The Lax-Wendroff Scheme

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The Lax-Wendroff scheme starts from the linear 1-d advection equation:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}, \tag{1}$$

and uses the Taylor series expansion of $u(x, t + \Delta t) = u(x_m, t^{n+1}) = u_m^{n+1}$:

$$u_m^{n+1} = u_m^n + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^3). \tag{2}$$

If we replace the time derivatives in (2) by the spatial derivative from (1), we get[1]:

$$u_m^{n+1} = u_m^n + \Delta t(-c\frac{\partial u}{\partial x}) + \frac{\Delta t^2}{2}c^2\frac{\partial^2 u}{\partial x^2} + O(\Delta t^3). \tag{3}$$

Now we can approximate the spatial derivatives with centred differences to obtain the Lax-Wendroff scheme:

$$u_m^{n+1} = u_m^n - c\frac{\Delta t}{2\Delta x}(u_{m+1}^n - u_{m-1}^n) + c^2\frac{\Delta t^2}{\Delta x^2}(u_{m+1}^n - 2u_m^n + u_{m-1}^n), \tag{4}$$

where we recognize that the last term in (4) is a dissipative term added to (1) to control what would otherwise be an unstable Euler method. To find out the order of the Lax-Wendroff scheme, we can return to (3) and rewrite it as:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = -c\frac{\partial u}{\partial x} + \frac{\Delta t}{2}c^2\frac{\partial^2 u}{\partial x^2} + O(\Delta t^2),$$

where we then replace the spatial derivatives with their second-order approximations:

$$\begin{aligned}\frac{u_m^{n+1} - u_m^n}{\Delta t} = -c\left[\frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} + O(\Delta x^2)\right] + \\ + \frac{\Delta t}{2}c^2\left[\frac{u_{m+1}^n - 2 * u_m^n - u_{m-1}^n}{\Delta x^2} + O(\Delta x^2)\right] + O(\Delta t^2),\end{aligned}$$

to find that the scheme is second order in space as in time.

[1] Note that from (1) we have: $\frac{\partial^2 u}{\partial t^2} = -c\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = -c\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = -c\frac{\partial}{\partial x}\left(-c\frac{\partial u}{\partial x}\right) = c^2\frac{\partial^2 u}{\partial x^2}$