|_{\partial^v M}$, and $\operatorname{BAut}_{\partial^v}^\ell(M, \partial^h)$ is the covering space of $\operatorname{BAut}_{\partial^v}^\ell(M, \partial^h)$ corresponding to the image on fundamental groups based at ℓ of the map $\operatorname{BAut}_{\partial^v}^\theta(TM, \partial^h; \ell^v) \to \operatorname{BAut}_{\partial^v}(M, \partial^h)$. As a final modification, we replace $\operatorname{BEmb}_{\partial^v}^\theta(M; \ell^v)^\times$ and $\operatorname{BAut}_{\partial^v}^\theta(TM, \partial^h; \ell^v)$ by their fibrewise rationalisation over $\operatorname{BAut}_{\partial^v}(M, \partial^h)^{\ell^v}$ in the sense of Section 2.1.1 (indicated by a $(-)_{fQ}$ -subscript), arriving at a variant of the sequence (26) that has the form

$$\operatorname{BEmb}_{\partial^{v}}^{\theta}(M; \ell^{v})_{\mathrm{fQ}}^{\times} \longrightarrow \operatorname{BAut}_{\partial^{v}}^{\theta}(TM, \partial^{h}; \ell^{v})_{\mathrm{fQ}} \longrightarrow \operatorname{BAut}_{\partial^{v}}(M, \partial^{h})^{\ell^{v}}. \tag{27}$$

5.1.2. Statement of the pullback decomposition. With these explanations out of the way, we can state the main result of this section. We say that a sequence $X \to Y \to Z$ of spaces can be completed to a pullback square if there exists a space W and a pullback square in S

$$\begin{array}{c} X \to W \\ \downarrow & \downarrow \\ Y \to Z \end{array}$$

which involves the two given maps. An *oriented tangential structure* is a map of the form $\theta \colon B \to BSTop(d)$. Postcomposition with $BSTop(d) \to BTop(d)$ gives a tangential structure in above sense.

Theorem 5.1. Let $(M, \partial^v M, \partial^h M)$ a smoothable compact d-manifold triad with $d \ge 5$, $\theta \colon B \to BSTop(d)$ an oriented tangential structure with B a nilpotent space, and ℓ^v a θ -structure on $TM|_{\partial^v}$. If

- (i) θ factors through the stabilisation map BSTop $(d-2) \to BSTop(d)$ after rationalisation,
- (ii) the inclusions $\partial^h M \subset M$ and $\partial^{vh} M \subset \partial^v M$ are both 2-connected, where $\partial^{vh} M = \partial^v M \cap \partial^h M$, then the sequence of spaces (27) can be completed to a pullback square.

We prove Theorem 5.1 in Section 5.2 below, after explaining why the variant of this result for spaces of homeomorphisms stated in the introduction as Theorem F follows from it.

Proof of Theorem F. We argue that Theorem F is equivalent to Theorem 5.1 applied to the triad $(M, \partial^v M, \partial^h M) = (M, \frac{1}{2}\partial M, D^{d-1})$. To see this, note that for $d \ge 5$ the inclusions $\partial^h M = D^{d-1} \subset M$ and $\partial^{vh} M = S^{d-2} \subset \frac{1}{2}\partial M = \partial^v M$ are 2-connected if and only if M and $\frac{1}{2}\partial M$ are 2-connected, and that the inclusion $\frac{1}{2}\partial M = \partial M \setminus I(D^{d-1}) \subset \partial M$ is (d-1)-connected. Thus, for $d \ge 5$, the conditions

(ii) in Theorem F and in Theorem 5.1 are equivalent. It remains to prove that the sequences in the two results, (3) and (27), are equivalent, which follows by showing that the forgetful maps

$$\operatorname{Homeo}_{\partial}(M) \longrightarrow \operatorname{Emb}_{1/2\partial}(M) \quad \text{ and } \quad \operatorname{Aut}_{\partial}(M) \longrightarrow \operatorname{Aut}_{1/2\partial}(M, D^{d-1})$$

are equivalences (this is even slightly stronger, since it shows $\mathrm{Emb}_{1/2\partial}(M) = \mathrm{Emb}_{1/2\partial}(M)^{\times}$). For the second map, this follows from the contractibility of $\mathrm{Aut}_{\partial}(D^{d-1})$. For the first map, we argue that all its fibres are contractible. By isotopy extension, the fibre at an embedding $e \in \mathrm{Emb}_{1/2\partial}(M)$ is equivalent to $\mathrm{Homeo}_{\partial}(D^d, M \backslash e(M \backslash D^{d-1}))$. Using $M \cong (M \backslash e(M \backslash D^{d-1})) \cup_{D^{d-1}} e(M)$ and the Seifert–van Kampen and Mayer–Vietoris theorem, one deduces that $(M \backslash e(M \backslash D^{d-1}))$ is 1-connected and acyclic, so homeomorphic to D^d by the topological Poincaré conjecture. We thus have $\mathrm{Homeo}_{\partial}(D^d, M \backslash e(M \backslash D^{d-1})) \cong \mathrm{Homeo}_{\partial}(D^d) \cong *$ by the Alexander trick, and the claim follows. \square

Remark 5.2. There is a subtlety in the statement of Theorem 5.1 (and thus also in Theorem F) having to do with rationalisations. These results involve the fibrewise rationalisations in the sense of Section 2.1 (i.e. fibrewise Q-completion) of the spaces $\operatorname{BEmb}_{\partial^v}^{\theta}(M;\ell^v)^{\times}$ and $\operatorname{BAut}_{\partial^v}^{\theta}(TM,\partial^h;\ell^v)$ over $\operatorname{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}$. Fibrewise rationalisations behaves well for $\operatorname{BAut}_{\partial^v}^{\theta}(TM,\partial^h;\ell^v)$ in that it induces isomorphisms on rational (co)homology and rationalised higher homotopy groups, since it is not hard to see that this space is fibrewise nilpotent over $\operatorname{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}$ (see the proof of Proposition 5.5). For $\operatorname{BEmb}_{\partial^v}^{\theta}(M;\ell^v)^{\times}$ this is not so clear. This issue can be addressed in two ways:

- (i) Extending the strategy of proof of [KuRW20, Theorem C], one can prove the missing fibrewise nilpotency statement of $\operatorname{BEmb}_{\partial^v}^{\theta}(M; \ell^v)^{\times}$ for all manifolds M as in Theorem 5.1 and Theorem F, so there is no issue after all. We intend to revisit this point as part of future work.
- (ii) For all results in this work, this nilpotency statement is actually not necessary. In particular, the proofs of Theorem 5.1 and Theorem F do not rely on it, and neither do the applications such as Theorems A or E. This is because all we use in the proofs (see e.g. Remark 6.7) is that the Q-cohomology of fibrewise rationalisation contains the Q-cohomology of the unrationalised space as a summand (see Section 2.1.3), without any assumption (see also Remark 6.7).
- 5.2. **The proof of Theorem 5.1.** To prove Theorem 5.1, we first formulate a variant of this result in the context of embedding calculus, then explain why Theorem 5.1 follows from this variant, and finally prove the variant. Throughout the section we fix a triad $(M, \partial^v M, \partial^h M)$ satisfying the assumptions in Theorem 5.1. To state the variant, we define a version $T_{\bullet}\text{Emb}_{\partial^v}(M, \partial^h)^{\times}$ of the embedding calculus tower $T_{\bullet}\text{Emb}_{\partial^v}(M)^{\times}$ from Section 4.3 by the pullback in towers of spaces

$$T_{\bullet} \operatorname{Emb}_{\partial^{v}}(M, \partial^{h})^{\times} \longrightarrow T_{\bullet} \operatorname{Emb}_{\partial^{v}}(M)^{\times}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{Aut}_{\partial^{v}}(M, \partial^{h}) \longrightarrow \operatorname{Aut}_{\partial^{v}}(M, \rho_{c(\bullet)} \partial^{h}), \qquad (28)$$

where the right vertical map is the one from Theorem 4.9 and the bottom horizontal map is induced by truncation. Here and henceforth, we adopt the convention that a space (in this case $\operatorname{Aut}_{\partial^{u}}(M, \partial^{h})$) is viewed as a constant tower. This definition depends on the choice of a function $c \colon \mathbb{N} \to \mathbb{N}$ as in Theorem 4.9 which we fix once and for all. Since $T_{1}\operatorname{Emb}_{\partial^{u}}(M)^{\times} \simeq \operatorname{Aut}_{\partial^{u}}(TM)$ (see [KK24c, Section 5]), we have $T_{1}\operatorname{Emb}_{\partial^{u}}(TM, \partial^{h})^{\times} \simeq \operatorname{Aut}_{\partial^{u}}(TM, \partial^{h})$. Using this and replacing the role of $\operatorname{Emb}_{\partial^{u}}(M)^{\times}$ in the construction of (27) by $T_{k}\operatorname{Emb}_{\partial^{u}}(M, \partial^{h})^{\times}$ results in a sequence of towers of spaces

$$\mathrm{B}T_{\bullet}\mathrm{Emb}_{\partial^{v}}^{\theta}(M,\partial^{h};\ell^{v})_{\mathrm{fQ}}^{\times}\longrightarrow\mathrm{BAut}_{\partial^{v}}^{\theta}(TM,\partial^{h};\ell^{v})_{\mathrm{fQ}}\longrightarrow\mathrm{BAut}_{\partial^{v}}(M,\partial^{h})^{\ell^{v}}.\tag{29}$$

The E_1 -map $\operatorname{Emb}_{\partial_0}(M)^{\times} \to \operatorname{Aut}_{\partial^v}(M, \partial^h)$ from (14) and the tower of E_1 -maps $\operatorname{Emb}_{\partial^v}(M)^{\times} \to T_{\bullet}\operatorname{Emb}_{\partial^v}(M)^{\times}$ combine to the tower of E_1 -map $\operatorname{Emb}_{\partial_0}(M)^{\times} \to T_{\bullet}\operatorname{Emb}_{\partial^v}(M, \partial^h)^{\times}$ which induces a map of towers of spaces

$$\operatorname{BEmb}_{\partial^{v}}^{\theta}(M; \ell^{v})_{\mathrm{fQ}}^{\times} \longrightarrow \operatorname{B}T_{\bullet}\operatorname{Emb}_{\partial^{v}}^{\theta}(M, \partial^{h}; \ell^{v})_{\mathrm{fQ}}^{\times}. \tag{30}$$

This map turns out to become an equivalence after taking limits:

Lemma 5.3. The connectivity of the maps in the map of tower (30) diverges with ●. In particular,

$$\mathrm{BEmb}_{\partial^v}^{\theta}(M;\ell^v)_{\mathrm{fO}}^{\times} \simeq \lim_k \left(\mathrm{B} T_k \mathrm{Emb}_{\partial^v}^{\theta}(M,\partial^h;\ell^v)_{\mathrm{fO}}^{\times} \right).$$

Proof. We first show that the connectivity of the analogous maps without the fQ-subscripts diverges. Both sides are the orbits of an action on the same space of θ -structures, so comparing the two fibre sequence involving the orbits, it suffices to prove the connectivity of the map $\operatorname{Emb}_{\partial^{v}}(M)^{\times} \to T_{k}\operatorname{Emb}_{\partial^{v}}(M,\partial^{h})^{\times}$ between the E_{1} -groups that act diverges with k. Moreover, since $\operatorname{Emb}_{\partial^{v}}(M)^{\times} \to T_{k}\operatorname{Emb}_{\partial^{v}}(M)^{\times}$ is at least (k-d+4)-connected by [KK24c, Theorem 6.3 and Remark 6.11], it is enough to show that the connectivity of the upper horizontal map $T_{k}\operatorname{Emb}_{\partial^{v}}(M,\partial^{h})^{\times} \to T_{k}\operatorname{Emb}_{\partial^{v}}(M)^{\times}$ in the pullback (28) diverges, or equivalently that the connectivity of the bottom map $\operatorname{Aut}_{\partial^{v}}(M,\partial^{h}) \to \operatorname{Aut}_{\partial^{v}}(M,\rho_{c(k)}\partial^{h})$ diverges. This map is obtained by taking group-like components of the map on horizontal fibres in the square of mapping spaces in $\mathbb{S}^{[1]}$

$$\operatorname{Map}_{\mathbb{S}^{[1]}}\left(\begin{array}{c} \partial^{h}M & \partial^{h}M \\ \downarrow & , & \downarrow \\ M & M \end{array}\right) \xrightarrow{\operatorname{inc}^{*}} \operatorname{Map}_{\mathbb{S}^{[1]}}\left(\begin{array}{c} \partial^{vh}M & \partial^{h}M \\ \downarrow & , & \downarrow \\ \partial^{v}M & M \end{array}\right)$$

$$\downarrow \rho_{c(k)} \qquad \qquad \downarrow \rho_{c(k)}$$

$$\operatorname{Map}_{\mathbb{S}^{[1]}}\left(\begin{array}{c} \rho_{c(k)}\partial^{h}M & \rho_{c(k)}\partial^{h}M \\ \downarrow & , & \downarrow \\ M & M \end{array}\right) \xrightarrow{(\rho_{c(k)}\operatorname{inc})^{*}} \operatorname{Map}_{\mathbb{S}^{[1]}}\left(\begin{array}{c} \rho_{c(k)}\partial^{vh}M & \rho_{c(k)}\partial^{h}M \\ \downarrow & , & \downarrow \\ M & M \end{array}\right).$$

Using that $\rho_{c(k)} : \mathbb{S}^{[1]} \to \mathbb{S}^{[1]}$ is a localisation, we can rewrite the two vertical maps as

$$\mathsf{Map}_{\mathbb{S}^{[1]}}\Big(\begin{smallmatrix} \partial^h M & \partial^h M \\ \downarrow & , & \downarrow \\ M & M \end{smallmatrix}\Big) \to \mathsf{Map}_{\mathbb{S}^{[1]}}\Big(\begin{smallmatrix} \partial^h M & \rho_{c(k)} \partial^h M \\ \downarrow & , & \downarrow \\ M & M \end{smallmatrix}\Big) \text{ and } \mathsf{Map}_{\mathbb{S}^{[1]}}\Big(\begin{smallmatrix} \partial^{vh} M & \partial^h M \\ \downarrow & , & \downarrow \\ \partial^v M & M \end{smallmatrix}\Big) \to \mathsf{Map}_{\mathbb{S}^{[1]}}\Big(\begin{smallmatrix} \partial^{vh} M & \rho_{c(k)} \partial^h M \\ \downarrow & , & \downarrow \\ \partial^v M & M \end{smallmatrix}\Big),$$

so it suffices to show that these two maps have diverging connectivity, for which it is in turn enough to show that the maps $\operatorname{Map}_{\mathcal{S}}(\partial^h M, \partial^h M) \to \operatorname{Map}_{\mathcal{S}}(\partial^h M, \rho_{c(k)}\partial^h M)$ and $\operatorname{Map}_{\mathcal{S}}(\partial^{vh} M, \partial^h M) \to \operatorname{Map}_{\mathcal{S}}(M, \rho_{c(k)}\partial^h M)$ induced by postcomposition with the c(k)-connected maps $\partial^h M \to \rho_{c(k)}\partial^h M$ and $\partial^{vh} M \to \rho_{c(k)}\partial^{vh} M$ have diverging connectivity. This is a direct consequence of obstruction theory: in general, postcomposition $\phi_* \colon \operatorname{Map}(A, X) \to \operatorname{Map}(A, Y)$ with an n-connected map $\phi \colon X \to Y$ is (n-a)-connected if A is homotopically a-dimensional.

To deduce the claim for the map with the fQ-subscripts, we consider the commutative triangle

$$\operatorname{BEmb}_{\partial^v}^{\theta}(M;\ell^v)^{\times} \xrightarrow{\hspace*{1cm}} \operatorname{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}. \xleftarrow{\hspace*{1cm}} \operatorname{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}.$$

We already showed that the connectivity of the horizontal map diverges with k, so the same holds for the map between the diagonal fibres at all basepoints. Since rationalisation preserves connectivity (see Section 2.1.4), this implies that the connectivity of the map between fibres after rationalisation diverges with k, which in turn implies that the connectivity of the horizontal map in the triangle after fibrewise rationalisation diverges with k, as claimed.

Given Lemma 5.3, Theorem 5.1 follows by taking limits from the following version of the statement for the sequence (29). Its proof occupies the remainder of this section.

Theorem 5.4. Under the assumptions of Theorem 5.1 the sequence (29) of towers can be completed to a levelwise pullback square of towers.

5.2.1. Proof of Theorem 5.4. We begin by considering the commutative diagram of towers

$$T_{\bullet} \operatorname{Emb}_{\partial^{v}}(M)^{\times} \longrightarrow T_{\bullet} \operatorname{Emb}_{\partial^{v}}^{p}(M)^{\times} \longrightarrow \operatorname{Aut}_{\partial^{v}}(M, \rho_{c(\bullet)} \partial^{h})$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{Aut}_{\partial^{v}}(TM) = \operatorname{Aut}_{\partial^{v}}^{/B\operatorname{Top}(d)}(M) \longrightarrow \operatorname{Aut}_{\partial^{v}}^{/B\operatorname{Aut}_{\leq \bullet}(E_{d})}(M) \xrightarrow{\operatorname{forget}} \operatorname{Aut}_{\partial^{v}}(M)$$

$$(31)$$

of E_1 -groups whose left-hand square is the pullback square relating topological and particle embedding calculus from [KK24c, Section 5.7]. Concretely, this pullback is the pullback of group-like components of the pasting of the pullbacks ③ and ④ in Equation (127) in loc.cit. in the case where M and N in loc.cit. are both given as the noncompact nullbordism $M \setminus (\partial^h M) : \operatorname{int}(\partial^v M) \to \emptyset$. Its bottom row is induced by postcomposition with the map of towers $\operatorname{BTop}(d) \to \operatorname{BAut}_{\leq \bullet}(E_d)$ that already featured in the proof of Lemma 4.8. The right-hand square in (31) results from Theorem 4.9. Replacing the upper row of the left square in (31) viewed as a map over $\operatorname{Aut}_{\partial^v}(M, \rho_{c(\bullet)}\partial^h)$ with the pullback along $\operatorname{Aut}_{\partial^v}(M, \partial^h) \to \operatorname{Aut}_{\partial^v}(M, \rho_{c(\bullet)}\partial^h)$ and the lower row viewed as a map over $\operatorname{Aut}_{\partial^v}(M)$ with the pullback along $\operatorname{Aut}_{\partial^v}(M, \partial^h) \to \operatorname{Aut}_{\partial^v}(M)$, we arrive at a pullback square with left vertical map $T_{\bullet} \operatorname{Emb}_{\partial^v}(M, \partial^h)^{\times} \to \operatorname{Aut}_{\partial^v}(M, \partial^h)$. Restricting $\operatorname{Aut}_{\partial^v}(M, \partial^h)$ and right vertical map $T_{\bullet} \operatorname{Emb}_{\partial^v}(M, \partial^h)^{\times} \to \operatorname{Aut}_{\partial^v}(M, \partial^h)$. Restricting $\operatorname{Aut}_{\partial^v}(M, \partial^h)$ to the components hit by $\operatorname{Aut}_{\partial^v}(M, \partial^h)$ and restricting $T_{\bullet} \operatorname{Emb}_{\partial^v}(M, \partial^h)^{\times}$ to the components that are mapped to these components in $\operatorname{Aut}_{\partial^v}(M, \partial^h)$, we obtain a pullback square

$$T_{\bullet} \operatorname{Emb}_{\partial^{v}}(M, \partial^{h})^{\times} \xrightarrow{} T_{\bullet} \operatorname{Emb}_{\partial^{v}}^{p}(M, \partial^{h})^{t}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{Aut}_{\partial^{v}}(TM, \partial^{h}) = \operatorname{Aut}_{\partial^{v}}^{/\operatorname{BTop}(d)}(M, \partial^{h}) \xrightarrow{} \operatorname{Aut}_{\partial^{v}}^{/\operatorname{BAut}_{\leq \bullet}(E_{d})}(M, \partial^{h})^{t}$$

$$(32)$$

in which the t-superscripts in the right column indicates the collection of components we just specified. The restriction to these components is done so that the square obtained from (32) by delooping remains a pullback. Adding tangential structures to the left column, we obtain a pullback

$$BT_{\bullet} \text{Emb}_{\partial^{v}}^{\theta}(M, \partial^{h}; \ell^{v})^{\times} \longrightarrow BT_{\bullet} \text{Emb}_{\partial^{v}}^{p}(M, \partial^{h})^{t}$$

$$\downarrow \qquad \qquad \downarrow$$

$$BAut_{\partial^{v}}^{\theta}(TM, \partial^{h}; \ell^{v}) \longrightarrow BAut_{\partial^{v}}^{/BAut_{\leq \bullet}(E_{d})}(M, \partial^{h})^{t}$$

which we factor as a pasting of two pullbacks by taking horizontal Moore-Postnikov 1-factorisations

$$BT_{\bullet}Emb_{\partial^{v}}^{\theta}(M, \partial^{h}; \ell^{v})^{\times} \longrightarrow BT_{\bullet}Emb_{\partial^{v}}^{\rho}(M, \partial^{h})^{\ell^{v}} \longrightarrow BT_{\bullet}Emb_{\partial^{v}}^{\rho}(M, \partial^{h})^{t}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad$$

here the rows are defined as the Moore–Postnikov 1-factorisations. By the functoriality of Moore–Postnikov 1-factorisations, the left pullback square maps to the middle space $\mathrm{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}$ in the Moore–Postnikov 1-factorisation of $\mathrm{BAut}_{\partial^v}^\theta(TM,\partial^h;\ell^v)\to\mathrm{BAut}_{\partial^v}(M,\partial^h)$, so we may take fibrewise rationalise the left square over $\mathrm{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}$ to arrive at a square

$$BT_{\bullet}Emb_{\partial^{v}}^{\theta}(M, \partial^{h}; \ell^{v})_{fQ}^{\times} \longrightarrow BT_{\bullet}Emb_{\partial^{v}}^{p}(M, \partial^{h})_{fQ}^{\ell^{v}}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$BAut_{\partial^{v}}^{\theta}(TM, \partial^{h}; \ell^{v})_{fQ} \longrightarrow BAut_{\partial^{v}}^{/BAut_{\leq \bullet}(E_{d})}(M, \partial^{h})_{fQ}^{\ell^{v}}.$$

$$(34)$$

Proposition 5.5. The square (34) is a pullback. Moreover, its bottom map considered as a map over $BAut_{\partial^{v}}(M, \partial^{h})^{\ell^{v}}$, admits a factorisation of the form

$$\mathrm{BAut}^{\theta}_{\partial^v}(TM,\partial^h;\ell^v)_{\mathrm{fQ}} \longrightarrow \mathrm{BAut}_{\partial^v}(M,\partial^h)^{\ell^v} \longrightarrow \mathrm{BAut}_{\partial^v}^{/\mathrm{BAut}_{\leq \bullet}(E_d)}(M,\partial^h)^{\ell^v}_{\mathrm{fQ}}.$$

Assuming Proposition 5.5, Theorem 5.4 follows by factoring the bottom map in the square (34) according to the second part of the proposition and take a pullback to write this square as the pasting of two squares. The right square of this pasting is a pullback by construction and the outer rectangle is one by the first part of the proposition, so the left square is a pullback as well. The latter is the pullback extension promised in Theorem 5.4. Hence we are left to show Proposition 5.5.

5.2.2. Proof of Proposition 5.5. Consider the commutative triangle

$$\operatorname{BAut}_{\partial^{v}}^{\theta}(TM, \partial^{h}; \ell^{v}) \xrightarrow{} \operatorname{BAut}_{\partial^{v}}^{\theta}(M, \partial^{h})^{\ell^{v}}$$

$$\longrightarrow \operatorname{BAut}_{\partial^{v}}(M, \partial^{h})^{\ell^{v}}$$

$$(35)$$

whose horizontal map and left diagonal map are 1-connected by construction, so the right diagonal map is as well. In particular, all maps have connected fibres and induce bijections on components, so we may split the diagram into its components (see Section 5.1.1 for the notation):

$$\bigsqcup_{[\ell] \in \theta\text{-st}} \operatorname{BAut}^{\theta}(TM, \partial^{h}; \ell^{v})_{\ell} \xrightarrow{\qquad} \bigsqcup_{[\ell] \in \theta\text{-st}} \operatorname{BAut}^{\ell}_{\partial^{v}}(M, \partial^{h}) \xrightarrow{\qquad}$$

where $\mathrm{BAut}^\theta(TM,\partial^h;\ell^v)_\ell\subset\mathrm{BAut}^\theta(TM,\partial^h;\ell^v)$ is the component corresponding to $[\ell]\in\theta$ -st , and the ℓ -superscripts on the other spaces indicate that we are passing to the covering spaces that make the map from $\mathrm{BAut}^\theta(TM,\partial^h;\ell^v)_\ell$ be 1-connected. By general bundle yoga, up to precomposing the horizontal map with a covering space, we may further rewrite the triangle as

$$\bigsqcup_{[\ell]\in\theta\text{-st}}\frac{\underset{\operatorname{Aut}_{\partial^{\upsilon}}^{\ell}(M,B)_{\ell}}{\operatorname{Aut}_{\partial^{\upsilon}}^{\ell}(M,\partial^{h})}}{+}\bigsqcup_{[\ell]\in\theta\text{-st}}\frac{\underset{\operatorname{Aut}_{\partial^{\upsilon}}^{\ell}(M,B\operatorname{Aut}_{\leq\bullet}(E_{d}))_{TM}}{\operatorname{Aut}_{\partial^{\upsilon}}^{\ell}(M,\partial^{h})}}$$

$$\downarrow \bigsqcup_{[\ell]\in\theta\text{-st}}\operatorname{BAut}_{\partial^{\upsilon}}^{\ell}(M,\partial^{h})$$

$$(36)$$

where the ℓ -subscript on $\operatorname{Map}_{\partial^v}(M,B)_\ell$ indicates the component of $\ell\colon M\to B$ and the TM-subscript on $\operatorname{Map}_{\partial^v}(M,\operatorname{BAut}_{\leq \bullet}(E_d))_{TM}$ the component of the composition of $M\to B$ with the composition $B\to\operatorname{BTop}(d)\to\operatorname{BAut}_{\leq \bullet}(E_d)$. Postcomposition with the latter induces the horizontal arrow. Since $B\to\operatorname{BTop}(d)$ factors by assumption through the classifying space of group $\operatorname{STop}(d)$ of orientation-preserving homeomorphisms of \mathbb{R}^d which agrees with the identity component of $\operatorname{Top}(d)$ as a result of the stable homeomorphism theorem, the composition $B\to\operatorname{BTop}(d)\to\operatorname{BAut}_{\leq \bullet}(E_d)$ factors as

$$B \longrightarrow \mathrm{BSTop}(d) \longrightarrow \mathrm{BAut}^{\mathrm{id}}_{<\bullet}(E_d) \longrightarrow \mathrm{BAut}_{<\bullet}(E_d),$$
 (37)

where $\operatorname{Aut}_{\leq \bullet}^{\operatorname{id}}(E_d) \subset \operatorname{Aut}_{\leq \bullet}(E_d)$ are the components of the identity. Using that $\operatorname{Map}_{\partial^v}(M, -)$ sends covering maps to componentwise equivalences, we may replace $\operatorname{BAut}_{\leq \bullet}(E_d)$ in (36) by $\operatorname{BAut}_{\leq \bullet}^{\operatorname{id}}(E_d)$. The diagonal arrows in (35) have thus connected fibres which are nilpotent as a result of [BK72, V.§5.1], being covering spaces of components of relative mapping spaces from the finite CW-pair $(M, \partial^v M)$ into the nilpotent spaces B and $\operatorname{BAut}_{\leq b}^{\operatorname{id}}(E_d)$.

Based on this discussion, we now prove the result. To show that (34) is a pullback it suffices to show that the square originating from (34) by taking fibres of the maps of each entry to BAut_{∂^v} (M, ∂^h)^{ℓ^v} at any fixed basepoint is a pullback. By construction, these squares are the rationalisations of the pullback squares obtained from the left pullback square in (33) by taking fibres of the maps to BAut_{∂^v} (M, ∂^h)^{ℓ^v}. By Lemma 2.1, it suffices to show that the bottom horizontal map in the latter square of fibres is 1-connected and nilpotent. This map is the map on diagonal fibres of (35), so its source and target are nilpotent by the above discussion, so the map is nilpotent by II.§4.5 loc.cit.. It is also 1-connected since the horizontal map in (35) is 1-connected.

To establish the asserted factorisation, note that the bottom map in (34) agrees by the above discussion, up to postcomposition with a covering space, with the map

$$\bigsqcup_{[\ell] \in \theta \text{-st}} \frac{(\operatorname{Map}_{\partial^{v}}(M, B)_{\ell})_{\mathbb{Q}}}{\operatorname{Aut}_{\partial^{v}}^{\ell}(M, \partial^{h})} \longrightarrow \bigsqcup_{[\ell] \in \theta \text{-st}} \frac{(\operatorname{Map}_{\partial^{v}}(M, B \operatorname{Aut}_{\leq \bullet}^{\operatorname{id}}(E_{d}))_{TM})_{\mathbb{Q}}}{\operatorname{Aut}_{\partial^{v}}^{\ell}(M, \partial^{h})}$$
(38)

induced by postcomposition with (37). The map (38) factors over

$$\bigsqcup_{[\ell] \in \theta\text{-st }\frac{*}{\operatorname{Aut}_{\partial^v}^\ell(M,\partial^h)}} \simeq \operatorname{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}$$

since we will show momentarily that the map $(\operatorname{Map}_{\partial^v}(M,B)_\ell)_Q \to (\operatorname{Map}_{\partial^v}(M,\operatorname{BAut}_{\leq \bullet}^{\operatorname{id}}(E_d)_{TM})_Q$ induced by postcomposition with (37) is $\operatorname{Aut}_{\partial^v}^\ell(M,\partial^h)$ -equivariantly nullhomotopic as a map of towers. Using the natural equivalence $(\operatorname{Map}_{\partial^v}(M,X)_f)_Q \simeq \operatorname{Map}_{\partial^v}(M,X_Q)_{f_Q}$ for nilpotent spaces

X and maps $f: M \to X$ resulting from [BK72, V.5.1], the latter follows by showing that the rationalisation of $B \to \mathrm{BSTop}(d) \to \mathrm{BAut}^{\mathrm{id}}_{\leq \bullet}(E_d)$ is nullhomotopic. By assumption the latter factors through the rationalisation of $\mathrm{BSTop}(d-2) \to \mathrm{BSTop}(d) \to \mathrm{BAut}^{\mathrm{id}}_{\leq \bullet}(E_d)$. To show that the latter is nullhomotopic, we consider the commutative diagram

whose upper two squares are induced by (5) and (10) and whose bottom-right square is induced by truncation. As ① is levelwise a rational equivalence as a result of [KK24a, Theorems 7.9-7.10], to show that BSTop(d-2) \to BAut $^{\mathrm{id}}_{\leq \bullet}(E_d)$ is rationally nullhomotopic as a map of towers it suffices to show that ② is nullhomotopic, which follows from Theorem G by taking universal covers.

This almost provides the factorisation of the bottom map in (34) as claimed in Proposition 5.5, in that it gives such a factorisation after postcomposition with a covering map. But as the map $\mathrm{BAut}_{\partial^v}^\theta(TM,\partial^h;\ell^v)_{\mathrm{fQ}}\to\mathrm{BAut}_{\partial^v}(M,\partial^h)^{\ell^v}$ is 1-connected by construction, obstruction theory ensures that there is a unique lift of the factorisation to one as required.

Remark 5.6. Recall that one of the conditions in Theorems F, 5.1, and 5.4 is that the tangential structure $\theta: B \to \mathrm{BSTop}(d)$ factors through $\mathrm{BSTop}(d-2) \to \mathrm{BSTop}(d)$ after rationalisation.

- (i) The only thing we used in the proof about this condition is that this ensures that the composition of θ with the map $\mathrm{BSTop}(d) \to \mathrm{BAut}^{\mathrm{id}}(E_{d,Q})$ is nullhomotopic, as a result of Theorem G. As a result, the result remains true if one replaces the assumption on θ by the condition that it factors rationally over the fibre $\mathrm{fib}(\mathrm{BSTop}(d)_Q \to \mathrm{BAut}^{\mathrm{id}}(E_{d,Q})_Q)$. This is a weaker condition, but in practice less easily verified.
- (ii) If d is even, then this condition can be replaced by the (neither stronger nor weaker) condition that $\theta \colon B \to \mathrm{BSTop}(d)$ factors through $\mathrm{BSO}(d-1) \to \mathrm{BSTop}(d)$ after rationalisation. The reason is that in the proofs of these results, the only place where the condition on the factorisation enters is when we used that the map $\mathrm{BSTop}(d-2) \to B\mathrm{Aut}^{\mathrm{id}}(E_{d,\mathbb{Q}})$ is rationally nullhomotopic. This implies by precomposition that the map $\mathrm{BSO}(d-2) \to B\mathrm{Aut}^{\mathrm{id}}(E_{d,\mathbb{Q}})$ is rationally nullhomotopic which in turn implies that $\mathrm{BSO}(d-1) \to B\mathrm{Aut}^{\mathrm{id}}(E_{d,\mathbb{Q}})$ is rationally nullhomotopic since for d even any map $\mathrm{BSO}(d-1) \to X$ to a 1-connected space X is rationally nullhomotopic if and only if the composition $\mathrm{BSO}(d-2) \to \mathrm{BSO}(d-1) \to X$ is rationally nullhomotopic, by an application of $[\mathrm{HR87}, \mathrm{Theorem}\ 2.3]$ using $\mathrm{BSO}(d-2) \to \mathrm{BSO}(d-1)$ is surjective on rational homology if d is even. Note that this final fact fails for $\mathrm{BSTop}(d-2) \to \mathrm{BSTop}(d-1)$, e.g. as a consequence of $[\mathrm{KrRW21}, \mathrm{Theorem}\ \mathrm{C}]$.

Remark 5.7. There is a variant of Theorem 5.1 which is in some situations more convenient to use. It is based on the observation that, since the second map in the sequence (27) that can be completed to a pullback is 1-connected by construction, the sequence of connected spaces resulting from (27) by restriction to the components induced by a fixed θ -structure $[\ell] \in \theta$ -st can be completed to a pullback too. By the discussion in Section 5.2.2 this sequence has the form

$$\operatorname{BEmb}_{\partial^{v}}^{\theta}(M; \ell^{v})_{\operatorname{fQ}, \ell}^{\times} \longrightarrow \operatorname{Map}_{\partial^{v}}(M, B_{\operatorname{Q}})_{\ell_{\operatorname{Q}}}/\operatorname{Aut}_{\partial^{v}}^{\ell}(M, \partial^{h}) \longrightarrow \operatorname{BAut}_{\partial^{v}}^{\ell}(M, \partial^{h}). \tag{39}$$

Here ℓ_Q is the composition of the map $\ell \colon M \to B$ with the rationalisation map $B \to B_Q$, and $\operatorname{Aut}^{\ell}_{\partial}(M) \subset \operatorname{Aut}_{\partial}(M)$ consists of those components that stabilise $[\ell] \in \pi_0(\operatorname{Map}_{1/2\partial}(M,B))$, before rationalisation. Since (39) can be completed to a pullback, the same applies to the variant

$$\operatorname{BEmb}_{\partial^{v}}^{\theta}(M; \ell^{v})_{\operatorname{fQ}, \ell}^{\times} \longrightarrow \operatorname{Map}_{\partial^{v}}(M, B_{\mathbb{Q}})_{\ell_{\mathbb{Q}}}/\operatorname{Aut}_{\partial^{v}}^{\ell_{\mathbb{Q}}}(M, \partial^{h}) \longrightarrow \operatorname{BAut}_{\partial^{v}}^{\ell_{\mathbb{Q}}}(M, \partial^{h}) \tag{40}$$

involving the stabiliser $\operatorname{Aut}_{\partial^v}^{\ell_Q}(M,\partial^h)$ of the rationalised map $[\ell_Q] \in \pi_0(\operatorname{Map}_{1/2\partial}(M,B_Q))$, since the second map in (39) is pulled back from the second map in (40) along the covering space $\operatorname{BAut}_{\partial^v}^\ell(M,\partial^h) \to \operatorname{BAut}_{\partial^v}^{\ell_Q}(M,\partial^h)$. The reason why the sequence (40) is sometimes more convenient is that its middle space agrees with the pathcomponent induced by ℓ_Q of the full orbit space $\operatorname{Map}_{\partial^v}(M,B_Q)/\operatorname{Aut}_{\partial^v}(M,\partial^h)$. In the special case of Theorem F, the variant (40) takes the form

$$\mathrm{BHomeo}_{\partial}^{\theta}(M;\ell_{1/2\partial})_{\mathrm{fQ},\ell} \longrightarrow \left(\mathrm{Map}_{1/2\partial}(M,B_{\mathrm{Q}})/\mathrm{Aut}_{\partial}(M)\right)_{\ell_{\mathrm{Q}}} \longrightarrow \mathrm{BAut}_{\partial}^{\ell_{\mathrm{Q}}}(M),\tag{41}$$

which is the version that is suited best for the applications presented in the next section.

6. Applications

We explain various applications of the pullback decomposition established in the previous section to the study of topological fibre bundles. For simplicity and since we consider it the most interesting case, we phrase them for spaces of homeomorphisms as in Theorem F, but similar arguments apply to spaces of self-embeddings as in the more general Theorem 5.1.

6.1. **Tautological classes**. As preparation for the applications of the pullback decomposition, we discuss a source of characteristic classes of fibrations with certain extra structure, inspired by [RW19, Section 4.3] and [Wei21, Section 2.2]. All cohomology groups are taken with coefficients in a fixed commutative ring \mathbf{k} which we omit from the notation.

Fix a compact oriented topological *d*-manifold *M* and a map $\ell \colon M \to B$ to a fixed space *B*. We first describe characteristic classes associated to the following three pieces of data:

- (i) an oriented relative fibration $(M, \partial M) \to (E, \partial M \times X) \xrightarrow{\pi} X$,
- (ii) a map $\ell^{\pi} : E \to B$,
- (iii) a homotopy h^{π} of maps $\partial M \times X \to B$ between $\ell^{\pi}|_{\partial M \times X}$ and $(\ell|_{\partial M} \circ \operatorname{pr}_1)$.

Given such a triple $(\pi, \ell^{\pi}, h^{\pi})$ and a cohomology class $c \in H^*(B)$, there is a characteristic class $\kappa_c(\pi) \in H^{*-d}(X)$ defined as follows: writing -M for M equipped with the opposite orientation, form the oriented fibration $\overline{\pi} \colon \overline{E} \coloneqq (E \cup_{\partial M \times X} (-M \times X)) \to X$ with fibre the double $M \cup_{\partial M} (-M)$, extend the map ℓ^{π} to a map $\overline{\ell} \colon \overline{E} \to B$ using the map $(\ell \circ \operatorname{pr}_1) \colon M \times X \to B$ and the homotopy h^{π} , pull back c to a class in $H^*(\overline{E})$, and fibre integrate this class along $\overline{\pi}$ using the orientation to obtain a class $\kappa_c(\pi) \in H^{*-d}(X)$. This gives rise to a morphism $\kappa_{(-)}(\pi) \colon H^*(B) \to H^{*-d}(X)$ which a priori depends on the choice of ℓ , but actually turns out to only depend on the restriction $\ell|_{\partial M}$ as long as *-d>0. Triples $(\pi, \ell^{\pi}, h^{\pi})$ as above are classified by the orbits $\operatorname{Map}_{\partial}(M, B)/\operatorname{Aut}_{\partial}^+(M)$ of the precomposition action of the E_1 -group $\operatorname{Aut}_{\partial}^+(M)$ of orientation-preserving homotopy automorphisms extending the boundary inclusion on the space of maps $M \to B$ extending $\ell|_{\partial M}$. Applied to the universal triple, the above construction thus gives a morphism

$$\kappa_{(-)}: H^*(B) \longrightarrow H^{*-d}(\operatorname{Map}_{\partial}(M, B)/\operatorname{Aut}_{\partial}^+(M)).$$

Now suppose we have fixed an embedded disc $D^{d-1} \subset \partial M$ in the boundary and the homotopy h^{π} is only defined on the subspace $^{1/2}\partial M \times X \subset \partial M \times X$ where $^{1/2}\partial M := \partial \setminus \operatorname{int}(D^{d-1})$; we denote such a homotopy on this subspace by $h^{\pi}_{^{1/2}}$. Given only the partial homotopy $h^{\pi}_{^{1/2}}$, the class $\kappa_c(\pi)$ for $c \in H^*(B)$ is a priori no longer defined, since we made use of the full homotopy h^{π} to extend ℓ^{π} to \overline{E} . However, it turns out that $\kappa_c(\pi)$ is defined as long as c comes with a cup-product decomposition which is trivialised over a point. More precisely, equipping B with the basepoint given as the image under ℓ of a fixed point $*\in \partial D^{d-1} \subset {}^{1/2}\partial M$, we construct a morphism $\kappa_{(-)\otimes(-)}(\pi): H^*(B,*)^{\otimes 2} \to H^{*-d}(X)$ which only requires the partial homotopy $h^{\pi}_{1/2}$ and agrees, whenever the full homotopy is defined, with the composition of the morphism $H^*(B,*)^{\otimes 2} \to H^*(B)$ induced by restriction and taking cup-product, with the morphism $\kappa_{(-)}(\pi): H^*(B) \to H^{*-d}(X)$ from above. Triples $(\pi, \ell^{\pi}, h^{\pi}_{1/2})$ are classified by the orbits $\operatorname{Map}_{1/2\partial}(M,B)/\operatorname{Aut}^+_{\partial}(M)$ of the space of maps from M to B extending $\ell|_{1/2\partial}$.

The universal instance of the construction thus gives the upper row in a commutative diagram

$$\begin{array}{ccc}
H^{*}(B,*)^{\otimes 2} & \xrightarrow{\kappa_{(-)\otimes(-)}} & H^{*-d}\left(\operatorname{Map}_{1/2\partial}(M,B)/\operatorname{Aut}_{\partial}^{+}(M)\right) \\
\downarrow & & \downarrow \\
H^{*}(B) & \xrightarrow{\kappa_{(-)}} & & H^{*-d}\left(\operatorname{Map}_{\partial}(M,B)/\operatorname{Aut}_{\partial}^{+}(M)\right)
\end{array} (42)$$

whose right column is induced by the forgetful map $\operatorname{Map}_{\partial}(M,B) \to \operatorname{Map}_{1/2\partial}(M,B)$.

We now explain the promised construction of the morphism $\kappa_{(-)\otimes(-)}(\pi): H^*(B,*)^{\otimes 2} \to H^{*-d}(X)$. Writing $\overline{E}_{1/2}$ for the total space of the oriented relative fibration

$$(M \cup_{1/2\partial M} (-M), D^{d-1} \cup_{\partial D^{d-1}} (-D^{d-1})) \to (E \cup_{1/2\partial M \times X} (-M \times X), (D^{d-1} \cup_{\partial D^{d-1}} (-D^{d-1})) \times X) \xrightarrow{\overline{\pi}_{1/2}} X$$

and viewing $\overline{E}_{1/2}$ as a subspace of \overline{E} with complement $D^d \times X$, the morphism is defined as the long composition in the commutative diagram

$$H^{*}(B,*)^{\otimes 2} \longrightarrow H^{*}(\overline{E}_{1/2},*\times X)^{\otimes 2}$$

$$\downarrow^{\operatorname{res}^{\otimes 2}} \longrightarrow H^{*}(\overline{E},*\times X)^{\otimes 2} \longrightarrow H^{*}(\overline{E},*\times X) \xrightarrow{\operatorname{res}} H^{*}(\overline{E}) \xrightarrow{\int_{\overline{\pi}}} H^{*-d}(X)$$

$$(43)$$

where the upper-left horizontal map is defined by pulling back along the extension of $\ell^{\pi} : E \to B$ to a map $\overline{E}_{1/2} \to B$ by $\ell \circ \operatorname{pr}_1 : M \times X \to B$ using the homotopy $h^{\pi}_{1/2}$. The dashed map results from:

Lemma 6.1. The middle vertical map $res^{\otimes 2}$ in (43) is surjective and there exists a (necessarily unique) dashed map as indicated that makes the diagram commutative.

Proof. To show surjectivity of res and thus of $\operatorname{res}^{\otimes 2}$, since $\operatorname{H}^*(\overline{E}, * \times X) \cong \operatorname{H}^*(\overline{E}, D^d \times X) \cong \operatorname{H}^*(\overline{E}_{1/2}, S^{d-1} \times X)$ by excision, it suffices by the long exact sequence of a triple to show that the upper horizontal map in the commutative diagram

is zero. Thinking of the slant product as fibrewise integration along the trivial bundle, the bottom composition is trivial by the "fibrewise Stokes theorem" (a direct consequence of the construction of fibrewise integration via the Serre spectral sequence), so the upper horizontal map is trivial as well. Using the surjectivity, the existence of the dashed map in (43) follows once we show that the kernel of res⁸² has trivial cup-products. Identifying $H^*(\overline{E}, * \times X)$ with $H^*(\overline{E}_{1/2}, S^{d-1} \times X)$ as above, this kernel is spanned by elements of the form $x \otimes \delta(y)$ and $\delta(y) \otimes x$ where $\delta \colon H^{*-1}(S^{d-1} \times X, * \times X) \to H^*(\overline{E}_{1/2}, S^{d-1} \times X)$ is the connecting homomorphism, so the claim follows from the general fact that cup products of two relative cohomology classes vanish if one is in the image of the connecting homomorphism or, equivalently, trivial in absolute cohomology.

Remark 6.2. Going through the construction, one sees that $\kappa_{(-)\otimes(-)}(\pi)$: $H^*(B,*)^{\otimes 2} \to H^{*-d}(X)$ descends to a morphism on $H^*(B,*)\otimes_{H^*(B,*)}H^*(B,*)$. Note that as $R:=H^*(B,*)$ does not have a unit unless it is trivial, $R\otimes_R R$ is typically not isomorphic to R.

Remark 6.3. For later applications, we record a formula for the pullback of the class $\kappa_{c\otimes c'} \in H^{*-d}(\operatorname{Map}_{1/2\partial}(M,B)/\operatorname{hAut}_{\partial}^+(M))$ for $c\otimes c'\in H^*(B,*)^{\otimes 2}$ along the quotient map $\operatorname{Map}_{1/2\partial}(M,B)\to \operatorname{Map}_{1/2\partial}(M,B)/\operatorname{hAut}_{\partial}^+(M)$, for simplicity in the case $\mathbf{k}=\mathbf{Q}$. The formula involves the map

$$\operatorname{ev}_{\ell} \colon (M \cup_{1/2\partial} (-M)) \times \operatorname{Map}_{1/2\partial} (M, B) \longrightarrow B$$

given by the evaluation map on M and $(\operatorname{pr}_1 \circ \ell)$ on -M, a choice of a homogenous basis $(b_i)_i \subset \widetilde{\operatorname{H}}_*(M \cup_{1/2\partial} (-M); \mathbf{Q})$, and its dual basis $(b_i^{\vee})_i \subset \widetilde{\operatorname{H}}^*(M \cup_{1/2\partial} (-M))$. In these terms, viewing

 $\widetilde{\mathrm{H}}_*(M \cup_{1/2\partial} (-M); \mathbf{Q})$ as subspace of $\widetilde{\mathrm{H}}_*(D(M); \mathbf{Q})$ via the inclusion $M \cup_{1/2\partial} (-M) \subset (M \cup_{1/2\partial} (-M) \cup_{S^{d-1}} D^d \cong D(M)$, we have the following identity in $\mathrm{H}^{*-d}(\mathrm{Map}_{1/2\partial}(M,B); \mathbf{Q})$

$$\kappa_{c \otimes c'} = \sum_{\{(i,j): |b_i| + |b_j| = d\}} (-1)^{|b_j||c| - d} \langle b_i^{\vee} \cup b_j^{\vee}, [D(M)] \rangle \cdot (\operatorname{ev}_{\ell}^*(c)/b_i \cup \operatorname{ev}_{\ell}^*(c')/b_j),$$

where -/- denotes the slant product. This follows from a straight-forward check, using that fibre integration along a trivial bundle is given by the slant product with the fundamental class of the fibre. Moreover, if $\partial M \cong S^{d-1}$, so $1/2\partial M \cong D^{d-1} \simeq *$ and $D(M) \cong \overline{M} \sharp (-\overline{M})$ for $\overline{M} := M \cup_{S^{d-1}} D^d$, the above formula can be rewritten in terms of a choice of homogenous basis $(a_i)_i \subset \widetilde{H}_*(M; \mathbb{Q})$ and the evaluation map ev: $M \times \operatorname{Map}_*(M, B) \to B$ as

$$\kappa_{c \otimes c'} = \sum_{\{(i,j): |a_i| + |a_j| = d\}} (-1)^{|a_j||c| - d} \langle a_i^{\vee} \cup a_j^{\vee}, [\overline{M}] \rangle \cdot (\text{ev}^*(c)/a_i \cup \text{ev}^*(c')/a_j.$$

6.2. **Modifying vertical tangent bundles.** Recall that Theorem **F** in its variant from (41) said that for any smoothable compact d-manifold M of dimension $d \ge 5$ with M and ∂M 2-connected, a choice of disc $D^{d-1} \subset \partial M$ in its boundary, an oriented tangential structure $\theta \colon B \to \mathrm{BSTop}(d)$ such that B is nilpotent and θ factors through $\mathrm{BSTop}(d-2) \to \mathrm{BSTop}(d)$ after rationalisation, and a θ -structure $\ell \colon M \to B$ on TM, the composition

$$\mathrm{BHomeo}_{\partial}^{\theta}(M; \ell_{1/2\partial})_{\mathrm{fQ},\ell} \longrightarrow \left(\mathrm{Map}_{1/2\partial}(M, B_{\mathrm{Q}})/\mathrm{Aut}_{\partial}(M)\right)_{\ell_{\mathrm{Q}}} \longrightarrow \mathrm{BAut}_{\partial}^{\ell_{\mathrm{Q}}}(M) \tag{44}$$

can be completed to a pullback square. The action of $\operatorname{Aut}_{\partial}(M)$ on the mapping space is by precomposition, so it commutes with the action by postcomposition of the endomorphisms $\operatorname{End}_{1/2\partial}(B_{\mathbb{Q}})$ of $(\ell_{1/2})_{\mathbb{Q}} \colon 1/2\partial M \to B_{\mathbb{Q}}$ in the undercategory $\mathcal{S}_{1/2\partial M/}$. Restricted to the components

$$\operatorname{End}_{1/2\partial}^{\ell_{\mathbb{Q}}}(B_{\mathbb{Q}}) \subset \operatorname{End}_{1/2\partial}(B_{\mathbb{Q}})$$

that stabilise $[\ell_{\mathbb{Q}}] \in \pi_0(\operatorname{Map}_{1/2\partial}(M,B_{\mathbb{Q}}))$, the action restricts to an action on the middle space of (44) in $\mathcal{S}_{/\operatorname{BAut}_\partial^{\ell_{\mathbb{Q}}}(M)}$. The action can also be described in terms what it classifies: the middle space in (44) classifies triples $(\pi,\ell^\pi,h^\pi_{1/2})$ as in Section 6.1 (set $B=B_{\mathbb{Q}}$ and $\ell=\ell_{\mathbb{Q}}$) and $\chi\in\operatorname{End}_{1/2\partial}^{\ell_{\mathbb{Q}}}(B_{\mathbb{Q}})$ acts by fixing π and postcomposing ℓ^π and $h^\pi_{1/2\partial}$ with χ . Since (44) can be completed to a pullback, we may use the map $\operatorname{End}_{\mathcal{S}_{/W}}(Z)\to\operatorname{End}_{\mathcal{S}_{/Y}}(X)$ for pullbacks $X\simeq Z\times_W Y$ to conclude the following:

Proposition 6.4. Under the assumptions of Theorem F, the action of the endomorphisms $\operatorname{End}_{1/2\partial}^{\ell_Q}(B_Q)$ on the space $(\operatorname{Map}_{1/2\partial}(M,B_Q)/\operatorname{Aut}_{\partial}(M))_{\ell_Q}$ by postcomposition lifts to an action on $\operatorname{BHomeo}_{\partial}^{\ell_Q}(M;\ell_{1/2})_{fQ,\ell}$.

Remark 6.5. Roughly speaking and ignoring boundary conditions, Proposition 6.4 allows us to modify vertical tangent bundles of fibre bundles. Suppose we are given a topological M-bundle $\pi \colon E \to X$ with θ -structure $\ell^{\pi} \colon E \to B$ on its vertical tangent bundle and a self-map $\chi \colon B_Q \to B_Q$ with $\chi \circ \ell_Q \simeq \ell_Q$. Then using the action we obtain a new topological M-bundle (in a certain rational sense), whose underlying fibration is the same as that of the original bundle but now whose rational vertical tangent bundle turned from $(\theta \circ \ell^{\pi}) \colon E \to \mathrm{BSTop}(d)_Q$ to $(\theta \circ \chi \circ \ell^{\pi}) \colon E \to \mathrm{BSTop}(d)_Q$.

6.2.1. Action on tautological classes. Next we explain the effect of the action on BHomeo $_{\partial}^{\theta}(M; \ell_{1/2})_{fQ,\ell}$ by $\operatorname{End}_{1/2\partial}^{\ell_Q}(B_Q)$ on various characteristic classes, such as those in the image of the composition

$$H^{*}(B, *; \mathbf{Q})^{\otimes 2} \xrightarrow{\kappa_{(-)\otimes(-)}} H^{*-d}(\operatorname{Map}_{1/2\partial}(M, B_{\mathbf{Q}})/\operatorname{Aut}_{\partial}(M); \mathbf{Q}) \longrightarrow H^{*}(\operatorname{BHomeo}_{\partial}^{\theta}(M; \ell_{1/2})_{fQ, \ell}; \mathbf{Q})$$

of the construction in Section 6.1 applied to $B=B_{\mathbb{Q}}$ and $\ell=\ell_{\mathbb{Q}}$, with the map induced by the second map in (44). By construction, both maps are equivariant with respect to the action of $\operatorname{End}_{1/2\partial}^{\ell_{\mathbb{Q}}}(B_{\mathbb{Q}})$, so for $c,c'\in H^*(B,*;\mathbb{Q})$ and $\chi\in\operatorname{End}_{1/2\partial}^{\ell_{\mathbb{Q}}}(B_{\mathbb{Q}})$ we have

$$\chi^*\kappa_{c\otimes c'}=\kappa_{(\chi^*c)\otimes(\chi^*c')}$$

in the rational cohomology of $\operatorname{Map}_{1/2\partial}(M,B_{\mathbb{Q}})/\operatorname{Aut}_{\partial}(M)$ and thus also in that of BHomeo $_{\partial}^{\theta}(M;\ell_{1/2})_{\ell,f\mathbb{Q}}$. Another source of characteristic classes for BHomeo $_{\partial}^{\theta}(M;\ell_{1/2})_{f\mathbb{Q},\ell}$ is the composition

$$\mathrm{H}^*(\mathrm{BSTop}(d); \mathbf{Q}) \xrightarrow{\kappa_{(-)}} \mathrm{H}^{*-d}(\mathrm{Map}_{\partial}(M, \mathrm{BSTop}(d)_{\mathbf{Q}}) / \mathrm{Aut}_{\partial}(M); \mathbf{Q}) \to \mathrm{H}^{*-d}(\mathrm{BHomeo}_{\partial}^{\theta}(M; \ell_{1/2})_{\mathrm{f}\mathbf{Q}, \ell}; \mathbf{Q})$$

where the first map is an instance of the first construction from Section 6.1 and the second map is induced by the fibrewise rationalisation over $BAut_{\partial}(M)$ of the forgetful composition

$$BHomeo_{\partial}^{\theta}(M; \ell_{1/2})_{\ell} \longrightarrow BHomeo_{\partial}(M) \longrightarrow Map_{\partial}(M, BSTop(d))/Aut_{\partial}(M)$$
 (45)

whose second map is induced by taking vertical tangent bundles. As a result of (42), the two sources of classes fit into a commutative diagram

$$H^*(B,*;\mathbf{Q})^{\otimes 2} \xrightarrow{\theta^* \otimes \theta^*} H^*(\mathrm{BSTop}(d),*;\mathbf{Q})^{\otimes 2} \xrightarrow{\cup} H^*(\mathrm{BSTop}(d);\mathbf{Q})$$

$$\kappa_{(-)\otimes (-)} \xrightarrow{H^{*-d}} H^*(\mathrm{BHomeo}_{\partial}^{\theta}(M;\ell_{1/2})_{\ell,\mathrm{f}\mathbf{Q}};\mathbf{Q}) \xleftarrow{\kappa_{(-)}}$$

which gives a procedure to compute the action of $\chi \in \operatorname{End}_{1/2\partial}^{\ell_Q}(B_Q)$ -action on those classes $c \in \operatorname{H}^*(\operatorname{BSTop}(d); \mathbb{Q})$ that are decomposable: for $c = \sum c_i \cup c_i'$ with $c_i, c_i' \in \operatorname{H}^{*>0}(\operatorname{BSTop}(d); \mathbb{Q})$, we have

$$\chi^* \kappa_c = \sum \kappa_{(\chi^* \theta^* c_i) \otimes (\chi^* \theta^* c_i')} \in H^{*-d}(BHomeo_{\partial}^{\theta}(M; \ell_{1/2})_{fQ, \ell}; \mathbf{Q}). \tag{46}$$

Example 6.6. The previous discussion allows us in particular to determine the action of $\operatorname{End}_{1/2\partial}^{\ell_Q}(B_Q)$ on all classes $\kappa_c \in \operatorname{H}^{*-d}(\operatorname{BHomeo}_{\partial}^{\theta}(M;\ell_{1/2})_{\mathbb{Q},\ell};\mathbb{Q})$ for which $c \in \operatorname{H}^*(\operatorname{BSTop}(d);\mathbb{Q})$ pulls back along the stabilisation map $\operatorname{BSTop}(d) \to \operatorname{BSTop}$. To explain how, recall that the ring $\operatorname{H}^*(\operatorname{BSTop};\mathbb{Q}) \cong \mathbb{Q}[\mathcal{L}_1,\mathcal{L}_2,\ldots]$ is polynomial on the Hirzebruch L-classes $\mathcal{L}_i \in \operatorname{H}^{4i}(\operatorname{BSTop};\mathbb{Q})$. If c is a sum of decomposable monomials in the classes \mathcal{L}_i of positive degrees, then the action is described by (46). Indecomposable classes $\kappa_{\mathcal{L}_i} \in \operatorname{H}^{4i-d}(\operatorname{BHomeo}_{\partial}^{\theta}(M;\ell_{1/2})_{f\mathbb{Q},\ell};\mathbb{Q})$ on the other hand are acted upon trivially, since these classes pulls back from $\operatorname{BAut}_{\partial}(M)$. To see the latter claim, note that $\kappa_{\mathcal{L}_i}$ is by construction pulled back along the composition (45), fibrewise rationalised over $\operatorname{BAut}_{\partial}(M)$. By the family signature theorem (see e.g. $[\operatorname{RW}24$, Theorem 2.6]), the class $\kappa_{\mathcal{L}_i} \in \operatorname{H}^{4i-d}(\operatorname{BHomeo}_{\partial}(M;\ell_{1/2})_{f\mathbb{Q},\ell};\mathbb{Q})$ is pulled back from $\operatorname{BAut}_{\partial}(M)$, so the same holds for its image in $\operatorname{H}^{4i-d}(\operatorname{BHomeo}_{\partial}^{\theta}(M;\ell_{1/2})_{f\mathbb{Q},\ell};\mathbb{Q})$.

Remark 6.7. There is a subtlety in the final part in Example 6.6 that we have not yet addressed. The family signature theorem applies to $\kappa_{\mathcal{L}_i}$ viewed as a class in $H^{4i-d}(BHomeo_{\partial}(M); \mathbf{Q})$, not when considered as a class in the cohomology $H^{4i-d}(BHomeo_{\partial}(M)_{f\mathbf{Q}}; \mathbf{Q})$ of the fibrewise rationalisation. A priori these groups differ since we have not yet shown that $BHomeo_{\partial}(M)$ is fibrewise nilpotent over $BAut_{\partial}(M)$. There are several ways to circumvent this issue, for example one could rely on Remark 5.2 (i) for the missing fibrewise nilpotency claim. Alternatively, one can argue along the lines of Remark 5.2 (ii) as follows: it suffices to show that $\kappa_{\mathcal{L}_i}$ lies in the canonical $H^{4i-d}(BHomeo_{\partial}(M); \mathbf{Q})$ -summand of $H^{4i-d}(BHomeo_{\partial}(M)_{f\mathbf{Q}}; \mathbf{Q})$ (see Section 2.1.3), which by the naturality of this summand follows from the same statement for any of the spaces in the composition (45) along which $\kappa_{\mathcal{L}_i}$ pulls back. The space $Map_{\partial}(M, BSTop(d))/Aut_{\partial}(M)$ is fibrewise nilpotent over $BAut_{\partial}(M)$, so its cohomology consists entirely of the summand in question.

6.3. **Pontryagin–Weiss classes.** One of the main application of Theorem **F** we carry out in this work is the following result. It is a reformulation of Theorem A.

Theorem 6.8. For all $d \ge 6$ and $k \ge 1$, the kth rational Pontryagin class $p_k \in H^{4k}(BSTop(d); \mathbb{Q})$ is nontrivial and evaluates nontrivially on the image of $\pi_{4k}(BSTop(d))$ under the Hurewicz morphism.

In addition to Theorem F, the proof of Theorem 6.8 relies on the work of Galatius and Randal-Williams on stable moduli spaces of even-dimensional manifolds [GRW17] and the following auxiliary lemma, inspired by [Wei21, Section 6].

Lemma 6.9. Fix a cohomology class $c \in H^k(BSTop(d); \mathbb{Q})$ of degree $k \geq 2$. If there is a tangential structure $\theta \colon B \to BSTop(d)$ with B nilpotent, a compact d-manifold M with $d \geq 3$, an embedded disc $D^{d-1} \subset \partial M$, and a θ -structure ℓ on M such that

- (i) the pullback of c along θ vanishes,
- (ii) the map $\theta: B \to BSTop(d)$ is injective on $\pi_{k-1}(-)_{\mathbb{Q}}$, and

(iii) the class
$$\kappa_c \in H^{k-d}(\operatorname{Map}_{\partial}(M,\operatorname{BSTop}(d))/\operatorname{Aut}_{\partial}(M); \mathbf{Q})$$
 is nontrivial when pulled back to
$$\frac{\operatorname{Map}_{\partial}(M,\operatorname{BSTop}(d)_{\mathbf{Q}}) \times_{\operatorname{Map}_{1/2\partial}(M,\operatorname{BSTop}(d)_{\mathbf{Q}})} \operatorname{Map}_{1/2\partial}(M,B_{\mathbf{Q}})}{\operatorname{Aut}_{\partial}(M)},$$

then the class c evaluates nontrivially on $\pi_k(BSTop(d))$.

Proof. Represent c by a map c: BSTop $(d) \to K(Q, k)$ and use (i) to choose a nullhomotopy of $(c \circ \theta)$ to get a class $\widetilde{c} \in H^{k-1}(\mathrm{fib}(\theta); Q)$ that transgresses to c. By (ii), the map $\pi_k(\mathrm{BSTop}(d))_Q \to \pi_{k-1}(\mathrm{fib}(\theta))_Q$ is surjective, so it suffices to show that \widetilde{c} is detected on $\pi_{k-1}(\mathrm{fib}(\theta))$. To do so, choose a map from a space X of finite rational type into the pullback in (iii) such that the pullback of κ_c along this map is nontrivial. This map classifies a relative fibration $\pi \colon (E, \partial M \times X) \to X$ with fibre $(M, \partial M)$, together with a map $\ell^\pi \colon E \to B_Q$, a homotopy $h^\pi_{1/2}$ of maps $1/2\partial M \times X \to B_Q$ between $\ell^\pi|_{1/2\partial M \times X}$ and $\ell|_{1/2\partial M} \circ \mathrm{pr}_1$, and an extension of the postcomposition of this homotopy with θ to a homotopy of maps $\partial M \times X \to \mathrm{BSTop}(d)_Q$ between $\theta \circ \ell^\pi|_{\partial M \times X}$ and $\theta \circ \ell|_{\partial M} \circ \mathrm{pr}_1$. From the construction in Section 6.1, we obtain a commutative diagram

together with a compatible nullhomotopy of the composition $* \times X \to B_Q$. By construction the class $\widetilde{c} \in H^{k-1}(\mathrm{fib}(\theta); \mathbf{Q})$ is represented by the map on vertical fibres of the rightmost square. Now recall that $\kappa_c \in H^{k-d}(X; \mathbf{Q})$ is obtained from the class in $H^k(E \cup_{\partial M \times X} (-M \times X); \mathbf{Q})$ by integration along the fibres, so in particular the latter is nontrivial and hence also the relative class given by the composition of the two rightmost squares in the diagram is nontrivial. As the second square is a pushout, also the relative class given by the composition of the three rightmost squares has to be nontrivial. Using the nullhomotopy of $* \times X \to B_Q$ and taking cofibres, we obtain a diagram

$$\Sigma^{d-1}(X_{+}) \simeq \operatorname{cofib}\left(*\times X \to (D^{d-1} \cup_{\partial D^{d-1}} D^{d-1}) \times X\right) \longrightarrow B_{\mathbb{Q}} \longrightarrow *$$

$$\downarrow \qquad \qquad \downarrow \theta \qquad \qquad \downarrow \downarrow$$

$$* \simeq \operatorname{cofib}\left(*\times X \to D^{d} \times X\right) \longrightarrow \operatorname{BSTop}(d)_{\mathbb{Q}} \xrightarrow{c} K(\mathbb{Q}, k)$$

whose outer rectangle is a nontrivial relative cohomology class. Taking vertical fibres yields a map $\Sigma^{d-1}(X_+) \to \mathrm{fib}(\theta)$ that detects $\widetilde{c} \in \mathrm{H}^{k-1}(\mathrm{fib}(\theta); \mathbf{Q})$. As $d \geq 3$, the suspension $\Sigma^{d-1}(X_+)$ is 1-connected and of finite rational type, so its rational Hurewicz morphism is surjective. Hence the class \widetilde{c} is nontrivial on $\pi_{k-1}(\Sigma^{d-1}(X_+))$, so also on $\pi_{k-1}(\mathrm{fib}(\theta))$ and the claim follows.

Proof of Theorem 6.8. We may assume $k \ge 3$ and d = 6: the former since p_k evaluates nontrivially on $\pi_{4k}(\mathrm{BSO}(6))$ for $k \le 2$, so also on $\pi_{4k}(\mathrm{BSTop}(6))_{\mathbb{Q}}$, and the latter since p_k pulls back along the stabilisation map BSTop(d) → BSTop. For $k \ge 3$ and d = 6, we show the claim by an application of Lemma 6.9 to $p_k \in \mathrm{H}^{4k}(\mathrm{BSTop}(6);\mathbb{Q})$. As tangential structure we choose the composition $\theta \colon \tau_{>3}\mathrm{BSO}(4) \to \mathrm{BSTop}(6)$ of the 3-connected cover $\tau_{>3}\mathrm{BSO}(4) \to \mathrm{BSO}(4)$ followed by the stabilisation and forgetful map BSO(4) → BSTop(6). This satisfies conditions (i) and (ii) in the lemma since p_k is trivial in BSO(4) as $k \ge 3$ and BSO(4)_Q ≈ $K(\mathbb{Q}, 4)^{\times 2}$. As the manifold M we choose $W_{g,1} := (S^3 \times S^3)^{\sharp g} \setminus \mathrm{int}(D^6)$ for some $g \ge 0$ that will be specified later. The manifold $W_{g,1}$ is parallelisable, so the constant map $\ell \colon W_{g,1} \to \tau_{>3}\mathrm{BSO}(4)$ can be enhanced to a θ -structure ℓ on $W_{g,1}$. We are left to show that κ_{p_k} is nontrivial in the pullback appearing in (iii). We will show the stronger claim that the pullback of κ_{p_k} along the map from BHomeo $_{\partial}^{\theta}(W_{g,1}; \ell_{1/2})_{\ell,f\mathbb{Q}}$ to the pullback in Lemma 6.9 (iii) induced by the first map in (44) and the composition (45) is nontrivial. To do so, we first argue that it suffices to prove instead that $\kappa_{\mathcal{L}_k} \in \mathrm{H}^{4k-6}(\mathrm{BHomeo}_{\partial}^{\theta}(W_{g,1}; \ell_{1/2})_{\ell,f\mathbb{Q}}; \mathbb{Q})$ is nontrivial. The manifold $W_{g,1}$ and the tangential structure θ satisfy the assumptions of Theorem F, so writing $B \coloneqq \tau_{>3} \mathrm{BSO}(4)$ Proposition 6.4 equips $\mathrm{H}^{4k-6}(\mathrm{BHomeo}_{\partial}^{\theta}(W_{g,1}; \ell_{1/2})_{\ell,f\mathbb{Q}}; \mathbb{Q})$ with an End $_{1/2}^{\ell_{Q}}(B_{\mathbb{Q}})$ -action. As the underlying map of ℓ is constant, the constant map $B_{\mathbb{Q}} \to B_{\mathbb{Q}}$ yields an

element $\chi \in \operatorname{End}_{1/2\partial}^{\ell_Q}(B_Q)$. Using that $\kappa_{\mathcal{L}_k}$ is fixed by the action of χ (see Example 6.6) and writing $\kappa_{\mathcal{L}_k} = A \cdot \kappa_{p_k} + B \cdot \kappa_c$ as a linear combination of κ_{p_k} and κ_c for c a sum of decomposable monomials in Pontryagin classes, we get $\kappa_{\mathcal{L}_k} = \chi^* \kappa_{\mathcal{L}_k} = A \cdot \chi^* \kappa_{p_k} + B \cdot \chi^* \kappa_c = A \cdot \chi^* \kappa_{p_k}$, since $\chi^* \kappa_c = 0$ by (46), so if $\kappa_{\mathcal{L}_k}$ is nontrivial, then κ_{p_k} has to be nontrivial as well.

This leaves us with showing that $\kappa_{\mathcal{L}_k} \in H^{4k-6}(\mathrm{BHomeo}_{\partial}^{\theta}(W_{g,1};\ell_{1/2})_{\ell,fQ};\mathbf{Q})$ is nontrivial. To do so, we write $\theta^{\mathrm{sm}} \colon \tau_{>3}\mathrm{BSO}(4) \to \mathrm{BSO}(6)$ for the composition of the 3-connected cover with the stabilisation map $\mathrm{BSO}(4) \to \mathrm{BSO}(6)$, and consider the composition

$$\mathrm{BDiff}_{\partial}^{\theta^{\mathrm{sm}}}(W_{g,1};\ell_{\partial})_{\ell} \longrightarrow \mathrm{BHomeo}_{\partial}^{\theta}(W_{g,1};\ell_{1/2})_{\ell} \longrightarrow \mathrm{BHomeo}_{\partial}^{\theta}(W_{g,1};\ell_{1/2})_{\ell,\mathrm{fQ}} \tag{47}$$

whose first space is the component induced by ℓ of the $\mathrm{Diff}_{\partial}(W_{g,1})$ -orbits of the space of θ^{sm} -structures on the smooth tangent bundle $T^{\mathrm{sm}}W_{g,1}$ that agree with ℓ on $T^{\mathrm{sm}}W_{g,1}|_{\partial W_{g,1}}$. The latter classifies smooth relative bundles $(W_{g,1},\partial W_{g,1})\to (E,\partial W_{g,1}\times X)\to X$ together with a lift $\ell^\pi\colon E\to \tau_{>3}\mathrm{BSO}(4)$ of its vertical tangent bundle $T_\pi\colon E\to \mathrm{BSO}(4)$ and a homotopy $h^\pi\colon \partial W_{g,1}\times X\to \tau_{>3}\mathrm{BSO}(4)$ between $\ell^\pi|_{\partial W_{g,1}\times X}$ and $\ell|_{\partial W_{g,1}}\circ\mathrm{pr}_1$ such that ℓ^π is on each fibre homotopic to ℓ relative to $\partial W_{g,1}$. In these terms the first map in (47) is given by remembering the underlying topological bundle, viewing ℓ^π as a lift of the *topological* vertical tangent bundle, and restricting the homotopy h^π to $1/2\partial W_{g,1}\times X\subset \partial W_{g,1}\times X$. The second map in (47) is the fibrewise rationalisation. We finish the proof by applying Galatius and Randal–Williams' work to show that $\kappa_{\mathcal{L}_k}\in H^*(\mathrm{BHomeo}_{\partial}^{\theta}(W_{g,1};\ell_{1/2})_{\ell}; f\mathbb{Q})$ is nontrivial for $g\gg 0$ by proving that its pullback along (47) is nontrivial.

The parametrised Pontryagin–Thom construction gives a map PT: BDiff $_{\partial}^{\theta^{\rm sm}}(W_{g,1})_{\ell} \to \Omega_0^{\infty} \mathbf{M} \mathbf{T} \theta^{\rm sm}$ to the 0-component of the infinite loop space of the Thom spectrum $\mathbf{M} \mathbf{T} \theta^{\rm sm}$ of the inverse of the vector bundle classified by $\theta^{\rm sm}$. By an application of [GRW17, Corollary 1.8], this map induces an cohomology isomorphism in a range increasing with g. Now consider the composition

$$H^*(\tau_{>3}\mathrm{BSO}(4);\mathbf{Q})\overset{\mathrm{Thom}}{\cong}H^{*-6}(\mathbf{MT}\theta^{\mathrm{sm}};\mathbf{Q})\to H^{*-6}(\Omega_0^\infty\mathbf{MT}\theta^{\mathrm{sm}};\mathbf{Q})\overset{\mathrm{PT}^*}{\to}H^{*-6}(\mathrm{BDiff}_{\partial}^{\theta^{\mathrm{sm}}}(W_{g,1})_\ell;\mathbf{Q})$$

whose middle map is induced by the counit of the $(\Sigma_+^\infty, \Omega^\infty)$ -adjunction. As for any spectrum, the latter is injective in positive degrees, so the composition is injective in a range of degrees growing with g. Moreover, by an exercise in the Pontryagin–Thom construction, given a class $c \in H^*(\mathrm{BSTop}(d); \mathbf{Q})$, the image of the pullback along θ under this composition agrees with the class $\kappa_c \in H^{*-6}(\mathrm{BDiff}_{\partial}^{\mathrm{gsm}}(W_{g,1})_{\ell}; \mathbf{Q})$, so the pullback of $\kappa_{\mathcal{L}_k}$ along (47) is nontrivial for large enough $g \gg 0$ if \mathcal{L}_k is nontrivial in $H^{4k}(\tau_{>3}\mathrm{BSO}(4); \mathbf{Q})$. But this is the case: it is even nontrivial in $H^{4k}(\tau_{>3}\mathrm{BSO}(3); \mathbf{Q})$ because all monomials in Pontryagin classes except p_1^k vanish in this group and the coefficient of p_1^k in $\mathcal{L}_k \in H^*(\mathrm{BSO}; \mathbf{Q})$ is nontrivial by [BB18, Corollary 3].

Proof of Corollary B. To construct a rational section of $STop(d) \to STop$ for $d \ge 6$, we use Theorem 6.8 to choose $x_i \in \pi_{4i}(BSTop(d)_Q)$ with $p_i(x) = 1$ for $i \ge 1$ and consider the composition

$$\vee_{i\geq 1}S_{\mathbf{Q}}^{4i} \stackrel{\vee_{i}x_{i}}{\longrightarrow} \mathrm{BSTop}(d)_{\mathbf{Q}} \stackrel{\mathrm{stab}}{\longrightarrow} \mathrm{BTop}_{\mathbf{Q}} \stackrel{\sqcap_{i}p_{i}}{\simeq} \prod_{i\geq 1} K(\mathbf{Q},4i)$$

whose composition represents the dual of the fundamental classes of the spheres. It suffices to show that the composition admits a section after looping once which follows from the fact that $\Omega(\vee_{i=1}^{\infty}S_{\mathbf{Q}}^{4i})$ splits as a product of Eilenberg–MacLane spaces.

Proof of Corollary C. We explain the proof in even dimensions; the one in odd dimensions is analogous. Consider for $2n \ge 6$ the maps of horizontal fibre sequences

where the upper map of fibre sequences consists of vertical 0-coconnected maps resulting from smoothing theory (see e.g. [KrRW21, Section 8.2]) and the bottom one is as in the proof of Corollary

8.3 loc.cit.. By the long exact sequence and the 5-lemma, it suffices to show that the middle lower vertical map is surjective on rational homotopy groups. This follows from Theorem 6.8.

6.4. A rational Burghelea-Lashof splitting. We now prove (a sharper version of) Theorem D. As mentioned in the introduction, a similar splitting was known after Postnikov truncation in a range depending on the dimension (see [BL82, Corollary D] or [HJ82, Theorem 2.5]). The novelty of Theorem 6.10 is that it does not require any truncations.

Theorem 6.10. For M a 2-connected smoothable compact manifold of dimension $d \geq 8$ with 2connected boundary ∂M , the rationalised forgetful map between the components of the identity

$$\left(\operatorname{Aut}_{\partial}(M)/\operatorname{Homeo}_{\partial}(M)\right)_{\operatorname{id}, \mathbf{Q}} \longrightarrow \left(\operatorname{Aut}_{\partial}(M)/\widetilde{\operatorname{Homeo}_{\partial}}(M)\right)_{\operatorname{id}, \mathbf{Q}}$$

has the following properties:

- (i) It admits a section if the rationalised inclusion $(1/2\partial M)_Q \to M_Q$ desuspends once.
- (ii) After taking loop spaces it admits a section if one of the following conditions are satisfied: (a) d = 2n + 1 is odd and the Pontryagin class $p_{n-i}(M) \in H^{4(n-i)}(M; \mathbb{Q})$ vanishes for $1 \le i \le 3$
 - (b) d = 2n is even and the Pontryagin class $p_{n-i}(M) \in H^{4(n-i)}(M; \mathbb{Q})$ vanishes for $1 \le i \le 3$ and is a square for i = 4.

These conditions are always satisfied for $d \geq 13$.

Example 6.11. If $\partial M \cong S^{d-1}$, then the condition in the first part of Theorem 6.10 is equivalent to the rational Hurewicz morphism $\pi_{*>0}(M) \otimes \mathbf{O} \to \mathrm{H}_{*>0}(M; \mathbf{O})$ being surjective, since $1/2\partial M \simeq *$ and a 1connected space is rationally equivalent to a suspension if and only if its rational Hurewicz morphism is surjective (see e.g. [FHT01, Theorem 24.5]). A concrete class of manifolds that satisfies this assumption are iterated connected sums of the form $(\sharp_{i=1}^g(S^{m_i}\times S^{d-m_i}))\setminus \operatorname{int}(D^d)$ for $3\leq m_i\leq d-3$.

Proof of Theorem 6.10. We consider the tangential structure θ : BSTop $(d) \times_{BSTop(d)_Q} BSTop(d-2)_Q \rightarrow$ BSTop(d) and first argue that M admits a θ -structure ℓ . Since M is smoothable, this follows from showing that the rationalised smooth tangent bundle $M \to BSO(d)_Q$ lifts along $BSO(d-2)_Q \to BSO(d)_Q$ BSO $(d)_Q$. By obstruction theory this follows if the Euler class e the Pontryagin class $p_{\lfloor \frac{d}{a} \rfloor}$ both rationally vanish. This is the case, since they lie in $H^i(M; \mathbb{Q}) \cong H_{d-i}(M, \partial M; \mathbb{Q})$ for $i \geq d$ which vanishes. Now choose an embedded disc $D^{d-1} \subset \partial M$, set $1/2\partial M := M \setminus \operatorname{int}(D^{d-1})$ and consider the commutative diagram

$$\begin{split} \operatorname{fib}_{\ell} \big(\operatorname{BHomeo}_{\partial}^{\theta}(M; \ell_{1/2}) &\to \operatorname{BAut}_{\partial}(M)^{\ell_{1/2}} \big)_{\ell} \to \operatorname{Map}_{1/2\partial}(M, \operatorname{BSTop}(d) \times_{\operatorname{BSTop}(d)_{\mathbb{Q}}} \operatorname{BSTop}(d-2)_{\mathbb{Q}})_{\ell} \\ & \qquad \qquad \qquad \qquad \downarrow^{\operatorname{forget}} & \qquad \qquad \downarrow^{\theta*} \\ & \left(\operatorname{Aut}_{\partial}(M) / \operatorname{Homeo}_{\partial}(M) \right)_{\operatorname{id}} & \qquad \qquad \to \operatorname{Map}_{1/2\partial}(M, \operatorname{BSTop}(d))_{TM} \\ & \qquad \qquad \downarrow^{\operatorname{forget}} & \qquad \qquad \downarrow^{s_{*}} \\ & \left(\operatorname{Aut}_{\partial}(M) / \operatorname{Homeo}_{\partial}(M) \right)_{\operatorname{id}} & \qquad \qquad \to \operatorname{Map}_{1/2\partial}(M, \operatorname{BSTop})_{T^{s}M}. \end{split}$$

Here s: BSTop $(d) \rightarrow$ BSTop is the stabilisation map and the subscripts indicate that we restrict to the respective path-components. The horizontal maps are induced by taking vertical tangent bundles; for the bottom map this uses that a block bundle still has a stable vertical tangent bundle. After rationalisation, the top horizontal arrow admits a dashed section as indicated since it agrees with the first map in (44) after taking fibres of the maps to BAut $_{a}^{t_{Q}}(M)$ and restricting to components, so it admits a section since (44) can be completed to a pullback square as a result of Theorem F and Remark 5.7. The fact that the bottom horizontal map is a rational equivalence is a consequence of surgery theory (see [Kra22, Section 2.1] for a fitting reminder of surgery theory in the smooth setting; the topological case is analogous).

A diagram chase shows that the (once looped) map in the statement admits a rational section if this holds for the (once looped) right-vertical composition. As $BSTop(d) \times_{BSTop(d)_Q} BSTop(d-2)_Q \rightarrow$

BSTop(d-2) is a rational equivalence this is the case if the map

$$\operatorname{Map}_{1/2\partial}(M, \operatorname{BSTop}(d-2)_{\mathbb{Q}})_{\ell_{\mathbb{Q}}} \xrightarrow{s_*} \operatorname{Map}_{1/2\partial}(M, \operatorname{BSTop}_{\mathbb{Q}})_{T^sM_{\mathbb{Q}}}$$
(48)

admits a rational section (after looping once). If the inclusion $^1/2\partial M\subset M$ is rationally the suspension of a map $A\to X$, then (48) has rationally the form $\operatorname{Map}_A(X,\operatorname{STop}(d-2)_{\mathbb{Q}})_{\overline{\ell}}\to\operatorname{Map}_A(X,\operatorname{STop}_{\mathbb{Q}})_{\overline{T^sM}}$ where $\overline{\ell}$ and $\overline{T^sM}$ are the adjoints of $\ell_{\mathbb{Q}}$ and $T^sM_{\mathbb{Q}}$. As $d\geq 8$, the stabilisation map $\operatorname{STop}(d-2)_{\mathbb{Q}}\to\operatorname{STop}_{\mathbb{Q}}$ admits by Corollary B a section $\sigma\colon\operatorname{STop}_{\mathbb{Q}}\to\operatorname{STop}(d-2)_{\mathbb{Q}}$. Using this and the E_1 -group-structure of $\operatorname{STop}(d-2)_{\mathbb{Q}}$, we see that (48) has indeed a section, namely

$$\operatorname{Map}_A(X,\operatorname{STop}_{\mathbb{Q}})_{\overline{T^sM}} \xrightarrow{\sigma_*} \operatorname{Map}_A(X,\operatorname{STop}(d-2)_{\mathbb{Q}})_{\sigma \circ \overline{T^sM}} \xrightarrow{\overline{\ell} \cdot (-) \cdot (\sigma \circ \overline{T^sM})^{-1}} \operatorname{Map}_A(X,\operatorname{STop}(d-2)_{\mathbb{Q}})_{\overline{\ell}},$$

so part (i) of the claim follows.

To see (ii), note since we assumed that M is smoothable we may, after possibly rechoosing ℓ assume that ℓ_Q factors over $BSO(d-2)_Q$. The assumptions in (ii) are made exactly to ensure that ℓ_Q factors further over $BSO(d-8)_Q$ and thus also over $BSTop(d-8)_Q$. In order to use this to show that (48) admits a section after looping once, we extend the looped map to a commutative square

$$\begin{split} \Omega\big(\mathrm{Map}_{^{1/2}\partial}(M,\mathrm{BSTop}(6)_{\mathbb{Q}})_{\mathrm{const}}\big) &\longrightarrow \Omega\big(\mathrm{Map}_{^{1/2}\partial}(M,\mathrm{BSTop}_{\mathbb{Q}})_{\mathrm{const}}\big) \\ & \downarrow^{(-)+\ell_{\mathbb{Q}}} & \simeq \downarrow^{(-)+T^sM} \\ \Omega\big(\mathrm{Map}_{^{1/2}\partial}(M,\mathrm{BSTop}(d-2)_{\mathbb{Q}})_{\ell_{\mathbb{Q}}}\big) & \to \Omega\big(\mathrm{Map}_{^{1/2}\partial}(M,\mathrm{BSTop}_{\mathbb{Q}})_{T^sM_{\mathbb{Q}}}\big) \end{split}$$

whose vertical maps are induced by the external multiplication maps $\mathrm{BSTop}(k) \times \mathrm{BSTop}(n) \to \mathrm{BSTop}(k+n)$ induced by taking direct products. By commutativity, it suffices to show that the top horizontal map has a section. As the loop spaces in base and target of this top map are based at the constant map, we may "pull Ω inside the mapping space" to see that this map is equivalent to the map $\mathrm{Map}_{1/2\partial}(M,\mathrm{STop}(6)_{\mathbb{Q}})_{\mathrm{const}} \to \mathrm{Map}_{1/2\partial}(M,\mathrm{STop}_{\mathbb{Q}})_{\mathrm{const}}$ induced by stabilisation. The latter admits a section since $\mathrm{STop}(6)_{\mathbb{Q}} \to \mathrm{STop}_{\mathbb{Q}}$ does, by Corollary B, so this gives the claim.

6.5. **Detecting tautological classes.** Theorem \mathbb{F} can also be used to show nontriviality of many tautological classes $\kappa_c \in \mathbb{H}^{*-d}(\operatorname{BHomeo}_{\partial}(M); \mathbb{Q})$ for decomposable classes $c \in \mathbb{H}^*(\operatorname{BTop}(d); \mathbb{Q})$, and of products of such. To explain the strategy, fix an oriented smoothable 2-connected d-manifold M with $d \geq 6$ and 2-connected boundary ∂M . Fix a lift $\ell \colon M \to \operatorname{BSTop}(d-2)_{\mathbb{Q}}$ of the rational tangent bundle $TM_{\mathbb{Q}} \colon M \to \operatorname{BSTop}(d)_{\mathbb{Q}}$. To see that such a lift exists, note that $TM_{\mathbb{Q}}$ lifts to $\operatorname{BSO}(d)_{\mathbb{Q}}$ as M is smoothable, then use obstruction theory to lift further to $\operatorname{BSO}(d-2)_{\mathbb{Q}}$ and thus to $\operatorname{BSTop}(d-2)_{\mathbb{Q}}$.

Theorem 6.12. Fix M and ℓ as above. There exists a dashed lift

$$\operatorname{Map}_{1/2\partial}(M,\operatorname{BSTop}(d-2)_{\mathbb{Q}})_{\ell} \xrightarrow{q} \operatorname{Map}_{1/2\partial}(M,\operatorname{BSTop}(d)_{\mathbb{Q}})/\operatorname{hAut}_{\partial}(M) \tag{49}$$

of the map q induced by stabilisation and taking quotients, along the map d induced by the topological derivative. In particular, if a decomposable class $c = c' \cup c'' \in H^*(BSTop(d); \mathbf{Q})$ has the property that

$$\kappa_{c'\otimes c''}\in \mathrm{H}^{*-d}(\mathrm{Map}_{1/2\partial}(M,\mathrm{BSTop}(d)_{\mathbb{Q}})/\mathrm{hAut}_{\partial}(M);\mathbb{Q})$$

pulls back nontrivially along the horizontal map in (49), then $\kappa_c \in H^{*-d}(BHomeo_{\partial}(M); \mathbf{Q})$ is nontrivial. The same applies to linear combinations of products of such classes.

Proof. The second part follows from the first, since by the discussion in Section 6.1, $\kappa_{c'\otimes c''}$ pulls back to κ_c along the vertical map in (49). To show the first part, we use the same tangential structure θ : BSTop $(d-2)_Q \times_{\text{BSTop}(d)_Q} \text{BSTop}(d) \to \text{BSTop}(d)_Q$ as in the proof of Theorem 6.10. The claim

then follows from the commutative diagram

where the existence of the dashed map results from Theorem \mathbf{F} in the form of Remark 5.7.

The condition in the second part of Theorem 6.12 can often be checked by using the explicit description the pullback of $\kappa_{c\otimes c'}$ to $\operatorname{Map}_{1/2\partial}(M,\operatorname{BSTop}(d-2)_{\mathbb{Q}})_\ell$ resulting from Remark 6.3, in the case $B=\operatorname{BSTop}(d-2)_{\mathbb{Q}}$. We will illustrate this method to detect tautological classes by proving Theorem E from the introduction, which said that for M as above that also contains $S^m\times S^n$ as a connected summand (with n=d-m), we have

- (i) The class $\kappa_{p_ip_j} \in H^{4(i+j)-d}(BHomeo_{\partial}(M); \mathbb{Q})$ is nontrivial for all $i, j > \max(m, n)/4$.
- (ii) If n and m are both even, then the total dimension of the cohomology $H^{*\leq k}(BHomeo_{\partial}(M); \mathbb{Q})$ grows with k faster than any polynomial.

Proof of Theorem E. By naturality, it suffices to prove the result for $M = D^d \sharp (S^m \times S^n)$, which is homotopy equivalent to $S^m \vee S^n$. Choosing classes $a_m \in H_m(M; \mathbb{Q})$ and $b_n \in H_n(M; \mathbb{Q})$ corresponding to the fundamental classes of the two spheres, for (i), it suffices by the second part of Theorem 6.12 and Remark 6.3 to show that the class

ev* $(p_i)/a_m \cup$ ev* $(p_j)/b_n +$ ev* $(p_i)/b_n \cup$ ev* $(p_j)/a_m \in$ H^{4(i+j)-d} (Map_{*} $(M, BSTop(d-2)_Q)_0; Q)$ (50) is nontrivial. To do so, we use Theorem 6.8 and the assumption that $d-2 \ge 6$ to find a map of the form $\rho: \bigvee_{k>1} S^{4k} \to BSTop(d-2)_Q$ with $\langle \rho^* p_k, [S^{4k}] \rangle = 1$ for all k. Pulling back (50) along

$$\rho_* \colon \Omega_0^m(\bigvee_{k \ge 1} S^{4k}) \times \Omega_0^n(\bigvee_{k \ge 1} S^{4k}) \simeq \operatorname{Map}_*(M, \bigvee_{k \ge 1} S^{4k})_0 \longrightarrow \operatorname{Map}_*(M, \operatorname{BSTop}(d-2)_{\mathbb{Q}})_0$$
 yields the class

$$\operatorname{ev}^*(\lambda_{4i})/a_m \otimes \operatorname{ev}^*(\lambda_{4j})/b_n + (-1)^{nm} \operatorname{ev}^*(\lambda_{4j})/a_m \otimes \operatorname{ev}^*(\lambda_{4i})/b_n$$
(51)

in $\mathrm{H}^{4(i+j)-d}(\Omega_0^m(\bigvee_{k\geq 1}S^{4k})\times\Omega_0^n(\bigvee_{k\geq 1}S^{4k});\mathbf{Q})$ where λ_{4k} is the dual of the fundamental class of S^{4k} . It follows from the Milnor–Moore theorem that the cohomology of $\Omega_0^m(\bigvee_{k\geq 1}S^{4k})$ is a free graded-commutative algebra on a set of generators which include the classes $\mathrm{ev}^*(\lambda_{4k})/a_m$ for k>m/4, and similarly for $\Omega_0^n(\bigvee_{k\geq 1}S^{4k})$, so this pullback is nonzero and (i) follows.

To show (ii), note that by using the first part of Theorem 6.12 it suffices to prove that the classes (51) for $i, j > \max(m, n)/4$ span a subalgebra whose total dimension has the claimed growth. To show this, we consider an algebra morphism $H^*(\Omega_0^m(\bigvee_{k\geq 1}S^{4k})\times\Omega_0^n(\bigvee_{k\geq 1}S^{4k}); \mathbf{Q})\to\mathbf{Q}[x_i,y_j\mid i,j>\max(m,n)/4]$ to a free graded algebra with generators of even degrees $|x_i|=4i-m$ and $|y_j|=4j-n$, by sending the generators $\mathrm{ev}^*(\lambda_{4i})/a_m\otimes 1$ to x_i and $1\otimes \mathrm{ev}^*(\lambda_{4j})/b_n$ to y_j , and all other generators to zero. It thus suffices to show that the subalgebra generated by the elements $x_iy_j+x_jy_i$ has the required growth. Assuming without loss of generality $m\leq n$, a crude lower bound for the growth of this subalgebra can be obtained by reindexing y_j to be in degree 4j-m instead, and defining an algebra morphism to the polynomial algebra $\mathbf{Q}[z_i\mid i>n/4]$ in infinitely many generators by sending both x_i and y_i to z_i . This is surjective onto the polynomial subalgebra on generators z_i^2 for i>n/4. The growth of its total dimension can be bounded from below by the partition function which is known to grow faster than any polynomial.

Remark 6.13.

(i) Note that by choosing $M = D^8 \sharp (S^4 \times S^4)$, the first part of Theorem E in particular implies that $p_i p_j \neq 0 \in H^*(\mathrm{BTop}(d); \mathbb{Q})$ for $d \geq 8$ and all i, j > 1. By picking different choices of M and extending the strategy of Theorem E, one can exclude more relations between products of Pontryagin classes in $H^*(\mathrm{BTop}(d); \mathbb{Q})$ and thus partially recover [GRW23, Theorem 1.1]. However it seems unlikely that this method can be extended to exclude all relations.

(ii) It may seem surprising at first sight that our method also shows that certain products of Pontryagin classes are nontrivial, even though the input Theorem 6.8 only shows that the individual p_i s are nontrivial and detected on the image of the Hurewicz map. The surprise comes from the implicit use of the Alexander trick and is similar to the following observation: suppose one only knew (i) surgery theory and (ii) that p_i and p_j for some i, j > 1 are nontrivial in $H^*(BTop; \mathbb{Q})$ and detected on the image of the Hurewicz map, then one can deduce that their product $p_i p_j \in H^*(BTop; \mathbb{Q})$ is nontrivial, by considering the map

$$\left(\operatorname{Aut}_{\partial}(M)/\widetilde{\operatorname{Homeo}}_{\partial}(M)\right)_{\operatorname{id}} \longrightarrow \operatorname{Map}_{1/2\partial}(M,\operatorname{BSTop})_{0}$$
 (52)

induced by taking stable topological derivatives for $M = D^d \sharp (S^n \times S^m)$. Arguing as in the proof of Theorem E, the class (50) with BTop(d-2) replaced by BTop is nontrivial and it pulls back to $\kappa_{p_ip_j}$ along (52). But it follows from surgery theory and the Alexander trick that (52) is a rational equivalence, so $\kappa_{p_ip_j}$ is nontrivial and hence so is p_ip_j .

(iii) The nonzero classes produced in the proof of Theorem E all lie in the subalgebra generated by κ_c for c monomials in Pontryagin classes. In the smooth situation, the corresponding subalgebra of H*(BDiff $_{\partial}(M)$; Q) is known to be finitely generated in many cases (so in particular to grow polynomially) by [RW18, Theorem A], in particular for $M = D^{2n} \# (S^n \times S^n)^{\# g}$ for all g and g (if g is odd, it is even finite-dimensional [GGRW17, Corollary 1.4]).

Remark 6.14. Theorem \mathbf{F} can also be used to produce weight decompositions on the rational cohomology of BHomeo $_{\partial}^{\theta}(M; \ell_{1/2\partial})_{fQ,\ell}$ for suitable choices of θ , similar to the weight decompositions of the rational cohomology of block diffeomorphisms outlined in [KK19]. We expect that these weight decompositions are useful to exclude relations between tautological classes and to resolve differentials in spectral sequences, but we do not pursue this further at this point.

Appendix A. Tensor products of truncated operads

In this appendix, we construct the symmetric monoidal lift of the tower (4) of categories of truncated operads from Section 2.2 and establish several properties of it, summarised in Theorem A.1. We adopt the notation of Section 2.2. We write $X * Y := X \cup_{X \times Y} Y$ for the *join* of spaces and $S^{\Sigma_n} = \operatorname{Fun}(\Sigma_n, S)$ for the category of spaces with an action of the symmetric group on n letters.

Theorem A.1. The tower of categories of truncated unital operads

$$\mathfrak{O}p^{un} \simeq \mathfrak{O}p^{\leq \infty, un} \to \cdots \to \mathfrak{O}p^{\leq 2, un} \to \mathfrak{O}p^{\leq 1, un} \simeq \mathfrak{C}at \tag{53}$$

can be lifted to a tower of symmetric monoidal categories such that

- (i) the symmetric monoidal structure on Opun is the one described in Section 2.2,
- (ii) objectwise, the symmetric monoidal structure on $\mathfrak{O}p^{\leq k,\mathrm{un}}$ is given by natural equivalences

$$\mathfrak{O} \otimes \mathfrak{P} \simeq \tau(\tau_*(\mathfrak{O}) \otimes \tau_*(\mathfrak{P})) \simeq \tau(\tau_!(\mathfrak{O}) \otimes \tau_!(\mathfrak{P}))$$

where $\tau \colon \mathbb{O}p^{\mathrm{un}} \to \mathbb{O}p^{\leq k,\mathrm{un}}$ is the truncation functor and $\tau_!$ and τ_* are its left and right adjoints.

- (iii) the symmetric monoidal structure on $\mathfrak{Op}^{\leq k,\mathrm{un}}$ preserves colimits in both variables for all $k \leq \infty$,
- (iv) the symmetric monoidal structure on $\mathbb{O}p^{\leq 1,un} \simeq \mathbb{C}$ at is the cartesian one. In particular, as $*\times * \simeq *$, the full subcategories of reduced operads

$$\mathbb{O}p^{\leq k, \text{red}} = \text{fib}_*(\mathbb{O}p^{\leq k, \text{un}} \to \mathbb{C}\text{at}_{\infty}) \subset \mathbb{O}p^{\leq k, \text{un}}$$

inherits symmetric monoidal structures for $1 \le k \le \infty$,

(v) with respect to the equivalence $\mathfrak{Op}^{\leq 2, \mathrm{red}} \simeq \mathbb{S}^{\Sigma_2}$ from Lemma 2.4, the binary tensor product $\mathfrak{Op}^{\leq 2, \mathrm{red}} \times \mathfrak{Op}^{\leq 2, \mathrm{red}} \to \mathfrak{Op}^{\leq 2, \mathrm{red}}$ is given by the join, i.e. there is a natural equivalence

$$(-)(2)*(-)(2) \simeq ((-) \otimes (-))(2)$$

between the join of the 2-ary operation spaces to the 2-ary operation space of the tensor product.

Remark A.2. We believe that the natural equivalence in (v) extends to an equivalence $\mathfrak{Op}^{\leq 2, \text{red}} \simeq \mathfrak{S}^{\Sigma_2}$ of symmetric monoidal categories where the source is equipped with the tensor product and the target with the join, but this will not be needed for our applications.

Throughout this appendix, we use the notation and terminology from [KK24c, Section 1.4].

A.1. **Monoidality of truncation.** Apart from Items (iv) and (v), which we deal with separately, Theorem A.1 will be proved by an application of the following general lemma:

Lemma A.3. Fix a tower of categories

$$\mathcal{C} = \mathcal{C}^{\leq \infty} \to \cdots \to \mathcal{C}^{\leq 2} \to \mathcal{C}^{\leq 1} \tag{54}$$

such that all functors in this tower admit fully faithful left and right adjoints. If \mathbb{C} admits a symmetric monoidal structure such that the following condition is satisfied for each of the functors $\tau \colon \mathbb{C} \to \mathbb{C}^{\leq k}$

whenever $\tau(\varphi)$ is an equivalence for a morphism φ in \mathbb{C} , then so is $\tau(\varphi \otimes \mathrm{id}_c)$ for all $c \in \mathbb{C}$, (55)

then the tower (54) can be lifted to a tower of symmetric monoidal categories such that

(i) objectwise the monoidal structure on $\mathbb{C}^{\leq k}$ is given by natural equivalences

$$c \otimes d \simeq \tau(\tau_*(c) \otimes \tau_*(d)) \simeq \tau(\tau_!(c) \otimes \tau_!(d))$$

where τ_* and $\tau_!$ are the right and left adjoints of τ , respectively,

- (ii) the right and left adjoints to the functors in the tower $\mathbb{C}^{\leq k} \to \mathbb{C}^{\leq j}$ for $1 \leq k \leq j \leq \infty$ lift to lax respectively oplax symmetric monoidal functors,
- (iii) If the monoidal structure on \mathbb{C} preserves colimits in both variables, then so does the one on $\mathbb{C}^{\leq k}$.

Proof. The first part of this proof is similar to the proof of [KK24c, Proposition 4.2]: One considers the symmetric monoidal structure $\mathbb{C}^{\otimes} \to \mathrm{Fin}_*$ on \mathbb{C} and the sequence $\cdots \subset \mathbb{C}^{\leq k, \otimes} \subset \mathbb{C}^{\leq k+1, \otimes} \subset \cdots \subset \mathbb{C}^{\otimes}$ of full subcategories $\mathbb{C}^{\leq k, \otimes} \subset \mathbb{C}^{\otimes}$ spanned by those objects $(c_s)_{s \in S}$ for finite sets S for which all c_s lie in the essential image of τ_{k*} . By [Lur17, 2.2.1.7, 2.2.1.9], condition (55) ensures that the restriction $\mathbb{C}^{\leq k,\otimes} \to \operatorname{Fin}_*$ is a symmetric monoidal category and the sequence of inclusions $\cdots \subset \mathcal{C}^{\leq k, \otimes} \subset \mathcal{C}^{\leq k+1, \otimes} \subset \cdots \subset \mathcal{C}^{\otimes} \text{ is a sequence of lax symmetric monoidal functors. The argument}$ in the proof of [KK24c, Proposition 4.2] then lifts (55) to a tower of symmetric monoidal categories with oplax symmetric monoidal functors between them. These oplax symmetric monoidal functors are in fact symmetric monoidal, by the argument in the proof of Lemma 3.3 (v) loc.cit., so we have produced the claimed symmetric monoidal lift of (54) and the lax symmetric monoidal lift of the right adjoints as claimed in (ii). Moreover, the argument in the proof of Lemma 3.3 (ii) loc.cit. yields the first equivalence in (i). Applying the same argument to the tower obtained from (54) by taking opposites and then taking opposite symmetric monoidal structures of the result, we obtain the claimed oplax symmetric monoidal lift of the left adjoints in (ii) and the second equivalence in (i) as long as we ensure that for all k the two resulting symmetric monoidal structures on $\mathbb{C}^{\leq k}$ (the one using the right adjoints and the one using the left adjoints) agree. Denoting by $\mathbb{C}^{\leq k, \otimes_L}$ respectively $\mathbb{C}^{\leq k, \otimes_R}$ the symmetric monoidal structures resulting from using the left respectively right adjoint, we have by construction lax monoidal lifts $\mathbb{C}^{\leq k, \otimes_R} \to \mathbb{C}^{\otimes}$ and $\mathbb{C}^{\otimes} \to \mathbb{C}^{\leq k, \otimes_L}$ of τ_* and τ respectively (the second one is even monoidal). Their composition is a lax symmetric monoidal lift of $\tau^*\tau_* \simeq \mathrm{id}_{e^{\leq k}}$, so it suffices to show that it is symmetric monoidal. To see this, recall that the tensor products are on objects given by $c \otimes_R d \simeq \tau(\tau_*(c) \otimes \tau_*(d))$ and $c \otimes_L d \simeq \tau(\tau_!(c) \otimes \tau_!(d))$. In terms of this, the components of the natural transformation which we have to show are equivalences, are

$$\tau \tau_* \big[c \otimes_R d \big] \simeq \tau \tau_* \big[\tau \big(\tau_*(c) \otimes \tau_*(d) \big) \big] \to \tau \big(\tau_*(c) \otimes \tau_*(d) \big) \to \tau \big[\tau_! \big(\tau \tau_*(c) \big) \otimes \tau_! \big(\tau \tau_*(d) \big) \big] \simeq \tau \tau_*(c) \otimes_L \tau \tau_*(d)$$

where the first map is induced by the counit $\tau_*\tau \to id$, which is an equivalence after composing with τ by the triangle identities, and where the second map is induced by the inverse of the counit $\tau_!\tau \to id$, which is an equivalence since $\tau_!$ is fully faithful. This concludes the proof of (i) and (iii). Part (iv) holds by the second formula in (i), since τ and $\tau_!$ are left adjoints, so preserve colimits. \square

A.1.1. The free operad on a category over Fin_{*}. To eventually verify the condition (55) for the truncation functors $\tau \colon \mathbb{O}p^{\mathrm{un}} \to \mathbb{O}p^{\leq k,\mathrm{un}}$, we will first establish some properties for the left adjoint

$$F : \operatorname{Cat}_{/\operatorname{Fin}_{*}} \longrightarrow \operatorname{Op}$$

to the forgetful functor inc: $\mathfrak{Op} \to \mathfrak{C}at_{/\mathrm{Fin}_*}$ which may be of independent interest as F can be interpreted as taking a "free operad" on a category over Fin_* (see e.g. [BHS22, Corollary 4.2.3] for the existence of this adjoint; alternatively see the proof of Lemma A.5 below). Given an operad \mathfrak{O} , we denote the underlying category over Fin_* as $\mathfrak{O}^\otimes \to \mathrm{Fin}_*$ and the fibre over $\langle r \rangle \in \mathrm{Fin}_*$ as $\mathfrak{O}^\otimes_{\langle r \rangle}$ which we often identify with $(\mathfrak{O}^\otimes_{\langle 1 \rangle})^{\times r}$ by taking cocartesian lifts over inerts (c.f. [Lur17, 2.1.1.14]). A special case is the *category of colours* $\mathfrak{O}^{\mathrm{col}} := \mathfrak{O}^\otimes_{\langle 1 \rangle}$

The first property of F we discuss is an explicit description of its values on functors of the form $\alpha \colon [n] \to \operatorname{Fin}_*$ for $[n] \in \Delta$, that is, simplices in Fin_* . This uses the inclusion $\ell \colon \operatorname{Op} \hookrightarrow \operatorname{Op}$ of the 1-category of coloured operads in sets and the full subcategories $\Omega \subset \Phi \subset \operatorname{Op}$ of trees and forests, as discussed in Section 2.2.3. There is an explicit construction of a forest out of simplex $\alpha \colon [n] \to \operatorname{Fin}_*$, which yields a functor

$$\omega \colon \Delta_{/\operatorname{Fin}_*} \longrightarrow \Phi \subset \operatorname{Op}$$

out of the category $\Delta_{/\mathrm{Fin}_*}$ of simplices in Fin $_*$; see [HM24, Section 3.1.1]. It follows from Proposition 3.4.1 loc.cit. that there is a natural map $\alpha \to \ell(\omega(\alpha))$ whose adjoint is an equivalence

$$F(\alpha) \stackrel{\simeq}{\longrightarrow} \ell(\omega(\alpha)). \tag{56}$$

In other words: restricted along $\Delta_{/\operatorname{Fin}_*} \subset \operatorname{Cat}_{/\operatorname{Fin}_*}$, the adjoint $F \colon \operatorname{Op} \to \operatorname{Cat}_{/\operatorname{Fin}_*}$ is given by $\ell \circ \omega$.

Example A.4. Examples of the natural map $\alpha \to \ell(\omega(\alpha))$ for a simplex $\alpha \colon [n] \to \operatorname{Fin}_*$ include:

(i) For $\alpha = e_r \colon [0] \to \operatorname{Fin}_*$ with image $\langle r \rangle \coloneqq \{1, \dots, r, *\}$, the operad $\omega(e_r)$ has r colours and only identity 1-ary operations. It corresponds to the forest $\sqcup_r \eta \in \Phi$ where $\eta \in \Omega$ is the unique tree with no vertices [HM22, p. 92]. The natural map $e_r \to \ell(\omega(e_r))$ sends [0] to the collection of colours in $\ell(\omega(e_r))^{\otimes}_{\langle r \rangle} \simeq (\ell(\omega(e_r))^{\operatorname{col}})^{\times r}$. For an operad $\mathbb O$, we have

$$\operatorname{Map}_{/\operatorname{Fin}_*}(e_r, \mathbb{O}^{\otimes}) \simeq (e_r \times_{\operatorname{Fin}_*} \mathbb{O}^{\otimes})^{\simeq} \simeq \mathbb{O}_{\langle r \rangle}^{\simeq} \simeq (\mathbb{O}^{\operatorname{col}, \simeq})^{\times r}.$$

(ii) For $\alpha = c_r \colon [1] \to \operatorname{Fin}_*$ given by the unique active map of the form $\langle r \rangle \to \langle 1 \rangle$, the operad $\omega(c_r)$ corresponds to the open corolla $C_r \in \Omega$ (see p. 93 loc.cit.) and $c_r \to \ell(\omega(c_r))$ is given by the unique r-ary operation, viewed as morphism in $\ell(\omega(c_r))^{\otimes}$. For an operad \mathbb{O} , we have

$$\operatorname{Map}_{\operatorname{Cat}_{\operatorname{Fin}_*}}(c_r, \mathbb{O}^{\otimes}) \simeq (c_r \times_{\operatorname{Fin}_*} \mathbb{O}^{\otimes})^{\simeq},$$

which is, informally speaking, the space of triples of r input colours, a single output colour, and an r-ary multi-operation. More precisely: the restriction of c_r : $[1] o Fin_*$ to the boundary of the 1-simplex [1] is given by $e_r \sqcup e_1$: $[0] \sqcup [0] o Fin_*$ and restriction along its inclusion gives a map $\operatorname{Map}_{/\operatorname{Fin}_*}(c_r, \mathbb{O}^\otimes) \to \operatorname{Map}_{/\operatorname{Fin}_*}(e_r, \mathbb{O}^\otimes) \times \operatorname{Map}_{/\operatorname{Fin}_*}(e_1, \mathbb{O}^\otimes) \simeq (\mathbb{O}^{\operatorname{col}_*})^r \times \mathbb{O}^{\operatorname{col}_*}$, whose fibre over $((x_s)_{s \in r}, x)$ is, by definition (see 2.1.1.16 loc.cit.), the space of multi-operations $\operatorname{Mul}^{\mathbb{O}}((x_s)_{s \in r}, x)$. In particular, if \mathbb{O} is reduced, then $\operatorname{Map}_{(r)}((x_r)_{s \in r}, x) \simeq \operatorname{Mul}_{\mathbb{O}}((*)_{i \in r}; *) =: \mathbb{O}(r)$

Mul $^{\mathbb{O}}((x_s)_{s\in r}, x)$. In particular, if \mathbb{O} is reduced, then $\operatorname{Map}_{/\operatorname{Fin}_*}(c_r, \mathbb{O}^{\otimes}) \cong \operatorname{Mul}_{\mathbb{O}}((*)_{i\in r}; *) =: \mathbb{O}(r)$. (iii) For $\alpha = \overline{c}_r : [2] \to \operatorname{Fin}_*$ given by the unique composition $\langle 0 \rangle \to \langle r \rangle \to \langle 1 \rangle$ where the second map is active, the operad $\omega(\overline{c}_r)$ corresponds to the closed corolla $\overline{C}_r \in \Omega$ (see p. 93 loc.cit.) and $\overline{c}_r \to \ell(\omega(\overline{c}_r))$ sends (0 < 1) to the unique 0-operation and (1 < 2) to the unique r-ary one. For a general operad \mathbb{O} , the space $\operatorname{Map}_{/\operatorname{Fin}_*}(\overline{c}_r, \mathbb{O}^{\otimes})$ is informally given by adding to the data of Item (ii) a 0-ary operations for each of the r input colours. More precisely, the restriction along the 1-simplex c_r in the 2-simplex \overline{c}_r yields a map $\operatorname{Map}_{/\operatorname{Fin}_*}(\overline{c}_r, \mathbb{O}^{\otimes}) \to \operatorname{Map}_{/\operatorname{Fin}_*}(c_r, \mathbb{O}^{\otimes})$ whose fibre over $m \in \operatorname{Mul}_{\mathbb{O}}((x_s)_{s\in r}; x)$ is given by $\bigcap_{s\in r} \operatorname{Mul}_{\mathbb{O}}(\emptyset, x_s)$. Note that the latter is contractible if \mathbb{O} is unital, so in this case the restriction map is an equivalence.

The second property of the adjoint F we discuss is how it interacts with tensor products of operads. This involves the smash product of finite sets \land : Fin_{*} \times Fin_{*} \rightarrow Fin_{*} (see [Lur17, 2.2.5.1]), the functor \land !: $\operatorname{Cat}_{/\operatorname{Fin}_* \times \operatorname{Fin}_*} \rightarrow \operatorname{Cat}_{/\operatorname{Fin}_*} \rightarrow \operatorname$

Lemma A.5. The functor $F: \operatorname{Cat}_{/\operatorname{Fin}_*} \to \operatorname{Op}$ lifts to a monoidal functor with respect to a monoidal structure on $\operatorname{Cat}_{/\operatorname{Fin}_*}$, whose monoidal unit is $\operatorname{id}_{\operatorname{Fin}_*}$ and whose binary tensor product is the composition

$$\operatorname{Cat}_{/\operatorname{Fin}_*} \times \operatorname{Cat}_{/\operatorname{Fin}_*} \xrightarrow{\times} \operatorname{Cat}_{/\operatorname{Fin}_* \times \operatorname{Fin}_*} \xrightarrow{\wedge_!} \operatorname{Cat}_{\operatorname{Fin}_*}.$$

The monoidal structure on the target $\mathfrak{O}p$ is the one underlying the symmetric monoidal structure on $\mathfrak{O}p$. In particular, we have for $X, Y \in \mathfrak{C}at_{/Fin_*}$ natural equivalences $F(\wedge_!(X \times Y)) \simeq F(X) \otimes F(Y)$.

Proof. We start by some model-categorical preliminaries, based on Lurie's theory of *categorical* patterns from [Lur17, Appendix B]. By B.0.20 loc.cit., associated to a categorical pattern $\mathfrak{P} = (M_{\mathbb{C}}, T, A)$ in the sense of B.0.19 on a simplicial set \mathbb{C} , there is a left proper combinatorial simplicial model category $\mathrm{sSet}^+_{/\mathfrak{P}}$ whose underlying category is the category $\mathrm{sSet}^+_{/(\mathfrak{C}, M_{\mathbb{C}})}$ of marked simplicial sets (X, M) over $(\mathcal{C}, M_{\mathbb{C}})$ (see [Lur09, 3.1]). The model structure is uniquely determined by the properties that (a) cofibrations are those maps that are monomorphisms on underlying simplicial sets (so in particular any object is cofibrant) and (b) fibrant objects are the \mathfrak{P} -fibred objects in the sense of [Lur17, B.0.19]. Given a map $f: \mathbb{C} \to \mathbb{C}'$ compatible with categorical patterns \mathfrak{P} and \mathfrak{P}' on domain and target in the sense of B.2.8 loc.cit., then by B.2.9. we have a Quillen adjunction

$$sSet^{+}_{/\mathfrak{P}} \xrightarrow{f!} sSet^{+}_{/\mathfrak{P}'}$$

whose left adjoint $f_!$ precomposes with f and right adjoint f^* pulls back along $(\mathfrak{C}, M_{\mathfrak{C}}) \to (\mathfrak{C}', M_{\mathfrak{C}'})$. Moreover, by B.2.5 loc.cit., taking external products yields a left Quillen bifunctor

$$sSet_{\mathfrak{P}}^{+} \times sSet_{\mathfrak{P}'}^{+} \xrightarrow{\times} sSet_{\mathfrak{P} \times \mathfrak{P}'}^{+}$$

$$(58)$$

where $\mathfrak{P} \times \mathfrak{P}'$ is the categorical pattern on $\mathfrak{C} \times \mathfrak{C}'$ from B.1.8..

For each quasi-category \mathcal{C} , there is categorical pattern $\mathfrak{P}_{slice} = (\text{equivalences}, \text{all 2-simplices}, \varnothing)$ on \mathcal{C} for which the \mathfrak{P}_{slice} -fibred objects are the categorical fibrations $\mathcal{E} \to \mathcal{C}$ where \mathcal{E} is marked by its equivalences. The underlying ∞ -category of $\mathrm{sSet}^+_{/\mathfrak{P}_{slice}}$ is the overcategory $\mathrm{Cat}_{/\mathcal{C}}$ (see [GH15, Examples 3.2.11]). It has the property that any $f \colon \mathcal{C} \to \mathcal{C}$ is compatible with the \mathfrak{P}_{slice} -patterns, and the resulting Quillen adjunction $f_! \dashv f^*$ gives rise to the analogous adjunction $\mathrm{Cat}_{/\mathcal{C}} \rightleftarrows \mathrm{Cat}_{/\mathcal{C}'}$ on ∞ -categories. Moreover, (58) in the situation of the slice patterns yields the external product functor $\mathrm{Cat}_{/\mathcal{C}} \times \mathrm{Cat}_{/\mathcal{C}'} \to \mathrm{Cat}_{/\mathcal{C} \times \mathcal{C}'}$ on ∞ -categories.

On $\mathcal{C} = \operatorname{Fin}_*$, there is also another categorical pattern \mathfrak{P}_{opd} such that the \mathfrak{P}_{opd} -fibred objects are ∞ -operads $\mathcal{O}^{\otimes} \to \operatorname{Fin}_*$ as in 2.2.1.10 loc.cit. marked by the inert morphisms in the sense of 2.1.2.3 loc.cit., and the underlying ∞ -category is $\mathfrak{O}p$ (see 2.1.4.6 and its proof). The identity functor id: $\operatorname{sSet}^+_{/\mathfrak{P}_{Shice}} \to \operatorname{sSet}^+_{/\mathfrak{P}_{opd}}$ is compatible with the indicated categorical pattern, so it is left Quillen. On fibrant objects the right adjoint id^* sends $(\mathfrak{O}^{\otimes}, \operatorname{inerts}) \to \operatorname{Fin}_*$ to $(\mathfrak{O}^{\otimes}, \operatorname{equivalences}) \to \operatorname{Fin}_*$ (use [Lur09, 2.4.1.5] to see this), so it models the forgetful functor $\mathfrak{O}p \to \operatorname{Cat}_{/\operatorname{Fin}_*}$ and thus $\operatorname{id}_!$ models the left adjoint F. Moreover, the map \wedge : $\operatorname{Fin}_* \times \operatorname{Fin}_* \to \operatorname{Fin}_*$ is compatible with $\mathfrak{P} \times \mathfrak{P}$ and \mathfrak{P} for either choices of $\mathfrak{P} \in \{\mathfrak{P}_{slice}, \mathfrak{P}_{opd}\}$, so (57) gives a Quillen adjunction $\wedge_! \dashv \wedge^*$. By the discussion above, for $\mathfrak{P} = \mathfrak{P}_{slice}$ this models the same-named adjunction $\wedge_! \dashv \wedge^*$ on ∞ -categories. Moreover, the fibrous object $\operatorname{id}_{\operatorname{Fin}_*}$ in $\operatorname{sSet}^+_{/\mathfrak{P}}$, marked with the equivalences in the slice case and the inerts in the operad case, together with the left Quillen functor

$$sSet_{/\mathfrak{P}}^{+}\times sSet_{/\mathfrak{P}}^{+}\overset{\times}{\longrightarrow} sSet_{/\mathfrak{P}\times\mathfrak{P}}^{+}\overset{\wedge_{!}}{\longrightarrow} sSet_{/\mathfrak{P}}^{+}$$

turns sSet $_{/\mathfrak{P}}^{+}$ into a simplicial monoidal model category (see the proofs of 2.2.5.7 and 4.1.7.18 loc.cit. for the proof in the operad case; the slice case is analogous). By construction, the identity id: sSet $_{/\mathfrak{P}_{\text{slice}}}^{+}$ \rightarrow sSet $_{/\mathfrak{P}_{\text{opd}}}^{+}$ is a (strictly monoidal) functor of simplicial monoidal model categories. Applying 4.1.7.18 loc.cit. , we obtain a lift of $F: \mathfrak{Cat}_{/\mathrm{Fin}_*} \rightarrow \mathfrak{Op}$ to a monoidal functor between monoidal ∞ -categories where the monoidal structure on the source has by construction the properties asserted in the claim. By 4.1.7.17 loc.cit. the monoidal structure on the target is the one underlying the symmetric monoidal structure on \mathfrak{Op} , so the claim follows.

The third and final property of *F* we establish is the following:

Lemma A.6. The category $\bigcirc p$ of operads is generated under colimits by the values of F at the 0-simplex $e_1 \colon [0] \to \operatorname{Fin}_*$ and the 1-simplices $c_r \colon [1] \to \operatorname{Fin}_*$ for $r \ge 0$ as described in Example A.4.

Proof. As a result of (56) and Example A.4, the operads in the statement are sent under the equivalence $\delta_{\Omega} \colon \mathcal{O}p \simeq \mathrm{PSh}(\Omega)^{\mathrm{seg},c}$ from Section 2.2.3 to $y(\eta)$ and $y(C_r)$ for $r \geq 0$ where y is the Yoneda embedding for Ω , and η as well as C_r are as in Example A.4. It thus suffices that these presheaves generate $\mathrm{PSh}(\Omega)^{\mathrm{seg},c}$ under colimits. As a localisation of the presheaf category $\mathrm{PSh}(\Omega)$, the category $\mathrm{PSh}(\Omega)^{\mathrm{seg},c}$ is generated by y(T) for $T \in \Phi$ under colimits. Moreover, the dendroidal Segal condition implies that any y(T) is a finite colimit of $y(\eta)$ and $y(C_r)$ for $r \geq 0$, so the claim follows. \square

A.1.2. The right adjoint to $(-) \otimes \mathcal{P}$. As a next step towards proving Theorem A.1, we establish two properties of the right adjoint

$$Alg_{\mathcal{D}}(-): \mathcal{O}p \longrightarrow \mathcal{O}p$$

to the tensor product functor $(-) \otimes \mathcal{P} \colon \mathcal{O}p \to \mathcal{O}p$ for a fixed operad \mathcal{P} . Its value $\underline{Alg}_{\mathcal{P}}(\mathcal{O})$ at $\mathcal{O} \in \mathcal{O}p$ has as category of colours the category $Alg_{\mathcal{P}}(\mathcal{O})$ of \mathcal{P} -algebras in \mathcal{O} from [Lur17, 2.1.3.1] (see [Lur17, 3.2.4.4] for the construction of $\underline{Alg}_{\mathcal{P}}(\mathcal{O})$; the fact that $\underline{Alg}_{\mathcal{P}}(-)$ is right adjoint to $(-) \otimes \mathcal{P}$ can be deduced by the universal property of $(-) \otimes \mathcal{P}$ in 2.2.5 loc.cit., but is also spelled out in a more general setting in [Ste25, Proposition 3.7]).

The first property is a description of $Alg_{\mathcal{P}}(-)$ if \mathcal{P} is in the image of the left adjoint F:

Lemma A.7. For $K, L \in \operatorname{Cat}_{/\operatorname{Fin}_*}$ and $\mathfrak{O} \in \operatorname{Op}$, there are natural equivalence

$$\operatorname{Map}_{\operatorname{Cat}/\operatorname{Fin}_{\bullet}}(K, \underline{\operatorname{Alg}}_{F(L)}(\mathcal{O})^{\otimes}) \simeq \operatorname{Map}_{\operatorname{Cat}/\operatorname{Fin}_{\bullet}}(\wedge_{!}(K \times L), \mathcal{O}^{\otimes}) \simeq \operatorname{Map}_{\operatorname{Cat}/\operatorname{Fin}_{\bullet} \times \operatorname{Fin}_{\bullet}}(K \times L, \wedge^{*}\mathcal{O}^{\otimes}).$$

Proof. Using the adjunctions $(-) \otimes \mathcal{P} \dashv \underline{\mathrm{Alg}}_{\mathcal{P}}(-)$ and $F \dashv \mathrm{inc}$, we compute

$$\operatorname{Map}_{\operatorname{\mathfrak{C}at}_{/\operatorname{Fin}_*}}(K,\underline{\operatorname{Alg}}_{F(L)}({\mathfrak{O}})^{\otimes}) \simeq \operatorname{Map}_{\operatorname{\mathfrak{O}p}}(F(K),\underline{\operatorname{Alg}}_{F(L)}({\mathfrak{O}})) \simeq \operatorname{Map}_{\operatorname{\mathfrak{O}p}}(F(K) \otimes F(L),{\mathfrak{O}})$$

which we can further rewrite, using Lemma A.7 as well as $F\dashv \mathrm{inc},$ as

$$\operatorname{Map}_{\mathcal{O}_{\mathcal{D}}}(F(K) \otimes F(L), \mathcal{O}) \simeq \operatorname{Map}_{\mathcal{O}_{\mathcal{D}}}(F(\wedge_{!}(K \times L)), \mathcal{O}) \simeq \operatorname{Map}_{\operatorname{Cat}_{\mathcal{D}_{\operatorname{in}}}}(\wedge_{!}(K \times L), \mathcal{O}^{\otimes}),$$

proving the first equivalence in the claim. The second one follows from the adjunction $\wedge_! \dashv \wedge^*$. \square

The second property of the right adjoint $\mathrm{Alg}_{\mathfrak{D}}(-)$ we establish is that it preserves k-truncated operads. Recall from [KK24c, Section 1.4.4] that an operad $\mathfrak{O} \in \mathfrak{Op}$ is k-truncated for some $k \geq 1$ if it is unital and if the counit $\mathfrak{O} \to \tau_* \tau(\mathfrak{O})$ of the adjunction involving the truncation functor $\tau \colon \mathfrak{Op}^{\mathrm{un}} \to \mathfrak{Op}^{\leq k,\mathrm{un}}$ is an equivalence. As explained in Section 1.4.4 loc.cit., this is equivalent to \mathfrak{O} being unital and having the property that for any finite collection $(x_s)_{s \in S}$ of colours in \mathfrak{O} and another colour c, the following map induced by inserting the unique 0-ary operation is an equivalence:

$$\operatorname{Mul}_{\mathcal{O}}((x_s)_{s \in S}; c) \longrightarrow \lim_{S' \subseteq S, |S'| \le k} \operatorname{Mul}_{\mathcal{O}}((x_s)_{s \in S'}; c).$$

Proposition A.8. *If an operad* \mathbb{O} *is k-truncated operad, then so is* $\underline{Alg}_{\mathbb{P}}(\mathbb{O})$ *for any operad* \mathbb{P} .

Proof. The functor $\underline{\mathrm{Alg}}_{(-)}(\mathbb{O})\colon \mathbb{O}\mathrm{p}^{\mathrm{op}}\to \mathbb{O}\mathrm{p}$ sends colimits to limits, since using $(-)\otimes \mathbb{P}\dashv \underline{\mathrm{Alg}}_{\mathbb{P}}(-)$ and the fact that the tensor product in $\mathbb{O}\mathrm{p}$ preserves colimits in both variables, we have

$$\operatorname{Map}_{\operatorname{\mathcal{O}p}}(\mathbb{Q}, \underline{\operatorname{Alg}}_{\operatorname{colim}_i \mathcal{P}_i}(\mathbb{O})) \simeq \operatorname{Map}_{\operatorname{\mathcal{O}p}}(\mathbb{Q} \otimes (\operatorname{colim}_i \mathcal{P}_i), \mathbb{O}) \simeq \operatorname{Map}_{\operatorname{\mathcal{O}p}}(\operatorname{colim}_i (\mathbb{Q} \otimes \mathcal{P}_i), \mathbb{O})$$
 which is turn equivalent to

$$\lim_{i} \operatorname{Map}_{\mathcal{O}_{\mathbf{p}}}(\mathbb{Q} \otimes \mathcal{P}_{i}, \mathbb{O}) \simeq \lim_{i} \operatorname{Map}_{\mathcal{O}_{\mathbf{p}}}(\mathbb{Q}, \operatorname{Alg}_{\mathcal{P}_{i}}(\mathbb{O})) \simeq \operatorname{Map}_{\mathcal{O}_{\mathbf{p}}}(\mathbb{Q}, \lim_{i} \underline{\operatorname{Alg}}_{\mathcal{P}_{i}}(\mathbb{O})).$$

Note furthermore that k-truncated operads are closed under taking limits, since all functors featuring in the unit id $\to \tau_* \tau$ are right adjoints and thus preserve limits. Combining these two properties with Lemma A.6, it suffices to prove the statement when $\mathcal{P} = F(\alpha)$ for α being the simplices $e_1 \colon [0] \to \operatorname{Fin}_*$ and $c_r \colon [1] \to \operatorname{Fin}_*$ for $r \geq 0$ from Example A.4. We prove this by a repeated application of Lemma A.7:

In the case $\alpha = e_1$, since $\wedge_!(-\times e_1) \simeq \mathrm{id}_{\operatorname{Cat}_{\operatorname{Fin}*}}$, Lemma A.7 implies $\operatorname{\underline{Alg}}_{F(e_1)}(\mathfrak{O}) \simeq \mathfrak{O}$, so there is nothing to prove. In the case $\alpha = c_r$, we first note that since $\wedge_!(e_1 \times c_r)$, the core of the category of colours of $\operatorname{\underline{Alg}}_{F(e_1)}(\mathfrak{O})$ is given by (cf. Example A.4 (i))

$$\operatorname{Map}_{\operatorname{Cat}_{/\operatorname{Fin}_{*}}}(e_{1}, \underline{\operatorname{Alg}}_{F(c_{r})}(\mathcal{O})^{\otimes}) \simeq \operatorname{Map}_{\operatorname{Cat}_{/\operatorname{Fin}_{*}}}(\wedge_{!}(e_{1} \times c_{r}), \mathcal{O}^{\otimes}) \simeq \operatorname{Map}_{\operatorname{Cat}_{/\operatorname{Fin}_{*}}}(c_{r}, \mathcal{O}^{\otimes}) \simeq (c_{r} \times_{\operatorname{Fin}_{*}} \mathcal{O}^{\otimes})^{\simeq},$$

so the colours of $\underline{\mathrm{Alg}}_{F(c_r)}(\mathcal{O})$ can be described (cf. Example A.4 (ii)) as triples (\underline{x}, x, m) of a collection $\underline{x} = (x^s)_{s \in \underline{r}}$ of colours in \mathcal{O} , a colour x in \mathcal{O} , and a multi-operation $m \in \mathrm{Mul}_{\mathcal{O}}((x^s)_{s \in \underline{r}}, x)$. To determine the n-ary multi-operations in $\underline{\mathrm{Alg}}_{F(c_r)}(\mathcal{O})$ we consider the 1-simplex $c_n \colon [1] \to \mathrm{Fin}_*$ given by the active map $\langle n \rangle \to \langle 1 \rangle$ whose two faces are given by the 0-simplices $e_n, e_1 \colon [0] \to \mathrm{Fin}_*$ given by $\langle n \rangle$, $\langle 1 \rangle \in \mathrm{Fin}_*$. The n-ary multi-operations in $\underline{\mathrm{Alg}}_{F(c_r)}(\mathcal{O})$ from colours $(\underline{x}_i, x_i, m_i)_{i \in \underline{n}}$ to a colour (\underline{x}, x, m) are, by definition (see [Lur17, 2.1.1.6]), the fibre of the upper horizontal map in

$$\begin{split} \operatorname{Map}_{/\operatorname{Fin}_*}(c_n, \operatorname{Alg}_{F(c_r)}(\mathfrak{O})^{\otimes}) & \longrightarrow \operatorname{Map}_{/\operatorname{Fin}_*}(e_n, \underline{\operatorname{Alg}}_{F(c_r)}(\mathfrak{O})^{\otimes}) \times \operatorname{Map}_{/\operatorname{Fin}_*}(e_1, \underline{\operatorname{Alg}}_{F(c_r)}(\mathfrak{O})^{\otimes}) \\ & \bowtie \\ \operatorname{Map}_{/\operatorname{Fin}_*}(\wedge_!(c_n \times c_r), \mathfrak{O}^{\otimes}) & \longrightarrow \operatorname{Map}_{/\operatorname{Fin}_*}(\wedge_!(e_n \times c_r), \mathfrak{O}^{\otimes}) \times \operatorname{Map}_{/\operatorname{Fin}_*}(\wedge_!(e_1 \times c_r), \mathfrak{O}^{\otimes}) \end{split}$$

at the point $((x_i, x_i, m_i)_{i \in n}, (x, x, m))$ in

$$\operatorname{Map}_{/\operatorname{Fin}_{*}}(e_{n}, \underline{\operatorname{Alg}}_{F(c_{r})}(\mathcal{O})^{\otimes}) \times \operatorname{Map}_{/\operatorname{Fin}_{*}}(e_{1}, \underline{\operatorname{Alg}}_{F(c_{r})}(\mathcal{O})^{\otimes} \simeq \left(\sqcap^{n} \operatorname{Alg}_{F(c_{r})}(\mathcal{O})\right) \times \operatorname{Alg}_{F(c_{r})}(\mathcal{O})$$

The fibre of the bottom horizontal map over the image of this point is the space of commutative squares in 0^{\otimes} of the form (the solid arrows are predetermined)

$$(x_i^s)_{(i,s)\in\underline{n}\times\underline{r}} \xrightarrow{(m^s)_{s\in\underline{r}}} (x_i)_{i\in\underline{n}}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

so we have a pullback of the form

$$\begin{split} \operatorname{Mul}_{\operatorname{\underline{Alg}}_{F(c_r)}(\mathcal{O})} \left((\underline{x}_i, x_i, m_i)_{i \in \underline{n}}, (\underline{x}, x, m) \right) &\longrightarrow \operatorname{Mul}_{\mathcal{O}} \left((x_i)_{i \in \underline{n}}, x \right) \\ \downarrow & \qquad \qquad \downarrow^{(-) \circ (m^1, \dots, m^r)} \\ & \qquad \qquad \qquad \qquad \downarrow^{(-) \circ (m^1, \dots, m^r)} \\ & \qquad \operatorname{Mul}_{\mathcal{O}} \left((x_i^s)_{i \in \underline{n}}, x^s \right) & \longrightarrow \operatorname{Mul}_{\mathcal{O}} \left((x_i^s)_{(i,s) \in \underline{n} \times \underline{r}}, x \right). \end{split}$$

Given this description of the multi-operations, we argue as follows: if n=0 then all terms are (products of) 0-ary operations, so contractible if $\mathfrak O$ is unital, which implies that $\mathrm{Alg}_{F(c_r)}(\mathfrak O)$ is also unital. For n>1, we map the previous pullback by inserting 0-ary operations to the pullback

$$\lim_{J\subseteq \underline{n}, |J|\leq k} \mathrm{Mul}_{\mathrm{Alg}_{F(c_r)}(\bigcirc)} \left((\underline{x}_j, x_j, m_j)_{j\in J}, (\underline{x}, x, m) \right) \longrightarrow \lim_{J\subseteq \underline{n}, |J|\leq k} \mathrm{Mul}_{\bigcirc} \left((x_j)_{j\in J}, x \right) \\ \downarrow \qquad \qquad \downarrow \\ \lim_{J\subseteq \underline{n}, |J|\leq k} \prod_{s\in \underline{r}} \mathrm{Mul}_{\bigcirc} \left((x_j^s)_{j\in J}, x^s \right) \longrightarrow \lim_{J\subseteq \underline{n}, |J|\leq k} \mathrm{Mul}_{\bigcirc} \left((x_j^s)_{(j,s)\in J\times\underline{r}}, x \right).$$

Since \circlearrowleft is k-truncated, the maps between all corners of the pullbacks except the top left one is an equivalence (for the bottom right one this uses Lemma A.9 below), so the map between the top left corners is an equivalence as well, which implies that $\underline{\mathrm{Alg}}_{F(c_r)}(\circlearrowleft)$ is k-truncated as claimed. \Box

Lemma A.9. Fix a functor $X : \mathcal{P}(S \times T)^{\mathrm{op}} \to \mathcal{S}$ for finite sets S and T where $\mathcal{P}(S \times T)$ is the poset of subsets of $S \times T$ ordered by inclusion. If for fixed $k \geq 1$ and all subsets $L \subseteq S \times T$ the left map in

$$X(L) \longrightarrow \lim_{K \subseteq L, |K| \le k} X(K)$$
 $X(S \times T) \longrightarrow \lim_{J \subseteq S, |J| \le k} X(J \times T).$

is an equivalence, then the right map is an equivalence as well.

Proof. Consider the commutative diagram

where the horizontal maps are equivalences by hypothesis and C is the poset of pairs of sets (J, K) with are both of cardinality $\leq k$ and satisfy $J \subset S$ and $K \subset J \times T$. It remains to show that the right vertical map is an equivalence. The latter is induced by the functor $\operatorname{pr}_2(J, K) = K$. This has a left adjoint sending $K \subseteq S \times T$ to $(\operatorname{pr}_1(K), K)$ so pr_2 is right cofinal and the claim follows.

We have everything in place to prove Theorem A.1, apart from Item (v).

Proof of Theorem A.1 except Item (v). As explained in Section 2.2, firstly, the functors in (53) admit fully faithful left and right adjoint, secondly, $\mathbb{O}p^{\mathrm{un}}$ inherits a symmetric monoidal structure from Lurie's symmetric monoidal structure on $\mathbb{O}p$, and, thirdly, this monoidal structure preserves colimits in both variables. Apart from Items (iv) and (v), the claim thus follows from Lemma A.3 once we establish the condition (55) for the truncation functors $\tau \colon \mathbb{O}p^{\mathrm{un}} \to \mathbb{O}p^{\leq k,\mathrm{un}}$. By the Yoneda lemma, this condition follows from showing that, given a map $\varphi \colon \mathbb{O} \to \mathbb{O}'$ in $\mathbb{O}p^{\mathrm{un}}$ with $\tau(\varphi)$ an equivalence, the map $\tau(\varphi \otimes \mathrm{id}_{\mathcal{P}})^* \colon \mathrm{Map}_{\mathbb{O}p^{\leq k,\mathrm{un}}}(\tau(\mathbb{O}' \otimes \mathcal{P}), \mathbb{Q}) \to \mathrm{Map}_{\mathbb{O}p^{\leq k,\mathrm{un}}}(\tau(\mathbb{O} \otimes \mathcal{P}), \mathbb{Q})$ is an equivalence. From the adjunctions $\tau \dashv \tau^*$ - and $(-) \otimes \mathcal{P} \dashv \underline{\mathrm{Alg}}_{\mathcal{P}}(-)$, we get natural equivalences

$$\mathrm{Map}_{\mathfrak{O}\mathrm{p}^{\leq k,\mathrm{un}}}(\tau(\mathfrak{O}\otimes \mathfrak{P}),\mathfrak{Q}) \simeq \mathrm{Map}_{\mathfrak{O}\mathrm{p}^{\mathrm{un}}}(\mathfrak{O}\otimes \mathfrak{P},\tau_*(\mathfrak{Q})) \simeq \mathrm{Map}_{\mathfrak{O}\mathrm{p}^{\mathrm{un}}}(\mathfrak{O},\underline{\mathrm{Alg}}_{\mathfrak{P}}(\tau_*(\mathfrak{Q}))).$$

Since $\tau_*(\mathbb{Q})$ is k-truncated as τ_* is fully faithful, the same holds for $\underline{\mathrm{Alg}}_{\mathbb{P}}(\tau_*(\mathbb{Q}))$ by Proposition A.8, so the unit $\mathrm{Alg}_{\mathbb{P}}(\tau_*(\mathbb{R})) \to \tau_*\tau(\mathrm{Alg}_{\mathbb{P}}(\tau_*(\mathbb{Q})))$ is an equivalence and we get natural isomorphisms

$$\mathrm{Map}_{\mathfrak{Op}^\mathrm{un}}(\mathbb{O},\mathrm{Alg}_{\mathfrak{P}}(\tau_*(\mathbb{Q}))) \simeq \mathrm{Map}_{\mathfrak{Op}^\mathrm{un}}(\mathbb{O},\tau_*\tau(\underline{\mathrm{Alg}}_{\mathfrak{P}}(\tau_*(\mathbb{Q})))) \simeq \mathrm{Map}_{\mathfrak{Op}^{\leq k,\mathrm{un}}}(\tau(\mathbb{O}),\tau(\underline{\mathrm{Alg}}_{\mathfrak{P}}(\tau_*(\mathbb{Q})))).$$

In total, we have $\operatorname{Map}_{\mathbb{O}_p^{\leq k, \operatorname{un}}}(\tau(\mathbb{O} \otimes \mathbb{P}), \mathbb{R}) \simeq \operatorname{Map}_{\mathbb{O}_p^{\leq k, \operatorname{un}}}(\tau(\mathbb{O}), \tau(\underline{A}\lg_{\mathbb{P}}(\tau_*(\mathbb{Q}))))$, so it follows that $\tau(\varphi \otimes \operatorname{id}_{\mathbb{P}})^*$ is indeed an equivalence since $\tau(\varphi)^*$ is one by assumption, so condition (55) follows.

To prove Item (iv), we write Cat^{\otimes} and Cat^{\times} for the category $Cat \simeq Op^{\leq 1,un}$ equipped with the symmetric monoidal structure produced in the first part, and with the cartesian structure induced by taking direct products as constructed in [Lur17, 2.4.1], respectively. The goal is to lift id_{Cat} to a symmetric monoidal equivalence $Cat^{\otimes} \simeq Cat^{\times}$. To do so, we consider the composition

$$\operatorname{Cat}^{\times} \xrightarrow{\iota} \operatorname{Op} \xrightarrow{(-)\otimes E_0} \operatorname{Op}^{\operatorname{un}} \xrightarrow{\tau^{\simeq}(-)^{\operatorname{col}}} \operatorname{Cat}^{\otimes_{\operatorname{BV}}}$$
(59)

of symmetric monoidal functors where τ is the symmetric monoidal functor from the first part (whose underlying functor is $(-)^{\mathrm{col}} \colon \mathfrak{Op}^{\mathrm{un}} \to \mathfrak{Cat}$ by construction) and $(-) \otimes E_0$ is the symmetric monoidal functor discussed in Section 2.2. On underlying categories ι is the left adjoint to $(-)^{\mathrm{col}} \colon \mathfrak{Op} \to \mathfrak{Cat}$ (see [Lur17, 2.1.4.11]) which admits a symmetric monoidal lift as indicated, as a result of 2.2.5.11 loc.cit.. We are left to show that this composition agrees with the identity on underlying categories. We can check this after taking right adjoints in which case (59) becomes on underlying categories

$$\operatorname{\mathfrak{C}at} \xrightarrow{\ \ (-)^\sqcup} \operatorname{\mathfrak{O}p}^{\operatorname{un}} \xrightarrow{\ \ \subset} \operatorname{\mathfrak{O}p} \xrightarrow{\ \ (-)^{\operatorname{col}} \ } \operatorname{\mathfrak{C}at}$$

where $(-)^{\sqcup}$ is the functor that takes cocartesian operads (see 2.4.3.3 loc.cit. for the construction and 2.4.3.9 loc.cit. for the fact that it is right adjoint to $(-)^{\operatorname{col}}$). But $((-)^{\sqcup})^{\operatorname{col}} \simeq \operatorname{id}_{\operatorname{Cat}}$ is clear from the construction of the cocartesian operad (see 2.4.3 loc.cit.), so the claim follows.

A.2. **Tensor products of 2-truncated reduced operads.** We now turn towards proving Theorem A.1 (v). In the process of this, for brevity we will not distinguish between ordinary coloured operads in sets and their induced operads in Op, i.e. we implicitly apply the fully faithful inclusion $\ell : Op \hookrightarrow Op$ from Section 2.2.3. For an operad Op, we write

$$\mathcal{O}(\overline{C}_n) := \mathrm{Map}_{\mathcal{O}_{\mathrm{p}}}(\overline{C}_n, \mathcal{O}) \simeq \mathrm{Map}_{/\mathrm{Fin}_*}(\overline{c}_n, \mathcal{O}^{\otimes}) \in \mathcal{S}^{\Sigma_n}$$

where the Σ_n -action is by precomposition, using that $\operatorname{Aut}_{\operatorname{Cat}/\operatorname{Fin}_*}(\overline{c}_n) \simeq \operatorname{Aut}_{\operatorname{Fin}_*}(\langle n \rangle) \simeq \Sigma_n$ (see Example A.4 (iii) for the definition of \overline{c}_n and \overline{C}_n). Note that when restricted to the unital operads the functor $(-)(\overline{C}_n) : \operatorname{Op}^{\operatorname{un}} \to \mathbb{S}^{\Sigma_n}$ canonically factors over the truncation functors $\tau : \operatorname{Op}^{\operatorname{un}} \to \operatorname{Op}^{\leq k, \operatorname{un}}$ as long as $n \leq k$, since \overline{C}_n is in the image of the left adjoint $\tau_!$ for $n \leq k$ (see Section 2.2.3).

A main step in the proof of Theorem A.1 (v) is the following lemma (see [FV15, Corollary 4.8] for a similar construction in the underived setting). The maps along which the pullbacks and pushouts in the statement are taken will be explained during the proof.

Lemma A.10. There is a natural transformation of functors $\mathfrak{Op}^{\leq 2,\mathrm{un}} \times \mathfrak{Op}^{\leq 2,\mathrm{un}} \to S^{\Sigma_2}$ of the form

$$\left[\mathfrak{O}(\overline{C}_2) \times \mathfrak{P}(\overline{C}_1) \times_{\mathfrak{P}(\overline{C}_0)} \mathfrak{P}(\overline{C}_1) \right] \sqcup_{\mathfrak{O}(\overline{C}_2) \times \mathfrak{P}(\overline{C}_2)} \left[\mathfrak{O}(\overline{C}_1) \times_{\mathfrak{O}(\overline{C}_0)} \mathfrak{O}(\overline{C}_1) \times \mathfrak{P}(\overline{C}_2) \right] \to (\mathfrak{O} \otimes \mathfrak{P})(\overline{C}_2)$$

where the Σ_2 -action on the target is through \overline{C}_2 and the one on the source is induced by the action on \overline{C}_2 , swapping the two $\mathfrak{O}(\overline{C}_2)$ -factors, and swapping the two $\mathfrak{P}(\overline{C}_2)$ -factors.

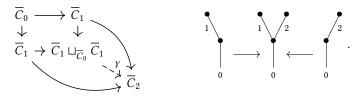
Proof. We first construct a natural transformation as indicated of functors $\mathfrak{Op} \times \mathfrak{Op} \to \mathfrak{S}^{\Sigma_2}$. Precomposing the result with $(\subset \circ \tau_1)^{\times 2} \colon \mathfrak{Op}^{\leq 2, \mathrm{un}} \to \mathfrak{Op}$ then yields the claimed natural transformation, once we show that both its source and target send the components of the counit $\tau_1 \tau \to \mathrm{id}$ to equivalences. For the source, this follows from the above mentioned fact that $(-)(\overline{C_n})$ factors over $\tau \colon \mathfrak{Op}^{\mathrm{un}} \to \mathfrak{Op}^{\leq 2, \mathrm{un}}$ for $n \leq 2$. Using this once more, for the target it follows by proving that the map $\tau((\tau_1\tau(\mathfrak{O})) \otimes (\tau_1\tau(\mathfrak{P}))) \to \tau(\mathfrak{O} \otimes \mathfrak{P})$ induced by the counit is an equivalence. This in turn is a consequence the symmetric monoidality of τ from Theorem A.1 and the description of the tensor product in $\mathfrak{Op}^{\leq 2, \mathrm{un}}$ in terms of τ_1 from Theorem A.1 (ii).

We start by forming the left-hand pushout in Op

of the map $\overline{C}_0 \to \overline{C}_1$ encoding the output colour of the unique non-identity 1-ary operation in \overline{C}_1 along itself (illustrated in the right-hand figure). The maps u and v encode 1-ary operations in the pushout with common output colour that we denote by 0. We next specify various operations in the tensor product $(\overline{C}_1 \sqcup_{\overline{C}_0} \overline{C}_1) \otimes \overline{C}_2$. We denote the unique 2-operation in \overline{C}_2 by m, its output colour by 0, and its two input colours by 1 and 2. Using that $E_0 \simeq \overline{C}_0$ is the unit of the tensor product in $\operatorname{Op^{un}}$, the map $(0 \otimes \operatorname{id}) : \overline{C}_2 \simeq \overline{C}_0 \otimes \overline{C}_2 \to (\overline{C}_1 \sqcup_{\overline{C}_0} \overline{C}_1) \otimes \overline{C}_2$ takes m to a 2-operation in the operad $(\overline{C}_1 \sqcup_{\overline{C}_0} \overline{C}_1) \otimes \overline{C}_2$, which we denote $0 \otimes m$, with input colours $0 \otimes 1$ and $0 \otimes 2$, and output colour $0 \otimes 0$. Using the unique 0-operation of the two input colours 1, 2 of m, we similarly obtain 1-ary operations $u \otimes 1$ and $v \otimes 2$ in $(\overline{C}_1 \sqcup_{\overline{C}_0} \overline{C}_1) \otimes \overline{C}_2$, and thus a 2-ary operation $\alpha := (u \otimes 1, v \otimes 2) \circ (0 \otimes m)$ in $(\overline{C}_1 \sqcup_{\overline{C}_0} \overline{C}_1) \otimes \overline{C}_2$ by composition. Analogously, we obtain a 2-ary operation $\beta := (1 \otimes u, 2 \otimes v) \circ (m \otimes 0)$ in $\overline{C}_2 \otimes (\overline{C}_1 \sqcup_{\overline{C}_0} \overline{C}_1)$. These are represented by maps

$$\overline{C}_2 \xrightarrow{\alpha} (\overline{C}_1 \sqcup_{\overline{C}_0} \overline{C}_1) \otimes \overline{C}_2 \quad \text{and} \quad \overline{C}_2 \xrightarrow{\beta} \overline{C}_2 \otimes (\overline{C}_1 \sqcup_{\overline{C}_0} \overline{C}_1).$$

Inserting the unique 0-ary operation into one of the input colours 1 or 2 of the unique 2-ary operation in \overline{C}_2 gives two 1-ary operations in \overline{C}_2 with the same output colour 0 (indicated by the right-hand figure), so we obtain a dashed map in the left-hand diagram



We next claim that the following square in Op is commutative:

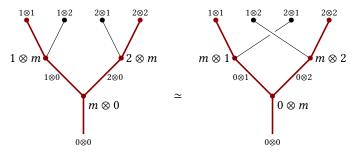
$$\overline{C}_{2} \xrightarrow{\beta} \overline{C}_{2} \otimes (\overline{C}_{1} \sqcup_{\overline{C}_{0}} \overline{C}_{1})$$

$$\downarrow^{id \otimes \gamma} \qquad \downarrow^{id \otimes \gamma}$$

$$(\overline{C}_{1} \sqcup_{\overline{C}_{0}} \overline{C}_{1}) \otimes \overline{C}_{2} \xrightarrow{\gamma \otimes id} \overline{C}_{2} \otimes \overline{C}_{2}.$$

$$(61)$$

For this we use that on operads encoded by trees, the tensor product of operads is given by the classical Boardman–Vogt tensor product of operads in sets, i.e. the functor ℓ : Op \rightarrow Op preserves tensor products of operads in $\Phi \subset$ Op (see the proof of [HM24, 5.4.1]). In particular, the two compositions encode two 2-ary operations in the classical Boardman–Vogt tensor product $\overline{C}_2 \otimes_{\text{BV}} \overline{C}_2$ and we have to show that they agree. This is a consequence of the *interchange relation* (see [HM22, p. 27 (5)]), as indicated in the following figure with the counterclockwise composition the 2-ary operation indicated in red on the left, and the clockwise composition that on the right:



Given operads O and P, we can then form the commutative diagram

The outermost part of the diagram induces a map out of the pushout of the two maps involving γ^* into $\operatorname{Map}_{\mathbb{Op}}(\overline{C}_2, \mathbb{O} \otimes \mathbb{P}) = (\mathbb{O} \otimes \mathbb{P})(\overline{C}_2)$ yielding the desired natural transformation.

If \mathbb{O} and \mathbb{P} are reduced operads then their values on \overline{C}_0 and \overline{C}_1 are contractible, so the left hand side in Lemma A.10 becomes the join $\mathbb{O}(2) * \mathbb{P}(2)$. We will show that the natural transformation is an equivalence in this case, which will in particular imply Theorem A.1 (v).

Lemma A.11. The natural transformation of functors $\mathfrak{O}p^{\leq 2, red} \times \mathfrak{O}p^{\leq 2, red} \to \mathfrak{S}^{C_2}$

$$\mathcal{O}(2) * \mathcal{P}(2) \longrightarrow (\mathcal{O} \otimes \mathcal{P})(2)$$

given by the restriction of Lemma A.10 to reduced operads, is a natural equivalence.

Before giving the proof, we record a few preparatory observations. Recall from [Lur17, 2.2.5.3] that, given operads \mathcal{O} , \mathcal{P} , and \mathcal{R} , the category of *bifunctors* $\mathcal{O} \times \mathcal{P} \to \mathcal{R}$ is the full subcategory

$$\mathrm{Bf}(\mathcal{O},\mathcal{P};\mathcal{R}) \coloneqq \mathrm{Fun}^{\mathrm{inert}}_{/\mathrm{Fin}_* \times \mathrm{Fin}_*} (\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}, \wedge^* \mathcal{R}^{\otimes}) \subset \mathrm{Fun}_{/\mathrm{Fin}_* \times \mathrm{Fin}_*} (\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}, \wedge^* \mathcal{R}^{\otimes})$$

on those functors over Fin* \times Fin* that preserve cocartesian lifts over inerts, where a morphism in Fin* \times Fin* is *inert* if both of its components are inert in Fin*. A bifunctor $\theta \colon \mathcal{O} \times \mathcal{P} \to \mathcal{R}$ exhibits \mathcal{R} as a tensor product of \mathcal{O} and \mathcal{P} if for any operad \mathcal{Q} the functor $\mathrm{Alg}_{\mathcal{R}}(\mathcal{Q}) = \mathrm{Fun}^{\mathrm{inert}}_{/\mathrm{Fin}}(\mathcal{R}^{\otimes}, \mathcal{Q}^{\otimes}) \to \mathrm{Bf}(\mathcal{O}, \mathcal{P}; \mathcal{Q})$

induced by θ is an equivalence, which on cores gives $\operatorname{Map}_{\mathbb{O}p}(\mathcal{R}, \mathbb{Q}) \simeq \operatorname{Bf}(\mathbb{O}, \mathcal{P}; \mathbb{Q})^{\simeq}$. Given a general bifunctor $\theta \colon \mathbb{O} \times \mathcal{P} \to \mathcal{R}$, we can form a commutative diagram similar to (62)

whose diagonal arrows are induced by the product functor $\boxtimes : \operatorname{Cat}_{/\operatorname{Fin}_*} \times \operatorname{Cat}_{/\operatorname{Fin}_*} \to \operatorname{Cat}_{/\operatorname{Fin}_*} \times \operatorname{Fin}_*$ and postcomposition with θ , using that $\mathcal{O}(\overline{C_n}) = \operatorname{Map}_{\mathcal{Op}}(\overline{C_n}, \mathcal{O}) = \operatorname{Alg}_{\overline{C_n}}(\mathcal{O})^{\simeq}$. If θ exhibits \mathcal{R} as the tensor product of \mathcal{O} and \mathcal{P} , then (63) recovers (62).

Next, we give a description of the space Bf $(\overline{C}_k, \overline{C}_\ell; \mathcal{R})^{\simeq}$ for $k, \ell \geq 0$, for simplicity in the case where \mathcal{R} is unital. Adopting the notation from Example A.4, the composition $c_n \to \overline{c}_n \to F(\overline{c}_n) = \overline{C}_n^{\otimes}$ for $n \geq 0$ induces by restriction maps

$$\mathrm{Bf}(\overline{C}_k,\overline{C}_\ell;\mathcal{R})^{\simeq} \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Cat}_{/\mathrm{Fin}_*} \times \mathrm{Fin}_*} (\overline{c}_k \times \overline{c}_\ell, \wedge^*(\mathcal{R}^{\otimes})) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Cat}_{/\mathrm{Fin}_*} \times \mathrm{Fin}_*} (c_k \times c_\ell, \wedge^*(\mathcal{R}^{\otimes})) \simeq \mathrm{Map}_{\mathrm{Cat}_{/\mathrm{Fin}_*} \times \mathrm{Fin}_*} (\wedge_! (c_k \times c_\ell), \mathcal{R}^{\otimes})$$

which are both equivalences: For the first, this follows from the universal property of the tensor product, the adjunction involving F, and the monoidality of F from Lemma A.5. For the second, it follows from unitality of \mathbb{R} by an argument as in Example A.4 (iii). The rightmost space $\operatorname{Map}_{\operatorname{Cat}_{/\operatorname{Fin}_*}}(\wedge_!(c_k \times c_\ell), \mathbb{R}^\otimes) = (\wedge_!(c_k \times c_\ell) \times_{\operatorname{Fin}_*} \mathbb{R}^\otimes)^\simeq$ is the space of commutative squares in \mathbb{R}^\otimes that cover the following square in Fin_* induced by the active maps $\langle k \rangle \to \langle 1 \rangle$ and $\langle \ell \rangle \to \langle 1 \rangle$

$$\begin{array}{c} \langle k \rangle \wedge \langle \ell \rangle \longrightarrow \langle k \rangle \wedge \langle 1 \rangle \\ \downarrow \qquad \qquad \downarrow \\ \langle 1 \rangle \wedge \langle \ell \rangle \longrightarrow \langle 1 \rangle \wedge \langle 1 \rangle \end{array}.$$

By an argument similar to the one in the proof of Proposition A.8, this space fits into a fibre sequence

$$P_{\mathcal{R}}\big((x_{ij})_{(i,j)\in\underline{k}\times\underline{\ell}},(x_{i0})_{i\in\underline{k}},(x_{0j})_{i\in\underline{\ell}},x_{00}\big) \to \operatorname{Map}_{\operatorname{Cat}_{/\operatorname{Fins}}}\big(\wedge_!(c_k\times c_\ell),\mathcal{R}^\otimes\big) \to \big(\mathcal{R}^{\operatorname{col},\simeq}\big)^{k}\times\big(\mathcal{R}^{\operatorname{col},\simeq}\big)^k\times\big(\mathcal{R}^{\operatorname{col},\simeq}\big)^{\ell}\times\mathcal{R}^{\operatorname{col},\simeq}$$

where the leftmost space is the fibre over $((x_{ij})_{(i,j)\in\underline{k}\times\underline{\ell}},(x_{i0})_{i\in\underline{k}},(x_{0j})_{i\in\underline{\ell}},x)\in(\mathbb{R}^{\mathrm{col},\simeq})^{k\cdot\ell}\times(\mathbb{R}^{\mathrm{col},\simeq})^k\times(\mathbb{R}^{\mathrm{col},\simeq})^k\times(\mathbb{R}^{\mathrm{col},\simeq})^\ell\times\mathbb{R}^{\mathrm{col},\simeq}$. This fibre in turn fits into a pullback of the form

$$P_{\mathcal{R}}((x_{ij})_{(i,j)\in\underline{k}\times\underline{\ell}},(x_{i0})_{i\in\underline{k}},(x_{0j})_{j\in\underline{\ell}};x_{00}) \to \operatorname{Mul}_{\mathcal{R}}((x_{i0})_{i\in\underline{k}};x) \times \sqcap_{i\in\underline{k}}\operatorname{Mul}_{\mathcal{R}}((x_{ij})_{j\in\underline{\ell}};x_{0i})$$

$$\downarrow^{\circ}$$

$$\operatorname{Mul}_{\mathcal{R}}((x_{0j})_{j\in\underline{\ell}};x_{00}) \times \sqcap_{j\in\underline{\ell}}\operatorname{Mul}_{\mathcal{R}}((x_{ij})_{i\in\underline{k}};x_{0j}) \xrightarrow{\circ} \operatorname{Mul}_{\mathcal{R}}((x_{ij})_{(i,j)\in\underline{k}\times\underline{\ell}};x_{00})$$

where the maps denoted by \circ are given by operadic composition. Loosely speaking, the space $\mathrm{Bf}(\overline{C_k},\overline{C_\ell};\mathcal{R}) \simeq \mathrm{Map}_{\mathrm{Cat}_{/\mathrm{Fin}*}}(\wedge_!(c_k\times c_\ell),\mathcal{R}^\otimes)$ is thus the space of the following data: two $(k\cdot\ell)$ -ary operations m and m' with the same input colours $(x_{ij})_{i,j}$ and output colour x_{00} , a path between m and m' in the space of such operations, a decomposition of m as k operations of arity ℓ inserted into a k-ary operation with input colours $(x_{i0})_i$, and a decomposition of m' as ℓ operations or arity k inserted into a ℓ -ary operation with input colours $(x_{0j})_j$; see the top-left box in Figure 3 for a schematic figure of an element in $\mathrm{Bf}(\overline{C_2},\overline{C_2};\mathcal{R})$. We can also describe the maps in the inner square in (63) in these terms: the map $(\gamma,\mathrm{id})^*: \mathrm{Bf}(\overline{C_2},\overline{C_2};\mathcal{R})^\cong \to \mathrm{Bf}(\overline{C_1},\overline{C_2};\mathcal{R})^\cong \times_{\mathrm{Bf}(\overline{C_0},\overline{C_2};\mathcal{R})^\cong} \mathrm{Bf}(\overline{C_1},\overline{C_2};\mathcal{R})^\cong$ is on the first factor given by inserting the unique 0-ary operation in x_{21} and x_{22} , and on the second factor by inserting the unique 0-ary operation in x_{11} and x_{21} on the second factor. The

map α and β are given by forming the unique operadic composition possible to obtain a operation with input colours x_{11} and x_{22} , and output colour x_{00} ; see Figure 3 for a schematic figure.

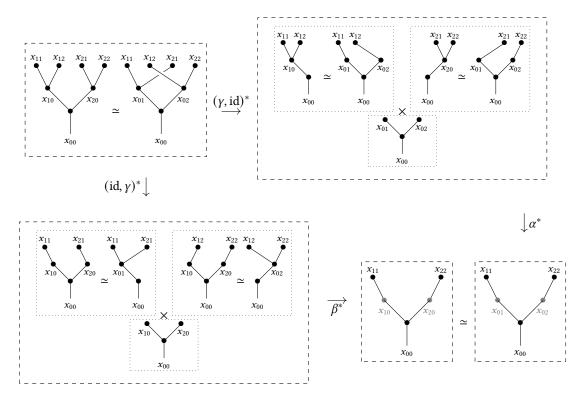


FIGURE 3. The inner square of (63). Each dashed box indicates an element in the respective space. The \simeq -signs in the dashed boxes indicate paths of operations. The \simeq -sign between the two bottom-right dashed boxes indicates the homotopy between the clockwise and counterclockwise composition of maps in the square.

Proof of Lemma A.11. We first prove that the natural transformation is an equivalence when $\mathcal{O}=E_n$ and $\mathcal{P}=E_m$ (we will actually only use the case n=m=1 later). We use use the model of the E_d -operad in terms of rectilinear embeddings of open unit cubes (see [Lur17, 5.1.0.2]) and the explicit bifunctor $\theta\colon E_n\times E_m\to E_{n+m}$ from 5.1.2.1 loc.cit. which expresses E_{n+m} as the tensor product of E_n and E_m , and is given by taking cartesian products of n- and m-cubes. Writing $E_d(k):=E_d(\overline{C}_k)$, id_d := id_(-1,1)^d, and using $E_d(0)=*$, the task in this case is to show that the left square in

$$E_{n}(2) \times E_{m}(2) \xrightarrow{\gamma^{*} \times \mathrm{id}} E_{n}(1) \times E_{n}(1) \times E_{m}(2) \qquad E_{n}(2) \times E_{m}(2) \xrightarrow{\mathrm{pr}_{2}} E_{m}(2)$$

$$\downarrow^{\alpha^{*} \circ \theta \circ \boxtimes} \qquad \downarrow^{\mathrm{pr}_{1}} \qquad \downarrow^{\mathrm{id}_{n} \times (-)} \qquad (64)$$

$$E_{n}(2) \times E_{m}(1) \times E_{m}(1) \xrightarrow{\beta^{*} \circ \theta \circ \boxtimes} E_{n+m}(2) \qquad \qquad E_{n}(2) \xrightarrow{(-) \times \mathrm{id}_{m}} E_{n+m}(2)$$

is a pushout. Explicitly, using that $E_d(k)$ is the space of rectilinear embeddings $\underline{k} \times (-1, 1)^d \hookrightarrow (-1, 1)^d$ and writing $\iota_i \colon (-1, 1)^d \subset \underline{k} \times (-1, 1)^d$ for the inclusion of the ith cube, the top horizontal arrow is given by $(e, e') \mapsto (e \circ \iota_1, e \circ \iota_2, e')$ and the left vertical arrow by $(e, e') \mapsto (e, e' \circ \iota_1, e' \circ \iota_2)$. From the description of α^* and β^* given above, together with fact that θ is given by direct products of rectilinear cubes, we see that the right vertical map is given by $(e_1, e_2, e') \mapsto (id_n \times e') \circ (e_1 \times id_m, e_2 \times id_m)$ and the left vertical map by $(e, e'_1, e'_2) \mapsto (e \times id_m) \circ (id_n \times e'_1, id_n \times e'_2)$. Using the deformation retraction of $E_d(1)$ onto $\{id_d\} \subset E_d(1)$ induced by scaling and translation, the strictly commutative

left square in (64) becomes the homotopy commutative right square where the homotopy sends $(e, e') \in E_n(2) \times E_m(2)$ at time $t \in [0, 2]$ to

$$\begin{cases} (\mathrm{id}_n \times e') \circ \Big((1-t)\mathrm{id}_{n+m} + t((e \circ \iota_1) \times \mathrm{id}_m), (1-t)\mathrm{id}_{n+m} + t((e \circ \iota_2) \times \mathrm{id}_m) \Big) & t \in [0,1] \\ (e \times \mathrm{id}_m) \circ \Big((t-1)\mathrm{id}_{n+m} + (2-t)(\mathrm{id}_n \times (e' \circ \iota_1)), (t-1)\mathrm{id}_{n+m} + (2-t)(\mathrm{id}_m \times (e' \circ \iota_2)) \Big) & t \in [1,2] \end{cases}$$

Identifying $E_d(2)$ with S^{d-1} via the homotopy equivalence $(e, e') \mapsto \frac{e'(0) - e(0)}{\|e'(0) - e(0)\|}$, one checks that the induced map $E_n(2) \cup_{E_n(2) \times E_m(2)} E_m(2) = E_n(2) * E_m(2) \rightarrow E_{n+m}(2)$ becomes the standard map $S^{n-1} * S^{m-1} \to S^{n+m-1}$, so it is in particular an equivalence.

We now show the general case. In view of the equivalence $\mathfrak{Op}^{\leq 2, \text{red}} \simeq S^{\Sigma_2}$ induced by taking 2-ary operations (see Lemma 2.4), both source and target are functors of the form $F: \mathbb{S}^{\Sigma_2} \times \mathbb{S}^{\Sigma_2} \to \mathbb{S}^{\Sigma_2}$ that satisfy (a) $F(-,\emptyset) \simeq \operatorname{id} \simeq F(\emptyset,-)$, (b) the functors $F(X,-),F(-,X): \mathcal{S}^{\Sigma_2} \simeq \mathcal{S}^{\Sigma_2}_{\varnothing/} \to \mathcal{S}^{\Sigma_2}_{X/}$ preserve colimits for all $X \in \mathcal{S}^{\Sigma_2}$. These properties can easily be verified for join *, and for the tensor product \otimes on $\mathfrak{Op}^{\leq 2, \text{red}}$ follow from (a) that E_0 is the unit, (b) Lemma A.12 below. Moreover, by the above argument for n = 1 = m the component of the natural transformation at $(E_1(2), E_1(2)) \simeq (S^0, S^0) \in \mathbb{S}^{\Sigma_2} \times \mathbb{S}^{\Sigma_2}$ is an equivalence where S^0 is equipped with the flip action, i.e. the free Σ_2 -space on a point. It turns out that any natural transformation $\eta\colon F\to F'$ with this property is an equivalence: it is an equivalence on (X, S^0) for any Σ_2 -space X, since $X \simeq \operatorname{colim}_{X_{\Sigma_2}} S^0$ where X_{Σ_2} are the orbits of the action, so the component of η at (X, S^0) has, using (b), the form $\operatorname{colim}_{X_{\Sigma_2}} F(S^0, S^0) \cup_{\operatorname{colim}_{X_{\Sigma_2}} S^0} S^0 \to \operatorname{colim}_{X_{\Sigma_2}} F'(S^0, S^1) \cup_{X_{\Sigma_2} S^0} S^0$, so is an equivalence as $F(S^0, S^0) \to F'(S^0, S^0)$ is one. For general (X, Y) the component has, using (b), the form $\operatorname{colim}_{Y_{\Sigma_2}} F(X, S^0) \cup_{\operatorname{colim}_{Y_{\Sigma_2}} X} X \to \operatorname{colim}_{Y_{\Sigma_2}} F'(X, S^0) \cup_{\operatorname{colim}_{Y_{\Sigma_2}} X} X$, so is an equivalence as we have already shows that $F(X, S^0) \to F'(X, S^0)$ is one.

Lemma A.12. $Fix \mathcal{P} \in \mathfrak{Op}^{red}$.

- (i) The operad E₀ is initial in Op^{red}, so the forgetful functor Op^{red}_{E₀/} → Op^{red} is an equivalence.
 (ii) The inclusion Op^{red}_{P/} ⊂ Op^{un}_{P/} is a fully faithful left adjoint, so preserves and detects colimits. In particular, colimits in the category of reduced operads Op^{red} ≃ Op^{red}_{E₀/} can be computed in Op^{un}_{E₀/}.
 (iii) The tensor product functor P⊗ (-): Op^{red} ≃ Op^{red}_{E₀/} → Op^{red}_{P//} preserves colimits.
 Moreover, (i)-(iii) holds when replacing Op^{red} by Op^{≤k,red} and Op^{un} by Op^{≤k,un} for any 1 ≤ k ≤ ∞.

Proof. Item (i) and Item (ii) for $\mathcal{P} = E_0$ hold by [SY19, 2.2.7–2.2.9]. Item (ii) for general \mathcal{P} follows from the case $\mathcal{P} = E_0$ in view of the equivalence of pairs

$$\left((\mathfrak{O}\mathrm{p}^{\mathrm{red}}_{E_0})_{E_0\to\mathcal{P}/}\subset(\mathfrak{O}\mathrm{p}^{\mathrm{un}}_{E_0/})_{E_0\to\mathcal{P}/}\right)\xrightarrow{\simeq}\left(\mathfrak{O}\mathrm{p}^{\mathrm{red}}_{\mathcal{P}/}\subset\mathfrak{O}\mathrm{p}^{\mathrm{un}}_{\mathcal{P}/}\right)$$

and the observation that if a functor $\varphi \colon \mathcal{C} \to \mathcal{D}$ is a fully faithful left adjoint, then so is the induced functor $\mathcal{C}_{/c} \to \mathcal{D}_{/\varphi(c)}$ on overcategories for any $c \in \mathcal{C}$ (this is a straightforward check). Using Item (ii), we compute

$$(\operatorname{colim}_{i}^{\mathcal{O}p^{\operatorname{red}}}\mathcal{O}_{i})\otimes\mathcal{P}\simeq(\operatorname{colim}_{i}^{\mathcal{O}p^{\operatorname{un}}_{/E_{0}}}\mathcal{O}_{i})\otimes\mathcal{P}\simeq(\operatorname{colim}_{i}^{\mathcal{O}p^{\operatorname{un}}}\mathcal{O}_{i}\sqcup_{\operatorname{colim}_{i}^{\mathcal{O}p^{\operatorname{un}}}E_{0}}E_{0})\otimes\mathcal{P}$$

which we can—using that tensor product of unital operads preserves colimits in each entry (see Theorem A.1 (iv))—further simplify to

$$(\operatorname{colim}_{i}^{\mathcal{O}p^{\operatorname{un}}}\mathcal{O}_{i}\sqcup_{\operatorname{colim}_{i}^{\mathcal{O}p^{\operatorname{un}}}E_{0}}E_{0})\otimes\mathcal{P}\simeq\operatorname{colim}_{i}^{\mathcal{O}p^{\operatorname{un}}}(\mathcal{O}_{i}\otimes\mathcal{P})\sqcup_{\operatorname{colim}_{i}^{\mathcal{O}p^{\operatorname{un}}}\mathcal{P}}\mathcal{P}\simeq\operatorname{colim}_{i}^{\mathcal{O}p^{\operatorname{un}}}(\mathcal{O}_{i}\otimes\mathcal{P}),$$

so Item (iii) follows. Moreover, the same argument applies in the k-truncated situation for any $1 \le k \le \infty$, once we establish Item (i) and Item (ii) for $\mathcal{P} = E_0$. Item (i) follows from the observation that $E_0 = \overline{C_0}$ is in the image of the fully faithful left adjoint $\tau_! : \mathcal{O}p^{\leq k, \mathrm{un}} \to \mathcal{O}p^{\mathrm{un}}$, as observed as part of Section 2.2.3. To see that $\mathcal{O}p_{E_0/}^{\leq k, \mathrm{red}} \subset \mathcal{O}p_{E_0/}^{\leq k, \mathrm{un}}$ is a left adjoint, we can argue as in the proof of Proposition 2.2.9 loc.cit. ((-)^{col} is denoted L in loc.cit.), using that the functor $\tau: : \mathfrak{Op}^{\leq k, \mathrm{un}} \to (\mathfrak{Op}^{\leq \hat{1}, \mathrm{un}} \simeq \mathfrak{C}at)$ is both a right adjoint (since it agrees with $(-)^{\mathrm{col}} \circ \tau_*$) and a left adjoint with fully faithful right adjoint (since it agrees with $(-)^{\text{col}} \circ \tau_!$).

Appendix B. Tensor products of rational pro-operads

The rationalisation of operads $(-)_Q \colon \mathcal{O}p^{red} \to \mathcal{O}p^{red}$ from Section 2.2.4 could potentially fail to preserve tensor products of operads. This issue occurs in Section 3.1 and to circumvent it, we will prove in this appendix that, once restricted to a certain full subcategory, the functor $(-)_Q$ factors as

$$\mathfrak{O}p^{\text{red}} \xrightarrow{(-)_{Q}^{\wedge}} \mathfrak{O}p_{Q}^{\text{red}} \xrightarrow{|-|} \mathfrak{O}p^{\text{red}} \tag{65}$$

through a certain category $\mathcal{O}p_Q^{red}$ of *reduced rational pro-operads*, such that the first functor in (65) preserves tensor products in suitable sense and the second functor is fully faithful on a large full subcategory. This will allow us in particular to construct the promised filler in (10). The content of this appendix is inspired by work of Boavida de Brito-Horel [BdBH21] (see also Remark 3.2).

B.1. **Pro-rationalisation of spaces.** To construct the factorisation (65), we will make use of a different description of the rationalisation of spaces $X \to X_Q$ from Section 2.1, using pro-homotopy theory. We write Pro(S) for the category of *pro-spaces* in the sense of [Lur18, A.8.1.1]. Its objects are given by cofiltered diagrams of spaces $\{X_{\alpha}\}_{{\alpha}\in A}$ and its mapping spaces are given by

$$\operatorname{Map}_{\operatorname{Pro}(\mathcal{S})}(\{X_{\alpha}\}_{\alpha \in A}, \{Y_{\beta}\}_{\alpha \in B}) \simeq \lim_{\beta} \operatorname{colim}_{\alpha} \operatorname{Map}_{\mathcal{S}}(X_{\alpha}, Y_{\beta}); \tag{66}$$

see A.8.1.5 loc.cit.. There is an adjunction $(-)^{\wedge}$: $\mathcal{S} \rightleftarrows \operatorname{Pro}(\mathcal{S})$: |-| where the left adjoint $(-)^{\wedge}$ takes the constant pro-space on a space and is fully faithful, and the right adjoint |-| takes the limit of a pro-space. We will also consider a rational version of $\operatorname{Pro}(\mathcal{S})$: writing $\mathcal{R} \subset \mathcal{S}$ for the full subcategory generated by $K(\mathbf{Q},n)$ for $n \geq 0$ under taking finite limits and retracts (see [BdBH21, Proposition 2.6] for an explicit description), we consider the full subcategory $\operatorname{Pro}(\mathcal{S})_{\mathbf{Q}} \subset \operatorname{Pro}(\mathcal{S})$ on those pro-spaces that are equivalent to $\{X_{\alpha}\}_{{\alpha}\in\mathcal{A}}$ with $X_{\alpha}\in\mathcal{R}$. As a result of [BdBH21, Theorem 2.3] (or [Hoy23, Proposition 3.7]), the inclusion $\operatorname{Pro}(\mathcal{S})_{\mathbf{Q}} \subset \operatorname{Pro}(\mathcal{S})$ has a left-adjoint which exhibits $\operatorname{Pro}(\mathcal{S})_{\mathbf{Q}}$ as the localisation of $\operatorname{Pro}(\mathcal{S})$ at the maps that induce isomorphisms on rational cohomology groups $\operatorname{H}^k(-;\mathbf{Q}) = \pi_0 \operatorname{Map}_{\operatorname{Pro}(\mathcal{S})}(-,K(\mathbf{Q},k))$ for $k \geq 0$. Composing the localisation adjunction $\operatorname{Pro}(\mathcal{S})_{\mathbf{Q}}$ with $(-)^{\wedge}$: $\mathcal{S} \rightleftarrows \operatorname{Pro}(\mathcal{S})$: |-|, we obtain an adjunction

$$(-)^{\wedge}_{\mathbf{Q}} \colon \mathbb{S} \Longrightarrow \operatorname{Pro}(\mathbb{S})_{\mathbf{Q}} \colon |-| \ .$$
 (67)

We record some properties of (67) in the next lemma. We write $\tau_{\leq k} \colon \mathcal{S} \to \mathcal{S}$ for the Postnikov k-truncation localisation functor (see [Lur09, 5.5.6.18]), which induces a localisation $\tau_{\leq k} \colon \operatorname{Pro}(\mathcal{S}) \to \operatorname{Pro}(\mathcal{S})$ that objectwise truncates pro-spaces (see [Lur18, A.8.1.8] for the existence of the functor and use [Lur09, 5.2.7.4] to see that it is a localisation). We call a space X of $finite\ \mathbf{Q}$ -type if $\dim(\mathbf{H}_k(X;\mathbf{Q})) < \infty$ for $k \geq 0$.

Lemma B.1.

- (i) The category $Pro(S)_Q$ has small limits and colimits.
- (ii) The left adjoint $(-)_{\mathbf{0}}^{\wedge}$ preserves finite products.
- (iii) On spaces X of finite Q-type, we have a natural equivalence $X_Q^{\wedge} \simeq \{\tau_{\leq n}(Q_n(X))\}_{n \in \mathbb{N}}$ in Pro(S) where $Q_{\bullet}(X)$ is the tower resulting from the cosimplicial resolution from Section 2.1. In particular, the unit $X \to |X_Q^{\wedge}|$ of (67) is equivalent to the rationalisation $X \to X_Q$ from Section 2.1.
- (iv) For connected spaces X with $\dim(H_1(X; \mathbb{Q})) < \infty$, we have a natural equivalence in $\operatorname{Pro}(S)$

$$\tau_{\leq 1}(X_{\mathbf{O}}^{\wedge}) \simeq \{K((\pi/\Gamma_n(\pi)) \otimes \mathbf{Q}, 1)\}_{n \in \mathbf{N}} \quad \text{ for } \pi \coloneqq \pi_1(X)$$

where $\Gamma_{\bullet}(\pi)$ is the lower central series, and $(-)\otimes Q$ denotes rationalisation of nilpotent groups.

Proof. Item (i) follows from the fact that $Pro(S)_Q$ is the underlying ∞-category of a model category (see [BdBH21, Theorem 2.3]). Item (ii) is [BdBH21, Proposition 2.9]. To show Item (iii), we need to say more about this model category: one starts with a model structure on the category Pro(SSE) of pro-objects in the 1-category of simplicial sets, called the strict model structure, whose underlying ∞-category is Pro(S) (see [Isa05, Section 2.2] and [BHH17, Section 7.1, Theorem 5.2.1]). There is a Bousfield left localisation of this model category at the maps of pro-simplicial sets that induce isomorphisms of rational *homology*, resulting in the *homological model structure* (see [Isa05, Theorem

6.7]). The latter can in turn be further Bousfield left localised at the maps of pro-simplicial sets that induce isomorphisms of rational *cohomology* to arrive at the *cohomological model structure* (see Theorem 6.3 loc.cit.). The underlying ∞ -category of the latter is $\operatorname{Pro}(\mathbb{S})_{\mathbb{Q}}$ by [BdBH21, Theorem 2.3]. Writing $\operatorname{Pro}(\mathbb{S})_{h,\mathbb{Q}}$ of the underlying ∞ -category of the homological model structure, this yields a factorisation of $(-)_{\mathbb{Q}}^{\wedge}$ into three left adjoints $\mathbb{S} \to \operatorname{Pro}(\mathbb{S}) \to \operatorname{Pro}(\mathbb{S})_{h,\mathbb{Q}} \to \operatorname{Pro}(\mathbb{S})_{\mathbb{Q}}$. It follows from [Isa05, Proposition 7.3] that the functor $\mathbb{S} \to \operatorname{Pro}(\mathbb{S})_{h,\mathbb{Q}} \subset \operatorname{Pro}(\mathbb{S})$ sends X to $\{\tau_{\leq n}(\mathbb{Q}_n(X))\}_{n\in\mathbb{N}}$. Since the functors $\mathbb{S} \to \operatorname{Pro}(\mathbb{S})_{h,\mathbb{Q}} \subset \operatorname{Pro}(\mathbb{S})$ and $\mathbb{S} \to \operatorname{Pro}(\mathbb{S})_{\mathbb{Q}} \subset \operatorname{Pro}(\mathbb{S})$ agree on spaces of finite \mathbb{Q} -type by the argument in [Hor24, Proposition 7.2, Remark 7.3], Item (iii) follows.

To prove Item (iv), we first observe that the setup described at the beginning of this appendix equally makes sense using the category S_* of pointed spaces, in particular we have adjunctions $\mathcal{S}_* \rightleftarrows \operatorname{Pro}(\mathcal{S}_*) \rightleftarrows \operatorname{Pro}(\mathcal{S}_*)_{\mathbf{O}}$ where $\operatorname{Pro}(\mathcal{S}_*)_{\mathbf{O}}$ is the localisation of $\operatorname{Pro}(\mathcal{S}_*)$ at the maps that are isomorphisms on reduced rational cohomology $\widetilde{H}^k(-; \mathbb{Q}) = \pi_0 \operatorname{Map}_{\operatorname{Pro}(\mathcal{S}_*)}(-, K(\mathbb{Q}, k))$. The composition of the two left adjoints $(-)_{0,*}^{\wedge} : \mathcal{S}_* \to \operatorname{Pro}(\mathcal{S}_*)_Q$ with the forgetful functor $\iota_Q : \operatorname{Pro}(\mathcal{S}_*)_Q \to \operatorname{Pro}(\mathcal{S})_Q$ is related to the composition of the forgetful map $\iota \colon \mathcal{S}_* \to \mathcal{S}$ with $(-)_{\mathbb{Q}}^{\wedge} \colon \mathcal{S} \to \operatorname{Pro}(\mathcal{S})_{\mathbb{Q}}$ by a Beck-Chevalley transformation $\iota(X)_{\mathbf{O}}^{\wedge} \to \iota_{\mathbf{Q}}(X_{\mathbf{O},*}^{\wedge})$. This turns out to be an equivalence: it suffices to prove that $\iota(X) \to \iota_{\mathbb{Q}}(X_{\mathbb{Q}^*}^{\wedge})$ is an isomorphism on rational cohomology, which, since $X \to X_{\mathbb{Q}^*}^{\wedge}$ is an isomorphism on reduced cohomology, follows from the fact that filtered colimits in abelian groups are exact. Together with the fact that truncation of (pointed) spaces commutes with the forgetful map $S_* \to S$, this in particular yields a pointed lift $\tau_{\leq 1}(X_{\mathbf{O},*}^{\wedge}) \in \operatorname{Pro}(S_*)$ of $\tau_{\leq 1}(X_{\mathbf{O}}^{\wedge}) \in \operatorname{Pro}(S)$. It thus suffices to show that $\tau_{\leq 1}(X_{\mathbf{Q},*}^{\wedge}) \simeq \{K(\pi/\Gamma_n(\pi) \otimes \mathbf{Q}, 1)\}_{n \in \mathbb{N}}^*$ in $\text{Pro}(\mathcal{S}_*)$. Both sides lie in the full subcategory $\text{Pro}(S_*)^{\leq 1}_{\mathbf{0}} \subset \widetilde{\text{Pro}}(S_*)$ of pointed pro-spaces that are equivalent to filtered diagrams $\{Y_{\beta}\}_{\beta\in B}$ of 1-truncated spaces that lie in \mathcal{R} (for the left-hand side this holds by definition and for the right-hand side it follows from the second part of the proof of [BdBH21, Proposition A.2] using the assumption on X). We will now prove the claimed equivalence by showing that the two sides have the same image under the Yoneda embedding of the opposite of $\operatorname{Pro}(S_*)_{\mathbb{Q}}^{\leq 1}$. For $\{Y_{\beta}\}_{\beta \in B} \in \operatorname{Pro}(S_*)_{\mathbb{Q}}^{\leq 1}$, it follows from adjunction and the localising property of truncation that $\operatorname{Map}_{\operatorname{Pro}(S_*)}(\tau_{\leq 1}(X_{\mathbb{Q}_*}^{\wedge}), \{Y_{\beta}\}_{\beta \in B}) \simeq \operatorname{Map}_{\operatorname{Pro}(S_*)}(\tau_{\leq 1}(X), \{Y_{\beta}\}_{\beta \in B})$ which using (66) yields $\operatorname{Map}_{\operatorname{Pro}(S_*)}(\tau_{\leq 1}(X_{\mathbf{0},*}^{\wedge}), \{Y_{\beta}\}_{\beta \in B}) \simeq \lim_{\beta \in B} \operatorname{Hom}(\pi_1(X), \pi_1(Y_{\beta})). \text{ On the other hand, using (66), we}$ $\operatorname{have} \operatorname{Map}_{\operatorname{Pro}(\mathbb{S}_*)}(\{\widetilde{K}(\pi/\Gamma_n(\pi)\otimes \mathbb{Q},1)\}_{n\in\mathbb{N}},\{Y_\beta\}_{\beta\in B}) \simeq \lim_{\beta\in B} \operatorname{colim}_n\operatorname{Hom}(K(\pi/\Gamma_n(\pi)\otimes \mathbb{Q},\pi_1(Y_\beta)),$ which combining with the first part of the proof of Proposition A.2 and Proposition 2.6 loc.cit. yields $\operatorname{Map}_{\operatorname{Pro}(S_n)}(\{K(\pi/\Gamma_n(\pi) \otimes \mathbf{Q}, 1)\}_{n \in \mathbb{N}}, \{Y_\beta\}_{\beta \in B}) \simeq \lim_{\beta \in B} \operatorname{Hom}(\pi, \pi_1(Y_\beta)).$ Since these equivalences are natural in the target, the claim follows from the Yoneda lemma.

Next, we discuss a full subcategory on which |-|: $\operatorname{Pro}(\mathcal{S})_{\mathbb{Q}} \to \mathcal{S}$ is fully faithful. We say that a space X is \mathbb{Q} -good if the unit $X \to |X_{\mathbb{Q}}^{\wedge}|$ induces an isomorphism on rational cohomology (if X is of finite \mathbb{Q} -type, this is by Lemma B.1 (iii) equivalent to \mathbb{Q} -goodness in the sense [BK72, I.5.1 (iii)], but note that it differs from the notion of \mathbb{Q} -goodness in [BdBH21]). We call a pro-space $P \in \operatorname{Pro}(\mathcal{S})$ k-truncated if the objectwise truncation map $P \to \tau_{\leq k}(P)$ in $\operatorname{Pro}(\mathcal{S})$ is an equivalence.

Lemma B.2. For a space X and a pro-space $P \in \text{Pro}(S)_{\mathbb{O}}$ such that either

- (i) X is Q-good, or
- (ii) $X_{\mathbb{Q}}^{\wedge}$ and P are 1-truncated as pro-spaces and we have $\dim(H_1(X;\mathbb{Q})) < \infty$,

the map $|-|: \operatorname{Map}_{\operatorname{Pro}(S)_{\mathbf{O}}}(X_{\mathbf{O}}^{\wedge}, P) \to \operatorname{Map}_{S}(|X_{\mathbf{O}}^{\wedge}|, |P|)$ is an equivalence.

Example B.3. Here are some situations where the assumptions in Lemma B.2 are met:

- (i) If a space *X* is of nilpotent of finite Q-type, then it is Q-good. This follows by combining Lemma B.1 (iii) with the discussion in Section 2.1.2).
- (ii) For a connected space X with $\dim(\mathrm{H}_1(X;\mathbf{Q})) < \infty$, it follows from Lemma B.1 (iv) that the prospace $X_{\mathbf{Q}}^{\wedge}$ is 1-truncated if and only if the map $\mathrm{colim}_n\mathrm{H}^k(\pi_1(X)/\Gamma_n(\pi_1(X))\otimes \mathbf{Q};\mathbf{Q})\to \mathrm{H}^k(X;\mathbf{Q})$ induced by the 1-truncation $X\to K(\pi_1(X),1)$ is an isomorphism for all $k\geq 0$. The case that is most relevant to us is $X=K(PB_m,1)$ for the pure braid group PB_m , where this isomorphism was verified in the proof of [BdBH21, Proposition A.4].

(iii) The conditions on *X* or *P* appearing in Lemma B.2 are closed under taking finite products (use Lemma B.1 (ii), the Künneth theorem, and that truncation preserves products).

Proof. The first case is straight-forward: by adjunction, the map in the claim is equivalent to the map $\operatorname{Map}_{\operatorname{Pro}(S)_{\mathbb{Q}}}(X_{\mathbb{Q}}^{\wedge},P) \to \operatorname{Map}_{\operatorname{Pro}(S)_{\mathbb{Q}}}(|X_{\mathbb{Q}}^{\wedge}|_{\mathbb{Q}}^{\wedge},P)$ induced by precomposition with the counit $|X_{\mathbb{Q}}^{\wedge}|_{\mathbb{Q}}^{\wedge} \to X_{\mathbb{Q}}^{\wedge}$. Using a triangle identity, the latter is an equivalence if the map $X_{\mathbb{Q}}^{\wedge} \to |X_{\mathbb{Q}}^{\wedge}|_{\mathbb{Q}}^{\wedge}$ induced by the unit is an equivalence, i.e. if the unit $X \to |X_{\mathbb{Q}}^{\wedge}|$ is a rational cohomology isomorphism. This is exactly the \mathbb{Q} -goodness of X, so the first case follows.

For the second case, as X_Q^{\wedge} is 1-truncated, it suffices to show that the map $\operatorname{Map}_{\operatorname{Pro}(\mathbb{S})}(\tau_{\leq 1}(X_Q^{\wedge}), P) \to \operatorname{Map}_{\mathbb{S}}(|\tau_{\leq 1}(X_Q^{\wedge})|, |P|)$ is an equivalence. By adjunction and the fact that P is 1-truncated, it suffices to show that the map $\tau_{\leq 1}(|\tau_{\leq 1}(X_Q^{\wedge})|_Q^{\wedge}) \to \tau_{\leq 1}(X_Q^{\wedge})$ induced by the counit is an equivalence. Using a triangle identity, this is an equivalence if the map $\tau_{\leq 1}(X_Q^{\wedge}) \to \tau_{\leq 1}(|\tau_{\leq 1}(X_Q^{\wedge})|_Q^{\wedge})$ induced by the unit is an equivalence. By Lemma B.1 (iv), we have $|\tau_{\leq 1}(X_Q^{\wedge})| \cong K(\operatorname{Mal}_Q(\pi_1(X)), 1)$ where we write $\operatorname{Mal}_Q(\pi) \coloneqq \lim_k ((\pi/\Gamma^k(\pi)) \otimes Q)$ for groups π . As a result of [Bou77, Theorem 13.3 (iv)], we have $\operatorname{H}_1(\operatorname{Mal}_Q(\pi_1(X); Q) \cong \operatorname{H}_1(X; Q)$, so these vector spaces are finite-dimensional by assumption. Using Lemma B.1 (iv) once more and writing $\pi \coloneqq \pi_1(X)$, the map $\tau_{\leq 1}(X_Q^{\wedge}) \to \tau_{\leq 1}(|\tau_{\leq 1}(X_Q^{\wedge})|_Q^{\wedge})$ is equivalent to the map $\{K(\pi/\Gamma^n(\pi), 1)\}_{n \in \mathbb{N}} \to \{K(\operatorname{Mal}_Q(\pi)/\Gamma^n(\operatorname{Mal}_Q(\pi)), 1)\}_{n \in \mathbb{N}}$ induced levelwise by the canonical morphism $\pi \to \operatorname{Mal}_Q(\pi)$. This map of pro-spaces is a levelwise equivalence by [Bou77, Theorem 13.3 (iv)], so the claim follows.

B.2. **Pro-rationalisation of operads.** With respect to the equivalence $\delta_{\Phi} \colon \mathcal{O}p \simeq \mathrm{PSh}(\Phi)^{\mathrm{seg},c}$ from Section 2.2.3, the subcategory of reduced operads $\mathcal{O}p^{\mathrm{red}} \subset \mathcal{O}p \subset \mathrm{PSh}(\Phi)$ (see Section 2.2) corresponds to those presheaves whose values at $\overline{C}_0, \overline{C}_1 \in \overline{\Omega}$ and at the unique tree without vertices $\eta \in \Omega$ are contractible, and which satisfy the dendroidal Segal and forest Segal condition, both of which only involve finite products (the completeness condition is automatic). This is a localisation of $\mathrm{PSh}(\Phi) = \mathrm{Fun}(\Phi^{\mathrm{op}}, \mathbb{S})$, so the inclusion $\iota \colon \mathcal{O}p^{\mathrm{red}} \subset \mathrm{PSh}(\Phi)$ admits left adjoint $L \colon \mathrm{PSh}(\Phi) \to \mathcal{O}p^{\mathrm{red}}$. Similarly, we can define the category of *reduced rational pro-operads* as the full subcategory

$$\mathfrak{O}_{\mathbf{O}}^{\mathrm{red}} \subset \mathrm{Fun}(\Phi^{\mathrm{op}}, \mathrm{Pro}(\mathbb{S})_{\mathbf{Q}}) \tag{68}$$

of those functors which have terminal value on \overline{C}_0 , \overline{C}_1 , and η , and which satisfy the analogue of the dendroidal Segal and forest Segal condition (phrased in terms of finite products). Since Φ has finite morphism sets, it follows from [BdBH21, Proposition 4.4] that the inclusion (68) is a localisation as well, that is, it admits a left adjoint $L_Q\colon \operatorname{Fun}(\Phi^{\operatorname{op}},\operatorname{Pro}(\mathcal{S})_Q)\to \mathcal{O}\mathrm{p}_Q^{\operatorname{red}}$, (c.f. the proof of Proposition 6.3 loc.cit.). Since $(-)_Q^{\wedge}$ and |-| preserve finite products (see Lemma B.1 (ii)), the adjunction $(-)_Q^{\wedge}\colon \operatorname{Fun}(\Phi^{\operatorname{op}},\mathcal{S}) \leftrightarrows \operatorname{Fun}(\Phi^{\operatorname{op}},\operatorname{Pro}(\mathcal{S})_Q)\colon |-|_Q$ restricts to an adjunction $(-)_Q^{\wedge}\colon \operatorname{Op}_Q \leftrightarrows \operatorname{Op}\colon |-|_Q$. This in particular explains the left-hand commutative square in

and the right-hand commutative square is obtained by taking left adjoints. By Lemma B.1 (iii), the composition $|(-)_Q^{\wedge}|$: $Op \rightarrow Op$ agrees with the rationalisation $(-)_Q: Op \rightarrow Op$ from Section 2.2.4 on the full subcategory of reduced operads whose multi-operation spaces have finite Q-type.

Lemma B.4. Let $\mathbb{O}, \mathbb{P} \in \mathbb{O}p^{red}$ and assume that their spaces of multi-operations satisfy one of the following conditions for all $n \geq 0$:

- (i) O(n) is **Q**-good, or
- (ii) $\mathfrak{O}(n)_{\mathbf{O}}^{\wedge}$ and $\mathfrak{P}(n)_{\mathbf{O}}^{\wedge}$ are 1-truncated as pro-spaces and $\dim(H_1(\mathfrak{O}(n);Q))<\infty$.

Then the map $|-|\colon \mathrm{Map}_{\mathbb{O}p_Q^{red}}(\mathbb{O}_Q^{\wedge}, \mathbb{P}_Q^{\wedge}) \to \mathrm{Map}_{\mathbb{O}p^{red}}(|\mathbb{O}_Q^{\wedge}|, |\mathbb{P}_Q^{\wedge}|)$ is an equivalence.

Proof. Using the natural description of mapping spaces in functor categories as ends (see e.g. [GHN17, Proposition 5.1]), the claim follows by showing that for all forests $T, U \in \Phi$ the map

 $\operatorname{Map}_{\operatorname{Pro}(\mathbb{S})_{\mathbb{Q}}}(\mathbb{O}(T)_{\mathbb{Q}}^{\wedge}, \mathcal{P}(U)_{\mathbb{Q}}^{\wedge}) \to \operatorname{Map}_{\mathbb{S}}(|\mathbb{O}(T)_{\mathbb{Q}}^{\wedge}|, |\mathcal{P}(U)_{\mathbb{Q}}^{\wedge}|)$ is an equivalence. The dendroidal Segal and the forest Segal conditon implies that $\mathbb{O}(T)$ is a finite product of spaces of the form $\mathbb{O}(n)$ for $n \geq 0$, and similarly for \mathcal{P} , so the claim follows from Example B.3 (iii) and Lemma B.2.

Example B.5. For $\mathfrak{O} = \mathfrak{P} = E_d$ the little *d*-discs operad (see Section 2.2.2), we have

$$\operatorname{Map}_{\operatorname{\mathcal{O}p}_{\mathbf{O}}}((E_d)^{\wedge}_{\mathbf{O}}, (E_d)^{\wedge}_{\mathbf{O}}) \simeq \operatorname{Map}_{\operatorname{\mathcal{O}p}}(|(E_d)^{\wedge}_{\mathbf{O}}|, |(E_d)^{\wedge}_{\mathbf{O}}|) \simeq \operatorname{Map}_{\operatorname{\mathcal{O}p}}(E_{d, \mathbf{Q}}, E_{d, \mathbf{Q}}) \quad \text{for all } d \geq 0.$$

Indeed, since $E_d(k)$ is equivalent to the ordered configuration space $F_k(\mathbf{R}^d)$ of k points in \mathbf{R}^d (see Section 2.2.2), it is always of finite Q-type, so the second equivalence follows from Lemma B.1 (i). For $d \neq 2$, the space $F_k(\mathbf{R}^d)$ is nilpotent since it is either homotopy discrete (for $d \leq 1$) or simply connected (for $d \geq 2$), so it is Q-good by Example B.3 (i) and thus condition (i) in Lemma B.4 is satisfied and yields the first equivalence. If d = 2, then $E_d(k) \simeq K(PB_k, 1)$, so condition (ii) in Lemma B.4 is satisfied by Example B.3 (ii) and yields the first equivalence.

B.3. **Tensor products of reduced rational pro-operads.** There is an analogue \otimes_Q of the tensor product functor $\otimes \colon \mathbb{O}p \times \mathbb{O}p \to \mathbb{O}p$ from Section 2.2.1 for the category $\mathbb{O}p_Q^{red}$, which is compatible with the tensor product on $\mathbb{O}p$ in that it fits into a commutative square

To construct $\otimes_{\mathbb{Q}}$, we will make use of a dendroidal description of the tensor product on \mathfrak{Op} : it follows from [HM24, Theorem 1.1.1, Section 5] that under the equivalence $\delta_{\Phi} \colon PSh(\Phi)^{seg,c} \simeq \mathfrak{Op}$ from Section 2.2.3, the restriction of the tensor product $\otimes \colon \mathfrak{Op} \times \mathfrak{Op} \to \mathfrak{Op}$ to reduced operads is given by the composition

$$\operatorname{\mathcal{O}p}^{\operatorname{red}} \times \operatorname{\mathcal{O}p}^{\operatorname{red}} \xrightarrow{i \times i} \operatorname{PSh}(\Phi) \times \operatorname{PSh}(\Phi) \xrightarrow{\otimes \operatorname{pre}} \operatorname{PSh}(\Phi) \xrightarrow{L} \operatorname{\mathcal{O}p},$$

where $\otimes^{\text{pre}} : \text{PSh}(\Phi) \times \text{PSh}(\Phi) \to \text{PSh}(\Phi)$ is the composition

$$PSh(\Phi)^{\times 2} \overset{(\subseteq_!)^{\times 2}}{\longrightarrow} PSh(Op)^{\times 2} \overset{\times}{\longrightarrow} PSh(Op^{\times 2}) \overset{\otimes_!}{\longrightarrow} PSh(Op) \overset{\subseteq^*}{\longrightarrow} PSh(\Phi)$$

where the first functor is induced by left Kan extension along the full subcategory inclusion $\Phi \subseteq Op$, the second functor is induced by taking products in S, the third functor by left Kan extension along the tensor product in Op given by the classical Boardman–Vogt tensor product of operads in sets, and the final functor by restriction along $\Phi \subseteq Op$. The only properties of the target category in $PSh(\Phi) = Fun(\Phi^{op}, S)$ this uses is that it has small colimits and finite products, so the same construction makes sense for $Fun(\Phi^{op}, Pro(S)_Q)$ using Lemma B.1 (i) and yields a functor $\bigotimes_Q^{pre} : Fun(\Phi^{op}, Pro(S)_Q)^{\times 2} \to Fun(\Phi^{op}, Pro(S)_Q)$. Moreover, since $(-)_Q^{\wedge}$ preserves finite products by Lemma B.1 (iii) and colimits since it is a left adjoint, the middle square in the diagram

commutes. The left-hand and right-hand squares commute by the discussion in Appendix B.2, so the bottom composition yields the promised dashed functor that makes (69) commute.

B.4. 2-ary operations of tensor products of rational pro-operads. Finally, we establish a partial analogue of Lemma A.11 in the context of reduced rational pro-operads:

Lemma B.6. There is a natural transformation of functors $\mathfrak{O}p_{\mathbf{O}}^{\mathrm{red}} \times \mathfrak{O}p_{\mathbf{O}}^{\mathrm{red}} \to \mathbb{S}^{C_2}$

$$|\mathfrak{X}|(2) * |\mathfrak{Y}|(2) \longrightarrow |\mathfrak{X} \otimes_{\mathbf{O}} \mathfrak{Y}|(2) \tag{70}$$

with the following properties:

- (i) Precomposed with $(-)^{\wedge}_{O} \colon \mathbb{O}p^{red} \to \mathbb{O}p^{red}_{O}$ in both arguments, it factors as
 - $|\mathcal{O}_Q^{\wedge}|(2)*|\mathcal{P}_Q^{\wedge}|(2)\simeq|\mathcal{O}(2)_Q^{\wedge}|*|\mathcal{P}(2)_Q^{\wedge}|\rightarrow|(\mathcal{O}(2)*\mathcal{P}(2))_Q^{\wedge}|\rightarrow|(\mathcal{O}\otimes\mathcal{P})(2)_Q^{\wedge}|\simeq|\mathcal{O}_Q^{\wedge}\otimes_Q\mathcal{P}_Q^{\wedge}|(2)\otimes\mathcal{P}(2)_Q^{\wedge}|$
 - where the first map is induced by the canonical maps from $O(2) \rightarrow O(2) * P(2)$ and $P(2) \rightarrow$
- $\mathcal{O}(2)*\mathcal{P}(2)$ and the second map is the value of the map from Lemma A.11 under $|(-)_Q^{\wedge}|$. (ii) If $\mathcal{X}=\mathcal{O}_Q^{\wedge}$ and $\mathcal{Y}=\mathcal{P}_Q^{\wedge}$ for $\mathcal{O},\mathcal{P}\in\mathcal{O}_Q^{\mathrm{red}}$ such that $\mathcal{O}(2)$ and $\mathcal{P}(2)$ are nilpotent of finite Q-type and at least one of them is connected, then (70) is an equivalence.

Proof. The construction is a minor modification of the one from the proof of Lemma A.10: first one applies $(-)_0^{\wedge}$ to the square (61) and then uses the result to form the analogue of (62) where the role of \emptyset , $\mathcal{P} \in \mathfrak{O}_{\mathbf{p}}^{\sim}$ is now played by $\mathcal{X}, \mathcal{Y} \in \mathfrak{O}_{\mathbf{p}_{\mathbf{0}}}$, the mapping spaces are formed in $\mathfrak{O}_{\mathbf{p}_{\mathbf{0}}}$, and the diagonal maps are induced by $\otimes_{\mathbb{Q}}$. One then considers the map from the pushout of the outer two maps involving $(\gamma_Q^{\wedge})^*$ to the bottom rightmost corner. Using $\operatorname{Map}_{\mathcal{O}p_Q}((-)_Q^{\wedge}, (-)) \simeq \operatorname{Map}_{\mathcal{O}p}((-), |(-)|)$ as well as $\operatorname{Map}_{\mathcal{O}p}(\overline{C}_n, |(-)|) \simeq |(-)|(n)$ and the fact that the latter space is contractible for $n \leq 1$ since the operads are reduced, this gives a natural transformation as required. Comparing this construction to the one in Equation (62) and using naturality as well as commutativity of (69) yields Property (i). By Lemma A.11, the second map in the factorisation from Property (i) is an equivalence, so to show (ii) it suffices that the first map is an equivalence under the stated conditions, which follows by showing that the canonical map $|A_{\mathbb{Q}}^{\wedge}| * |B_{\mathbb{Q}}^{\wedge}| \to |(A * B)_{\mathbb{Q}}^{\wedge}|$ is an equivalence for nilpotent spaces A, B of finite Q-type such that at least one of them is connected. By Lemma B.1 (iii), we may show this for |-0| replaced by the rationalisation (-0) from Section 2.1. Both sides are 1-connected since $(-)_{O}$ preserves connectivity (see Section 2.1.4) and the join with a connected spaces is 1-connected, so it suffices to show that the map induces an isomorphism on homology, which follows by using that $H_*((-)_O; Z) \cong H_*(-; Q)$ on nilpotent spaces (see Section 2.1.2) together with the Mayer-Vietoris sequence.

B.5. The construction of the filler in (10). Equipped with the previous discussion on tensor products of rational pro-operads, we construct the asserted filler in the diagram (10). To this end, we first consider the commutative diagram

$$\operatorname{Aut}_{\operatorname{\mathcal{O}p^{red}}}(E_n) \times \operatorname{Aut}_{\operatorname{\mathcal{O}p^{red}}}(E_m) \xrightarrow{(-)_Q^{\wedge}} \operatorname{Aut}_{\operatorname{\mathcal{O}p^{red}}}((E_n)_Q^{\wedge}) \times \operatorname{Aut}_{\operatorname{\mathcal{O}p^{red}}}((E_m)_Q^{\wedge}) \xrightarrow{|-|(2)|} \operatorname{Aut}_{\operatorname{\mathcal{S}}}(S_Q^{n-1}) \times \operatorname{Aut}_{\operatorname{\mathcal{S}}}(S_Q^{m-1})$$

$$\xrightarrow{(-) \otimes (-)} \downarrow \qquad \qquad \downarrow^{(-) * (-)}$$

$$\operatorname{Aut}_{\operatorname{\mathcal{O}p^{red}}}(E_{n+m}) \xrightarrow{(-)_Q^{\wedge}} \operatorname{Aut}_{\operatorname{\mathcal{O}p^{red}}}((E_{n+m})_Q^{\wedge}) \xrightarrow{|-|(2)|} \operatorname{Aut}_{\operatorname{\mathcal{S}}}(S_Q^{n+m-1})$$

where the commutativity of the left-hand square is induced by (69) and the additivity equivalence $E_n \otimes E_m \simeq E_{n+m}$ from Section 2.2.2 and commutativity of the right-hand square is induced by the natural transformation from Equation (70) which is an equivalence in the relevant case by (ii) of that lemma since $E_d(2) \simeq S^{d-1}$ is nilpotent of finite Q-type. Extending the diagram to the right by taking reduced homology of the rational spheres to map from the leftmost column to $\otimes : GL(Q) \times GL(Q) \to GL(Q)$, we see that the middle vertical map almost provides a filler as in (10), except that the diagram now involves $\operatorname{Aut}(|(E_d)_{\mathbb{Q}}^{\wedge}|)$ instead of $\operatorname{Aut}(E_{d,\mathbb{Q}})$. However, they are equivalent by Example B.5, so we obtain the desired filler.

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