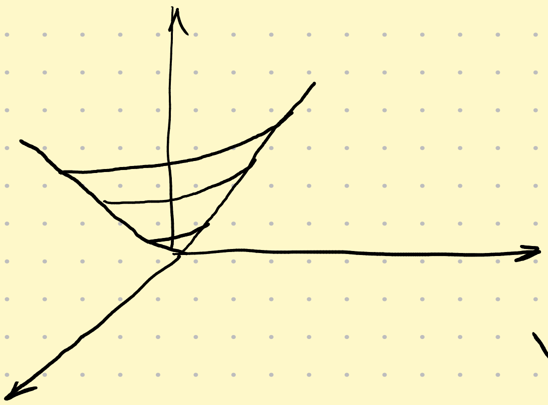
Exercícios Superfícies Parametrizadas

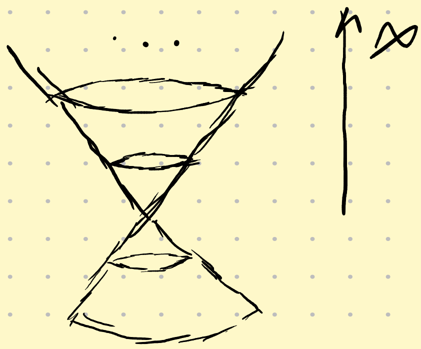
Ex 1) $\sigma(u,v) = (u\cos v, u\sin v, u)$ $\quad 0 \leq v \leq 2\pi$, $u \in \mathbb{R}$

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = u \end{cases}$$

$x^2 + y^2 = u^2$

$x^2 + y^2 = z^2 \leftarrow$ representa um cone





$x^2 + y^2 = z^2$

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = r \end{cases}$$

$0 \leq \theta \leq 2\pi$

$-\infty \leq r \leq +\infty$

Parametrização por "gráfico"

$x^2 + y^2 = z^2$

$$\begin{cases} x = x \\ y = y \\ z = \sqrt{x^2+y^2} = f(x,y) \end{cases}$$

Ex 2) $\sigma(u,v) = (v\cos u, v\sin u, 1 - v^2)$

$x = v\cos u$

$y = v\sin u$

$z = 1 - v^2$

$v^2 = x^2 + y^2$

$z = 1 - x^2 + y^2$ (Parabolóide ao contrário)

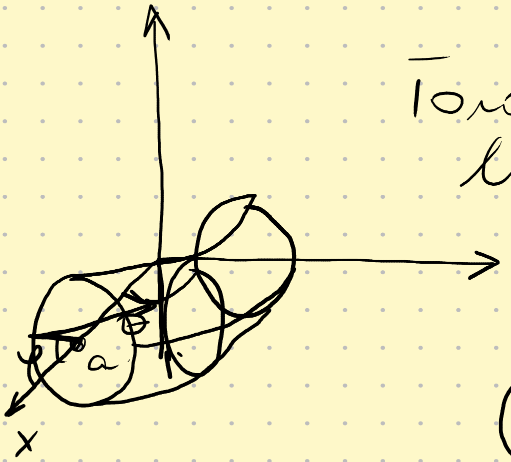
$$X_u = (-v \operatorname{sen} u, v \cos u, 0)$$

$$X_v = (\cos u, \operatorname{sen} u, -2v)$$

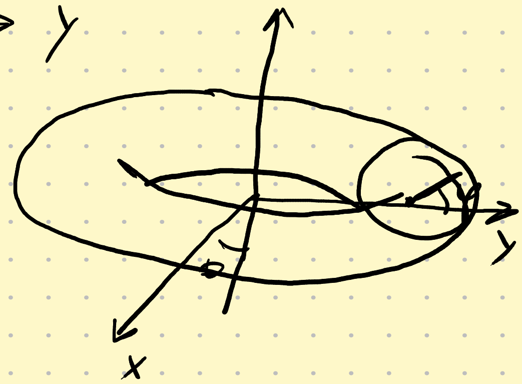
$$|X_u \wedge X_v| = i(-2v^2 \cos u) + j(-2v^2 \operatorname{sen} u) + k(2v^2 \operatorname{sen} u + v\cos^2 u)$$

$v = 0 \Rightarrow$ ponto $(0,0,1)$

Ex3)



Toro de revolução



Parametrização do toro

$$\begin{cases} x = (a + b\cos\varphi)\cos\theta \\ y = (a + b\cos\varphi)\operatorname{sen}\theta \\ z = b \operatorname{sen}\varphi \end{cases}$$

$$\|X_\theta \wedge X_\varphi\| = \sqrt{EG - F^2}$$

$E = \langle X_\theta, X_\theta \rangle$

$F = \langle X_\theta, X_\varphi \rangle$

$G = \langle X_\varphi, X_\varphi \rangle$

$$\|X_\theta \wedge X_\varphi\| = b(a + b\cos\varphi) \leftarrow$$

Regular em todos os pontos

Ex 4) encontre a parametrização

a) Plano $3x-2y+z=2$

$$\begin{cases} x=u \\ y=v \\ z=2-3u+2v \end{cases}$$

b) Porção do paraboloide $z=x^2+y^2$ $0\leq z\leq 9$

$$\begin{cases} x=r\cos\theta \\ y=r\cos\theta \\ z=r^2 \end{cases} \quad \begin{matrix} 0\leq r\leq 3 \\ 0\leq\theta\leq 2\pi \end{matrix}$$

c) tronco do cone $z^2=x^2+y^2$ $2\leq z\leq 8$

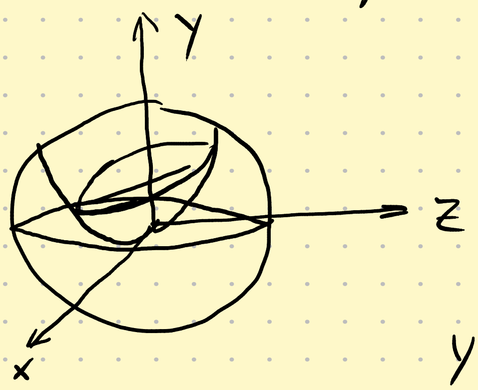
$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=r \end{cases} \quad \begin{matrix} 0\leq\theta\leq 2\pi \\ 2\leq r\leq 8 \end{matrix}$$

d) Porção do cilindro $x^2+y^2=9$ no ~~1~~ 1° octante, c/ $\cancel{\alpha} z\leq 3$

$$\begin{cases} x=3\cos\theta \\ y=3\sin\theta \\ z=z \end{cases} \quad \begin{matrix} 0\leq\theta\leq\pi/2 \\ 0\leq z\leq 3 \end{matrix}$$

c) porção da parabolóide contido no $(y=x^2+z^2)$ (I)

interior da esfera $x^2+y^2+z^2=2$



$y = 2-y^2 \qquad +y^2+y-2=0$

$\Delta = 1+8 = \underline{9}$

$y = \dfrac{-1 \pm 3}{2}$ $\quad y' = 1$ $\quad y'' = -2$ ← NÃO FAZ sentido

intersec no circ $x^2+z^2 = 1$ em $y=1$:

• parametrização

$$\begin{cases} x = r\cos\theta \\ z = r\operatorname{sen}\theta \\ y = r^2 \end{cases} \qquad \begin{array}{l} 0 \leq y \leq 1 \\ 0 \leq r^2 \leq 1 \\ 0 \leq r \leq 1 \end{array} \qquad 0 \leq \theta \leq 2\pi$$

$X_\theta = (-r\operatorname{sen}\theta,\ r\cos\theta,\ 0)$

$X_r = (\cos\theta,\ \operatorname{sen}\theta,\ 2r)$

$X_\theta \wedge X_r = i(2r^2\cos\theta) + j(2r^2\operatorname{sen}\theta) + k(\underbrace{-r\operatorname{sen}^2\theta - r\cos^2\theta}_{-r})$

$\|X_\theta \wedge X_r\| = \sqrt{4r^4+r^2}$

$E = r^2\operatorname{sen}^2\theta + r^2\cos^2\theta + 0 \to r^2$

$F = -r\operatorname{sen}\theta\cos\theta + r\cos\theta\operatorname{sen}\theta + 0$

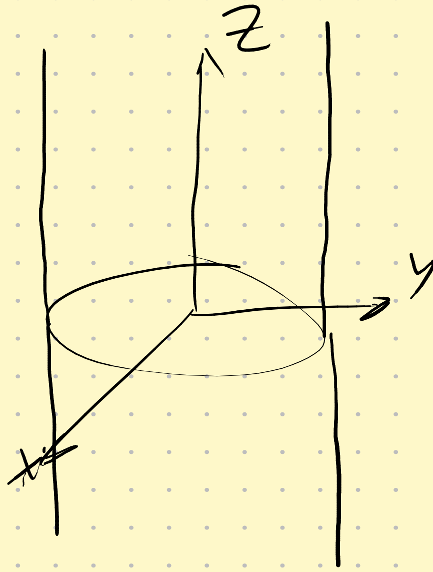
$G = \underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{1} + 4r^2$

$\|X_\theta \wedge X_r\| = \sqrt{EG-F^2}$

$(4r^2+1)(r^2) = \sqrt{4r^4+r^2}$

Ex 5) Area do cilindro $2\pi r h$

$0 \le z \le h$



$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases} \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le z \le h \end{array}$$

$X_\theta = (-r\,\text{sen}\,\theta,\ r\cos\theta,\ 0)$

$X_z = (0,\ 0,\ 1)$

$E = r^2\,\text{sen}\,\theta + r^2\cos\theta = r^2$

$F = 0$ $\qquad X_\theta \wedge X_z = \sqrt{r^2+1}$

$G = 1$

$$\int_0^{2\pi}\int_0^{h} r \quad dz\, d\theta = (2\pi)(h)(r)$$

$= 2\pi r h$ cqd

$(-r\,\cancel{\text{sen}\,\theta}\ \cos)i + j(r\,\text{sen}\,\theta)$

$r^2\cos^2\theta + r^2\,\text{sen}^2\theta$

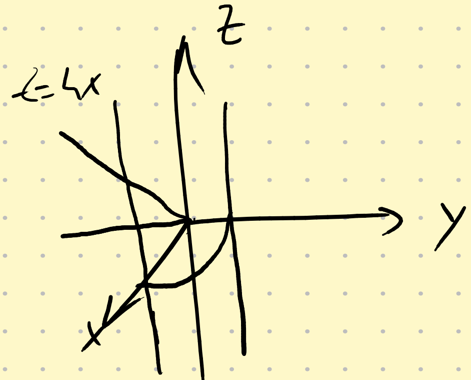
$\sqrt{r^2} = r$

Exemplo 10)

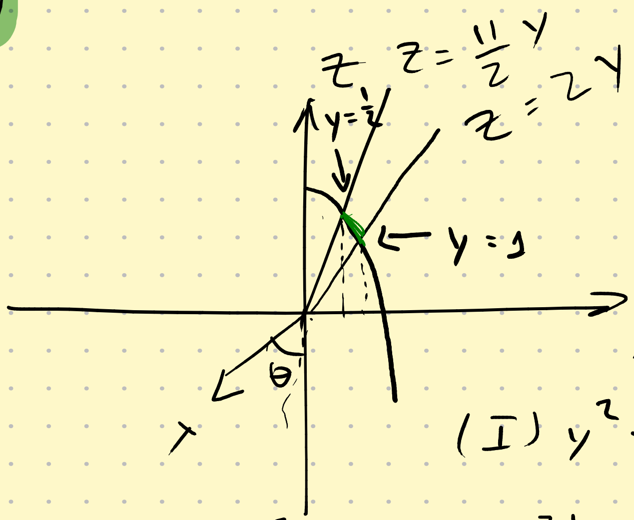
$$\int_0^{\pi/2}\int_{2a\cos\theta}^{4a\cos\theta} a\, dz\, d\theta = \int_0^{\pi/2} 4a^2\cos\theta - 2a^2\cos\theta$$

$2a^2 \int_0^{\pi/2} \cos\theta \qquad + \cancel{\text{sen}\,\pi/2}\ 1 + \text{sen}\,\cancel{\theta}$

$4 \cdot 2a^2 = 8a^2$



Ex 6)



revolução no eixo z

$z=3-y^2$

(I) $y^2+\frac{11}{2}y-3=0$

$\Delta=\frac{121}{4}+12=\frac{169}{4}$ $\quad y=\frac{-\frac{11}{2}\pm\frac{13}{2}}{2}$

$\boxed{y'=\frac{1}{2}}$ $\quad \cancel{y^2<0}$

parametrização R^2

$\gamma(t)=(\underbrace{t}_{y},\underbrace{3-t^2}_{z})$ *

(II) $y^2+2y-3=0$

$\Delta=4-12=16$

$y=\frac{-2\pm4}{2}$ =>

$\boxed{y'=1}$ $\quad \cancel{y^4<0}$

Parametrização revolução

$$\begin{cases} x=t\cos\theta & 0\leq\theta\leq2\pi \\ y=t\,\text{sen}\,\theta & \frac{1}{2}\leq t\leq1 \\ z=3-t^2 \end{cases}$$

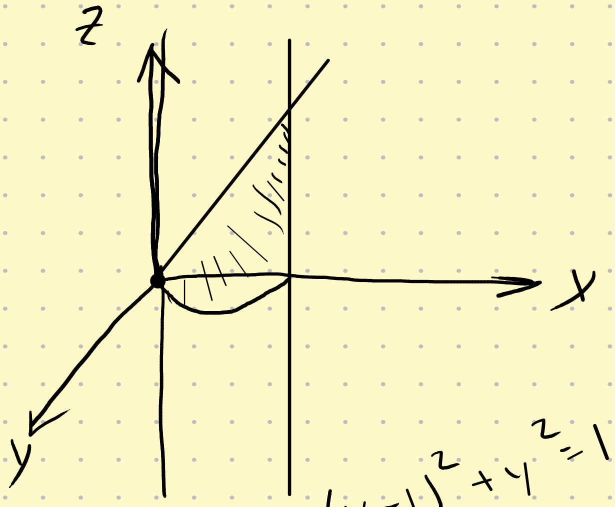
$$\begin{array}{cccc} & i & j & k \\ X_\theta=(& -t\,\text{sen}\,\theta, & t\cos\theta, & 0) \\ X_z=(& \cos\theta, & \text{sen}\,\theta, & -2t) \end{array}$$

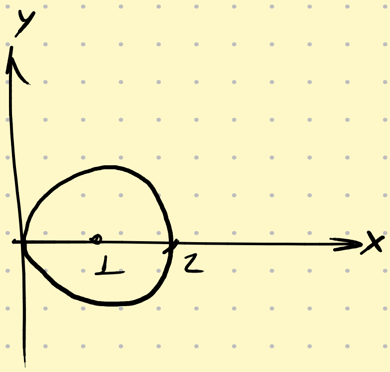
$X_\theta\wedge X_z=(-2t^2\cos\theta,\ -2t^2\,\text{sen}\,\theta,\ -t)$

$X_\theta\wedge X_z=\sqrt{4t^4+t^2}$

$$\int_0^{2\pi}\int_{1/2}^{1}\sqrt{4t^4+t^2}\,dt\,d\theta=\frac{1}{6}(5\sqrt{5}-2\sqrt{2})\pi$$

7)





$(x-1)^2+y^2=1 \longleftarrow x^2+y^2=2x$ limitado pelo cone $z=\sqrt{x^2+y^2}$

$S(\theta,z)$

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases} \qquad r = 2\cos\theta \qquad \Rightarrow \qquad \begin{cases} x = 2\cos^2\theta \\ y = 2\cos\theta\,\text{sen}\,\theta \\ z = z \end{cases} \qquad \begin{aligned} -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2} \\ 0 \le z \le 2\cos\theta \end{aligned}$$

$$S(\theta,z) = (2\cos^2\theta,\ 2\cos\theta\,\text{sen}\,\theta,\ z)$$

$$X_\theta = (4\cos\theta\cdot(-\text{sen}\,\theta),\ 2\cos^2\theta - 2\,\text{sen}^2\theta,\ 0)$$

$$X_z = (0,\ 0,\ 1)$$

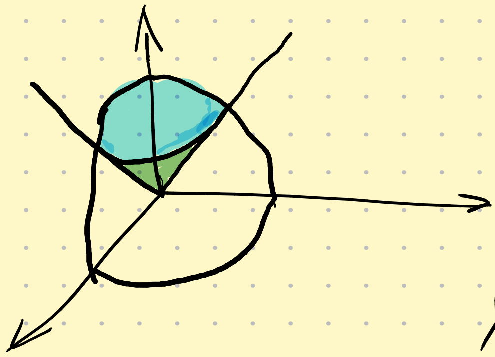
$$X_\theta \wedge X_z = (2\cos(2\theta),\ 4\cos\theta\,\text{sen}\,\theta,\ 0)$$

$$\|X_\theta \wedge X_z\| = \sqrt{(4\cos^2(2\theta) + 16\cos^2\theta\,\text{sen}^2\theta)}$$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \|X_\theta \wedge X_z\|\, dz\, d\theta = 8$$

8) Superfície do hem sup de esfera limitada pelo cone $z^2 = x^2 + y^2$

esf: $x^2 + y^2 + z^2 = 1$



param esfera

$$\begin{cases} x = 1 \operatorname{sen}\varphi \cos\theta \\ y = 1 \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = 1 \cos\varphi \end{cases} \qquad \begin{aligned} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{aligned}$$

$$X_\theta \wedge X_\varphi = \operatorname{sen}\varphi$$

intersecção: $2z^2 = 1 \quad z = \frac{\sqrt{2}}{2} \to x^2 + y^2 = \frac{1}{2}$

Sup 1: area do cone

$z^2 = x^2 + y^2$

$$\begin{cases} x = r\cos\theta & 0 \le \theta \le 2\pi \\ y = r\operatorname{sen}\theta & 0 \le z \le \sqrt{2}/2 \\ z = r \end{cases}$$

$X_\theta = (-r\operatorname{sen}\theta, r\cos\theta, 0)$

$X_r = (\cos\theta, \operatorname{sen}\theta, 1)$

$$X_\theta \wedge X_\varphi = (r\cos\theta, r\operatorname{sen}\theta, -r)$$

$$= \sqrt{r^2 + r^2} = r\sqrt{2}$$

$$\int_0^{2\pi} \int_0^{\sqrt{2}/2} r\sqrt{2} \, dr \, d\theta$$

$$\int_0^{2\pi} \frac{r^2}{2}\sqrt{2} \Big|_0^{\sqrt{2}/2} = \int_0^{2\pi} \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{2}}{4} \cdot 2\pi$$

$$= \frac{\sqrt{2}\pi}{2}$$

Sup 2: area no topo da esfera

$$\begin{cases} x = \operatorname{sen}\varphi\cos\theta \\ y = \operatorname{sen}\varphi\operatorname{sen}\theta \\ z = \cos\varphi \end{cases}$$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi/4$

$X_\theta = (-\operatorname{sen}\varphi\operatorname{sen}\theta, \operatorname{sen}\varphi\cos\theta, 0)$

$X_\varphi = (\cos\varphi\cos\theta, \cos\varphi\operatorname{sen}\theta, -\operatorname{sen}\varphi)$

$X_\theta \wedge X_\varphi = (-\operatorname{sen}^2\varphi\cos\theta, -\operatorname{sen}^2\operatorname{sen}\theta,$
$\operatorname{sen}\varphi\cos\varphi)$

$\|X_\theta \wedge X_\varphi\| = r^2 \operatorname{sen}\varphi = 1 \operatorname{sen}\varphi$

$$\int_0^{2\pi}\int_0^{\pi/4} \operatorname{sen}\varphi \, d\varphi \, d\theta = \int_0^{2\pi} -\cos\pi/4 + \cos 0$$

$$\sqrt{2}/2 \quad , \quad 2\pi \, \frac{2-\sqrt{2}}{2}$$

$$= (2-\sqrt{2})\pi$$

$$(2-\sqrt{2})\pi + \frac{\sqrt{2}}{2}\pi \Rightarrow \frac{\pi}{2}\left(4-2\sqrt{2}+\sqrt{2}\right) =$$

$$\frac{\pi}{2}\left(4-\sqrt{2}\right)$$

Ex 9) Sup do parabolóide $z = 5 - \frac{x^2}{2} - y^2$ que está no interior de $x^2+4y^2=4$

condição

$$x^2+4y^2 \leq 4$$

$$z = f(x,y) = 5 - \frac{x^2}{2} - y^2$$

$$S(x,y) = (x, y, f(x,y))$$

$$X_x \wedge X_y = \left(-\frac{\partial f(x,y)}{x}, +\frac{\partial f(x,y)}{y}, 1\right)$$

$$= (x, -2y, 1)$$

$$\begin{cases} x = u \\ y = v/2 \end{cases}$$

$$\|X_x \wedge X_y\| = \sqrt{x^2+4y^2+1}$$

$$Jac_1 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1/2 \end{vmatrix} = +\frac{1}{2}$$

$$\sqrt{u^2+v^2+1}$$

$$u^2+v^2 \leq 4$$

Mudança polares

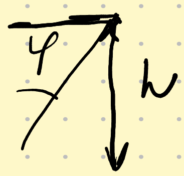
$$\begin{cases} u = r\cos\theta \\ v = r\,\text{sen}\,\theta \end{cases} \qquad Jac_2 = r \qquad \begin{matrix} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{matrix}$$

$$\int_0^{2\pi}\int_0^2 \sqrt{r^2+1}\; Jac_2\; Jac_1 \; dr\,d\theta$$

$$\int_0^{2\pi}\int_0^2 \sqrt{r^2+1}\; r \cdot \frac{1}{2} = \frac{1}{3}\left(5\sqrt{5}-1\right)\pi$$

Ex 10)

$$\begin{cases} x = a\,\text{sen}\,\varphi\cos\theta \\ y = a\,\text{sen}\,\varphi\,\text{sen}\,\theta \\ z = a\cos\varphi \end{cases} \qquad 0 \leq \theta \leq \alpha$$



$$\text{arcsen}\varphi \leq \varphi \leq$$

$$h = \text{arcsen}\,\varphi$$

$$\varphi = \text{arcsen}\,h$$

$$z = a\cos\varphi$$

$$h = h_2 - h_1$$

$$h_1 = a\cos\varphi_1 \qquad \varphi_1 = \arccos\frac{h_1}{a}$$

$$h_2 = a\cos\varphi_2 \qquad \varphi_2 = \arccos\frac{h_2}{a}$$

$$\int_0^{\alpha}\int_{\varphi_2}^{\varphi_1} a^2\,\text{sen}\,\varphi\,d\varphi\,d\theta$$

$$\int_0^{\alpha} -a^2\cos\varphi_1 + a^2\cos\varphi_2 = a\alpha\big(\overbrace{a\cos\varphi_2}^{h_2} - \overbrace{a\cos\varphi_1}^{h_1}\big)$$

$$= a\alpha\,h \;//$$