MODULE *tictactoe*

EXTENDS *Naturals*

VARIABLES

board, *board*[1 . . 3][1 . . 3] A 3*x*3 tic-tac-toe board

nextTurn who goes next

$BoardIs(coordinate, player) \triangleq$

$board[coordinate[1]][coordinate[2]] = player$

$BoardFilled \triangleq$

There does not exist

$\neg\exists\, i \in 1 \,..\, 3, j \in 1 \,..\, 3 :$

an empty space

LET $space \triangleq board[i][j]$ IN

$space =$ "␣"

$BoardEmpty \triangleq$

There does not exist

$\forall\, i \in 1 \,..\, 3, j \in 1 \,..\, 3 :$

an empty space

LET $space \triangleq board[i][j]$ IN

$space =$ "␣"

$WinningPositions \triangleq \{$

Horizonal wins

$\langle\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle\rangle,$

$\langle\langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle\rangle,$

$\langle\langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle\rangle,$

Vertical wins

$\langle\langle 1, 1\rangle, \langle 2, 1\rangle, \langle 3, 1\rangle\rangle,$

$\langle\langle 1, 2\rangle, \langle 2, 2\rangle, \langle 3, 2\rangle\rangle,$

$\langle\langle 1, 3\rangle, \langle 2, 3\rangle, \langle 3, 3\rangle\rangle,$

Diagonal wins

$\langle\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\rangle,$

$\langle\langle 3, 1\rangle, \langle 2, 2\rangle, \langle 1, 3\rangle\rangle$

$\}$

$Won(player) \triangleq$

A player has won if there exists a winning position

$\exists\, winningPosition \in WinningPositions :$

Where all the needed spaces

$\forall\, i \in 1 \,..\, 3 :$

are occupied by one player

$board[winningPosition[i][1]][winningPosition[i][2]] = player$

$Move(player,\ coordinate) \triangleq$
$\quad \land\ board[coordinate[1]][coordinate[2]] = \text{“\_”}$
$\quad \land\ board' = [board \text{ EXCEPT}$
$\qquad\qquad\qquad\qquad ![coordinate[1]][coordinate[2]] = player]$

$MoveToEmpty(player) \triangleq$
$\quad \land \exists\, i \in 1\,..\,3 : \exists\, j \in 1\,..\,3 :$ There exists a position on the board
$\qquad \land\ board[i][j] = \text{“\_”}$ Where the board is currently empty
$\qquad \land\ Move(player, \langle i, j \rangle)$

$MoveO \triangleq$
$\quad \land\ nextTurn = \text{“O”}$ Only enabled on O's turn
$\quad \land \neg Won(\text{“X”})$ And X has not won
$\quad \land\ MoveToEmpty(\text{“O”})$ O still tries every empty space
$\quad \land\ nextTurn' = \text{“X”}$ The future state of next turn is X

$Corners \triangleq \{$
$\quad \langle 1, 1 \rangle,$
$\quad \langle 3, 1 \rangle,$
$\quad \langle 1, 3 \rangle,$
$\quad \langle 3, 3 \rangle$
$\}$

$StartInCorner \triangleq$
$\quad \exists\, corner \in Corners :$
$\qquad Move(\text{“X”}, corner)$

$PartialWins \triangleq \{$
$\quad \langle 1, 2, 3 \rangle,$
$\quad \langle 2, 3, 1 \rangle,$
$\quad \langle 3, 1, 2 \rangle$
$\}$

$CanWin \triangleq \exists\, winningPostion \in WinningPositions,\ partialWin \in PartialWins :$
$\qquad\qquad \land\ BoardIs(winningPostion[partialWin[1]],\ \text{“X”})$
$\qquad\qquad \land\ BoardIs(winningPostion[partialWin[2]],\ \text{“X”})$
$\qquad\qquad \land\ BoardIs(winningPostion[partialWin[3]],\ \text{“\_”})$

$CanBlockWin \triangleq \exists\, winningPostion \in WinningPositions,\ partialWin \in PartialWins :$
$\qquad\qquad\qquad \land\ BoardIs(winningPostion[partialWin[1]],\ \text{“O”})$
$\qquad\qquad\qquad \land\ BoardIs(winningPostion[partialWin[2]],\ \text{“O”})$
$\qquad\qquad\qquad \land\ BoardIs(winningPostion[partialWin[3]],\ \text{“\_”})$

$CanTakeCenter \triangleq\ board[2][2] = \text{“\_”}$

$CanSetupWin \triangleq$
$\quad \exists\, winningPostion \in WinningPositions,\ partialWin \in PartialWins :$
$\qquad \land\ BoardIs(winningPostion[partialWin[1]],\ \text{“X”})$

$\land\ BoardIs(winningPostion[partialWin[2]],\ \text{“\_”})$
$\land\ BoardIs(winningPostion[partialWin[3]],\ \text{“\_”})$

$Win \triangleq \exists\, winningPostion \in WinningPositions,\ partialWin \in PartialWins :$
$\land\ BoardIs(winningPostion[partialWin[1]],\ \text{“X”})$
$\land\ BoardIs(winningPostion[partialWin[2]],\ \text{“X”})$
$\land\ BoardIs(winningPostion[partialWin[3]],\ \text{“\_”})$
$\land\ Move(\text{“X”},\ winningPostion[partialWin[3]])$

$BlockWin \triangleq \exists\, winningPostion \in WinningPositions,\ partialWin \in PartialWins :$
$\land\ BoardIs(winningPostion[partialWin[1]],\ \text{“O”})$
$\land\ BoardIs(winningPostion[partialWin[2]],\ \text{“O”})$
$\land\ BoardIs(winningPostion[partialWin[3]],\ \text{“\_”})$
$\land\ Move(\text{“X”},\ winningPostion[partialWin[3]])$

$TakeCenter \triangleq$
$\land\ Move(\text{“X”},\ \langle 2,\ 2\rangle)$

$SetupWin \triangleq$
$\exists\, winningPostion \in WinningPositions,\ partialWin \in PartialWins :$
$\land\ BoardIs(winningPostion[partialWin[1]],\ \text{“X”})$
$\land\ BoardIs(winningPostion[partialWin[2]],\ \text{“\_”})$
$\land\ BoardIs(winningPostion[partialWin[3]],\ \text{“\_”})$
$\land\ \exists\, i \in 2\,..\,3 :$
$Move(\text{“X”},\ winningPostion[partialWin[i]])$

$MoveX \triangleq$
$\land\ nextTurn = \text{“X”}$ Only enabled on X's turn
$\land\ \neg\, Won(\text{“O”})$ And X has not won
This specifies the spots X will move on X's turn
$\land\ \lor\ \land\ BoardEmpty$
$\land\ StartInCorner$
$\lor\ \land\ \neg BoardEmpty$ If its not the start
$\land\ \lor\ \land\ CanWin$
$\land\ Win$
$\lor\ \land\ \neg CanWin$
$\land\ \lor\ \land\ CanBlockWin$
$\land\ BlockWin$
$\lor\ \land\ \neg CanBlockWin$
$\land\ \lor\ \land\ CanTakeCenter$
$\land\ TakeCenter$
$\lor\ \land\ \neg CanTakeCenter$
$\land\ \lor\ \land\ CanSetupWin$
$\land\ SetupWin$
$\lor\ \land\ \neg CanSetupWin$
$\land\ MoveToEmpty(\text{“X”})$ No more strategies. Pick spot

$\land\ nextTurn' = \text{“O”}$ The future state of next turn is O

$Init \triangleq$
$\land\ nextTurn = \text{“X”}$ X always goes first
Every space in the board states blank
$\land\ board = [i \in 1 \,..\, 3 \mapsto [j \in 1 \,..\, 3 \mapsto \text{“\_”}]]$

$GameOver \triangleq Won(\text{“X”}) \land Won(\text{“O”}) \land BoardFilled$

Every state, X will move if X's turn, O will move on O's turn
$Next \triangleq MoveX \lor MoveO \lor (GameOver \land \text{UNCHANGED } \langle board, nextTurn \rangle)$

$XHasNotWon \triangleq \neg Won(\text{“X”})$
$OHasNotWon \triangleq \neg Won(\text{“O”})$

It's not a stalemate if one player has won or the board is not filled
$NotStalemate \triangleq$
$\lor\ Won(\text{“X”})$
$\lor\ Won(\text{“O”})$
$\lor\ \neg BoardFilled$

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