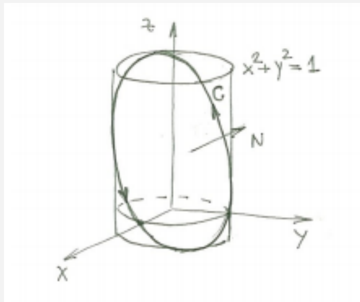
Teorema de Stokes I

$$\iint_S \text{rot}\, F \cdot \vec{n}\, dS = \int_{\partial S} \vec{F}\, dr$$

$$F = (P, Q, R) \qquad \text{rot}\, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Exemplo 1:

$$\oint F\, dr$$



$$F = (yz + x^3,\ 2xz + 3y^2,\ xy + 4)$$

$$C = \begin{cases} x^2 + y^2 = 1 \\ x + y + z = 1 \end{cases}$$

$$\int_C \vec{F}\, dr = \iint_S \text{rot}\, F \cdot \vec{n}\, dS$$

S = plano com bordo C

$$\text{rot}\, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz + x^3 & 2xz + 3y^2 & xy + 4 \end{vmatrix}$$

$$\text{rot}\, F = (x - 2x,\ y - y,\ 2z - z) = (-x,\ 0,\ +z)$$

Disco c/ bordo C:

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = 1 - r\cos\theta - r\,\text{sen}\,\theta \end{cases} \qquad \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}$$

$$\begin{array}{l} \quad\ \ i \qquad\quad\ \ j \qquad\qquad\quad k \\ X_r = (\cos\theta,\ \text{sen}\,\theta,\ -\cos\theta - \text{sen}\,\theta) \end{array}$$

$$X_\theta = (-r\,\text{sen}\,\theta,\ r\cos\theta,\ r\,\text{sen}\,\theta - r\cos\theta)$$

$$X_r \wedge X_\theta = (r\,\text{sen}^2\theta - r\cancel{\text{sen}\,\theta\cos\theta} + r\cos^2\theta + r\cancel{\text{sen}\,\theta\cos\theta},$$
$$-r\cancel{\text{sen}\,\theta}\cos\theta + r\cos^2\theta + r\cancel{\text{sen}\,\theta}\cos\theta + r\,\text{sen}^2\theta,$$
$$r\cos^2 + r\,\text{sen}^2\theta)$$

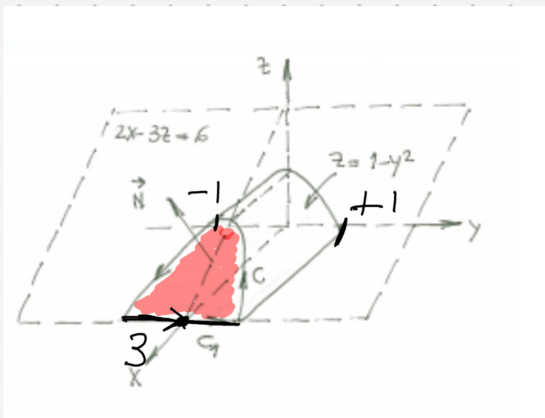
$X_r \wedge X_\theta = (r, r, r)$

$\text{rot } F = (-x, 0, +z) = (-r\cos\theta, 0, 1 - r\cos\theta - r\,\text{sen}\,\theta)$

$$\int_0^{2\pi}\int_0^1 = -r\cos\theta + 1 - r\cos\theta - r\,\text{sen}\,\theta \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -2r\cos\theta - r\,\text{sen}\,\theta + 1$$

Exemplo 2)



$F = (z^2, xz, 2xy)$

$$C \begin{cases} z = 1 - y^2, & z \geq 0 \\ 2x + 3z = 6 \end{cases}$$

$C_1 = (3, y, 0)\ -1 \leq y \leq 1$

$C_1' = (0, 1, 0)$

$$\text{rot } F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ z^2 & xz & 2xy \end{vmatrix}$$

$$= (2x - x, 2z - 2y, z)$$

$$\int_C \vec{F}\,d\vec{r} = \overset{\text{I}}{\iint_S \vec{F}\cdot\vec{n}} - \overset{\text{II}}{\int_{C_1} \vec{F}\,d\vec{r}}$$

II $\displaystyle\int_{-1}^1 F(x, y, z)\cdot(0, 1, 0) \Rightarrow \int_{-1}^1 0 + (1)(3\cdot 0) + 0$

$= 0$

(I) Parametrizando a sup:

$$\begin{cases} z = 1 - y^2 \,, \; z \geq 0 \\ 2x + 3z = 6 \end{cases}$$

Exercícios

Ex 1)

a) $$\oint_C y\,dx + z\,dy + x\,dz = -2\pi\sqrt{2}$$

$$\begin{cases} x + y = 2 \\ x^2 + y^2 + z^2 = 2(x+y) \end{cases}$$

$$\downarrow$$

$$(x-1)^2 + (y-1)^2 + z^2 = 2$$

$$F = (y, z, x)$$

$$\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y & z & x \end{vmatrix} = (-1, -1, -1)$$

$\to (1, 1, 0)$

Plano $(x, 2-x, z)$

$\vec{n} = (1, 1, 0)$

$\vec{F} = (y, z, x)$

$$\text{rot} F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y & z & x \end{vmatrix}$$

$$(-1, -1, -1)$$

$$N = \frac{(1, 1, 0)}{\sqrt{2}}$$

$$\int_\gamma F\,dr = \iint \text{rot} F \cdot \vec{N}\,dS$$

$$\Rightarrow \iint \frac{1}{\sqrt{2}}(-1, -1 + 0)\,dS = -\frac{2}{\sqrt{2}} \cdot \text{área}(S) = -\sqrt{2} \cdot (\sqrt{2})^2 \pi$$

$$= -2\sqrt{2}\pi$$

- Normal unitária, utilizado na imagem (para relacionar c/ a área)

  $$\iint F \cdot \vec{N}\,dS$$

- Normal $X_u \wedge X_v$ quando estamos utilizando o domínio da sup-

b) $\oint 2xy\,dx + \left[(1-y)z + x^2 + x\right]dy + \left(\frac{x^2}{2} + e^z\right)dz$

$$C \begin{cases} x^2 + y^2 = 1 & , z \geq 0 \\ z^2 = x^2 + (y-1)^2 \end{cases}$$

$$F = \left(2xy, \ (1-y)z + x^2 + x, \ \frac{x^2}{2} + e^z\right)$$

Plano de interseccão
$$\begin{cases} x = u \\ y = v \\ z = \sqrt{u^2 + (v-1)^2} \end{cases}$$

$$X_u = \left(1, 0, \frac{\cancel{2}u}{\cancel{2}\sqrt{u^2+(v-1)^2}}\right)$$

$$X_v = \left(0, 1, \frac{\cancel{2}(v-1)\cdot(1)}{\cancel{2}\sqrt{u^2+(v-1)^2}}\right)$$

$$X_u \wedge X_v = \left(\frac{-u}{\sqrt{u^2+(v-1)^2}}, \ \frac{-(v-1)}{\sqrt{u^2+(v-1)^2}}, \ 1\right)$$

$$\text{rot}\,F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 2xy & (1-y)z + x^2 + x & \frac{x^2}{2} + e^z \end{vmatrix}$$

$$\text{rot}\,F = \left(-1+y, -x, \cancel{2x} + 1 - \cancel{2x}\right) = (y-1, -x, +1)$$

$$= (v-1, -u, +1)$$

$$\iint \cancel{\frac{(-u)(v-1)}{\sqrt{\ }}} + \cancel{\frac{(-v-1)(-u)}{\sqrt{\ }}} + 1 \cdot 1$$

$$\iint 1 \, du \, dv = 1 \cdot \text{área}(D) = \text{área da circunferência que ama o cilindro}$$

$$= 1 \cdot (1)^2 \pi = \pi$$

c) $\oint (y+z)dx + (z+x)dy + (x+y)dz = 0$

$$C \begin{cases} x^2 + y^2 = 2y \\ y = z \end{cases}$$

$$\text{rot } F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y+z & z+x & x+y \end{vmatrix} = (0, 0, 0)$$

Superfície

$$\begin{cases} x = u \\ y = v \\ z = v \end{cases} \quad 0 \leq u^2 + (\cancel{y}v-1)^2 \leq 1$$

Coord polares

$$\begin{cases} u = r\cos\theta \\ v = r \,\text{sen}\,\theta \end{cases} \qquad \text{Jac} = r$$

$$0 \leq \theta \leq \pi \qquad 0 \leq r \leq 2\,\text{sen}\,\theta$$

$$u^2 + v^2 - 2\cancel{v} \cancel{+1} \leq \cancel{1}$$

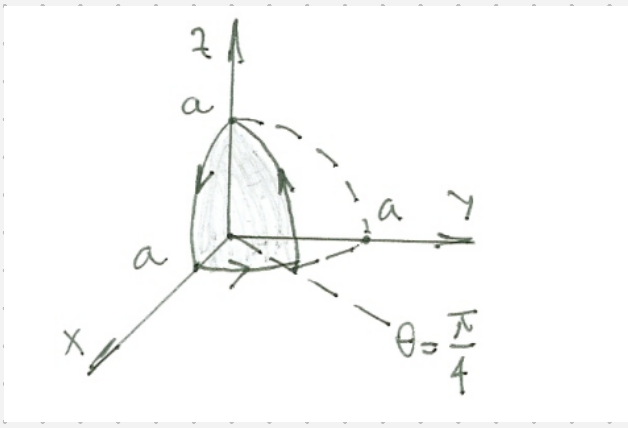
$$r^2 \leq 2\cancel{r}\,\text{sen}\,\theta$$

$$\iint 0 \, du \, dv$$

$$= \int_0^{\pi} \int_0^{2\,\text{sen}\,\theta} 0 \, r \, dr \, d\theta = 0$$

2) $\text{rot}\, F = (x, -2y, z)$

$\oint \vec{F}\, d\vec{r}$



Superficie

$$\begin{cases} x = a \,\text{sen}\varphi \cos\theta & 0 \leq \theta \leq \pi/4 \\ y = a\,\text{sen}\varphi\,\text{sen}\theta & 0 \leq \varphi \leq \pi/2 \\ z = a\cos\varphi \end{cases}$$

$$X_\theta = (-a\,\text{sen}\varphi\,\text{sen}\theta,\ a\,\text{sen}\varphi\cos\theta,\ 0)$$

$$X_\varphi = (a\cos\varphi\cos\theta,\ a\cos\varphi\,\text{sen}\theta,\ -a\,\text{sen}\varphi)$$

$$X_\theta \wedge X_\varphi = (-a\,\text{sen}^2\varphi\cos\theta,\ -a\,\text{sen}^2\varphi\,\text{sen}\theta,\ -a\,\text{sen}\varphi\cos\varphi\,\text{sen}^2\theta - a\,\text{sen}\varphi\cos\varphi\cos^2\theta)$$

$$X_\theta \wedge X_\varphi = (-a\,\text{sen}^2\varphi\cos\theta,\ -a\,\text{sen}^2\varphi\,\text{sen}\theta,\ -a\,\text{sen}\varphi\cos\varphi)$$

Normal correcto:

$$X_\theta \wedge X_\varphi = (a^2\,\text{sen}^2\varphi\cos\theta,\ a^2\,\text{sen}^2\varphi\,\text{sen}\theta,\ a^2\,\text{sen}\varphi\cos\varphi)$$

$$\text{rot}\, F = (x, -2y, z)$$

$$\int_0^{\pi/4}\int_0^{\frac{\pi}{2}} a^3\,\text{sen}^3\varphi\cos^2\theta - 2\cdot a^3\,\text{sen}^3\varphi\,\text{sen}^2\theta + a^3\,\text{sen}\varphi\cos^2\varphi$$

$$a^3 \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}} \text{sen}^3\varphi\cos^2\theta - 2\,\text{sen}^3\varphi\,\text{sen}^2\theta + \text{sen}\varphi\cos^2\varphi$$

$$= a^3/2$$

3) Circunferência de raio a no plano

$2x+2y+z=4$ centrado em $(1,2,-2)$

$F=(y-x,\ z-x,\ x-y)$ $\quad z=4-2x-2y$

Determine a p/:

$$\oint_C \vec{F}\,dr = -\frac{8\pi}{3} = \iint_S \operatorname{rot} F \cdot \underset{\text{unitário}}{\vec{n}}\, dS$$

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y-x & z-x & x-y \end{vmatrix} = (-2,-1,-2)$$

$$\vec{n} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = (2,2,1)$$

$$\vec{N} = \frac{(2,2,1)}{\sqrt{9}} = \frac{1}{3}(2,2,1)$$

$$\frac{1}{3}\iint_S (-2,-1,-2)(2,2,1)\,dS = \frac{(-4-2-2)}{3} \cdot \text{area (circ)}$$

$$= -\frac{8}{3}\pi a^2 \qquad (\text{fluxo}) = -\frac{8\cancel{\pi}}{\cancel{3}}$$

$$= a^2 = \frac{8}{8} = \boxed{1 = a}$$

Exercício 4)

$$\int_C (2xyz + \sin x)\,dx + (x^2 z + e^y)\,dy + \left(x^2 y + \frac{1}{z}\right) dz$$

C é a curva parametrizada por

$\gamma(t) = (\cos^3 t, \sin^2 t, (t+1)^2)$, $0 \leq t \leq \pi$

$$\text{rot } F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 2xyz + \sin x & x^2 z + e^y & x^2 y + \frac{1}{z} \end{vmatrix}$$

$= (0, 0, 0) = 0$

• achando função potencial:

$\frac{\partial \varphi}{\partial x} = 2xyz + \sin x \qquad \varphi = x^2 yz - \cos(x)$

$\frac{\partial \varphi}{\partial y} = x^2 z + e^y \qquad \varphi = x^2 zy + e^y$

$\frac{\partial \varphi}{\partial z} = x^2 y + \frac{1}{z} \qquad \varphi = x^2 zy + \ln(z)$

$\varphi = x^2 yz + e^y + \ln(z) - \cos(x)$

$\gamma(t) = (\cos^3 t, \sin^2 t, (t+1)^2)$, $0 \leq t \leq \pi$

$\varphi(\gamma(\pi)) - \varphi(\gamma(0)) = \varphi(-1, 0, (\pi+1)^2) - \varphi(1, 0, 1)$

$[\cancel{1} + \ln(\pi+1)^2 - \cancel{\cos}(-1)] - [\cancel{1} + \ln(1) - \cancel{\cos} 1]$

$= \ln((\pi+1)^2) = 2\ln(\pi+1)$

Ex 5) $\int_C (2xyz + 2x)\,dx + x^2z\,dy + x^2y\,dz$

C é a intersecção da superfície:

$z = \sqrt{4 - x^2 - y^2}$ com o plano $x + y = 2$

$$\text{rot } F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 2xyz + 2x & x^2z & x^2y \end{vmatrix}$$

$$= (0, 0, 0)$$

φ

$\frac{\partial \varphi}{\partial x} = 2xyz + 2x$ $\qquad \varphi = x^2yz + x^2$

$\frac{\partial \varphi}{\partial y} = x^2z$ $\qquad \varphi = x^2yz$

$\frac{\partial \varphi}{\partial z} = x^2y$ $\qquad \varphi = x^2yz$

de $(0,2,0)$ para $(2,0,0)$:

$\varphi(2,0,0) - \varphi(0,2,0) = 0 + 4 - 0 + 0 = 4$ //

Ex 6)

$$F = \left(e^x \sin(y) + \frac{x}{x^2+y^2},\ e^x \cos(y) + \frac{y}{x^2+y^2},\ z^2 \right)$$

$e^x \sin(y) + \frac{1}{2}\ln(x^2+y^2)$ $\quad$ $e^x \operatorname{sen}(y) + \frac{1}{2}\ln(x^2+y^2)$ $\quad$ $z^3/3$

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ P & Q & R \end{vmatrix}$$

$$\varphi = e^x \sin(y) + \frac{1}{2}\ln(x^2+y^2) + z^3/3$$

$$= \left(0,\ 0,\ e^x\cos(y) + y\ln(x^2+y^2)\cdot 2x - e^x\cos(y) - x\ln(x^2+y^2)\cdot 2y\right)$$

$$= \vec{0}$$

O campo possui restrição de existência para o eixo z. Porém, como possui função potencial

$$\varphi = e^x \sin(y) + \frac{1}{2}\ln(x^2+y^2) + \frac{z^3}{3}$$

então qq curva fechada q não contenha o eixo z possui integral de linha $\underline{\underline{0}}$.