① Demonstrate that the minimum of cross-entropy is obtained when the two probabilities are equal.

Cross-Entropy: $\quad \mathcal{H}(p,q) = \mathcal{H}(q) = -\sum_{x} p(x) \log_2[1-p(x)] = 0$

$\log_2(1-p) - p \dfrac{1}{1-p} = 0 \qquad$ Solutions: $\quad p=1 \Rightarrow$ critical point

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad p=0 \Rightarrow \qquad\quad "$

and $\quad p = 1/2$:

$$\log_2\left(1-1/2\right) - \underbrace{\dfrac{1}{2}}_{} \cdot \underbrace{\dfrac{1}{1-1/2}}_{} = 0 \quad\Rightarrow\quad \underbrace{\log_2(1/2)}_{=-1/2} - \underbrace{\dfrac{1}{2} \cdot \dfrac{1}{1/2}}_{=1} = 0 \qquad \checkmark$$

thus, $\quad p = \dfrac{1}{2} \quad$ is a minimal of cross-entropy