1997 European Control Conference (ECC)
1-4 July 1997, Brussels, Belgium

AN LMI APPROACH FOR STABILITY ANALYSIS OF NONLINEAR SYSTEMS

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Keywords: Nonlinear systems, Fuzzy systems, Stability, Linear Matrix Inequalities (LMIs).

Abstract

This paper presents a constructive method for showing stability of nonlinear systems consisting of state-dependent weighted linear systems. This kind of system representation is common in for instance fuzzy systems or when local linear models of a nonlinear system are weighted together. Stability is shown by joining multiple local quadratic Lyapunov functions properly in the state space. The stability conditions are formulated as a linear matrix inequality (LMI) problem. Hence, these can be verified efficiently by computerized methods.

1 Introduction

In this paper we present a constructive method for showing stability of nonlinear systems consisting of state-dependent weighted linear systems. A large class of nonlinear systems are represented in this way, e.g. fuzzy systems [11, 12] or weighted linearized systems [5]. The second kind of systems is the result of a linearization around several states, which gives a number of linear systems. However, contrary to approaches that approximate a nonlinear system by only one of these linearized systems in a specific region, the local linear models are weighted together. A nonlinear vector field can be approximated arbitrary well (in $\sup$-norm) by a weighted sum of linearized models using enough local linear models [4].

To show stability of state-dependent weighted linear systems, we begin by presenting stability conditions for nonlinear systems that are similar to Lyapunov's direct method [10]. However, we relax the condition requiring differentiability of the Lyapunov function at all states. This means that a Lyapunov function is allowed to be *discontinuous*. The reason for this relaxation is that local Lyapunov functions can be used in different regions and stability is ensured if these functions are joined properly in the state space.

When applying the stability result to state-dependent weighted linear systems, the local Lyapunov functions are common for all linear systems that contribute to the nonlinear vector field in a region, that depends on the state-dependent weights of the corresponding linear systems. By utilizing *quadratic* local Lyapunov functions, the stability conditions can be formulated as a linear matrix inequality (LMI) problem. This is attractive since efficient computerized methods to solve such problems are available [1, 3].

The outline of the paper is: In the next section we formulate the model under consideration. In Section 3 the stability results are formulated. The LMI formulation is described in Section 4. Finally, we conclude with an example that illustrates the theory.

2 Nonlinear model

The model considered in this paper is nonlinear systems

$$\dot{x}(t) = f(x(t)) \tag{1}$$

represented on the form

$$\dot{x}(t) = \sum_{i=1}^{r} w_i(x(t))(A_i x(t) + B_i), \tag{2}$$

where $w_i : \Re^n \to \Re^+$, $i \in I_r = \{1, \ldots, r\}$, are *non-negative* scalar *continuous* weighting functions, $w_i(x) \geq 0$, $i \in I_r$. Further on, without loss of generality, it is assumed that the origin is an equilibrium point to (2).

By the *support* Ω_i of $w_i(x)$ we mean the set of states fulfilling:

$$\Omega_i = \{x \in \Re^n \mid w_i(x) > 0\}. \tag{3}$$

Thus, Ω_i is a set of states indicating where the local linear system i contributes to the vector field in (2). It is required that $\Omega_1 \cup \ldots \cup \Omega_r = \Omega \subseteq \Re^n$, where $0 \in \Omega$. In this paper we consider $\Omega = \Re^n$; the local analysis can however be performed in an obvious way [10]. Note that since the functions w_i, $i \in I_r$, are continuous and $\Omega_1 \cup \ldots \cup \Omega_r = \Omega \subseteq \Re^n$, there is an overlap between the different local linear systems, cf. Figure 1.

A large class of nonlinear systems in the literature are represented in the form (2). For example, if a nonlinear system (1) is linearized about a point x_i, the result is a linear system:

$$\dot{x}(t) = f(x_i) + A_i(x - x_i) = A_i x + B_i,$$

ISBN 978-3-9524269-0-6

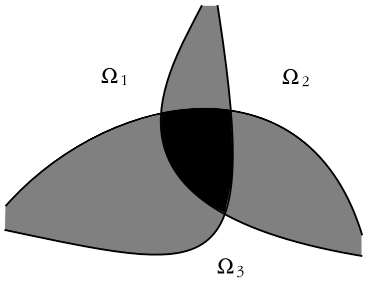

Figure 1: The different local linear systems overlap each other in the state space. Each Ω_i represents a region where the local linear system i contributes to the vector field (2). In the shaded and black regions, there are two respectively three overlapping local linear systems.

where $A_i = \partial f/\partial x|_{x=x_i}$ and $B_i = f(x_i) - A_i x_i$. This *local* linear model is valid only within a certain operating region $\Omega_i \subseteq \Re^n$, and will be more or less invalid outside this region. However, if local linear models are used in different operating regions, and smooth *model validity functions* $w_i : \Re^n \to [0\ 1]$ are constructed such that its value is close to 1 for states within Ω_i, and close to 0 otherwise, then f in (1) can be approximated arbitrary well (in $\sup$-norm) by (2) using enough linearized models [4, 5].

Furthermore, a model in the form (2) can be interpreted as a fuzzy system consisting of r IF-THEN rules according to:

Rule i: IF $x(t) \in \Omega_i$ THEN $\dot{x}(t) = A_i x(t) + B_i$,

where $\Omega_i \subseteq \Re^n$ is a *fuzzy set*. The output of the system is a weighted average of the different local linear systems in the form (2), where the weight $w_i : \Re^n \to [0\ 1]$ is the *membership function* associated with the fuzzy set Ω_i. This kind of model can be found in [11, 12] in the case when all $B_i = 0$.

Note that, contrary to the above mentioned representations, we do not require that the range of the functions w_i, $i \in I_r$, have to be upper bounded by 1 but can take any value in $\Re^+$. As a result of this possibility, some nonlinear systems (1) can compactly be represented on the form (2), which the following example illustrates:

Example 1 *Consider the nonlinear system*

$$\dot{x} = x + x^3.$$

The system can be written on the form (2) as:

$$\dot{x} = w_1(x)x + w_2(x)x,$$

where $w_1(x) = 1$ and $w_2(x) = x^2$ take any value in $\Re^+$.

In fact, it can easily be shown that every nonlinear system (1) in $\Re^n$ can be represented on the form $\dot{x} = \sum_{i=1}^{2n} w_i(x) B_i$, which is a special case of (2). However, this form of the vector field is mostly not suitable when stability is investigated, since too much necessary knowledge about the vector field has been included in the weighting functions.

If the weighting functions $w_i(x)$, $i \in I_r$, are step functions, then (2) becomes a switched or variable structure system. It is then more common to write the nonlinear system (1) as:

$$\dot{x}(t) = f_i(x(t)),$$

where $f_i : \Re^n \to \Re^n$ represents the different vector fields. Furthermore, if a piecewise constant function $i(t)$ is included that determines the next vector field as a function of the previous one and the continuous state x according to:

$$i(t) = \phi(x(t), i(t^-)),$$

then a hybrid system is obtained. In this paper we specially deal with overlapping Ω_i's and continuous w_i's. For a stability treatment of switched and hybrid systems cf. [7, 2, 13, 8, 6, 9].

3 Stability analysis

The presented stability theorem in Section 3.1 is valid for continuous vector fields described by (1). To show stability, the state space (or a subset of the state space) is partitioned into ℓ *disjoint* regions $\tilde{\Omega}_q$, $q \in I_\ell$, cf. Figure 2. Each region is not necessarily connected. In each region $\tilde{\Omega}_q$, $q \in I_\ell$, a scalar function $V_q(x)$ is used as a measure of the system's (abstract) energy. Let

$$V(x) = V_q(x) \quad \text{when} \quad x \in \tilde{\Omega}_q. \tag{4}$$

Furthermore, let the *switch region* $\tilde{\Lambda}_{ij}$ be the set of continuous states for which the trajectory $x(t)$ passes from $\tilde{\Omega}_q$ to $\tilde{\Omega}_r$, i.e.:

$$\tilde{\Lambda}_{qr} = \{x \in \Re^n \mid x(t^-) \in \tilde{\Omega}_q,\ x(t) \in \tilde{\Omega}_r\}.$$

Note that $\tilde{\Lambda}_{qr}$ are given by *hyper-surfaces*. It is necessary that $\tilde{\Omega}_q$ and $\tilde{\Omega}_r$ are neighboring sets if $\tilde{\Lambda}_{qr} \neq \emptyset$. Let

$$I_\Lambda = \{(q,r) | \tilde{\Lambda}_{qr} \neq \emptyset\}.$$

The switch regions $\tilde{\Lambda}_{qr}$ are obtained by studying the vector field direction $f(x)$ at the neighboring points between the sets $\tilde{\Omega}_q$ and $\tilde{\Omega}_r$, i.e. at the states $x \in \partial\tilde{\Omega}_q \cap \partial\tilde{\Omega}_r$. For instance, the left white circle on the trajectory $x(t)$ between region $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ in Figure 2 is included in $\tilde{\Lambda}_{12}$.

3.1 Stability

Theorem 1 *Assume that the state trajectory evolves according to (1). If there exist scalar functions $V_q : \tilde{\Omega}_q \to \Re$, each $V_q(x)$ differentiable $\forall x \in \tilde{\Omega}_q$, $q \in I_\ell$, and class K functions $\alpha : \Re^+ \to \Re^+$ and $\beta : \Re^+ \to \Re^+$ such that*

- $\forall x \in \tilde{\Omega}_q$, $\alpha(||x||) \leq V_q(x) \leq \beta(||x||)$, $\quad q \in I_\ell$
- $\forall x \in \tilde{\Omega}_q$, $\dot{V}_q(x) \leq 0$, $\quad q \in I_\ell$
- $\forall x \in \tilde{\Lambda}_{qr}$, $V_r(x) \leq V_q(x)$, $\quad (q,r) \in I_\Lambda$

then the equilibrium point 0 is (uniformly) stable in the sense of Lyapunov.

A class K function is a continuous function $\kappa : \Re^+ \to \Re^+$ fulfilling: $\kappa(0) = 0$, $\kappa(z) > 0$, $z > 0$ and $\kappa(z_1) \leq \kappa(z_2)$, $z_1 < z_2$.

The proof of this theorem is the same as the proof of Theorem 4.1 in [10] which is a common stability result for non-autonomous systems. However, there is one main difference, namely that we allow the Lyapunov function to be discontinuous. We can allow this since continuity of $V(x)$ is not used in the proof. Instead, the proof is based on continuity of the class K functions $\alpha(\cdot)$ and $\beta(\cdot)$.

The discontinuous function $V(x)$ defined in (4) is called a *discontinuous Lyapunov function*, or shortly a *Lyapunov function*, if the conditions in Theorem 1 are satisfied. Each $V_q(x)$, $q \in I_\ell$, is called a *local Lyapunov function*. Note that if there is only one region $\tilde{\Omega} = \Re^n$, the above stability theorem is the usual one found in the literature [10].

Figure 2 illustrates the stability conditions. In each region $\tilde{\Omega}_q$, the corresponding function $V_q(x)$ is bounded by the class K functions, and the energy, measured by $V_q(x)$, is non-increasing in this region. Furthermore, when another region $\tilde{\Omega}_r$ is entered, $V_r(x) \leq V_q(x)$ when the trajectory pass from $\tilde{\Omega}_q$ to $\tilde{\Omega}_r$. Hence, the system's energy $V(x)$ in (4) non-increases at all time instants.

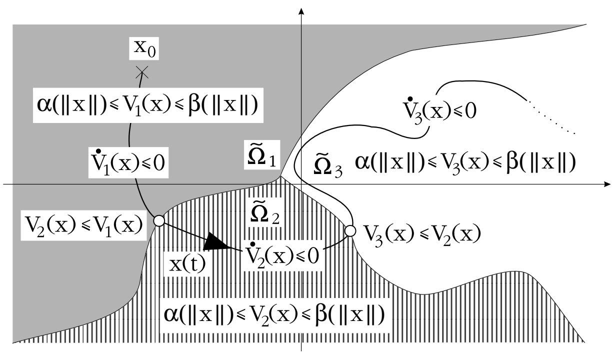

Figure 2: Requirements on the discontinuous Lyapunov function (4). If $\forall x \in \tilde{\Omega}_q$, $\alpha(||x||) \leq V_q(x) \leq \beta(||x||)$, $\dot{V}_q(x) \leq 0$ and $V_r(x) \leq V_q(x)$ when the trajectory pass from $\tilde{\Omega}_q$ to $\tilde{\Omega}_r$, then the system is stable.

One may wonder if the third condition in Theorem 1 is necessary for stability. The answer is yes, which can be shown by constructing counterexamples which are unstable but satisfy all conditions except the third one. Hence, it is important also to have conditions that join the local Lyapunov functions properly.

Note that if for some $x \in \partial\tilde{\Omega}_q \cap \partial\tilde{\Omega}_r$, $V_q(x) = V_r(x)$, it is not necessary to study the vector field direction $f(x)$ at this state since the third condition in Theorem 1 is satisfied regardless of the vector field direction. Hence, if $V_q(x) = V_r(x)$ $\forall x \in \partial\tilde{\Omega}_q \cap \partial\tilde{\Omega}_r$ no vector field directions have to be investigated.

3.2 Stability of fuzzy systems

In this section it is shown how Theorem 1 can be used to show stability for systems on the form (2). We define

$$I_\Omega = \{(q, i) \mid \tilde{\Omega}_q \cap \Omega_i \neq \emptyset\}, \tag{5}$$

which is the set of pairs (q, i) indicating that the local linear system i contributes to the vector field (2) in region $\tilde{\Omega}_q$. Remember that Ω_i is defined according to (3). As an example, a partitioning of Figure 1 as in Figure 3 results in the following set:

$$I_\Omega = \{(1,1), (1,2), (2,1), (2,2), (2,3)\}.$$

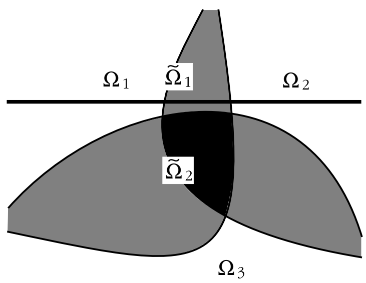

Figure 3: The state space is partitioned into two regions $\tilde{\Omega}_q$, $q = 1, 2$, in which several local linear systems contribute to the vector field (2).

We now state the following lemma.

Lemma 1 *Assume that the state trajectory evolves according to (2). If there exist scalar functions $V_q : \tilde{\Omega}_q \to \Re$, each $V_q(x)$ differentiable $\forall x \in \tilde{\Omega}_q$, $q \in I_\ell$, and class K functions $\alpha : \Re^+ \to \Re^+$ and $\beta : \Re^+ \to \Re^+$ such that*

- $\forall x \in \tilde{\Omega}_q$, $\alpha(||x||) \leq V_q(x) \leq \beta(||x||)$, $\quad q \in I_\ell$
- $\forall x \in \tilde{\Omega}_q$, $\frac{\partial V_q}{\partial x}(A_i x + B_i) \leq 0$, $\quad (q, i) \in I_\Omega$
- $\forall x \in \tilde{\Lambda}_{qr}$, $V_r(x) \leq V_q(x)$, $\quad (q, r) \in I_\Lambda$

then the equilibrium point 0 *is stable in the sense of Lyapunov.*

Theorem 1 can be stated as an "if and only if" theorem, meaning that converse stability theorems can be stated. The reason for this is that there are converse theorems for stable systems on the form (2), or more general, for arbitrary nonlinear systems (1), implying the existence of differentiable Lyapunov functions [10]. Hence, since differentiable Lyapunov functions are special forms of piecewise differentiable functions this implies also the necessity of the theorem. However, Lemma 1 gives only sufficient conditions for stability. This is due to several facts. First, even if the local linear system i only contributes to the vector field (2) in a subset of the region $\tilde{\Omega}_q$, the second condition of the lemma states that $\frac{\partial V_q}{\partial x}(A_i x + B_i) \leq 0$ has to be valid in the entire region $\tilde{\Omega}_q$. This conservatism can be reduced by a finer partitioning of the state space.

A second fact is that there must be a *common* function $V_q(x)$ in region $\tilde{\Omega}_q$ that satisfies the second condition $\forall(q,i) \in I_\Omega$. Since a lot of knowledge about the vector field is included in the weighting functions, such a common function will not always exist which the following example illustrates.

Example 2 *Consider the following system in* $\Re$

$$\dot{x} = -x,$$

which is obviously stable. By writing the system on the form

$$\dot{x} = w_1(x)(-2x) + w_2(x)x,$$

where $w_1(x) = w_2(x) = 1$*, it is not possible, despite any possible partitioning, to show stability by Lemma 1. The reason for this is that the linear system* x *has to satisfy* $\frac{\partial V_q}{\partial x}x \le 0$ *in an arbitrary region* $\tilde{\Omega}_q$*. However, arbitrarily close to* 0*, this means that* $\frac{\partial V}{\partial x} \le 0$ *for* $x > 0$ *and* $\frac{\partial V}{\partial x} \ge 0$ *for* $x < 0$*, which cannot be satisfied if the first condition of Lemma 1 has to be fulfilled.*

Even though this example shows that Lemma 1 not always can be applied to show stability for all systems on the form (2), they are indeed useful in many cases. Specifically, the lemma may successfully be applied to fuzzy systems, cf. Section 5. In fact, the idea to find common Lyapunov functions for the different local linear systems has been inspired by the work in [12] which presents stability results for fuzzy systems. However, that paper only considers the case when each $B_i = 0$, $i \in I_r$, and there is only one region $\tilde{\Omega} = \Re^n$ for which there is a search for a common quadratic Lyapunov function.

4 Linear matrix inequalities

Applying Lemma 1 means that we have to construct the different functions V_q, $q \in I_\ell$, satisfying the stability conditions in the specified $\tilde{\Omega}_q$, $q \in I_\ell$, $\tilde{\Lambda}_{qr}$, $(q,r) \in I_\Lambda$. In this section we will show how this can be achieved by stating the conditions as linear matrix inequalities (LMIs) [1]. This formulation is attractive since efficient methods to solve such problems are available [1, 3].

4.1 Quadratic local Lyapunov functions

In order to obtain LMIs, we express V_q, $q \in I_\ell$, with quadratic local Lyapunov functions according to:

$$V_q(x) = x^T P_q x + 2p_q^T x + \pi_q = \tilde{x}^T \tilde{P}_q \tilde{x},$$

where

$$\tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad \tilde{P}_q = \begin{bmatrix} P_q & p_q \\ p_q^T & \pi_q \end{bmatrix}, \tag{6}$$

and $P_q \in \Re^{n\times n}$, $p_q \in \Re^{n\times 1}$ and $\pi_q \in \Re$. Without loss of generality $P_q = P_q^T$. This compact representation is given on page 23 in [1]. With the following definitions:

$$\tilde{\alpha} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{\beta} = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_i = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix},$$

Lemma 1 becomes:

Corollary 1 *Assume that the state trajectory evolves according to (2). If there exist* $\tilde{P}_q$, $q \in I_\ell$*, and constants* $\alpha > 0$, $\beta > 0$*, such that*

- $\forall x \in \tilde{\Omega}_q$, $\tilde{x}^T\tilde{\alpha}\tilde{x} \le \tilde{x}^T\tilde{P}_q\tilde{x} \le \tilde{x}^T\tilde{\beta}\tilde{x}, \quad q \in I_\ell$
- $\forall x \in \tilde{\Omega}_q$, $\tilde{x}^T\left(\tilde{A}_i^T\tilde{P}_q + \tilde{P}_q\tilde{A}_i\right)\tilde{x} \le 0, \quad (q,i) \in I_\Omega$
- $\forall x \in \tilde{\Lambda}_{qr}$, $\tilde{x}^T\tilde{P}_r\tilde{x} \le \tilde{x}^T\tilde{P}_q\tilde{x}, \quad (q,r) \in I_\Lambda$

then the equilibrium point 0 *is stable in the sense of Lyapunov.*

4.2 The $\mathcal{S}$-procedure

The conditions in Corollary 1 have to be satisfied in specified regions. If these are contained in regions given by quadratic forms, we can utilize the so called $\mathcal{S}$-procedure to express the different conditions as LMIs. The $\mathcal{S}$-procedure is a simple technique to replace a condition on a function with constraints by a condition without constraints.

To explain the $\mathcal{S}$-procedure for quadratic functions and nonstrict inequalities, let $F^0, \ldots, F^s$, be quadratic functions of the variable $x \in \Re^n$ on the form:

$$F^k(x) = x^T Q^k x + 2(c^k)^T x + d^k = \tilde{x}^T\tilde{Q}^k\tilde{x}, \quad k = 0,\ldots,s, \tag{7}$$

where $\tilde{x}$ is defined as in (6) and

$$\tilde{Q}^k = \begin{bmatrix} Q^k & c^k \\ (c^k)^T & d^k \end{bmatrix}, \tag{8}$$

where $Q^k = (Q^k)^T$, $Q^k \in \Re^{n\times n}$, $c^k \in \Re^{n\times 1}$ and $d^k \in \Re$. We consider the following condition on F^0:

$$F^0(x) \ge 0 \ \text{ in the region } \{x \in \Re^n | F^k(x) \ge 0, \ k \in I_s\}. \tag{9}$$

Regions $\tilde{\Omega}_q$

In our case, the first two conditions in Corollary 1 are on the form

$$F^0(x) \ge 0 \ \text{ in the region } \tilde{\Omega}. \tag{10}$$

where $\tilde{\Omega}$ is a region $\tilde{\Omega}_q$ and $F^0(x) \ge 0$ is the corresponding condition in the region. By finding quadratic functions $F^k(x) \ge 0$, $k \in I_s$, such that $\tilde{\Omega} \subseteq \{x \in \Re^n \mid F^k(x) \ge 0, k \in I_s\}$ it is obvious that if (9) is satisfied, so is (10). This means that it is required that $F^0(x) \ge 0$ in a region $\{x \in \Re^n | F^k(x) \ge 0, \ k \in I_s\}$ that is possibly larger than the actual region $\tilde{\Omega}$; the extreme case is to let $F^0(x) \ge 0$ in the entire state-space. However, it should be avoided specifying $\{x \in \Re^n | F^k(x) \ge 0, \ k \in I_s\}$ larger than necessary, since this conservatism may result in that we cannot show that the original condition (10) has a solution.

By including $\tilde{\Omega}$ in a region specified by quadratic functions, we can replace the constrained condition (9), by a condition without constraints, in the following way:

Lemma 2 [1] *If there exist* $\delta^k \ge 0$, $k = 1,\ldots,s$*, such that*

$$\forall x \in \Re^n, \ F^0(x) \ge \sum_{k=1}^{s} \delta^k F^k(x), \tag{11}$$

then (9) holds.

Hence, by introducing additional variables $\delta^k \geq 0$, $k \in I_s$, the condition (10) have been turned into an LMI which can be written as

$$\tilde{x}^T\tilde{Q}^0\tilde{x} \geq \sum_{k=1}^{s}\delta^k\tilde{x}^T\tilde{Q}^k\tilde{x}. \quad (12)$$

The replacement of (9) by Lemma 2 may be conservative. However, it can be shown that the converse is true in the case of a single quadratic form, $s = 1$ [1]. When applying Lemma 1, it means that we also decide the region partitioning $\tilde{\Omega}_q$, $q \in I_\ell$. Hence, it is possible to avoid conservatism by specifying each of the regions $\tilde{\Omega}_q$, $q \in I_\ell$, by a single quadratic form. Then, both (9) and (10), and (9) and Lemma 2 are equivalent.

Switch regions $\tilde{\Lambda}_{qr}$

The switch regions $\tilde{\Lambda}_{qr}$, $(q,r) \in I_\Lambda$, are given by *hyper surfaces* (regions with $\dim < \mathrm{n}$). When these are defined by $F^k(x) = 0$, $k \in I_s$, where each $F^k(x)$ has the form (7), it is not necessary to require that the different δ^k, $k \in I_s$, have to be greater or equal to zero in Lemma 2, since this lemma holds despite the sign of these constants. However, if some switch surface cannot exactly be described by $F^k(x) = 0$, $k \in I_s$, then it is possible to conservatively describe such a switch surface by larger switch regions, cf. the discussion after (10). Note that a switch region described by both hyper surfaces $F^k(x) = 0$ and regions $F^k(x) \geq 0$, implies that we only state conditions that the different δ^k, $k \in I_s$, are greater or equal to zero for those k described by regions $F^k(x) \geq 0$.

LMI formulation

By assuming that the regions $\tilde{\Omega}_q$, $q \in I_\ell$, are given by $F^k(x) \geq 0$, $k \in I_s$, and $\tilde{\Lambda}_{qr}$, $(q,r) \in I_\Lambda$, are given by $F^k(x) = 0$, $k \in I_s$, and by replacing the conditions given in the different regions by conditions on the form (12), where subscript q corresponds to region $\tilde{\Omega}_q$ and qr corresponds to region $\tilde{\Lambda}_{qr}$, the following corollary is obtained:

Corollary 2 *Assume that the state trajectory evolves according to (2). If there exists $\tilde{P}_q$, $q \in I_\ell$, and constants $\alpha > 0$ and $\beta > 0$, and $\mu_q^k \geq 0$, $\nu_{qi}^k \geq 0$, and η_{qr}^k such that*

- $\tilde{\alpha} + \sum_{k=1}^{s_q}\mu_q^k\tilde{Q}_q^k \leq \tilde{P}_q \leq \tilde{\beta} - \sum_{k=1}^{s_q}\mu_q^k\tilde{Q}_q^k, \quad q \in I_\ell$
- $\tilde{A}_i^T\tilde{P}_q + \tilde{P}_q\tilde{A}_i + \sum_{k=1}^{s_{qi}}\nu_{qi}^k\tilde{Q}_{qi}^k \leq 0, \quad (q,i) \in I_\Omega,$
- $\tilde{P}_r \leq \tilde{P}_q - \sum_{k=1}^{s_{qr}}\eta_{qr}^k\tilde{Q}_{qr}^k, \quad (q,r) \in I_\Lambda$

then the equilibrium point 0 *is stable in the sense of Lyapunov.*

Corollary 2 gives stability conditions formulated as LMIs. Hence, it is possible to find the unknown variables in the corollary by efficient convex optimization algorithms [1, 3]. In some cases there is a search for all variables in the corollary, in other cases some of them are specified beforehand (for example $\mu_q^k = 0$) and the remaining unknowns are searched for. The flexibility is large since in general there is no unique, if possible, solution to the stability conditions.

Similar stability conditions for switched and hybrid systems stated as LMIs can be found in [8, 6, 9].

4.3 Robustness

Several robustness properties can be obtained when using Lemma 1. First, knowledge about the actual shape of the weighting functions $w_i(x)$, $i \in I_r$, is not utilized in the lemma; it concerns only about the support of the different weighting functions (3). Hence, if there exist local energy functions and class K functions that satisfies the lemma, the system is also stable for all other forms of the weighting functions with the same support. Secondly, even if the local linear system i only contributes to the vector field (2) in a strict subset of the region $\tilde{\Omega}_q$, the second condition of the lemma states that $\frac{\partial V_q}{\partial x}(A_ix + B_i) \leq 0$ has to be valid in the entire region $\tilde{\Omega}_q$. Hence, the support Ω_i of w_i may as well be extended to $\tilde{\Omega}_q$. These robustness properties may be useful when to conclude stability for systems where it is not possible to determine the exact shape and location of the weighting functions, which is the case when these are identified from an experiment.

5 Example

Consider (2) with the following three local linear systems:

$$A_1 = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The weighting functions are illustrated in Figure 4 for a specific value of x_1. Another value of x_1 only changes the location of the vertices of the weighting functions, see the figure. The corresponding support Ω_i, $i = 1, 2, 3$, of the weighting functions (3) overlap each other in the shaded regions. Note that this system may be interpreted as a fuzzy system.

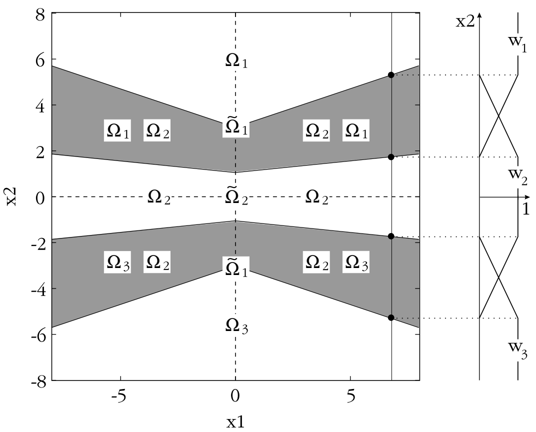

Figure 4: Weighting functions $w_i(x)$, $i = 1, 2, 3$, and the corresponding support Ω_i, $i = 1, 2, 3$. The state space is partitioned into two regions $\tilde{\Omega}_q$, $q = 1, 2$.

Figure 5 shows four trajectory simulations of the system. The system is stable which will be shown next.

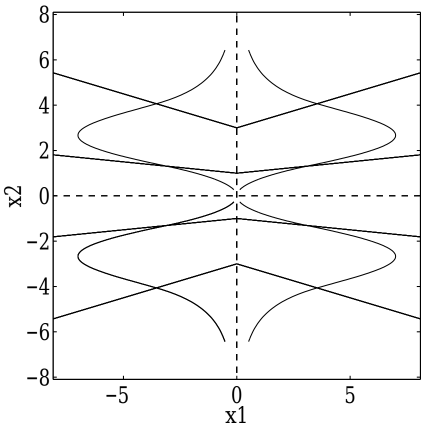

Figure 5: Four simulations of the system.

To show stability, we partitioning the state space into two regions according to Figure 4. With this partitioning we have

$$I_\Omega = \{(1,1),(1,2),(1,3),(2,2)\}$$

It should be pointed out that it is not possible to find a common Lyapunov function for the different local linear systems in the entire state space, since the local linear system 1 and 3 have unstable eigenvalues.

Stating the stability conditions in Corollary 2 for the partitioning in Figure 4, we find a solution with the optimization routine in [3]. The two obtained quadratic local Lyapunov functions are:

$$\tilde{P}_1 = \begin{bmatrix} 98.1 & -3.67\cdot 10^{-12} & -2.58\cdot 10^{-12} \\ -3.67\cdot 10^{-12} & 1.18\cdot 10^{5} & -1.57\cdot 10^{-11} \\ -2.58\cdot 10^{-12} & -1.57\cdot 10^{-11} & 2.58\cdot 10^{4} \end{bmatrix},$$
$$\tilde{P}_2 = \begin{bmatrix} 69.6 & -1.82\cdot 10^{-12} & 0 \\ -1.82\cdot 10^{-12} & 5.20\cdot 10^{4} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, note that the system is stable even if we change the actual shape and location of the weighting functions in the different regions $\tilde{\Omega}_q$, $q = 1, 2$, cf. the discussion in Section 4.3.

6 Conclusions

We have presented a constructive method for showing stability for nonlinear systems consisting of state-dependent weighted linear systems. By allowing the Lyapunov function to be discontinuous, we can utilize local Lyapunov functions and stability is ensured if these are joined properly in the state-space. In case of quadratic local Lyapunov functions, the stability conditions can be formulated as linear matrix inequalities (LMIs). This is attractive since they can be verified by computerized methods.

Acknowledgements: This work has been financially supported by the Swedish Research Council for Engineering Sciences (TFR) under the project number 92-185. Valuable discussions with Prof. Bo Egardt and Prof. Anders Rantzer are very much appreciated.

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