GEOPHYSICAL INSTITUTE - UNIVERSITY OF BERGEN

GEOF211
Spring 2023

Numerical Modelling

Intro

João Bettencourt 01.02.2023

UNIVERSITY OF BERGEN

GEOF211 course topics

1. The three classes of partial differential equations, and the division of diagnostic and prognostic equations.
2. The finite difference method (FDM).
3. The questions of consistency, numerical stability, and convergence.
4. The analytical solution of one-dimensional advection with constant speed, t-x-diagram and characteristics.

GEOF211 course topics

5. FDM formulations of the advection problem: truncation error, stability, damping and phase.
6. Application of FDM to the wave equation.
7. Non-linear equations and non-linear instability, the model resolution, aliasing.
8. Transport error.
9. The oscillation equation, explicit versus implicit treatment.

GEOF211 course topics

10. Shallow water gravity waves, staggered space grids and staggered time grids.
11. Semi-Lagrangian and semi-implicit methods.
12. Arakawa grids and geostrophic adjustment.
13. Total variation diminishing (TVD) schemes for advection.
14. Diffusive initial value problems.
15. Relaxation method for solving boundary value problems.

Course staff

João Bettencourt (GFI)

Øyvind Breivik (Met)

Knut Barthel (GFI)

Course info

- Bi-weekly lectures:
  - Tuesdays 08:15 – 10:00
  - Fridays 14:15 – 16:00
  - Room 234 (GFI building, 2nd floor)
- Assessment:
  - Portfolio assessment (5 assignments during the term)
  - Grading A to F.
- Practical work sessions:
  - mandatory to attend 80%
  - announced 7-14 days in advance

Model Tasks and practical sessions:

- Tentative dates:
  - Week 5: Model task 1 handed out (3 week deadline)
  - Week 8: Model task 2 handed out (3 week deadline)
  - Week 10: Model task 4 (5 week deadline)
  - Week 13: Model task 6 (4 week deadline)
  - Week 16: Model task 33 (3 week deadline)
- Model tasks: deliver short report and code (w/ comments)
- Practical work sessions during lecture hours (tba in advance)

Reading list

- Haltiner, G.J., Williams, R.T., 1980. ***Numerical prediction and dynamic meteorology***, 2d ed. ed. Wiley, New York.
- Press, W.H. (Ed.), 1996. ***FORTRAN numerical recipes***, 2nd ed. ed. Cambridge University Press, Cambridge [England] ; New York.
- Cushman-Roisin, B., Beckers, J.-M., 2011. ***Introduction to geophysical fluid dynamics: physical and numerical aspects***, 2nd ed. ed, International geophysics series. Academic Press, Waltham, MA.
- Selected ***papers from the literature on numerical modelling*** (mostly informative)

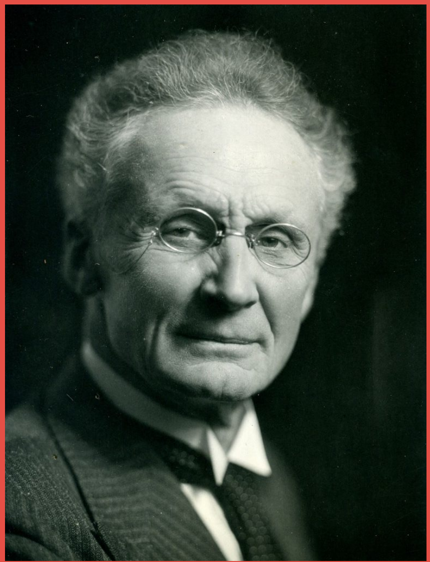
More information

- mitt.uib.no
- https://www.uib.no/en/course/GEOF211

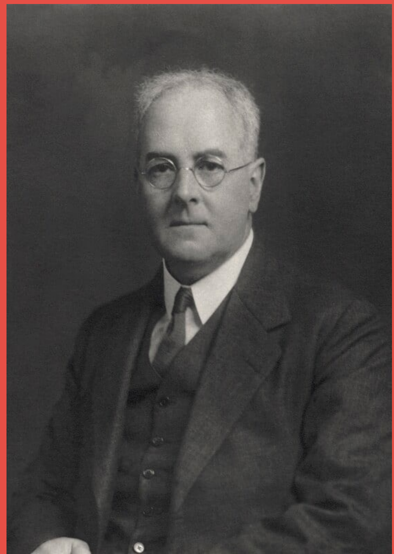
Why numerical modelling?

Most laws of nature are expressed as ***partial differential equations that cannot be solved analytically****. We need numerical methods to find solutions to the equations that* ***allows us to make useful forecasts****.*

Numerical weather forecasting

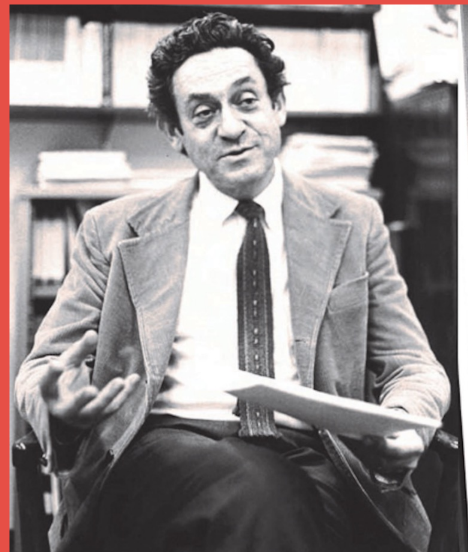


Vilhelm Bjerknes
(1862–1951)



Lewis Fry Richardson
(1881–1953)

John von Neumann
(1903-1957)



Jule Charney
(1917–1981)

Numerical weather forecasting

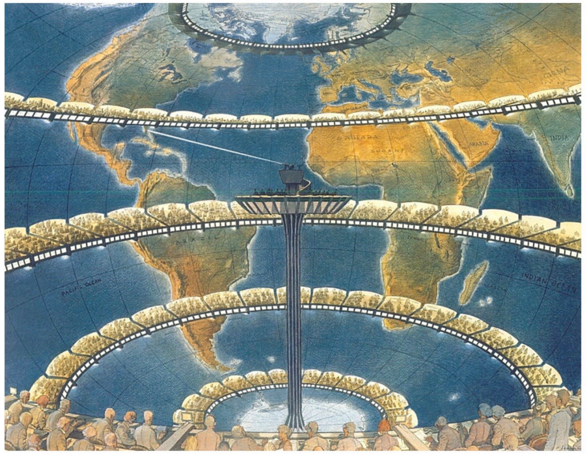


Fig. 5. An artist's impression of Richardson's Forecast Factory (© François Schuiten).

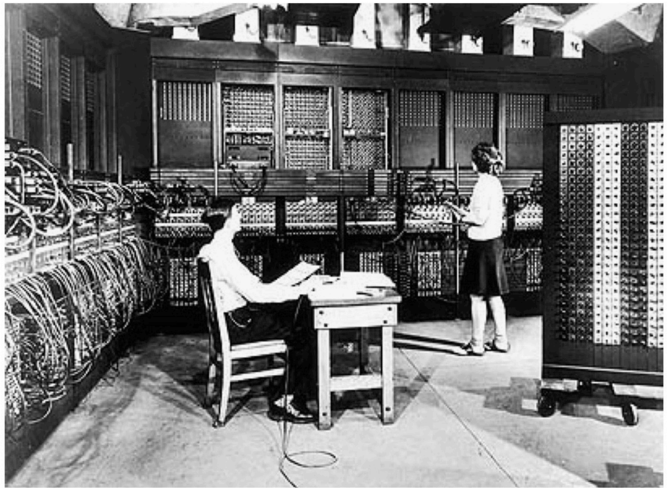
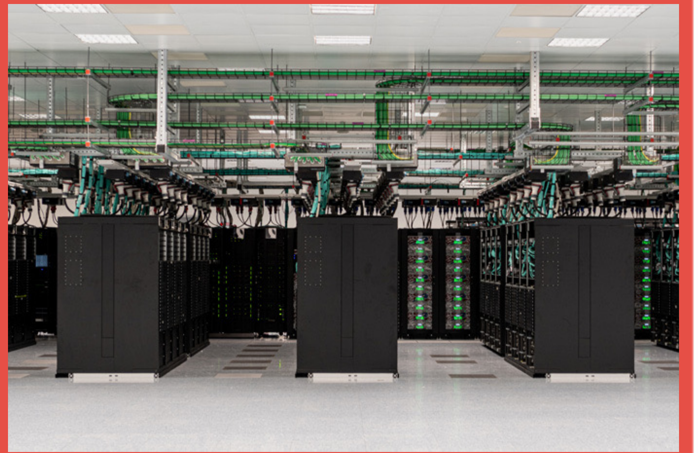


Fig. 3. The Electronic Numerical Integrator and Computer (ENIAC).



"Richardson's fantasy is an early example of Massively Parallel Processing. Each computer is responsible for a specific gridpoint, receives information required for calculation at that point and passes to neighboring computers the data required by them. Such message passing and memory distribution are features of modern machines, such as the HPCF in use at the European Centre. ***But, at peak computational power, the HPCF is about 15 orders of magnitude faster than a single human computer, and equivalent in purely number-crunching terms to about 10 billion of Richardson's Forecast Factories****."* *(Lynch, 2008)*

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Forecast skill improvement:

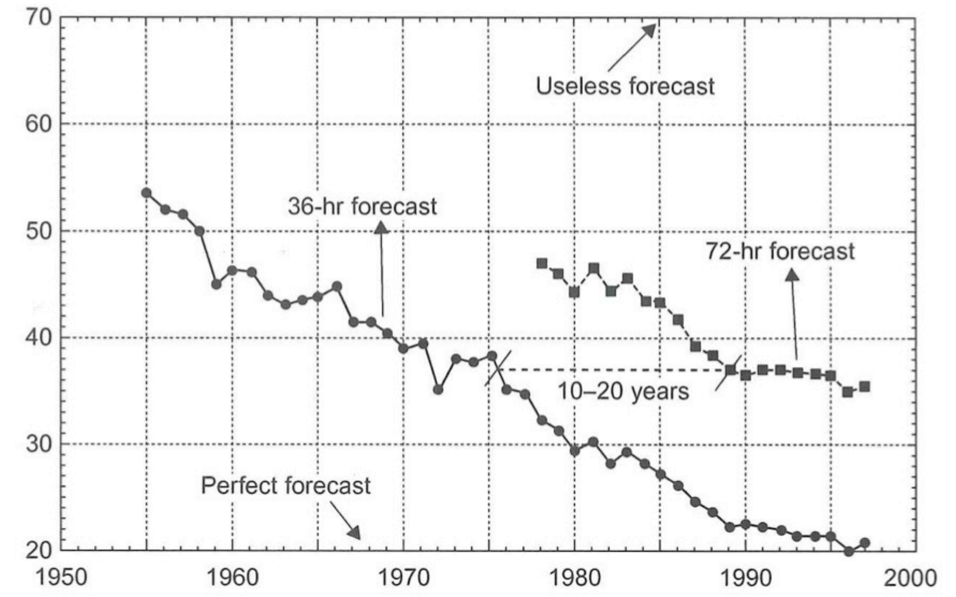
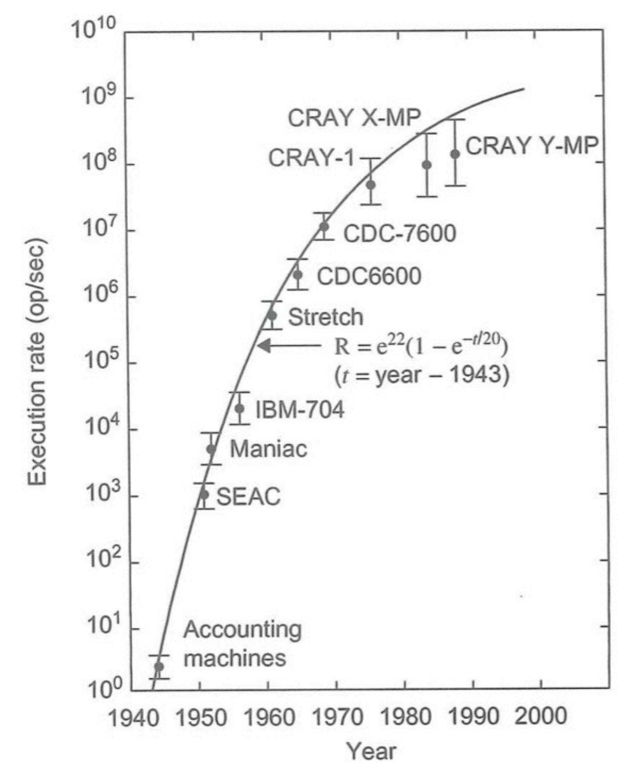


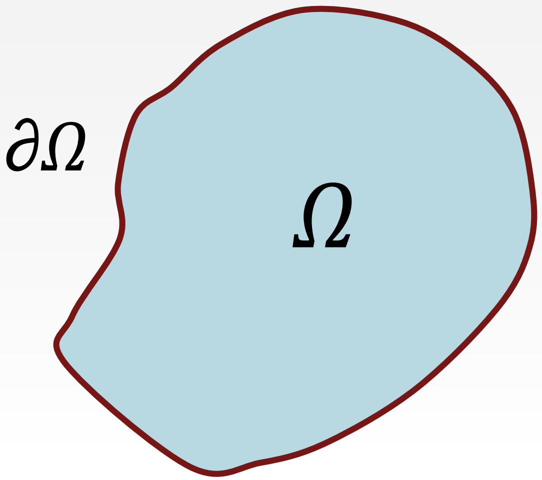
FIGURE 1.9 Historical improvement of weather forecasting skill over North America. The S1 score shown here is a measure of the relative error in the pressure gradient predictions at mid-height in the troposphere. (*From Kalnay, Lord & McPherson, 1998, reproduction with the kind permission of the American Meteorological Society*)



A k^{th} order partial differential equation (PDE) is an expression involving a function u(x), its derivatives $D^{\alpha}u(x)$ and (possibly) x:

(1) $$F\left(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0, \;\; x \in \Omega \subset \mathbb{R}^n$$

PDE domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$



1. $u(x) : \Omega \rightarrow \mathbb{R}$ is the solution if it satisfies (1)
2. Boundary conditions applied on $\partial\Omega$
3. Initial conditions applied in Ω at t=0

A k^{th} order partial differential equation (PDE) is an expression involving a function u(x), its derivatives $D^{\alpha}u(x)$ and (possibly) x:

(1) $$F\left(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0,\ \ x \in \Omega \subset \mathbb{R}^n$$

Gradient vector: $$Du = \left(u_{x_1}, \ldots, u_{x_n}\right)$$

Hessian matrix: $$D^2 u = \begin{pmatrix} u_{x_1 x_1} & \cdots & u_{x_1 x_n} \\ \vdots & \ddots & \vdots \\ u_{x_n x_1} & \cdots & u_{x_n x_n} \end{pmatrix}$$

A k^{th} order partial differential equation (PDE) is an expression involving a function u(x), its derivatives $D^{\alpha}u(x)$ and (possibly) x:

(1) $$F\left(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x\right) = 0, \;\; x \in \Omega \subset \mathbb{R}^n$$

PDE Classification I:

i. Order: is the order of the highest derivative that occurs in the PDE

a) First-order: $u_t + c u_x = 0$

b) Second-order: $u_{tt} = a^2 u_{xx}$

A k[th] order partial differential equation (PDE) is an expression involving a function u(x), its derivatives $D^{\alpha}u(x)$ and (possibly) x:

(1) $$F\left(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x\right) = 0, \; x \in \Omega \subset \mathbb{R}^n$$

PDE Classification II:

i. Linear: $\sum_{\alpha=1}^{k} a_{\alpha}(x) D^{\alpha} u + b(x) u = f(x)$

ii. Semilinear: $a_k(x) D^k u + F_0\left(D^{k-1} u, \dots, Du, u, x\right) = 0$

iii. Quasi-linear: $a_k\left(D^{k-1} u, \dots, Du, u, x\right) D^k u + F_0\left(D^{k-1} u, \dots, Du, u, x\right) = 0$

iv. Fully non-linear PDE: if neither i, nor ii nor iii.

A k^{th} order partial differential equation (PDE) is an expression involving a function u(x), its derivatives $D^{\alpha}u(x)$ and (possibly) x:

(1) $$F\left(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x\right) = 0,\ \ x \in \Omega \subset \mathbb{R}^n$$

<u>PDE Classification III:</u>

Linear 2nd order 2D PDEs:

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0$$

<u>where *a,b,c,d,e,f,g* are real and (possibly) functions of (x,y)</u>, are classified by the value of

$$D_\lambda = b^2 - 4ac$$

A k^{th} order partial differential equation (PDE) is an expression involving a function u(x), its derivatives $D^{\alpha}u(x)$ and (possibly) x:

(1) $$F\left(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0, \;\; x \in \Omega \subset \mathbb{R}^n$$

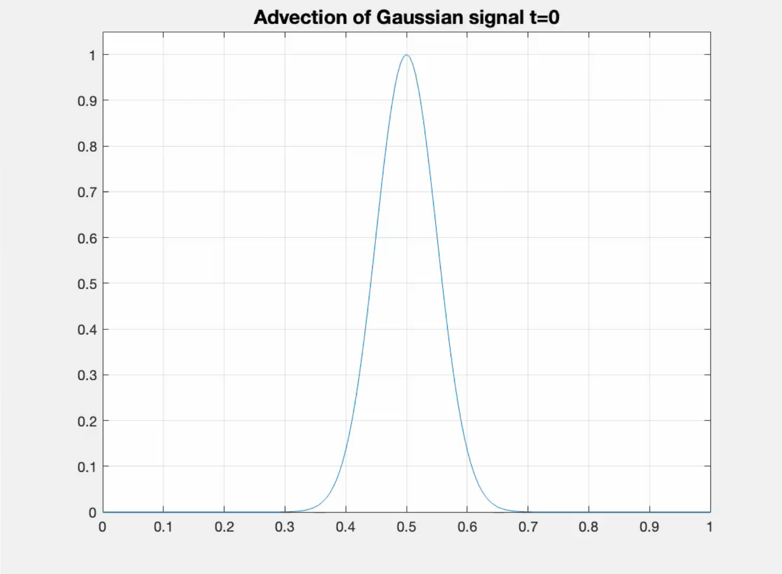
PDE Classification III:

i. $D_\lambda<0$: **elliptic** (PDEs that describe processes that have already reached steady state, and hence are time-independent)
ii. $D_\lambda>0$: **hyperbolic** (PDEs that describe time-dependent, conservative processes, such as convection, that are not evolving toward steady state)
iii. $D_\lambda=0$: **parabolic** (PDEs that describe time-dependent, dissipative processes, such as diffusion, that are evolving toward steady state)

PDE Example:

Linear advection equation: $u_t + cu_x = 0$

Solution in x ∈ [0,1] w/ periodic b.c. and u(x,0) = u_0: $u(x,t) = u_0(x - ct)$



ECMWF IFS atmospheric model (Cy47r3):

The momentum equations are

$$\frac{\partial U}{\partial t}+\frac{1}{a\cos^2\theta}\left\{U\frac{\partial U}{\partial \lambda}+V\cos\theta\frac{\partial U}{\partial\theta}\right\}+\dot{\eta}\frac{\partial U}{\partial\eta}-fV+\frac{1}{a}\left\{\frac{\partial\phi}{\partial\lambda}+R_{\mathrm{dry}}T_{\mathrm{v}}\frac{\partial}{\partial\lambda}(\ln p)\right\}=P_U+K_U \qquad (2.1)$$

$$\frac{\partial V}{\partial t}+\frac{1}{a\cos^2\theta}\left\{U\frac{\partial V}{\partial \lambda}+V\cos\theta\frac{\partial V}{\partial\theta}+\sin\theta(U^2+V^2)\right\}+\dot{\eta}\frac{\partial V}{\partial\eta}$$
$$+fU+\frac{\cos\theta}{a}\left\{\frac{\partial\phi}{\partial\theta}+R_{\mathrm{dry}}T_{\mathrm{v}}\frac{\partial}{\partial\theta}(\ln p)\right\}=P_V+K_V \qquad (2.2)$$

where a is the radius of the earth, $\dot{\eta}$ is the η-coordinate vertical velocity ($\dot{\eta}=\mathrm{d}\eta/\mathrm{d}t$), ϕ is geopotential, R_{dry} is the gas constant for dry air, and T_{v} is the virtual temperature defined by

$$T_{\mathrm{v}}=T[1+\{(R_{\mathrm{vap}}/R_{\mathrm{dry}})-1\}q-\sum_k q_k]$$

The thermodynamic equation is

$$\frac{\partial T}{\partial t}+\frac{1}{a\cos^2\theta}\left\{U\frac{\partial T}{\partial \lambda}+V\cos\theta\frac{\partial T}{\partial\theta}\right\}+\dot{\eta}\frac{\partial T}{\partial\eta}-\frac{\kappa T_{\mathrm{v}}\omega}{(1+(\delta-1)q)p}=P_T+K_T \qquad (2.3)$$

where $\kappa=R_{\mathrm{dry}}/c_{p_{\mathrm{dry}}}$ (with $c_{p_{\mathrm{dry}}}$ the specific heat of dry air at constant pressure), ω is the pressure-coordinate vertical velocity ($\omega=\mathrm{d}p/\mathrm{d}t$), and $\delta=c_{p_{\mathrm{vap}}}/c_{p_{\mathrm{dry}}}$ (with $c_{p_{\mathrm{vap}}}$ the specific heat of water vapour at constant pressure).

The moisture equation is

$$\frac{\partial q}{\partial t}+\frac{1}{a\cos^2\theta}\left\{U\frac{\partial q}{\partial \lambda}+V\cos\theta\frac{\partial q}{\partial\theta}\right\}+\dot{\eta}\frac{\partial q}{\partial\eta}=P_q+K_q \qquad (2.4)$$

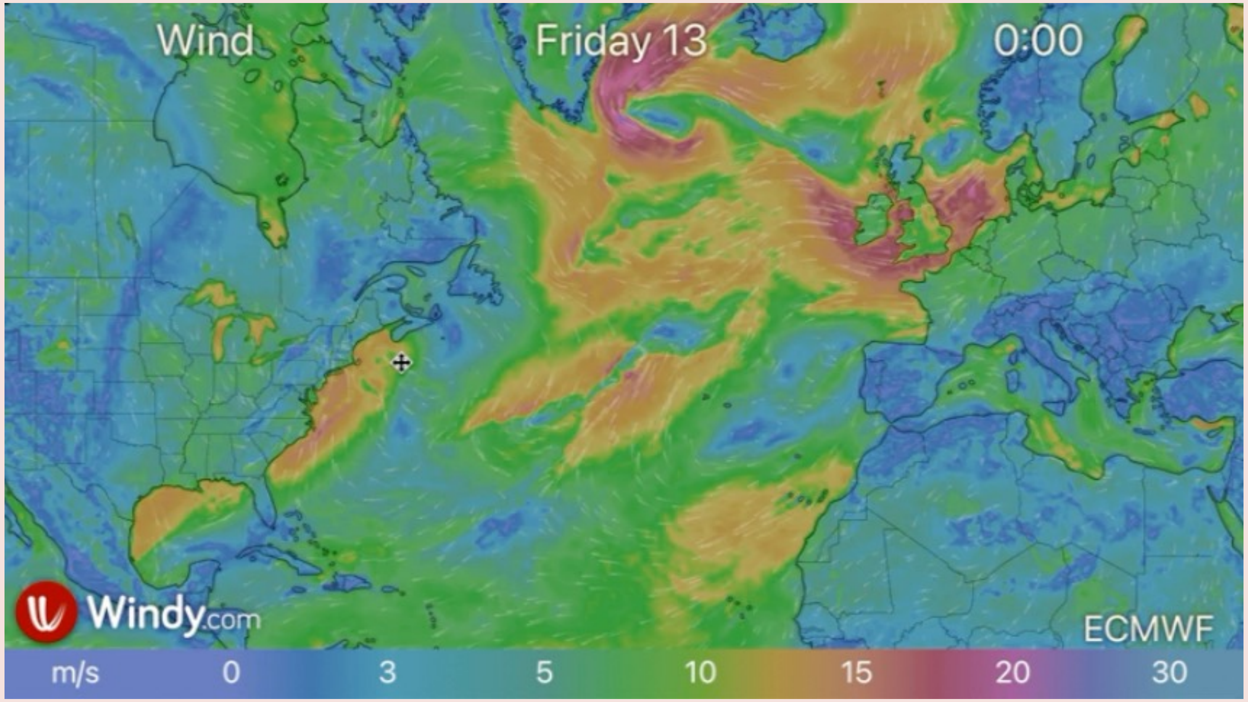
In (2.2) and (2.3), P_T and P_q represent the contributions of the parameterised physical processes, while K_T and K_q are the horizontal diffusion terms.

The continuity equation is

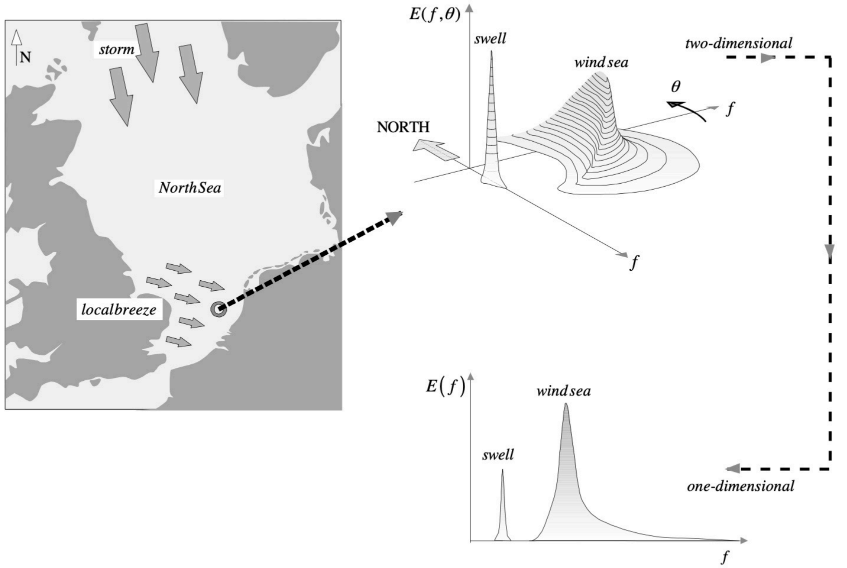
$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial\eta}\right)+\nabla\cdot\left(\mathbf{v}_{\mathrm{H}}\frac{\partial p}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left(\dot{\eta}\frac{\partial p}{\partial\eta}\right)=0 \qquad (2.5)$$

where ∇ is the horizontal gradient operator in spherical coordinates and $\mathbf{v}_{\mathrm{H}}=(u,v)$ is the horizontal wind.

ECMWF IFS atmospheric model (Cy47r3) surface wind speed



ECMWF wave model (Cy47r3):



Wave action *N=E(f)/f*

mbining previous results of this chapter, the action balance equation becomes

$$\frac{\partial}{\partial t}N + (\cos\phi)^{-1}\frac{\partial}{\partial \phi}(\dot{\phi}\cos\phi N) + \frac{\partial}{\partial \lambda}(\dot{\lambda}N) + \frac{\partial}{\partial \omega}(\dot{\omega}N) + \frac{\partial}{\partial \theta}(\dot{\theta}N) = S \qquad (2.24)$$

$$\dot{\phi} = (c_g\cos\theta - \mathbf{U}|_{\text{north}})R^{-1} \qquad (2.25a)$$

$$\dot{\lambda} = (c_g\sin\theta - \mathbf{U}|_{\text{east}})(R\cos\phi)^{-1} \qquad (2.25b)$$

$$\dot{\theta} = c_g\sin\theta\tan\phi R^{-1} + \dot{\theta}_D \qquad (2.25c)$$

$$\dot{\omega} = \partial\Omega/\partial t \qquad (2.25d)$$

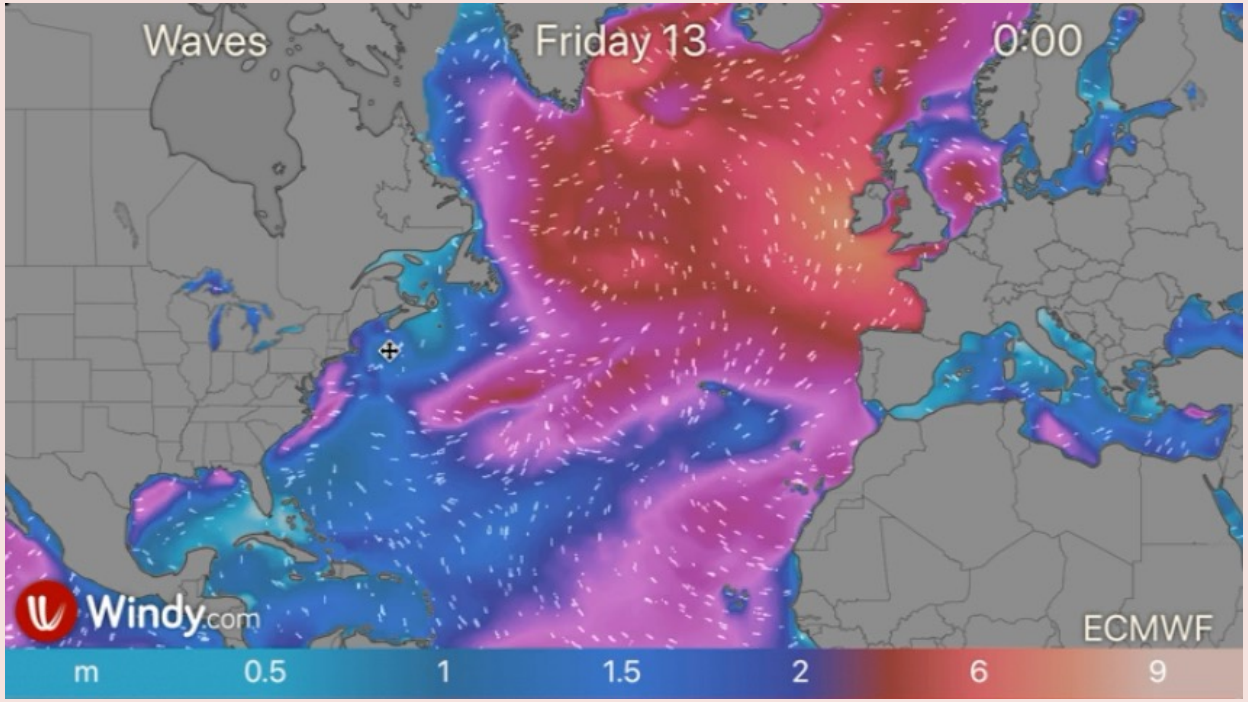
$$\dot{\theta}_D = \left(\sin\theta\frac{\partial}{\partial\phi}\Omega - \frac{\cos\theta}{\cos\phi}\frac{\partial}{\partial\lambda}\Omega\right)(kR)^{-1} \qquad (2.26)$$

and Ω is the dispersion relation given in (2.4). Before discussing possible numerical schemes to approximate the left-hand side of (2.24) we shall first discuss the parametrization of the source term S, where S is given by

$$S = S_{\text{in}} + S_{\text{nl}} + S_{\text{ds}} + S_{\text{bot}} \qquad (2.27)$$

These terms represent the physics of wind input, wave–wave interactions, dissipation due to whitecapping, and bottom friction.

ECMWF wave model (Cy47r3) significant wave height Hs



NEMO Ocean Model (v3.6):

in situ density. The vector invariant form of the primitive equations in the (**i**,**j**,**k**) vector system provides the following six equations (namely the momentum balance, the hydrostatic equilibrium, the incompressibility equation, the heat and salt conservation equations and an equation of state):

$$\frac{\partial \mathbf{U}_h}{\partial t} = -\left[(\nabla \times \mathbf{U}) \times \mathbf{U} + \frac{1}{2} \nabla \left(\mathbf{U}^2 \right) \right]_h - f\, \mathbf{k} \times \mathbf{U}_h - \frac{1}{\rho_o} \nabla_h p + \mathbf{D}^{\mathbf{U}} + \mathbf{F}^{\mathbf{U}} \tag{2.1a}$$

$$\frac{\partial p}{\partial z} = -\rho\, g \tag{2.1b}$$

$$\nabla \cdot \mathbf{U} = 0$$

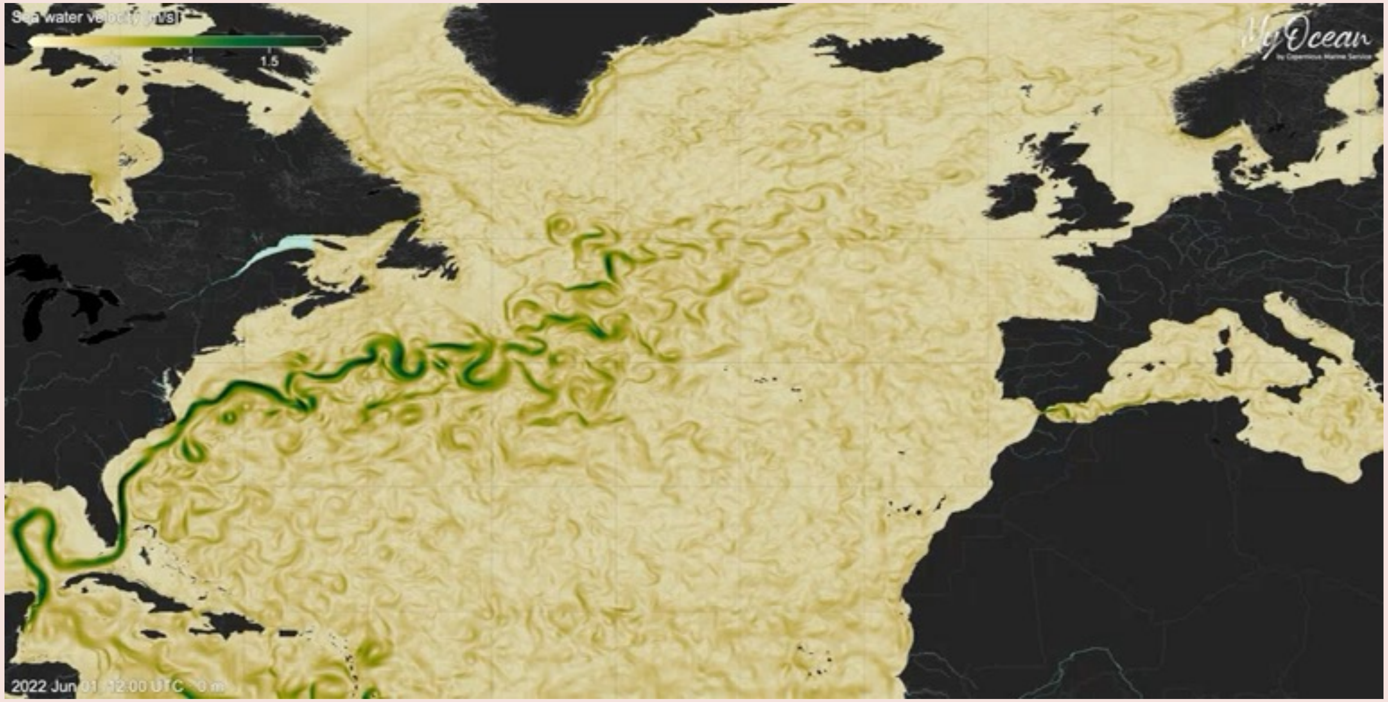
$$\frac{\partial T}{\partial t} = -\nabla \cdot (T\, \mathbf{U}) + D^T + F^T \tag{2.1d}$$

$$\frac{\partial S}{\partial t} = -\nabla \cdot (S\, \mathbf{U}) + D^S + F^S \tag{2.1e}$$

$$\rho = \rho\, (T, S, p) \tag{2.1f}$$

where ∇ is the generalised derivative vector operator in $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ directions, t is the time, z is the vertical coordinate, ρ is the *in situ* density given by the equation of state (2.1f), ρ_o is a reference density, p the pressure, $f = 2\mathbf{\Omega} \cdot \mathbf{k}$ is the Coriolis acceleration (where $\mathbf{\Omega}$ is the Earth's angular velocity vector), and g is the gravitational acceleration. $\mathbf{D}^{\mathbf{U}}$, D^T and D^S are the parameterisations of small-scale physics for momentum, temperature and salinity, and $\mathbf{F}^{\mathbf{U}}$, F^T and F^S surface forcing terms. Their nature and formulation are discussed in §2.5 and page §2.1.2.

NEMO Ocean Model (v3.6) surface currents



Atmosphere vs. Ocean:

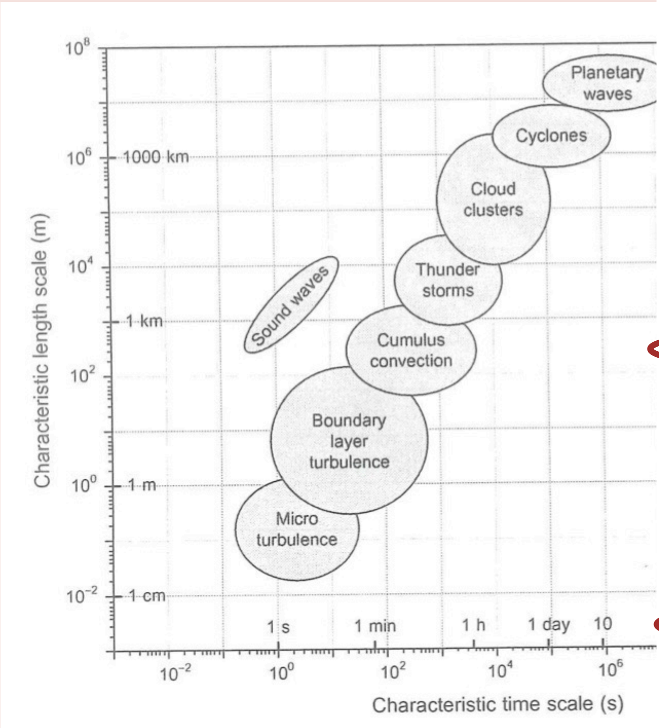
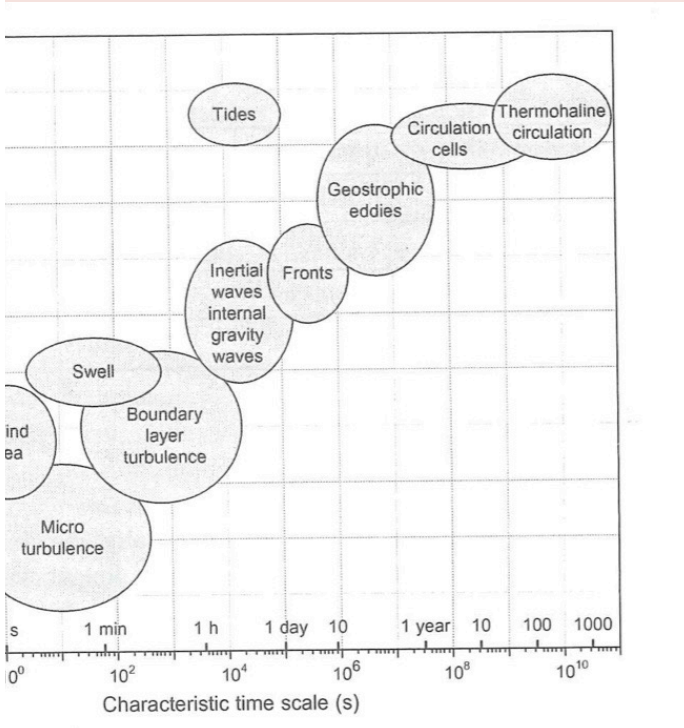


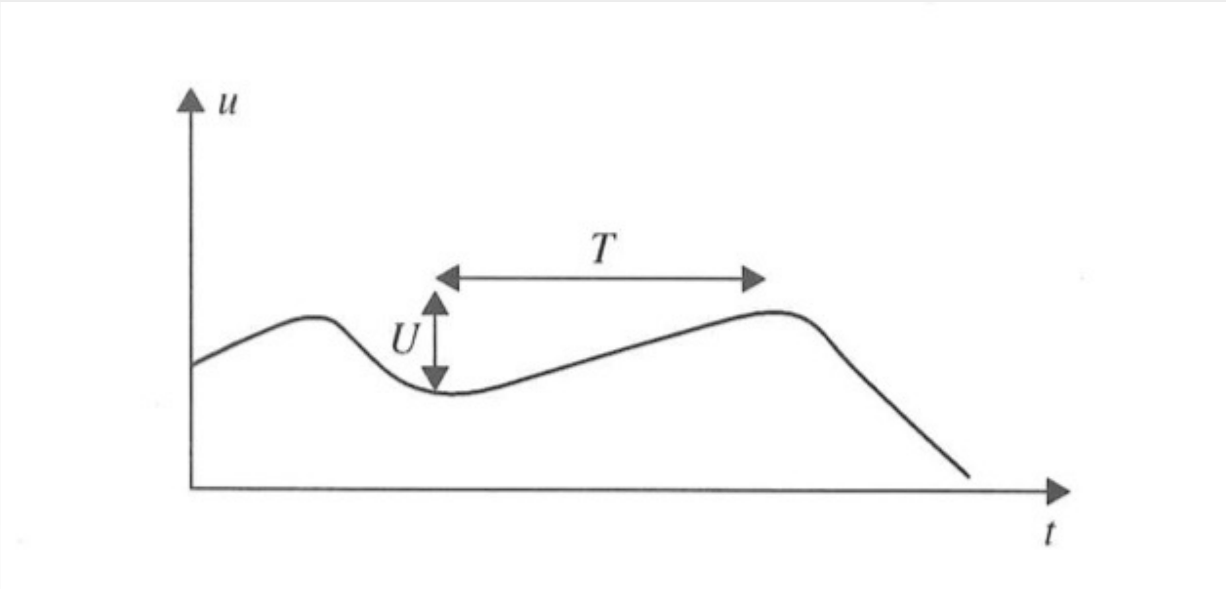
TABLE 1.2 Length, Velocity and Time Scales in the Earth's Atmosphere and Oceans

Phenomenon	Length Scale *L*	Velocity Scale *U*	Timescale *T*
Atmosphere			
Microturbulence	10–100 cm	5–50 cm/s	few seconds
Thunderstorms	few km	1–10 m/s	few hours
Sea breeze	5–50 km	1–10 m/s	6 h
Tornado	10–500 m	30–100 m/s	10–60 min
Hurricane	300–500 km	30–60 m/s	Days to weeks
Mountain waves	10–100 km	1–20 m/s	Days
Weather patterns	100–5000 km	1–50 m/s	Days to weeks
Prevailing winds	Global	5–50 m/s	Seasons to years
Climatic variations	Global	1–50 m/s	Decades and beyond
Ocean			
Microturbulence	1–100 cm	1–10 cm/s	10–100 s
Internal waves	1–20 km	0.05–0.5 m/s	Minutes to hours
Tides	Basin scale	1–100 m/s	Hours
Coastal upwelling	1–10 km	0.1–1 m/s	Several days
Fronts	1–20 km	0.5–5 m/s	Few days
Eddies	5–100 km	0.1–1 m/s	Days to weeks
Major currents	50–500 km	0.5–2 m/s	Weeks to seasons
Large-scale gyres	Basin scale	0.01–0.1 m/s	Decades and beyond



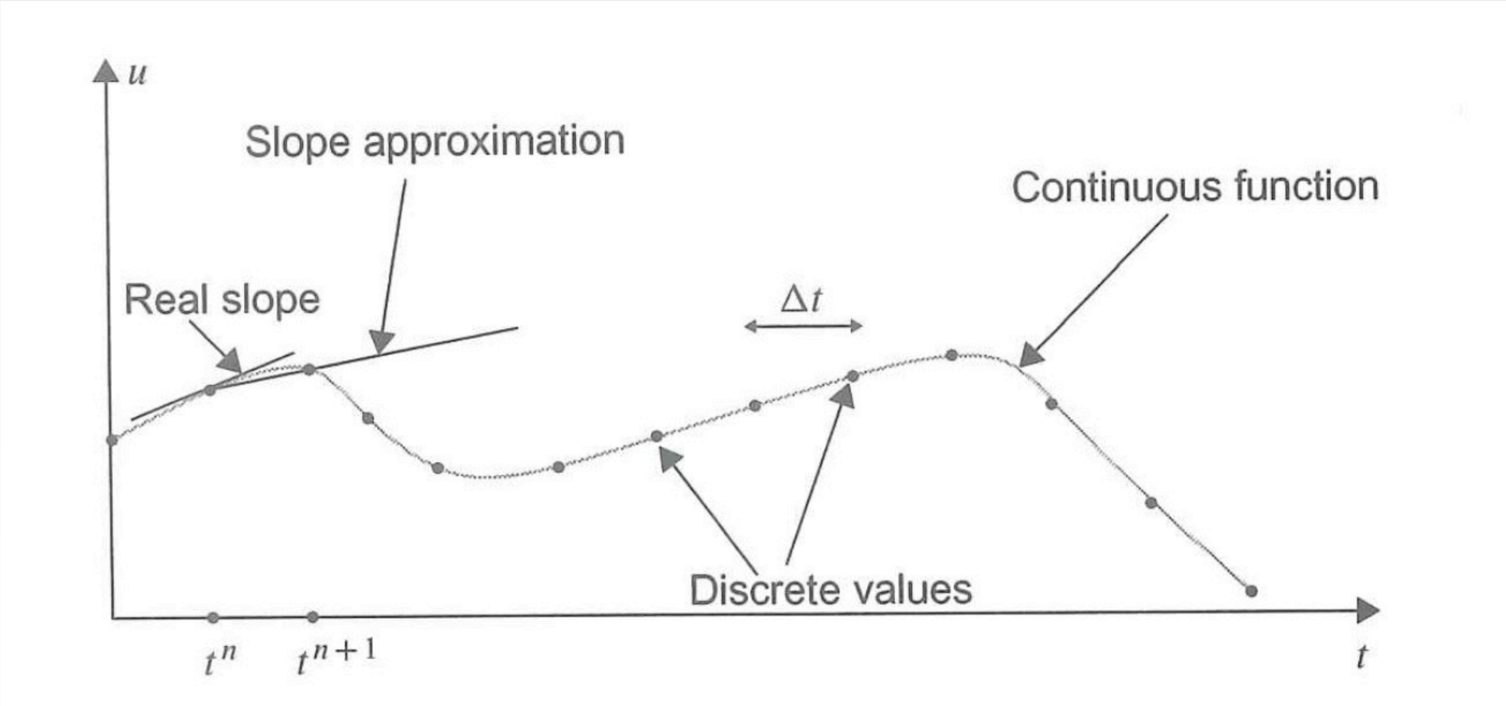
In general, oceanic motions are slower and smaller than atmospheric motions

Time and space scales



Determined by fluid properties (e.g. density), external properties and geometric constraints. In turn they determine the properties of the numerical method used.

Discretization

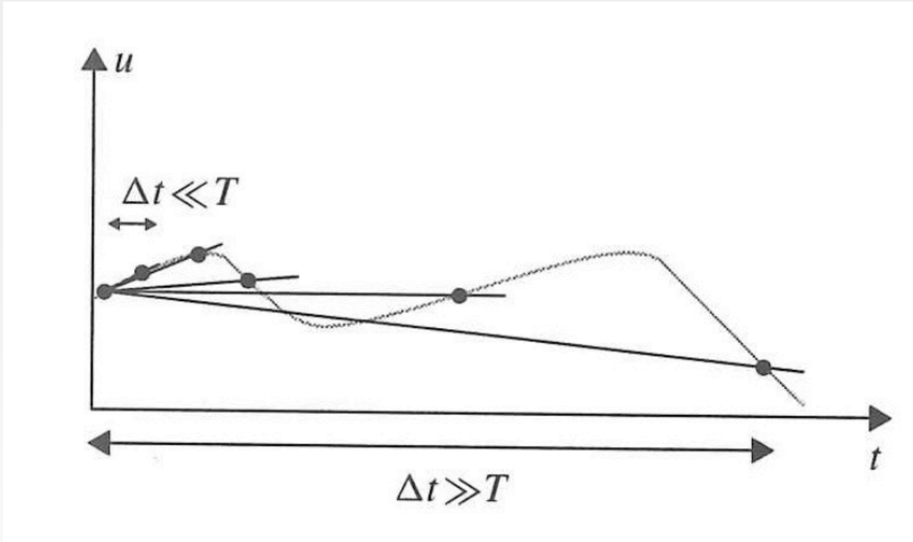


$t^n = t^0 + n\Delta t$, $n = 1, 2, \ldots$ Slope $du/dt \sim (u^{n+1} - u^n)/(t^{n+1} - t^n)$

Discretization

- $\Delta t << T$
- $\Delta x << L = U*T$
- Courant – Friderichs – Levy (CFL) number: $c\Delta t/\Delta x < 1$

Distance travelled during one time step ($c\Delta t$) should be smaller than the spatial step size (Δx)



GEOF211
Spring 2023

Numerical Modelling

Finite Differences

The finite difference method (FDM)

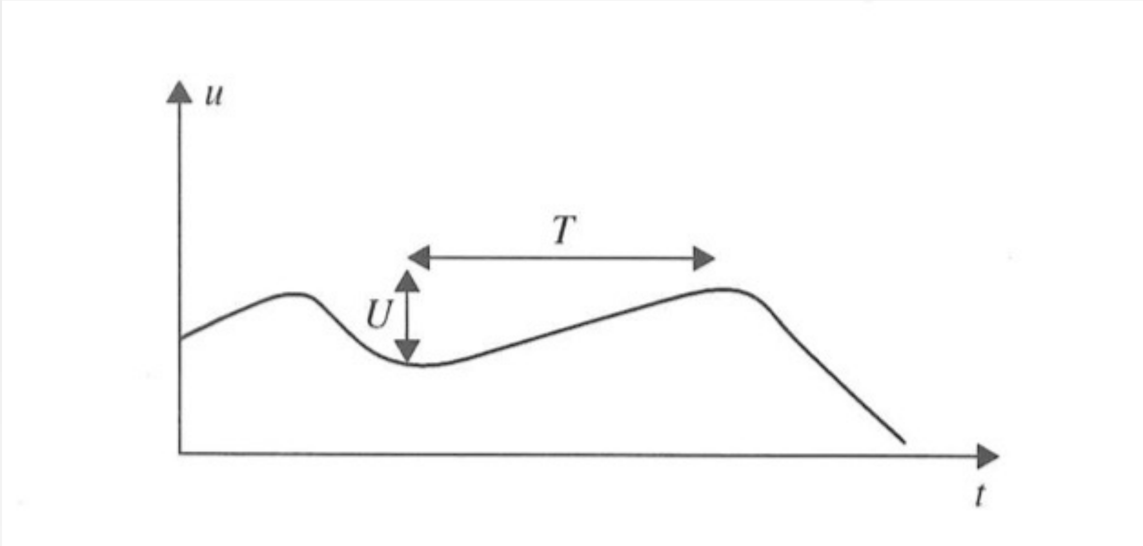
To solve numerically a PDE for the unknown function u(x,t) we **approximate all derivatives by finite differences**.

We use uniform partitions of the domain in time and space, i.e:

$x_i = x_0 + i\Delta x$, i=0,1,….,M
$t^n = t^0 + n\Delta t$, n=0,1,….,N.

Now consider the Taylor Series expansion of an analytical function *u(x)*:

$$u(x \pm \Delta x) = u(x) + \sum_{k=1}^{\infty} \frac{(\pm 1)^k \Delta x^k}{k!} \frac{\partial^k u}{\partial x^k} = u(x) \pm \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{3!} \frac{\partial^3 u}{\partial x^3} + \cdots$$

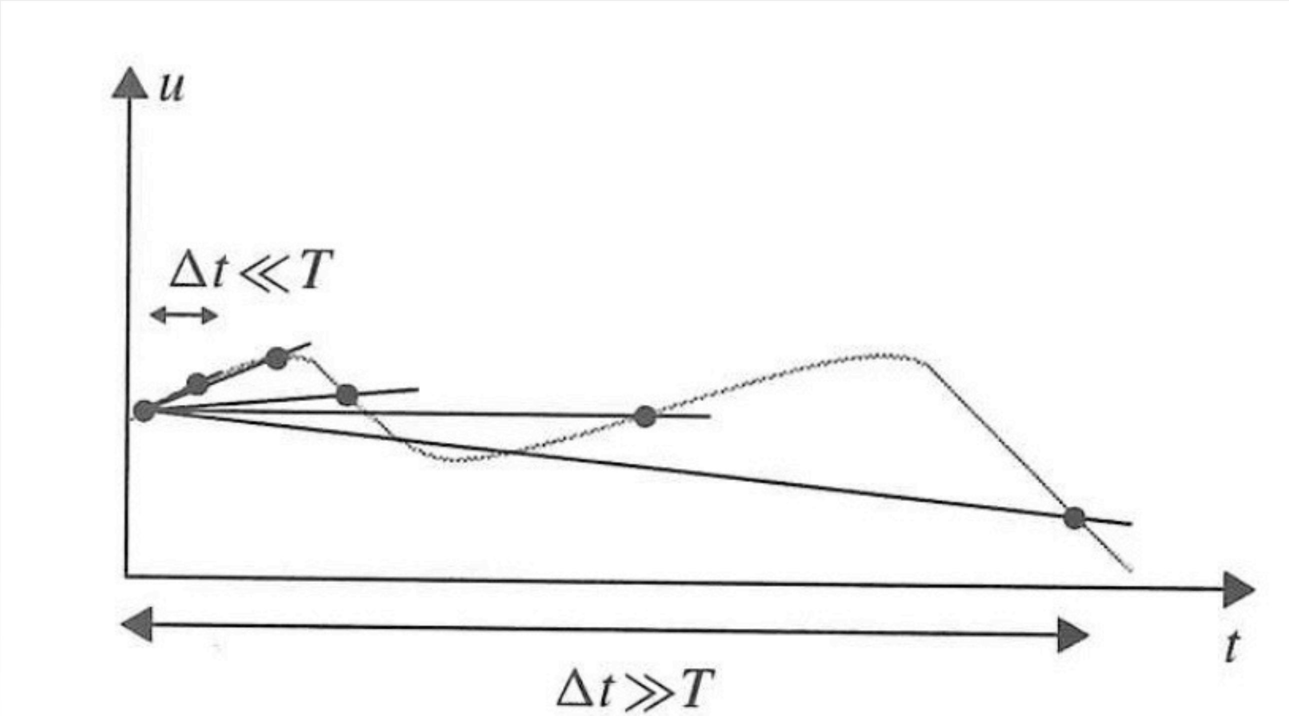


The magnitude of the time/space derivatives of u can be estimated by the magnitude of the variation of u during the characteristic time/space scale of u:

$$\frac{du}{dt} \sim \frac{U}{T}, \ \frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) \sim \frac{U}{T^2}, \ \ldots$$

Discretization

- $\Delta t << T$
- $\Delta x << L = U^*T$



1st order finite difference formulas for the 1st derivative

Forward:

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^{n} - u_i^n}{\Delta x} + O(\Delta x)$$

Backward:

$$\frac{\partial u}{\partial t} = \frac{u_i^{n} - u_i^{n-1}}{\Delta t} + O(\Delta t)$$

$$\frac{\partial u}{\partial x} = \frac{u_i^{n} - u_{i-1}^n}{\Delta x} + O(\Delta x)$$

2nd order finite difference formulas for the 1st derivative

Taylor series expansions of u(x) for x±Δx:

$$u(x+\Delta x) = u(x) + \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

$$u(x-\Delta x) = u(x) - \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

Subtracting and rearranging:

$$u(x+\Delta x) - u(x-\Delta x) = 2\Delta x\frac{\partial u}{\partial x} + 2\frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

$$\frac{\partial u}{\partial x} = \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} + \frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

2nd order finite difference formulas for the 1st derivative

Centred:

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + O(\Delta t^2)$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2\Delta x} + O(\Delta x^2)$$

Errors of FD formulas for du/dt:

$$u(t) = U \sin \omega t, \ \omega = \frac{2\pi}{T}$$

$$\frac{du}{dt} = U\omega \cos \omega t$$

Forward:

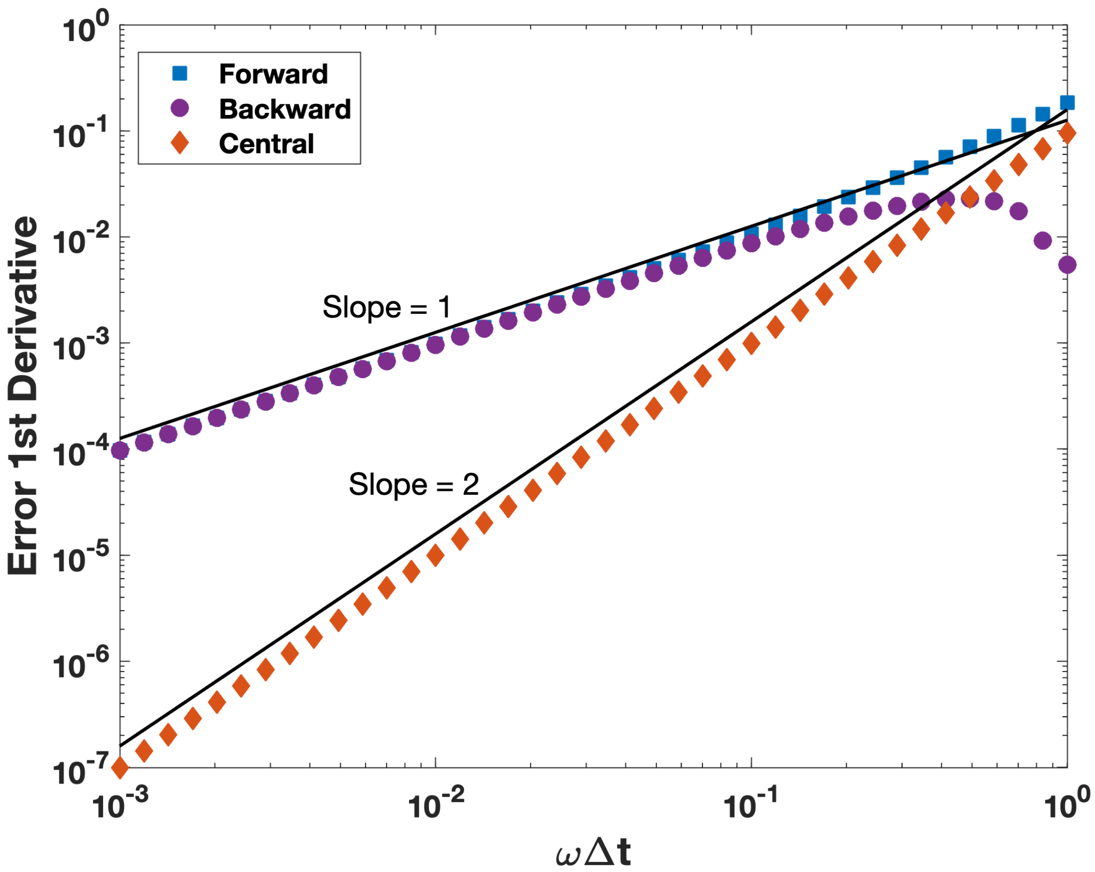
$$\frac{du}{dt} = \frac{u^{n+1} - u^n}{\Delta t} + O(\Delta t)$$

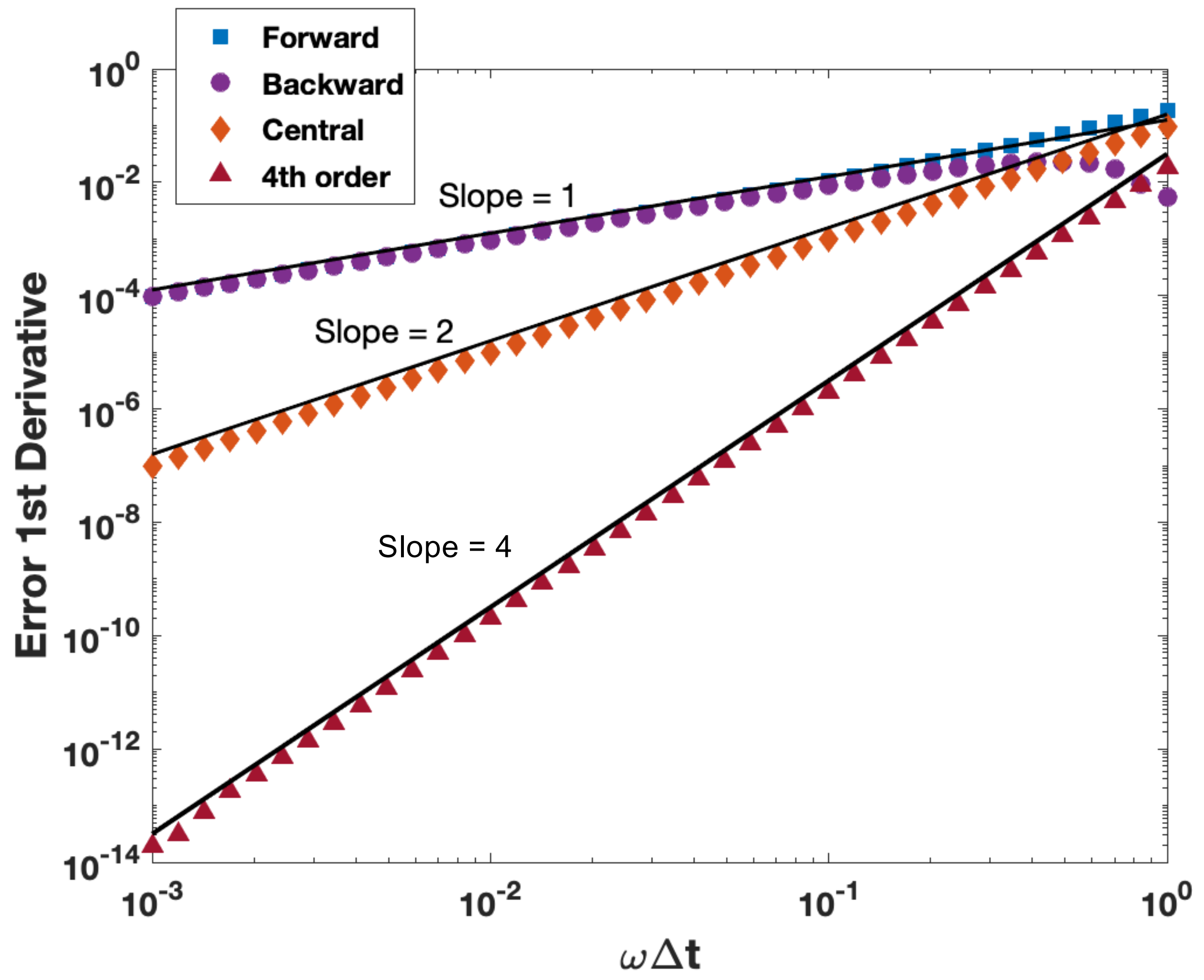
Backward:

$$\frac{du}{dt} = \frac{u^n - u^{n-1}}{\Delta t} + O(\Delta t)$$

Central:

$$\frac{du}{dt} = \frac{u^{n+1} - u^{n-1}}{2\Delta t} + O(\Delta t^2)$$




Forward
Backward
Central
4th order
Slope = 1
Slope = 2
Slope = 4
Error 1st Derivative
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-3}
10^{-2}
10^{-1}
10^{0}
$\omega \Delta t$

Finite difference formula for the 2nd derivative

Taylor series expansions of u(t) for t±Δt:

$$u(t+\Delta t) = u(t) + \Delta t \frac{du}{dt} + \frac{\Delta t^2}{2!}\frac{d^2u}{dt^2} + \frac{\Delta t^3}{3!}\frac{d^3u}{dt^3} + \frac{\Delta t^4}{4!}\frac{d^4u}{dt^4}$$

$$u(t-\Delta t) = u(t) - \Delta t \frac{du}{dt} + \frac{\Delta t^2}{2!}\frac{d^2u}{dt^2} - \frac{\Delta t^3}{3!}\frac{d^3u}{dt^3} + \frac{\Delta t^4}{4!}\frac{d^4u}{dt^4}$$

Sum and rearrange for d^2u/dt^2:

$$u(t+\Delta t) + u(t-\Delta t) = 2u(t) + \Delta t^2 \frac{d^2u}{dt^2} + \frac{\Delta t^4}{12}\frac{d^4u}{dt^4}$$

$$\frac{d^2u}{dt^2} = \frac{u(t+\Delta t) - 2u(t) + u(t-\Delta t)}{\Delta t^2} + \frac{\Delta t^2}{12}\frac{d^4u}{dt^4} + \cdots$$

Generalized method to obtain FD formulas:

Obtain the p^{th} order derivative at t^n using $u^{n-m},\ldots,u^n,\ldots,u^{n+m}$:

$$\frac{d^p u}{dt^p} = a_{-m}u^{n-m} + \cdots + a_0 u^n + \cdots + a_m u^{n+m}$$

by determination of the unknown coefficients a_q:

1. Express u^{n+q} ($q=-m,\ldots,m$) as a Taylor series expansion of u around u^n:

$$u^{n+q} = u^n + q\Delta t\frac{du}{dt} + q^2\Delta t^2\frac{d^2u}{dt^2} + \cdots + q^p\frac{\Delta t^p}{p!}\frac{d^p u}{dt^p} + O(\Delta t^{p+1})$$

2. Group the terms according to the order of the derivative in which they appear.

Generalized method to obtain FD formulas:

1. Express u^{n+q} (q=-m,…,m) as a Taylor series expansion of u around u^n:

$$u^{n+q} = u^n + q\Delta t \frac{du}{dt} + q^2 \Delta t^2 \frac{d^2u}{dt^2} + \cdots + q^p \frac{\Delta t^p}{p!} \frac{d^p u}{dt^p} + O(\Delta t^{p+1})$$

2. Group the terms according to the order of the derivative in which they appear.
3. Require that:
    1. The sum of coefficients a_q multiplying all derivatives of order $k<p$ be zero
    2. The sum of coefficients a_q multiplting the derivative of order p be unity
4. Solve the resulting system to obtain the values of a_q (q=-m,…,m)

5. This only works if there are 2m+1 ≥ p+1 points, e.g we must have 2m ≥ p

Readings

- Cushman-Roisin chapter 1 sec. 1.10 and 1.11
- Cushman-Roisin appendix C.2 for a table of FD formulas
- Finite-difference formula calculator: https://web.media.mit.edu/~crtaylor/calculator.html

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