Integral de sup de função escalar/vetorial

Ex 1)

a) $\iint_S (x^2+y^2)\,dS$ S é a esfera $x^2+y^2+z^2=a^2$

Parametrização

$x = a\,\text{sen}\,\varphi\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$y = a\,\text{sen}\,\varphi\,\text{sen}\,\varphi$ $\quad 0 \le \varphi \le \pi$

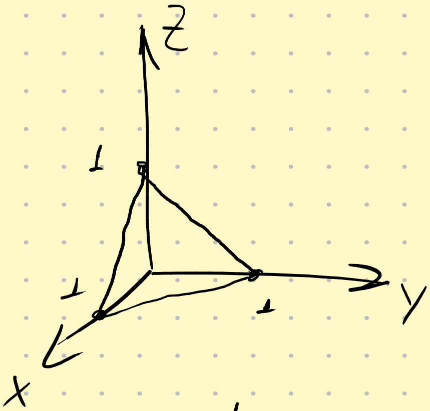
$z = a\cos\varphi$

$\|X_\theta \wedge X_\varphi\| = a^2\,\text{sen}\,\varphi$

$$\int_0^{2\pi}\int_0^{\pi} a^2\,\text{sen}^2\varphi\, a^2\,\text{sen}\,\varphi \Rightarrow a^4\int_0^{2\pi}\int_0^{\pi}\text{sen}^3\varphi\,d\varphi\,d\theta$$

$$\Rightarrow a^4\cdot 2\pi\cdot\frac{4}{3} = \frac{8\pi a^4}{3}$$

b) $\iint_S xyz\,dS$ S triang $(1,0,0),(0,1,0),(0,0,1)$



eq do plano

$$\begin{vmatrix} x-1 & y & z \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (x-1)(1)+(y)(1)+z$$

$x-1+y+z=0$

$z = 1-x-y$

parametrização por gráfico

$$\begin{cases} x = x \\ y = y \\ z = 1-x-y \end{cases} \qquad \begin{aligned} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{aligned}$$

$\leftarrow$ Parametrização do segue a ideia fubini

$$X_x \wedge X_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = (1, 1, 1)$$

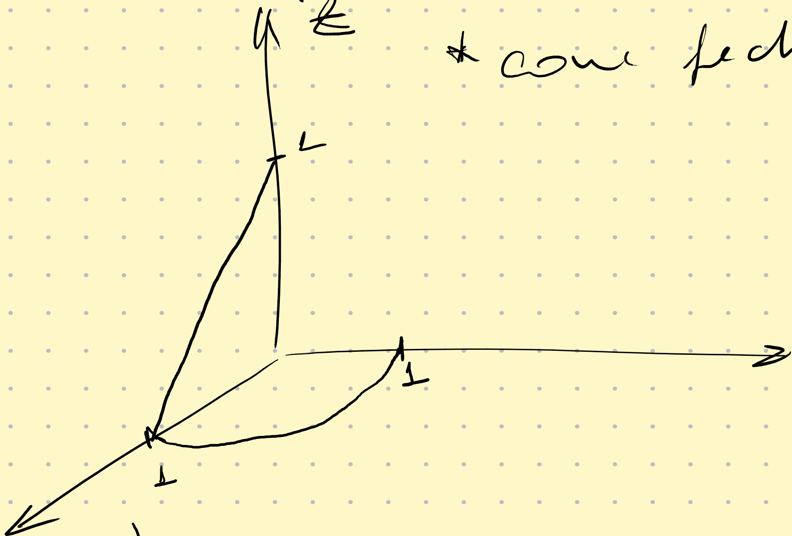
$$\|X_x \wedge X_y\| = \sqrt{3}$$

$$\int_0^1 \int_0^{1-x} xy(1-x-y)\sqrt{3}\, dy\, dx$$

$$\sqrt{3}\int_0^1 \int_0^{1-x} xy - x^2y - xy^2\, dy\, dx = \frac{\sqrt{3}}{120}$$

Ex 2)

* cone fechado *



S_1 : cone (I)

+

S_2 : disco (II)

$\gamma(t) = (t, 1-t)$

(I) $\begin{cases} x = t\cos\theta \\ y = t\,\text{sen}\,\theta \\ z = 1-t \end{cases}$ $\quad 0 \leq t \leq 1$, $\quad 0 \leq \theta \leq 2\pi$

$X_\theta = (-t\,\text{sen}\,\theta, t\cos\theta, 0)$

$X_t = (\cos\theta, \text{sen}\,\theta, -1)$

$X_\theta \wedge X_t = (-t\cos\theta, -t\,\text{sen}\,\theta, -t)$

$\|X_\theta \wedge X_t\| \sqrt{t^2+t^2} = t\sqrt{2}$

$$\int_0^{2\pi}\int_0^1 t\sqrt{2}\cdot t\, dt\, d\theta = \frac{2\pi\sqrt{2}}{3}\, t^3\Big|_0^1 = \frac{2\pi\sqrt{2}}{3}$$

(II) $\begin{cases} x = t\cos\theta \\ y = t\,\mathrm{sen}\,\theta \\ z = 0 \end{cases}$ $\quad 0 \le t \le 1$, $0 \le \theta \le 2\pi$

$X_\theta = (-t\,\mathrm{sen}\,\theta,\ t\cos\theta,\ 0)$ $\quad X_\theta \wedge X_t = (0,\ 0,\ -t)$

$X_t = (\cos\theta,\ \mathrm{sen}\,\theta,\ 0)$ $\quad \|X_\theta \wedge X_t\| = t$

$$\int_0^{2\pi}\int_0^1 t \cdot t \, dt\, d\theta = 2\pi \frac{t^3}{3}\Big|_0^1 = \frac{2\pi}{3}$$

Massa total: $\dfrac{2\sqrt{2}\pi}{3} + \dfrac{2\pi}{3} = \boxed{\dfrac{2\pi(1+\sqrt{2})}{3}}$

Ex 3) $\iint_S F \cdot \vec{N}\, dS \quad F = (x, y, -2z)$

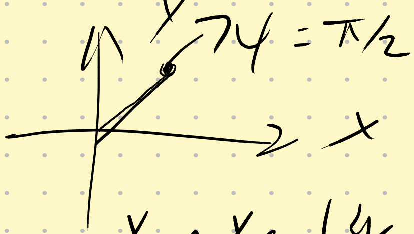
S esfera $x^2 + y^2 + z^2 = 4$

$\begin{cases} x = 2\,\mathrm{sen}\,\varphi\cos\theta \\ y = 2\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta \\ z = 2\cos\varphi \end{cases}$

$X_\theta = (-2\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta,\ 2\,\mathrm{sen}\,\varphi\cos\theta,\ 0)$

$X_\varphi = (2\cos\varphi\cos\theta,\ 2\cos\varphi\,\mathrm{sen}\,\theta,\ -2\,\mathrm{sen}\,\varphi)$

$X_\theta \wedge X_\varphi = (-4\,\mathrm{sen}^2\varphi\cos\theta,\ -4\,\mathrm{sen}^2\varphi\,\mathrm{sen}\,\theta,\ -4\,\mathrm{sen}\,\varphi\cos\varphi)$



$\to \varphi = \pi/2 \quad \theta = \pi/4$

vetor normal pedido:

$$X_\theta \wedge X_\varphi = (4\,\text{sen}^2\varphi\cos\theta,\ 4\,\text{sen}^2\varphi\,\text{sen}\,\theta,\ 4\,\text{sen}\,\varphi\cos\varphi)$$

$$F = (2\,\text{sen}\,\varphi\cos\theta,\ 2\,\text{sen}\,\varphi\,\text{sen}\,\theta,\ -4\cos\varphi)$$

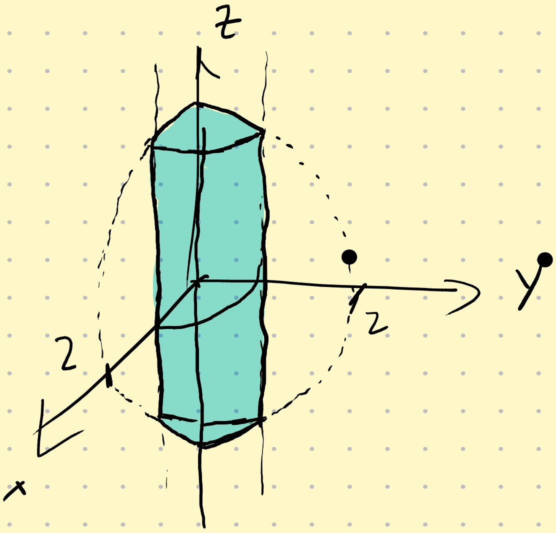
$$\langle F, X_\theta \wedge X_\varphi\rangle = 8\,\text{sen}^3\varphi\cos^2\theta + 8\,\text{sen}^3\varphi\,\text{sen}^2\theta - 16\,\text{sen}\,\varphi\cos^2\varphi$$

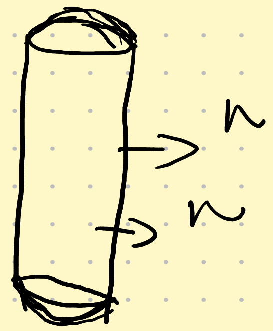
$$8\,\text{sen}^3\varphi - 16\,\text{sen}\,\varphi\cos^2\varphi$$

$$\int_0^{2\pi}\int_0^{\pi} 8\,\text{sen}^3\varphi - 16\,\text{sen}\,\varphi\cos^2\varphi\, d\varphi\, d\theta = 0$$

Ex5) $\iint_S F\cdot N\, dS \qquad F = (x, y, z)$

$W = \{ x^2+y^2 \leq 1 \text{ e } x^2+y^2+z^2 \leq 4 \}$ $\quad \vec{n}$ exterior



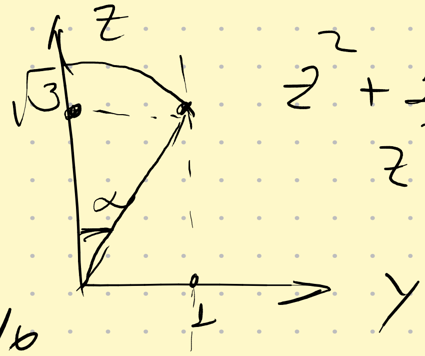


2 sup: cilindro (I) + topo esf (II) + inferior esf (III)

(II)

cilindro raio 2

$$\begin{cases} x = 2\,\text{sen}\,\varphi \cos\theta \\ y = 2\,\text{sen}\,\varphi\,\text{sen}\,\theta \\ z = 2\cos\theta \end{cases} \qquad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi/6 \end{matrix}$$



$z^2 + 1 = 4$

$z^2 = 3$

$tg\,\alpha = \sqrt{3}/3$

$\alpha = 30°$

$$X_\theta \wedge X_\varphi = (4\,\text{sen}^2\varphi\cos\theta,\ 4\,\text{sen}^2\varphi\,\text{sen}\,\theta,\ 4\,\text{sen}\,\varphi\cos\varphi)$$

$$F = (2\,\text{sen}\,\varphi\cos\theta,\ 2\,\text{sen}\,\varphi\,\text{sen}\,\theta,\ 2\cos\varphi)$$

$$\langle F, X_\theta \wedge X_\varphi \rangle = 8\,\text{sen}^3\varphi\cos^2\theta + 8\,\text{sen}^3\varphi\,\text{sen}^2\theta + 8\,\text{sen}\,\varphi\cos^2\varphi$$

$$8\,\text{sen}^3\varphi + 8\,\text{sen}\,\varphi\cos^2\varphi \Rightarrow 8\,\text{sen}\,\varphi$$

$$\int_0^{2\pi}\int_0^{\pi/6} 8\,\text{sen}\,\varphi\, d\varphi\, d\theta = -8(\sqrt{3}-2)\pi$$

III $\displaystyle\int_0^{2\pi}\int_{7\pi/6}^{\pi} 8\,\text{sen}\,\varphi\, d\varphi\, d\theta = -8(\sqrt{3}-2)\pi$

I Cilindro

$$\begin{cases} x = \cos\theta \\ y = \text{sen}\,\theta \\ z = z \end{cases} \qquad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le z \le \sqrt{3} \end{matrix} \qquad \begin{matrix} X_\theta = (-\text{sen}\,\theta, \cos\theta, 0) \\ X_z = (0, 0, 1) \end{matrix}$$

$$X_\theta \wedge X_z = (\cos\theta, \text{sen}\,\theta, 0)$$ vetor radial no plano Oxy

$$F = (\cos\theta, \text{sen}\,\theta, z)$$

$$\langle F, X_\theta \wedge X_z \rangle = \cos^2\theta + \text{sen}^2\theta = 1$$

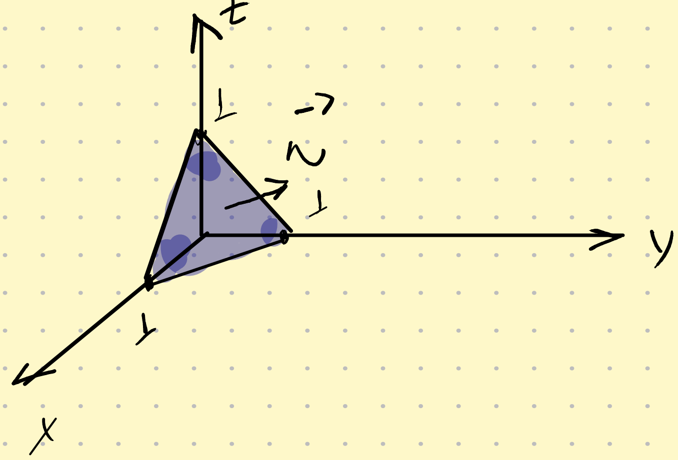
$$\int_0^{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} 1\, dp\, d\theta = 4\sqrt{3}\pi$$

$$I + II + III = 4\sqrt{3}\pi - 16(\sqrt{3}-2)\pi$$
$$4\sqrt{3}\pi - 16\sqrt{3}\pi + 32\pi$$
$$4\pi(8 - 3\sqrt{3})$$

Ex 4) $\iint_S \vec{F}\,\vec{N}\,dS \qquad F = (x,y,z)$

S é o triang de vértices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

vet normal se afastando da origem



parametrização do plano:

$$\begin{vmatrix} x-1 & y & z \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$(x-1) + y + z = 0 \qquad z = 1 - x - y$$

$$S(x,y) = \begin{cases} x = x \\ y = y \\ z = 1 - x - y \end{cases} \qquad f(x,y) = 1 - x - y$$

$$X_x \wedge X_y = (1, 1, 1)$$

$$F = (x, y, 1-x-y)$$

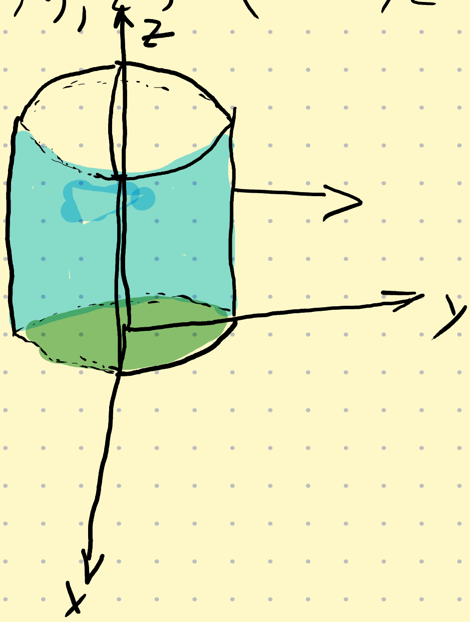
$$\langle F, X_x \wedge X_y \rangle = x + y + 1 - x - y = 1$$

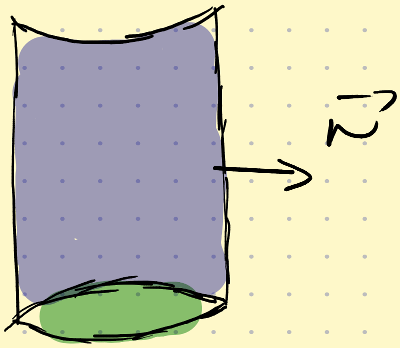
$$\int_0^1 \int_0^{1-x} 1 \, dy\,dx = \boxed{1/2}$$

Ex 6)

$$\iint_S F\,\vec{N}\,ds$$

$$F(x,y,z) = (-3xyz^2,\ x + 2yz - 2xz^4,\ yz^3 - z^2)$$





calculamos:

$$\begin{cases} x = \cos\theta \\ y = \operatorname{sen}\theta \\ z = z \end{cases} \quad \begin{matrix} 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq 1 \end{matrix}$$

$$X_\theta = (-\operatorname{sen}\theta,\ \cos\theta,\ 0)$$
$$X_z = (0,\ 0,\ 1)$$

$$X_\theta \wedge X_z = (\cos\theta,\ \operatorname{sen}\theta,\ 0)$$

$$F = (-3\cos\theta\operatorname{sen}\theta z^2,\ \cos\theta + 2\operatorname{sen}\theta z - 2\cos\theta z^4,\ \operatorname{sen}\theta z^3 - z^2)$$

$$\langle F, X_\theta \wedge X_z \rangle = -3\cos^2\theta\operatorname{sen}\theta z^2 + \cos\theta\operatorname{sen}\theta + 2\operatorname{sen}^2\theta z - 2\cos\theta\operatorname{sen}\theta z^4$$

$$\int_0^{2\pi}\int_0^1 \langle F, X_\theta \wedge X_z \rangle = \pi$$

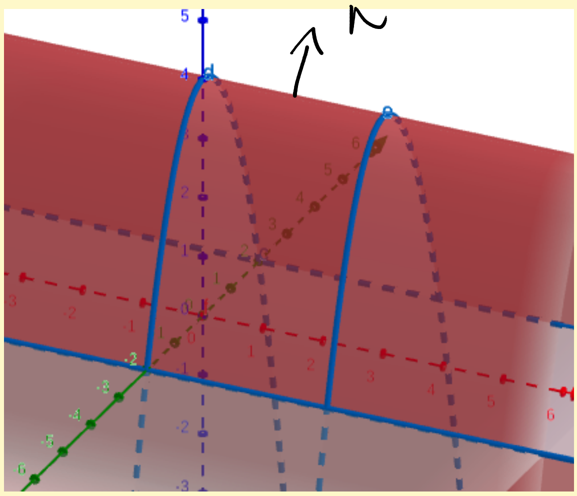
Ex 7) $F = (z^2 - x, -xy, 3z)$

$S: z = 4 - y^2, x = 0, x = 3, z = 0$

Parte da calha

$$S_1(x,y) = \begin{cases} x = x \\ y = y \\ z = 4 - y^2 \end{cases} \quad \begin{matrix} -2 \leq y \leq 2 \\ 0 \leq x \leq 3 \end{matrix}$$

$\vec{n}$



$X_x \wedge X_y = (0, +2y, 1)$

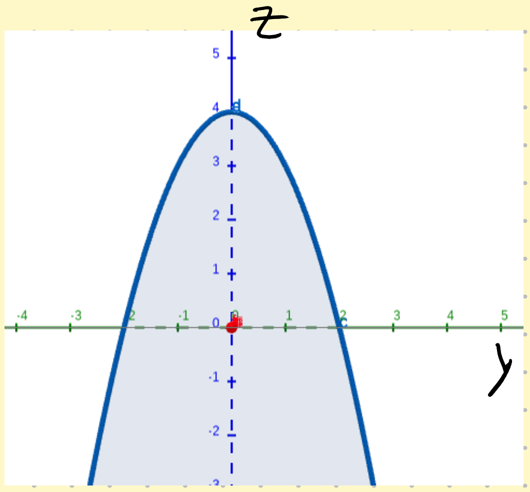
Azul:

$F = ((4-y^2)^2 - x, -xy, 3(4-y^2))$

$= (16 - 8y^2 + y^4 - x, -xy, 12 - 3y^2)$

$\langle F, X_x \wedge X_y \rangle = -2xy^2 + 12 - 3y^2$

$$\int_0^3 \int_{-2}^{2} -2xy^2 + 12 - 3y^2 \, dy \, dx = 48$$



$S_2: x = 0$

$$S_2(y,z) \begin{cases} x = 0 \\ y = y \\ z = z \end{cases} \quad \begin{matrix} -2 \leq y \leq 2 \\ 0 \leq z \leq 4 - y^2 \end{matrix}$$

$X_y = (0, 1, 0)$ $X_z = (0, 0, 1)$

$X_y \wedge X_z = (1, 0, 0)$

vetor n $(-1, 0, 0)$

$\langle F, X_y \wedge X_z \rangle = -z^2$

$F = (z^2, 0, 3z)$

$$-\int_{-2}^{2} \int_0^{4-y^2} z^2 \, dz \, dy = -\frac{4096}{105}$$

$S_3 \quad x=3 \qquad -2 \leq y \leq 2$

$\begin{cases} x=3 \\ y=y \\ z=z \end{cases} \quad S(y,z) \qquad 0 \leq z \leq 4-y^2$

$X_y \wedge X_z = (1,0,0)$

$F = (z^2-3, -3y, 3z)$

$\langle F, X_y \wedge X_z \rangle = z^2-3$

$$\int_{-2}^{2} \int_{0}^{4-y^2} z^2-3 \, dz\, dy = \frac{736}{105}$$

$S_4 : z=0$

$\begin{cases} x=x \\ y=y \\ z=0 \end{cases} \qquad X_x = (1,0,0) \quad X_y = (0,1,0)$

$X_x \wedge X_y = (0,0,1)$

$F = (-x, -xy, 0)$

$\langle F, X_x \wedge X_y \rangle = 0$

$\iint 0 = 0$ //

$$S_1 + S_2 + S_3 + S_4 = 48 - \frac{4096}{105} + \frac{736}{105}$$

$$= 48 - 32 = 16 //$$

Ex 8) $E(x,y,z) = \dfrac{\varepsilon q}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$

$x^2+y^2+z^2 = R^2$

$$\begin{cases} x = R\,\text{sen}\,\varphi\cos\theta \\ y = R\,\text{sen}\,\varphi\,\text{sen}\,\theta \\ z = R\cos\varphi \end{cases}$$

$X_\theta = (-R\,\text{sen}\,\varphi\,\text{sen}\,\theta,\ R\,\text{sen}\,\varphi\cos\theta,\ 0)$

$X_\varphi = (R\cos\varphi\cos\theta,\ R\cos\varphi\,\text{sen}\,\theta,\ -R\,\text{sen}\,\varphi)$

* outro vetor normal *

$X_\theta \wedge X_\varphi = (+R^2\text{sen}^2\varphi\cos\theta,\ +R^2\text{sen}^2\varphi\,\text{sen}\,\theta,\ +R^2\text{sen}\,\varphi\cos\varphi)$;

$\vec{F} = \dfrac{\varepsilon q}{R^3}(R\,\text{sen}\,\varphi\cos\theta,\ R\,\text{sen}\,\varphi\,\text{sen}\,\theta,\ R\cos\varphi)$

$\langle F, X_\theta\wedge X_\varphi\rangle = +\varepsilon q\,\text{sen}^3\varphi\cos^2\theta + \varepsilon q\,\text{sen}^3\varphi\,\text{sen}^2\theta + \varepsilon q\,\text{sen}\,\varphi\cos^2\varphi$

$\langle F, X_\theta\wedge X_\varphi\rangle = +\varepsilon q(\text{sen}^3\varphi + \text{sen}\,\varphi\cos^2\varphi)$
$= +\varepsilon q\,\text{sen}\,\varphi$

$$\int_0^{2\pi}\int_0^{\pi} \varepsilon q\,\text{sen}\,\varphi\,d\varphi\,d\theta = 4\pi\varepsilon q$$