Probability Theory Refresher (and a few more things)

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Lisboa, **Portugal**

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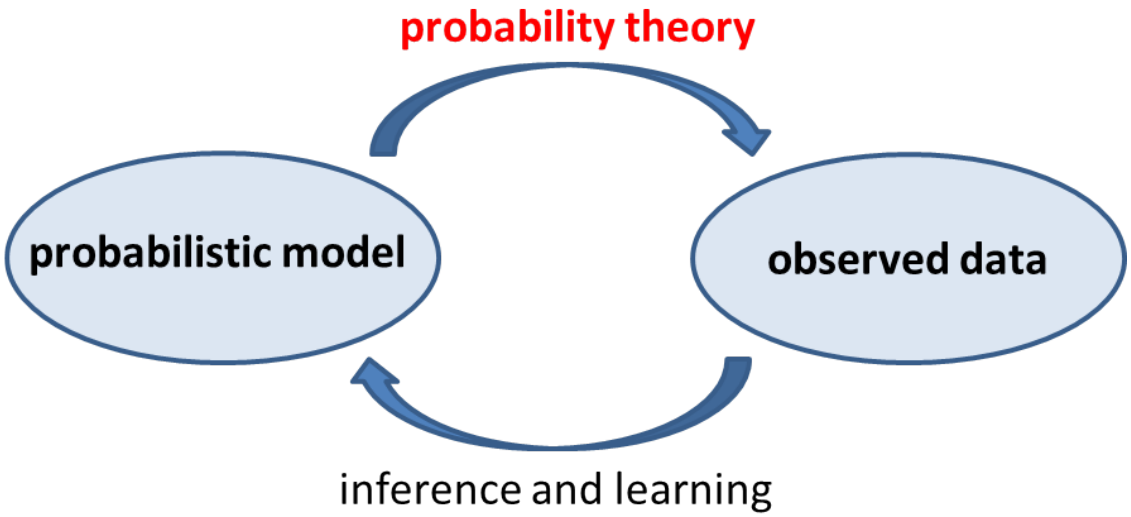
Probability theory



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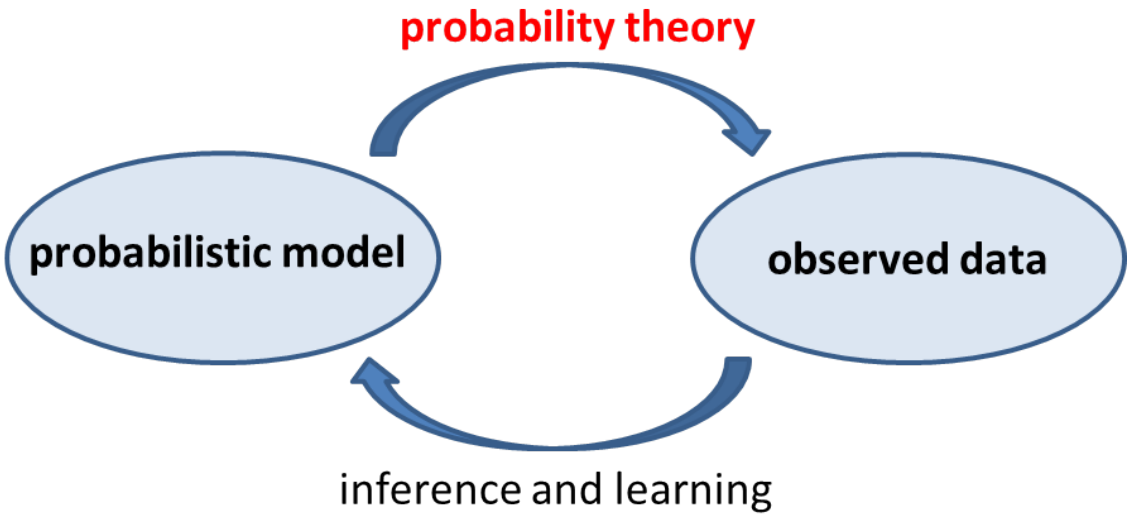

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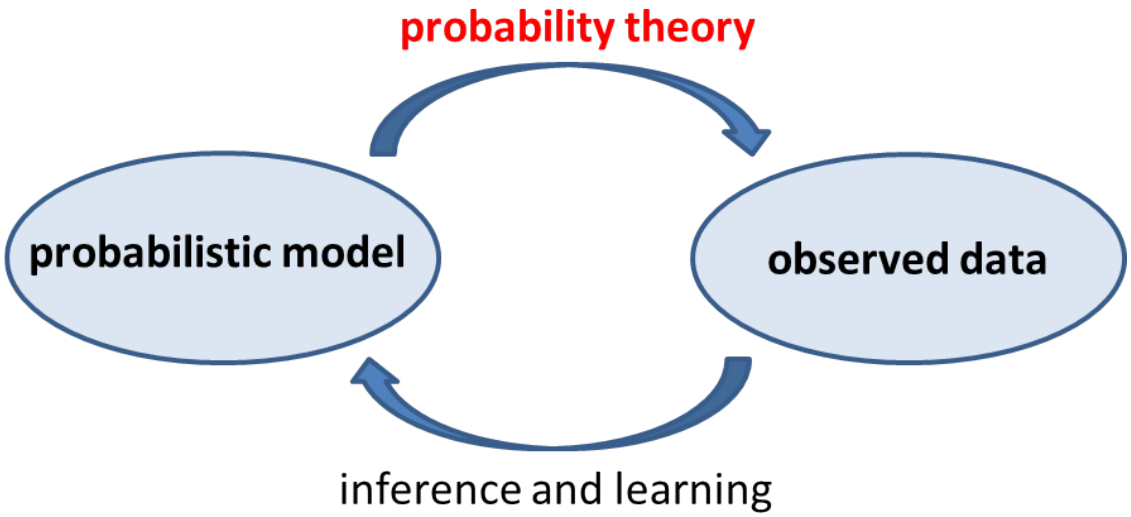
Probability theory



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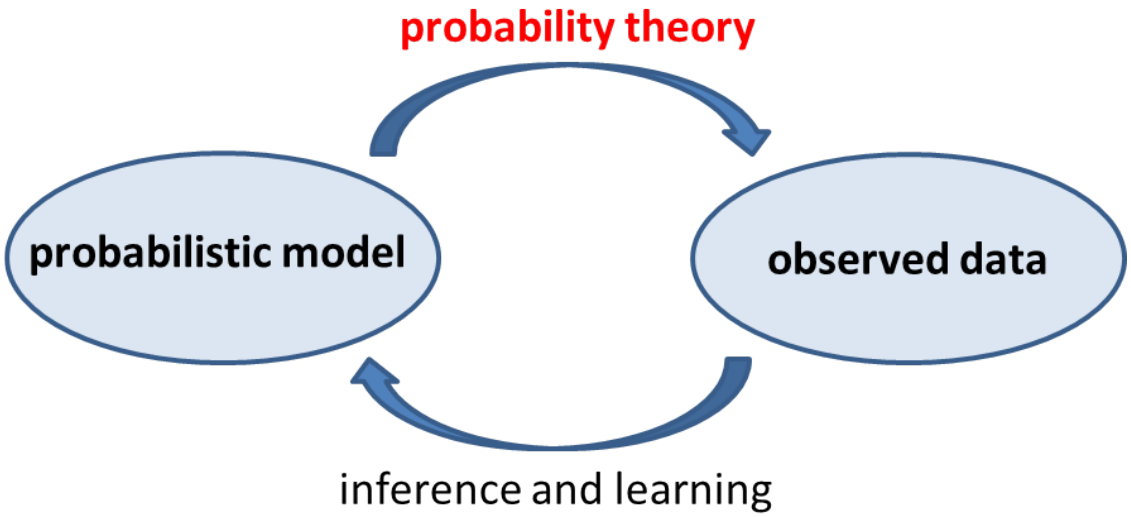

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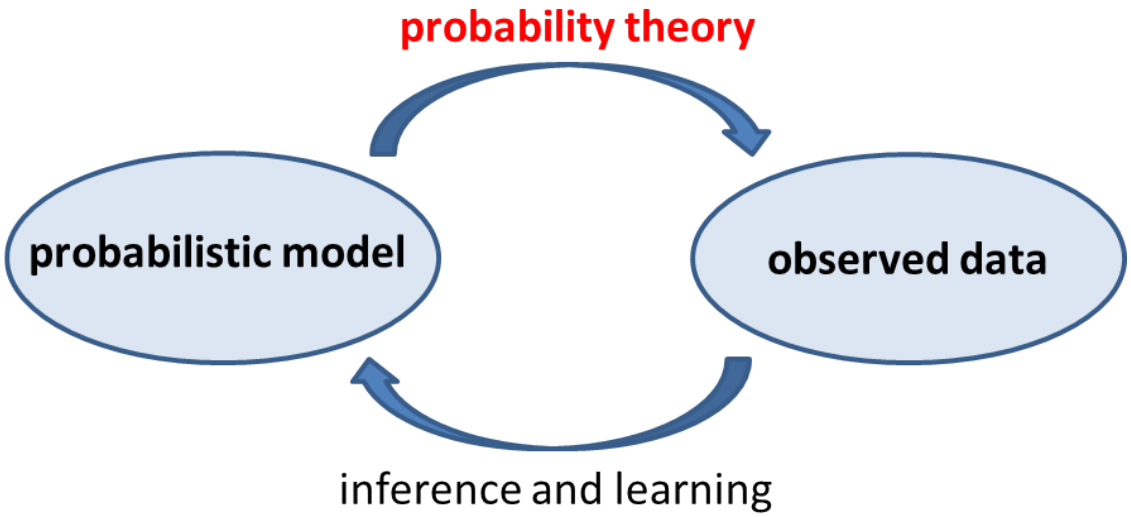

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- Great names of science: Bayes, Cardano, Fermat, Pascal, Laplace, Kolmogorov, Bernoulli, Poisson, Cauchy, Boltzman, ...
- Tool to handle uncertainty, information, knowledge, observations, ...
- ...thus also learning, decision making, inference, science,...

Do we still need this?

CONTENTS

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What book is this from?

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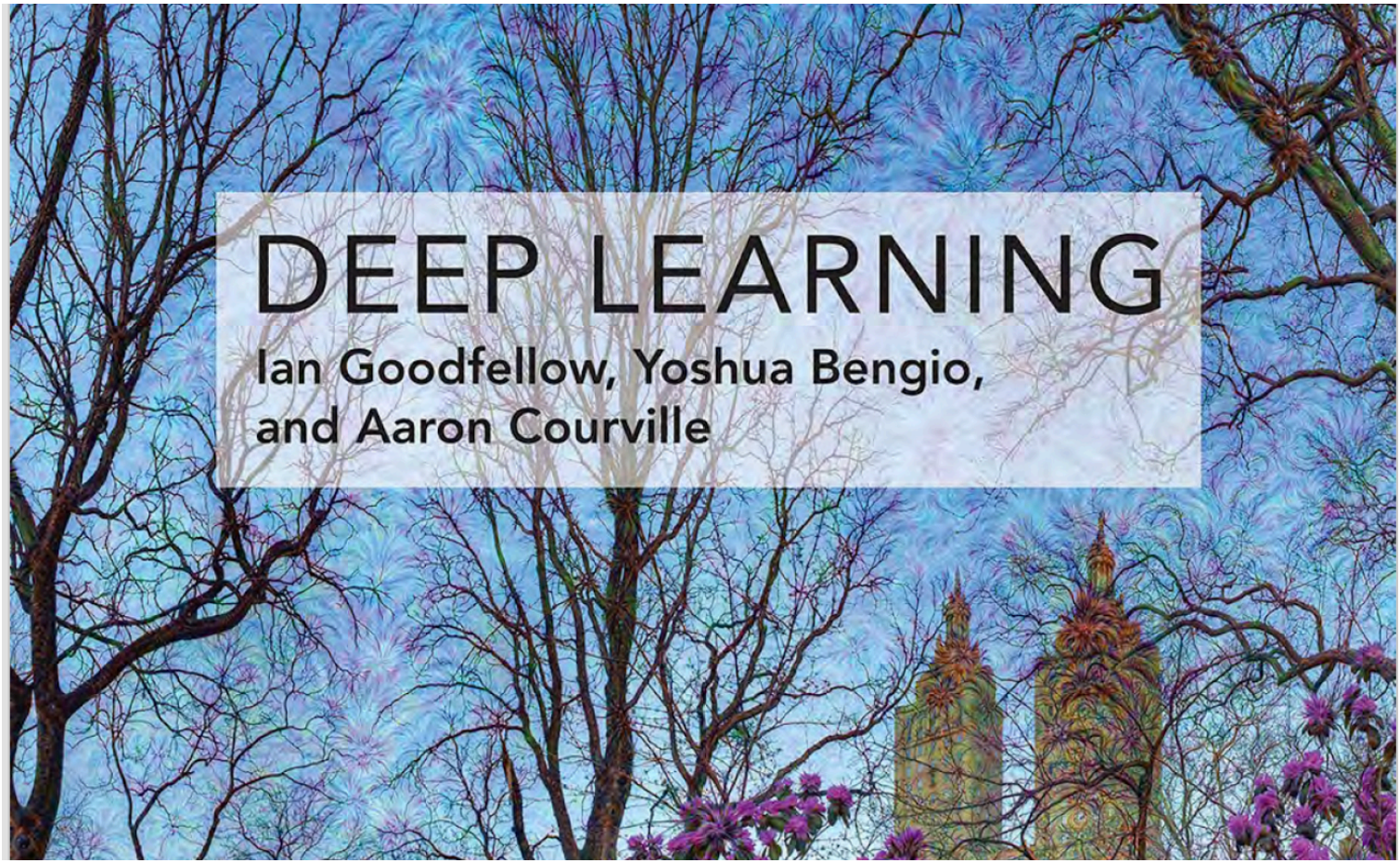

What is probability?

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- Subjective probability: $\mathbb{P}(A)$ is a degree of belief. *de Finetti, 1930s*

  .

  ...gives meaning to $\mathbb{P}(\text{“it will rain today”})$, or
  $\mathbb{P}(\text{“I'll have the flue next winter”})$

Key concepts: Sample space and events

- Sample space $\mathcal{X}$ = set of possible outcomes of a random experiment.

  Examples:
  - Tossing two coins: $\mathcal{X} = \{HH, TH, HT, TT\}$
  - Roulette: $\mathcal{X} = \{1, 2, ..., 36\}$
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- An event A is a subset of $\mathcal{X}$: $A \subseteq \mathcal{X}$ (also written $A \in 2^{\mathcal{X}}$).

  Examples:
  - "exactly one H in 2-coin toss": $A = \{TH, HT\}$.
  - "odd number in the roulette": $B = \{1, 3, ..., 35\}$.
  - "drawn a $\heartsuit$ card": $C = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\}$

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...heavier stuff, not covered here

Kolmogorov's Axioms for Probability

- Probability is a function that maps events A into the interval $[0, 1]$.

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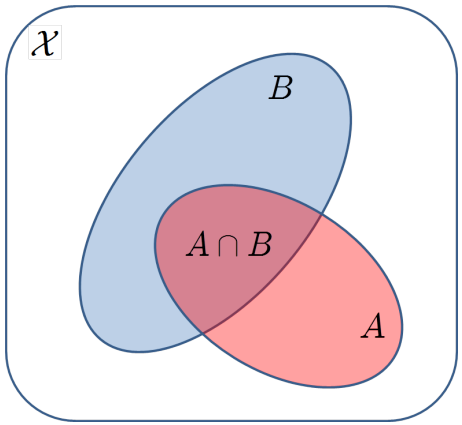

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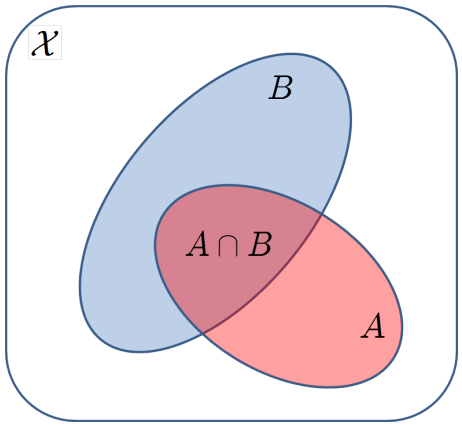
  - $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ (union bound)



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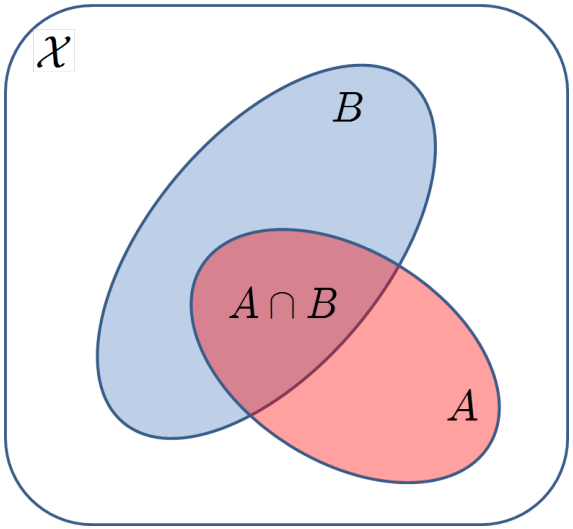

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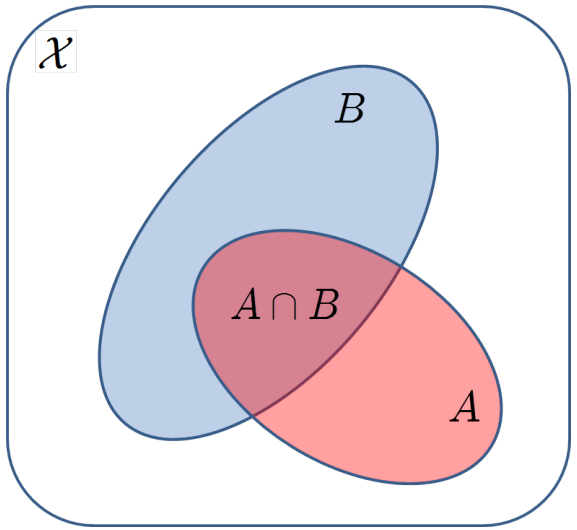

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- Example: $\mathcal{X} =$ "52 cards", $A = \{4\heartsuit, 4\clubsuit, 4\diamondsuit, 4\clubsuit\}$, and $B = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\}$; then, $\mathbb{P}(A) = 1/13$, $\mathbb{P}(B) = 1/4$

$$\mathbb{P}(A \cap B) \quad = \quad \mathbb{P}(\{4\heartsuit\}) = \frac{1}{52}$$

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Bayes Theorem

- Law of total probability: if $A_1, ..., A_n$ are a partition of $\mathcal{X}$

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$
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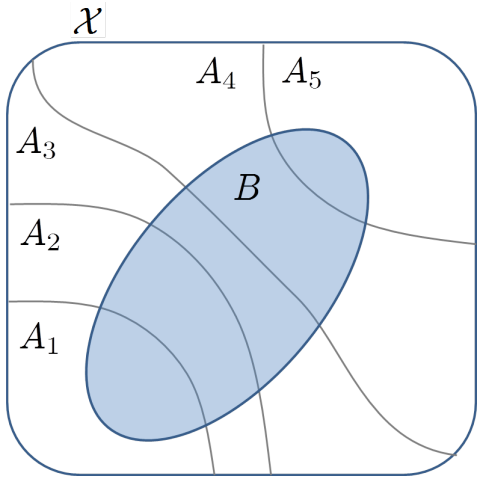

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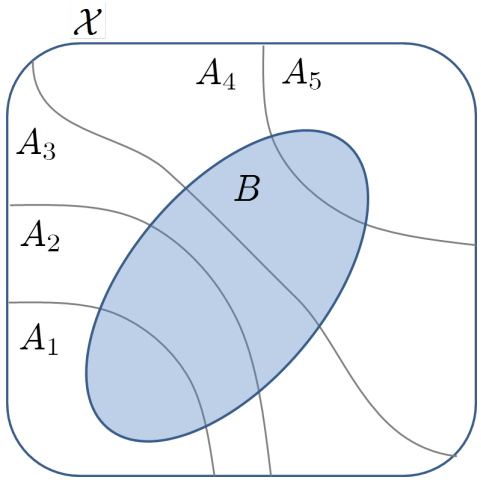
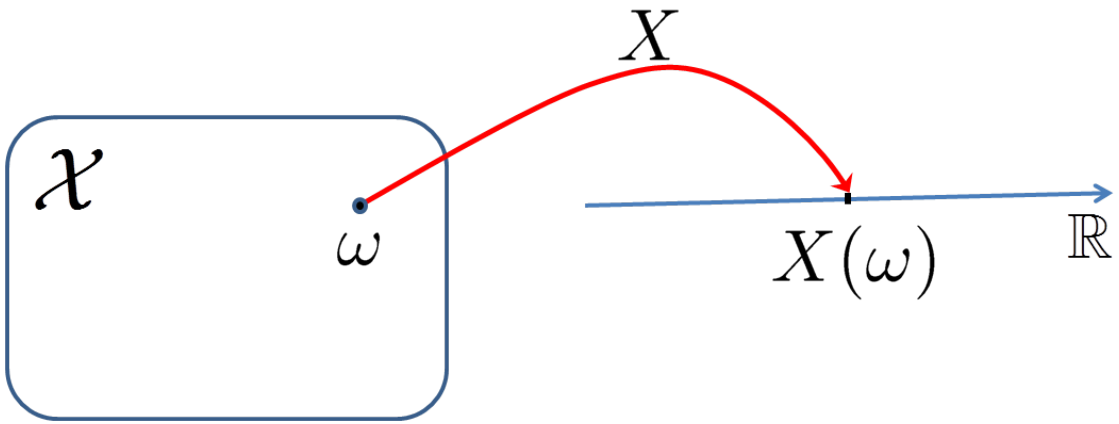


- Bayes' theorem: if $\{A_1, ..., A_n\}$ is a partition of $\mathcal{X}$

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B \cap A_i)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A_i)\ \mathbb{P}(A_i)}{\mathbb{P}(B)}$$

Random Variables

- A (real) random variable (RV) is a function: $X : \mathcal{X} \rightarrow \mathbb{R}$



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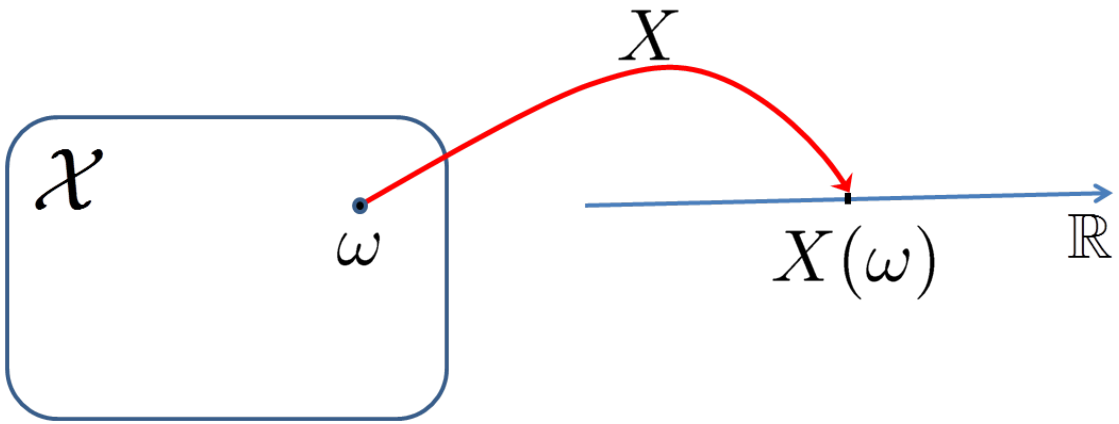

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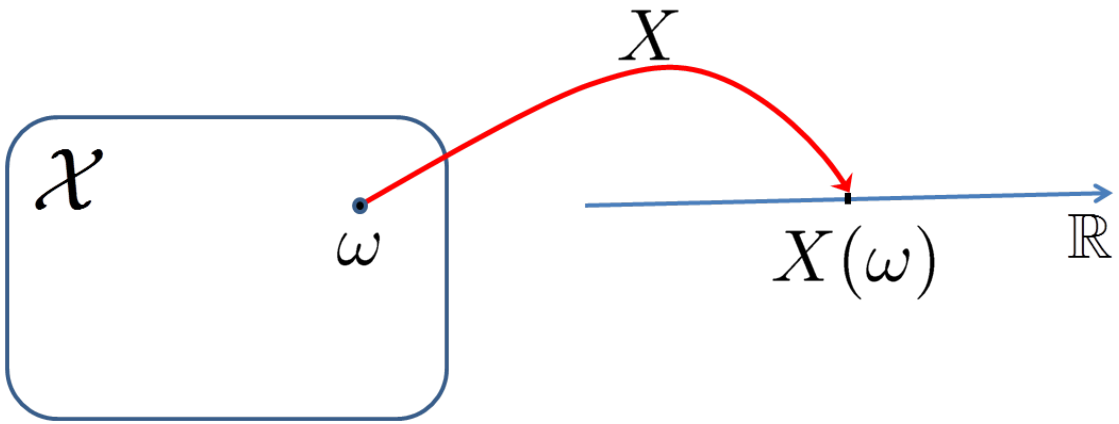

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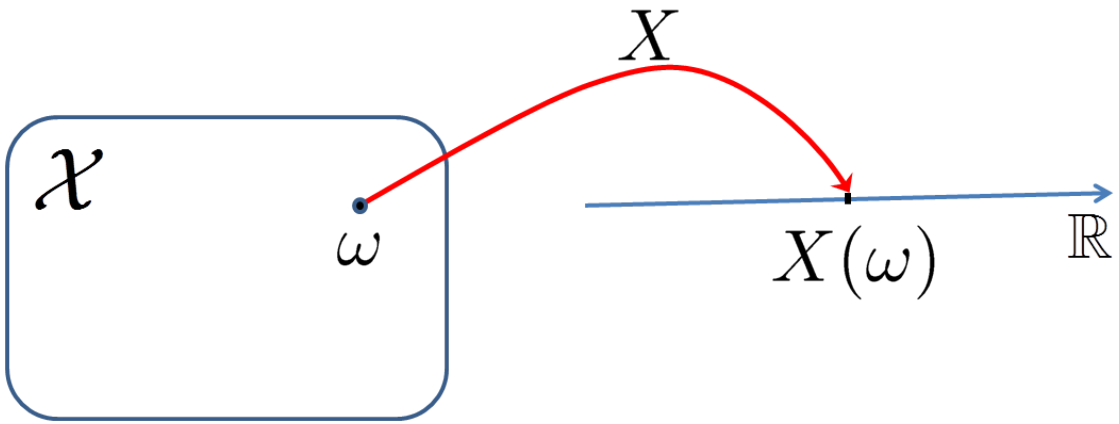

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  - Continuous RV: range of X is uncountable (*e.g.*, $\mathbb{R}$ or $[0,\ 1]$)
  - Example: number of heads in tossing two coins,
    $\mathcal{X} = \{HH, HT, TH, TT\}$,
    $X(HH) = 2$, $X(HT) = X(TH) = 1$, $X(TT) = 0$.
    Range of $X = \{0, 1, 2\}$.

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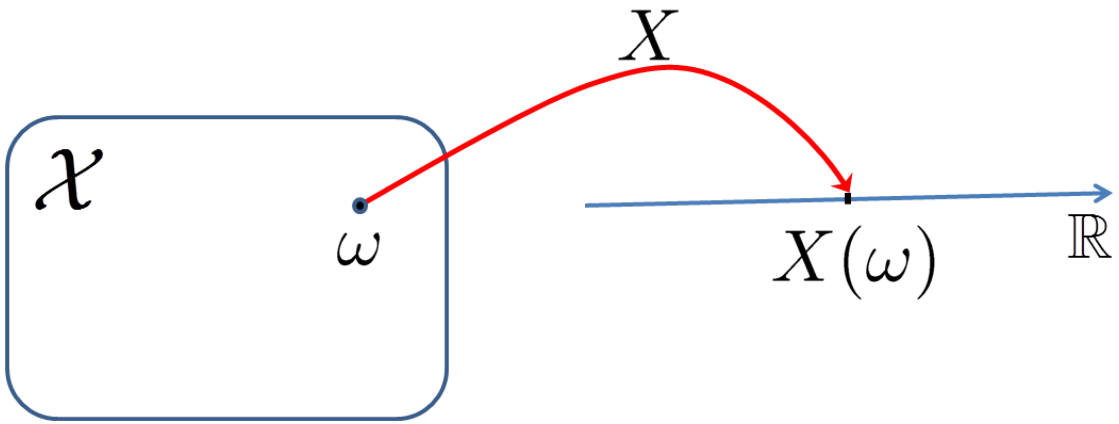

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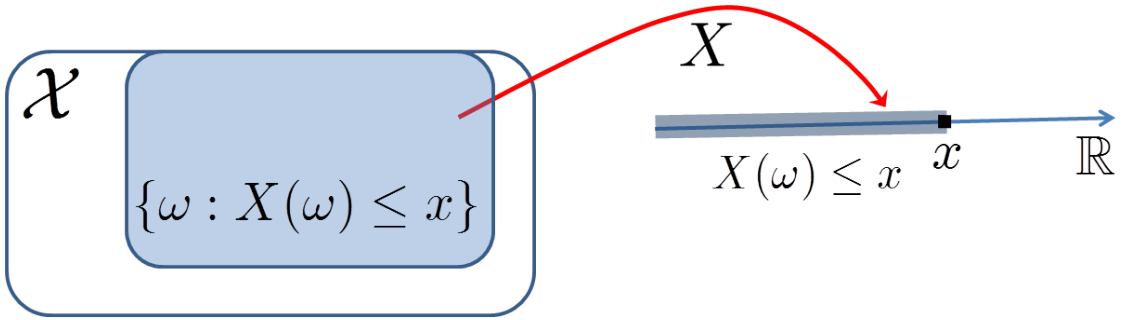
  - Example: distance traveled by a tossed coin; range of $X = \mathbb{R}_+$.

Random Variables: Distribution Function

- Distribution function: $F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\})$



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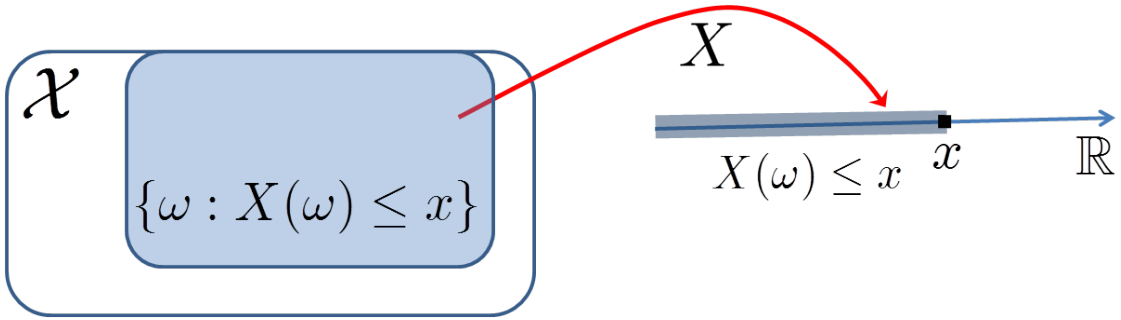
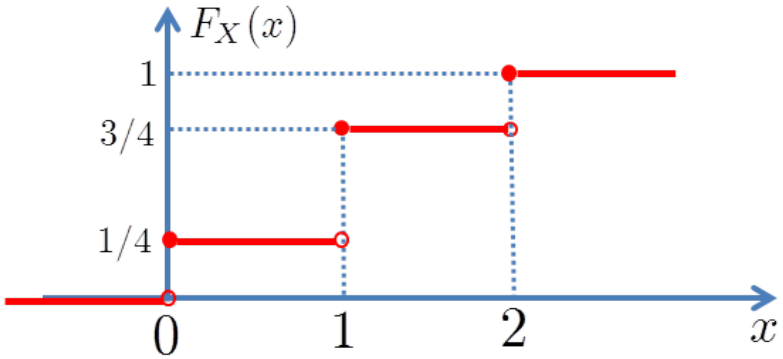
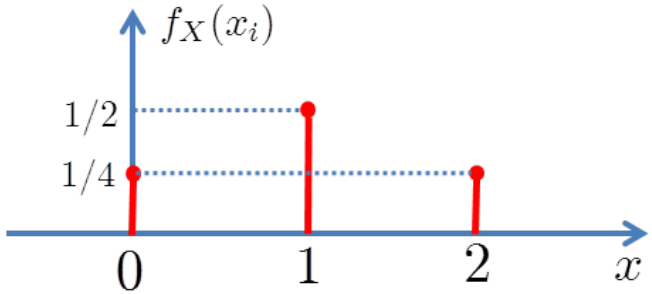


- Example: number of heads in tossing 2 coins; range$(X) = \{0, 1, 2\}$.





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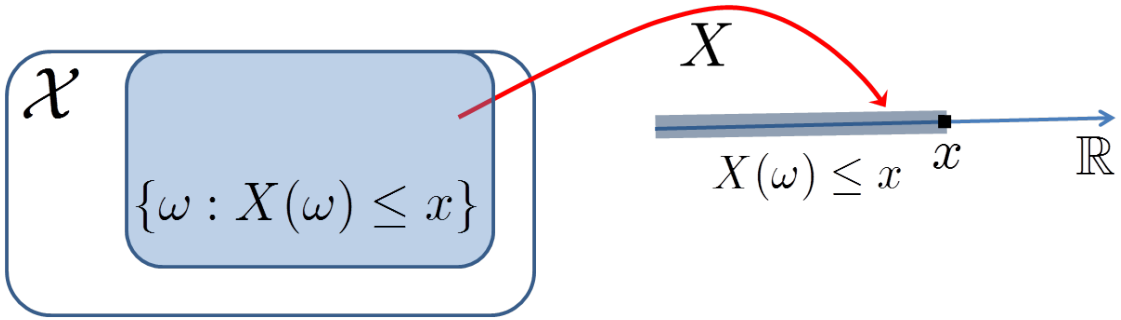

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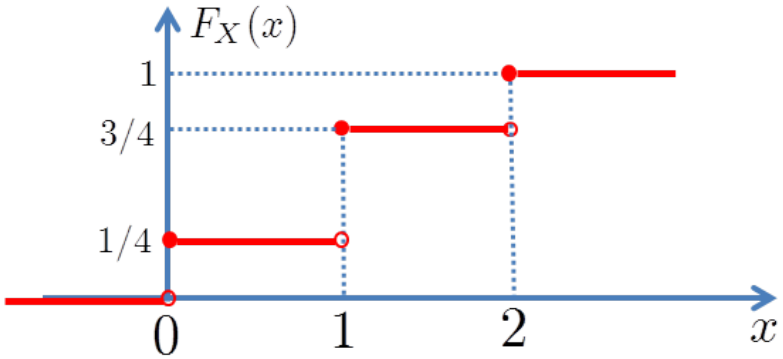

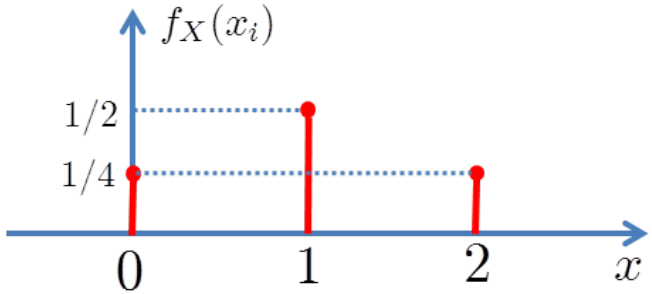

- Probability mass function (discrete RV): $f_X(x) = \mathbb{P}(X = x)$,

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i).$$

Important Discrete Random Variables

- Uniform: $X \in \{x_1, ..., x_K\}$, pmf $f_X(x_i) = 1/K$.

  Example: a fair roulette $X \in \{1, ..., 36\}$, with $f_X(x) = 1/36$

  Example: a fair die $X \in \{1, ..., 6\}$, with $f_X(x) = 1/6$

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- Bernoulli RV: $X \in \{0, 1\}$, pmf $f_X(x) = \begin{cases} p & \Leftarrow \quad x = 1 \\ 1-p & \Leftarrow \quad x = 0 \end{cases}$

  Compact form: $f_X(x) = p^x (1-p)^{1-x}$.

  Example: an unfair coin (heads $= 0$, tails $= 1$), with $p \neq 1/2$.

Important Discrete Random Variables

- Binomial RV: $X \in \{0, 1, ..., n\}$ (sum of n Bernoulli RVs)

$$f_X(x) = \text{Binomial}(x; n, p) = \binom{n}{x} p^x \ (1-p)^{(n-x)}$$

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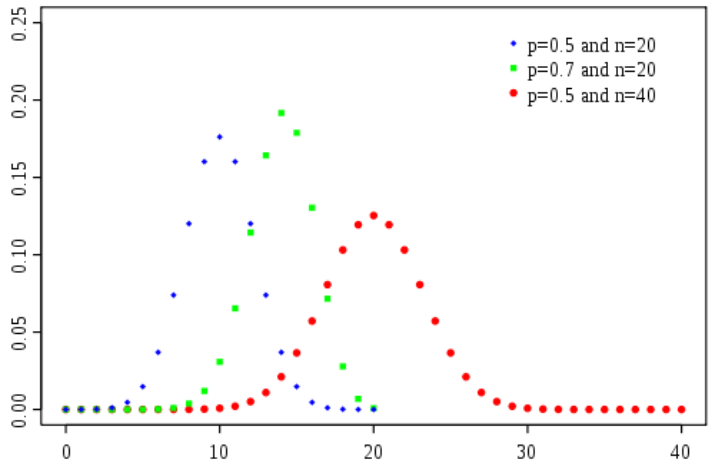

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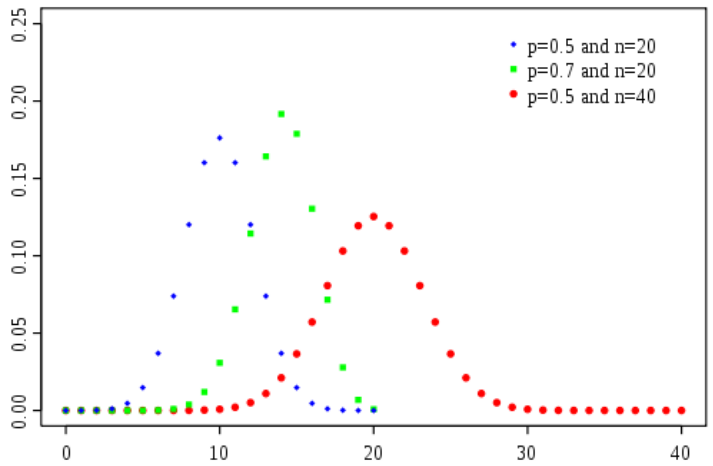

Example: number of heads in n coin tosses.

Other Important Discrete Random Variables

- Geometric(p): $X \in \mathbb{N}$, pmf $f_X(x) = p(1-p)^{x-1}$.

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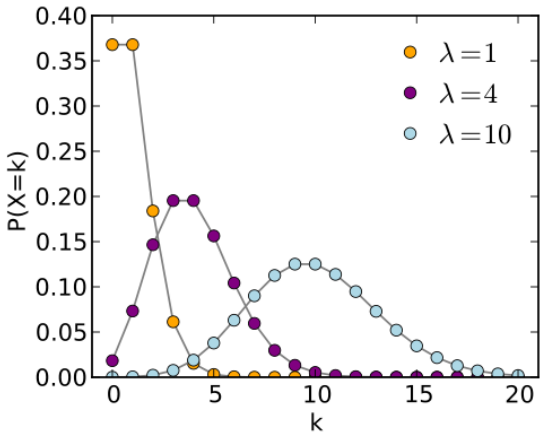

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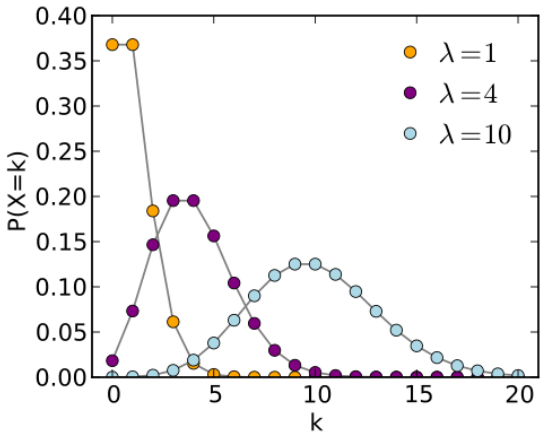


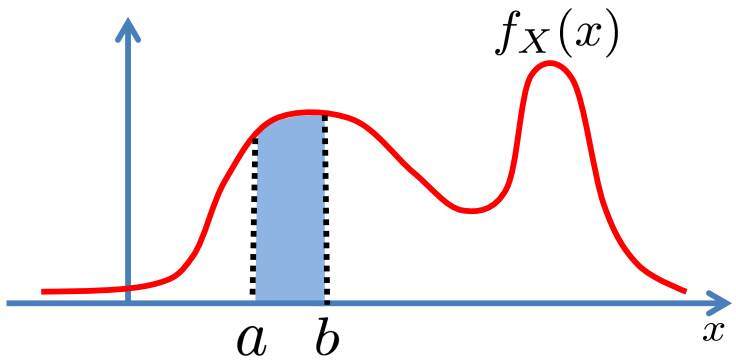
"...probability of the number of independent occurrences in a fixed (time/space) interval, if these occurrences have known average rate"

Examples: number of rain drops per second on a given area, number of calls per hour in a call center, number of tweets per day by DT, ...

Continuous Random Variables

- Probability density function (pdf, continuous RV): $f_X(x)$

$$\int_{-\infty}^{\infty} f_X(x) = 1 \qquad \mathbb{P}(X \in [a,b]) = \int_a^b f_X(x)\, dx$$



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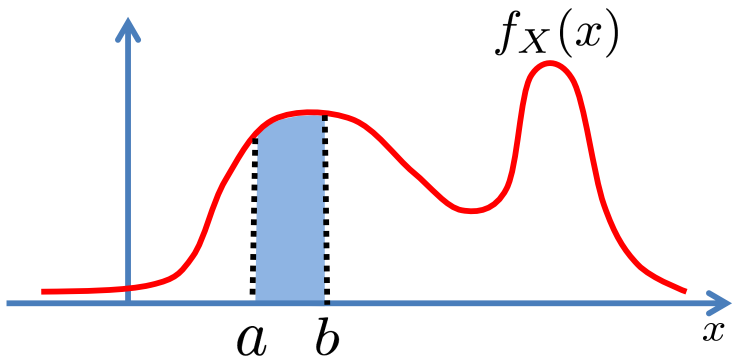

- Notice: $\mathbb{P}(X = c) = 0$

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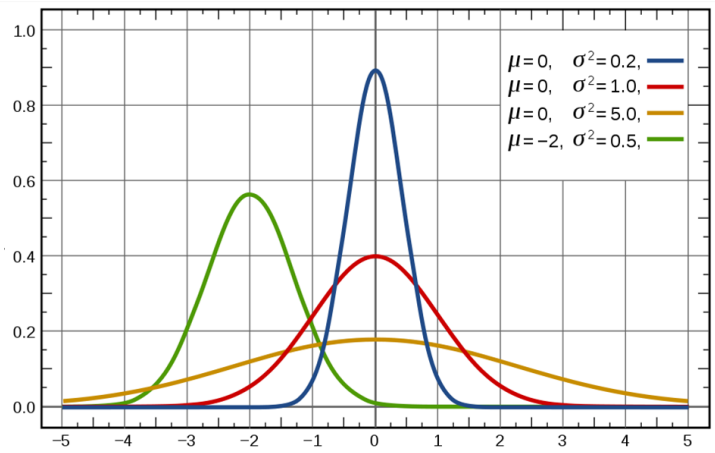

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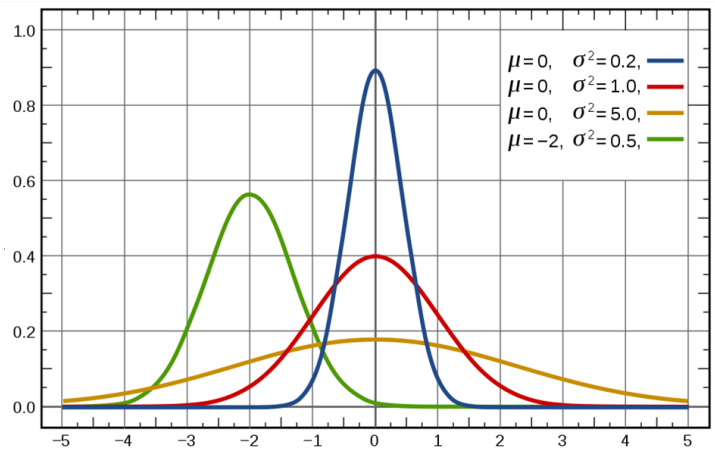

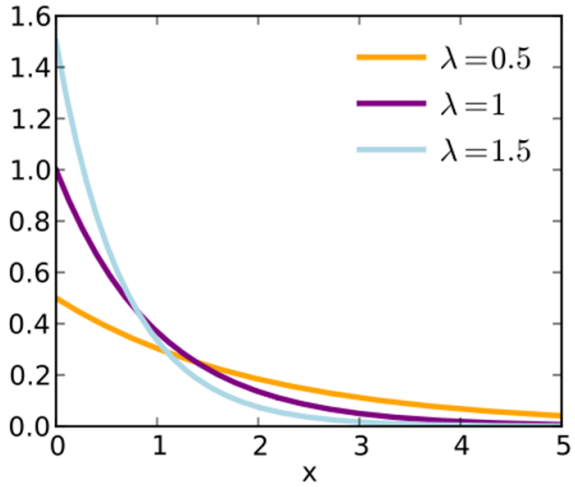

- Exponential: $f_X(x) = \mathsf{Exp}(x; \lambda) = \begin{cases} \lambda e^{-\lambda\, x} & \Leftarrow \quad x \geq 0 \\ 0 & \Leftarrow \quad x < 0 \end{cases}$

Expectation of (Real) Random Variables

- Expectation: $\mathbb{E}(X) = \begin{cases} \sum_i x_i \, f_X(x_i) & X \in \{x_1, ...x_K\} \subset \mathbb{R} \\ \int_{-\infty}^{\infty} x \, f_X(x) \, dx & X \text{ continuous} \end{cases}$

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- Linearity of expectation:

  $$\mathbb{E}(\alpha \, X + \beta \, Y) = \alpha \, \mathbb{E}(X) + \beta \, \mathbb{E}(Y), \quad \alpha, \beta \in \mathbb{R}$$

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- Probability as expectation of indicator, $\mathbf{1}_A(x) = \begin{cases} 1 & \Leftarrow \quad x \in A \\ 0 & \Leftarrow \quad x \notin A \end{cases}$

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- Central limit theorem: under mild conditions,

$$\lim_{n \to \infty} Z_n \sim \mathcal{N}(0, 1)$$

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Conditionals and Bayes' Theorem

- Conditional pmf (discrete RVs):

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- Also valid in the mixed case (*e.g.*, X continuous, Y discrete).

Joint, Marginal, and Conditional Probabilities: An Example

- A pair of binary variables $X, Y \in \{0, 1\}$, with joint pmf:

$f_{X,Y}(x,y)$	$Y=0$	$Y=1$
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An Important Multivariate RV: Multinomial

- Multinomial: $X = (X_1, ..., X_K)$, $X_i \in \{0, ..., n\}$, such that $\sum_i X_i = n$,

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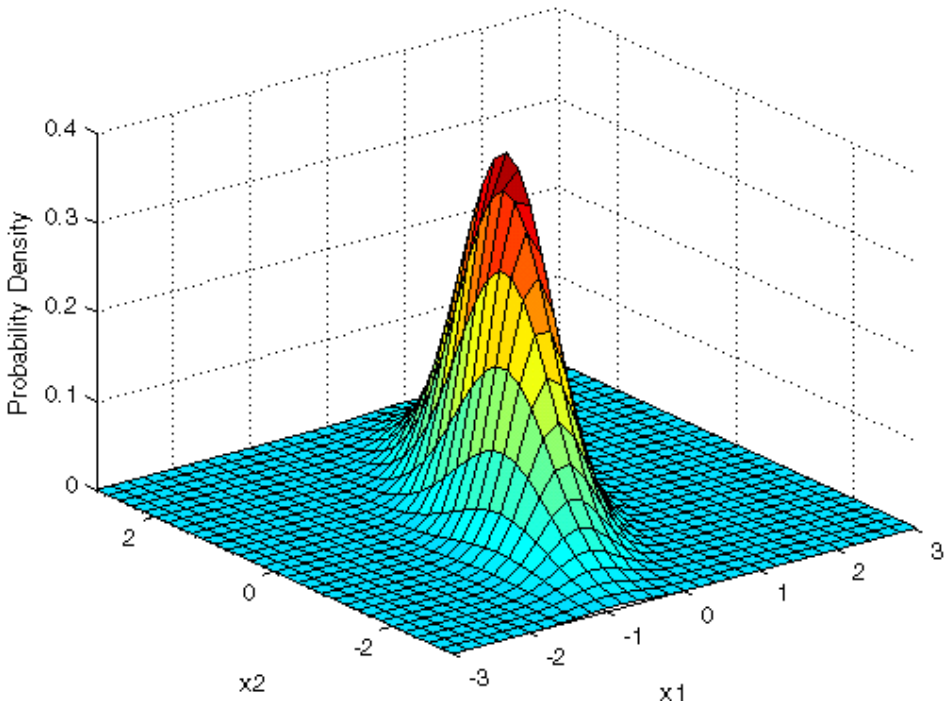

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- Covariance of Gaussian RV, $f_X(x) = \mathcal{N}(x;\mu,C) \;\;\Rightarrow\;\; \text{cov}(X) = C$

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Combining the 2-nd and the 4-th facts is called standardization

Exponential Families

A pdf or pmf $f_X(x|\eta)$, with parameter(s) η, for $X \in \mathcal{X}$, is in an exponential family if

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- Canonical parameter(s): η
- Sufficient statistics: $\phi(x)$
- Partition function: $Z(\eta)$

Examples: Bernoulli, Poisson, binomial, multinomial, Gaussian, exponential, beta, Dirichlet, Laplacian, log-normal, Wishart, ...

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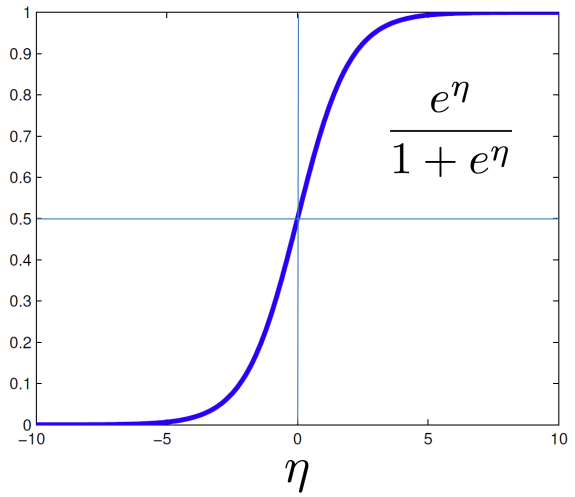
Notice that $p = \frac{e^\eta}{1+e^\eta}$

(logistic transformation)



More on Exponential Families

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- Expected sufficient statistics:

$$\frac{d \log Z(\eta)}{d\eta} = \frac{\frac{dZ(\eta)}{d\eta}}{Z(\eta)} = \frac{1}{Z(\eta)} \int \phi(x) h(x) \exp\left(\eta^T \phi(x)\right) dx = \mathbb{E}\left(\phi(X)\right)$$

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- Chebyshev's inequality: $\mu = \mathbb{E}(Y)$ and $\sigma^2 = \text{var}(Y)$, then

$$\mathbb{P}(|Y - \mu| \geq s) \leq \frac{\sigma^2}{s^2}$$

...simple corollary of Markov's inequality, with $X = |Y - \mu|^2,\ t = s^2$

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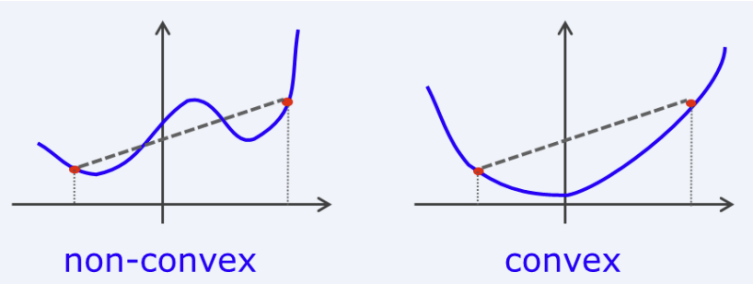

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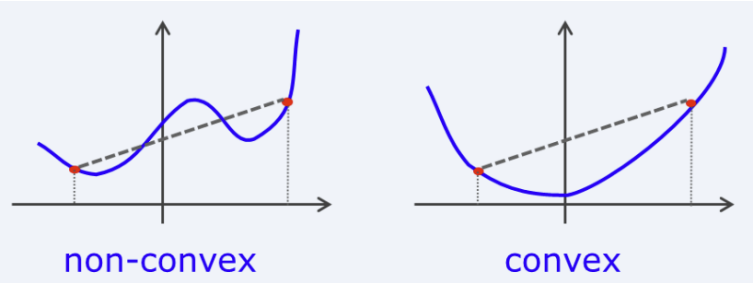

Jensen's inequality: if g is a real convex function, then

$$\mathbb{E}(g(X)) \geq g(\mathbb{E}(X))$$

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- Recall that a real function g is convex if, for any x, y, and $\alpha \in [0, 1]$

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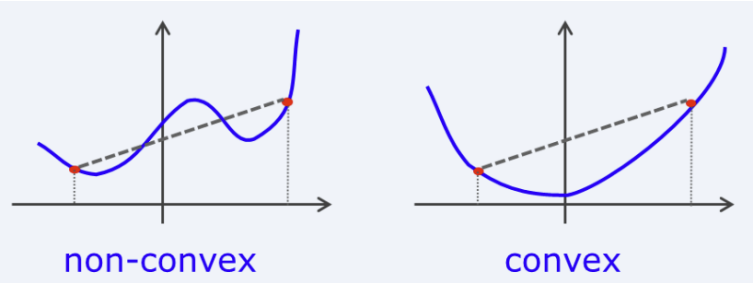

Jensen's inequality: if g is a real convex function, then

$$\mathbb{E}(g(X)) \geq g(\mathbb{E}(X))$$

Examples: $\mathbb{E}(X)^2 \leq \mathbb{E}(X^2) \;\Rightarrow\; \text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \geq 0.$

$\mathbb{E}(\log X) \leq \log \mathbb{E}(X)$, for X a positive RV.

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- With $\log_2$, units are bits/symbol
- Central role in information/coding theory: lower bound on expected number of bits to code X
- Widely used: physics, biological sciences (computational biology, neurosciences, ecology, ...), economics, finances, social sciences, ...

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- Gaussian upper bound: $f_X(x) = \mathcal{N}(x; \mu, \sigma^2)$, then

  $$h(X) = \frac{1}{2} \log(2\pi e \sigma^2).$$

  For any RV Y with $\text{var}(Y) = \sigma^2$, then $h(Y) \leq \frac{1}{2} \log(2\pi e \sigma^2)$.

  ...yet another reason for why the Gaussian is important.

Kullback-Leibler divergence

Kullback-Leibler divergence (KLD) between two pmf:

$$D(f_X \| g_X) = \sum_{x=1}^{K} f_X(x) \log \frac{f_X(x)}{g_X(x)}$$

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Issues: not symmetric; $D(f_X \| g_X) = +\infty$ if $g_X(x) = 0$ and $f_X(x) \neq 0$

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Deterministic function: if $Y = \phi(X)$, then $I(X;Y) = H(Y) \leq H(X)$

Recommended Reading (Probability and Statistics)

- K. Murphy, "Machine Learning: A Probabilistic Perspective", MIT Press, 2012 (Chapter 2).

- L. Wasserman, "All of Statistics: A Concise Course in Statistical Inference", Springer, 2004.

Concluding...

Enjoy LxMLS 2017!

Linear Algebra

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- Example: the system

$$4\,x_1 - 5\,x_2 = -13$$
$$-2\,x_1 + 3\,x_2 = 9$$

can be written compactly as $\boxed{Ax = b}$, where

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix},\ b = \begin{bmatrix} -13 \\ 9 \end{bmatrix},$$

and can be solved as

$$x = A^{-1}b = \begin{bmatrix} 1.5 & 2.5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -13 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Notation: Matrices and Vectors

- $A \in \mathbb{R}^{m \times n}$ is a matrix with m rows and n columns.

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Norms

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- Notable case: the ℓ_0 "norm" (not): $\|x\|_0 = |\{i : x_i \neq 0\}|$.

Special Matrices

- The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix such that

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

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- There are several algorithms to compute A^{-1}; general case, computational cost $O(n^3)$.

Quadratic Forms and Positive (Semi-)Definite Matrices

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