Teorema de gauss

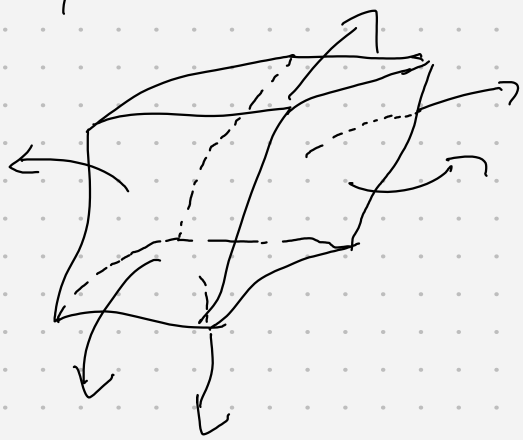
Divergente

$F = (P, Q, R)$

$\operatorname{div} F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$ (número real)

representa a saída de massa em um ponto

$\operatorname{div} F(x,y,z)$



Teo. Gauss

$$\iiint_V \operatorname{div} \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot \vec{n}\, dS$$

→ produto escalar do campo com o vetor normal da sup.

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\, dy\, dz = \iint_{\partial V} P \cdot dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$$

$F \cdot (X_u \wedge X_v)$ ↙ vetor normal à sup parametrizada

$\operatorname{div} F = 0$ (campo incompressível)

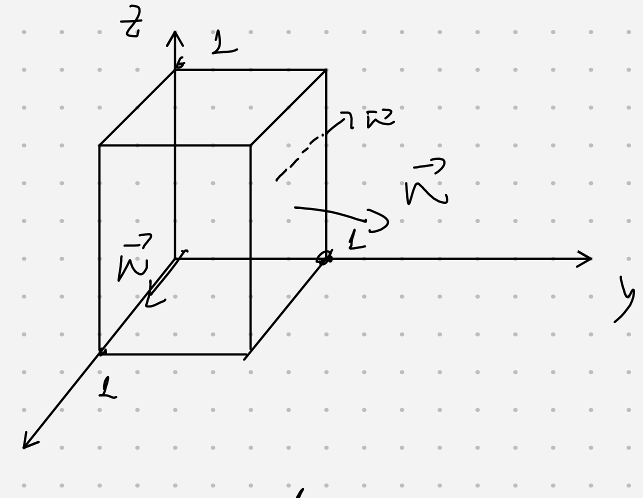
↳ integral de superfície $== 0$

Exemplo 1:

fluxo $F = (xy,\ 4yz^2,\ -yz)$

no cubo S

$S =$



em cada uma das superfícies

$$\iint_S \vec{F}\cdot\vec{n} = \iiint_{\text{cubo}} \operatorname{div} F$$

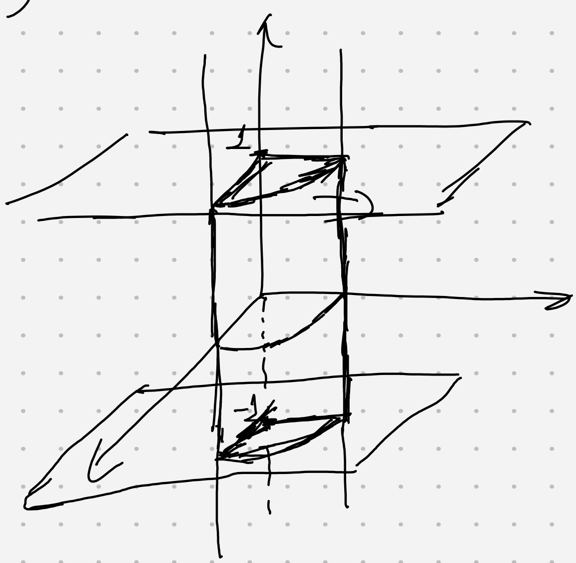
$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \cancel{y} + 4z^2 - \cancel{y} = 4z^2$$

$$\int_0^1\int_0^1\int_0^1 4z^2\,dz\,dy\,dx = \frac{4z^3}{3}\Big|_0^1 = \boxed{\frac{4}{3}}$$

Ex 2: fluxo $F = (xy^2,\ x^2y,\ y)$

$\vec{n}$: normal exterior

$$S = \begin{cases} x^2+y^2=1 \\ -1<z<1 \end{cases}$$



$\operatorname{div} = y^2 + x^2 + 0 = x^2+y^2$

coord cilíndricas

$$\begin{cases} x = r\cos\theta & 0\leq r\leq 1 \\ y = r\,\text{sen}\,\theta & 0\leq\theta\leq 2\pi \\ z = z & -1\leq z\leq 1 \end{cases}$$

$\text{Jac} = r$

$$\operatorname{div} F = x^2+y^2 = (r\cos\theta)^2 + (r\,\text{sen}\,\theta)^2 = r^2$$

Mudança de coordenadas

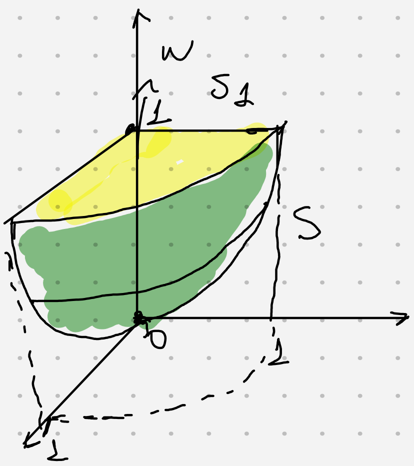
$$\int_0^{2\pi}\int_{-1}^{1}\int_0^1 r^2 \cdot r \, dr\, dz\, d\theta = 2\pi \int_{-1}^{1}\int_0^1 r^3\, dr = \left.\frac{r^4}{4}\right|_0^1 2\pi$$

$$= 2\pi \cdot \frac{1}{4} \cdot \int_{-1}^{1} 1\, dz = \frac{2\pi}{4}(1+1)$$

$$= \boxed{\pi}$$

Exemplo 3: $F = (x, y, -2z)$

S = parabolóide $z = x^2 + y^2$ $\quad 0 \leq z \leq 1$ $\quad \vec{n}$: vetor normal se afasta de z



$\operatorname{div} F = 1 + 1 - 2 = 0$

$$\iiint_V \operatorname{div} F = 0$$

$$\iint_{S \cup S_1} F \cdot \vec{n} = \iiint_V \operatorname{div} F$$

$$\iint_S F \cdot \vec{n} - \iint_{S_1} F \cdot \vec{n} \quad (*)$$

Parametrização superfície S_1:

$$\begin{cases} x = r\cos\theta & 0 \leq r \leq 1 \\ y = r\,\text{sen}\,\theta & 0 \leq \theta \leq 2\pi \\ z = 1 \end{cases}$$

$$\begin{array}{cccc} & i & j & k \\ X_\theta = & (-r\,\text{sen}\,\theta, & r\cos\theta, & 0) \\ X_r = & (\cos\theta, & \text{sen}\,\theta, & 0) \end{array}$$

$X_\theta \wedge X_{r} = (0, 0, -r)$

$\neq$ Normal errada, portanto normal utilizada $(0, 0, r)$

$$\circledast \iint_{S_1} \vec{F}\cdot\vec{n} = \int_0^{2\pi}\int_0^1 (r\cos\theta, r\,\text{sen}\,\theta, -2)\cdot(0,0,r)$$

$$= \int_0^{2\pi}\int_0^1 -2r\,dr\,d\theta = 2\pi \; -\frac{\cancel{2}r^2}{\cancel{2}}\Big|_0^1 = -2\pi$$

$$\iint_S F\cdot\vec{n} = -\iint_{S_1} \vec{F}\cdot\vec{n} = -(-2\pi) = 2\pi$$

Exemplo 4:

$$E(x,y,z) = \frac{\varepsilon q}{(x^2+y^2+z^2)^{3/2}}(x,y,z) \qquad (x^2+y^2+z^2) = (\;)$$

$$\text{div}\,E = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \Rightarrow \varepsilon q\left[\frac{(\;)^{3/2} - 3(\;)^{1/2}x^2}{(\;)^3} + \ldots\right.$$

$$\frac{(\;)^{3/2} - 3(\;)^{1/2}x^2}{(\;)^3} + \frac{(\;)^{3/2} - 3(\;)^{1/2}y^2}{(\;)^3} + \frac{(\;)^{3/2} - 3(\;)^{1/2}z^2}{(\;)^3}$$

$$x^2+y^2+z^2 = (\;)$$

$$\frac{3(\;)^{3/2} - 3(\;)^{3/2}}{(\;)^3} = 0$$

Exercícios

Exercício 2) $F = \left(e^y + \cos yz, -2zy + \sin(xz), z^2 + \frac{3}{\sqrt{2}}\right)$

sup S com $\vec{n}$ exterior

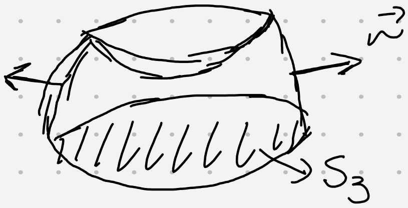
$S = S_1 \cup S_2$

$\operatorname{div} F = 0 - 2z + 2z = 0$

$S_1: z = 4 - 2x^2 - y^2, \quad 0 \leq z \leq 2$

$S_2: z = 1 + x^2 + \frac{y^2}{2}, \quad 1 \leq z \leq 2$

Sólido:



$S_3 = 2x^2 + y^2 = 4, \; z = 0$
$= \frac{x^2}{2} + \frac{y^2}{4} = 1$

$\underbrace{\iint_{S_1 \cup S_2} F \cdot \vec{n}}_{*} + \iint_{S_3} F \cdot \vec{n} = \iiint_V \cancel{\operatorname{div} F}^{\,0}$

s/ singularidade

$0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$

$S = \begin{cases} x = \sqrt{2}\, r\cos\theta \\ y = 2r\,\text{sen}\,\theta \\ z = 0 \end{cases}$

$* = -\iint_{S_3} F \cdot \vec{n}$

$X_r = (+\sqrt{2}\cos\theta, \; 2\sin\theta, 0)$ $\quad X_r \wedge X_\theta = (0, 0, 2\sqrt{2}r)$

$X_\theta = (-\sqrt{2}r\sin\theta, \; 2r\cos\theta, 0)$ Normal correta:

$= (0, 0, -2\sqrt{2}r)$

$F = (P, Q, z^2 + \frac{3}{\sqrt{2}})$

$F = (P, Q, 3/\sqrt{2})$

$$\int_0^{2\pi}\int_0^1 F\cdot\vec{n} = \int_0^{2\pi}\int_0^1 -2\cancel{\sqrt{2}}\cdot\frac{3}{\cancel{\sqrt{2}}}r = -6\cdot 2\pi\int_0^1 r\,dr$$

$$= -\sqrt{6}\cdot 2\pi\left.\frac{r^2}{2}\right|_0^1 = -6\pi$$

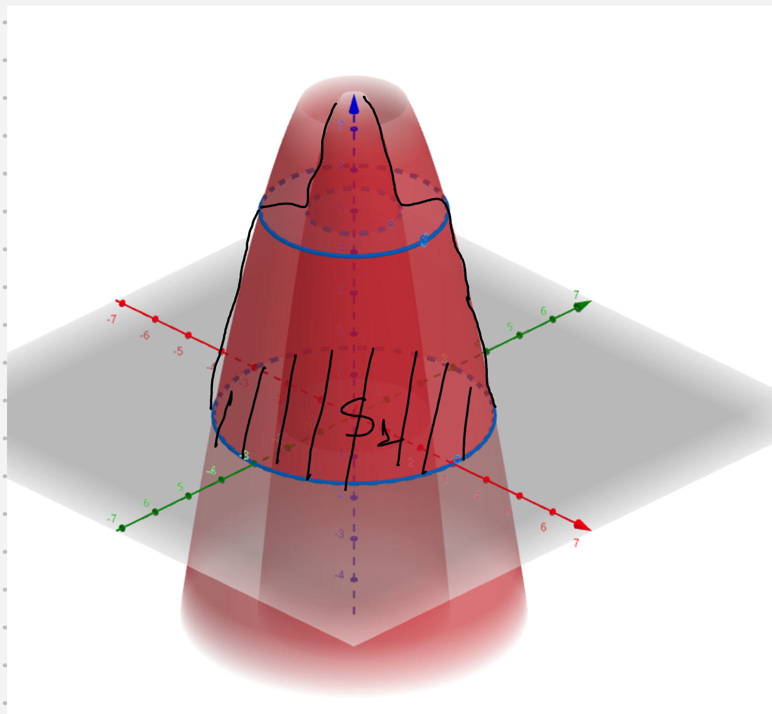
resp $= -(-6\pi)$

Exercício 3)

S1 singularidade

$F = (x, -2y + e^x\cos z, z + \cancel{x}^2)$ $\quad \operatorname{div} F = 0$

Superfície $S: \begin{cases} z = 9 - (x^2+y^2) \\ z = 5 \\ z = 8 - 3(x^2+y^2) \end{cases}$ $\quad \begin{matrix} 0 \le z \le 5 \\ 1 \le x^2+y^2 \le 4 \\ x^2+y^2 \le 1 \end{matrix}$



Definindo S_1 como $X_\theta = (-r\operatorname{sen}\theta, r\cos\theta, 0)$

$$\begin{cases} x = r\cos\theta \\ y = r\operatorname{sen}\theta \\ z = 0 \end{cases} \quad \begin{matrix} 0 \leq r \leq 3 \\ 0 \leq \theta \leq 2\pi \end{matrix} \qquad X_r = (\cos\theta, \operatorname{sen}\theta, 0)$$

$X_\theta \wedge X_r = (0, 0, -r)$ $\quad F = (x, -2y + e^x\cos z, z + x^2)$

$$\int_0^3 \int_0^{2\pi} 0 + 0 + (-r)(0 + r^2\cos^2\theta)\, dr\, d\theta$$

$$\int_0^{2\pi} \int_0^3 -r^3\cos\theta = -\int_0^{2\pi} \cos^2\theta \cdot \left(\frac{r^4}{4} \Big|_0^3 \right)$$

$$= -\frac{81}{4} \int_0^{2\pi} \cos^2\theta\, d\theta = \left(-\frac{81}{4}\right)\left(\frac{1}{2}\left(x + \sin(x)\cos(x)\right)\Big|_0^{2\pi}\right)$$

$$= -\left(\frac{81}{4}\right)\left(\frac{1}{2} \cdot 2\pi\right) = -81\pi/4$$

$$* \iint_S F \cdot \vec{n} = -\iint_{S_1} \vec{F} \cdot \vec{n} = -\left(-\frac{81\pi}{4}\right)$$

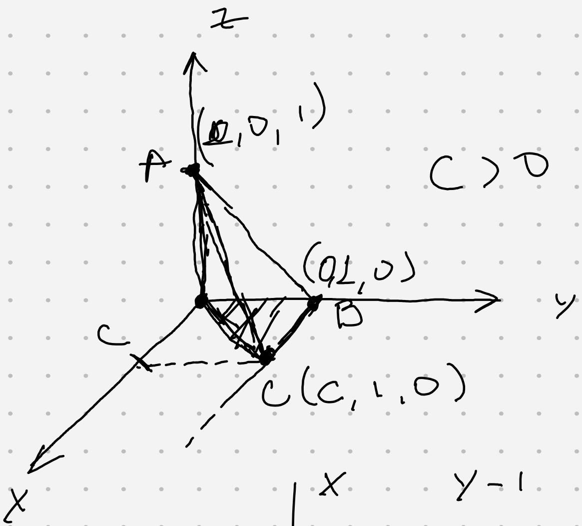
Exercício 4)

$F = (x, y, z)$

$\operatorname{div} F = 3$

Volume da pirâmide

$$\frac{\frac{1}{2} \cdot c}{3} = \frac{c}{6}$$



$$\iint_{S_{ABC}} F \cdot N$$

$$\begin{vmatrix} x & y-1 & z \\ c & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0$$

$$x\,0 + (y-1)(-c) + z(-c) = 0$$

$-cy + c - cz = 0$ $\quad X_u = (1,0,0)$ $\quad X_u \wedge X_v = (0,1,1)$

$\begin{cases} x = u & 0 \leq u \leq cv \\ y = v & 0 \leq v \leq 1 \\ z = 1 - v \end{cases}$ $\quad X_v = (0,1,-1)$ $\quad F = (x,y,z)$

$\iint \cancel{v} + 1 - \cancel{v}$

$$\int_0^1 \int_0^{cv} 1 \, du\, dv = \int_0^1 cv \, dv = \boxed{\frac{c}{2}}$$

$$\underset{\text{piramide}}{\iiint} 3 = 3 \cdot \text{Volume piramide} = 3 \frac{c}{6} = \boxed{\frac{c}{2}}$$

$$\frac{c}{2} + \frac{c}{2} = 1 \Rightarrow \boxed{c = 1}$$

Ex 3) Fluxo

$$F = (x, y, z) \frac{1}{(x^2+y^2+z^2)}$$

Singularidade em $(0,0,0)$

$$\operatorname{div} F = \frac{(\ ) - 2x^2}{(\ )^2} + \frac{(\ ) - 2y^2}{(\ )^2} + \frac{(\ ) - 2z^2}{(\ )^2}$$

$$\frac{3(\ ) - 2(\ )}{(\ )^2} = \frac{(\ )}{(\ )^2} = \frac{1}{(\ )} = \frac{1}{x^2+y^2+z^2}$$

Mudança de coordenadas

$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta & a \leq \rho \leq b \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta & 0 \leq \varphi \leq \pi \\ z = \rho \cos\varphi & 0 \leq \theta \leq 2\pi \end{cases}$ $\quad J_{ac} = \rho^2 \operatorname{sen}\varphi$

$$\int_0^{2\pi} \int_0^{\pi} \int_a^b \frac{1}{\cancel{\rho^2}} \cancel{\rho^2} \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta$$

$$2\pi(b-a) \underbrace{\left(-\cos\varphi \Big|_0^{\pi}\right)}_{-(-1)+1 = 2} = 4\pi(b-a)$$