Macroeconomics II

Lecture Notes: Session 1

You can access this lecture note and codes in: https://github.com/joaoblf0/Macroeconomics-II

1 Notation

Let X_t be a stationary variable, we define:

- X^* : steady state value
- $\hat{x}_t \equiv \frac{X_t - X^*}{X^*}$: percent deviation from steady state
- $x_t \equiv \ln(X_t)$

This is the most usual notation, but sometimes texts use $x_t = \frac{X_t - X^*}{X^*}$.

2 First-order Approximations

If $X_t \approx X^*$

$$\hat{x}_t = \frac{X_t}{X_t^*} - 1 \Longrightarrow 1 + \hat{x}_t = \frac{X_t}{X_t^*}$$

Since $\hat{x}_t \approx 0$, then

$$\ln(1 + \hat{x}_t) \approx \hat{x}_t \Longrightarrow \hat{x}_t \approx \ln(X_t) - \ln(X_t^*)$$

Hint: Remember that $\varepsilon \approx 0 \Rightarrow \ln(1 + \varepsilon) = \varepsilon$

3 Taylor Expansion

Let $f(\boldsymbol{X}_t)$ with $\boldsymbol{X}_t \in \mathbb{R}^K$, the Taylor expansion of f around $\boldsymbol{X}^*$ is

$$f(\boldsymbol{X}_t) = f(\boldsymbol{X}^*) + \sum_{k=1}^{K} \frac{\partial f}{\partial X_k}\Big|_{\boldsymbol{X}_t = \boldsymbol{X}^*} (X_k - X_k^*)$$

4 Log-linearization Steps

1. Obtain the equation's steady state
2. Define the target variables

   Usually quantitative variables as output, consumption, capital etc. are defined as percent deviations ($\hat{x}_t$) and rates, as inflation or interest, as $\ln(x_t)$ (e.g. $\pi_t = \ln(\Pi_t) = \ln(1+\pi_t)$ for $\pi_t \approx 0$).
3. Apply natural logarithm in both sides
4. Apply first-order Taylor expansion in both sides
5. Cancel out steady-state terms and simplify the resulting equation

5 Examples

1. (Labor Supply) Log-linearize the expression bellow as function of $\hat{w}_t^R$, $\hat{c}_t$ and $\hat{l}_t$:

   $$W_t^R = C_t^\sigma L_t^\varphi$$

   Step 1: Equation in steady state

   $$W^R = C^\sigma L^\varphi$$

   Step 2: Target variables

   $$\hat{x}_t^R = \frac{1}{X^R}(X_t^R - X^R)$$

   Step 3: Apply natural log

   $$\ln(W_t^R) = \sigma \ln(C_t) + \varphi \ln(L_t)$$

   $$\ln(W^R) = \sigma \ln(C) + \varphi \ln(L)$$

   Step 4: Apply F.O. Taylor expansion

   $$\ln(W^R) + \frac{1}{W^R}(W_t^R - W^R) = \sigma \left[\ln(C) + \frac{1}{C}(C_t - C)\right] + \varphi \left[\ln(L) + \frac{1}{L}(L_t - L)\right]$$

   Step 5: Simplifying

   $$\hat{w}_t^R = \sigma \hat{c}_t + \varphi \hat{l}_t$$

2. (Capital Evolution) Log-linearize the expression bellow as function of $\hat{k}_t$, $\hat{k}_{t+1}$ and $\hat{i}_t$:

   $$K_{t+1} = (1-\delta)K_t + I_t$$

The equation in SS:

$$K = (1-\delta)K + I \Longrightarrow \frac{I}{K} = \delta$$

Applying ln:

$$\ln(K_{t+1}) = \ln[(1-\delta)K_t + I_t]$$

F.O. Taylor expansion:

$$\ln(K) + \frac{1}{K}(K_{t+1} - K) = \ln[(1-\delta)K + I] + \frac{1-\delta}{(1-\delta)K + I}(K_t - K) + \frac{1}{(1-\delta)K + I}(I_t - I)$$

Simplifying

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \delta\hat{i}_t$$

3. (Euler Equation) Log-linearize the expression bellow as function of r_t, $\hat{c}_t$, $\hat{c}_{t+1}$ and π_t. Consider $\Pi^* = 1$ and $\rho = \ln(1/\beta)$:

$$\beta R_t \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+1}} \right) \right] = 1$$

Let's rewrite the equation in the form:

$$\beta R_t \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{1}{\Pi_{t+1}} \right) \right] = 1$$

The equation in steady state:

$$\beta R = 1 \Longrightarrow R = \frac{1}{\beta}$$

Applying ln:

$$\ln(\beta) + \ln(R_t) + \mathbb{E}_t \left[-\sigma \left(\ln(C_{t+1}) - \ln(C_t) \right) - \ln(\Pi_{t+1}) \right] = 0$$

F.O. Taylor expansion:

$$\ln(\beta) + r_t - \sigma\mathbb{E}_t \left[\ln(C) + \frac{1}{C}(C_{t+1} - C) - \ln(C) - \frac{1}{C}(C_t - C) \right] - \mathbb{E}_t \left[\pi_{t+1} \right] = 0$$

Simplifying:

$$-\rho + r_t - \sigma\mathbb{E}_t \left[\hat{c}_{t+1} \right] + \sigma\hat{c}_t - \mathbb{E}_t \left[\pi_{t+1} \right] = 0$$

4. (Dixit–Stiglitz Aggregator) Log-linearize the expression bellow as function of $\hat{c}_t$ and $\hat{c}_{i,t}$:

$$C_t = \left(\int C_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

In steady state all items i have the same steady state, thus $C_i = C_j, \forall i, j$. The equation in steady state:

$$C = C_i$$

Applying ln:

$$\ln(C_t) = \frac{\varepsilon}{1-\varepsilon} \ln\left(\int C_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di\right)$$

F.O. Taylor:

$$\ln(C) + \frac{1}{C}(C_t - C) = \frac{\varepsilon}{1-\varepsilon}\frac{1-\varepsilon}{\varepsilon}\ln(C) + \frac{\varepsilon}{1-\varepsilon}\frac{1}{C^{\frac{\varepsilon-1}{\varepsilon}}}\left[\int \frac{\varepsilon-1}{\varepsilon}C^{-1/\varepsilon}(C_{i,t} - C)di\right]$$

Simplifying

$$\hat{c}_t = \frac{\varepsilon}{1-\varepsilon}\frac{1}{C^{\frac{\varepsilon-1}{\varepsilon}}}\left[\int \frac{\varepsilon-1}{\varepsilon}C^{-1/\varepsilon}(C_{i,t} - C)di\right]$$

$$\hat{c}_t = \int \hat{c}_{i,t} di$$

5. (Public Debt Dynamics) Log-linearize the expression bellow as function of $\hat{b}_t$, $\hat{b}_{t-1}$ $\hat{g}_t$ and r_{t-1} and assume $R^* = \frac{1}{\beta}$:

$$B_t = G_t + B_{t-1}R_{t-1}$$

Equation in steady state:

$$B = G + BR \Longrightarrow \frac{G}{B} = 1 - R$$

Applying ln:

$$\ln(B_t) = \ln(G_t + B_{t-1}R_{t-1})$$

F.O. Taylor expansion

$$\ln(B) + \hat{b}_t = \ln(G + BR) + \frac{1}{G+BR}[(G_t - G) + B(R_{t-1} - R) + R(B_{t-1} - B)]$$

Simplifying

$$\hat{b}_t = \left(1 - \frac{1}{\beta}\right)\hat{g}_t + r_{t-1} + \frac{1}{\beta}\hat{b}_{t-1} + 1 - \frac{1}{\beta}$$

6. (Consumption Basket) Log-linearize the expression bellow as function of $\hat{c}_t$, $\hat{c}_{H,t}$ and $\hat{c}_{F,t}$:

$$C_t = \left[(1-\alpha)^{\frac{1}{\eta}}(C_{H,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}(C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$\hat{c}_t = (1-\alpha)^{\frac{1}{\eta}} \left(\frac{C_H}{C}\right)^{\frac{\eta-1}{\eta}} \hat{c}_{H,t} + \alpha^{\frac{1}{\eta}} \left(\frac{C_F}{C}\right)^{\frac{\eta-1}{\eta}} \hat{c}_{F,t}$$

6 Blanchard-Khan Conditions and IRFs

6.1 Definitions

Jump variables: determined in current period, often in a forward-looking optimization process. Examples:

- C_t in: $\beta R_t \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \left(\frac{P_t}{P_{t+1}}\right) \right] = 1$
- π_t in $\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \kappa \hat{y}_t$

Predetermined variables: depend only on past values and actual shocks. Are determined once the jump variables are given. Examples:

- A_t in: $A_t = \rho_a A_{t-1} + \varepsilon_t^a$
- r_t in $r_t = \rho + \phi \pi_t + \varepsilon_t^m$
- K_{t+1} in $K_{t+1} = (1-\delta)K_t + I_t$ and household maximizes in C_t

6.2 Intuition

Let the predetermined variable A_t that follows $A_t = \rho_a A_{t-1} + \varepsilon_t^a$. We can iterate backwards:

$$A_t = \rho_a(\rho_a A_{t-2} + \varepsilon_{t-1}^a) + \varepsilon_t^a = \rho_a(\rho_a(\rho_a A_{t-3} + \varepsilon_{t-2}^a) + \varepsilon_{t-1}^a) + \varepsilon_t^a = \dots$$

Repeating until $t = 0$:

$$X_t = \rho_a^t X_0 + \sum_{k=0}^{t-1} \rho_a^k \varepsilon_{t-k}^a$$

Note that A_t have a stationary equilibrium if $|\rho_a| < 1$. **Predetermined variable $\rightarrow$ stable root** $|a| < 1$

Now let the jump variable X_t that follows: $\mathbb{E}_t[X_{t+1}] = bX_t + u_t$. We can iterate forward:

$$X_t = \frac{1}{b}\mathbb{E}_t[X_{t+1}] + \frac{1}{b}u_t = \frac{1}{b}\mathbb{E}_t\left[\frac{1}{b}\mathbb{E}_t[X_{t+2}] + \frac{1}{b}u_{t+1}\right] + \frac{1}{b}u_t = \dots$$

Repeating until $t = \infty$:

$$X_t = \lim_{T\to\infty}\left[\left(\frac{1}{b}\right)^{T}\mathbb{E}_t[X_T] - \sum_{k=0}^{T-1}\left(\frac{1}{b}\right)^{-(k+1)}\mathbb{E}_t[u_{t+k}]\right]$$

Note that X_t have a stationary equilibrium if $|1/b| < 1$ or $|b| > 1$. **Jump variable → "explosive" root** $|b| > 1$

6.3 Blanchard-Khan Conditions

Let a system of linear equations:

$$\begin{bmatrix} \boldsymbol{X}_{t+1} \\ \mathbb{E}_t[\boldsymbol{Y}_{t+1}] \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} \boldsymbol{X}_t \\ \boldsymbol{Y}_t \end{bmatrix} + B\boldsymbol{u}_t$$

- If matrix A has many explosive eigenvalues ($|\lambda| > 1$) than jump variables: unique stationary solution
- If matrix A has more explosive eigenvalues ($|\lambda| > 1$) than jump variables: no stationary solution
- If matrix A has few explosive eigenvalues ($|\lambda| > 1$) than jump variables: infinite stationary solutions

Note. If the system is written as $\boldsymbol{A}\begin{bmatrix} \boldsymbol{X}_{t+1} \\ \mathbb{E}_t[\boldsymbol{Y}_{t+1}] \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_t \\ \boldsymbol{Y}_t \end{bmatrix} + B\boldsymbol{u}_t$ use the opposite rules (explosive eigenvalues must be equal to predetermined variables).

6.4 Step-by-step Example: Taylor Principle

Let the NK system

$$\hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}] - \frac{1}{\sigma}(r_t - \mathbb{E}_t[\pi_{t+1}] - \rho)$$

$$\pi_t = \beta\mathbb{E}_t[\pi_{t+1}] + \kappa\hat{y}_t$$

$$r_t = \rho + \phi\pi_t + \varepsilon_t$$

We need to write the system in the form above. Plugging r_t in the first equation:

$$\hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}] - \frac{1}{\sigma}(\phi\pi_t + \varepsilon_t - \mathbb{E}_t[\pi_{t+1}])$$

$$\pi_t = \beta\mathbb{E}_t[\pi_{t+1}] + \kappa\hat{y}_t$$

Isolating $\mathbb{E}_t[\pi_{t+1}]$ in the second equation, plugging it in the first equation and rearranging we obtain:

$$\mathbb{E}_t[\hat{y}_{t+1}] = \left(\frac{\kappa + \sigma\beta}{\sigma\beta}\right)\hat{y}_t + \left(\frac{\beta\phi - 1}{\beta\sigma}\right)\pi_t + \frac{1}{\sigma}\varepsilon_t$$

$$\mathbb{E}_t[\pi_{t+1}] = -\frac{\kappa}{\beta}\hat{y}_t + \frac{1}{\beta}\pi_t$$

Writing in matrix form:

$$\mathbb{E}_t\begin{bmatrix}\hat{y}_{t+1}\\ \pi_{t+1}\end{bmatrix} = \begin{bmatrix}\frac{\kappa+\sigma\beta}{\sigma\beta} & \frac{\beta\phi-1}{\beta\sigma}\\ -\frac{\kappa}{\beta} & \frac{1}{\beta}\end{bmatrix}\begin{bmatrix}\hat{y}_t\\ \pi_t\end{bmatrix} + \begin{bmatrix}\frac{1}{\sigma}\\ 0\end{bmatrix}\varepsilon_t$$

Calculating the eigenvalues:

$$\lambda_1 = \frac{\kappa + \sigma + \beta\sigma + \sqrt{[\kappa + \sigma - \sigma\beta]^2 + 4\beta\kappa\sigma(\phi - 1)}}{2\beta\sigma} \qquad \lambda_2 = \frac{\kappa + \sigma + \beta\sigma - \sqrt{[\kappa + \sigma - \sigma\beta]^2 + 4\beta\kappa\sigma(\phi - 1)}}{2\beta\sigma}$$

It is clear that $\lambda_1 > 1$. Now, if $\phi > 1$ then

$$\sqrt{[\kappa + \sigma - \sigma\beta]^2 + 4\beta\kappa\sigma(\phi - 1)} > \sqrt{[\kappa + \sigma - \sigma\beta]^2} = \kappa + \sigma - \sigma\beta$$

leading to

$$\lambda_2 > \frac{\kappa + \sigma + \beta\sigma - (\kappa + \sigma - \sigma\beta)}{2\beta\sigma} = 1$$

Thus, as there is 2 jump variables $|\lambda_1|$ and $|\lambda_2|$ must be greater than 1, requiring $\phi > 1$.

6.5 BK Conditions Numerically

In practice is hard (and often impossible) to obtain analytical eigenvalues in most models, specially in large ones. We proceed directly with Sims (2002) method to solve linear models with rational expectations.

Let a linear equilibrium system:

$$\boldsymbol{\Gamma}_0\boldsymbol{Z}_t = \boldsymbol{\Gamma}_1\boldsymbol{Z}_{t-1} + \boldsymbol{\Psi}u_t + \boldsymbol{\Pi}\eta_t + \boldsymbol{Const}$$

Where $\boldsymbol{Z}_t$ is the variable vector, $\boldsymbol{u}_t$ is the shocks vector, $\boldsymbol{\eta}_t$ is the expectation errors vector ($\boldsymbol{Z}_t - E_{t-1}[\boldsymbol{Z}_t]$) and $\boldsymbol{Const}$ is the constant vectors.

The solution is given by a VAR:

$$\boldsymbol{Z}_t = \boldsymbol{G}_1\boldsymbol{Z}_{t-1} + \boldsymbol{Impact}\ u_t + \boldsymbol{C}$$

Returning to New-Keynesian model above with a persistence structure to u_t^m:

$$\begin{aligned}
\hat{y}_t &= \mathbb{E}_t[\hat{y}_{t+1}] - \frac{1}{\sigma}(r_t - \mathbb{E}_t[\pi_{t+1}] - \rho) \\
\pi_t &= \beta \mathbb{E}_t[\pi_{t+1}] + \kappa \hat{y}_t \\
r_t &= \rho + \phi \pi_t + u_t^m \\
u_t^m &= \rho_m u_{t-1}^m + \sigma_m \varepsilon_t
\end{aligned}$$

We can rewrite in the form below:

$$\begin{aligned}
\mathbb{E}_{t-1}[\hat{y}_t] + \frac{1}{\sigma}\mathbb{E}_{t-1}[\pi_t] &= \hat{y}_{t-1} + \frac{1}{\sigma} r_{t-1} - \frac{1}{\sigma}\rho \\
\beta \mathbb{E}_{t-1}[\pi_t] &= \pi_{t-1} - \kappa \hat{y}_{t-1} \\
r_t - \phi \pi_t - u_t^m &= \rho \\
u_t^m &= \rho_m u_{t-1}^m + \sigma_m \varepsilon_t
\end{aligned}$$

Let $\eta_{1,t} = \hat{y}_t - \mathbb{E}_{t-1}[\hat{y}_t]$ and $\eta_{2,t} = \pi_t - \mathbb{E}_{t-1}[\pi_t]$. We can write:

$$\begin{aligned}
\hat{y}_t + \frac{1}{\sigma}\pi_t &= \hat{y}_{t-1} + \frac{1}{\sigma} r_{t-1} - \frac{\rho}{\sigma} + \eta_{1,t} + \frac{1}{\sigma}\eta_{2,t} \\
\beta \pi_t &= \pi_{t-1} - \kappa \hat{y}_{t-1} + \beta \eta_{1,t} \\
r_t - \phi \pi_t - u_t^m &= \rho \\
u_t^m &= \rho_m u_{t-1}^m + \sigma_m \varepsilon_t
\end{aligned}$$

In matrix form:

$$\underbrace{\begin{bmatrix} 1 & \frac{1}{\sigma} & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & -\phi & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{\Gamma}_0} \underbrace{\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \\ r_t \\ u_t^m \end{bmatrix}}_{\boldsymbol{Z}_t} = \underbrace{\begin{bmatrix} 1 & 0 & \frac{1}{\sigma} & 0 \\ -\kappa & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_m \end{bmatrix}}_{\mathbf{\Gamma}_1} \underbrace{\begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ r_{t-1} \\ u_{t-1}^m \end{bmatrix}}_{\boldsymbol{Z}_{t-1}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma_m \end{bmatrix}}_{\mathbf{\Psi}} \underbrace{\begin{bmatrix} \varepsilon_t^m \end{bmatrix}}_{\boldsymbol{u}_t} + \underbrace{\begin{bmatrix} 1 & \frac{1}{\sigma} \\ \beta & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{\Pi}} \underbrace{\begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix}}_{\boldsymbol{\eta}_t} + \underbrace{\begin{bmatrix} -\frac{\rho}{\sigma} \\ 0 \\ \rho \\ 0 \end{bmatrix}}_{\boldsymbol{Const}}$$

Plugging in the Gensys code we obtain the matrices $\boldsymbol{G}_1$, $\boldsymbol{Impact}$ and $\boldsymbol{C}$ such that

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \\ r_t \\ u_t^m \end{bmatrix} = \boldsymbol{G}_1 \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ r_{t-1} \\ u_{t-1}^m \end{bmatrix} + \boldsymbol{Impact}\ [\varepsilon_t^m] + \boldsymbol{C}$$

We also obtain an array “eu” indicating the Blanchard-Khan conditions:

- $[1, 1]$: one stationary equilibrium
- $[1, 0]$: infinite stationary equilibriums
- $[0, 1]$: no stationary equilibrium
- $[-2, -2]$: coincident zeros (redundant equations)

7 Exercises

1. Verify if the Blanchard-Khan conditions for equilibrium existence and uniqueness are satisfied and plot the IFRs to ε_t^m shock in the model above. Define $\beta = 0.99$, $\sigma = 5$, $\kappa = 1.2$, $\phi = 1.5$, $\rho = 0.05$, $\rho_m = 0.5$ and $\sigma_m = 0.025$.

   Run the codes with the parameters above in Matlab, you should obtain $eu = [1\ 1]$, indicating unique stable equilibrium. IRFs should indicate a temporary drop in output and inflation in response to contractionist shock.

2. Now, define $\phi = 0.9$, verify the Blanchard-Khan conditions. Explain.

   Run the codes with the parameters above in Matlab, you should obtain $eu = [1\ 0]$, indicating infinite stable equilibrium, it occurs because now monetary policy is "too weak" to control the inflation, thus monetary authority could not ensure that prices will follow the desirable path, allowing other equilibrium with different price levels.

3. Now, return $\phi = 1.5$ and define $\rho_m = 1.1$, verify the Blanchard-Khan conditions. Explain.

   Run the codes with the parameters above in Matlab, you should obtain $eu = [0\ 1]$, indicating no stable equilibrium, it occurs because now monetary shock is not stationary and any impact from ε_t will be amplified ahead, generating an explosive solution.

4. Simulate item 1 using Dynare. Consider the same system of equations given in section 6.5.