Trigonometria

A trigonometria estuda as relações existentes no triângulo retângulo, descobrindo e explorando as conveniências e proporcionalidades existentes entre suas grandezas.

Fórmulas Básicas das Funções Trigonométricas

Seja um triângulo retângulo com os catetos a e b, e a hipotenusa c. Para nossa análise vamos escolher o ângulo θ desse triângulo cujo cateto oposto a esse ângulo será o lado a do triângulo e o cateto adjacente será o lado b.

$$\boxed{\sin(\theta) = \frac{cat\ op}{hip} = \frac{a}{c}}$$

$$\boxed{\cos(\theta) = \frac{cat\ adj}{hip} = \frac{b}{c}}$$

$$\boxed{\tan(\theta) = \frac{cat\ op}{cat\ adj} = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin(\theta)}{\cos(\theta)}}$$

$$\boxed{\sec(\theta) = \frac{hip}{cat\ adj} = \frac{c}{b} = \frac{1}{\frac{b}{c}} = \frac{1}{\cos(\theta)}}$$

$$\boxed{\csc(\theta) = \frac{hip}{cat\ op} = \frac{c}{a} = \frac{1}{\frac{a}{c}} = \frac{1}{\sin(\theta)}}$$

$$\boxed{\cot(\theta) = \frac{cat\ adj}{cat\ op} = \frac{b}{a} = \frac{1}{\frac{a}{b}} = \frac{1}{\tan(\theta)}}$$

Relação Fundamental Trigonométrica

$$\boxed{\sin^2(\theta) + \cos^2(\theta) = 1}$$

A relação fundamental trigonométrica é facilmente deduzida ao se analisar o ciclo trigonométrico. Ao analisarmos o triângulo retângulo formado por qualquer ângulo dentro do ciclo chegamos à relação fundamental por meio de Pitágoras ($a^2 + b^2 = c^2$), em que os valores nos eixos do seno e cosseno desse ângulo são os catetos, e o raio ($R = 1$) do ciclo é a hipotenusa do triângulo.

Soma e Diferença de Arcos

Seno

$$\boxed{\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)\\ \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha)\end{aligned}}$$

Cosseno

$$\boxed{\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\\ \cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin\,(\beta)\end{aligned}}$$

Tangente

$$\boxed{\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\\ &= \frac{\dfrac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\dfrac{\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\end{aligned}}$$

$$\boxed{\begin{aligned}\tan(\alpha-\beta) &= \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{\sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)}\\ &= \frac{\dfrac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\dfrac{\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} = \frac{\tan(\alpha)-\tan\,(\beta)}{1+\tan(\alpha)\tan(\beta)}\end{aligned}}$$

Secante, Cossecante e Cotangente

$$\boxed{\begin{aligned} \sec(\alpha+\beta) &= \frac{1}{\cos(\alpha+\beta)} \\ \sec(\alpha-\beta) &= \frac{1}{\cos(\alpha-\beta)} \end{aligned}}$$

$$\boxed{\begin{aligned} \csc(\alpha+\beta) &= \frac{1}{\sin(\alpha+\beta)} \\ \csc(\alpha-\beta) &= \frac{1}{\sin(\alpha-\beta)} \end{aligned}}$$

$$\boxed{\begin{aligned} \cot(\alpha+\beta) &= \frac{1}{\tan(\alpha+\beta)} = \frac{1}{\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}} = \frac{1-\tan(\alpha)\tan(\beta)}{\tan(\alpha)+\tan(\beta)} \\ \cot(\alpha-\beta) &= \frac{1}{\tan(\alpha-\beta)} = \frac{1}{\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}} = \frac{1+\tan(\alpha)\tan(\beta)}{\tan(\alpha)-\tan(\beta)} \end{aligned}}$$

Por meio de uma análise de ângulos usando geometria plana é possível deduzir as fórmulas da soma e diferença de arcos da função seno e cosseno, todas as outras derivam dessas duas.

Arco duplo

Seno

$$\boxed{\begin{gathered}\sin(2\alpha) = \sin(\alpha + \alpha) = \sin(\alpha)\cos(\alpha) + \sin(\alpha)\cos(\alpha) = 2\sin(\alpha)\cos(\alpha) \\ \therefore \sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)\end{gathered}}$$

Cosseno

$$\boxed{\begin{gathered}\cos(2\alpha) = \cos(\alpha + \alpha) = \cos(\alpha)\cos(\alpha) - \sin(\alpha)\sin(\alpha) = \\ \cos^2(\alpha) - \sin^2(\alpha) = [1 - \sin^2(\alpha)] - \sin^2(\alpha) = 1 - 2\sin^2(\alpha) \\ = \cos^2(\alpha) - [1 - \cos^2(\alpha)] = 2\cos^2(\alpha) - 1 \\ \therefore \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha) = 2\cos^2(\alpha) - 1\end{gathered}}$$

Tangente

$$\boxed{\tan(2\alpha) = \tan(\alpha + \alpha) = \frac{\tan(\alpha) + \tan\,(\alpha)}{1 - \tan(\alpha)\tan(\alpha)} = \frac{2\tan\,(\alpha)}{1 - \tan^2(\alpha)}}$$

Secante, Cossecante e Cotangente

$$\boxed{\sec(2\alpha) = \frac{1}{\cos(2\alpha)}}$$

$$\boxed{\csc(2\alpha) = \frac{1}{\sin(2\alpha)}}$$

$$\boxed{\cot(2\alpha) = \frac{1}{\tan(2\alpha)} = \frac{1}{\frac{2\tan(\alpha)}{1-\tan^2(\alpha)}} = \frac{1-\tan^2(\alpha)}{2\tan(\alpha)}}$$

As equações de arco duplo derivam das já vistas equações de soma de arcos.

Arco metade

Cosseno

$$\boxed{\begin{gathered}\theta = \frac{\alpha}{2} \rightarrow \cos(2\theta) = 2\cos^2(\theta) - 1 \rightarrow \cos(\alpha) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1 \\ \therefore \cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}\end{gathered}}$$

Seno

$$\boxed{\begin{gathered}\theta = \frac{\alpha}{2} \rightarrow \cos(2\theta) = 1 - 2\sin^2(\theta) \rightarrow \cos(\alpha) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right) \\ \therefore \sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}\end{gathered}}$$

Tangente

$$\boxed{\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)} = \pm\frac{\sqrt{\frac{1-\cos(\alpha)}{2}}}{\sqrt{\frac{1+\cos(\alpha)}{2}}} = \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}}$$

Secante, Cossecante e Cotangente

$$\boxed{\sec\left(\frac{\alpha}{2}\right) = \frac{1}{\cos\left(\frac{\alpha}{2}\right)} = \pm\frac{1}{\sqrt{\frac{1+\cos(\alpha)}{2}}} = \pm\sqrt{\frac{2}{1+\cos(\alpha)}}}$$

$$\boxed{\csc\left(\frac{\alpha}{2}\right) = \frac{1}{\sin\left(\frac{\alpha}{2}\right)} = \pm\frac{1}{\sqrt{\frac{1-\cos(\alpha)}{2}}} = \pm\sqrt{\frac{2}{1-\cos(\alpha)}}}$$

$$\boxed{\cot\left(\frac{\alpha}{2}\right) = \frac{1}{\tan\left(\frac{\alpha}{2}\right)} = \pm\frac{1}{\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}} = \pm\sqrt{\frac{1+\cos(\alpha)}{1-\cos(\alpha)}}}$$

Por meio de substituições ($\theta = \frac{\alpha}{2}$) e manipulações algébricas, principalmente na fórmula do cosseno do arco duplo, encontra-se o seno e o cosseno do arco metade, todas as outras equações, novamente, derivam dessas duas.

Soma e Diferença de Senos e Cossenos (Transformação em Produto)

Seno

$$\boxed{\begin{gathered}\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)\qquad (I)\\ \sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha)\qquad (II)\\ (I)+(II)\rightarrow \sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin(\alpha)\cos(\beta)\\ \beta=\frac{\theta-\gamma}{2}\qquad \alpha=\frac{\theta+\gamma}{2}\\ \sin\left(\frac{\theta+\gamma}{2}+\frac{\theta-\gamma}{2}\right)+\sin\left(\frac{\theta+\gamma}{2}-\frac{\theta-\gamma}{2}\right)=2\sin\left(\frac{\theta+\gamma}{2}\right)\cos\left(\frac{\theta-\gamma}{2}\right)\\ \therefore \sin(\theta)+\sin(\gamma)=2\sin\left(\frac{\theta+\gamma}{2}\right)\cos\left(\frac{\theta-\gamma}{2}\right)\end{gathered}}$$

$$\boxed{\begin{gathered}\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)\qquad (I)\\ \sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha)\qquad (II)\\ (I)-(II)\rightarrow \sin(\alpha+\beta)-\sin(\alpha-\beta)=2\sin(\beta)\cos(\alpha)\\ \beta=\frac{\theta-\gamma}{2}\qquad \alpha=\frac{\theta+\gamma}{2}\\ \sin\left(\frac{\theta+\gamma}{2}+\frac{\theta-\gamma}{2}\right)-\sin\left(\frac{\theta+\gamma}{2}-\frac{\theta-\gamma}{2}\right)=2\sin\left(\frac{\theta-\gamma}{2}\right)\cos\left(\frac{\theta+\gamma}{2}\right)\\ \therefore \sin(\theta)-\sin(\gamma)=2\sin\left(\frac{\theta-\gamma}{2}\right)\cos\left(\frac{\theta+\gamma}{2}\right)\end{gathered}}$$

Cosseno

$$\boxed{\begin{gathered}\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\qquad (I)\\ \cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)\qquad (II)\\ (I)+(II)\rightarrow \cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos(\alpha)\cos(\beta)\\ \beta=\frac{\theta-\gamma}{2}\qquad \alpha=\frac{\theta+\gamma}{2}\\ \cos\left(\frac{\theta+\gamma}{2}+\frac{\theta-\gamma}{2}\right)+\cos\left(\frac{\theta+\gamma}{2}-\frac{\theta-\gamma}{2}\right)=2\cos\left(\frac{\theta+\gamma}{2}\right)\cos\left(\frac{\theta-\gamma}{2}\right)\\ \therefore \cos(\theta)+\cos(\gamma)=2\cos\left(\frac{\theta+\gamma}{2}\right)\cos\left(\frac{\theta-\gamma}{2}\right)\end{gathered}}$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \qquad (I)$$
$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \qquad (II)$$
$$(I) - (II) \rightarrow \cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin(\alpha)\sin(\beta)$$
$$\beta = \frac{\theta - \gamma}{2} \qquad \alpha = \frac{\theta + \gamma}{2}$$
$$\cos\left(\frac{\theta + \gamma}{2} + \frac{\theta - \gamma}{2}\right) - \cos\left(\frac{\theta + \gamma}{2} - \frac{\theta - \gamma}{2}\right) = -2\sin\left(\frac{\theta + \gamma}{2}\right)\sin\left(\frac{\theta - \gamma}{2}\right)$$
$$\therefore \cos(\theta) - \cos(\gamma) = -2\sin\left(\frac{\theta + \gamma}{2}\right)\sin\left(\frac{\theta - \gamma}{2}\right)$$

As transformações acima mostradas são muito uteis quando se precisa resolver problemas envolvendo somas e diferenças de senos e cossenos, uma vez que é muito mais fácil simplificar e isolar variáveis em expressões compostas apenas por produtos.

Lei dos Senos

Seja um triângulo qualquer com lados a, b e c. Os ângulos opostos a cada lado são respectivamente A, B e C. Considerando também que esse triângulo esteja inscrito dentro de uma circunferência de raio R. A lei dos senos nos diz que:

$$\boxed{\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R}$$

É possível deduzir essa lei usando um triângulo qualquer e conceitos relacionados a arcos e ângulos.

Lei dos Cossenos

Seja um triângulo qualquer com lados a, b e c. Os ângulos opostos a cada lado são respectivamente A, B e C. Por meio de substituições chega-se a lei dos cossenos, que diz o seguinte:

$$\boxed{\begin{array}{l} a^2 = b^2 + c^2 - 2bc\cos(A) \\ b^2 = a^2 + c^2 - 2ac\cos(B) \\ c^2 = a^2 + b^2 - 2ab\cos(C) \end{array}}$$

Ângulos Complementares

$$\boxed{\begin{gathered}\sin\left(\frac{\pi}{2}-\alpha\right) = \sin\left(\frac{\pi}{2}\right)\cos(\alpha) - \sin(\alpha)\cos\left(\frac{\pi}{2}\right) = \cos(\alpha) \\ \therefore \sin\left(\frac{\pi}{2}-\alpha\right) = \cos(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\cos\left(\frac{\pi}{2}-\alpha\right) = \cos\left(\frac{\pi}{2}\right)\cos(\alpha) + \sin\left(\frac{\pi}{2}\right)\sin(\alpha) = \sin(\alpha) \\ \therefore \cos\left(\frac{\pi}{2}-\alpha\right) = \sin(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\tan\left(\frac{\pi}{2}-\alpha\right) = \frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\cos\left(\frac{\pi}{2}-\alpha\right)} = \frac{\cos(\alpha)}{\sin(\alpha)} = \cot(\alpha) \\ \therefore \tan\left(\frac{\pi}{2}-\alpha\right) = \cot(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\cot\left(\frac{\pi}{2}-\alpha\right) = \frac{1}{\tan\left(\frac{\pi}{2}-\alpha\right)} = \frac{1}{\cot(\alpha)} = \frac{1}{\frac{1}{\tan(\alpha)}} = \tan(\alpha) \\ \therefore \cot\left(\frac{\pi}{2}-\alpha\right) = \tan(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\sec\left(\frac{\pi}{2}-\alpha\right) = \frac{1}{\cos\left(\frac{\pi}{2}-\alpha\right)} = \frac{1}{\sin(\alpha)} = \csc(\alpha) \\ \therefore \sec\left(\frac{\pi}{2}-\alpha\right) = \csc(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\csc\left(\frac{\pi}{2}-\alpha\right) = \frac{1}{\sin\left(\frac{\pi}{2}-\alpha\right)} = \frac{1}{\cos(\alpha)} = \sec(\alpha) \\ \therefore \csc\left(\frac{\pi}{2}-\alpha\right) = \sec(\alpha)\end{gathered}}$$

Ângulos Suplementares

$$\boxed{\begin{gathered}\sin(\pi-\alpha)=\sin(\pi)\cos(\alpha)-\sin(\alpha)\cos(\pi)=\sin(\alpha)\\ \therefore \sin(\pi-\alpha)=\sin(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\cos(\pi-\alpha)=\cos(\pi)\cos(\alpha)+\sin(\pi)\sin(\alpha)=-\cos(\alpha)\\ \therefore \cos(\pi-\alpha)=-\cos(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\tan(\pi-\alpha)=\frac{\sin(\pi-\alpha)}{\cos(\pi-\alpha)}=-\frac{\sin(\alpha)}{\cos(\alpha)}=-\tan(\alpha)\\ \therefore \tan(\pi-\alpha)=-\tan(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\cot(\pi-\alpha)=\frac{1}{\tan(\pi-\alpha)}=-\frac{1}{\tan(\alpha)}=-\cot(\alpha)\\ \therefore \cot(\pi-\alpha)=-\cot(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\sec(\pi-\alpha)=\frac{1}{\cos(\pi-\alpha)}=-\frac{1}{\cos(\alpha)}=-\sec(\alpha)\\ \therefore \sec(\pi-\alpha)=-\sec(\alpha)\end{gathered}}$$

$$\boxed{\begin{gathered}\csc(\pi-\alpha)=\frac{1}{\sin(\pi-\alpha)}=\frac{1}{\sin(\alpha)}=\csc(\alpha)\\ \therefore \csc(\pi-\alpha)=\csc(\alpha)\end{gathered}}$$