CA320 - Computability & Complexity

Context-Sensitive Languages

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Context-Sensitive Grammars

An *unrestricted grammar* is a 4-tuple $G = (V, \Sigma, S, P)$, where V and Σ are disjoint sets of variables and terminals respectively. $S \in V$ is called the *start symbol* and P is a set of production rules of the form $\alpha \rightarrow \beta$.

A *Context-Sensitive Grammar* (CSG) is an *unrestricted grammar* in which every production is of the form $\alpha \rightarrow \beta$ and $|\beta| \geq |\alpha|$, i.e. no production rule is length-decreasing.

An Example CSG

An example CSG is:

$$\begin{array}{rcl} S & \rightarrow & aBCT|aBC \\ T & \rightarrow & ABCT|ABC \\ BA & \rightarrow & AB \\ CA & \rightarrow & AC \\ CB & \rightarrow & BC \\ aA & \rightarrow & aa \\ aB & \rightarrow & ab \\ bB & \rightarrow & bb \\ bC & \rightarrow & bc \\ cC & \rightarrow & cc \end{array}$$

How a variable is derived depends on the context!

An Example CSG (2)

Here are some derivations from this CSG.

$$\begin{array}{rcl} S & \Rightarrow & aBC \\ & \Rightarrow & abC \\ & \Rightarrow & \text{abc} \end{array}$$

$$\begin{array}{rcl} S & \Rightarrow & aBCT \\ & \Rightarrow & aBCABC \\ & \Rightarrow & \text{aBACBC} \\ & \Rightarrow & \text{aABCBC} \\ & \Rightarrow & \text{aABBCC} \\ & \Rightarrow & \text{aaBBCC} \\ & \Rightarrow & \text{aabBCC} \\ & \Rightarrow & \text{aabbCC} \\ & \Rightarrow & \text{aabbcC} \\ & \Rightarrow & \text{aabbcc} \end{array}$$

$$L(G) = \{a^n b^n c^n | n \geq 1\}$$

We have already shown that $AnBnCn$ is not a context-free language using the CFG pumping lemma. But it is a context-sensitive language.

CSLs are not a generalisation of CFGs as CSLs cannot have any Λ-productions.

Linear Bounded Automata

A *linear bounded automaton* (LBA) is a finite state machine with a finite length data store called a *tape*. The tape consists of a sequence of cells, where each cell can store a symbol from the machine's alphabet. Symbols can be written or read from any position on this tape and therefore the LBA has a *read-write head* that can be moved left or right one cell.

The tape is used both to store the input and any ongoing calculations. There are 2 special symbols, [and], that are used to mark the finite bounds of the tape. The read-write head cannot move beyond either of these symbols and it cannot overwrite these symbols.

Linear Bounded Automata (2)

At each step the LBA read the symbol under the read-write head, replaces the by another symbol (could be the same symbol) and then perform one of four possible actions $\mathcal{A} \in \{Y, N, L, R\}$, where:

- Y denotes "Yes", accept the input string
- N denotes "No", reject the input string
- L denotes "Left", move the read-write head one cell to the left
- R denotes "Right", move the read-write head one cell to the right

Linear Bounded Automata (2)

Formally, a linear bounded automaton is a 5-tuple $M = \{Q, \Sigma, \Gamma, q_0, \delta\}$ where:

- Q is a finite set of *states*;
- Σ is a finite alphabet (*input symbols*);
- Γ is a finite alphabet (*store symbols*);
- $q_0 \in Q$ is the *initial state*; and
- $\delta : Q \times (\Gamma \cup \{[,]\}) \to Q \times (\Gamma \cup \{[,]\} \times \mathcal{A}$, is the *transition function*.

If $((q, \sigma), (q', \psi, \mathcal{A})) \in \delta$ then when in state q with σ at the current read-write head position, M will replace σ by ψ and perform action $\mathcal{A}$ and enter state q'.

M accepts $w \in \Sigma^*$ iff it starts with configuration $(q_0, [w])$ and the action Y is taken.

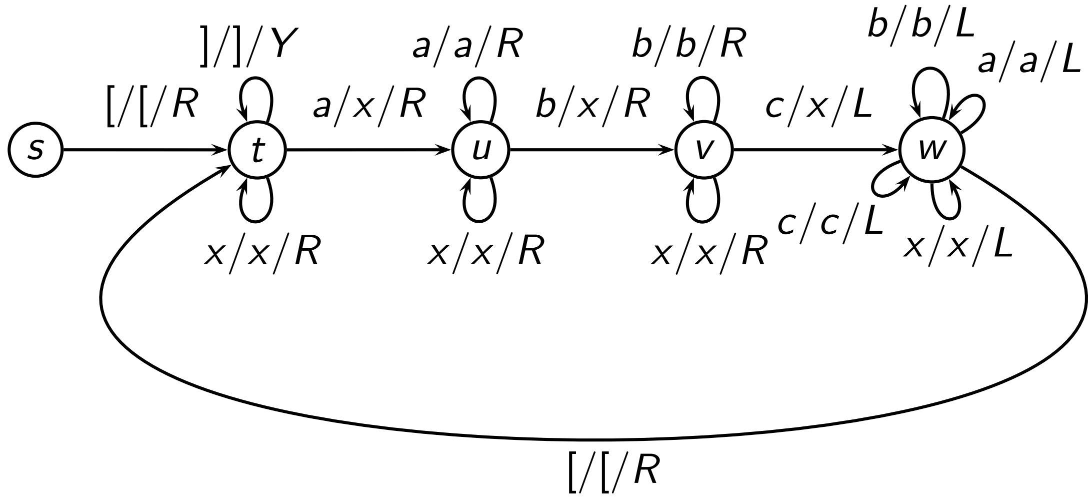
Linear Bounded Automaton Example

An LBA to accept $AnBnCn = \{a^n b^n c^n | n \geq 0\}$ is:

$$
\begin{array}{lcl}
Q & = & \{s, t, u, v, w\} \\
\Sigma & = & \{a, b, c\} \\
\Gamma & = & \{a, b, c, x\} \\
q_0 & = & \mathrm{s} \\
\delta & = & \{((s, [), (t, [, R)), \\
 & & ((t,]), (t,], Y)), ((t, x), (t, x, R)), ((t, a), (u, x, r)), \\
 & & ((u, a), (u, a, R)), ((u, x), (u, x, R)), ((u, b), (v, x, R)) \\
 & & ((v, b), (v, b, R)), ((v, x), (v, x, R)), ((v, c), (w, x, L)) \\
 & & ((w, c), (w, c, L)), ((w, b), (w, b, L)), ((w, a), (w, a, L)) \\
 & & ((w, x), (w, x, L)), ((w, [), (t, [, R)) \\
 & & \}
\end{array}
$$

Linear Bounded Automaton Example (2)

Or as a transition diagram;



where $\sigma/\psi/\mathcal{A}$ denotes reading symbol σ, writing symbol ψ and performing action $\mathcal{A}$.

Linear Bounded Automaton Example (3)

Intuitively the previous LBA behaves as follows.

- The LBA performs multiple passes over the input string.
- On each pass (from left to right) starting at the start symbol [in state t, the LBA coverts the first a into an x, and then the first b into an x and finally the first c into an x.
- After converting a c into an x (after matching it with an a and b) the LBA moves right to left until it reaches the start symbol [and goes into state t.
- If the LBA in state t only encounters x symbols from the start symbol [to the end symbol], then the LBA performs a Y action.
- the LBA gets stuck in either state u, v or w if there is not a "matching" a, b or c symbol respectively.

Can you design a better version of this LBA that actually rejects string that are not in the language $AnBnCn$?