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# Aircraft Structures for Engineering Students

Seventh Edition



T.H.G. Megson



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*To the memory of my darling wife, Margaret.*

# Preface to the Seventh Edition of *Aircraft Structures*

The reviews of the sixth edition of *Aircraft Structures* were encouraging, although it was pointed out that some students had experienced difficulty in following the derivation of some of the formulae. In earlier editions, some steps in the derivations were omitted for the sake of brevity, but in response to these comments, I have looked carefully at all such derivations, and where I could see possible problems arising, I have expanded the theory.

I have reexamined the contents of the book and felt that it would benefit from some additional topics. In [Chapter 16](#), I have included the moment-area method for the calculation of beam deflections, since this is particularly useful when only the deflection of a particular point in a beam is required rather than the complete deflected shape. To illustrate this, I have included a problem where the deflection of the wingtip of an aircraft is calculated at touchdown. Furthermore, in [Chapter 22](#), in which the stress analysis of fuselages is presented, I have extended the theory to the calculation of stresses in fuselages subjected to internal pressure—an important feature in modern passenger carrying aircraft.

A major change in the seventh edition is an introduction to spacecraft design. Initially, in [Chapter 12](#), this takes the form of a description of the structural components of spacecraft and the loads to which they are subjected. Airworthiness and factors of safety are considered in [Chapter 13](#) while the whole of [Chapter 25](#) is devoted to strength, stiffness, and stability considerations in which the stability analysis presented in [Chapter 8](#) and the theory of beam vibrations in [Chapter 10](#) are applicable. Also, in [Chapter 12](#), I have extended the work on connections to describe the use of welding and adhesives in aircraft construction.

In [Chapter 20](#), I have added the idealization of unsymmetrical sections to the existing symmetrical cases and then in [Chapter 26](#) presented a method of idealizing composite beam sections.

Finally, I have included more examples and problems with an emphasis on making these as practical as possible.

**T.H.G. Megson**

# Basic elasticity

# 1

We consider, in this chapter, the basic ideas and relationships of the theory of elasticity. The treatment is divided into three broad sections: stress, strain, and stress-strain relationships. The third section is deferred until the end of the chapter to emphasize the fact that the analysis of stress and strain—for example, the equations of equilibrium and compatibility—does not assume a particular stress-strain law. In other words, the relationships derived in [Sections 1.1–1.14](#) are applicable to nonlinear as well as linearly elastic bodies.

## 1.1 STRESS

Consider the arbitrarily shaped, three-dimensional body shown in [Fig. 1.1](#). The body is in equilibrium under the action of externally applied forces  $P_1, P_2, \dots$  and is assumed to constitute a continuous and deformable material, so that the forces are transmitted throughout its volume. It follows that, at any internal point O, there is a resultant force  $\delta P$ . The particle of material at O subjected to the force  $\delta P$  is in equilibrium, so that there must be an equal but opposite force  $\delta P$  (shown dotted in [Fig. 1.1](#)) acting on the particle at the same time. If we now divide the body by any plane  $nn$  containing O, then these two forces  $\delta P$  may be considered as being uniformly distributed over a small area  $\delta A$  of each face of the plane at the corresponding point O, as in [Fig. 1.2](#). The *stress* at O is defined by the equation

$$\text{Stress} = \lim_{\delta A \rightarrow 0} \frac{\delta P}{\delta A} \quad (1.1)$$

The directions of the forces  $\delta P$  in [Fig. 1.2](#) are such as to produce *tensile* stresses on the faces of the plane  $nn$ . It must be realized here that, while the direction of  $\delta P$  is absolute, the choice of plane is arbitrary, so that, although the direction of the stress at O is always in the direction of  $\delta P$ , its magnitude depends upon the actual plane chosen, since a different plane has a different inclination and therefore a different value for the area  $\delta A$ . This may be more easily understood by reference to the bar in simple tension in [Fig. 1.3](#). On the cross-sectional plane  $mm$ , the uniform stress is given by  $P/A$ , while on the *inclined plane*  $m'm'$ , the stress is of magnitude  $P/A'$ . In both cases, the stresses are parallel to the direction of  $P$ .

Generally, the direction of  $\delta P$  is not normal to the area  $\delta A$ , in which case it is usual to resolve  $\delta P$  into two components: one,  $\delta P_n$ , normal to the plane, and the other,  $\delta P_s$ , acting in the plane itself (see [Fig. 1.2](#)). Note that, in [Fig. 1.2](#), the plane containing  $\delta P$  is perpendicular to  $\delta A$ . The stresses associated with these components are a *normal* or *direct stress* defined as

$$\sigma = \lim_{\delta A \rightarrow 0} \frac{\delta P_n}{\delta A} \quad (1.2)$$

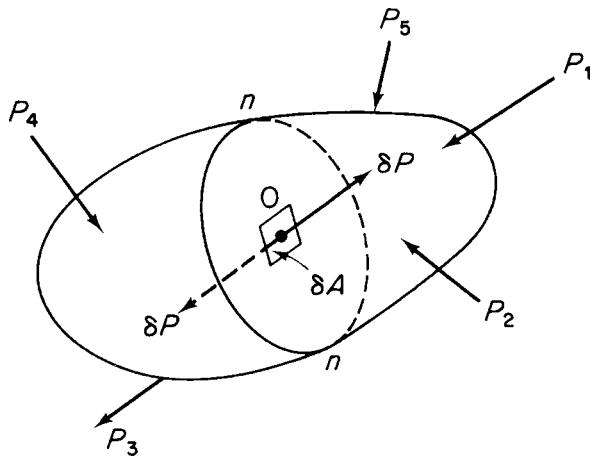


FIGURE 1.1 Internal Force at a Point in an Arbitrarily Shaped Body

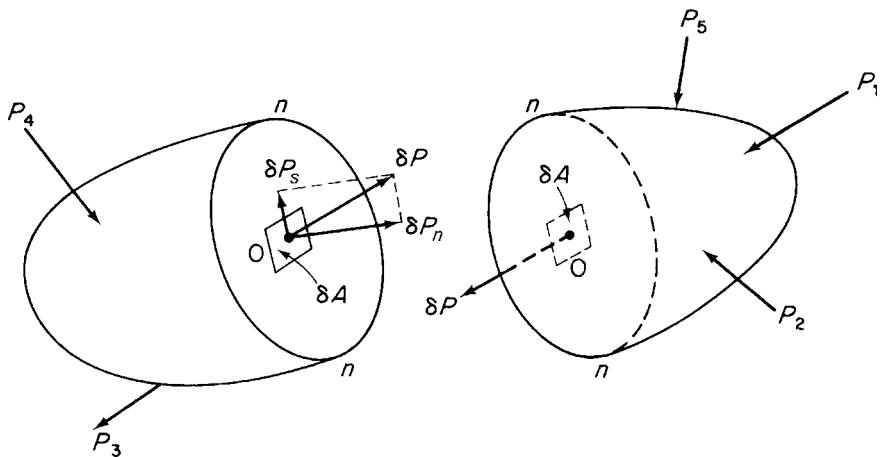


FIGURE 1.2 Internal Force Components at the Point O

and a *shear stress* defined as

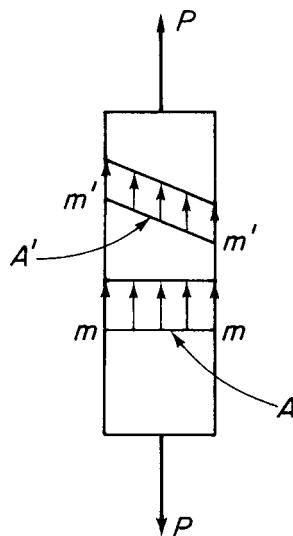
$$\tau = \lim_{\delta A \rightarrow 0} \frac{\delta P_s}{\delta A} \quad (1.3)$$

The resultant stress is computed from its components by the normal rules of vector addition, i.e.,

$$\text{Resultant stress} = \sqrt{\sigma^2 + \tau^2}$$

Generally, however, as indicated previously, we are interested in the separate effects of  $\sigma$  and  $\tau$ .

However, to be strictly accurate, stress is not a vector quantity, for, in addition to magnitude and direction, we must specify the plane on which the stress acts. Stress is therefore a *tensor*, its complete description depending on the two vectors of force and surface of action.



**FIGURE 1.3 Values of Stress on Different Planes in a Uniform Bar**

## 1.2 NOTATION FOR FORCES AND STRESSES

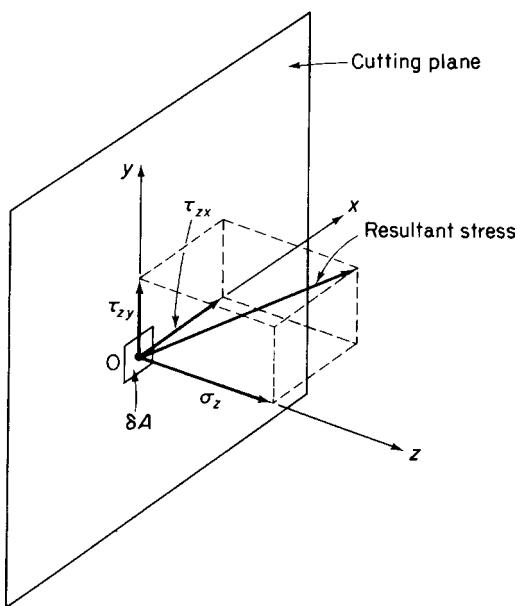
It is usually convenient to refer the state of stress at a point in a body to an orthogonal set of axes Oxyz. In this case we cut the body by planes parallel to the direction of the axes. The resultant force  $\delta P$  acting at the point O on one of these planes may then be resolved into a normal component and two in-plane components, as shown in Fig. 1.4, thereby producing one component of direct stress and two components of shear stress.

The direct stress component is specified by reference to the plane on which it acts, but the shear stress components require a specification of direction in addition to the plane. We therefore allocate a single subscript to direct stress to denote the plane on which it acts and two subscripts to shear stress, the first specifying the plane, the second, direction. Therefore, in Fig. 1.4, the shear stress components are  $\tau_{zx}$  and  $\tau_{zy}$  acting on the z plane and in the x and y directions, respectively, while the direct stress component is  $\sigma_z$ .

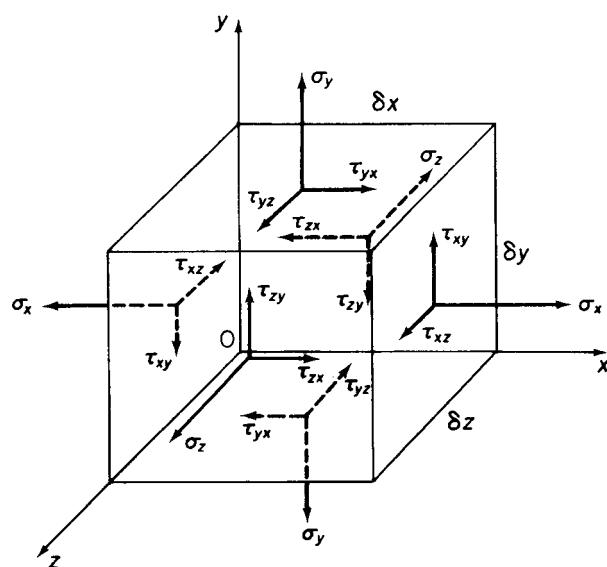
We may now completely describe the state of stress at a point O in a body by specifying components of shear and direct stress on the faces of an element of side  $\delta x$ ,  $\delta y$ ,  $\delta z$ , formed at O by the cutting planes as indicated in Fig. 1.5.

The sides of the element are infinitesimally small, so that the stresses may be assumed to be uniformly distributed over the surface of each face. On each of the opposite faces there will be, to a first simplification, equal but opposite stresses.

We now define the directions of the stresses in Fig. 1.5 as positive, so that normal stresses directed away from their related surfaces are tensile and positive; opposite compressive stresses are negative. Shear stresses are positive when they act in the positive direction of the relevant axis in a plane on which the direct tensile stress is in the positive direction of the axis. If the tensile stress is in the



**FIGURE 1.4 Components of Stress at a Point in a Body**



**FIGURE 1.5 Sign Conventions and Notation for Stresses at a Point in a Body**

opposite direction, then positive shear stresses are in directions opposite to the positive directions of the appropriate axes.

Two types of external force may act on a body to produce the internal stress system we have already discussed. Of these, *surface forces* such as  $P_1, P_2, \dots$ , or hydrostatic pressure, are distributed over the surface area of the body. The surface force per unit area may be resolved into components parallel to our orthogonal system of axes, and these are generally given the symbols  $\bar{X}, \bar{Y}$ , and  $\bar{Z}$ . The second force system derives from gravitational and inertia effects, and the forces are known as *body forces*. These are distributed over the volume of the body, and the components of body force per unit volume are designated  $X, Y$ , and  $Z$ .

### 1.3 EQUATIONS OF EQUILIBRIUM

Generally, except in cases of uniform stress, the direct and shear stresses on opposite faces of an element are not equal, as indicated in Fig. 1.5, but differ by small amounts. Therefore if, say, the direct stress acting on the  $z$  plane is  $\sigma_z$ , then the direct stress acting on the  $z + \delta z$  plane is, from the first two terms of a Taylor's series expansion,  $\sigma_z + (\partial\sigma_z/\partial z)\delta z$ .

We now investigate the equilibrium of an element at some internal point in an elastic body where the stress system is obtained by the method just described.

In Fig. 1.6, the element is in equilibrium under forces corresponding to the stresses shown and the components of body forces (not shown). Surface forces acting on the boundary of the body, although contributing to the production of the internal stress system, do not directly feature in the equilibrium equations.

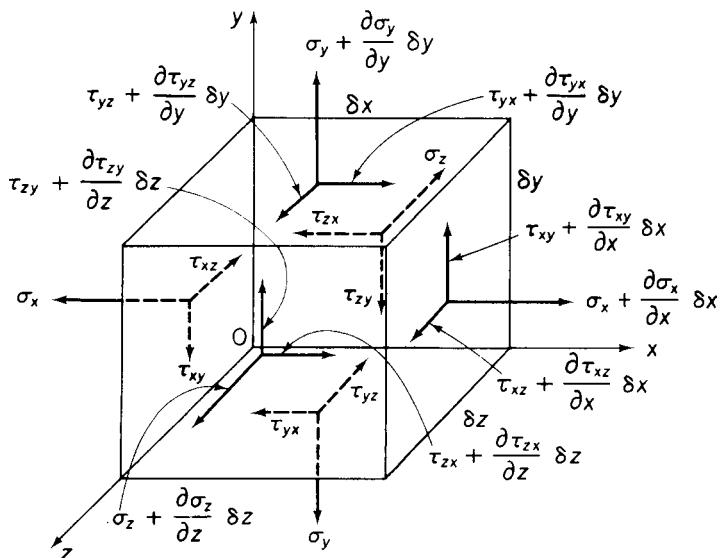


FIGURE 1.6 Stresses on the Faces of an Element at a Point in an Elastic Body

Taking moments about an axis through the center of the element parallel to the  $z$  axis,

$$\begin{aligned} \tau_{xy}\delta y\delta z \frac{\delta x}{2} + \left( \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \delta x \right) \delta y \delta z \frac{\delta x}{2} - \tau_{yx}\delta x\delta z \frac{\delta y}{2} \\ - \left( \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \delta y \right) \delta x \delta z \frac{\delta y}{2} = 0 \end{aligned}$$

which simplifies to

$$\tau_{xy}\delta y\delta z\delta x + \frac{\partial \tau_{xy}}{\partial x} \delta y \delta z \frac{(\delta x)^2}{2} - \tau_{yx}\delta x\delta z\delta y - \frac{\partial \tau_{yx}}{\partial y} \delta x \delta z \frac{(\delta y)^2}{2} = 0$$

dividing through by  $\delta x\delta y\delta z$  and taking the limit as  $\delta x$  and  $\delta y$  approach zero.

$$\left. \begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{xz} &= \tau_{zx} \\ \tau_{yz} &= \tau_{zy} \end{aligned} \right\} \quad (1.4)$$

Similarly,

We see, therefore, that a shear stress acting on a given plane ( $\tau_{xy}, \tau_{xz}, \tau_{yz}$ ) is always accompanied by an equal *complementary shear stress* ( $\tau_{yx}, \tau_{zx}, \tau_{zy}$ ) acting on a plane perpendicular to the given plane and in the opposite sense.

Now, considering the equilibrium of the element in the  $x$  direction,

$$\begin{aligned} \left( \sigma_x + \frac{\partial \sigma_x}{\partial x} \delta x \right) \delta y \delta z - \sigma_x \delta y \delta z + \left( \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \delta y \right) \delta x \delta z \\ - \tau_{xy} \delta x \delta z + \left( \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \delta z \right) \delta x \delta y \\ - \tau_{xz} \delta x \delta y + X \delta x \delta y \delta z = 0 \end{aligned}$$

which gives

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

Or, writing  $\tau_{xy} = \tau_{yx}$  and  $\tau_{xz} = \tau_{zx}$  from Eq. (1.4),

$$\left. \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y &= 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z &= 0 \end{aligned} \right\} \quad (1.5)$$

similarly,

The *equations of equilibrium* must be satisfied at all interior points in a deformable body under a three-dimensional force system.

## 1.4 PLANE STRESS

Most aircraft structural components are fabricated from thin metal sheet, so that stresses across the thickness of the sheet are usually negligible. Assuming, say, that the  $z$  axis is in the direction of the thickness, then the three-dimensional case of [Section 1.3](#) reduces to a two-dimensional case in which  $\sigma_z$ ,  $\tau_{xz}$ , and  $\tau_{yz}$  are all zero. This condition is known as *plane stress*; the equilibrium equations then simplify to

$$\left. \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + Y &= 0 \end{aligned} \right\} \quad (1.6)$$

## 1.5 BOUNDARY CONDITIONS

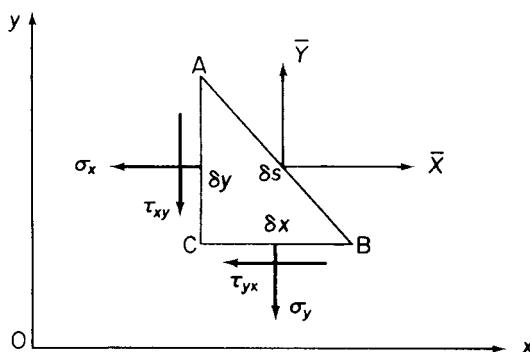
The equations of equilibrium (1.5)—and also (1.6), for a two-dimensional system—satisfy the requirements of equilibrium at all internal points of the body. Equilibrium must also be satisfied at all positions on the boundary of the body, where the components of the surface force per unit area are  $\bar{X}$ ,  $\bar{Y}$ , and  $\bar{Z}$ . The triangular element of [Fig. 1.7](#) at the boundary of a two-dimensional body of unit thickness is then in equilibrium under the action of surface forces on the elemental length AB of the boundary and internal forces on internal faces AC and CB.

Summation of forces in the  $x$  direction gives

$$\bar{X}\delta s - \sigma_x \delta y - \tau_{yx} \delta x + X \frac{1}{2} \delta x \delta y = 0$$

which, by taking the limit as  $\delta x$  approaches zero and neglecting second order terms, becomes

$$\bar{X} = \sigma_x \frac{dy}{ds} + \tau_{yx} \frac{dx}{ds}$$



**FIGURE 1.7 Stresses on the Faces of an Element at the Boundary of a Two-Dimensional Body**

The derivatives  $dy/ds$  and  $dx/ds$  are the direction cosines  $l$  and  $m$  of the angles that a normal to AB makes with the  $x$  and  $y$  axes, respectively. It follows that

$$\bar{X} = \sigma_x l + \tau_{yx} m$$

and in a similar manner

$$\bar{Y} = \sigma_y m + \tau_{xy} l$$

A relatively simple extension of this analysis produces the boundary conditions for a three-dimensional body, namely

$$\begin{aligned}\bar{X} &= \sigma_x l + \tau_{yx} m + \tau_{zx} n \\ \bar{Y} &= \sigma_y m + \tau_{xy} l + \tau_{zy} n \\ \bar{Z} &= \sigma_z n + \tau_{yz} m + \tau_{xz} l\end{aligned}\left.\right\} \quad (1.7)$$

where  $l$ ,  $m$ , and  $n$  become the direction cosines of the angles that a normal to the surface of the body makes with the  $x$ ,  $y$ , and  $z$  axes, respectively.

## 1.6 DETERMINATION OF STRESSES ON INCLINED PLANES

The complex stress system of Fig. 1.6 is derived from a consideration of the actual loads applied to a body and is referred to a predetermined, though arbitrary, system of axes. The values of these stresses may not give a true picture of the severity of stress at that point, so that it is necessary to investigate the state of stress on other planes on which the direct and shear stresses may be greater.

We restrict the analysis to the two-dimensional system of plane stress defined in Section 1.4.

Figure 1.8(a) shows a complex stress system at a point in a body referred to axes  $Ox$ ,  $Oy$ . All stresses are positive, as defined in Section 1.2. The shear stresses  $\tau_{xy}$  and  $\tau_{yx}$  were shown to be equal in Section 1.3. We now, therefore, designate them both  $\tau_{xy}$ . The element of side  $\delta x$ ,  $\delta y$ , and of unit

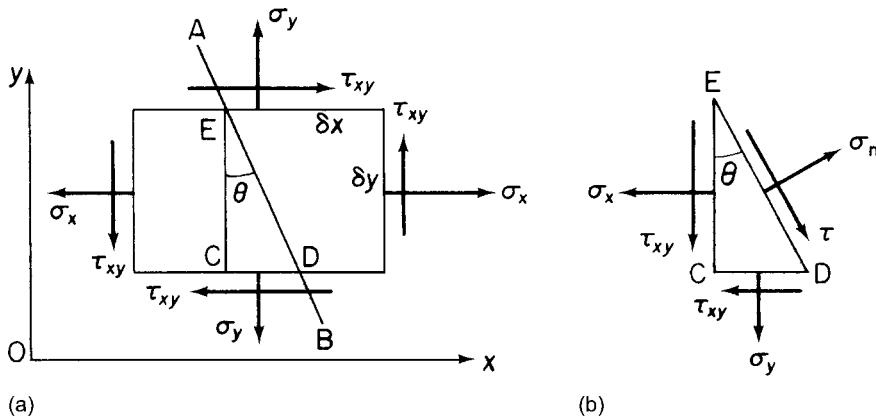


FIGURE 1.8 (a) Stresses on a Two-Dimensional Element; (b) Stresses on an Inclined Plane at the Point

thickness is small, so that stress distributions over the sides of the element may be assumed to be uniform. Body forces are ignored, since their contribution is a second-order term.

Suppose that we need to find the state of stress on a plane AB inclined at an angle  $\theta$  to the vertical. The triangular element EDC formed by the plane and the vertical through E is in equilibrium under the action of the forces corresponding to the stresses shown in Fig. 1.8(b), where  $\sigma_n$  and  $\tau$  are the direct and shear components of the resultant stress on AB. Then, resolving forces in a direction perpendicular to ED, we have

$$\sigma_n ED = \sigma_x EC \cos\theta + \sigma_y CD \sin\theta + \tau_{xy} EC \sin\theta + \tau_{xy} CD \cos\theta$$

Dividing through by ED,

$$\sigma_n = \sigma_x \frac{EC}{ED} \cos\theta + \sigma_y \frac{CD}{ED} \sin\theta + \tau_{xy} \frac{EC}{ED} \sin\theta + \tau_{xy} \frac{CD}{ED} \cos\theta$$

But

$$\frac{EC}{ED} = \cos\theta \quad \text{and} \quad \frac{CD}{ED} = \sin\theta$$

so that

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

or

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta \quad (1.8)$$

Now, resolving forces parallel to ED,

$$\tau ED = \sigma_x EC \sin\theta - \sigma_y CD \cos\theta - \tau_{xy} EC \cos\theta + \tau_{xy} CD \sin\theta$$

Again, dividing through by ED,

$$\tau = \sigma_x \frac{EC}{ED} \sin\theta - \sigma_y \frac{CD}{ED} \cos\theta - \tau_{xy} \frac{EC}{ED} \cos\theta + \tau_{xy} \frac{CD}{ED} \sin\theta$$

which becomes

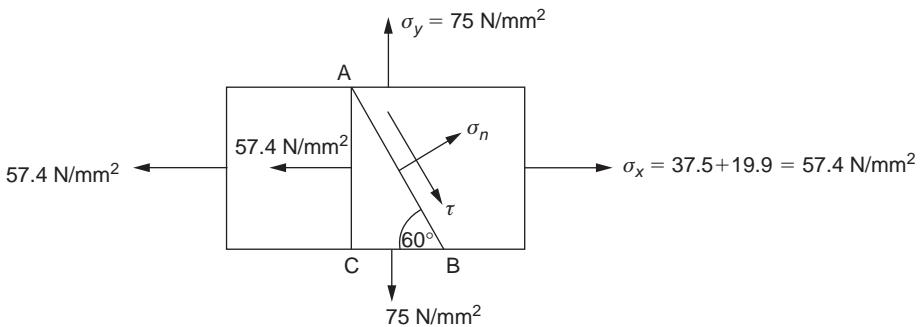
$$\tau = \sigma_x \cos\theta \sin\theta - \sigma_y \sin\theta \cos\theta - \tau_{xy} \cos^2\theta + \tau_{xy} \sin^2\theta$$

or

$$\tau = \frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta - \tau_{xy} \cos 2\theta \quad (1.9)$$

### Example 1.1

A cylindrical pressure vessel has an internal diameter of 2 m and is fabricated from plates 20 mm thick. If the pressure inside the vessel is 1.5 N/mm<sup>2</sup> and, in addition, the vessel is subjected to an axial tensile load of 2500 kN, calculate the direct and shear stresses on a plane inclined at an angle of 60° to the axis of the vessel. Calculate also the maximum shear stress.

**FIGURE 1.9 Element of Example 1.1**

The expressions for the longitudinal and circumferential stresses produced by the internal pressure may be found in any text on stress analysis<sup>1</sup> (or refer to Section 22.4) and are

$$\text{Longitudinal stress } (\sigma_x) = \frac{pd}{4t} = 1.5 \times 2 \times 10^3 / 4 \times 20 = 37.5 \text{ N/mm}^2$$

$$\text{Circumferential stress } (\sigma_y) = \frac{pd}{2t} = 1.5 \times 2 \times 10^3 / 2 \times 20 = 75 \text{ N/mm}^2$$

The direct stress due to the axial load will contribute to  $\sigma_x$  and is given by

$$\sigma_x \text{ (axial load)} = 2500 \times 10^3 / \pi \times 2 \times 10^3 \times 20 = 19.9 \text{ N/mm}^2$$

A rectangular element in the wall of the pressure vessel is then subjected to the stress system shown in Fig. 1.9. Note that no shear stresses act on the  $x$  and  $y$  planes; in this case,  $\sigma_x$  and  $\sigma_y$  form a *biaxial* stress system.

The direct stress,  $\sigma_n$ , and shear stress,  $\tau$ , on the plane AB, which makes an angle of  $60^\circ$  with the axis of the vessel, may be found from first principles by considering the equilibrium of the triangular element ABC or by direct substitution in Eqs. (1.8) and (1.9). Note that, in the latter case,  $\theta = 30^\circ$  and  $\tau_{xy} = 0$ . Then,

$$\begin{aligned}\sigma_n &= 57.4 \cos^2 30^\circ + 75 \sin^2 30^\circ = 61.8 \text{ N/mm}^2 \\ \tau &= (57.4 - 75)[\sin(2 \times 30^\circ)]/2 = -7.6 \text{ N/mm}^2\end{aligned}$$

The negative sign for  $\tau$  indicates that the shear stress is in the direction BA and not AB.

From Eq. (1.9), when  $\tau_{xy} = 0$ ,

$$\tau = (\sigma_x - \sigma_y)(\sin 2\theta)/2 \quad (i)$$

The maximum value of  $\tau$  therefore occurs when  $\sin 2\theta$  is a maximum, that is, when  $\sin 2\theta = 1$  and  $\theta = 45^\circ$ . Then, substituting the values of  $\sigma_x$  and  $\sigma_y$  in Eq. (i),

$$\tau_{\max} = (57.4 - 75)/2 = -8.8 \text{ N/mm}^2$$

### Example 1.2

A cantilever beam of solid, circular cross-section supports a compressive load of 50 kN applied to its free end at a point 1.5 mm below a horizontal diameter in the vertical plane of symmetry together with a torque of 1200 Nm

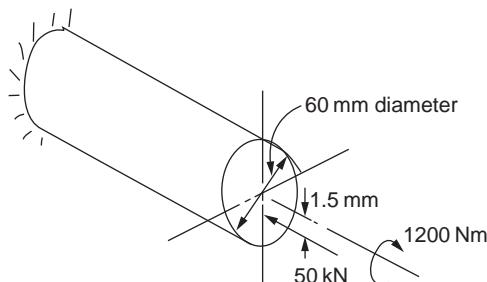


FIGURE 1.10 Cantilever Beam of Example 1.2

(Fig. 1.10). Calculate the direct and shear stresses on a plane inclined at  $60^\circ$  to the axis of the cantilever at a point on the lower edge of the vertical plane of symmetry.

The direct loading system is equivalent to an axial load of 50 kN together with a bending moment of  $50 \times 10^3 \times 1.5 = 75,000$  Nmm in a vertical plane. Therefore, at any point on the lower edge of the vertical plane of symmetry, there are compressive stresses due to the axial load and bending moment that act on planes perpendicular to the axis of the beam and are given, respectively, by Eqs. (1.2) and (16.9); that is,

$$\begin{aligned}\sigma_x \text{ (axial load)} &= 50 \times 10^3 / \pi \times (60^2 / 4) = 17.7 \text{ N/mm}^2 \\ \sigma_x \text{ (bending moment)} &= 75,000 \times 30 / \pi \times (60^4 / 64) = 3.5 \text{ N/mm}^2\end{aligned}$$

The shear stress,  $\tau_{xy}$ , at the same point due to the torque is obtained from Eq. (iv) in Example 3.1; that is,

$$\tau_{xy} = 1200 \times 10^3 \times 30 / \pi \times (60^4 / 32) = 28.3 \text{ N/mm}^2$$

The stress system acting on a two-dimensional rectangular element at the point is shown in Fig. 1.11. Note that, since the element is positioned at the bottom of the beam, the shear stress due to the torque is in the direction shown and is negative (see Fig. 1.8).

Again,  $\sigma_n$  and  $\tau$  may be found from first principles or by direct substitution in Eqs. (1.8) and (1.9). Note that  $\theta = 30^\circ$ ,  $\sigma_y = 0$ , and  $\tau_{xy} = -28.3 \text{ N/mm}^2$ , the negative sign arising from the fact that it is in the opposite direction to  $\tau_{xy}$  in Fig. 1.8.

Then,

$$\sigma_n = -21.2 \cos^2 30^\circ - 28.3 \sin 60^\circ = -40.4 \text{ N/mm}^2 \text{ (compression)}$$

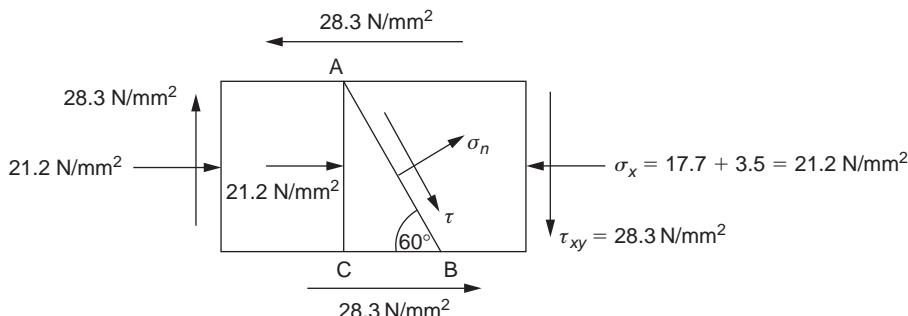


FIGURE 1.11 Stress System on a Two-Dimensional Element of the Beam of Example 1.2

$$\tau = (-21.2/2) \sin 60^\circ + 28.3 \cos 60^\circ = 5.0 \text{ N/mm}^2 \text{ (acting in the direction AB)}$$

Different answers are obtained if the plane AB is chosen on the opposite side of AC.

## 1.7 PRINCIPAL STRESSES

For given values of  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$ , in other words, given loading conditions,  $\sigma_n$  varies with the angle  $\theta$  and attains a maximum or minimum value when  $d\sigma_n/d\theta = 0$ . From Eq. (1.8),

$$\frac{d\sigma_n}{d\theta} = -2\sigma_x \cos\theta \sin\theta + 2\sigma_y \sin\theta \cos\theta + 2\tau_{xy} \cos 2\theta = 0$$

Hence,

$$-(\sigma_x - \sigma_y) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0$$

or

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad (1.10)$$

Two solutions,  $\theta$  and  $\theta + \pi/2$ , are obtained from Eq. (1.10), so that there are two mutually perpendicular planes on which the direct stress is either a maximum or a minimum. Further, by comparison of Eqs. (1.9) and (1.10), it will be observed that these planes correspond to those on which there is no shear stress. The direct stresses on these planes are called *principal stresses*, and the planes themselves, *principal planes*.

Equation (1.10) may be regarded as forming a triangular system as shown in Fig. 1.12 where, in the first quadrant, the angle of the triangle is  $2\theta$ , while in the third quadrant, the angle is  $2\theta + \pi$ .

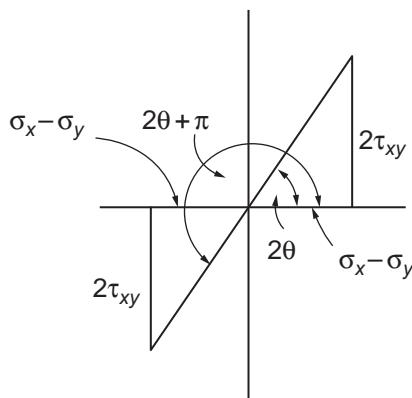


FIGURE 1.12 Representation of Eq. (1.10)

Further, in the first quadrant, both  $2\tau_{xy}$  and  $(\sigma_x - \sigma_y)$  are positive, while in the third quadrant, they are negative. Also, in both quadrants, the hypotenuse of the triangle is

$$\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

Then

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}, \quad \cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

and

$$\sin 2\left(\theta + \frac{\pi}{2}\right) = \frac{-2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}, \quad \cos 2\left(\theta + \frac{\pi}{2}\right) = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Equation (1.8) may be rewritten using the well-known trigonometrical relationships

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta), \quad \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

Then

$$\sigma_n = \frac{\sigma_x}{2}(1 + \cos 2\theta) + \frac{\sigma_y}{2}(1 - \cos 2\theta) + \tau_{xy} \sin 2\theta$$

Substituting for  $\cos 2\theta$  and  $\sin 2\theta$  from the above we obtain the *maximum or major principal stress*, i.e.,

$$\sigma_I = \frac{\sigma_x}{2} \left[ 1 + \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \right] + \frac{\sigma_y}{2} \left[ 1 - \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \right] + \tau_{xy} \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Expanding

$$\sigma_I = \frac{\sigma_x}{2} + \frac{\sigma_x}{2} \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} + \frac{\sigma_y}{2} - \frac{\sigma_y}{2} \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} + \frac{2\tau_{xy}^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

so that

$$\sigma_I = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} (\sigma_x^2 - \sigma_x \sigma_y - \sigma_y \sigma_x + \sigma_y^2 + 4\tau_{xy}^2)$$

or

$$\sigma_I = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} [(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]$$

from which

$$\sigma_I = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad (1.11)$$

Similarly by substituting for  $\sin^2(\theta + \pi/2)$  and  $\cos^2(\theta + \pi/2)$  we obtain the *minimum* or *minor principal stress*, i.e.,

$$\sigma_{II} = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad (1.12)$$

Note that  $\sigma_I$  is algebraically the greatest direct stress at the point while  $\sigma_{II}$  is algebraically the least. Therefore, when  $\sigma_{II}$  is negative—that is, compressive—it is possible for  $\sigma_{II}$  to be numerically greater than  $\sigma_I$ .

The maximum shear stress at this point in the body may be determined in an identical manner. From Eq. (1.9),

$$\frac{d\tau}{d\theta} = (\sigma_x - \sigma_y) \cos 2\theta + 2\tau_{xy} \sin 2\theta = 0$$

giving

$$\tan 2\theta = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} \quad (1.13)$$

It follows, in a similar manner as before, that

$$\begin{aligned} \sin 2\theta &= \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} & \cos 2\theta &= \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \\ \sin 2(\theta + \pi/2) &= \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} & \cos 2(\theta + \pi/2) &= \frac{-2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \end{aligned}$$

Substituting these values in Eq. (1.9) gives

$$\tau_{max,min} = \pm \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad (1.14)$$

Here, as in the case of principal stresses, we take the maximum value as being the greater algebraic value.

Comparing Eq. (1.14) with Eqs. (1.11) and (1.12), we see that

$$\tau_{max} = \frac{\sigma_I - \sigma_{II}}{2} \quad (1.15)$$

Equations (1.14) and (1.15) give the maximum shear stress at the point in the body in *the plane of the given stresses*. For a three-dimensional body supporting a two-dimensional stress system, this is not necessarily the maximum shear stress at the point.

Since Eq. (1.13) is the negative reciprocal of Eq. (1.10), the angles  $2\theta$  given by these two equations differ by  $90^\circ$  or, alternatively, the planes of maximum shear stress are inclined at  $45^\circ$  to the principal planes.

## 1.8 MOHR'S CIRCLE OF STRESS

The state of stress at a point in a deformable body may be determined graphically by *Mohr's circle of stress*.

In Section 1.6, the direct and shear stresses on an inclined plane were shown to be given by

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta \quad (1.8)$$

and

$$\tau = \frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta - \tau_{xy} \cos 2\theta \quad (1.9)$$

respectively. The positive directions of these stresses and the angle  $\theta$  are defined in Fig. 1.13(a). Equation (1.8) may be rewritten in the form

$$\sigma_n = \frac{\sigma_x}{2}(1 + \cos 2\theta) + \frac{\sigma_y}{2}(1 - \cos 2\theta) + \tau_{xy} \sin 2\theta$$

or

$$\sigma_n - \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta$$

Squaring and adding this equation to Eq. (1.9), we obtain

$$\left[ \sigma_n - \left( \frac{\sigma_x + \sigma_y}{2} \right) \right]^2 + \tau^2 = \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta + \tau_{xy} \sin 2\theta \right]^2 + \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta - \tau_{xy} \cos 2\theta \right]^2$$

which, since  $\cos^2 2\theta + \sin^2 2\theta = 1$  and the central terms cancel out, reduces to

$$\left[ \sigma_n - \frac{1}{2}(\sigma_x + \sigma_y) \right]^2 + \tau^2 = \left[ \frac{1}{2}(\sigma_x - \sigma_y) \right]^2 + \tau_{xy}^2$$

which represents the equation of a circle of radius  $\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$  and having its center at the point  $[(\sigma_x - \sigma_y)/2, 0]$ .

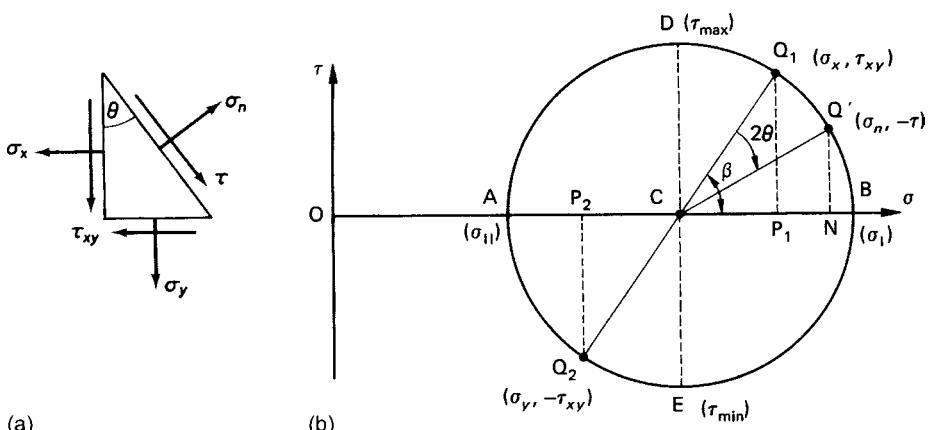


FIGURE 1.13 (a) Stresses on a Triangular Element; (b) Mohr's Circle of Stress for the Stress System Shown in (a)

The circle is constructed by locating the points  $Q_1(\sigma_x, \tau_{xy})$  and  $Q_2(\sigma_y, -\tau_{xy})$  referred to axes  $O\sigma\tau$ , as shown in Fig. 1.13(b). The center of the circle then lies at C, the intersection of  $Q_1Q_2$  and the  $O\sigma$  axis; clearly C is the point  $[(\sigma_x - \sigma_y)/2, 0]$  and the radius of the circle is  $\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$ , as required.  $CQ'$  is now set off at an angle  $2\theta$  (positive clockwise) to  $CQ_1$ ,  $Q'$  is then the point  $(\sigma_n, -\tau)$ , as demonstrated next. From Fig. 1.13(b), we see that

$$\sigma_n = ON = OC + CN$$

or, since  $OC = (\sigma_x + \sigma_y)/2$ ,  $CN = CQ' \cos(\beta - 2\theta)$ , and  $CQ' = CQ_1$ , we have

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + CQ_1(\cos\beta \cos 2\theta + \sin\beta \sin 2\theta)$$

But,

$$CQ_1 = \frac{CP_1}{\cos\beta} \quad \text{and} \quad CP_1 = \frac{(\sigma_x - \sigma_y)}{2}$$

Hence,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta + CP_1 \tan\beta \sin 2\theta$$

which, on rearranging, becomes

$$\sigma_n = \frac{\sigma_x}{2}(1 + \cos 2\theta) + \frac{\sigma_y}{2}(1 - \cos 2\theta) + \tau_{xy} \sin 2\theta$$

But  $(1 + \cos 2\theta)/2 = \cos^2\theta$  and  $(1 - \cos 2\theta)/2 = \sin^2\theta$  so that

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta$$

as in Eq. (1.8). Similarly, it may be shown that

$$Q'N = \tau_{xy} \cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta = -\tau$$

as in Eq. (1.9). Note that the construction of Fig. 1.13(b) corresponds to the stress system of Fig. 1.13(a), so that any sign reversal must be allowed for. Also, the  $O\sigma$  and  $O\tau$  axes must be constructed to the same scale or the equation of the circle is not represented.

The maximum and minimum values of the direct stress—that is, the major and minor principal stresses  $\sigma_I$  and  $\sigma_{II}$ —occur when N (and  $Q'$ ) coincide with B and A, respectively. Thus,

$$\sigma_I = OC + \text{radius of circle}$$

$$= \frac{(\sigma_x + \sigma_y)}{2} + \sqrt{CP_1^2 + P_1Q_1^2}$$

or

$$\sigma_I = \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

and, in the same fashion,

$$\sigma_{II} = \frac{(\sigma_x + \sigma_y)}{2} - \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

The principal planes are then given by  $2\theta = \beta(\sigma_I)$  and  $2\theta = \beta + \pi(\sigma_{II})$ .

Also, the maximum and minimum values of shear stress occur when Q' coincides with D and E at the upper and lower extremities of the circle.

At these points, Q'N is equal to the radius of the circle, which is given by

$$CQ_1 = \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2}$$

Hence,  $\tau_{max,min} = \pm \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$ , as before. The planes of maximum and minimum shear stress are given by  $2\theta = \beta + \pi/2$  and  $2\theta = \beta + 3\pi/2$ , these being inclined at  $45^\circ$  to the principal planes.

### Example 1.3

Direct stresses of  $160 \text{ N/mm}^2$  (tension) and  $120 \text{ N/mm}^2$  (compression) are applied at a particular point in an elastic material on two mutually perpendicular planes. The principal stress in the material is limited to  $200 \text{ N/mm}^2$  (tension). Calculate the allowable value of shear stress at the point on the given planes. Determine also the value of the other principal stress and the maximum value of shear stress at the point. Verify your answer using Mohr's circle.

The stress system at the point in the material may be represented as shown in Fig. 1.14 by considering the stresses to act uniformly over the sides of a triangular element ABC of unit thickness. Suppose that the direct stress on the principal plane AB is  $\sigma$ . For horizontal equilibrium of the element,

$$\sigma_{AB} \cos\theta = \sigma_x BC + \tau_{xy} AC$$

which simplifies to

$$\tau_{xy} \tan\theta = \sigma - \sigma_x \quad (i)$$

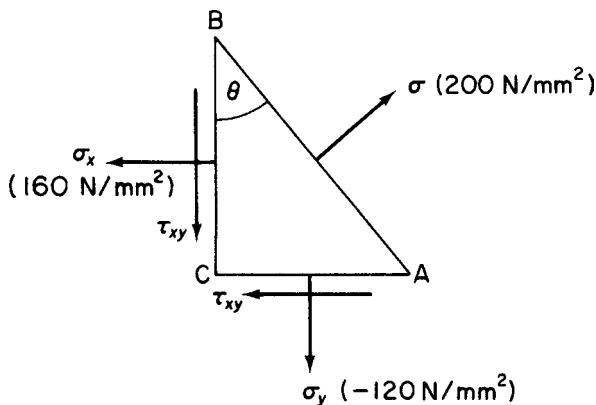


FIGURE 1.14 Stress System for Example 1.3

Considering vertical equilibrium gives

$$\sigma AB \sin\theta = \sigma_y AC + \tau_{xy} BC$$

or

$$\tau_{xy} \cot\theta = \sigma - \sigma_y \quad (\text{ii})$$

Hence, from the product of Eqs. (i) and (ii),

$$\tau_{xy}^2 = (\sigma - \sigma_x)(\sigma - \sigma_y)$$

Now, substituting the values  $\sigma_x = 160 \text{ N/mm}^2$ ,  $\sigma_y = -120 \text{ N/mm}^2$ , and  $\sigma = \sigma_1 = 200 \text{ N/mm}^2$ , we have

$$\tau_{xy} = \pm 113 \text{ N/mm}^2$$

Replacing  $\cot\theta$  in Eq. (ii) with  $1/\tan\theta$  from Eq. (i) yields a quadratic equation in  $\sigma$ :

$$\sigma^2 - \sigma(\sigma_x - \sigma_y) + \sigma_x \sigma_y - \tau_{xy}^2 = 0 \quad (\text{iii})$$

The numerical solutions of Eq. (iii) corresponding to the given values of  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  are the principal stresses at the point, namely

$$\sigma_1 = 200 \text{ N/mm}^2$$

given

$$\sigma_{II} = -160 \text{ N/mm}^2$$

Having obtained the principal stresses, we now use Eq. (1.15) to find the maximum shear stress, thus

$$\tau_{\max} = \frac{200 + 160}{2} = 180 \text{ N/mm}^2$$

The solution is rapidly verified from Mohr's circle of stress (Fig. 1.15). From the arbitrary origin O, OP<sub>1</sub> and OP<sub>2</sub> are drawn to represent  $\sigma_x = 160 \text{ N/mm}^2$  and  $\sigma_y = -120 \text{ N/mm}^2$ . The mid-point C of P<sub>1</sub>P<sub>2</sub> is then located. Next,

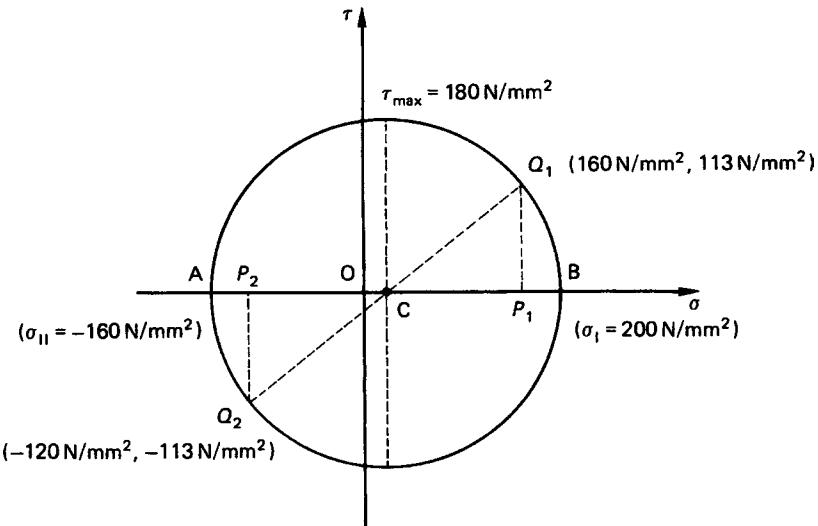


FIGURE 1.15 Solution of Example 1.3 Using Mohr's Circle of Stress

$OB = \sigma_1 = 200 \text{ N/mm}^2$  is marked out and the radius of the circle is then  $CB$ .  $OA$  is the required principal stress. Perpendiculars  $P_1Q_1$  and  $P_2Q_2$  to the circumference of the circle are equal to  $\pm \tau_{xy}$  (to scale), and the radius of the circle is the maximum shear stress.

### Example 1.3 MATLAB®

Repeat the derivations presented in [Example 1.3](#) using the Symbolic Math Toolbox in MATLAB®. Do not recreate Mohr's circle.

Using the element shown in [Fig. 1.14](#), derivations of the principal stresses and maximum shear stress are obtained through the following MATLAB file:

```
% Declare any needed symbolic variables
syms sig tau_xy sig_x sig_y theta AB BC AC

% Define known stress values
sig_x = sym(160);
sig_y = sym(-120);
sig_val = sym(200);

% Define relationships between AB, BC, and AC
BC = AB*cos(theta);
AC = AB*sin(theta);

% For horizontal equilibrium of the element
eqI = sig*AB*cos(theta)-sig_x*BC-tau_xy*AC;

% For vertical equilibrium of the element
eqII = sig*AB*sin(theta)-sig_y*AC-tau_xy*BC;

% Solve eqI and eqII for tau_xy
tau_xyI = solve(eqI,tau_xy);
tau_xyII = solve(eqII,tau_xy);

% Take the square-root of tau_xyI times tau_xyII to get tau_xy
tau_xy_val = sqrt(tau_xyI*tau_xyII);

% Substitute the given value of sig into tau_xy
tau_xy_val = subs(tau_xy_val,sig,sig_val);

% Solve eqI for theta and substitute into eqII
eqI = simplify(eqI/cos(theta));
theta_I = solve(eqI,theta,'IgnoreAnalyticConstraints',true);
eqIII = subs(eqII,theta,theta_I);
```

```
% Substitute the value of tau_xy into eqIII and solve for the principal stresses (sig_p)
sig_p = solve(subs(eqIII,tau_xy,tau_xy_val),sig);
sig_I = max(double(sig_p));
sig_II = min(double(sig_p));

% Calculate the maximum shear stress using Eq. (1.15)
tau_max = (sig_I-sig_II)/2;

% Output tau_xy, the principal stresses, and tau_max to the Command Window
disp(['tau_xy = +/- ' num2str(double(tau_xy_val)) ' N/mm^2'])
disp(['sig_I = ' num2str(sig_I) ' N/mm^2'])
disp(['sig_II = ' num2str(sig_II) ' N/mm^2'])
disp(['tau_max = ' num2str(tau_max) ' N/mm^2'])
```

The Command Window outputs resulting from this MATLAB file are as follows:

```
tau_xy = +/- 113.1371 N/mm^2
sig_I = 200 N/mm^2
sig_II = -160 N/mm^2
tau_max = 180 N/mm^2
```

## 1.9 STRAIN

The external and internal forces described in the previous sections cause linear and angular displacements in a deformable body. These displacements are generally defined in terms of *strain*. *Longitudinal* or *direct strains* are associated with direct stresses  $\sigma$  and relate to changes in length, while *shear strains* define changes in angle produced by shear stresses. These strains are designated, with appropriate suffixes, by the symbols  $\epsilon$  and  $\gamma$ , respectively, and have the same sign as the associated stresses.

Consider three mutually perpendicular line elements OA, OB, and OC at a point O in a deformable body. Their original or unstrained lengths are  $\delta x$ ,  $\delta y$ , and  $\delta z$ , respectively. If, now, the body is subjected to forces that produce a complex system of direct and shear stresses at O, such as that in Fig. 1.6, then the line elements deform to the positions O'A', O'B', and O'C' shown in Fig. 1.16.

The coordinates of O in the unstrained body are  $(x, y, z)$  so that those of A, B, and C are  $(x + \delta x, y, z)$ ,  $(x, y + \delta y, z)$ , and  $(x, y, z + \delta z)$ . The components of the displacement of O to O' parallel to the  $x$ ,  $y$ , and  $z$  axes are  $u$ ,  $v$ , and  $w$ . These symbols are used to designate these displacements throughout the book and are defined as positive in the positive directions of the axes. We again employ the first two terms of a Taylor's series expansion to determine the components of the displacements of A, B, and C. Thus, the displacement of A in a direction parallel to the  $x$  axis is  $u + (\partial u / \partial x) \delta x$ . The remaining components are found in an identical manner and are shown in Fig. 1.16.

We now define direct strain in more quantitative terms. If a line element of length  $L$  at a point in a body suffers a change in length  $\Delta L$ , then the longitudinal strain at that point in the body in the direction of the line element is

$$\epsilon = \lim_{L \rightarrow 0} \frac{\Delta L}{L}$$

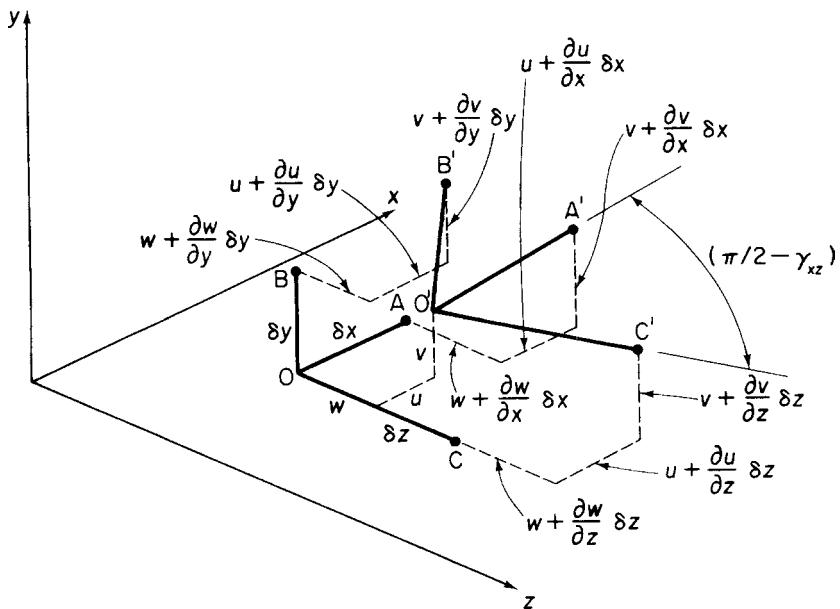


FIGURE 1.16 Displacement of Line Elements OA, OB, and OC

The change in length of the element OA is  $(O'A' - OA)$ , so that the direct strain at O in the  $x$  direction is obtained from the equation

$$\epsilon_x = \frac{OA' - OA}{OA} = \frac{O'A' - \delta x}{\delta x} \quad (1.16)$$

Now, the projection of  $O'A'$  in the  $xz$  plane parallel to the  $x$  axis is

$$\delta x + u + \frac{\partial u}{\partial x} \delta x - u = \delta x \left( 1 + \frac{\partial u}{\partial x} \right)$$

and its projection in the  $xy$  and  $yz$  planes are, respectively

$$v + \frac{\partial v}{\partial x} \delta x - v = \frac{\partial v}{\partial x} \delta x$$

$$w + \frac{\partial w}{\partial x} \delta x - w = \frac{\partial w}{\partial x} \delta x$$

Then

$$(O'A')^2 = \delta x^2 \left( 1 + \frac{\partial u}{\partial x} \right)^2 + \delta x^2 \left( \frac{\partial v}{\partial x} \right)^2 + \delta x^2 \left( \frac{\partial w}{\partial x} \right)^2$$

$$(O'A')^2 = \left( \delta x + u + \frac{\partial u}{\partial x} \delta x - u \right)^2 + \left( v + \frac{\partial v}{\partial x} \delta x - v \right)^2 + \left( w + \frac{\partial w}{\partial x} \delta x - w \right)^2$$

or

$$O'A' = \delta x \sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}$$

which may be written, when second-order terms are neglected, as

$$O'A' = \delta x \left(1 + 2 \frac{\partial u}{\partial x}\right)^{\frac{1}{2}}$$

Applying the binomial expansion to this expression, we have

$$O'A' = \delta x \left(1 + \frac{\partial u}{\partial x}\right) \quad (1.17)$$

in which squares and higher powers of  $\partial u / \partial x$  are ignored. Substituting for  $O'A'$  in Eq. (1.16), we have

$$\left. \begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \varepsilon_z &= \frac{\partial w}{\partial z} \end{aligned} \right\} \quad (1.18)$$

It follows that

The shear strain at a point in a body is defined as the change in the angle between two mutually perpendicular lines at the point. Therefore, if the shear strain in the  $xz$  plane is  $\gamma_{xz}$ , then the angle between the displaced line elements  $O'A'$  and  $O'C'$  in Fig. 1.16 is  $\pi/2 - \gamma_{xz}$  radians.

Now,  $\cos A'O'C' = \cos(\pi/2 - \gamma_{xz}) = \sin \gamma_{xz}$  and as  $\gamma_{xz}$  is small,  $\cos A'O'C' = \gamma_{xz}$ . From the trigonometrical relationships for a triangle,

$$\cos A'O'C' = \frac{(O'A')^2 + (O'C')^2 - (A'C')^2}{2(O'A')(O'C')} \quad (1.19)$$

We showed in Eq. (1.17) that

$$O'A' = \delta x \left(1 + \frac{\partial u}{\partial x}\right)$$

Similarly,

$$O'C' = \delta z \left(1 + \frac{\partial w}{\partial z}\right)$$

But, for small displacements, the derivatives of  $u$ ,  $v$ , and  $w$  are small compared with 1, so that, as we are concerned here with actual length rather than change in length, we may use the approximations

$$O'A' \approx \delta x, \quad O'C' \approx \delta z$$

Referring to Fig. 1.16 the projection of A'C' in the  $xz$  plane parallel to the  $x$  axis is

$$\delta x + u + \frac{\partial u}{\partial x} \delta x - u - \frac{\partial u}{\partial z} \delta z = \delta x \left( 1 + \frac{\partial u}{\partial x} \right) - \frac{\partial u}{\partial z} \delta z$$

The projection of A'C' in the  $yz$  plane parallel to the  $z$  axis is

$$\delta z + w + \frac{\partial w}{\partial z} \delta z - w - \frac{\partial w}{\partial x} \delta x = \delta z \left( 1 + \frac{\partial w}{\partial z} \right) - \frac{\partial w}{\partial x} \delta x$$

The projection of A'C' in the  $xy$  plane parallel to the  $y$  axis is

$$v + \frac{\partial v}{\partial x} \delta x - v - \frac{\partial v}{\partial z} \delta z = \frac{\partial v}{\partial x} \delta x - \frac{\partial v}{\partial z} \delta z$$

Then

$$(A'C')^2 = \left[ \delta x \left( 1 + \frac{\partial u}{\partial x} \right) - \frac{\partial u}{\partial z} \delta z \right]^2 + \left[ \delta z \left( 1 + \frac{\partial w}{\partial z} \right) - \frac{\partial w}{\partial x} \delta x \right]^2 + \left( \frac{\partial v}{\partial x} \delta x - \frac{\partial v}{\partial z} \delta z \right)^2$$

Expanding and simplifying, we obtain

$$(A'C')^2 = \left( \delta z - \frac{\partial w}{\partial x} \delta x \right)^2 + \left( \delta x - \frac{\partial u}{\partial z} \delta z \right)^2$$

Substituting for O'A', O'C', and A'C' in Eq. (1.19), we have

$$\cos A'O'C' = \frac{(\delta x^2) + (\delta z)^2 - [\delta z - (\partial w/\partial x)\delta x]^2 - [\delta x - (\partial u/\partial z)\delta z]^2}{2\delta x \delta z}$$

Expanding and neglecting fourth-order powers gives

$$\cos A'O'C' = \frac{2(\partial w/\partial x)\delta x \delta z + 2(\partial u/\partial z)\delta x \delta z}{2\delta x \delta z}$$

or,

$$\left. \begin{aligned} \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned} \right\} \quad (1.20)$$

Similarly,

It must be emphasized that Eqs. (1.18) and (1.20) are derived on the assumption that the displacements involved are small. Normally, these linearized equations are adequate for most types of structural problem, but in cases where deflections are large—for example, types of suspension cable—the full, nonlinear, large deflection equations, given in many books on elasticity, must be employed.

## 1.10 COMPATIBILITY EQUATIONS

In Section 1.9, we expressed the six components of strain at a point in a deformable body in terms of the three components of displacement at that point,  $u$ ,  $v$ , and  $w$ . We supposed that the body remains continuous during the deformation, so that no voids are formed. It follows that each component,  $u$ ,  $v$ , and  $w$ , must be a continuous, single-valued function or, in quantitative terms,

$$u = f_1(x, y, z), \quad v = f_2(x, y, z), \quad w = f_3(x, y, z)$$

If voids are formed, then displacements in regions of the body separated by the voids are expressed as different functions of  $x$ ,  $y$ , and  $z$ . The existence, therefore, of just three single-valued functions for displacement is an expression of the continuity or *compatibility* of displacement, which we presupposed.

Since the six strains are defined in terms of three displacement functions, they must bear some relationship to each other and cannot have arbitrary values. These relationships are found as follows. Differentiating  $\gamma_{xy}$  from Eq. (1.20) with respect to  $x$  and  $y$  gives

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} \frac{\partial v}{\partial x} + \frac{\partial^2}{\partial x \partial y} \frac{\partial u}{\partial y}$$

or, since the functions of  $u$  and  $v$  are continuous,

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x^2} \frac{\partial v}{\partial y} + \frac{\partial^2}{\partial y^2} \frac{\partial u}{\partial x}$$

which may be written, using Eq. (1.18), as

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} \quad (1.21)$$

In a similar manner,

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} \quad (1.22)$$

$$\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} \quad (1.23)$$

If we now differentiate  $\gamma_{xy}$  with respect to  $x$  and  $z$  and add the result to  $\gamma_{xz}$ , differentiated with respect to  $y$  and  $x$ , we obtain

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} + \frac{\partial^2 \gamma_{xz}}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial z} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial^2}{\partial y \partial x} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

or

$$\frac{\partial}{\partial x} \left( \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} \right) = \frac{\partial^2}{\partial z \partial y} \frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) + \frac{\partial^2}{\partial y \partial z} \frac{\partial u}{\partial x}$$

Substituting from Eqs. (1.18) and (1.21) and rearranging,

$$2 \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x} \left( - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (1.24)$$

Similarly,

$$2 \frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y} \left( \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (1.25)$$

and

$$2 \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (1.26)$$

Equations (1.21)–(1.26) are the six equations of *strain compatibility* that must be satisfied in the solution of three-dimensional problems in elasticity.

## 1.11 PLANE STRAIN

Although we derived the compatibility equations and the expressions for strain for the general three-dimensional state of strain, we shall be concerned mainly with the two-dimensional case described in Section 1.4. The corresponding state of strain, in which it is assumed that particles of the body suffer displacements in one plane only, is known as *plane strain*. We shall suppose that this plane is, as for plane stress, the  $xy$  plane. Then,  $\varepsilon_z$ ,  $\gamma_{xz}$ , and  $\gamma_{yz}$  become zero and Eqs. (1.18) and (1.20) reduce to

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad (1.27)$$

and

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (1.28)$$

Further, by substituting  $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$  in the six equations of compatibility and noting that  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\gamma_{xy}$  are now purely functions of  $x$  and  $y$ , we are left with Eq. (1.21), namely

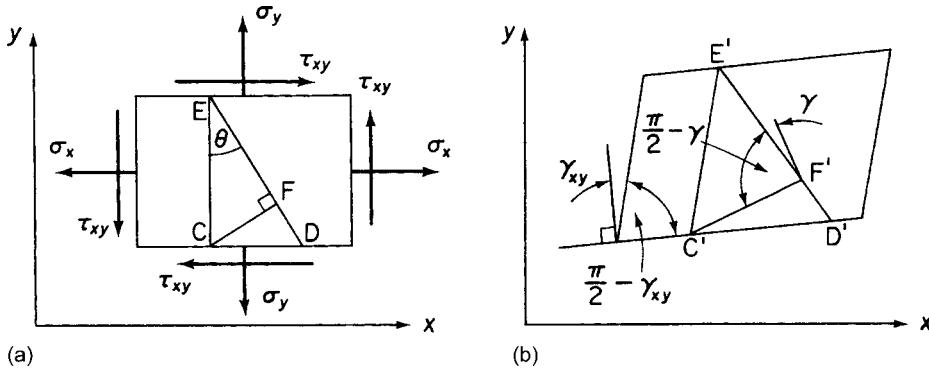
$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2}$$

as the only equation of compatibility in the two-dimensional or plane strain case.

## 1.12 DETERMINATION OF STRAINS ON INCLINED PLANES

Having defined the strain at a point in a deformable body with reference to an arbitrary system of coordinate axes, we may calculate direct strains in any given direction and the change in the angle (shear strain) between any two originally perpendicular directions at that point. We shall consider the two-dimensional case of plane strain described in Section 1.11.

An element in a two-dimensional body subjected to the complex stress system of Fig. 1.17(a) distorts into the shape shown in Fig. 1.17(b). In particular, the triangular element ECD suffers distortion to the shape E'C'D' with corresponding changes in the length FC and angle EFC. Suppose that the



**FIGURE 1.17 (a) Stress System on a Rectangular Element; (b) Distorted Shape of the Element Due to Stress System in (a)**

known direct and shear strains associated with the given stress system are  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\gamma_{xy}$  (the actual relationships are investigated later) and we are required to find the direct strain  $\varepsilon_n$  in a direction normal to the plane ED and the shear strain  $\gamma$  produced by the shear stress acting on the plane ED.

To a first order of approximation,

$$\left. \begin{aligned} C'D' &= CD(1 + \varepsilon_x) \\ C'E' &= CE(1 + \varepsilon_y) \\ E'D' &= ED(1 + \varepsilon_{n+\pi/2}) \end{aligned} \right\} \quad (1.29)$$

where  $\varepsilon_{n+\pi/2}$  is the direct strain in the direction ED. From the geometry of the triangle E'C'D' in which angle  $E'C'D' = \pi/2 - \gamma_{xy}$ ,

$$(E'D')^2 = (C'D')^2 + (C'E')^2 - 2(C'D')(C'E') \cos(\pi/2 - \gamma_{xy})$$

or, substituting from Eqs. (1.29),

$$\begin{aligned} (ED)^2(1 + \varepsilon_{n+\pi/2})^2 &= (CD)^2(1 + \varepsilon_x)^2 + (CE)^2(1 + \varepsilon_y)^2 \\ &\quad - 2(CD)(CE)(1 + \varepsilon_x)(1 + \varepsilon_y) \sin \gamma_{xy} \end{aligned}$$

Noting that  $(ED)^2 = (CD)^2 + (CE)^2$  and neglecting squares and higher powers of small quantities, this equation may be rewritten as

$$2(ED)^2 \varepsilon_{n+\pi/2} = 2(CD)^2 \varepsilon_x + 2(CE)^2 \varepsilon_y - 2(CD)(CE) \gamma_{xy}$$

Dividing through by  $2(ED)^2$  gives

$$\varepsilon_{n+\pi/2} = \varepsilon_x \sin^2 \theta + \varepsilon_y \cos^2 \theta - \cos \theta \sin \theta \gamma_{xy} \quad (1.30)$$

The strain  $\varepsilon_n$  in the direction normal to the plane ED is found by replacing the angle  $\theta$  in Eq. (1.30) by  $\theta - \pi/2$ . Hence,

$$\varepsilon_n = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta + \frac{\gamma_{xy}}{2} \sin 2\theta \quad (1.31)$$

Turning our attention to the triangle C'F'E', we have

$$(C'E')^2 = (C'F')^2 + (F'E')^2 - 2(C'F')(F'E') \cos(\pi/2 - \gamma) \quad (1.32)$$

in which

$$\begin{aligned} C'E' &= CE(1 + \varepsilon_y) \\ C'F' &= CF(1 + \varepsilon_n) \\ F'E' &= FE(1 + \varepsilon_{n+\pi/2}) \end{aligned}$$

Substituting for C'E', C'F', and F'E' in Eq. (1.32) and writing  $\cos(\pi/2 - \gamma) = \sin\gamma$ , we find

$$\begin{aligned} (CE)^2(1 + \varepsilon_y)^2 &= (CF)^2(1 + \varepsilon_n)^2 + (FE)^2(1 + \varepsilon_{n+\pi/2})^2 \\ &\quad - 2(CF)(FE)(1 + \varepsilon_n)(1 + \varepsilon_{n+\pi/2}) \sin\gamma \end{aligned} \quad (1.33)$$

All the strains are assumed to be small, so that their squares and higher powers may be ignored. Further,  $\sin\gamma \approx \gamma$  and Eq. (1.33) becomes

$$(CE)^2(1 + 2\varepsilon_y) = (CF)^2(1 + 2\varepsilon_n) + (FE)^2(1 + 2\varepsilon_{n+\pi/2}) - 2(CF)(FE)\gamma$$

From Fig. 1.17(a),  $(CE)^2 = (CF)^2 + (FE)^2$  and the preceding equation simplifies to

$$2(CE)^2\varepsilon_y = 2(CF)^2\varepsilon_n + 2(FE)^2\varepsilon_{n+\pi/2} - 2(CF)(FE)\gamma$$

Dividing through by  $2(CE)^2$  and transposing,

$$\gamma = \frac{\varepsilon_n \sin^2\theta + \varepsilon_{n+\pi/2} \cos^2\theta - \varepsilon_y}{\sin\theta \cos\theta}$$

Substitution of  $\varepsilon_n$  and  $\varepsilon_{n+\pi/2}$  from Eqs. (1.31) and (1.30) yields

$$\gamma = \frac{(\varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta) \sin^2\theta + (\varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \frac{\gamma_{xy}}{2} \sin 2\theta) \cos^2\theta - \varepsilon_y}{\sin\theta \cos\theta}$$

Rearranging gives

$$\gamma = \frac{2\varepsilon_x \cos^2\theta \sin^2\theta + \varepsilon_y (\sin^4\theta + \cos^4\theta) + \frac{\gamma_{xy}}{2} \sin 2\theta \sin^2\theta - \frac{\gamma_{xy}}{2} \sin 2\theta \cos^2\theta - \varepsilon_y}{\sin\theta \cos\theta}$$

which simplifies to

$$\gamma = \varepsilon_x \sin 2\theta + \varepsilon_y \frac{(\sin^4\theta + \cos^4\theta - 1)}{\sin\theta \cos\theta} + \gamma_{xy} (\sin^2\theta - \cos^2\theta)$$

But, using standard trigonometrical relationships,

$$\sin^4\theta + \cos^4\theta - 1 = \frac{-\sin^2 2\theta}{2}, \quad \sin^2\theta - \cos^2\theta = -\cos 2\theta$$

Substituting in the above,

$$\frac{\gamma}{2} = \frac{(\varepsilon_x - \varepsilon_y)}{2} \sin 2\theta - \frac{\gamma_{xy}}{2} \cos 2\theta \quad (1.34)$$

### 1.13 PRINCIPAL STRAINS

If we compare Eqs. (1.31) and (1.34) with Eqs. (1.8) and (1.9), we observe that they may be obtained from Eqs. (1.8) and (1.9) by replacing  $\sigma_n$  with  $\varepsilon_n$ ,  $\sigma_x$  by  $\varepsilon_x$ ,  $\sigma_y$  by  $\varepsilon_y$ ,  $\tau_{xy}$  by  $\gamma_{xy}/2$ , and  $\tau$  by  $\gamma/2$ . Therefore, for each deduction made from Eqs. (1.8) and (1.9) concerning  $\sigma_n$  and  $\tau$ , there is a corresponding deduction from Eqs. (1.31) and (1.34) regarding  $\varepsilon_n$  and  $\gamma/2$ .

Therefore, at a point in a deformable body, there are two mutually perpendicular planes on which the shear strain  $\gamma$  is zero and normal to which the direct strain is a maximum or minimum. These strains are the *principal strains* at that point and are given (from comparison with Eqs. (1.11) and (1.12)) by

$$\varepsilon_I = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{1}{2} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \quad (1.35)$$

and

$$\varepsilon_{II} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{1}{2} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \quad (1.36)$$

If the shear strain is zero on these planes, it follows that the shear stress must also be zero; and we deduce, from Section 1.7, that the directions of the principal strains and principal stresses coincide. The related planes are then determined from Eq. (1.10) or from

$$\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \quad (1.37)$$

In addition, the maximum shear strain at the point is

$$\left(\frac{\gamma}{2}\right)_{\max} = \frac{1}{2} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \quad (1.38)$$

or

$$\left(\frac{\gamma}{2}\right)_{\max} = \frac{\varepsilon_I - \varepsilon_{II}}{2} \quad (1.39)$$

(compare with Eqs. (1.14) and (1.15)).

### 1.14 MOHR'S CIRCLE OF STRAIN

We now apply the arguments of Section 1.13 to the Mohr's circle of stress described in Section 1.8. A circle of strain, analogous to that shown in Fig. 1.13(b), may be drawn when  $\sigma_x$ ,  $\sigma_y$ , etc., are replaced by  $\varepsilon_x$ ,  $\varepsilon_y$ , etc., as specified in Section 1.13. The horizontal extremities of the circle represent the principal strains, the radius of the circle, half the maximum shear strain, and so on.

### 1.15 STRESS–STRAIN RELATIONSHIPS

In the preceding sections, we developed, for a three-dimensional deformable body, three equations of equilibrium (Eqs. (1.5)) and six strain-displacement relationships (Eqs. (1.18) and (1.20)). From the latter, we eliminated displacements, thereby deriving six auxiliary equations relating strains. These compatibility equations are an expression of the continuity of displacement, which we have assumed

as a prerequisite of the analysis. At this stage, therefore, we have obtained nine independent equations toward the solution of the three-dimensional stress problem. However, the number of unknowns totals 15, comprising six stresses, six strains, and three displacements. An additional six equations are therefore necessary to obtain a solution.

So far we have made no assumptions regarding the force-displacement or stress-strain relationship in the body. This will, in fact, provide us with the required six equations, but before these are derived, it is worthwhile considering some general aspects of the analysis.

The derivation of the equilibrium, strain-displacement, and compatibility equations does not involve any assumption as to the stress-strain behavior of the material of the body. It follows that these basic equations are applicable to any type of continuous, deformable body, no matter how complex its behavior under stress. In fact, we shall consider only the simple case of linearly elastic, *isotropic* materials, for which stress is directly proportional to strain and whose elastic properties are the same in all directions. A material possessing the same properties at all points is said to be *homogeneous*.

Particular cases arise where some of the stress components are known to be zero and the number of unknowns may then be no greater than the remaining equilibrium equations that have not identically vanished. The unknown stresses are then found from the conditions of equilibrium alone, and the problem is said to be *statically determinate*. For example, the uniform stress in the member supporting a tensile load  $P$  in Fig. 1.3 is found by applying one equation of equilibrium and a boundary condition. This system is therefore statically determinate.

*Statically indeterminate* systems require the use of some, if not all, of the other equations involving strain-displacement and stress-strain relationships. However, whether the system be statically determinate or not, stress-strain relationships are necessary to determine deflections. The role of the six auxiliary compatibility equations will be discussed when actual elasticity problems are formulated in Chapter 2.

We now proceed to investigate the relationship of stress and strain in a three-dimensional, linearly elastic, isotropic body.

Experiments show that the application of a uniform direct stress, say  $\sigma_x$ , does not produce any shear distortion of the material, and that the direct strain  $\varepsilon_x$  is given by the equation

$$\varepsilon_x = \frac{\sigma_x}{E} \quad (1.40)$$

where  $E$  is a constant known as the *modulus of elasticity* or *Young's modulus*. Equation (1.40) is an expression of *Hooke's law*.<sup>1</sup> Further,  $\varepsilon_x$  is accompanied by lateral strains<sup>1</sup>

$$\varepsilon_y = -v \frac{\sigma_x}{E}, \quad \varepsilon_z = -v \frac{\sigma_x}{E} \quad (1.41)$$

in which  $v$  is a constant termed *Poisson's ratio*.

For a body subjected to direct stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$ , the direct strains are, from Eqs. (1.40) and (1.41) and the principle of superposition (see Chapter 5, Section 5.9),

$$\left. \begin{aligned} \varepsilon_x &= \frac{1}{E} [\sigma_x - v(\sigma_y + \sigma_z)] \\ \varepsilon_y &= \frac{1}{E} [\sigma_y - v(\sigma_x + \sigma_z)] \\ \varepsilon_z &= \frac{1}{E} [\sigma_z - v(\sigma_x + \sigma_y)] \end{aligned} \right\} \quad (1.42)$$

Equations (1.42) may be transposed to obtain expressions for each stress in terms of the strains.

From Eqs. (1.42),

$$\sigma_x = E\epsilon_x + v\sigma_y + v\sigma_z \quad (\text{i})$$

$$\sigma_y = E\epsilon_y + v\sigma_x + v\sigma_z \quad (\text{ii})$$

$$\sigma_z = E\epsilon_z + v\sigma_x + v\sigma_y \quad (\text{iii})$$

Adding Eq. (i) to Eq. (ii)  $\times v$  and rearranging gives

$$\sigma_x (1 - v^2) = E (\epsilon_x + v\epsilon_y) + v\sigma_z (1 + v) \quad (\text{iv})$$

Now subtract Eq. (iii) from Eq. (i) and rearrange,

$$\sigma_x (1 + v) = E (\epsilon_x - \epsilon_z) + \sigma_z (1 + v) \quad (\text{v})$$

Now subtract Eq. (v)  $\times v$  from Eq. (iv),

$$\sigma_x (1 - v^2 - v - v^2) = E (\epsilon_x + v\epsilon_y - v\epsilon_x + v\epsilon_z) + v\sigma_z (1 + v) - v\sigma_z (1 + v)$$

Then,

$$\sigma_x (1 + v)(1 - 2v) = E [\epsilon_x + v(\epsilon_x + \epsilon_y + \epsilon_z) - 2v\epsilon_x]$$

from which

$$\sigma_x = \frac{vE}{(1 + v)(1 - 2v)} e + \frac{E}{(1 + v)} \epsilon_x \quad (1.43)$$

where  $e = \epsilon_x + \epsilon_y + \epsilon_z$ .

Similarly,

$$\sigma_y = \frac{vE}{(1 + v)(1 - 2v)} e + \frac{E}{(1 + v)} \epsilon_y \quad (1.44)$$

and

$$\sigma_z = \frac{vE}{(1 + v)(1 - 2v)} e + \frac{E}{(1 + v)} \epsilon_z \quad (1.45)$$

See Eq. (1.53).

For the case of plane stress in which  $\sigma_z = 0$ , the first two of Eqs. (1.42) reduce to

$$\epsilon_x = \frac{1}{E} (\sigma_x - v\sigma_y) \quad (\text{vi})$$

$$\epsilon_y = \frac{1}{E} (\sigma_y - v\sigma_x) \quad (\text{vii})$$

Adding Eq. (vii)  $\times v$  to Eq. (vi) gives

$$\epsilon_x + v\epsilon_y = \frac{1}{E} (1 - v^2) \sigma_x$$

so that

$$\sigma_x = \frac{E}{1 - v^2} (\epsilon_x + v\epsilon_y) \quad (1.46)$$

Similarly,

$$\sigma_y = \frac{E}{1-v^2} (\varepsilon_y + v\varepsilon_x) \quad (1.47)$$

Suppose now that, at some arbitrary point in a material, there are principal strains  $\varepsilon_I$  and  $\varepsilon_{II}$  corresponding to principal stresses  $\sigma_I$  and  $\sigma_{II}$ . If these stresses (and strains) are in the direction of the coordinate axes  $x$  and  $y$ , respectively, then  $\tau_{xy} = \gamma_{xy} = 0$  and, from Eq. (1.34), the shear strain on an arbitrary plane at the point inclined at an angle  $\theta$  to the principal planes is

$$\gamma = (\varepsilon_I - \varepsilon_{II}) \sin 2\theta \quad (1.48)$$

Using the relationships of Eqs. (1.42) in which  $\sigma_z = \sigma_{III} = 0$  and substituting in Eq. (1.48), we have

$$\gamma = \frac{1}{E} [(\sigma_I - v\sigma_{II}) - (\sigma_{II} - v\sigma_I)] \sin 2\theta$$

or

$$\gamma = \frac{(1+v)}{E} (\sigma_I - \sigma_{II}) \sin 2\theta \quad (1.49)$$

Using Eq. (1.9) and noting that for this particular case  $\tau_{xy} = 0$ ,  $\sigma_x = \sigma_I$ , and  $\sigma_y = \sigma_{II}$ ,

$$2\tau = (\sigma_I - \sigma_{II}) \sin 2\theta$$

from which we may rewrite Eq. (1.49) in terms of  $\tau$  as

$$\gamma = \frac{2(1+v)}{E} \tau \quad (1.50)$$

The term  $E/2(1+v)$  is a constant known as the *modulus of rigidity*  $G$ . Hence,

$$\gamma = \tau/G$$

and the shear strains  $\gamma_{xy}$ ,  $\gamma_{xz}$ , and  $\gamma_{yz}$  are expressed in terms of their associated shear stresses as follows:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{xz} = \frac{\tau_{xz}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G} \quad (1.51)$$

Equations (1.51), together with Eqs. (1.42), provide the additional six equations required to determine the 15 unknowns in a general three-dimensional problem in elasticity. They are, however, limited in use to a linearly elastic, isotropic body.

For the case of plane stress, they simplify to

$$\left. \begin{aligned} \varepsilon_x &= \frac{1}{E} (\sigma_x - v\sigma_y) \\ \varepsilon_y &= \frac{1}{E} (\sigma_y - v\sigma_x) \\ \varepsilon_z &= \frac{-v}{E} (\sigma_x - \sigma_y) \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \end{aligned} \right\} \quad (1.52)$$

It may be seen from the third of Eqs. (1.52) that the conditions of plane stress and plane strain do not necessarily describe identical situations.

Changes in the linear dimensions of a strained body may lead to a change in volume. Suppose that a small element of a body has dimensions  $\delta x$ ,  $\delta y$ , and  $\delta z$ . When subjected to a three-dimensional stress system, the element sustains a volumetric strain  $e$  (change in volume/unit volume) equal to

$$e = \frac{(1 + \varepsilon_x)\delta x(1 + \varepsilon_y)\delta y(1 + \varepsilon_z)\delta z - \delta x\delta y\delta z}{\delta x\delta y\delta z}$$

Neglecting products of small quantities in the expansion of the right-hand side of this equation yields

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z \quad (1.53)$$

Substituting for  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\varepsilon_z$  from Eqs. (1.42), we find, for a linearly elastic, isotropic body,

$$e = \frac{1}{E}[\sigma_x + \sigma_y + \sigma_z - 2v(\sigma_x + \sigma_y + \sigma_z)]$$

or

$$e = \frac{(1 - 2v)}{E}(\sigma_x + \sigma_y + \sigma_z)$$

In the case of a uniform hydrostatic pressure,  $\sigma_x = \sigma_y = \sigma_z = -p$  and

$$e = -\frac{3(1 - 2v)}{E}p \quad (1.54)$$

The constant  $E/3(1 - 2v)$  is known as the *bulk modulus* or *modulus of volume expansion* and is often given the symbol  $K$ .

An examination of Eq. (1.54) shows that  $v \leq 0.5$ , since a body cannot increase in volume under pressure. Also, the lateral dimensions of a body subjected to uniaxial tension cannot increase, so that  $v > 0$ . Therefore, for an isotropic material  $0 \leq v \leq 0.5$  and for most isotropic materials,  $v$  is in the range 0.25–0.33 below the elastic limit. Above the limit of proportionality,  $v$  increases and approaches 0.5.

### Example 1.4

A rectangular element in a linearly elastic, isotropic material is subjected to tensile stresses of 83 and 65 N/mm<sup>2</sup> on mutually perpendicular planes. Determine the strain in the direction of each stress and in the direction perpendicular to both stresses. Find also the principal strains, the maximum shear stress, the maximum shear strain, and their directions at the point. Take  $E = 200,000$  N/mm<sup>2</sup> and  $v = 0.3$ .

If we assume that  $\sigma_x = 83$  N/mm<sup>2</sup> and  $\sigma_y = 65$  N/mm<sup>2</sup>, then from Eqs. (1.52),

$$\varepsilon_x = \frac{1}{200,000}(83 - 0.3 \times 65) = 3.175 \times 10^{-4}$$

$$\varepsilon_y = \frac{1}{200,000}(65 - 0.3 \times 83) = 2.005 \times 10^{-4}$$

$$\varepsilon_z = \frac{-0.3}{200,000}(83 + 65) = -2.220 \times 10^{-4}$$

In this case, since there are no shear stresses on the given planes,  $\sigma_x$  and  $\sigma_y$  are principal stresses, so that  $\varepsilon_x$  and  $\varepsilon_y$  are the principal strains and are in the directions of  $\sigma_x$  and  $\sigma_y$ . It follows from Eq. (1.15) that the maximum shear stress (in the plane of the stresses) is

$$\tau_{\max} = \frac{83 - 65}{2} = 9 \text{ N/mm}^2$$

acting on planes at  $45^\circ$  to the principal planes.

Further, using Eq. (1.50), the maximum shear strain is

$$\gamma_{\max} = \frac{2 \times (1 + 0.3) \times 9}{200,000}$$

so that  $\gamma_{\max} = 1.17 \times 10^{-4}$  on the planes of maximum shear stress. ■

### Example 1.5

At a particular point in a structural member, a two-dimensional stress system exists where  $\sigma_x = 60 \text{ N/mm}^2$ ,  $\sigma_y = -40 \text{ N/mm}^2$ , and  $\tau_{xy} = 50 \text{ N/mm}^2$ . If Young's modulus  $E = 200,000 \text{ N/mm}^2$  and Poisson's ratio  $v = 0.3$ , calculate the direct strain in the  $x$  and  $y$  directions and the shear strain at the point. Also calculate the principal strains at the point and their inclination to the plane on which  $\sigma_x$  acts; verify these answers using a graphical method.

From Eqs. (1.52),

$$\varepsilon_x = \frac{1}{200,000} (60 + 0.3 \times 40) = 360 \times 10^{-6}$$

$$\varepsilon_y = \frac{1}{200,000} (-40 - 0.3 \times 60) = -290 \times 10^{-6}$$

From Eq. (1.50), the shear modulus,  $G$ , is given by

$$G = \frac{E}{2(1+v)} = \frac{200,000}{2(1+0.3)} = 76923 \text{ N/mm}^2$$

Hence, from Eqs. (1.52),

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{50}{76,923} = 650 \times 10^{-6}$$

Now substituting in Eq. (1.35) for  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\gamma_{xy}$ ,

$$\varepsilon_I = 10^{-6} \left[ \frac{360 - 290}{2} + \frac{1}{2} \sqrt{(360 + 290)^2 + 650^2} \right]$$

which gives

$$\varepsilon_I = 495 \times 10^{-6}$$

Similarly, from Eq. (1.36),

$$\varepsilon_{II} = -425 \times 10^{-6}$$

From Eq. (1.37),

$$\tan 2\theta = \frac{650 \times 10^{-6}}{360 \times 10^{-6} + 290 \times 10^{-6}} = 1$$

Therefore,

$$2\theta = 45^\circ \text{ or } 225^\circ$$

so that

$$\theta = 22.5^\circ \text{ or } 112.5^\circ$$

■

The values of  $\varepsilon_I$ ,  $\varepsilon_{II}$ , and  $\theta$  are verified using Mohr's circle of strain (Fig. 1.18). Axes  $O\varepsilon$  and  $O\gamma$  are set up and the points  $Q_1(360 \times 10^{-6}, \frac{1}{2} \times 650 \times 10^{-6})$  and  $Q_2(-290 \times 10^{-6}, -\frac{1}{2} \times 650 \times 10^{-6})$  located. The center C of the circle is the intersection of  $Q_1Q_2$  and the  $O\varepsilon$  axis. The circle is then drawn with radius  $CQ_1$  and the points B( $\varepsilon_I$ ) and A( $\varepsilon_{II}$ ) located. Finally, angle  $Q_1CB = 2\theta$  and angle  $Q_1CA = 2\theta + \pi$ .

### 1.15.1 Temperature effects

The stress-strain relationships of Eqs. (1.43)–(1.47) apply to a body or structural member at a constant uniform temperature. A temperature rise (or fall) generally results in an expansion (or contraction) of the body or structural member so that there is a change in size—that is, a strain.

Consider a bar of uniform section, of original length  $L_o$ , and suppose that it is subjected to a temperature change  $\Delta T$  along its length;  $\Delta T$  can be a rise (+ve) or fall (-ve). If the coefficient of linear expansion of the material of the bar is  $\alpha$ , the final length of the bar is, from elementary physics,

$$L = L_o(1 + \alpha\Delta T)$$

so that the strain,  $\varepsilon$ , is given by

$$\varepsilon = \frac{L - L_o}{L_o} = \alpha\Delta T \quad (1.55)$$

Suppose now that a compressive axial force is applied to each end of the bar, such that the bar returns to its original length. The *mechanical* strain produced by the axial force is therefore just large enough to

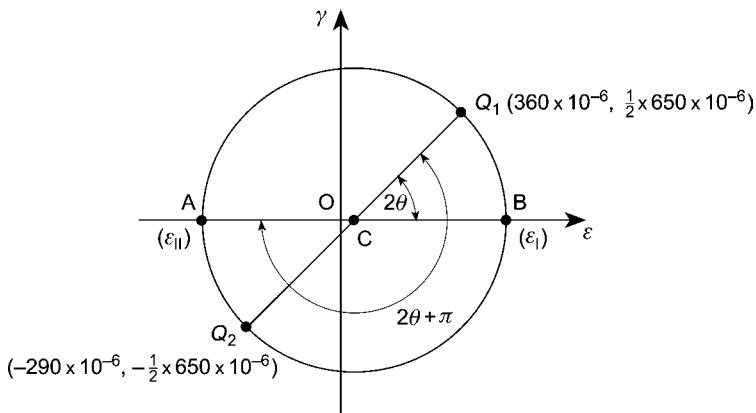


FIGURE 1.18 Mohr's Circle of Strain for Example 1.5

offset the *thermal* strain due to the temperature change, making the *total* strain zero. In general terms, the total strain,  $\varepsilon$ , is the sum of the mechanical and thermal strains. Therefore, from Eqs. (1.40) and (1.55),

$$\varepsilon = \frac{\sigma}{E} + \alpha\Delta T \quad (1.56)$$

In the case where the bar is returned to its original length or if the bar had not been allowed to expand at all, the total strain is zero and, from Eq. (1.56),

$$\sigma = -E\alpha\Delta T \quad (1.57)$$

Equations (1.42) may now be modified to include the contribution of thermal strain. Therefore, by comparison with Eq. (1.56),

$$\left. \begin{aligned} \varepsilon_x &= \frac{1}{E} [\sigma_x - v(\sigma_y + \sigma_z)] + \alpha\Delta T \\ \varepsilon_y &= \frac{1}{E} [\sigma_y - v(\sigma_x + \sigma_z)] + \alpha\Delta T \\ \varepsilon_z &= \frac{1}{E} [\sigma_z - v(\sigma_x + \sigma_y)] + \alpha\Delta T \end{aligned} \right\} \quad (1.58)$$

Equations (1.58) may be transposed in the same way as Eqs. (1.42) to give stress-strain relationships rather than strain-stress relationships; that is,

$$\left. \begin{aligned} \sigma_x &= \frac{vE}{(1+v)(1-2v)} e + \frac{E}{(1+v)} \varepsilon_x - \frac{E}{(1-2v)} \alpha\Delta T \\ \sigma_y &= \frac{vE}{(1+v)(1-2v)} e + \frac{E}{(1+v)} \varepsilon_y - \frac{E}{(1-2v)} \alpha\Delta T \\ \sigma_z &= \frac{vE}{(1+v)(1-2v)} e + \frac{E}{(1+v)} \varepsilon_z - \frac{E}{(1-2v)} \alpha\Delta T \end{aligned} \right\} \quad (1.59)$$

For the case of plane stress in which  $\sigma_z = 0$ , these equations reduce to

$$\left. \begin{aligned} \sigma_x &= \frac{E}{(1-v^2)} (\varepsilon_x + v\varepsilon_y) - \frac{E}{(1-v)} \alpha\Delta T \\ \sigma_y &= \frac{E}{(1-v^2)} (\varepsilon_y + v\varepsilon_x) - \frac{E}{(1-v)} \alpha\Delta T \end{aligned} \right\} \quad (1.60)$$

### Example 1.6

A composite bar of length  $L$  has a central core of copper loosely inserted in a sleeve of steel; the ends of the steel and copper are attached to each other by rigid plates. If the bar is subjected to a temperature rise  $\Delta T$ , determine the stress in the steel and in the copper and the extension of the composite bar. The copper core has a Young's modulus  $E_c$ , a cross-sectional area  $A_c$ , and a coefficient of linear expansion  $\alpha_c$ ; the corresponding values for the steel are

$E_s$ ,  $A_s$ , and  $\alpha_s$ . Further, if the bar is then subjected to a compressive axial load  $P$ , calculate the final stress in the steel and the copper.

Assume that  $\alpha_c > \alpha_s$ . If the copper core and steel sleeve are allowed to expand freely, their final lengths would be different, since they have different values of the coefficient of linear expansion. However, since they are rigidly attached at their ends, one restrains the other and an axial stress is induced in each. Suppose that this stress is  $\sigma_x$ . Then, in Eqs. (1.58).  $\sigma_x = \sigma_c$  or  $\sigma_s$  and  $\sigma_y = \sigma_z = 0$ ; the total strain in the copper and steel is then, respectively,

$$\varepsilon_c = \frac{\sigma_c}{E_c} + \alpha_c \Delta T \quad (\text{i})$$

$$\varepsilon_s = \frac{\sigma_s}{E_s} + \alpha_s \Delta T \quad (\text{ii})$$

The total strain in the copper and steel is the same, since their ends are rigidly attached to each other. Therefore, from compatibility of displacement,

$$\frac{\sigma_c}{E_c} + \alpha_c \Delta T = \frac{\sigma_s}{E_s} + \alpha_s \Delta T \quad (\text{iii})$$

No external axial load is applied to the bar, so that

$$\sigma_c A_c + \sigma_s A_s = 0$$

that is,

$$\sigma_s = -\frac{A_c}{A_s} \sigma_c \quad (\text{iv})$$

Substituting for  $\sigma_s$  in Eq. (iii) gives

$$\sigma_c \left( \frac{1}{E_c} + \frac{A_c}{A_s E_s} \right) = \Delta T (\alpha_s - \alpha_c)$$

from which

$$\sigma_c = \frac{\Delta T (\alpha_s - \alpha_c) A_s E_s}{A_s E_s + A_c E_c} \quad (\text{v})$$

Also  $\alpha_c > \alpha_s$ , so that  $\sigma_c$  is negative and therefore compressive. Now substituting for  $\sigma_c$  in Eq. (iv),

$$\sigma_s = -\frac{\Delta T (\alpha_s - \alpha_c) A_c E_s E_c}{A_s E_s + A_c E_c} \quad (\text{vi})$$

which is positive and therefore tensile, as would be expected by a physical appreciation of the situation.

Finally, the extension of the compound bar,  $\delta$ , is found by substituting for  $\sigma_c$  in Eq. (i) or for  $\sigma_s$  in Eq. (ii). Then, from Eqs. (i) and (iv),

$$\delta = L \varepsilon_c = \frac{L \Delta T (\alpha_s - \alpha_c) A_s E_s E_c}{E_c (A_s E_s + A_c E_c)} + L \alpha_c \Delta T$$

so that

$$\delta = \frac{\Delta T L (\alpha_s A_s E_s E_c - \alpha_c A_s E_s E_c + \alpha_c E_c A_s E_s + \alpha_c E_c A_c E_c)}{E_c (A_s E_s + A_c E_c)}$$

From which

$$\delta = \Delta TL \left( \frac{\alpha_c A_c E_c + \alpha_s A_s E_s}{A_s E_s + A_c E_c} \right) \quad (\text{vii})$$

Suppose that the loads in the copper and steel are  $P_c$  and  $P_s$ , respectively. Then, from equilibrium,

$$P_c + P_s = P \quad (\text{viii})$$

From Eqs. (1.40),

$$\varepsilon_c = \frac{P_c}{A_c E_c}, \quad \varepsilon_s = \frac{P_s}{A_s E_s}$$

Since the copper and steel tubes are firmly attached to each other at their ends  $\varepsilon_c = \varepsilon_s$ , so that

$$\frac{P_c}{A_c E_c} = \frac{P_s}{A_s E_s} \quad (\text{ix})$$

Then, from Eqs. (viii) and (ix),

$$P_c = \frac{A_c E_c}{A_c E_c + A_s E_s} P, \quad P_s = \frac{A_s E_s}{A_c E_c + A_s E_s} P$$

The additional stresses due to the compressive load  $P$  are then

$$\sigma_c = \frac{P_c}{A_c} = \frac{E_c}{A_c E_c + A_s E_s} P \quad (\text{x})$$

and

$$\sigma_s = \frac{P_s}{A_s} = \frac{E_s}{A_c E_c + A_s E_s} P \quad (\text{xi})$$

Both these stresses are compressive and are added algebraically to the stresses caused by the temperature change given by Eqs. (v) and (vi).

### Example 1.7

A strut, 2m long, supporting the cantilevered wing of a light aircraft, consists of an aluminum tube of elliptical cross-section reinforced by a titanium tube also of elliptical cross-section and which fits snugly inside the aluminum alloy tube; the tubes are firmly attached to each other at their ends. At sea level, the tubes are unstressed. If the areas of cross-section are, respectively,  $22 \text{ cm}^2$  and  $6 \text{ cm}^2$  for the aluminum alloy and titanium tubes, calculate the stress in each tube at an altitude of 4500 m where the temperature is  $25^\circ\text{C}$  below that at sea level. Determine also the contraction of the strut.

$$E(\text{aluminum}) = 70,000 \text{ N/mm}^2, \alpha(\text{aluminum}) = 2.1 \times 10^{-5}/^\circ\text{C}.$$

$$E(\text{titanium}) = 110,000 \text{ N/mm}^2, \alpha(\text{titanium}) = 1.1 \times 10^{-5}/^\circ\text{C}.$$

From Eq. (v) of Example 1.6,

$$\sigma_t = \frac{-25(\alpha_{\text{al}} - \alpha_t)A_{\text{al}}E_{\text{al}}E_t}{A_t E_t + A_{\text{al}} E_{\text{al}}}$$

That is

$$\sigma_t = \frac{-25(2.1 - 1.1) \times 10^{-5} \times 22 \times 10^2 \times 70,000 \times 110,000}{6 \times 10^2 \times 110,000 + 22 \times 10^2 \times 70,000}$$

which gives

$$\sigma_t = -19.25 \text{ N/mm}^2 \text{ (compression)}$$

and

$$\sigma_{al} = \frac{-25(1.1 - 2.1) \times 10^{-5} \times 6 \times 10^2 \times 70,000 \times 110,000}{6 \times 10^2 \times 110,000 + 22 \times 10^2 \times 70,000}$$

so that

$$\sigma_{al} = 5.25 \text{ N/mm}^2 \text{ (tension)}$$

Using Eq. (vii) of Example 1.6, the contraction of the strut is

$$\delta = \frac{-25 \times 2 \times 10^3 (2.1 \times 22 \times 10^2 \times 70,000 + 1.1 \times 6 \times 10^2 \times 110,000) \times 10^{-5}}{6 \times 10^2 \times 110,000 + 22 \times 10^2 \times 70,000}$$

that is,  $\delta = 9.0 \times 10^{-3}$  mm.



## 1.16 EXPERIMENTAL MEASUREMENT OF SURFACE STRAINS

Stresses at a point on the surface of a piece of material may be determined by measuring the strains at the point, usually by electrical resistance strain gauges arranged in the form of a rosette, as shown in Fig. 1.19. Suppose that  $\varepsilon_I$  and  $\varepsilon_{II}$  are the principal strains at the point; then if  $\varepsilon_a$ ,  $\varepsilon_b$ , and  $\varepsilon_c$  are the measured strains in the directions  $\theta$ ,  $(\theta + \alpha)$ ,  $(\theta + \alpha + \beta)$  to  $\varepsilon_I$ , we have, from the general direct strain relationship of Eq. (1.31),

$$\varepsilon_a = \varepsilon_I \cos^2 \theta + \varepsilon_{II} \sin^2 \theta \quad (1.61)$$

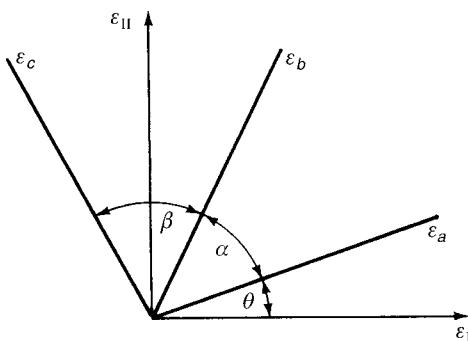


FIGURE 1.19 Strain Gauge Rosette

since  $\varepsilon_x$  becomes  $\varepsilon_I$ ,  $\varepsilon_y$  becomes  $\varepsilon_{II}$ , and  $\gamma_{xy}$  is zero, since the  $x$  and  $y$  directions have become principal directions. Rewriting Eq. (1.61), we have

$$\varepsilon_a = \varepsilon_I \left( \frac{1 + \cos 2\theta}{2} \right) + \varepsilon_{II} \left( \frac{1 - \cos 2\theta}{2} \right)$$

or

$$\varepsilon_a = \frac{1}{2}(\varepsilon_I + \varepsilon_{II}) + \frac{1}{2}(\varepsilon_I - \varepsilon_{II}) \cos 2\theta \quad (1.62)$$

Similarly,

$$\varepsilon_b = \frac{1}{2}(\varepsilon_I + \varepsilon_{II}) + \frac{1}{2}(\varepsilon_I - \varepsilon_{II}) \cos 2(\theta + \alpha) \quad (1.63)$$

and

$$\varepsilon_c = \frac{1}{2}(\varepsilon_I + \varepsilon_{II}) + \frac{1}{2}(\varepsilon_I - \varepsilon_{II}) \cos 2(\theta + \alpha + \beta) \quad (1.64)$$

Therefore, if  $\varepsilon_a$ ,  $\varepsilon_b$ , and  $\varepsilon_c$  are measured in given directions—that is, given angles  $\alpha$  and  $\beta$ —then  $\varepsilon_I$ ,  $\varepsilon_{II}$ , and  $\theta$  are the only unknowns in Eqs. (1.62)–(1.64).

The principal stresses are now obtained by substitution of  $\varepsilon_I$  and  $\varepsilon_{II}$  in Eqs. (1.52).

Thus,

$$\varepsilon_I = \frac{1}{E}(\sigma_I - v\sigma_{II}) \quad (1.65)$$

and

$$\varepsilon_{II} = \frac{1}{E}(\sigma_{II} - v\sigma_I) \quad (1.66)$$

Add Eq. (1.65) to Eq. (1.66) multiplied by  $v$ ,

$$\varepsilon_I + v\varepsilon_{II} = \frac{1}{E}(\sigma_I - v\sigma_{II} + v\sigma_{II} - v^2\sigma_I)$$

from which

$$\sigma_I = \frac{E}{1-v^2}(\varepsilon_I + v\varepsilon_{II}) \quad (1.67)$$

Similarly,

$$\sigma_{II} = \frac{E}{1-v^2}(\varepsilon_{II} + v\varepsilon_I) \quad (1.68)$$

A typical rosette would have  $\alpha = \beta = 45^\circ$ , in which case the principal strains are most conveniently found using the geometry of Mohr's circle of strain. Suppose that the arm  $a$  of the rosette is inclined at some unknown angle  $\theta$  to the maximum principal strain, as in Fig. 1.19. Then, Mohr's circle of strain is as shown in Fig. 1.20; the shear strains  $\gamma_a$ ,  $\gamma_b$ , and  $\gamma_c$  do not feature in the analysis and are therefore ignored. From Fig. 1.20,

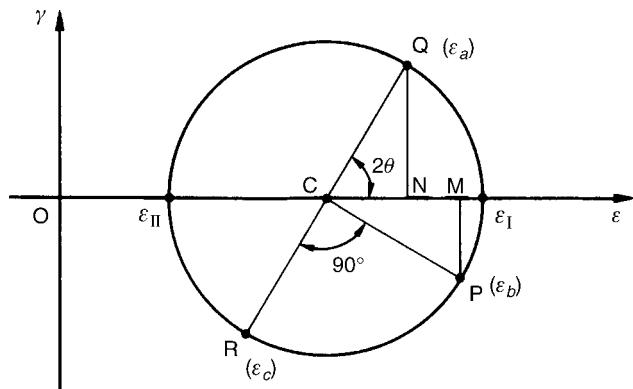


FIGURE 1.20 Experimental Values of Principal Strain Using Mohr's Circle

$$OC = \frac{1}{2}(\varepsilon_a + \varepsilon_c)$$

$$CN = \varepsilon_a - OC = \frac{1}{2}(\varepsilon_a - \varepsilon_c)$$

$$QN = CM = \varepsilon_b - OC = \varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c)$$

The radius of the circle is CQ, and

$$CQ = \sqrt{CN^2 + QN^2}$$

Hence,

$$CQ = \sqrt{\left[\frac{1}{2}(\varepsilon_a - \varepsilon_c)\right]^2 + \left[\varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c)\right]^2}$$

Expanding gives

$$CQ = \sqrt{\frac{1}{4}(\varepsilon_a^2 - 2\varepsilon_a\varepsilon_c + \varepsilon_c^2) + (\varepsilon_b^2 - \varepsilon_b\varepsilon_a - \varepsilon_b\varepsilon_c) + \frac{1}{4}(\varepsilon_a^2 + 1(\varepsilon_a^2 + 2\varepsilon_a\varepsilon_c + \varepsilon_c^2))}$$

which gives

$$CQ = \sqrt{\frac{1}{2}(\varepsilon_a^2 - 2\varepsilon_a\varepsilon_b + \varepsilon_b^2) + \frac{1}{2}(\varepsilon_c^2 - \varepsilon_c\varepsilon_b + \varepsilon_b^2)}$$

which simplifies to

$$CQ = \frac{1}{\sqrt{2}} \sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2}$$

Therefore,  $\varepsilon_I$ , which is given by

$$\varepsilon_I = OC + \text{radius of circle}$$

is

$$\varepsilon_I = \frac{1}{2}(\varepsilon_a + \varepsilon_c) + \frac{1}{\sqrt{2}} \sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2} \quad (1.69)$$

Also,

$$\varepsilon_{II} = OC - \text{radius of circle}$$

that is,

$$\varepsilon_{II} = \frac{1}{2}(\varepsilon_a + \varepsilon_c) - \frac{1}{\sqrt{2}} \sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2} \quad (1.70)$$

Finally, the angle  $\theta$  is given by

$$\tan 2\theta = \frac{QN}{CN} = \frac{\varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c)}{\frac{1}{2}(\varepsilon_a - \varepsilon_c)}$$

that is,

$$\tan 2\theta = \frac{2\varepsilon_b - \varepsilon_a - \varepsilon_c}{\varepsilon_a - \varepsilon_c} \quad (1.71)$$

A similar approach may be adopted for a  $60^\circ$  rosette.

### Example 1.8

A bar of solid circular cross-section has a diameter of 50 mm and carries a torque,  $T$ , together with an axial tensile load,  $P$ . A rectangular strain gauge rosette attached to the surface of the bar gives the following strain readings:  $\varepsilon_a = 1000 \times 10^{-6}$ ,  $\varepsilon_b = -200 \times 10^{-6}$ , and  $\varepsilon_c = -300 \times 10^{-6}$ , where the gauges  $a$  and  $c$  are in line with, and perpendicular to, the axis of the bar, respectively. If Young's modulus,  $E$ , for the bar is  $70,000 \text{ N/mm}^2$  and Poisson's ratio,  $\nu$ , is 0.3; calculate the values of  $T$  and  $P$ .

Substituting the values of  $\varepsilon_a$ ,  $\varepsilon_b$ , and  $\varepsilon_c$  in Eq. (1.69),

$$\varepsilon_I = \frac{10^{-6}}{2}(1000 - 300) + \frac{10^{-6}}{\sqrt{2}} \sqrt{(1000 + 200)^2 + (-200 + 300)^2}$$

which gives

$$\varepsilon_I = 1202 \times 10^{-6}$$

Similarly, from Eq. (1.70),

$$\varepsilon_{II} = -502 \times 10^{-6}$$

Now, substituting for  $\varepsilon_I$  and  $\varepsilon_{II}$  in Eq. (1.67),

$$\sigma_I = 70,000 \times 10^{-6} [1202 - 0.3(502)]/[1 - (0.3)^2] = 80.9 \text{ N/mm}^2$$

Similarly, from Eq. (1.68),

$$\sigma_{II} = -10.9 \text{ N/mm}^2$$

Since  $\sigma_y = 0$ , Eqs. (1.11) and (1.12) reduce to

$$\sigma_I = \frac{\sigma_x}{2} + \frac{1}{2} \sqrt{\sigma_x^2 + 4\tau_{xy}^2} \quad (\text{i})$$

and

$$\sigma_{II} = \frac{\sigma_x}{2} - \frac{1}{2} \sqrt{\sigma_x^2 + 4\tau_{xy}^2} \quad (\text{ii})$$

respectively. Adding Eqs. (i) and (ii), we obtain

$$\sigma_I + \sigma_{II} = \sigma_x$$

Therefore,

$$\sigma_x = 80.9 - 10.9 = 70 \text{ N/mm}^2$$

For an axial load  $P$ ,

$$\sigma_x = 70 \text{ N/mm}^2 = \frac{P}{A} = \frac{P}{\pi \times 50^2/4}$$

from which

$$P = 137.4 \text{ kN}$$

Substituting for  $\sigma_x$  in either of Eq. (i) or (ii) gives

$$\tau_{xy} = 29.7 \text{ N/mm}^2$$

From the theory of torsion of circular section bars (see Eq. (iv) in Example 3.1),

$$\tau_{xy} = 29.7 \text{ N/mm}^2 = \frac{Tr}{J} = \frac{T \times 25}{\pi \times 50^4/32}$$

from which

$$T = 0.7 \text{ kNm}$$

Note that  $P$  could have been found directly in this particular case from the axial strain. Thus, from the first of Eqs. (1.52),

$$\sigma_x = E\varepsilon_a = 70000 \times 1000 \times 10^{-6} = 70 \text{ N/mm}^2$$

as before.

### Example 1.8 MATLAB

Repeat the derivations presented in Example 1.8 using the Symbolic Math Toolbox in MATLAB. To obtain the same results as Example 1.8, set the number of significant digits used in calculations to four.

Calculations of  $T$  and  $P$  are obtained through the following MATLAB file:

```
% Declare any needed symbolic variables
syms e_a e_b e_c E v sig_y sig_x A J r T tau_xy
```

```
% Set the significant digits
digits(4);

% Define known variable values
e_a = sym(1000*10^(-6));
e_b = sym(-200*10^(-6));
e_c = sym(-300*10^(-6));
E = sym(70000);
v = sym(0.3);
sig_y = sym(0);
A = sym(pi*50^2/4,'d');
J = sym(pi*50^4/32,'d');
r = sym(25);

% Evaluate Eqs (1.69) and (1.70)
e_I = vpa((e_a + e_c)/2) + vpa(sqrt(((e_a - e_b)^2 + (e_c - e_c)^2)))/vpa(sqrt(2));
e_II = vpa((e_a + e_c)/2) - vpa(sqrt(((e_a - e_b)^2 + (e_c - e_c)^2)))/vpa(sqrt(2));

% Substitute e_I and e_II into Eqs (1.67) and (1.68)
sig_I = vpa(E*(e_I+v*e_II)/(1-v^2));
sig_II = vpa(E*(e_II+v*e_I)/(1-v^2));

% Evaluate Eqs (1.11) and (1.12)
eqI = vpa(-sig_I + (sig_x+sig_y)/2 + sqrt((sig_x-sig_y)^2 + 4*tau_xy^2)/2);
eqII = vpa(-sig_II + (sig_x+sig_y)/2 - sqrt((sig_x-sig_y)^2 + 4*tau_xy^2)/2);

% Add eqI and eqII and solve for the value of sig_x
sig_x_val = vpa(solve(eqI+eqII,sig_x));

% Calculate the axial load (P in kN)
P = vpa(sig_x_val*A/1000);
P = round(double(P)*10)/10;

% Substitute sig_x back into eqI and solve for tau_xy
tau_xy_val = sym(max(double(solve(subs(eqI,sig_x,sig_x_val),tau_xy))));

% Calculate the applied torsion (T in kN-m) using Eq. (iv) in Example 3.1
T = vpa(tau_xy_val*J/r/1000/1000);
T = round(double(T)*10)/10;

% Output values for P and T to the Command Window
disp(['P =' num2str(P) ' kN'])
disp(['T =' num2str(T) ' kN m'])
```

The Command Window outputs resulting from this MATLAB file are as follows:

P = 137.4 kN  
T = 0.7 kN m

## Reference

[1] Megson THG. Structural and stress analysis. 4th ed. Oxford: Elsevier; 2019.

## Further reading

Timoshenko S, Goodier JN. Theory of elasticity. 2nd ed. New York: McGraw-Hill; 1951.

Wang CT. Applied elasticity. New York: McGraw-Hill; 1953.

## PROBLEMS

**P.1.1.** A structural member supports loads that produce, at a particular point, a direct tensile stress of  $80 \text{ N/mm}^2$  and a shear stress of  $45 \text{ N/mm}^2$  on the same plane. Calculate the values and directions of the principal stresses at the point and also the maximum shear stress, stating on which planes this acts.

Answer:  $\sigma_I = 100.2 \text{ N/mm}^2, \theta = 24^\circ 11'$   
 $\sigma_{II} = -20.2 \text{ N/mm}^2, \theta = 114^\circ 11'$   
 $\tau_{\max} = 60.2 \text{ N/mm}^2$  at  $45^\circ$  to principal planes

**P.1.2.** At a point in an elastic material, there are two mutually perpendicular planes, one of which carries a direct tensile stress of  $50 \text{ N/mm}^2$  and a shear stress of  $40 \text{ N/mm}^2$ , while the other plane is subjected to a direct compressive stress of  $35 \text{ N/mm}^2$  and a complementary shear stress of  $40 \text{ N/mm}^2$ . Determine the principal stresses at the point, the position of the planes on which they act, and the position of the planes on which there is no normal stress.

Answer:  $\sigma_I = 65.9 \text{ N/mm}^2, \theta = 21^\circ 38'$   
 $\sigma_{II} = -50.9 \text{ N/mm}^2, \theta = 111^\circ 38'$

No normal stress on planes at  $70^\circ 21'$  and  $-27^\circ 5'$  to vertical.

**P.1.3.** The following are varying combinations of stresses acting at a point and referred to axes  $x$  and  $y$  in an elastic material. Using Mohr's circle of stress, determine the principal stresses at the point and their directions for each combination.

	$\sigma_x(\text{N/mm}^2)$	$\sigma_y(\text{N/mm}^2)$	$\tau_{xy}(\text{N/mm}^2)$
(i)	+54	+30	+5
(ii)	+30	+54	-5
(iii)	-60	-36	+5
(iv)	+30	-50	+30

Answer: (i)  $\sigma_I = +55 \text{ N/mm}^2$     $\sigma_{II} = +29 \text{ N/mm}^2$     $\sigma_I$  at  $11.5^\circ$  to  $x$  axis.  
(ii)  $\sigma_I = +55 \text{ N/mm}^2$     $\sigma_{II} = +29 \text{ N/mm}^2$     $\sigma_{II}$  at  $11.5^\circ$  to  $x$  axis.  
(iii)  $\sigma_I = -34.5 \text{ N/mm}^2$     $\sigma_{II} = -61 \text{ N/mm}^2$     $\sigma_I$  at  $79.5^\circ$  to  $x$  axis.  
(iv)  $\sigma_I = +40 \text{ N/mm}^2$     $\sigma_{II} = -60 \text{ N/mm}^2$     $\sigma_I$  at  $18.5^\circ$  to  $x$  axis.

**P.1.3 MATLAB** Repeat P.1.3 by creating a script in MATLAB in place of constructing Mohr's circle. In addition to calculating the principal stresses, also calculate the maximum shear stress at the point for each combination. Do not repeat the calculation of the principal stress directions.

- Answer: (i)  $\tau_{\max} = 13 \text{ N/mm}^2$ , (ii)  $\tau_{\max} = 13 \text{ N/mm}^2$   
 (ii)  $\tau_{\max} = 13 \text{ N/mm}^2$ , (iv)  $\tau_{\max} = 50 \text{ N/mm}^2$

$\sigma_I$  and  $\sigma_{II}$  for each combination are as shown in P.1.3

**P.1.4.** The state of stress at a point is caused by three separate actions, each of which produces a pure, unidirectional tension of  $10 \text{ N/mm}^2$  individually but in three different directions, as shown in Fig. P.1.4. By transforming the individual stresses to a common set of axes ( $x, y$ ), determine the principal stresses at the point and their directions.

Answer:  $\sigma_I = \sigma_{II} = 15 \text{ N/mm}^2$ . All directions are principal directions.

**P.1.5.** A shear stress  $\tau_{xy}$  acts in a two-dimensional field in which the maximum allowable shear stress is denoted by  $\tau_{\max}$  and the major principal stress by  $\sigma_I$ . Derive, using the geometry of Mohr's circle of stress, expressions for the maximum values of direct stress that may be applied to the  $x$  and  $y$  planes in terms of these three parameters.

$$\text{Answer: } \sigma_x = \sigma_I - \tau_{\max} + \sqrt{\tau_{\max}^2 - \tau_{xy}^2}$$

$$\sigma_y = \sigma_I - \tau_{\max} - \sqrt{\tau_{\max}^2 - \tau_{xy}^2}$$

**P.1.6.** A solid shaft of circular cross-section supports a torque of  $50 \text{ kNm}$  and a bending moment of  $25 \text{ kNm}$ . If the diameter of the shaft is  $150 \text{ mm}$ , calculate the values of the principal stresses and their directions at a point on the surface of the shaft.

$$\text{Answer: } \sigma_I = 121.4 \text{ N/mm}^2, \quad \theta = 31^\circ 43'$$

$$\sigma_{II} = -46.4 \text{ N/mm}^2, \quad \theta = 121^\circ 43'$$

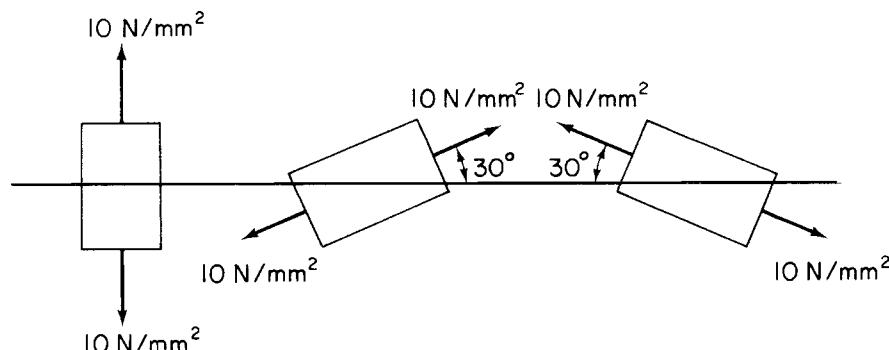


FIGURE P.1.4

**P.1.7.** An element of an elastic body is subjected to a three-dimensional stress system  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$ . Show that, if the direct strains in the directions  $x$ ,  $y$ , and  $z$  are  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\varepsilon_z$ , then

$$\sigma_x = \lambda e + 2G\varepsilon_x, \quad \sigma_y = \lambda e + 2G\varepsilon_y, \quad \sigma_z = \lambda e + 2G\varepsilon_z$$

where

$$\lambda = \frac{vE}{(1+v)(1-2v)} \text{ and } e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

the volumetric strain.

**P.1.8.** Show that the compatibility equation for the case of plane strain, namely

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2}$$

may be expressed in terms of direct stresses  $\sigma_x$  and  $\sigma_y$  in the form

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0$$

**P.1.9.** A bar of mild steel has a diameter of 75 mm and is placed inside a hollow aluminum cylinder of internal diameter 75 mm and external diameter 100 mm; both bar and cylinder are the same length. The resulting composite bar is subjected to an axial compressive load of 1000 kN. If the bar and cylinder contract by the same amount, calculate the stress in each. The temperature of the compressed composite bar is then reduced by 150°C but no change in length is permitted. Calculate the final stress in the bar and in the cylinder if  $E$  (steel) = 200,000 N/mm<sup>2</sup>,  $E$  (aluminum) = 80,000 N/mm<sup>2</sup>,  $\alpha$  (steel) = 0.000012/°C, and  $\alpha$  (aluminum) = 0.000005/°C.

Answer: Due to load:  $\sigma$  (steel) = 172.6 N/mm<sup>2</sup> (compression)  
 $\sigma$  (aluminum) = 69.1 N/mm<sup>2</sup> (compression)

Final stress:  $\sigma$  (steel) = 187.4 N/mm<sup>2</sup> (tension)  
 $\sigma$  (aluminum) = 9.1 N/mm<sup>2</sup> (compression)

**P.1.9 MATLAB** Repeat P.1.9 by creating a script in MATLAB using the Symbolic Math Toolbox for an axial tension load of 1000 kN.

Answer: Due to load:  $\sigma$  (steel) = 172.6 N/mm<sup>2</sup> (tension)  
 $\sigma$  (aluminum) = 69.1 N/mm<sup>2</sup> (tension)

Final stress:  $\sigma$  (steel) = 532.6 N/mm<sup>2</sup> (tension)  
 $\sigma$  (aluminum) = 129.1 N/mm<sup>2</sup> (tension)

**P.1.10.** If the wing lift in the light aircraft of Example 1.7 produces a tensile load of 10 kN in the strut, calculate the resultant stress in the aluminum and titanium when the aircraft is at an altitude of 4500 m.

Answer:  $\sigma_{al} = 8.45$  N/mm<sup>2</sup> (tension),  $\sigma_t = -18.75$  N/mm<sup>2</sup> (compression).

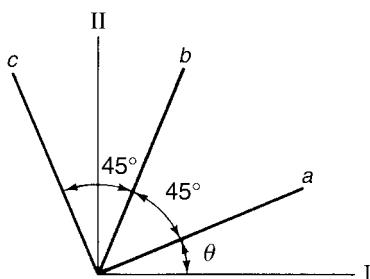


FIGURE P.1.11

**P.1.11.** In Fig. P.1.11, the direct strains in the directions  $a$ ,  $b$ ,  $c$  are  $-0.002$ ,  $-0.002$ , and  $+0.002$ , respectively. If I and II denote principal directions, find  $\varepsilon_I$ ,  $\varepsilon_{II}$ , and  $\theta$ .

Answer:  $\varepsilon_I = +0.00283$ ,  $\varepsilon_{II} = -0.00283$ ,  $\theta = -22.5^\circ$  or  $+67.5^\circ$

**P.1.12.** The simply supported rectangular beam shown in Fig. P.1.12 is subjected to two symmetrically placed transverse loads each of magnitude  $Q$ . A rectangular strain gauge rosette located at a point P on the centroidal axis on one vertical face of the beam gives strain readings as follows:  $\varepsilon_a = -222 \times 10^{-6}$ ,  $\varepsilon_b = -213 \times 10^{-6}$ , and  $\varepsilon_c = +45 \times 10^{-6}$ . The longitudinal stress  $\sigma_x$  at the point P due to an external compressive force is  $7 \text{ N/mm}^2$ . Calculate the shear stress  $\tau$  at the point P in the vertical plane and hence the transverse load  $Q$ :

$$(Q = 2bd\tau/3, \text{ where } b = \text{breadth}, d = \text{depth of beam})$$

$$E = 31,000 \text{ N/mm}^2 \quad v = 0.2$$

Answer:  $\tau_{xy} = 3.16 \text{ N/mm}^2$ ,  $Q = 94.8 \text{ kN}$

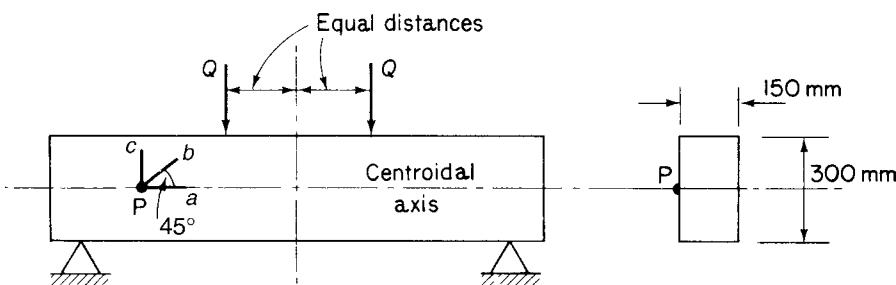


FIGURE P.1.12

**P.1.12. MATLAB** Repeat P.1.12 by creating a script in MATLAB using the Symbolic Math Toolbox for the following cross-section dimensions and longitudinal stress combinations:

	$b$	$d$	$\sigma_x$
(i)	150 mm	300 mm	7 N/mm <sup>2</sup>
(ii)	100 mm	250 mm	11 N/mm <sup>2</sup>
(iii)	270 mm	270 mm	9 N/mm <sup>2</sup>

Answer: (i)  $\tau_{xy} = 3.16 \text{ N/mm}^2$   $Q = 94.8 \text{ kN}$

(ii)  $\tau_{xy} = 2.5 \text{ N/mm}^2$   $Q = 41.7 \text{ kN}$

(iii)  $\tau_{xy} = 1.41 \text{ N/mm}^2$   $Q = 68.5 \text{ kN}$

# Two-dimensional problems in elasticity

# 2

Theoretically, we are now in a position to solve any three-dimensional problem in elasticity having derived three equilibrium conditions, Eqs. (1.5); six strain–displacement equations, Eqs. (1.18) and (1.20); and six stress–strain relationships, Eqs. (1.42) and (1.46). These equations are sufficient, when supplemented by appropriate boundary conditions, to obtain unique solutions for the six stress, six strain, and three displacement functions. It is found, however, that exact solutions are obtainable only for some simple problems. For bodies of arbitrary shape and loading, approximate solutions may be found by numerical methods (e.g., finite differences) or by the Rayleigh–Ritz method based on energy principles (Chapter 7).

Two approaches are possible in the solution of elasticity problems. We may solve initially for either the three unknown displacements or the six unknown stresses. In the former method, the equilibrium equations are written in terms of strain by expressing the six stresses as functions of strain (see Problem P.1.7). The strain–displacement relationships are then used to form three equations involving the three displacements  $u$ ,  $v$ , and  $w$ . The boundary conditions for this method of solution must be specified as displacements. Determination of  $u$ ,  $v$ , and  $w$  enables the six strains to be computed from Eqs. (1.18) and (1.20); the six unknown stresses follow from the equations expressing stress as functions of strain. It should be noted here that no use has been made of the compatibility equations. The fact that  $u$ ,  $v$ , and  $w$  are determined directly ensures that they are single-valued functions, thereby satisfying the requirement of compatibility.

In most structural problems the object is usually to find the distribution of stress in an elastic body produced by an external loading system. It is therefore more convenient in this case to determine the six stresses before calculating any required strains or displacements. This is accomplished by using Eqs. (1.42) and (1.46) to rewrite the six equations of compatibility in terms of stress. The resulting equations, in turn, are simplified by making use of the stress relationships developed in the equations of equilibrium. The solution of these equations automatically satisfies the conditions of compatibility and equilibrium throughout the body.

---

## 2.1 TWO-DIMENSIONAL PROBLEMS

For the reasons discussed in Chapter 1, we shall confine our actual analysis to the two-dimensional cases of plane stress and plane strain. The appropriate equilibrium conditions for plane stress are given by Eqs. (1.6); that is,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + Y = 0$$

and the required stress-strain relationships obtained from Eqs. (1.50) and (1.52):

$$\varepsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - v\sigma_x)$$

$$\gamma_{xy} = \frac{2(1+v)}{E}\tau_{xy}$$

We find that, although  $\varepsilon_z$  exists, Eqs. (1.22)–(1.26) are identically satisfied, leaving Eq. (1.21) as the required compatibility condition. Substitution in Eq. (1.21) of the preceding strains gives

$$2(1+v)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x^2}(\sigma_y - v\sigma_x) + \frac{\partial^2}{\partial y^2}(\sigma_x - v\sigma_y) \quad (2.1)$$

From Eqs. (1.6),

$$\frac{\partial^2 \tau_{xy}}{\partial y \partial x} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial X}{\partial x} \quad (2.2)$$

and

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial Y}{\partial y} \quad (\tau_{yx} = \tau_{xy}) \quad (2.3)$$

Adding Eqs. (2.2) and (2.3), then substituting in Eq. (2.1) for  $2\partial^2 \tau_{xy}/\partial x \partial y$ , we have

$$-(1+v)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) = \frac{\partial^2 \sigma_y}{\partial x^2} - \frac{v\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} - \frac{v\partial^2 \sigma_y}{\partial y^2}$$

which simplifies to

$$-(1+v)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2}$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+v)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \quad (2.4)$$

The alternative two-dimensional problem of plane strain may also be formulated in the same manner. We saw in Section 1.11 that the six equations of compatibility reduce to the single equation (1.21) for the plane strain condition. Further, from the third of Eqs. (1.42),

$$\sigma_z = v(\sigma_x + \sigma_y) \quad (\text{since } \varepsilon_z = 0 \text{ for plane strain})$$

so that substituting in the first of Eqs. (1.42) for  $\sigma_z$  and rearranging, we have

$$\varepsilon_x = \frac{1}{E} [(1 - v^2)\sigma_x - v(1 + v)\sigma_y]$$

similarly,

$$\varepsilon_y = \frac{1}{E} [(1 - v^2)\sigma_y - v(1 + v)\sigma_x]$$

Also,

$$\gamma_{xy} = \frac{2(1 + v)}{E} \tau_{xy}$$

Substituting as before in Eq. (1.21), we have:

$$2(1 + v) \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = (1 - v^2) \frac{\partial^2 \sigma_y}{\partial x^2} - v(1 + v) \frac{\partial^2 \sigma_x}{\partial x^2} + (1 - v^2) \frac{\partial^2 \sigma_x}{\partial y^2} - v(1 + v) \frac{\partial^2 \sigma_y}{\partial y^2}$$

Dividing through by  $(1 + v)$

$$2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = (1 - v) \frac{\partial^2 \sigma_y}{\partial x^2} - v \frac{\partial^2 \sigma_x}{\partial x^2} + (1 - v) \frac{\partial^2 \sigma_x}{\partial y^2} - v \frac{\partial^2 \sigma_y}{\partial y^2}$$

From Eqs. (2.2) and (2.3),

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial X}{\partial x}, \quad \frac{\partial^2 \tau_{yx}}{\partial x \partial y} = -\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial Y}{\partial y}$$

Since  $\tau_{xy} = \tau_{yx}$ , these two equations may be added to give

$$2 \frac{\tau_{xy}^2}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y}$$

Now equating the two expressions for  $2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y}$  and rearranging, we have

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = -\frac{1}{1 - v} \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \quad (2.5)$$

which is the compatibility equation for plane strain.

The two equations of equilibrium together with the boundary conditions, from Eqs. (1.7), and one of the compatibility equations, (2.4) or (2.5), are generally sufficient for the determination of the stress distribution in a two-dimensional problem.

## 2.2 STRESS FUNCTIONS

The solution of problems in elasticity presents difficulties, but the procedure may be simplified by the introduction of a *stress function*. For a particular two-dimensional case, the stresses are related to a single function of  $x$  and  $y$  such that substitution for the stresses in terms of this function automatically satisfies the equations of equilibrium no matter what form the function may take. However, a large

proportion of the infinite number of functions that fulfill this condition are eliminated by the requirement that the form of the stress function must also satisfy the two-dimensional equations of compatibility, (2.4) and (2.5), plus the appropriate boundary conditions.

For simplicity, let us consider the two-dimensional case for which the body forces are zero. The problem is now to determine a stress–stress function relationship that satisfies the equilibrium conditions of (see Eqs. (1.6))

$$\left. \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} &= 0 \end{aligned} \right\} \quad (2.6)$$

and a form for the stress function giving stresses which satisfy the compatibility equations (see Eq. (2.5))

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0 \quad (2.7)$$

The English mathematician Airy proposed a stress function  $\phi$  defined by the equations

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (2.8)$$

Clearly, substitution of Eqs. (2.8) into Eq. (2.6) verifies that the equations of equilibrium are satisfied by this particular stress–stress function relationship. Further substitution into Eq. (2.7) restricts the possible forms of the stress function to those satisfying the *biharmonic equation*:

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (2.9)$$

The final form of the stress function is then determined by the boundary conditions relating to the actual problem. Therefore, a two-dimensional problem in elasticity with zero body forces reduces to the determination of a function  $\phi$  of  $x$  and  $y$ , which satisfies Eq. (2.9) at all points in the body and Eqs. (1.7) reduced to two dimensions at all points on the boundary of the body.

## 2.3 INVERSE AND SEMI-INVERSE METHODS

The task of finding a stress function satisfying the preceding conditions is extremely difficult in the majority of elasticity problems, although some important classical solutions have been obtained in this way. An alternative approach, known as the *inverse method*, is to specify a form of the function  $\phi$  satisfying Eq. (2.9), assume an arbitrary boundary, and then determine the loading conditions that fit the assumed stress function and chosen boundary. Obvious solutions arise in which  $\phi$  is expressed as a polynomial. Timoshenko and Goodier<sup>1</sup> consider a variety of polynomials for  $\phi$  and determine the associated loading conditions for a variety of rectangular sheets. Some of these cases are quoted here.

**Example 2.1**

Consider the stress function

$$\phi = Ax^2 + Bxy + Cy^2$$

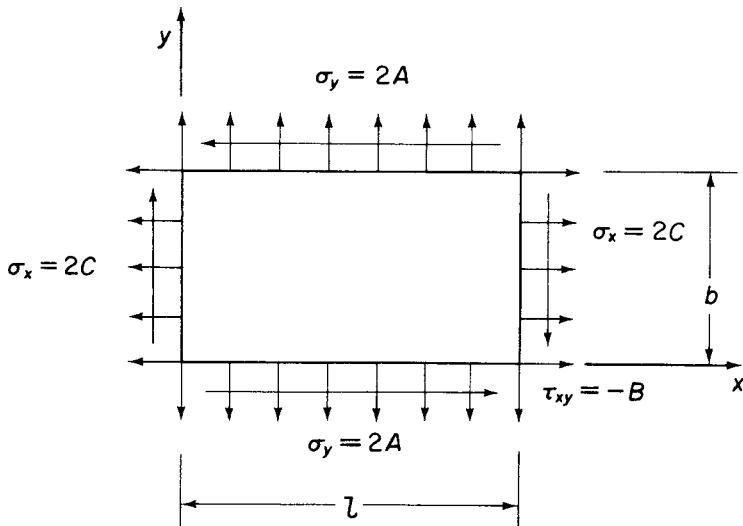
where  $A$ ,  $B$ , and  $C$  are constants. [Equation \(2.9\)](#) is identically satisfied, since each term becomes zero on substituting for  $\phi$ . The stresses follow from

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2C$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 2A$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -B$$

To produce these stresses at any point in a rectangular sheet, we require loading conditions providing the boundary stresses shown in [Fig. 2.1](#).



**FIGURE 2.1** Required Loading Conditions on Rectangular Sheet in Example 2.1

**Example 2.2**

A more complex polynomial for the stress function is

$$\phi = \frac{Ax^3}{6} + \frac{Bx^2y}{2} + \frac{Cxy^2}{2} + \frac{Dy^3}{6}$$

As before,

$$\frac{\partial^4 \phi}{\partial x^4} = \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = \frac{\partial^4 \phi}{\partial y^4} = 0$$

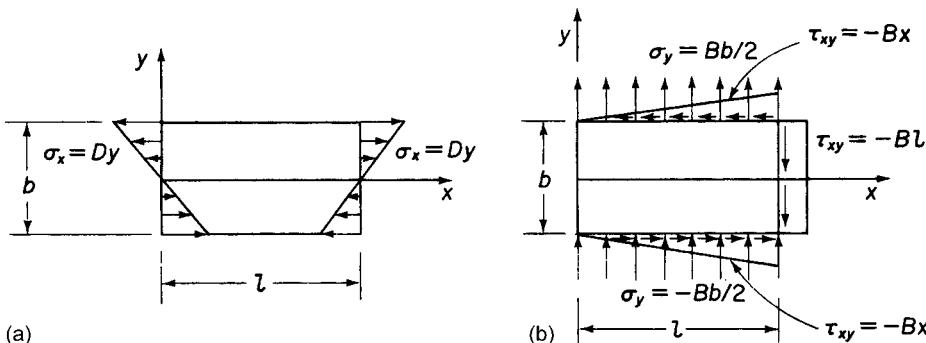
so that the compatibility equation (2.9) is identically satisfied. The stresses are given by

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = Cx + Dy$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = Ax + By$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -Bx - Cy$$

We may choose any number of values of the coefficients  $A, B, C$ , and  $D$  to produce a variety of loading conditions on a rectangular plate. For example, if we assume  $A = B = C = 0$ , then  $\sigma_x = Dy$ ,  $\sigma_y = 0$ , and  $\tau_{xy} = 0$ , so that, for axes referred to an origin at the mid-point of a vertical side of the plate, we obtain the state of pure bending shown in Fig. 2.2(a). Alternatively, Fig. 2.2(b) shows the loading conditions corresponding to  $A = C = D = 0$  in which  $\sigma_x = 0$ ,  $\sigma_y = By$ , and  $\tau_{xy} = -Bx$ .



**FIGURE 2.2 (a) Required Loading Conditions on Rectangular Sheet in Example 2.2 for  $A = B = C = 0$ ; (b) as in (a) but  $A = C = D = 0$**

By assuming polynomials of the second or third degree for the stress function, we ensure that the compatibility equation is identically satisfied whatever the values of the coefficients. For polynomials of higher degrees, compatibility is satisfied only if the coefficients are related in a certain way. For example, for a stress function in the form of a polynomial of the fourth degree,

$$\phi = \frac{Ax^4}{12} + \frac{Bx^3y}{6} + \frac{Cx^2y^2}{2} + \frac{Dxy^3}{6} + \frac{Ey^4}{12}$$

and

$$\frac{\partial^4 \phi}{\partial x^4} = 2A, \quad 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 4C, \quad \frac{\partial^4 \phi}{\partial y^4} = 2E$$

Substituting these values in Eq. (2.9), we have

$$E = -(2C + A)$$

The stress components are then

$$\begin{aligned}\sigma_x &= \frac{\partial^2 \phi}{\partial y^2} = Cx^2 + Dxy - (2C + A)y^2 \\ \sigma_y &= \frac{\partial^2 \phi}{\partial x^2} = Ax^2 + Bxy + Cy^2 \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{Bx^2}{2} - 2Cxy - \frac{Dy^2}{2}\end{aligned}$$

The coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are arbitrary and may be chosen to produce various loading conditions, as in the previous examples.

### Example 2.3

A cantilever of length  $L$  and depth  $2h$  is in a state of plane stress. The cantilever is of unit thickness, is rigidly supported at the end  $x = L$ , and loaded as shown in Fig. 2.3. Show that the stress function

$$\phi = Ax^2 + Bx^2y + Cy^3 + D(5x^2y^3 - y^5)$$

is valid for the beam and evaluate the constants  $A$ ,  $B$ ,  $C$ , and  $D$ .

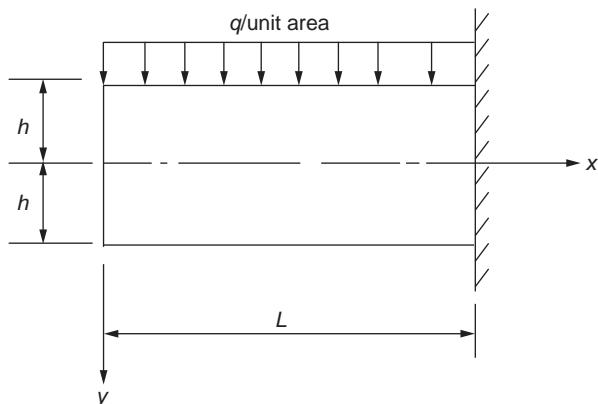


FIGURE 2.3 Beam of Example 2.3

The stress function must satisfy Eq. (2.9). From the expression for  $\phi$ ,

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= 2Ax + 2Bxy + 10Dxy^3 \\ \frac{\partial^2 \phi}{\partial x^2} &= 2A + 2By + 10Dy^3 = \sigma_y\end{aligned}\quad (\text{i})$$

Also,

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= Bx^2 + 3Cy^2 + 15Dx^2y^2 - 5Dy^4 \\ \frac{\partial^2 \phi}{\partial y^2} &= 6Cy + 30Dx^2y - 20Dy^3 = \sigma_x\end{aligned}\quad (\text{ii})$$

and

$$\frac{\partial^2 \phi}{\partial x \partial y} = 2Bx + 30Dxy^2 = -\tau_{xy}\quad (\text{iii})$$

Further,

$$\frac{\partial^4 \phi}{\partial x^4} = 0, \quad \frac{\partial^4 \phi}{\partial y^4} = -120Dy, \quad \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 60Dy$$

Substituting in Eq. (2.9) gives

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 2 \times 60Dy - 120Dy = 0$$

The biharmonic equation is therefore satisfied and the stress function is valid.

From Fig. 2.3,  $\sigma_y = 0$  at  $y = h$ , so that, from Eq. (i),

$$2A + 2BH + 10Dh^3 = 0 \quad (\text{iv})$$

Also, from Fig. 2.3,  $\sigma_y = -q$  at  $y = -h$ , so that, from Eq. (i),

$$2A - 2BH - 10Dh^3 = -q \quad (\text{v})$$

Again, from Fig. 2.3,  $\tau_{xy} = 0$  at  $y = \pm h$ , giving, from Eq. (iii),

$$2Bx + 30Dxh^2 = 0$$

so that

$$2B + 30Dh^2 = 0 \quad (\text{vi})$$

At  $x = 0$ , no resultant moment is applied to the beam; that is,

$$M_{x=0} = \int_{-h}^h \sigma_x y \, dy = \int_{-h}^h (6Cy^2 - 20Dy^4) \, dy = 0$$

so that

$$M_{x=0} = [2Cy^3 - 4Dy^5]_{-h}^h = 0$$

or

$$C - 2Dh^2 = 0 \quad (\text{vii})$$

Subtracting Eq. (v) from (iv),

$$4Bh + 20Dh^3 = q$$

or

$$B + 5Dh^2 = \frac{q}{4h} \quad (\text{viii})$$

From Eq. (vi),

$$B + 15Dh^2 = 0 \quad (\text{ix})$$

so that, subtracting Eq. (viii) from Eq. (ix),

$$D = -\frac{q}{40h^3}$$

Then,

$$B = \frac{3q}{8h}, \quad A = -\frac{q}{4}, \quad C = -\frac{q}{20h}$$

and

$$\phi = \frac{q}{40h^3} [-10h^3x^2 + 15h^2x^2y - 2h^2y^3 - (5x^2y^3 - y^5)]$$

The obvious disadvantage of the inverse method is that we are determining problems to fit assumed solutions, whereas in structural analysis the reverse is the case. However, in some problems, the shape of the body and the applied loading allow simplifying assumptions to be made, thereby enabling a solution to be obtained. St. Venant suggested a *semi-inverse method* for the solution of this type of problem, in which assumptions are made as to stress or displacement components. These assumptions may be based on experimental evidence or intuition. St. Venant first applied the method to the torsion of solid sections (Chapter 3) and to the problem of a beam supporting shear loads (Section 2.6).

### Example 2.3 MATLAB

Repeat Example 2.3 using the Symbolic Math Toolbox in MATLAB.

Using Fig. 2.3, derivations of the constants  $A$ ,  $B$ ,  $C$ , and  $D$ , along with validation of the stress function  $\phi$ , are obtained through the following MATLAB file:

```
% Declare any needed symbolic variables
syms A B C D x y L h q sig_y sig_x tau_xy

% Define the given stress function
phi = A*x^2 + B*x^2*y + C*y^3 + D*(5*x^2*y^3 - y^5);
```

```
% Check Eq. (2.9)
check = diff(phi,x,4) + 2*diff(diff(phi,x,2),y,2) + diff(phi,y,4);
if double(check) == 0
    disp('The stress function is valid')
    disp('')

% Calculate expressions for sig_y, sig_x, and tau_xy
eqI = diff(phi,x,2) - sig_y;
eqII = diff(phi,y,2) - sig_x;
eqIII = diff(diff(phi,x),y) + tau_xy;

% From Fig. 2.3, sig_y=0 at y=h so that from eqI
eqIV = subs(subs(eqI,y,h),sig_y,0);

% From Fig. 2.3, sig_y=-q at y=-h so that from eqI
eqV = subs(subs(eqI,y,-h),sig_y,-q);

% From Fig. 2.3, tau_xy=0 at y=+/-h so that from eqIII
eqVI = subs(subs(eqIII,y,h),tau_xy,0);

% Calculate the expression of the applied moment
M = int(solve(eqII,sig_x)*y, y, -h, h);

% From Fig. 2.3, M=0 at x=0
eqVII = subs(M,x,0);

% Solve eqIV, eqV, eqVI, and eqVII for A, B, C, and D
D_expr = solve(subs(eqIV-eqV,B,solve(eqVI,B)), D);
C_expr = solve(subs(eqVII,D,D_expr), C);
B_expr = solve(subs(eqVI,D,D_expr), B);
A_expr = solve(subs(subs(eqIV,D,D_expr),B,B_expr), A);

% Substitute the expressions for A, B, C, and D into phi
phi = subs(subs(subs(phi,A,A_expr),B,B_expr),C,C_expr),D,D_expr);

% Output the expression for phi to the Command Window
phi = factor(phi)
else
    disp('The stress function does not satisfy the biharmonic equation (Eq. (2.9))')
    disp('')
end
```

The Command Window outputs resulting from this MATLAB file are as follows.

The stress function is valid.

$\phi = -(q * (10 * h^3 * x^2 - 15 * h^2 * x^2 * y + 2 * h^2 * y^3 + 5 * x^2 * y^3 - y^5)) / (40 * h^3)$

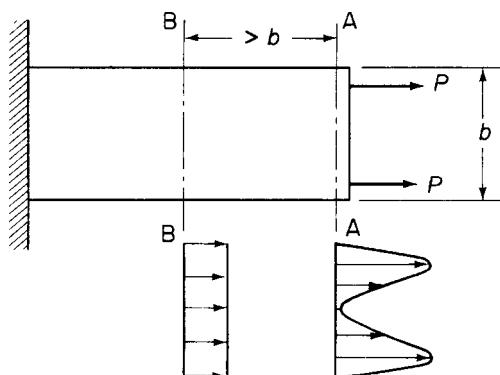
## 2.4 ST. VENANT'S PRINCIPLE

In the examples of [Section 2.3](#), we have seen that a particular stress function form may be applicable to a variety of problems. Different problems are deduced from a given stress function by specifying, in the first instance, the shape of the body and then assigning a variety of values to the coefficients. The resulting stress functions give stresses that satisfy the equations of equilibrium and compatibility *at all points within and on the boundary of the body*. It follows that the applied loads must be distributed around the boundary of the body in the same manner as the internal stresses at the boundary. In the case of pure bending, for example ([Fig. 2.2\(a\)](#)), the applied bending moment must be produced by tensile and compressive forces on the ends of the plate, their magnitudes being dependent on their distance from the neutral axis. If this condition is invalidated by the application of loads in an arbitrary fashion or by preventing the free distortion of any section of the body, then the solution of the problem is no longer exact. As this is the case in practically every structural problem, it would appear that the usefulness of the theory is strictly limited. To surmount this obstacle, we turn to the important *principle of St. Venant*, which may be summarized as stating

*that while statically equivalent systems of forces acting on a body produce substantially different local effects, the stresses at sections distant from the surface of loading are essentially the same.*

Therefore, at a section AA close to the end of a beam supporting two point loads  $P$ , the stress distribution varies as shown in [Fig. 2.4](#), while at the section BB, a distance usually taken to be greater than the dimension of the surface to which the load is applied, the stress distribution is uniform.

We may therefore apply the theory to sections of bodies away from points of applied loading or constraint. The determination of stresses in these regions requires, for some problems, separate calculation (see [Chapters 27](#) and [28](#)).



**FIGURE 2.4 Stress Distributions Illustrating St. Venant's Principle**

## 2.5 DISPLACEMENTS

Having found the components of stress, Eqs. (1.46) and (1.47) (for the case of plane stress) are used to determine the components of strain. The displacements follow from Eqs. (1.27) and (1.28). The integration of Eqs. (1.27) yields solutions of the form

$$u = \varepsilon_x x + a - by \quad (2.10)$$

$$v = \varepsilon_y y + c + bx \quad (2.11)$$

in which  $a$ ,  $b$ , and  $c$  are constants representing movement of the body as a whole or *rigid body* displacements. Of these,  $a$  and  $c$  represent pure translatory motions of the body, while  $b$  is a small angular rotation of the body in the  $xy$  plane. If we assume that  $b$  is positive in a counterclockwise sense, then in Fig. 2.5, the displacement  $v'$  due to the rotation is given by

$$\begin{aligned} v' &= P'Q' - PQ \\ &= OP \sin(\theta + b) - OP \sin\theta \end{aligned}$$

which, since  $b$  is a small angle, reduces to

$$v' = bx$$

Similarly,

$$u' = -by, \text{ as stated}$$

## 2.6 BENDING OF AN END-LOADED CANTILEVER

In his semi-inverse solution of this problem, St. Venant based his choice of stress function on the reasonable assumptions that the direct stress is directly proportional to the bending moment (and therefore distance from the free end) and height above the neutral axis. The portion of the stress function giving

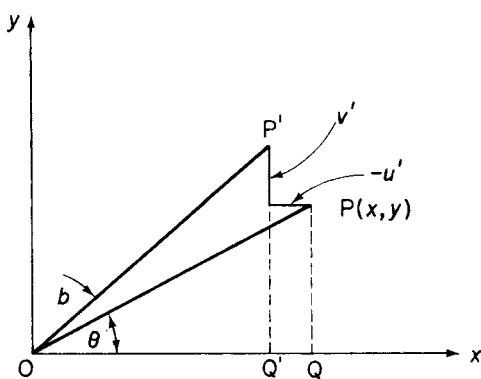
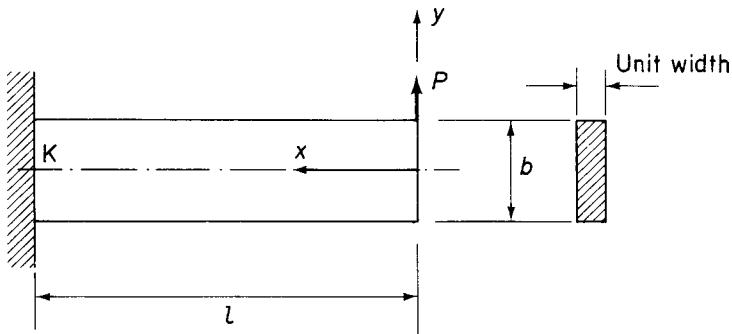


FIGURE 2.5 Displacements Produced by Rigid Body Rotation



**FIGURE 2.6 Bending of an End-Loaded Cantilever**

shear stress follows from the equilibrium condition relating  $\sigma_x$  and  $\tau_{xy}$  (see the first of Eqs. (1.6)). The appropriate stress function for the cantilever beam shown in Fig. 2.6 is then

$$\phi = Axy + \frac{Bxy^3}{6} \quad (2.12)$$

where  $A$  and  $B$  are unknown constants. Hence,

$$\left. \begin{aligned} \sigma_x &= \frac{\partial^2 \phi}{\partial y^2} = Bxy \\ \sigma_y &= \frac{\partial^2 \phi}{\partial x^2} = 0 \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} = -A - \frac{By^2}{2} \end{aligned} \right\} \quad (2.13)$$

Substitution for  $\phi$  in the biharmonic equation (Eq. (2.9)) shows that the form of the stress function satisfies compatibility for all values of the constants  $A$  and  $B$ . The actual values of  $A$  and  $B$  are chosen to satisfy the boundary condition, namely  $\tau_{xy} = 0$  along the upper and lower edges of the beam, and the resultant shear load over the free end is equal to  $P$ .

From the first of these,

$$\tau_{xy} = -A - \frac{By^2}{2} = 0 \text{ at } y = \pm \frac{b}{2}$$

giving

$$A = -\frac{Bb^2}{8}$$

From the second,

$$-\int_{-b/2}^{b/2} \tau_{xy} dy = P \text{ (see sign convention for } \tau_{xy})$$

or

$$-\int_{-b/2}^{b/2} \left( \frac{Bb^2}{8} - \frac{By^2}{2} \right) dy = P$$

from which

$$B = \frac{12P}{b^3}$$

The stresses follow from Eqs. (2.13):

$$\left. \begin{aligned} \sigma_x &= -\frac{12Pxy}{b^3} = -\frac{Px}{I}y \\ \sigma_y &= 0 \\ \tau_{xy} &= -\frac{12P}{8b^3}(b^2 - 4y^2) = -\frac{P}{8I}(b^2 - 4y^2) \end{aligned} \right\} \quad (2.14)$$

where  $I = b^3/12$ , the second moment of area of the beam cross-section (see Section 16.4).

We note from the discussion of Section 2.4 that Eqs. (2.14) represent an exact solution subject to the following conditions:

1. The shear force  $P$  is distributed over the free end in the same manner as the shear stress  $\tau_{xy}$  given by Eqs. (2.14);
2. The distribution of shear and direct stresses at the built-in end is the same as those given by Eqs. (2.14);
3. All sections of the beam, including the built-in end, are free to distort.

In practical cases, none of these conditions is satisfied, but by virtue of St. Venant's principle, we may assume that the solution is exact for regions of the beam away from the built-in end and the applied load. For many solid sections, the inaccuracies in these regions are small. However, for thin-walled structures, with which we are primarily concerned, significant changes occur, and we consider the effects of structural and loading discontinuities on this type of structure in Chapters 27 and 28.

We now proceed to determine the displacements corresponding to the stress system of Eqs. (2.14). Applying the strain-displacement and stress-strain relationships, Eqs. (1.27), (1.28), and (1.52) in which  $\sigma_y = \sigma_z = 0$ , we have

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{Pxy}{EI} \quad (2.15)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -\frac{v\sigma_x}{E} = \frac{vPxy}{EI} \quad (2.16)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = -\frac{P}{8IG}(b^2 - 4y^2) \quad (2.17)$$

Integrating Eqs. (2.15) and (2.16) and noting that  $\varepsilon_x$  and  $\varepsilon_y$  are partial derivatives of the displacements, we find

$$u = -\frac{Px^2y}{2EI} + f_1(y), \quad v = \frac{vPy^2}{2EI} + f_2(x) \quad (2.18)$$

where  $f_1(y)$  and  $f_2(x)$  are unknown functions of  $x$  and  $y$ . Substituting these values of  $u$  and  $v$  in Eq. (2.17),

$$-\frac{Px^2}{2EI} + \frac{\partial f_1(y)}{\partial y} + \frac{vPy^2}{2EI} + \frac{\partial f_2(x)}{\partial x} = -\frac{P}{8IG}(b^2 - 4y^2)$$

Separating the terms containing  $x$  and  $y$  in this equation and writing

$$F_1(x) = -\frac{Px^2}{2EI} + \frac{\partial f_2(x)}{\partial x} \quad F_2(y) = \frac{vPy^2}{2EI} - \frac{Py^2}{2IG} + \frac{\partial f_1(y)}{\partial y}$$

we have

$$F_1(x) + F_2(y) = -\frac{Pb^2}{8IG}$$

The term on the right-hand side of this equation is a constant, which means that  $F_1(x)$  and  $F_2(y)$  must be constants; otherwise, a variation in either  $x$  or  $y$  destroys the equality. Denoting  $F_1(x)$  by  $C$  and  $F_2(y)$  by  $D$  gives

$$C + D = -\frac{Pb^2}{8IG} \quad (2.19)$$

and

$$\frac{\partial f_2(x)}{\partial x} = \frac{Px^2}{2EI} + C \quad \frac{\partial f_1(y)}{\partial y} = \frac{Py^2}{2IG} - \frac{vPy^2}{2EI} + D$$

so that

$$f_2(x) = \frac{Px^3}{6EI} + Cx + F$$

and

$$f_1(y) = \frac{Py^3}{6IG} - \frac{vPy^3}{6EI} + Dy + H$$

Therefore, from Eqs. (2.18),

$$u = -\frac{Px^2y}{2EI} - \frac{vPy^3}{6EI} + \frac{Py^3}{6IG} + Dy + H \quad (2.20)$$

$$v = \frac{vPy^2}{2EI} + \frac{Px^3}{6EI} + Cx + F \quad (2.21)$$

The constants  $C$ ,  $D$ ,  $F$ , and  $H$  are now determined from Eq. (2.19) and the displacement boundary conditions imposed by the support system. Assuming that the support prevents movement of the point K in the beam cross-section at the built-in end,  $u = v = 0$  at  $x = l$ ,  $y = 0$ , and from Eqs. (2.20) and (2.21),

$$H = 0, \quad F = -\frac{Pl^3}{6EI} - Cl$$

If we now assume that the slope of the neutral plane is zero at the built-in end, then  $\partial v/\partial x = 0$  at  $x = l$ ,  $y = 0$  and, from Eq. (2.21),

$$C = -\frac{Pl^2}{2EI}$$

It follows immediately that

$$F = \frac{Pl^3}{3EI}$$

and, from Eq. (2.19),

$$D = \frac{Pl^2}{2EI} - \frac{Pb^2}{8IG}$$

Substitution for the constants  $C$ ,  $D$ ,  $F$ , and  $H$  in Eqs. (2.20) and (2.21) now produces the equations for the components of displacement at any point in the beam:

$$u = -\frac{Px^2y}{2EI} - \frac{vPy^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{Pl^2}{2EI} - \frac{Pb^2}{8IG}\right)y \quad (2.22)$$

$$v = \frac{vPxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \quad (2.23)$$

The deflection curve for the neutral plane is

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \quad (2.24)$$

from which the tip deflection ( $x = 0$ ) is  $Pl^3/3EI$ . This value is predicted by simple beam theory (Chapter 16) and does not include the contribution to deflection of the shear strain. This was eliminated when we assumed that the slope of the neutral plane at the built-in end was zero. A more detailed examination of this effect is instructive. The shear strain at any point in the beam is given by Eq. (2.17):

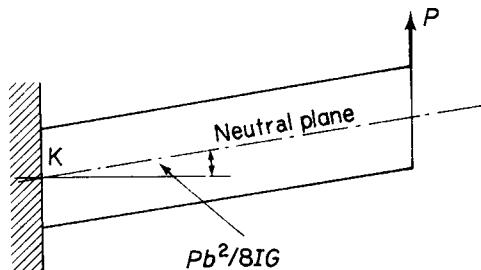
$$\gamma_{xy} = -\frac{P}{8IG}(b^2 - 4y^2)$$

and is obviously independent of  $x$ . Therefore, at all points on the neutral plane, the shear strain is constant and equal to

$$\gamma_{xy} = -\frac{Pb^2}{8IG}$$

which amounts to a rotation of the neutral plane, as shown in Fig. 2.7. The deflection of the neutral plane due to this shear strain at any section of the beam is therefore equal to

$$\frac{Pb^2}{8IG}(l - x)$$



**FIGURE 2.7 Rotation of Neutral Plane Due to Shear in an End-Loaded Cantilever**

and Eq. (2.24) may be rewritten to include the effect of shear as

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} + \frac{Pb^2}{8IG}(l-x) \quad (2.25)$$

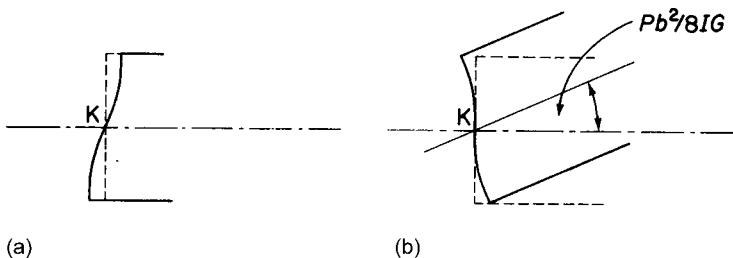
Let us now examine the distorted shape of the beam section, which the analysis assumes is free to take place. At the built-in end, when  $x = l$ , the displacement of any point is, from Eq. (2.22),

$$u = -\frac{vPy^3}{6EI} + \frac{Py^3}{6IG} - \frac{Pb^2y}{8IG} \quad (2.26)$$

The cross-section would, therefore, if allowed, take the shape of the shallow reversed S shown in Fig. 2.8(a). We have not included in Eq. (2.26) the previously discussed effect of rotation of the neutral plane caused by shear. However, this merely rotates the beam section as indicated in Fig. 2.8(b).

The distortion of the cross-section is produced by the variation of shear stress over the depth of the beam. Therefore, the basic assumption of simple beam theory that plane sections remain plane is not valid when shear loads are present, although for long, slender beams, bending stresses are much greater than shear stresses, and the effect may be ignored.

We observe from Fig. 2.8 that an additional direct stress system is imposed on the beam at the support, where the section is constrained to remain plane. For most engineering structures, this effect is small but, as mentioned previously, may be significant in thin-walled sections.



**FIGURE 2.8 (a) Distortion of the Cross-section Due to Shear; (b) Effect on Distortion of the Rotation Due to Shear**

## Reference

- [1] Timoshenko S, Goodier JN. Theory of elasticity. 2nd ed. New York: McGraw-Hill; 1951.

## PROBLEMS

**P.2.1** A metal plate has rectangular axes  $Ox$ ,  $Oy$  marked on its surface. The point  $O$  and the direction of  $Ox$  are fixed in space and the plate is subjected to the following uniform stresses:

Compressive,  $3p$ , parallel to  $Ox$ ,

Tensile,  $2p$ , parallel to  $Oy$ ,

Shearing,  $4p$ , in planes parallel to  $Ox$  and  $Oy$ , in a sense tending to decrease the angle  $xOy$ .

Determine the direction in which a certain point on the plate is displaced; the coordinates of the point are  $(2, 3)$  before straining. Poisson's ratio is 0.25.

Answer:  $19.73^\circ$  to  $Ox$ .

**P.2.2** What do you understand by an Airy stress function in two dimensions? A beam of length  $l$ , with a thin rectangular cross-section, is built in at the end  $x = 0$  and loaded at the tip by a vertical force  $P$  (Fig. P.2.2). Show that the stress distribution, as calculated by simple beam theory, can be represented by the expression

$$\phi = Ay^3 + By^3x + Cyx$$

as an Airy stress function and determine the coefficients  $A$ ,  $B$ , and  $C$ .

Answer:  $A = 2Pl/td^3$ ,  $B = -2P/td^3$ ,  $C = 3P/2td$

**P.2.3** The cantilever beam shown in Fig. P.2.3 is in a state of plane strain and is rigidly supported at  $x = L$ . Examine the following stress function in relation to this problem:

$$\phi = \frac{w}{20h^3} (15h^2x^2y - 5x^2y^3 - 2h^2y^3 + y^5)$$

Show that the stresses acting on the boundaries satisfy the conditions except for a distributed direct stress at the free end of the beam, which exerts no resultant force or bending moment.

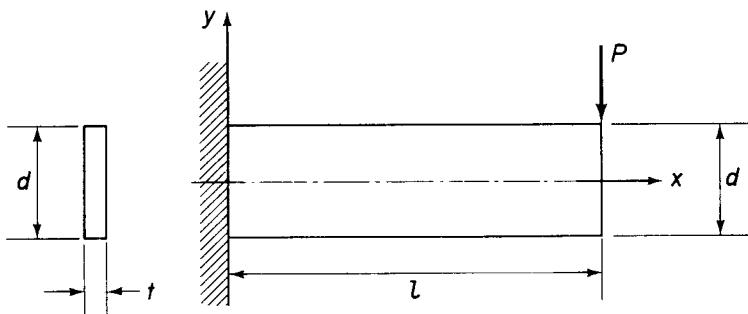


FIGURE P.2.2

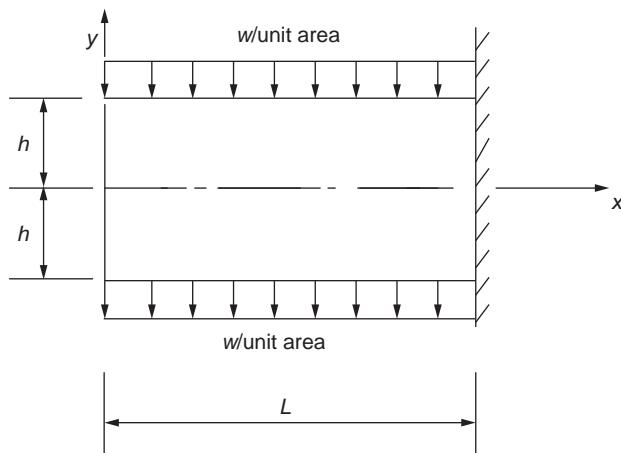


FIGURE P.2.3

Answer: The stress function satisfies the biharmonic equation:

- At  $y = h$ ,  $\sigma_y = w$  and  $\tau_{xy} = 0$ , boundary conditions satisfied,
- At  $y = -h$ ,  $\sigma_y = -w$  and  $\tau_{xy} = 0$ , boundary conditions satisfied.

Direct stress at free end of beam is not zero, there is no resultant force or bending moment at the free end.

**P.2.4** A thin rectangular plate of unit thickness (Fig. P.2.4) is loaded along the edge  $y = +d$  by a linearly varying distributed load of intensity  $w = px$  with corresponding equilibrating shears along the vertical edges at  $x = 0$  and  $l$ . As a solution to the stress analysis problem, an Airy stress function  $\phi$  is proposed, where

$$\phi = \frac{p}{120d^3} \left[ 5(x^3 - l^2x)(y+d)^2(y-2d) - 3yx(y^2 - d^2)^2 \right]$$

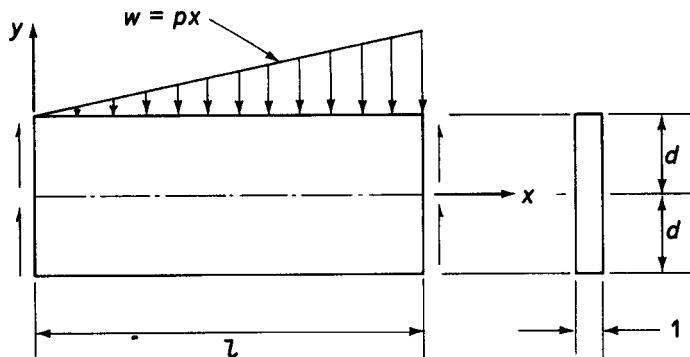


FIGURE P.2.4

Show that  $\phi$  satisfies the internal compatibility conditions and obtain the distribution of stresses within the plate. Determine also the extent to which the static boundary conditions are satisfied.

Answer:  $\sigma_x = \frac{px}{20d^3} [5y(x^2 - l^2) - 10y^3 + 6d^2y]$   
 $\sigma_y = \frac{px}{4d^3} (y^3 - 3yd^2 - 2d^3)$   
 $\tau_{xy} = \frac{-p}{40d^3} [5(3x^2 - l^2)(y^2 - d^2) - 5y^4 + 6y^2d^2 - d^4]$

The boundary stress function values of  $\tau_{xy}$  do not agree with the assumed constant equilibrating shears at  $x = 0$  and  $l$ .

**P.2.4 MATLAB** Use the Symbolic Math Toolbox in MATLAB to show the Airy stress function proposed in Problem P.2.4 satisfies the internal compatibility conditions. Also repeat the derivation of the distribution of stresses within the plate, assuming that  $y = -d, -d/2, 0, d/2$ , and  $d$ . Assume that for all cases  $x = l$ .

Answer:

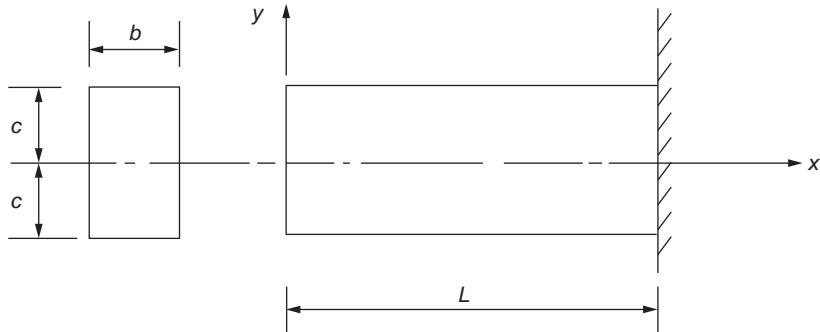
(i)	$\sigma_y = 0,$	$\sigma_x = lp/5,$
	$\tau_{xy} = 0$	
(ii)	$\sigma_y = -5lp/32,$	$\sigma_x = -7lp/80,$
	$\tau_{xy} = -p((9d^4)/16 - (45d^2l^2)/2)/(120d^3)$	
(iii)	$\sigma_y = -lp/2,$	$\sigma_x = 0,$
	$\tau_{xy} = p(3d^4 + 30d^2l^2)/(120d^3)$	
(iv)	$\sigma_y = -27lp/32,$	$\sigma_x = 7lp/80,$
	$\tau_{xy} = -p((9d^4)/16 - (45d^2l^2)/2)/(120d^3)$	
(v)	$\sigma_y = -lp,$	$\sigma_x = -lp/5,$
	$\tau_{xy} = 0$	

**P.2.5** The cantilever beam shown in Fig. P.2.5 is rigidly fixed at  $x = L$  and carries loading such that the Airy stress function relating to the problem is

$$\phi = \frac{w}{40bc^3} (-10c^3x^2 - 15c^2x^2y + 2c^2y^3 + 5x^2y^3 - y^5)$$

Find the loading pattern corresponding to the function and check its validity with respect to the boundary conditions.

Answer: The stress function satisfies the biharmonic equation. The beam is a cantilever under a uniformly distributed load of intensity  $w/\text{unit area}$  with a self-equilibrating stress application given by  $\sigma_x = w(12c^3y - 20y^3)/40bc^3$  at  $x = 0$ . There is zero shear stress at  $y = \pm c$  and  $x = 0$ . At  $y = +c$ ,  $\sigma_y = -w/b$ ; and at  $y = -c$ ,  $\sigma_y = 0$ .

**FIGURE P.2.5**

**P.2.6** A two-dimensional isotropic sheet, having a Young's modulus  $E$  and linear coefficient of expansion  $\alpha$ , is heated nonuniformly, the temperature being  $T(x, y)$ . Show that the Airy stress function  $\phi$  satisfies the differential equation

$$\nabla^2(\nabla^2\phi + E\alpha T) = 0$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplace operator.

**P.2.7** Investigate the state of plane stress described by the following Airy stress function:

$$\phi = \frac{3Qxy}{4a} - \frac{Qxy^3}{4a^3}$$

over the square region  $x = -a$  to  $x = +a$ ,  $y = -a$  to  $y = +a$ . Calculate the stress resultants per unit thickness over each boundary of the region.

Answer: The stress function satisfies the biharmonic equation. Also,

when  $x = a$ ,

$$\sigma_x = \frac{-3Qy}{2a^2}$$

when  $x = -a$ ,

$$\sigma_x = \frac{3Qy}{2a^2}$$

and

$$\tau_{xy} = \frac{-3Q}{4a} \left(1 - \frac{y^2}{a^2}\right)$$

# Torsion of solid sections

# 3

The elasticity solution of the torsion problem for bars of arbitrary but uniform cross-section is accomplished by the semi-inverse method (Section 2.3), in which assumptions are made regarding either stress or displacement components. The former method owes its derivation to Prandtl, the latter to St. Venant. Both methods are presented in this chapter, together with the useful membrane analogy introduced by Prandtl.

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## 3.1 PRANDTL STRESS FUNCTION SOLUTION

Consider the straight bar of uniform cross-section shown in Fig. 3.1. It is subjected to equal but opposite torques  $T$  at each end, both of which are assumed to be free from restraint so that warping displacements  $w$ —that is, displacements of cross-sections normal to and out of their original planes—are unrestrained. Further, we make the reasonable assumptions that, since no direct loads are applied to the bar,

$$\sigma_x = \sigma_y = \sigma_z = 0$$

and that the torque is resisted solely by shear stresses in the plane of the cross-section, i.e., the  $z$  plane, giving

$$\tau_{xy} = 0$$

To verify these assumptions, we must show that the remaining stresses satisfy the conditions of equilibrium and compatibility at all points throughout the bar and, in addition, fulfill the equilibrium boundary conditions at all points on the surface of the bar.

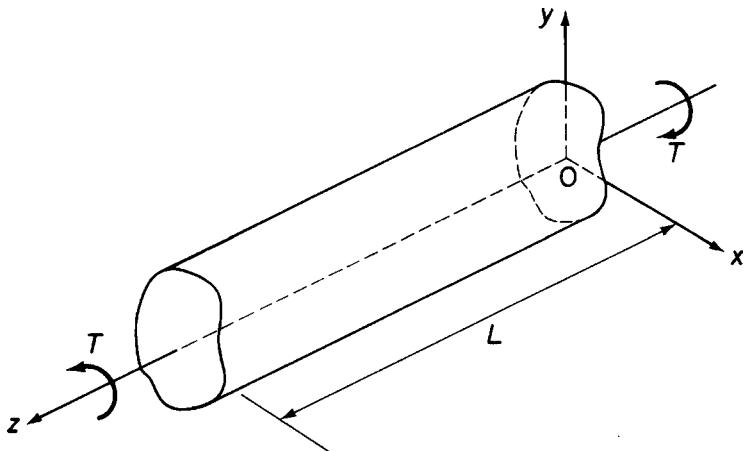
If we ignore body forces, the equations of equilibrium (1.5) reduce, as a result of our assumptions, to

$$\frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \quad (3.1)$$

The first two equations of Eqs. (3.1) show that the shear stresses  $\tau_{xz}$  and  $\tau_{yz}$  are functions of  $x$  and  $y$  only. They are therefore constant at all points along the length of the bar that have the same  $x$  and  $y$  coordinates. At this stage, we turn to the stress function to simplify the process of solution. Prandtl introduced a stress function  $\phi$  defined by

$$\frac{\partial \phi}{\partial x} = -\tau_{zy}, \quad \frac{\partial \phi}{\partial y} = \tau_{zx} \quad (3.2)$$

which identically satisfies the third of the equilibrium equations (3.1) whatever form  $\phi$  may take. We therefore have to find the possible forms of  $\phi$  that satisfy the compatibility equations



**FIGURE 3.1 Torsion of a Bar of Uniform, Arbitrary Cross-section**

(Eqs. (1.21)–(1.26)) and the boundary conditions, the latter being, in fact, the requirement that distinguishes one torsion problem from another.

From the assumed state of stress in the bar, we deduce that

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$$

(see Eqs. (1.42) and (1.51)). Further, since  $\tau_{xz}$  and  $\tau_{yz}$ , and hence  $\gamma_{xz}$  and  $\gamma_{yz}$  are functions of  $x$  and  $y$  only, the compatibility equations (1.21)–(1.23) are identically satisfied, as is Eq. (1.26). The remaining compatibility equations, (1.24) and (1.25), are then reduced to

$$\begin{aligned} \frac{\partial}{\partial x} \left( -\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} \right) &= 0 \\ \frac{\partial}{\partial y} \left( \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} \right) &= 0 \end{aligned}$$

Substituting initially for  $\gamma_{yz}$  and  $\gamma_{xz}$  from Eqs. (1.51) and for  $\tau_{zy}$  ( $= \tau_{yz}$ ) and  $\tau_{zx}$  ( $= \tau_{xz}$ ) from Eqs. (3.2) gives

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) &= 0 \\ -\frac{\partial}{\partial y} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) &= 0 \end{aligned}$$

or

$$\frac{\partial}{\partial x} \nabla^2 \phi = 0, \quad -\frac{\partial}{\partial y} \nabla^2 \phi = 0 \quad (3.3)$$

where  $\nabla^2$  is the two-dimensional Laplacian operator

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

The parameter  $\nabla^2\phi$  is therefore constant at any section of the bar, so that the function  $\phi$  must satisfy the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \text{constant} = F(\text{say}) \quad (3.4)$$

at all points within the bar.

Finally, we must ensure that  $\phi$  fulfills the boundary conditions specified by Eqs. (1.7). On the cylindrical surface of the bar, there are no externally applied forces, so that  $\bar{X} = \bar{Y} = \bar{Z} = 0$ . The direction cosine  $n$  is also zero, and therefore the first two equations of Eqs. (1.7) are identically satisfied, leaving the third equation as the boundary condition; that is,

$$\tau_{yz}m + \tau_{xz}l = 0 \quad (3.5)$$

The direction cosines  $l$  and  $m$  of the normal  $N$  to any point on the surface of the bar are, by reference to Fig. 3.2,

$$l = \frac{dy}{ds}, \quad m = -\frac{dx}{ds} \quad (3.6)$$

Substituting Eqs. (3.2) and (3.6) into (3.5), we have

$$\frac{\partial\phi}{\partial x} \frac{dx}{ds} + \frac{\partial\phi}{\partial y} \frac{dy}{ds} = 0$$

or

$$\frac{\partial\phi}{\partial s} = 0$$

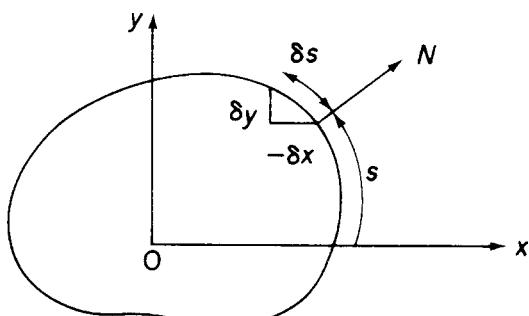


FIGURE 3.2 Formation of the Direction Cosines / and  $m$  of the Normal to the Surface of the Bar

Therefore,  $\phi$  is constant on the surface of the bar and, since the actual value of this constant does not affect the stresses of Eq. (3.2), we may conveniently take the constant to be zero. Hence, on the cylindrical surface of the bar, we have the boundary condition

$$\phi = 0 \quad (3.7)$$

On the ends of the bar, the direction cosines of the normal to the surface have the values  $l = 0, m = 0$ , and  $n = 1$ . The related boundary conditions, from Eqs. (1.7), are then

$$\begin{aligned}\bar{X} &= \tau_{zx} \\ \bar{Y} &= \tau_{zy} \\ \bar{Z} &= 0\end{aligned}$$

We now observe that the forces on each end of the bar are shear forces that are distributed over the ends of the bar in the same manner as the shear stresses are distributed over the cross-section. The resultant shear force in the positive direction of the  $x$  axis, which we shall call  $S_x$ , is

$$S_x = \iint \bar{X} dx dy = \iint \tau_{zx} dx dy$$

or, using the relationship of Eqs. (3.2),

$$S_x = \iint \frac{\partial \phi}{\partial y} dx dy = \int dx \int \frac{\partial \phi}{\partial y} dy = 0$$

as  $\phi = 0$  at the boundary. In a similar manner,  $S_y$ , the resultant shear force in the  $y$  direction, is

$$S_y = - \int dy \int \frac{\partial \phi}{\partial x} dx = 0$$

It follows that there is no resultant shear force on the ends of the bar and the forces represent a torque of magnitude, referring to Fig. 3.3,

$$T = \iint (\tau_{zy} x - \tau_{zx} y) dx dy$$

in which we take the sign of  $T$  as being positive in the counterclockwise sense.

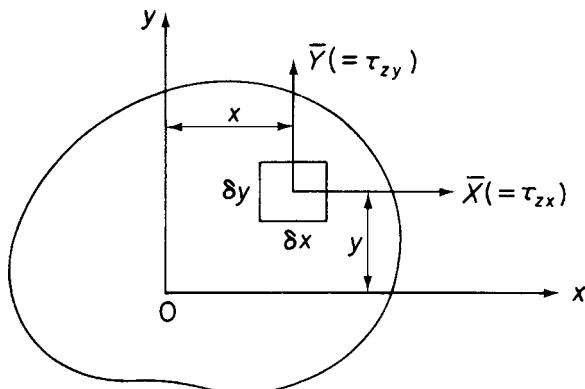


FIGURE 3.3 Derivation of Torque on Cross-section of Bar

Rewriting this equation in terms of the stress function  $\phi$ ,

$$T = - \iint \frac{\partial \phi}{\partial x} x \, dx \, dy - \iint \frac{\partial \phi}{\partial y} y \, dx \, dy$$

Each term on the right-hand side of this equation may be integrated by parts, i.e.,

$$\iint \frac{\partial \phi}{\partial x} x \, dx \, dy = \int \left[ x\phi - \int \phi \, dx \right] dy = [xy\phi]_{y_1}^{y_2} - \iint \phi \, dx \, dy$$

and

$$\iint \frac{\partial \phi}{\partial y} y \, dx \, dy = \int \left[ y\phi - \int \phi \, dy \right] dx = [yx\phi]_{x_1}^{x_2} - \iint \phi \, dx \, dy$$

where  $x_1, x_2, y_1$ , and  $y_2$  are limits of integration on the boundary of the bar. Since  $\phi = 0$  at all points on the boundary of the bar, we have

$$T = 2 \iint \phi \, dx \, dy \quad (3.8)$$

We are therefore in a position to obtain an exact solution to a torsion problem if a stress function  $\phi(x, y)$  can be found that satisfies Eq. (3.4) at all points within the bar and vanishes on the surface of the bar, and providing that the external torques are distributed over the ends of the bar in an identical manner to the distribution of internal stress over the cross-section. Although the last proviso is generally impracticable, we know from St. Venant's principle that only stresses in the end regions are affected; therefore, the solution is applicable to sections at distances from the ends usually taken to be greater than the largest cross-sectional dimension. We have now satisfied all the conditions of the problem without the use of stresses other than  $\tau_{zy}$  and  $\tau_{xz}$ , demonstrating that our original assumptions were justified.

Usually, in addition to the stress distribution in the bar, we need to know the angle of twist and the warping displacement of the cross-section. First, however, we shall investigate the mode of displacement of the cross-section. We have seen that, as a result of our assumed values of stress,

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$$

It follows, from Eqs. (1.18) and the second of Eqs. (1.20), that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

which result leads to the conclusions that each cross-section rotates as a rigid body in its own plane about a center of rotation or twist and that, although cross-sections suffer warping displacements normal to their planes, the values of this displacement at points having the same coordinates along the length of the bar are equal. Each longitudinal fiber of the bar therefore remains unstrained, as we have in fact assumed.

Let us suppose that a cross-section of the bar rotates through a small angle  $\theta$  about its center of twist assumed coincident with the origin of the axes Oxy (see Fig. 3.4). Some point P( $r, \alpha$ ) will be displaced to P'( $r, \alpha + \theta$ ), the components of its displacement being

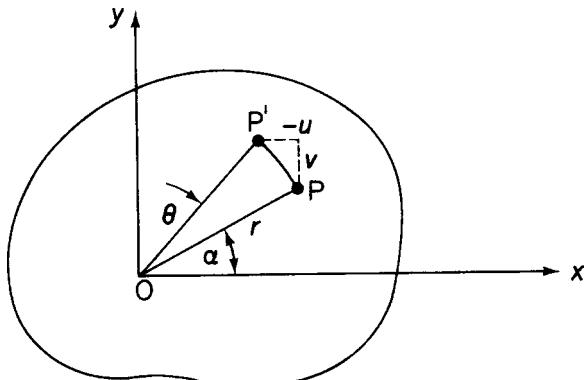


FIGURE 3.4 Rigid Body Displacement in the Cross-section of the Bar

$$u = -r\theta \sin\alpha, \quad v = r\theta \cos\alpha$$

or

$$u = -\theta y, \quad v = \theta x \quad (3.9)$$

Referring to Eqs. (1.20) and (1.51),

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G}, \quad \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\tau_{zy}}{G}$$

Rearranging and substituting for  $u$  and  $v$  from Eqs. (3.9),

$$\frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} + \frac{d\theta}{dz}y, \quad \frac{\partial w}{\partial y} = \frac{\tau_{zy}}{G} - \frac{d\theta}{dz}x \quad (3.10)$$

For a particular torsion problem, Eqs. (3.10) enable the warping displacement  $w$  of the originally plane cross-section to be determined. Note that, since each cross-section rotates as a rigid body,  $\theta$  is a function of  $z$  only.

Differentiating the first of Eqs. (3.10) with respect to  $y$ , the second with respect to  $x$ , and subtracting, we have

$$0 = \frac{1}{G} \left( \frac{\partial \tau_{zx}}{\partial y} - \frac{\partial \tau_{zy}}{\partial x} \right) + 2 \frac{d\theta}{dz}$$

Expressing  $\tau_{zx}$  and  $\tau_{zy}$  in terms of  $\phi$  from Eqs. (3.2) gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G \frac{d\theta}{dz}$$

or, from Eq. (3.4),

$$-2G \frac{d\theta}{dz} = \nabla^2 \phi = F(\text{constant}) \quad (3.11)$$

It is convenient to introduce a *torsion constant*  $J$ , defined by the general torsion equation

$$T = GJ \frac{d\theta}{dz} \quad (3.12)$$

The product  $GJ$  is known as the *torsional rigidity* of the bar and may be written, from Eqs. (3.8) and (3.11),

$$GJ = -\frac{4G}{\nabla^2 \phi} \iint \phi \, dx \, dy \quad (3.13)$$

Consider now the line of constant  $\phi$  in Fig. 3.5. If  $s$  is the distance measured along this line from some arbitrary point, then

$$\frac{\partial \phi}{\partial s} = 0 = \frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial x} \frac{dx}{ds}$$

Using Eqs. (3.2) and (3.6), we may rewrite this equation as

$$\frac{\partial \phi}{\partial s} = \tau_{zx}l + \tau_{zy}m = 0 \quad (3.14)$$

From Fig. 3.5, the normal and tangential components of shear stress are

$$\tau_{zn} = \tau_{zx}l + \tau_{zy}m \quad \tau_{zs} = \tau_{zy}l - \tau_{zx}m \quad (3.15)$$

Comparing the first of Eqs. (3.15) with Eq. (3.14), we see that the normal shear stress is zero, so that the resultant shear stress at any point is tangential to a line of constant  $\phi$ . These are known as *lines of shear stress* or *shear lines*.

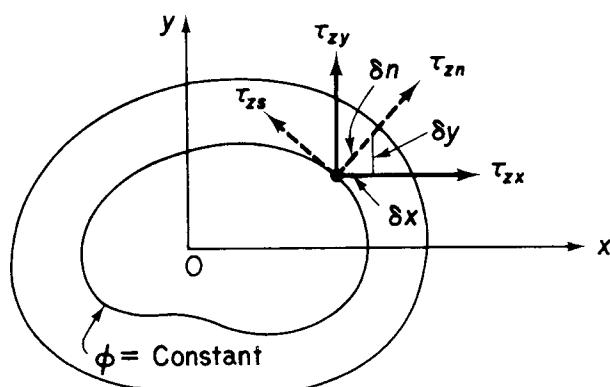


FIGURE 3.5 Lines of Shear Stress

Substituting  $\phi$  in the second of Eqs. (3.15), we have

$$\tau_{zs} = -\frac{\partial \phi}{\partial x} l - \frac{\partial \phi}{\partial y} m$$

which may be written, from Fig. 3.5, as

$$\tau_{zx} = \frac{\partial \phi}{\partial x} \frac{dx}{dn} - \frac{\partial \phi}{\partial y} \frac{dy}{dn} = -\frac{\partial \phi}{\partial n} \quad (3.16)$$

where, in this case, the direction cosines  $l$  and  $m$  are defined in terms of an elemental normal of length  $\delta n$ .

We have therefore shown that the resultant shear stress at any point is tangential to the line of shear stress through the point and has a value equal to minus the derivative of  $\phi$  in a direction normal to the line.

### Example 3.1

Determine the rate of twist and the stress distribution in a circular section bar of radius  $R$  that is subjected to equal and opposite torques  $T$  at each of its free ends.

If we assume an origin of axes at the center of the bar, the equation of its surface is given by

$$x^2 + y^2 = R^2$$

If we now choose a stress function of the form

$$\phi = C(x^2 + y^2 - R^2) \quad (\text{i})$$

the boundary condition  $\phi = 0$  is satisfied at every point on the boundary of the bar, and the constant  $C$  may be chosen to fulfill the remaining requirement of compatibility. Therefore, from Eqs. (3.11) and (i),

$$4C = -2G \frac{d\theta}{dz}$$

so that

$$C = -\frac{G}{2} \frac{d\theta}{dz}$$

and

$$\phi = -G \frac{d\theta}{dz} (x^2 + y^2 - R^2)/2 \quad (\text{ii})$$

Substituting for  $\phi$  in Eq. (3.8),

$$T = -G \frac{d\theta}{dz} \left( \iint x^2 dx dy + \iint y^2 dx dy - R^2 \iint dx dy \right)$$

The first integral in this expression may be evaluated as follows. Working in terms of angular coordinates,

$$\begin{aligned} \int_{-R}^{+R} \int_{-R}^{+R} x^2 dx dy &= 2 \int_{\pi/2}^{-\pi/2} 2 \int_0^{\pi} (R^2 \cos^2 \theta) (-R \sin \theta d\theta) (R \cos \theta d\theta) \\ &= -4R^4 \int_{-\pi/2}^{\pi/2} \int_0^{\pi} (\cos^3 \theta \sin \theta) d\theta \\ &= +4R^4 \int_{-\pi/2}^{\pi/2} \left[ \frac{\cos^4 \theta}{4} \right]_0^\pi d\theta \end{aligned}$$

Rewriting  $(\cos^4 \theta)$  as  $(1 + \cos 2\theta)(1 + \cos 2\theta)/4$  and integrating gives

$$\int_{-R}^{+R} \int_{-R}^{+R} x^2 dx dy = \frac{\pi R^4}{4}$$

Similarly

$$\iint y^2 dx dy = \frac{\pi R^4}{4}$$

and

$$\iint dx dy = \pi R^2$$

so that

$$T = -G \frac{d\theta}{dz} \left( \frac{\pi R^4}{4} + \frac{\pi R^4}{4} - \pi R^4 \right)$$

which gives

$$T = \frac{\pi R^4}{2} G \frac{d\theta}{dz}$$

that is,

$$T = GJ \frac{d\theta}{dz} \quad (\text{iii})$$

in which  $J = \pi R^4/2 = \pi D^4/32$  ( $D$  is the diameter), the *polar second moment of area* of the bar's cross-section.

Substituting for  $G(d\theta/dz)$  in Eq. (ii) from (iii),

$$\phi = -\frac{T}{2J} (x^2 + y^2 - R^2)$$

and from Eqs. (3.2),

$$\tau_{zy} = -\frac{\partial \phi}{\partial x} = \frac{Tx}{J}, \quad \tau_{zx} = \frac{\partial \phi}{\partial y} = -\frac{Ty}{J}$$

The resultant shear stress at any point on the surface of the bar is then given by

$$\tau = \sqrt{\tau_{zy}^2 + \tau_{zx}^2}$$

that is,

$$\tau = \frac{T}{J} \sqrt{x^2 + y^2}$$

so that

$$\tau = \frac{TR}{J} \quad (\text{iv})$$

This argument may be applied to any annulus of radius  $r$  within the cross-section of the bar, so that the stress distribution is given by

$$\tau = \frac{Tr}{J}$$

and therefore increases linearly from zero at the center of the bar to a maximum  $TR/J$  at the surface.

### Example 3.2

A uniform bar has the elliptical cross-section shown in Fig. 3.6(a) and is subjected to equal and opposite torques  $T$  at each of its free ends. Derive expressions for the rate of twist in the bar, the shear stress distribution, and the warping displacement of its cross-section.

The semi-major and semi-minor axes are  $a$  and  $b$ , respectively, so that the equation of its boundary is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we choose a stress function of the form

$$\phi = C \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \quad (\text{i})$$

then the boundary condition  $\phi = 0$  is satisfied at every point on the boundary, and the constant  $C$  may be chosen to fulfill the remaining requirement of compatibility. Thus, from Eqs. (3.11) and (i),

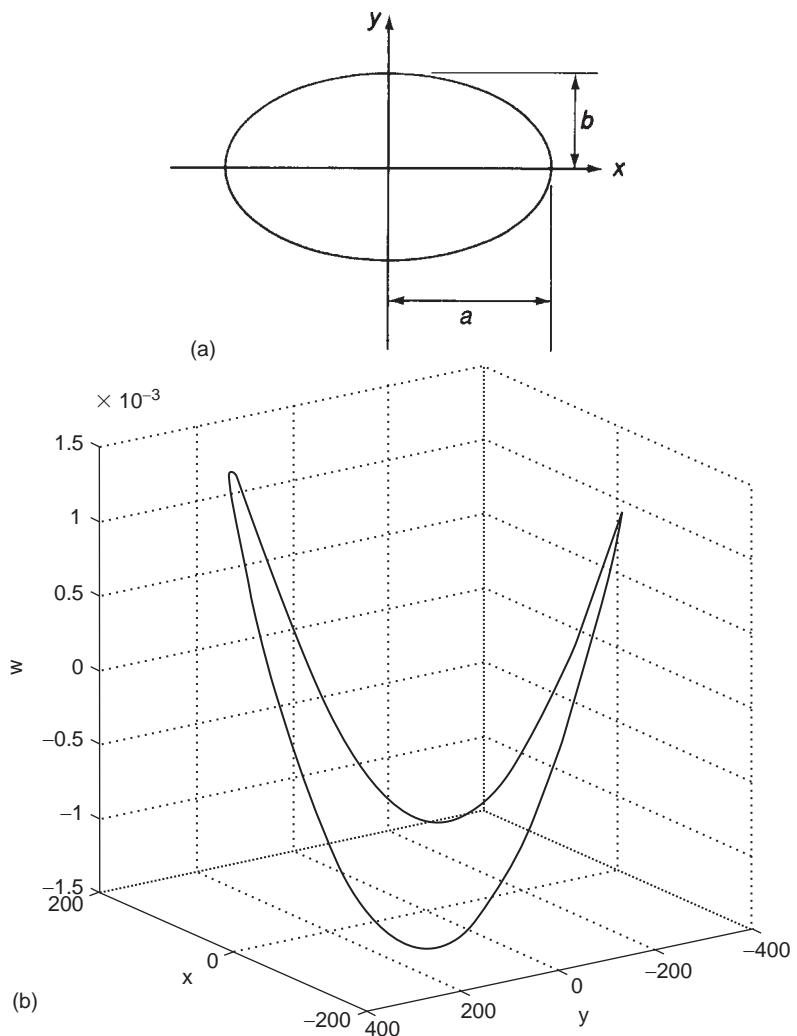
$$2C \left( \frac{1}{a^2} + \frac{1}{b^2} \right) = -2G \frac{d\theta}{dz}$$

or

$$C = -G \frac{d\theta}{dz} \frac{a^2 b^2}{(a^2 + b^2)} \quad (\text{ii})$$

giving

$$\phi = -G \frac{d\theta}{dz} \frac{a^2 b^2}{(a^2 + b^2)} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \quad (\text{iii})$$



**FIGURE 3.6 (a) Torsion of a Bar of Elliptical Cross-section; (b) (MATLAB) Warping Displacement Distribution**

Substituting this expression for  $\phi$  in Eq. (3.8) establishes the relationship between the torque  $T$  and the rate of twist:

$$T = -2G \frac{d\theta}{dz} \frac{a^2 b^2}{(a^2 + b^2)} \left( \frac{1}{a^2} \iint x^2 dx dy + \frac{1}{b^2} \iint y^2 dx dy - \iint dx dy \right)$$

The first and second integrals in this equation may be shown, in a similar manner to those in Example 3.1, to be the second moments of area  $I_{yy} = \pi a^3 b / 4$  and  $I_{xx} = \pi a b^3 / 4$ , while the third integral is the area of the cross-section  $A = \pi a b$ . Replacing the integrals by these values gives

$$T = G \frac{d\theta}{dz} \frac{\pi a^3 b^3}{(a^2 + b^2)} \quad (\text{iv})$$

from which (see Eq. (3.12))

$$J = \frac{\pi a^3 b^3}{(a^2 + b^2)} \quad (\text{v})$$

The shear stress distribution is obtained in terms of the torque by substituting for the product  $G(d\theta/dz)$  in Eq. (iii) from (iv) and then differentiating as indicated by the relationships of Eqs. (3.2). Thus,

$$\tau_{zx} = -\frac{2Ty}{\pi ab^3}, \quad \tau_{zy} = \frac{2Tx}{\pi a^3 b} \quad (\text{vi})$$

So far we have solved for the stress distribution, Eqs. (vi), and the rate of twist, Eq. (iv). It remains to determine the warping distribution  $w$  over the cross-section. For this, we return to Eqs. (3.10), which become, on substituting from the preceding for  $\tau_{zx}$ ,  $\tau_{zy}$ , and  $d\theta/dz$ ,

$$\frac{\partial w}{\partial x} = -\frac{2Ty}{\pi ab^3 G} + \frac{T(a^2 + b^2)}{G \pi a^3 b^3} y, \quad \frac{\partial w}{\partial y} = \frac{2Tx}{\pi a^3 b G} - \frac{T(a^2 + b^2)}{G \pi a^3 b^3} x$$

or

$$\frac{\partial w}{\partial x} = \frac{T}{\pi a^3 b^3 G} (b^2 - a^2) y, \quad \frac{\partial w}{\partial y} = \frac{T}{\pi a^3 b^3 G} (b^2 - a^2) x \quad (\text{vii})$$

Integrating both of Eqs. (vii),

$$w = \frac{T(b^2 - a^2)}{\pi a^3 b^3 G} yx + f_1(y) \quad w = \frac{T(b^2 - a^2)}{\pi a^3 b^3 G} xy + f_2(x)$$

The warping displacement given by each of these equations must have the same value at identical points  $(x, y)$ . It follows that  $f_1(y) = f_2(x) = 0$ . Hence,

$$w = \frac{T(b^2 - a^2)}{\pi a^3 b^3 G} xy \quad (\text{viii})$$

Lines of constant  $w$  therefore describe hyperbolas with the major and minor axes of the elliptical cross-section as asymptotes. Further, for a positive (counterclockwise) torque, the warping is negative in the first and third quadrants ( $a > b$ ) and positive in the second and fourth.

### Example 3.2 MATLAB

Calculate the warping displacement ( $w$ ) along the boundary of the cross-section illustrated in Fig. 3.6(a) using the equation for  $w$  derived in Example 3.2. Plot the resulting values of  $w$  using the plot3 function in MATLAB. Assume the following variable values:  $a = 200$  mm;  $b = 300$  mm;  $G = 25,000$  N/mm $^2$ ;  $T = 15$  kNm.

Calculations of  $w$  along the cross-section boundary and the code required to create the needed plot are obtained through the following MATLAB file:

```
% Declare any needed variables
a = 200;
b = 300;
G = 25000;
T = 15*10^6;

% Define (x,y) values for 1001 points along the cross-section boundary
steps = 1000;
x = zeros(steps+1,1);
y = zeros(steps+1,1);
w = zeros(steps+1,1);
for i=1:1:steps/2
    x(i) = a - (4*a/steps)*(i-1);
    y(i) = b*sqrt(1-(x(i)/a)^2);
end
for i=1:1:steps/2+1
    x(steps/2+i) = -a + (4*a/steps)*(i-1);
    y(steps/2+i) = -b*sqrt(1-(x(i)/a)^2);
end

% Calculate the warping displacement (w) at each point (x,y)
for i=1:1:steps+1
    w(i) = T*(b^2-a^2)*x(i)*y(i)/(pi*a^3*b^3*G);
end

% Plot the values of w
figure(1)
plot3(x,y,w)
grid on
axis square
xlabel('x')
ylabel('y')
zlabel('w')
view(-125,18)
```

The plot resulting from this MATLAB file is illustrated in Fig. 3.6(b) (MATLAB).

## 3.2 ST. VENANT WARPING FUNCTION SOLUTION

In formulating his stress function solution, Prandtl made assumptions concerning the stress distribution in the bar. The alternative approach presented by St. Venant involves assumptions as to the mode of displacement of the bar, namely that cross-sections of a bar subjected to torsion maintain their original unloaded shape, although they may suffer warping displacements normal to their plane. The first of these assumptions leads to the conclusion that cross-sections rotate as rigid bodies about a center of rotation or twist. This fact was also found to derive from the stress function approach of Section 3.1, so that, referring to Fig. 3.4 and Eq. (3.9), the components of displacement in the  $x$  and  $y$  directions of a point  $P$  in the cross-section are

$$u = -\theta y, v = \theta x$$

It is also reasonable to assume that the warping displacement  $w$  is proportional to the rate of twist and is therefore constant along the length of the bar. Hence, we may define  $w$  by the equation

$$w = \frac{d\theta}{dz} \psi(x, y) \quad (3.17)$$

where  $\psi(x, y)$  is the *warping function*.

The assumed form of the displacements  $u$ ,  $v$ , and  $w$  must satisfy the equilibrium and force boundary conditions of the bar. We note here that it is unnecessary to investigate compatibility, as we are concerned with displacement forms that are single-valued functions and therefore automatically satisfy the compatibility requirement.

The components of strain corresponding to the assumed displacements are obtained from Eqs. (1.18) and (1.20) and are

$$\left. \begin{aligned} \varepsilon_x &= \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0 \\ \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{d\theta}{dz} \left( \frac{\partial \psi}{\partial x} - y \right) \\ \gamma_{zy} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{d\theta}{dz} \left( \frac{\partial \psi}{\partial y} + x \right) \end{aligned} \right\} \quad (3.18)$$

The corresponding components of stress are, from Eqs. (1.43)–(1.45) and (1.51),

$$\left. \begin{aligned} \sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0 \\ \tau_{zx} &= G \frac{d\theta}{dz} \left( \frac{\partial \psi}{\partial x} - y \right) \\ \tau_{zy} &= G \frac{d\theta}{dz} \left( \frac{\partial \psi}{\partial y} + x \right) \end{aligned} \right\} \quad (3.19)$$

Ignoring body forces, we see that these equations identically satisfy the first two of the equilibrium equations (1.5) and also that the third is fulfilled if the warping function satisfies the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0 \quad (3.20)$$

The direction cosine  $n$  is zero on the cylindrical surface of the bar, and so the first two of the boundary conditions (Eqs. (1.7)) are identically satisfied by the stresses of Eqs. (3.19). The third of Eqs. (1.7) simplifies to

$$\left( \frac{\partial \psi}{\partial y} + x \right) m + \left( \frac{\partial \psi}{\partial x} - y \right) l = 0 \quad (3.21)$$

It may be shown, but not as easily as in the stress function solution, that the shear stresses defined in terms of the warping function in Eqs. (3.19) produce zero resultant shear force over each end of the bar.<sup>1</sup> The torque is found in a similar manner to that in Section 3.1, where, by reference to Fig. 3.3, we have

$$T = \iint (\tau_{zy}x - \tau_{zx}y) dx dy$$

or

$$T = G \frac{d\theta}{dz} \iint \left[ \left( \frac{\partial \psi}{\partial y} + x \right) x - \left( \frac{\partial \psi}{\partial x} - y \right) y \right] dx dy \quad (3.22)$$

By comparison with Eq. (3.12), the torsion constant  $J$  is now, in terms of  $\psi$ ,

$$J = \iint \left[ \left( \frac{\partial \psi}{\partial y} + x \right) x - \left( \frac{\partial \psi}{\partial x} - y \right) y \right] dx dy \quad (3.23)$$

The warping function solution to the torsion problem reduces to the determination of the warping function  $\psi$ , which satisfies Eqs. (3.20) and (3.21). The torsion constant and the rate of twist follow from Eqs. (3.23) and (3.22); the stresses and strains from Eqs. (3.19) and (3.18); and finally, the warping distribution from Eq. (3.17).

### 3.3 THE MEMBRANE ANALOGY

Prandtl suggested an extremely useful analogy relating the torsion of an arbitrarily shaped bar to the deflected shape of a membrane. The latter is a thin sheet of material that relies for its resistance to transverse loads on internal in-plane or membrane forces.

Suppose that a membrane has the same external shape as the cross-section of a torsion bar (Fig. 3.7(a)). It supports a transverse uniform pressure  $q$  and is restrained along its edges by a uniform tensile force  $N/\text{unit length}$ , as shown in Figs. 3.7(a) and (b). It is assumed that the transverse displacements of the membrane are small, so that  $N$  remains unchanged as the membrane deflects. Consider the equilibrium of an element  $\delta x \delta y$  of the membrane. Referring to Fig. 3.8 and summing forces in the  $z$  direction, we have

$$-N\delta y \frac{\partial w}{\partial x} - N\delta y \left( -\frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2} \delta x \right) - N\delta x \frac{\partial w}{\partial y} - N\delta x \left( -\frac{\partial w}{\partial y} - \frac{\partial^2 w}{\partial y^2} \delta y \right) + q\delta x \delta y = 0$$

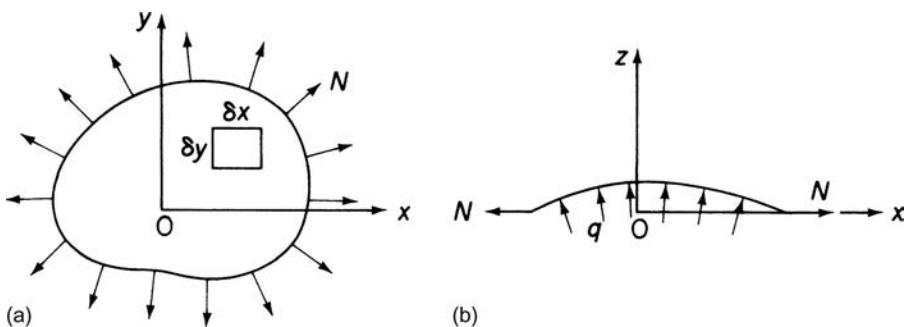


FIGURE 3.7 Membrane Analogy: (a) In-plane and (b) Transverse Loading

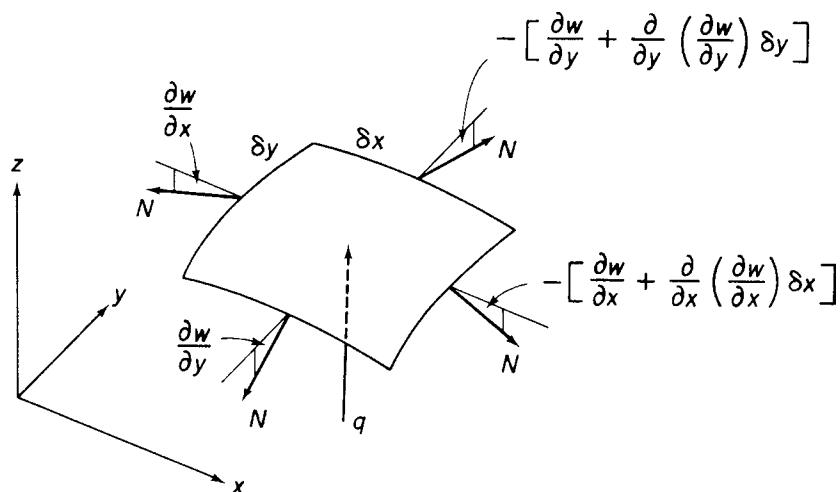


FIGURE 3.8 Equilibrium of Element of Membrane

or

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w = -\frac{q}{N} \quad (3.24)$$

Equation (3.24) must be satisfied at all points within the boundary of the membrane. Furthermore, at all points on the boundary,

$$w = 0 \quad (3.25)$$

and we see that, by comparing Eqs. (3.24) and (3.25) with Eqs. (3.11) and (3.7),  $w$  is analogous to  $\phi$  when  $q$  is constant. Thus, if the membrane has the same external shape as the cross-section of the bar, then

$$w(x, y) = \phi(x, y)$$

and

$$\frac{q}{N} = -F = 2G \frac{d\theta}{dz}$$

The analogy now being established, we may make several useful deductions relating the deflected form of the membrane to the state of stress in the bar.

Contour lines or lines of constant  $w$  correspond to lines of constant  $\phi$  or lines of shear stress in the bar. The resultant shear stress at any point is tangential to the membrane contour line and equal in value to the negative of the membrane slope,  $\partial w / \partial n$ , at that point, the direction  $n$  being normal to the contour line (see Eq. (3.16)). The volume between the membrane and the  $xy$  plane is

$$\text{Vol} = \iint w \, dx \, dy$$

and we see that, by comparison with Eq. (3.8),

$$T = 2 \text{ Vol}$$

The analogy therefore provides an extremely useful method of analyzing torsion bars possessing irregular cross-sections for which stress function forms are not known. Hetényi<sup>2</sup> describes experimental techniques for this approach. In addition to the strictly experimental use of the analogy, it is also helpful in the visual appreciation of a particular torsion problem. The contour lines often indicate a form for the stress function, enabling a solution to be obtained by the method of Section 3.1. Stress concentrations are made apparent by the closeness of contour lines where the slope of the membrane is large. These are in evidence at sharp internal corners, cut-outs, discontinuities, and the like.

### 3.4 TORSION OF A NARROW RECTANGULAR STRIP

In Chapter 18, we investigate the torsion of thin-walled open-section beams; the development of the theory being based on the analysis of a narrow rectangular strip subjected to torque. We now conveniently apply the membrane analogy to the torsion of such a strip, shown in Fig. 3.9. The corresponding membrane surface has the same cross-sectional shape at all points along its length except for small regions near its ends, where it flattens out. If we ignore these regions and assume that the shape of the membrane is independent of  $y$ , then Eq. (3.11) simplifies to

$$\frac{d^2\phi}{dx^2} = -2G \frac{d\theta}{dz}$$

Integrating twice,

$$\phi = -G \frac{d\theta}{dz} x^2 + Bx + C$$

Substituting the boundary conditions  $\phi = 0$  at  $x = \pm t/2$ , we have

$$\phi = -G \frac{d\theta}{dz} \left[ x^2 - \left( \frac{t}{2} \right)^2 \right] \quad (3.26)$$

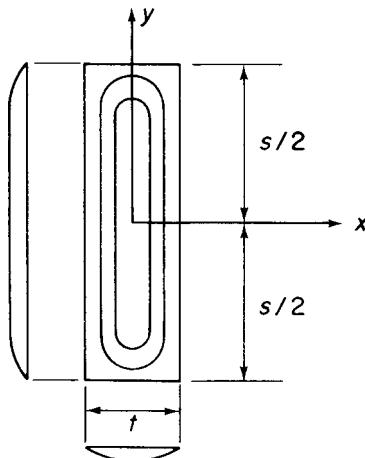


FIGURE 3.9 Torsion of a Narrow Rectangular Strip

Although  $\phi$  does not disappear along the short edges of the strip and therefore does not give an exact solution, the actual volume of the membrane differs only slightly from the assumed volume, so that the corresponding torque and shear stresses are reasonably accurate. Also, the maximum shear stress occurs along the long sides of the strip, where the contours are closely spaced, indicating, in any case, that conditions in the end region of the strip are relatively unimportant.

The stress distribution is obtained by substituting Eq. (3.26) in Eqs. (3.2), then

$$\tau_{zy} = 2Gx \frac{d\theta}{dz} \quad \tau_{zx} = 0 \quad (3.27)$$

the shear stress varying linearly across the thickness and attaining a maximum

$$\tau_{zy,\max} = \pm Gt \frac{d\theta}{dz} \quad (3.28)$$

at the outside of the long edges, as predicted. The torsion constant  $J$  follows from the substitution of Eq. (3.26) into (3.13), giving

$$J = \frac{st^3}{3} \quad (3.29)$$

and

$$\tau_{zy,\max} = \frac{3T}{st^3}$$

These equations represent exact solutions when the assumed shape of the deflected membrane is the actual shape. This condition arises only when the ratio  $s/t$  approaches infinity; however, for ratios in excess of 10, the error is on the order of only 6%. Obviously, the approximate nature of the solution increases as  $s/t$  decreases. Therefore, to retain the usefulness of the analysis, a factor  $\mu$  is included in the torsion constant; that is,

$$J = \frac{\mu s t^3}{3}$$

Values of  $\mu$  for different types of section are found experimentally and quoted in various references.<sup>3,4</sup> We observe that, as  $s/t$  approaches infinity,  $\mu$  approaches unity.

The cross-section of the narrow rectangular strip of Fig. 3.9 does not remain plane after loading but suffers warping displacements normal to its plane; this warping may be determined using either of Eqs. (3.10). From the first of these equations,

$$\frac{\partial w}{\partial x} = y \frac{d\theta}{dz} \quad (3.30)$$

since  $\tau_{zx} = 0$  (see Eqs. (3.27)). Integrating Eq. (3.30), we obtain

$$w = xy \frac{d\theta}{dz} + \text{constant} \quad (3.31)$$

Since the cross-section is doubly symmetrical,  $w = 0$  at  $x = y = 0$ , so that the constant in Eq. (3.31) is zero. Therefore,

$$w = xy \frac{d\theta}{dz} \quad (3.32)$$

and the warping distribution at any cross-section is as shown in Fig. 3.10.

We should not close this chapter without mentioning alternative methods of solution of the torsion problem. These in fact provide approximate solutions for the wide range of problems for which exact solutions are not known. Examples of this approach are the numerical finite difference method and the Rayleigh–Ritz method based on energy principles.<sup>5</sup>

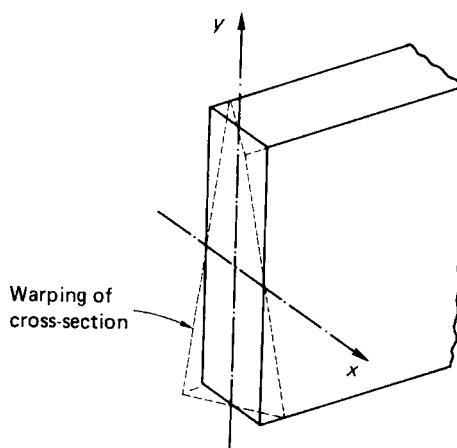


FIGURE 3.10 Warping of a Thin Rectangular Strip

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## References

- [1] Wang CT. Applied elasticity. New York: McGraw-Hill; 1953.
  - [2] Hetényi M. Handbook of experimental stress analysis. New York: John Wiley and Sons; 1950.
  - [3] Roark RJ. Formulas for stress and strain. 4th ed. New York: McGraw-Hill; 1965.
  - [4] Handbook of aeronautics, no. 1, Structural principles and data. 4th ed. Published under the authority of the Royal Aeronautical Society. London: New Era Publishing; 1952.
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## PROBLEMS

**P.3.1** Show that the stress function  $\phi = k(r^2 - a^2)$  is applicable to the solution of a solid circular section bar of radius  $a$ . Determine the stress distribution in the bar in terms of the applied torque, the rate of twist, and the warping of the cross-section. Is it possible to use this stress function in the solution for a circular bar of hollow section?

Answer:  $\tau = Tr/I_p$ , where  $I_p = \pi a^4/2$ ,  
 $d\theta/dz = 2T/G\pi a^4$ ,  $w = 0$  everywhere.

**P.3.2** Deduce a suitable warping function for the circular section bar of P.3.1 and derive the expressions for stress distribution and rate of twist.

Answer:  $\psi = 0$ ,  $\tau_{zx} = -\frac{Ty}{I_p}$ ,  $\tau_{zy} = \frac{Tx}{I_p}$ ,  $\tau_{zs} = \frac{Tr}{I_p}$ ,  $\frac{d\theta}{dz} = \frac{T}{GI_p}$

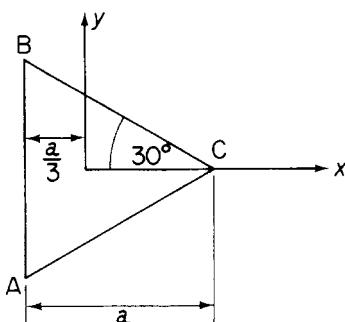
**P.3.3** Show that the warping function  $\psi = kxy$ , in which  $k$  is an unknown constant, may be used to solve the torsion problem for the elliptical section of Example 3.2.

**P.3.4** Show that the stress function

$$\phi = -G \frac{d\theta}{dz} \left[ \frac{1}{2}(x^2 + y^2) - \frac{1}{2a}(x^3 - 3xy^2) - \frac{2}{27}a^2 \right]$$

is the correct solution for a bar having a cross-section in the form of the equilateral triangle shown in Fig. P.3.4. Determine the shear stress distribution, the rate of twist, and the warping of the cross-section. Find the position and magnitude of the maximum shear stress.

Answer:  $\tau_{zy} = G \frac{d\theta}{dz} \left( x - \frac{3x^2}{2a} + \frac{3y^2}{2a} \right)$   
 $\tau_{zx} = -G \frac{d\theta}{dz} \left( y + \frac{3xy}{a} \right)$

**FIGURE P.3.4**

$$\tau_{\max} \text{ (at center of each side)} = -\frac{a}{2} G \frac{d\theta}{dz}$$

$$\frac{d\theta}{dz} = \frac{15\sqrt{3}T}{Ga^4}$$

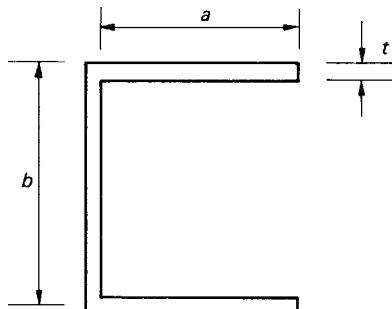
$$w = \frac{1}{2a} \frac{d\theta}{dz} (y^3 - 3x^2y)$$

**P.3.5** Determine the maximum shear stress and the rate of twist in terms of the applied torque  $T$  for the section made up of narrow rectangular strips shown in Fig. P.3.5.

Answer:  $\tau_{\max} = 3T/(2a + b)t^2$ ,  $d\theta/dz = 3T/G(2a + b)t^3$

**P.3.5 MATLAB** Use the Symbolic Math Toolbox in MATLAB to repeat Problem P.3.5, assuming that the vertical rectangular strip of length  $b$  in Fig. P.3.5 has a thickness of  $2t$ .

Answer:  $\tau_{\max} = 3T/[2t^2(a + 4b)]$ ,  $d\theta/dz = 3T/[2Gt^3(a + 4b)]$

**FIGURE P.3.5**

# Virtual work and energy methods

# 4

Many structural problems are statically determinate; that is, the support reactions and internal force systems may be found using simple statics, where the number of unknowns is equal to the number of equations of equilibrium available. In cases where the number of unknowns exceeds the possible number of equations of equilibrium—for example, a propped cantilever beam—other methods of analysis are required.

The methods fall into two categories and are based on two important concepts; the first, which is presented in this chapter, is *the principle of virtual work*. This is the most fundamental and powerful tool available for the analysis of statically indeterminate structures and has the advantage of being able to deal with conditions other than those in the elastic range. The second, based on *strain energy*, can provide approximate solutions of complex problems for which exact solutions do not exist and is discussed in [Chapter 5](#). In some cases, the two methods are equivalent, since, although the governing equations differ, the equations themselves are identical.

In modern structural analysis, computer-based techniques are widely used; these include the flexibility and stiffness methods (see [Chapter 6](#)). However, the formulation of, say, stiffness matrices for the elements of a complex structure is based on one of the previous approaches, so that a knowledge and understanding of their application is advantageous.

## 4.1 WORK

Before we consider the principle of virtual work in detail, it is important to clarify exactly what is meant by *work*. The basic definition of *work* in elementary mechanics is that “work is done when a force moves its point of application.” However, we require a more exact definition, since we are concerned with work done by both forces and moments and with the work done by a force when the body on which it acts is given a displacement that is not coincident with the line of action of the force.

Consider the force,  $F$ , acting on a particle, A, in [Fig. 4.1\(a\)](#). If the particle is given a displacement,  $\Delta$ , by some external agency so that it moves to A' in a direction at an angle  $\alpha$  to the line of action of  $F$ , the work,  $W_F$ , done by  $F$  is given by

$$W_F = F (\Delta \cos\alpha) \quad (4.1)$$

or

$$W_F = (F \cos\alpha)\Delta \quad (4.2)$$

We see therefore that the work done by the force,  $F$ , as the particle moves from A to A' may be regarded as either the product of  $F$  and the component of  $\Delta$  in the direction of  $F$  ([Eq. \(4.1\)](#)) or as the product of the component of  $F$  in the direction of  $\Delta$  and  $\Delta$  ([Eq. \(4.2\)](#)).

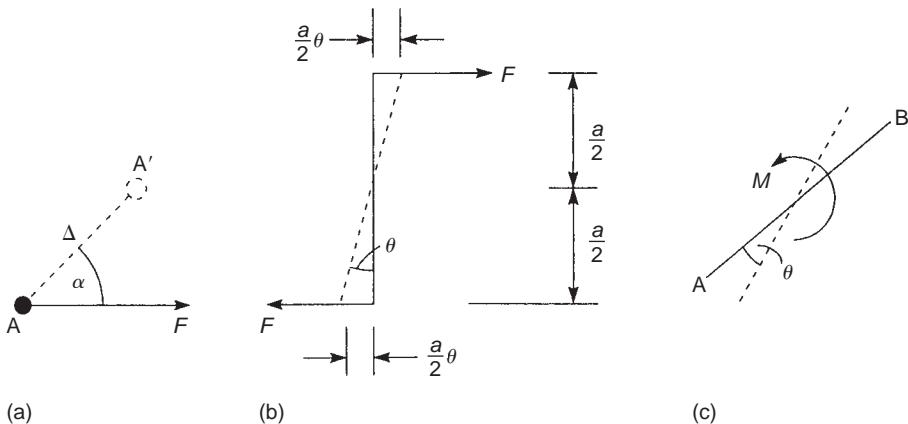


FIGURE 4.1 Work Done by a Force and a Moment

Now, consider the couple (pure moment) in Fig. 4.1(b) and suppose that the couple is given a small rotation of  $\theta$  radians. The work done by each force  $F$  is then  $F(a/2)\theta$ , so that the total work done,  $W_C$ , by the couple is

$$W_C = F \frac{a}{2} \theta + F \frac{a}{2} \theta = Fa\theta$$

It follows that the work done,  $W_M$ , by the pure moment,  $M$ , acting on the bar AB in Fig. 4.1(c) as it is given a small rotation,  $\theta$ , is

$$W_M = M\theta \quad (4.3)$$

Note that in this equation the force,  $F$ , and moment,  $M$ , are in position before the displacements take place and are not the cause of them. Also, in Fig. 4.1(a), the component of  $\Delta$  parallel to the direction of  $F$  is in the same direction as  $F$ ; if it had been in the opposite direction, the work done would have been negative. The same argument applies to the work done by the moment,  $M$ , where we see in Fig. 4.1(c) that the rotation,  $\theta$ , is in the same sense as  $M$ . Note also that if the displacement,  $\Delta$ , had been perpendicular to the force,  $F$ , no work would have been done by  $F$ .

Finally, it should be remembered that work is a scalar quantity, since it is not associated with direction. In Fig. 4.1(a), the force  $F$  does work if the particle is moved in any direction. Therefore, the work done by a series of forces is the algebraic sum of the work done by each force.

## 4.2 PRINCIPLE OF VIRTUAL WORK

The establishment of the principle is carried out in stages. First, we consider a particle, then a rigid body, and finally a deformable body, which is the practical application we require when analyzing structures.

### 4.2.1 Principle of virtual work for a particle

In Fig. 4.2, a particle, A, is acted upon by a number of concurrent forces,  $F_1, F_2, \dots, F_k, \dots, F_r$ ; the resultant of these forces is  $R$ . Suppose that the particle is given a small arbitrary displacement,  $\Delta_v$ , to  $A'$  in some specified direction;  $\Delta_v$  is an imaginary or *virtual* displacement and is sufficiently small so that the directions of  $F_1, F_2$ , etc., are unchanged. Let  $\theta_R$  be the angle that the resultant,  $R$ , of the forces makes with the direction of  $\Delta_v$  and  $\theta_1, \theta_2, \dots, \theta_k, \dots, \theta_r$  the angles that  $F_1, F_2, \dots, F_k, \dots, F_r$  make with the direction of  $\Delta_v$ , respectively. Then, from either of Eqs. (4.1) or (4.2), the total virtual work,  $W_F$ , done by the forces  $F$  as the particle moves through the virtual displacement,  $\Delta_v$ , is given by

$$W_F = F_1\Delta_v \cos \theta_1 + F_2\Delta_v \cos \theta_2 + \dots + F_k\Delta_v \cos \theta_k + \dots + F_r\Delta_v \cos \theta_r$$

Therefore,

$$W_F = \sum_{k=1}^r F_k \Delta_v \cos \theta_k$$

or, since  $\Delta_v$  is a fixed, although imaginary, displacement,

$$W_F = \Delta_v \sum_{k=1}^r F_k \cos \theta_k \quad (4.4)$$

In Eq. (4.4),  $\sum_{k=1}^r F_k \cos \theta_k$  is the sum of all the components of the forces,  $F$ , in the direction of  $\Delta_v$  and therefore must be equal to the component of the resultant,  $R$ , of the forces,  $F$ , in the direction of  $\Delta_v$ ; that is,

$$W_F = \Delta_v \sum_{k=1}^r F_k \cos \theta_k = \Delta_v R \cos \theta_R \quad (4.5)$$

If the particle, A, is in equilibrium under the action of the forces,  $F_1, F_2, \dots, F_k, \dots, F_r$ , the resultant,  $R$ , of the forces is zero. It follows from Eq. (4.5) that the virtual work done by the forces,  $F$ , during the virtual displacement,  $\Delta_v$ , is zero.

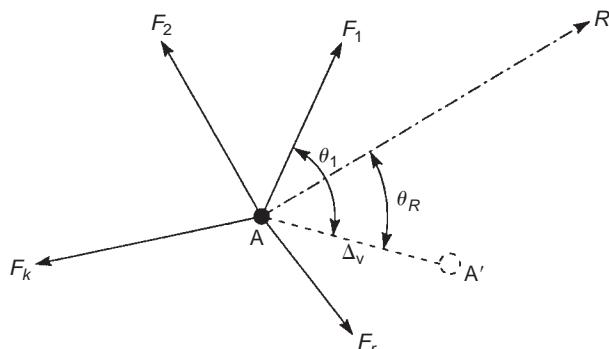


FIGURE 4.2 Virtual Work for a System of Forces Acting on a Particle

We can therefore state the *principle of virtual work* for a particle as follows:

*If a particle is in equilibrium under the action of a number of forces, the total work done by the forces for a small arbitrary displacement of the particle is zero.*

It is possible for the total work done by the forces to be zero even though the particle is not in equilibrium, if the virtual displacement is taken to be in a direction perpendicular to their resultant,  $R$ . We cannot, therefore, state the converse of this principle unless we specify that the total work done must be zero for *any* arbitrary displacement. Thus,

*A particle is in equilibrium under the action of a system of forces if the total work done by the forces is zero for any virtual displacement of the particle.*

Note that in this,  $\Delta_v$  is a purely imaginary displacement and is not related in any way to the possible displacement of the particle under the action of the forces,  $F$ . The virtual displacement  $\Delta_v$  has been introduced purely as a device for setting up the work-equilibrium relationship of Eq. (4.5). The forces,  $F$ , therefore remain unchanged in magnitude and direction during this imaginary displacement; this would not be the case if the displacement were real.

#### 4.2.2 Principle of virtual work for a rigid body

Consider the rigid body shown in Fig. 4.3, which is acted upon by a system of external forces,  $F_1, F_2, \dots, F_k, \dots, F_r$ . These external forces induce internal forces in the body, which may be regarded as comprising an infinite number of particles; on adjacent particles, such as  $A_1$  and  $A_2$ , these internal forces are equal and opposite; in other words, self-equilibrating. Suppose now that the rigid body is given a small, imaginary—that is, virtual—displacement,  $\Delta_v$  (or a rotation or a combination of both) in some specified direction. The external and internal forces then do virtual work and the total virtual work done,  $W_t$ , is the sum of the virtual work,  $W_e$ , done by the external forces and the virtual work,  $W_i$ , done by the internal forces. Thus,

$$W_t = W_e + W_i \quad (4.6)$$

Since the body is rigid, all the particles in the body move through the same displacement,  $\Delta_v$ , so that the virtual work done on all the particles is numerically the same. However, for a pair of adjacent particles,

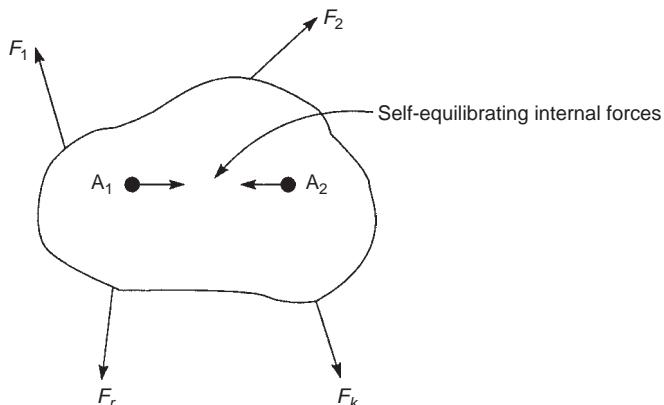


FIGURE 4.3 Virtual Work for a Rigid Body

such as  $A_1$  and  $A_2$  in Fig. 4.3, the self-equilibrating forces are in opposite directions, which means that the work done on  $A_1$  is opposite in sign to the work done on  $A_2$ . Therefore, the sum of the virtual work done on  $A_1$  and  $A_2$  is zero. The argument can be extended to the infinite number of pairs of particles in the body, from which we conclude that the internal virtual work produced by a virtual displacement in a rigid body is zero. Equation (4.6) then reduces to

$$W_t = W_e \quad (4.7)$$

Since the body is rigid and the internal virtual work is therefore zero, we may regard the body as a large particle. It follows that if the body is in equilibrium under the action of a set of forces,  $F_1, F_2, \dots, F_k, \dots, F_r$ , the total virtual work done by the external forces during an arbitrary virtual displacement of the body is zero.

### Example 4.1

Calculate the support reactions in the cantilever beam AB shown in Fig. 4.4(a).

The concentrated load,  $W$ , induces a vertical reaction,  $R_A$ , and also one of moment,  $M_A$ , at A.

Suppose that the beam is given a small imaginary—that is, virtual—rotation,  $\theta_{v,A}$ , at A, as shown in Fig. 4.4(b). Since we are concerned here with only external forces, we may regard the beam as a rigid body, so that the beam remains straight and B is displaced to  $B'$ . The vertical displacement of B,  $\Delta_{v,B}$ , is then given by

$$\Delta_{v,B} = \theta_{v,A}L$$

or

$$\theta_{v,A} = \Delta_{v,B}/L \quad (i)$$

The total virtual work,  $W_t$ , done by all the forces acting on the beam is given by

$$W_t = W\Delta_{v,B} - M_A\theta_{v,A} \quad (ii)$$

Note that the contribution of  $M_A$  to the total virtual work done is negative, since the assumed direction of  $M_A$  is in the opposite sense to the virtual displacement,  $\theta_{v,A}$ . Note also that there is no linear movement of the beam at A so that  $R_A$  does no work. Substituting in Eq. (ii) for  $\theta_{v,A}$  from Eq. (i), we have

$$W_t = W\Delta_{v,B} - M_A\Delta_{v,B}/L \quad (iii)$$

Since the beam is in equilibrium,  $W_t = 0$ , from the principle of virtual work. Therefore,

$$0 = W\Delta_{v,B} - M_A\Delta_{v,B}/L$$

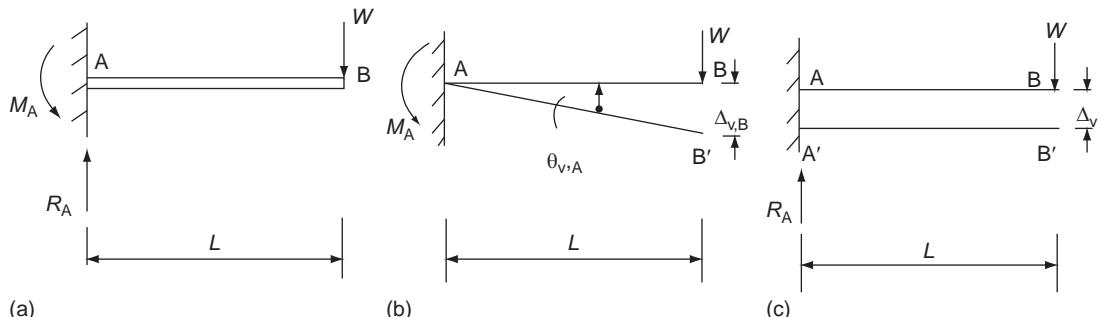


FIGURE 4.4 Use of the Principle of Virtual Work to Calculate Support Reactions

so that

$$M_A = WL$$

which is the result that would have been obtained from considering the moment equilibrium of the beam about A.

Suppose now that the complete beam is given a virtual displacement,  $\Delta_v$ , as shown in Fig. 4.4(c). There is no rotation of the beam, so that  $M_A$  does no work. The total virtual work done is then given by

$$W_t = W\Delta_v - R_A\Delta_v \quad (\text{iv})$$

The contribution of  $R_A$  is negative, since its line of action is in the direction opposite to  $\Delta_v$ . The beam is in equilibrium, so that  $W_t = 0$ . Therefore, from Eq. (iv),

$$R_A = W$$

which is the result we would have obtained by resolving forces vertically. ■

### Example 4.2

Calculate the support reactions in the cantilever beam shown in Fig. 4.5(a).

In this case, we obtain a solution by simultaneously giving the beam a virtual displacement,  $\Delta_{v,A}$ , at A and a virtual rotation,  $\theta_{v,A}$ , at A. The total deflection at B is then  $\Delta_{v,A} + \theta_{v,A}L$  and at a distance  $x$  from A is  $\Delta_{v,A} + \theta_{v,A}x$ .

Since the beam carries a uniformly distributed load, we find the virtual work done by the load by first considering an elemental length,  $\delta x$ , of the load a distance  $x$  from A. The load on the element is  $w\delta x$  and the virtual work done by this elemental load is given by

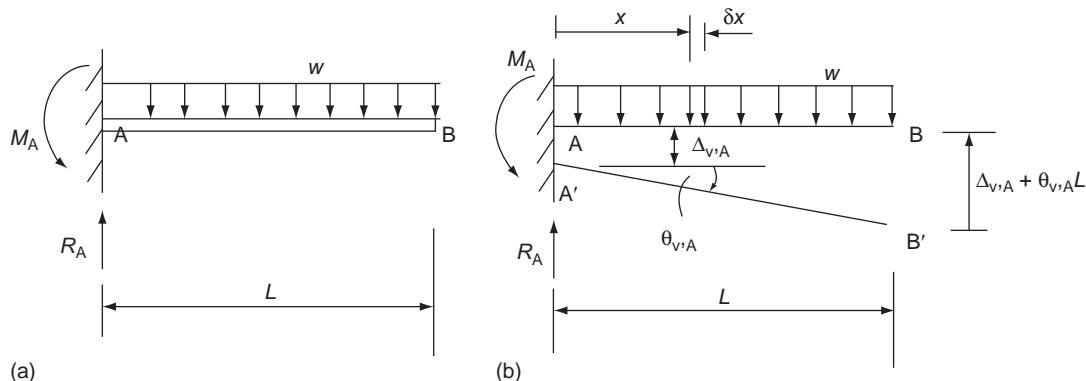
$$\delta W = w\delta x(\Delta_{v,A} + \theta_{v,A}x)$$

The total virtual work done on the beam is given by

$$W_t = \int_0^L w(\Delta_{v,A} + \theta_{v,A}x)dx - M_A\theta_{v,A} - R_A\Delta_{v,A}$$

which simplifies to

$$W_t = (wL - R_A)\Delta_{v,A} + [(wL^2/2) - M_A]\theta_{v,A} = 0 \quad (\text{i})$$



**FIGURE 4.5 Calculation of Support Reactions Using the Principle of Virtual Work**

since the beam is in equilibrium. [Equation \(i\)](#) is valid for all values of  $\Delta_{v,A}$  and  $\theta_{v,A}$ , so that

$$wL - R_A = 0 \text{ and } (wL^2/2) - M_A = 0$$

Therefore,

$$R_A = wL \text{ and } M_A = wL^2/2$$

the results that would have been obtained by resolving forces vertically and taking moments about A.

### Example 4.3

Calculate the reactions at the built-in end of the cantilever beam shown in [Fig. 4.6](#).

In this example the load,  $W$ , produces reactions of vertical force, moment, and torque at the built-in end. The vertical force and moment are the same as in [Example 4.1](#). To determine the torque reaction, we impose a small, virtual displacement,  $\Delta_{v,C}$ , vertically downward at C. This causes the beam AB to rotate as a rigid body through an angle,  $\theta_{v,AB}$ , which is given by

$$\theta_{v,AB} = \Delta_{v,C}/a \quad (\text{i})$$

Alternatively, we could have imposed a small virtual rotation,  $\theta_{v,AB}$ , on the beam, which would have resulted in a virtual vertical displacement of C equal to  $a\theta_{v,AB}$ ; clearly the two approaches produce identical results.

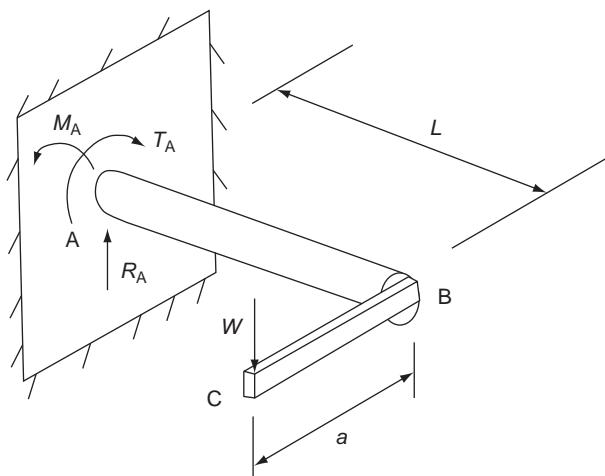
The total virtual work done on the beam is then given by

$$W_t = W\Delta_{v,C} - T_A \theta_{v,AB} = 0 \quad (\text{ii})$$

since the beam is in equilibrium. Substituting for  $\theta_{v,AB}$ , in [Eq. \(ii\)](#) from [Eq. \(i\)](#), we have

$$T_A = Wa$$

which is the result that would have been obtained by considering the statical equilibrium of the beam.



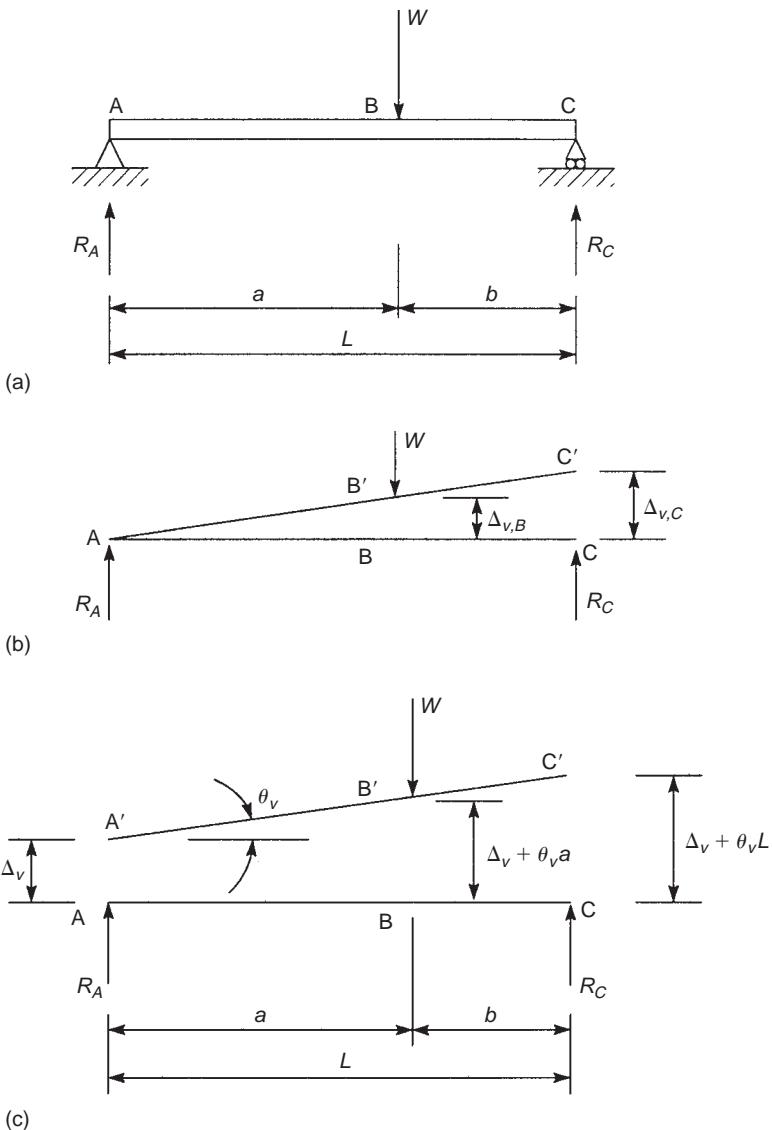
**FIGURE 4.6 Beam of Example 4.3**

**Example 4.4**

Calculate the support reactions in the simply supported beam shown in Fig. 4.7.

Only a vertical load is applied to the beam, so that only vertical reactions,  $R_A$  and  $R_C$ , are produced.

Suppose that the beam at C is given a small imaginary—that is, a virtual—displacement,  $\Delta_{v,C}$ , in the direction of  $R_C$  as shown in Fig. 4.7(b). Since we are concerned here solely with the *external* forces acting on the beam, we



**FIGURE 4.7 Use of the Principle of Virtual Work to Calculate Support Reactions**

may regard the beam as a rigid body. The beam therefore rotates about A so that C moves to C' and B moves to B'. From similar triangles, we see that

$$\Delta_{v,B} = \frac{a}{a+b} \Delta_{v,C} = \frac{a}{L} \Delta_{v,C} \quad (\text{i})$$

The total virtual work,  $W_t$ , done by all the forces acting on the beam is given by

$$W_t = R_C \Delta_{v,C} - W \Delta_{v,B} \quad (\text{ii})$$

Note that the work done by the load,  $W$ , is negative, since  $\Delta_{v,B}$  is in the opposite direction to its line of action. Note also that the support reaction,  $R_A$ , does no work, since the beam rotates only about A. Now substituting for  $\Delta_{v,B}$  in Eq. (ii) from Eq. (i), we have

$$W_t = R_C \Delta_{v,C} - W \frac{a}{L} \Delta_{v,C} \quad (\text{iii})$$

Since the beam is in equilibrium,  $W_t$  is zero from the principle of virtual work. Hence, from Eq. (iii),

$$R_C \Delta_{v,C} - W \frac{a}{L} \Delta_{v,C} = 0$$

which gives

$$R_C = W \frac{a}{L}$$

which is the result that would have been obtained from a consideration of the moment equilibrium of the beam about A. The determination of  $R_A$  follows in a similar manner. Suppose now that, instead of the single displacement  $\Delta_{v,C}$ , the complete beam is given a vertical virtual displacement,  $\Delta_v$ , together with a virtual rotation,  $\theta_v$ , about A, as shown in Fig. 4.7(c). The total virtual work,  $W_t$ , done by the forces acting on the beam is now given by

$$W_t = R_A \Delta_v - W (\Delta_v + a\theta_v) + R_C (\Delta_v + L\theta_v) = 0 \quad (\text{iv})$$

since the beam is in equilibrium. Rearranging Eq. (iv),

$$(R_A + R_C - W)\Delta_v + (R_C L - Wa)\theta_v = 0 \quad (\text{v})$$

Equation (v) is valid for all values of  $\Delta_v$  and  $\theta_v$  so that

$$R_A + R_C - W = 0, \quad R_C L - Wa = 0$$

which are the equations of equilibrium we would have obtained by resolving forces vertically and taking moments about A.

It is not being suggested here that the application of the principles of statics should be abandoned in favor of the principle of virtual work. The purpose of Examples 4.1–4.4 is to illustrate the application of a virtual displacement and the manner in which the principle is used.

### 4.2.3 Virtual work in a deformable body

In structural analysis, we are not generally concerned with forces acting on a rigid body. Structures and structural members deform under load, which means that if we assign a virtual displacement to a particular point in a structure, not all points in the structure suffer the same virtual displacement, as would be the case if the structure were rigid. This means that the virtual work produced by the internal forces is

not zero, as it is in the rigid body case, since the virtual work produced by the self-equilibrating forces on adjacent particles does not cancel out. The total virtual work produced by applying a virtual displacement to a deformable body acted upon by a system of external forces is therefore given by Eq. (4.6).

If the body is in equilibrium under the action of the external force system, then every particle in the body is also in equilibrium. Therefore, from the principle of virtual work, the virtual work done by the forces acting on the particle is zero, irrespective of whether the forces are external or internal. It follows that since the virtual work is zero for all particles in the body, it is zero for the complete body, and Eq. (4.6) becomes

$$W_e + W_i = 0 \quad (4.8)$$

Note that in this argument, only the conditions of equilibrium and the concept of work are employed. Equation (4.8) therefore does not require the deformable body to be linearly elastic (i.e., it need not obey Hooke's law), so that the principle of virtual work may be applied to any body or structure that is rigid, elastic, or plastic. The principle does require that displacements, whether real or imaginary, must be small, so that we may assume that external and internal forces are unchanged in magnitude and direction during the displacements. In addition, the virtual displacements must be compatible with the geometry of the structure and the constraints that are applied, such as those at a support. The exception is the situation we have in Examples 4.1–4.4, where we apply a virtual displacement at a support. This approach is valid, since we include the work done by the support reactions in the total virtual work equation.

#### 4.2.4 Work done by internal force systems

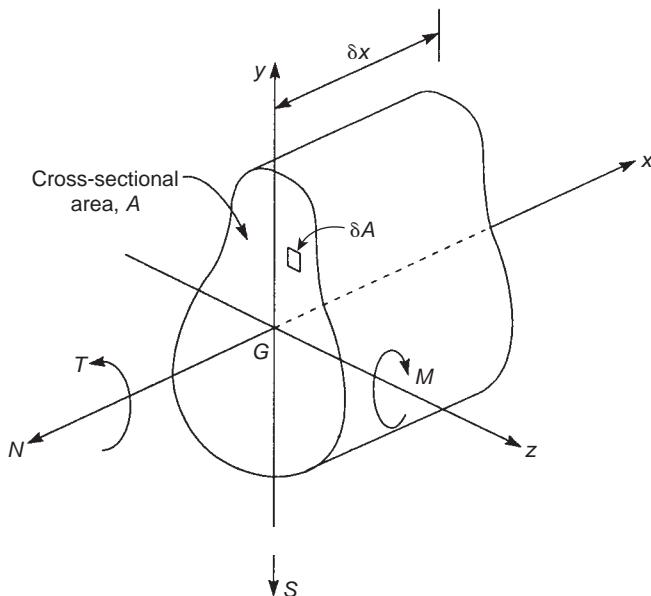
The calculation of the work done by an external force is straightforward in that it is the product of the force and the displacement of its point of application in its own line of action (Eqs. (4.1), (4.2), or (4.3)), whereas the calculation of the work done by an internal force system during a displacement is much more complicated. Generally, no matter how complex a loading system is, it may be simplified to a combination of up to four load types: axial load, shear force, bending moment, and torsion; these in turn produce corresponding internal force systems. We now consider the work done by these internal force systems during arbitrary virtual displacements.

##### **Axial force**

Consider the elemental length,  $\delta x$ , of a structural member as shown in Fig. 4.8 and suppose that it is subjected to a positive internal force system comprising a normal force (i.e., axial force),  $N$ ; a shear force,  $S$ ; a bending moment,  $M$ ; and a torque,  $T$ , produced by some external loading system acting on the structure of which the member is part. The stress distributions corresponding to these internal forces are related to an axis system whose origin coincides with the centroid of area of the cross-section. We are, in fact, using these stress distributions in the derivation of expressions for internal virtual work in linearly elastic structures, so that it is logical to assume the same origin of axes here; we also assume that the  $y$  axis is an axis of symmetry. Initially, we consider the normal force,  $N$ .

The direct stress,  $\sigma$ , at any point in the cross-section of the member is given by  $\sigma = N/A$ . Therefore the normal force on the element  $\delta A$  at the point  $(z, y)$  is

$$\delta N = \sigma \delta A = \frac{N}{A} \delta A$$



**FIGURE 4.8 Virtual Work Due to Internal Force System**

Suppose now that the structure is given an arbitrary virtual displacement, which produces a virtual axial strain,  $\varepsilon_v$ , in the element. The internal virtual work,  $\delta w_{i,N}$ , done by the axial force on the elemental length of the member is given by

$$\delta w_{i,N} = \int_A \frac{N}{A} dA \varepsilon_v \delta x$$

which, since  $\int_A dA = A$ , reduces to

$$\delta w_{i,N} = N \varepsilon_v \delta x \quad (4.9)$$

In other words, the virtual work done by  $N$  is the product of  $N$  and the virtual axial displacement of the element of the member. For a member of length  $L$ , the virtual work,  $w_{i,N}$ , done during the arbitrary virtual strain is then

$$w_{i,N} = \int_L N \varepsilon_v dx \quad (4.10)$$

For a structure comprising a number of members, the total internal virtual work,  $W_{i,N}$ , done by axial force is the sum of the virtual work of each of the members. Therefore,

$$w_{i,N} = \sum \int_L N \varepsilon_v dx \quad (4.11)$$

Note that in the derivation of Eq. (4.11), we make no assumption regarding the material properties of the structure, so that the relationship holds for nonelastic as well as elastic materials. However, for a

linearly elastic material—that is, one that obeys Hooke's law—we can express the virtual strain in terms of an equivalent virtual normal force:

$$\varepsilon_v = \frac{\sigma_v}{E} = \frac{N_v}{EA}$$

Therefore, if we designate the *actual* normal force in a member by  $N_A$ , Eq. (4.11) may be expressed in the form

$$w_{i,N} = \sum \int_L \frac{N_A N_v}{EA} dx \quad (4.12)$$

### **Shear force**

The shear force,  $S$ , acting on the member section in Fig. 4.8 produces a distribution of vertical shear stress that depends upon the geometry of the cross-section. However, since the element,  $\delta A$ , is infinitesimally small, we may regard the shear stress,  $\tau$ , as constant over the element. The shear force,  $\delta S$ , on the element is then

$$\delta S = \tau \delta A \quad (4.13)$$

Suppose that the structure is given an arbitrary virtual displacement that produces a virtual shear strain,  $\gamma_v$ , at the element. This shear strain represents the angular rotation in a vertical plane of the element  $\delta A \times \delta x$  relative to the longitudinal centroidal axis of the member. The vertical displacement at the section being considered is therefore  $\gamma_v \delta x$ . The internal virtual work,  $\delta w_{i,S}$ , done by the shear force,  $S$ , on the elemental length of the member is given by

$$\delta w_{i,S} = \int_A \tau dA \gamma_v \delta x$$

A uniform shear stress through the cross-section of a beam may be assumed if we allow for the actual variation by including a form factor,  $\beta$ .<sup>1</sup> The expression for the internal virtual work in the member may then be written

$$\delta w_{i,S} = \int_A \beta \left( \frac{S}{A} \right) dA \gamma_v \delta x$$

or

$$\delta w_{i,S} = \beta S \gamma_v \delta x \quad (4.14)$$

Hence, the virtual work done by the shear force during the arbitrary virtual strain in a member of length  $L$  is

$$w_{i,S} = \beta \int_L S \gamma_v dx \quad (4.15)$$

For a linearly elastic member, as in the case of axial force, we may express the virtual shear strain,  $\gamma_v$ , in terms of an equivalent virtual shear force,  $S_v$ :

$$\gamma_v = \frac{\tau_v}{G} = \frac{S_v}{GA}$$

so that, from Eq. (4.15),

$$w_{i,S} = \beta \int_L \frac{S_A S_V}{GA} dx \quad (4.16)$$

For a structure comprising a number of linearly elastic members, the total internal work,  $W_{i,S}$ , done by the shear forces is

$$W_{i,S} = \sum \beta \int_L \frac{S_A S_V}{GA} dx \quad (4.17)$$

### Bending moment

The bending moment,  $M$ , acting on the member section in Fig. 4.8 produces a distribution of direct stress,  $\sigma$ , through the depth of the member cross-section. The normal force on the element,  $\delta A$ , corresponding to this stress is therefore  $\sigma \delta A$ . Again, we suppose that the structure is given a small arbitrary virtual displacement that produces a virtual direct strain,  $\varepsilon_V$ , in the element  $\delta A \times \delta x$ . Therefore, the virtual work done by the normal force acting on the element  $\delta A$  is  $\sigma \delta A \varepsilon_V \delta x$ . Hence, integrating over the complete cross-section of the member, we obtain the internal virtual work,  $\delta w_{i,M}$ , done by the bending moment,  $M$ , on the elemental length of member:

$$\delta w_{i,M} = \int_A \sigma dA \varepsilon_V \delta x \quad (4.18)$$

The virtual strain,  $\varepsilon_V$ , in the element  $\delta A \times \delta x$  is, from Eq. (16.1), given by

$$\varepsilon_V = \frac{y}{R_V}$$

where  $R_V$  is the radius of curvature of the member produced by the virtual displacement. Thus, substituting for  $\varepsilon_V$  in Eq. (4.18), we obtain

$$\delta w_{i,M} = \int_A \sigma \frac{y}{R_V} dA \delta x$$

or, since  $\sigma y \delta A$  is the moment of the normal force on the element,  $\delta A$ , about the  $z$  axis,

$$\delta w_{i,M} = \frac{M}{R_V} \delta x$$

Therefore, for a member of length  $L$ , the internal virtual work done by an actual bending moment,  $M_A$ , is given by

$$w_{i,M} = \int_L \frac{M}{R_V} dx \quad (4.19)$$

In the derivation of Eq. (4.19), no specific stress-strain relationship has been assumed, so that it is applicable to a nonlinear system. For the particular case of a linearly elastic system, the virtual curvature  $1/R_V$  may be expressed in terms of an equivalent virtual bending moment,  $M_V$ , using the relationship of Eq. (16.8):

$$\frac{1}{R_V} = \frac{M_V}{EI}$$

Substituting for  $1/R_v$  in Eq. (4.19), we have

$$w_{i,M} = \int_L \frac{M_A M_v}{EI} dx \quad (4.20)$$

so that, for a structure comprising a number of members, the total internal virtual work,  $W_{i,M}$ , produced by bending is

$$W_{i,M} = \sum \int_L \frac{M_A M_v}{EI} dx \quad (4.21)$$

### Torsion

The internal virtual work,  $w_{i,T}$ , due to torsion in the particular case of a linearly elastic circular section bar may be found in a similar manner and is given by

$$w_{i,T} = \int_L \frac{T_A T_v}{GI_o} dx \quad (4.22)$$

in which  $I_o$  is the polar second moment of area of the cross-section of the bar (see Example 3.1). For beams of a noncircular cross-section,  $I_o$  is replaced by a torsion constant,  $J$ , which, for many practical beam sections, is determined empirically.

### Hinges

In some cases, it is convenient to impose a virtual rotation,  $\theta_v$ , at some point in a structural member where, say, the actual bending moment is  $M_A$ . The internal virtual work done by  $M_A$  is then  $M_A \theta_v$  (see Eq. (4.3)); physically this situation is equivalent to inserting a hinge at the point.

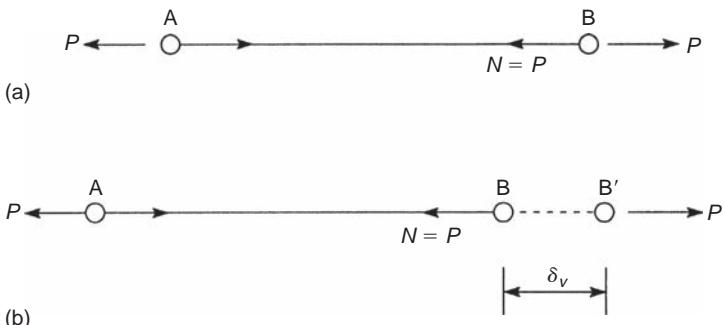
### Sign of internal virtual work

So far, we have derived expressions for internal work without considering whether it is positive or negative in relation to external virtual work.

Suppose that the structural member, AB, in Fig. 4.9(a) is a member of a truss and that it is in equilibrium under the action of two externally applied axial tensile loads,  $P$ ; clearly the internal axial—that is, normal—force at any section of the member is  $P$ . Suppose now that the member is given a virtual extension,  $\delta_v$ , such that B moves to  $B'$ . Then, the virtual work done by the applied load,  $P$ , is positive, since the displacement,  $\delta_v$ , is in the same direction as its line of action. However, the virtual work done by the internal force,  $N (= P)$ , is negative, since the displacement of B is in the opposite direction to its line of action; in other words, work is done *on* the member. Thus, from Eq. (4.8), we see that in this case,

$$W_e = W_i \quad (4.23)$$

Equation (4.23) applies if the virtual displacement is a contraction and not an extension, in which case the signs of the external and internal virtual work in Eq. (4.8) are reversed. Clearly this applies equally if  $P$  is a compressive load. The arguments may be extended to structural members subjected to shear, bending, and torsional loads, so that Eq. (4.23) is generally applicable.



**FIGURE 4.9 Sign of the Internal Virtual Work in an Axially Loaded Member**

#### 4.2.5 Virtual work due to external force systems

So far in our discussion, we have only considered the virtual work produced by externally applied concentrated loads. For completeness, we must also consider the virtual work produced by moments, torques, and distributed loads.

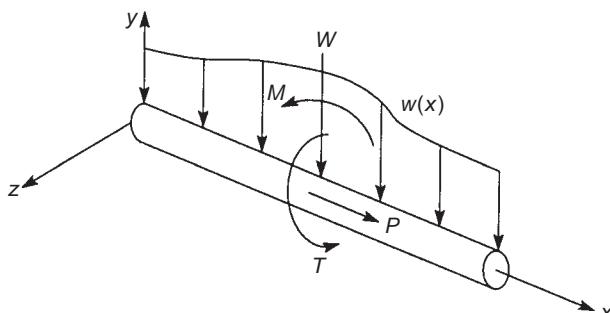
In Fig. 4.10, a structural member carries a distributed load,  $w(x)$ , and, at a particular point, a concentrated load,  $W$ ; a moment,  $M$ ; and a torque,  $T$ ; together with an axial force,  $P$ . Suppose that, at the point, a virtual displacement is imposed having translational components,  $\Delta_{v,y}$  and  $\Delta_{v,x}$ , parallel to the  $y$  and  $x$  axes, respectively, and rotational components,  $\theta_v$  and  $\phi_v$ , in the  $yx$  and  $zy$  planes, respectively.

If we consider a small element,  $\delta x$ , of the member at the point, the distributed load may be regarded as constant over the length  $\delta x$  and acting, in effect, as a concentrated load  $w(x)\delta x$ . The virtual work,  $w_e$ , done by the complete external force system is therefore given by

$$w_e = W\Delta_{v,y} + P\Delta_{v,x} + M\theta_v + T\phi_v + \int_L w(x)\Delta_{v,y} \, dx$$

For a structure comprising a number of load positions, the total external virtual work done is then

$$W_e = \sum \left[ W\Delta_{v,y} + P\Delta_{v,x} + M\theta_v + T\phi_v + \int_L w(x)\Delta_{v,y} \, dx \right] \quad (4.24)$$



**FIGURE 4.10 Virtual Work Due to Externally Applied Loads**

In Eq. (4.24), a complete set of external loads need not be applied at every loading point so, in fact, the summation is for the appropriate number of loads. Further, the virtual displacements in the equation are related to forces and moments applied in a vertical plane. We could, of course, have forces and moments and components of the virtual displacement in a horizontal plane, in which case Eq. (4.24) is extended to include their contribution.

The internal virtual work equivalent of Eq. (4.24) for a linear system is, from Eqs. (4.12), (4.17), (4.21), and (4.22),

$$W_i = \sum \left[ \int_L \frac{N_A N_v}{EA} dx + \beta \int_L \frac{S_A S_v}{GA} dx + \int_L \frac{M_A M_v}{EI} dx + \int_L \frac{T_A T_v}{GJ} dx + M_A \theta_v \right] \quad (4.25)$$

in which the last term on the right-hand side is the virtual work produced by an actual internal moment at a hinge (see previous text). Note that the summation in Eq. (4.25) is taken over all the *members* of the structure.

#### 4.2.6 Use of virtual force systems

So far, in all the structural systems we have considered, virtual work is produced by actual forces moving through imposed virtual displacements. However, the actual forces are not related to the virtual displacements in any way, since, as we have seen, the magnitudes and directions of the actual forces are unchanged by the virtual displacements so long as the displacements are small. Thus, the principle of virtual work applies for *any* set of forces in equilibrium and *any* set of displacements. Equally, therefore, we could specify that the forces are a set of virtual forces *in equilibrium* and that the displacements are actual displacements. Therefore, instead of relating actual external and internal force systems through virtual displacements, we can relate actual external and internal displacements through virtual forces.

If we apply a virtual force system to a deformable body, it induces an internal virtual force system that moves through the actual displacements; internal virtual work therefore is produced. In this case, for example, Eq. (4.10) becomes

$$w_{i,N} = \int_L N_v \epsilon_A dx$$

in which  $N_v$  is the internal virtual normal force and  $\epsilon_A$  is the actual strain. Then, for a linear system, in which the actual internal normal force is  $N_A$ ,  $\epsilon_A = N_A/EA$ , so that, for a structure comprising a number of members, the total internal virtual work due to a virtual normal force is

$$W_{i,N} = \sum \int_L \frac{N_v N_A}{EA} dx$$

which is identical to Eq. (4.12). Equations (4.17), (4.21), and (4.22) may be shown to apply to virtual force systems in a similar manner.

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### 4.3 APPLICATIONS OF THE PRINCIPLE OF VIRTUAL WORK

We have now seen that the principle of virtual work may be used in the form of either imposed virtual displacements or imposed virtual forces. Generally, the former approach, as we saw in Example 4.4, is used to determine forces, while the latter is used to obtain displacements.

For statically determinate structures, the use of virtual displacements to determine force systems is a relatively trivial use of the principle, although problems of this type provide a useful illustration of the method. The real power of this approach lies in its application to the solution of statically indeterminate structures. However, the use of virtual forces is particularly useful in determining actual displacements of structures. We shall illustrate both approaches by examples.

### Example 4.5

Determine the bending moment at the point B in the simply supported beam ABC shown in Fig. 4.11(a).

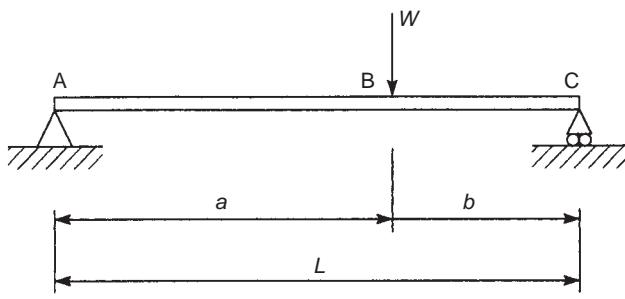
We determined the support reactions for this particular beam in Example 4.4. In this example, however, we are interested in the actual internal moment,  $M_B$ , at the point of application of the load. We must therefore impose a virtual displacement that relates the internal moment at B to the applied load and excludes other unknown external forces, such as the support reactions, and unknown internal force systems, such as the bending moment distribution along the length of the beam. Therefore, if we imagine that the beam is hinged at B and that the lengths AB and BC are rigid, a virtual displacement,  $\Delta_{v,B}$ , at B results in the displaced shape shown in Fig. 4.11(b).

Note that the support reactions at A and C do no work and that the internal moments in AB and BC do no work because AB and BC are rigid links. From Fig. 4.11(b),

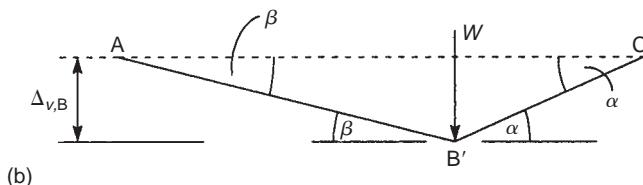
$$\Delta_{v,B} = a\beta = b\alpha \quad (i)$$

Hence,

$$\alpha = \frac{a}{b}\beta$$



(a)



(b)

FIGURE 4.11 Determination of Bending Moment at a Point in the Beam of Example 4.5 Using Virtual Work

and the angle of rotation of BC relative to AB is then

$$\theta_B = \beta + \alpha = \beta \left(1 + \frac{a}{b}\right) = \frac{L}{b} \beta \quad (\text{ii})$$

Now, equating the external virtual work done by  $W$  to the internal virtual work done by  $M_B$  (see Eq. (4.23)), we have

$$W\Delta_{v,B} = M_B\theta_B \quad (\text{iii})$$

Substituting in Eq. (iii) for  $\Delta_{v,B}$  from Eq. (i) and for  $\theta_B$  from Eq. (ii), we have

$$Wa\beta = M_B \frac{L}{b} \beta$$

which gives

$$M_B = \frac{Wab}{L}$$

which is the result we would have obtained by calculating the moment of  $R_C$  ( $= Wa/L$  from Example 4.4) about B.

### Example 4.6

Determine the force in the member AB of the truss shown in Fig. 4.12(a).

We are required to calculate the force in the member AB, so that again we need to relate this internal force to the externally applied loads without involving the internal forces in the remaining members of the truss. We therefore impose a virtual extension,  $\Delta_{v,B}$ , at B in the member AB, such that B moves to  $B'$ . If we assume that the remaining members are rigid, the forces in them do no work. Further, the triangle BCD rotates as a rigid body about D to  $B'C'D$ , as shown in Fig. 4.12(b). The horizontal displacement of C,  $\Delta_C$ , is then given by

$$\Delta_C = 4\alpha$$

while

$$\Delta_{v,B} = 3\alpha$$

Hence,

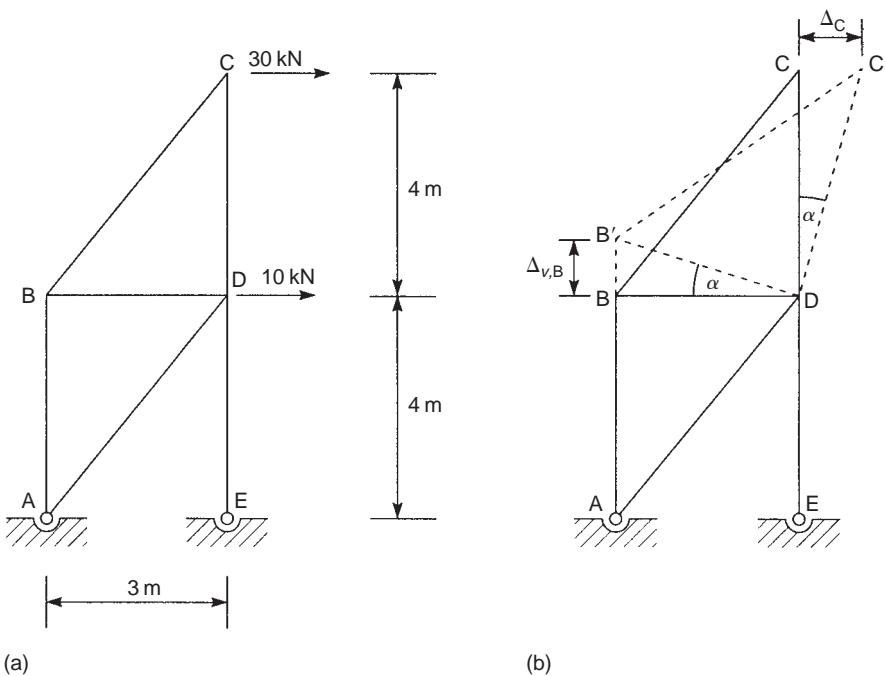
$$\Delta_C = \frac{4\Delta_{v,B}}{3} \quad (\text{i})$$

Equating the external virtual work done by the 30 kN load to the internal virtual work done by the force,  $F_{BA}$ , in the member, AB, we have (see Eq. (4.23) and Fig. 4.9)

$$30\Delta_C = F_{BA}\Delta_{v,B} \quad (\text{ii})$$

Substituting for  $\Delta_C$  from Eq. (i) in Eq. (ii),

$$30 \times \frac{4}{3} \Delta_{v,B} = F_{BA}\Delta_{v,B}$$



**FIGURE 4.12 Determination of the Internal Force in a Member of a Truss Using Virtual Work**

from which

$$F_{BA} = +40 \text{ kN}$$

In the preceding, we are, in effect, assigning a positive (that is,  $F_{BA}$  is tensile) sign to  $F_{BA}$  by imposing a virtual extension on the member AB.

The actual sign of  $F_{BA}$  is then governed by the sign of the external virtual work. Thus, if the 30 kN load were in the opposite direction to  $\Delta_C$ , the external work done would have been negative, so that  $F_{BA}$  would be negative and therefore compressive. This situation can be verified by inspection. Alternatively, for the loading shown in Fig. 4.12(a), a contraction in AB implies that  $F_{BA}$  is compressive. In this case, DC would have rotated in a counterclockwise sense,  $\Delta_C$  would have been in the opposite direction to the 30 kN load so that the external virtual work done would be negative, resulting in a negative value for the compressive force  $F_{BA}$ ;  $F_{BA}$  would therefore be tensile, as before. Note also that the 10 kN load at D does no work, since D remains undisplaced. ■

We now consider problems involving the use of virtual forces. Generally, we require the displacement of a particular point in a structure, so that if we apply a virtual force to the structure at the point and in the direction of the required displacement, the external virtual work done will be the product of the virtual force and the actual displacement, which may then be equated to the internal virtual work produced by the internal virtual force system moving through actual displacements. Since the choice of the virtual force is arbitrary, we may give it any convenient value; the simplest type of virtual force is therefore a unit load, and the method then becomes the *unit load method* (see also Section 5.5).

**Example 4.7**

Determine the vertical deflection of the free end of the cantilever beam shown in Fig. 4.13(a).

Let us suppose that the actual deflection of the cantilever at B produced by the uniformly distributed load is  $v_B$  and that a vertically downward virtual unit load is applied at B before the actual deflection takes place. The external virtual work done by the unit load is, from Fig. 4.13(b),  $1v_B$ . The deflection,  $v_B$ , is assumed to be caused by bending only; that is, we ignore any deflections due to shear. The internal virtual work is given by Eq. (4.21), which, since only one member is involved, becomes

$$W_{i,M} = \int_0^L \frac{M_A M_v}{EI} dx \quad (i)$$

The virtual moments,  $M_v$ , are produced by a unit load, so we replace  $M_v$  by  $M_1$ . Then,

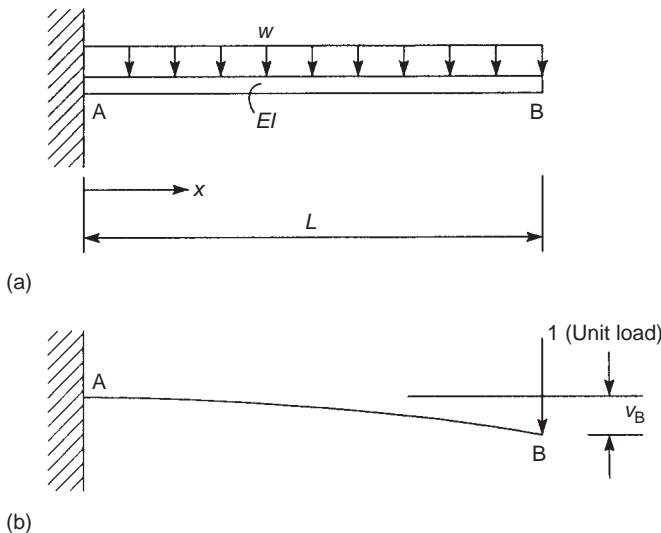
$$W_{i,M} = \int_0^L \frac{M_A M_1}{EI} dx \quad (ii)$$

At any section of the beam a distance  $x$  from the built-in end,

$$M_A = -\frac{w}{2}(L-x)^2, \quad M_1 = -1(L-x)$$

Substituting for  $M_A$  and  $M_1$  in Eq. (ii) and equating the external virtual work done by the unit load to the internal virtual work, we have

$$1 v_B = \int_0^L \frac{w}{2EI} (L-x)^3 dx$$



**FIGURE 4.13 Deflection of the Free End of a Cantilever Beam Using the Unit Load Method**

which gives

$$\int v_B = -\frac{w}{2EI} \left[ \frac{1}{4} (L-x)^4 \right]_0^L$$

so that

$$v_B = \frac{wL^4}{8EI}$$

Note that  $v_B$  is in fact negative, but the positive sign here indicates that it is in the same direction as the unit load.

### Example 4.7 MATLAB

Repeat Example 4.7 using the Symbolic Math Toolbox in MATLAB.

Calculation of the vertical deflection of the free end of the cantilever shown in Fig. 4.13(a) is obtained through the following MATLAB file:

```
% Declare any needed variables
syms M_A M_V W_e W_i E I w x L v_B

% Define the moments due to the applied load (M_A) and the unit virtual load (M_V)
M_A = -0.5*w*(L-x)^2;
M_V = -1*(L-x);

% Define equations for the external (W_e) and internal (W_i) virtual work
W_e = 1*v_B;
W_i = int(M_A*M_V/(E*I),x,0,L); % From Eq. (4.21)

% Equate W_e and W_i, and solve for the free end vertical displacement (v_B)
v_B = solve(W_e-W_i,v_B);

% Output v_B to the Command Window
disp(['v_B = ' char(v_B)])
```

The Command Window output resulting from this MATLAB file is as follows:

```
v_B = (L^4*w)/(8*E*I)
```

### Example 4.8

Determine the rotation—that is, the slope—of the beam ABC shown in Fig. 4.14(a) at A.

The actual rotation of the beam at A produced by the actual concentrated load,  $W$ , is  $\theta_A$ . Let us suppose that a virtual unit moment is applied at A before the actual rotation takes place, as shown in Fig. 4.14(b). The virtual unit moment induces virtual support reactions of  $R_{v,A}$  ( $= 1/L$ ) acting downward and  $R_{v,C}$  ( $= 1/L$ ) acting upward. The actual internal bending moments are

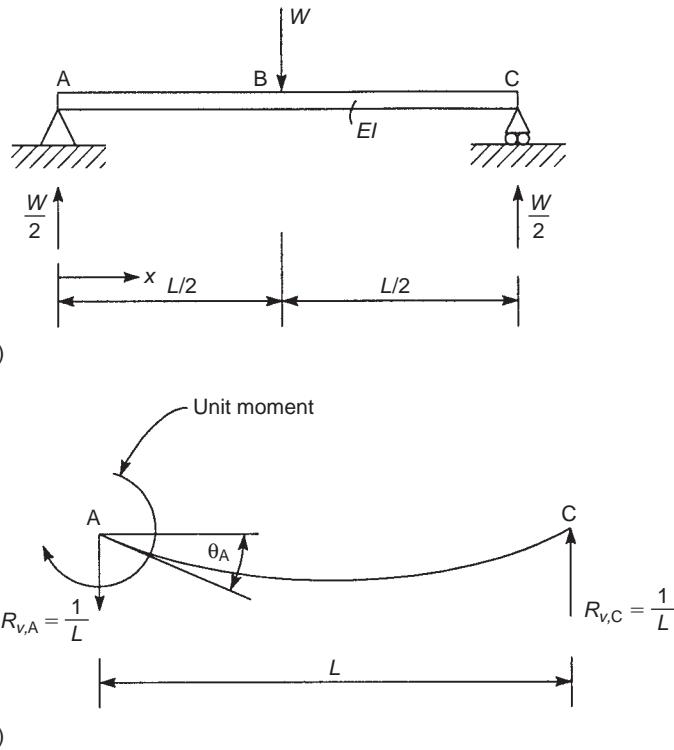


FIGURE 4.14 Determination of the Rotation of a Simply Supported Beam at a Support Using the Unit Load Method

$$M_A = +\frac{W}{2}x, \quad 0 \leq x \leq L/2$$

$$M_A = \frac{W}{2}x - W\left(x - \frac{L}{2}\right) = +\frac{W}{2}(L-x) \quad L/2 \leq x \leq L$$

The internal virtual bending moment is

$$M_v = 1 - \frac{1}{L}x, \quad 0 \leq x \leq L$$

The external virtual work done is  $1\theta_A$  (the virtual support reactions do no work, as there is no vertical displacement of the beam at the supports), and the internal virtual work done is given by Eq. (4.21). Hence,

$$1\theta_A = \frac{1}{EI} \left[ \int_0^{L/2} \frac{W}{2}x \left(1 - \frac{x}{L}\right) dx + \int_{L/2}^L \frac{W}{2}(L-x) \left(1 - \frac{x}{L}\right) dx \right] \quad (i)$$

Simplifying Eq. (i), we have

$$\theta_A = \frac{W}{2EIL} \left[ \int_0^{L/2} (Lx - x^2) dx + \int_{L/2}^L (L-x)^2 dx \right] \quad (ii)$$

Hence,

$$\theta_A = \frac{W}{2EI} \left\{ \left[ L \frac{x^2}{2} - \frac{x^3}{3} \right]_0^{L/2} - \frac{1}{3} \left[ (L-x)^3 \right]_{L/2}^L \right\}$$

from which

$$\theta_A = \frac{WL^2}{16EI}$$

### Example 4.8 MATLAB

Repeat Example 4.8 using the Symbolic Math Toolbox in MATLAB.

Calculation of the slope of the beam shown in Fig. 4.14(a) at A is obtained through the following MATLAB file:

```
% Declare any needed variables
syms M_A M_V W x L theta_A E I

% Define the moments due to the applied load (M_A) and the unit virtual load (M_V)
M_A = [W/2*x; % 0 >= x >= L/2
W/2*(L-x)]; % L/2 >= x >= L
M_V = 1 - x/L; % 0 >= x >= L

% Define equations for the external (W_e) and internal (W_i) virtual work
W_e = 1*theta_A;
W_i = (int(M_A(1)*M_V,x,0,L/2) + int(M_A(2)*M_V,x,L/2,L))/(E*I); % From Eq. (4.21)

% Equate W_e and W_i, and solve for the slope of the beam (theta_A)
theta_A = solve(W_e-W_i,theta_A);

% Output theta_A to the Command Window
disp(['theta_A =' char(theta_A)])
```

The Command Window output resulting from this MATLAB file is as follows:

```
theta_A=(L^2*W)/(16*E*I)
```

### Example 4.9

Calculate the vertical deflection of the joint B and the horizontal movement of the support D in the truss shown in Fig. 4.15(a). The cross-sectional area of each member is 1800 mm<sup>2</sup> and Young's modulus, E, for the material of the members is 200,000 N/mm<sup>2</sup>.

The virtual force systems—that is, unit loads—required to determine the vertical deflection of B and the horizontal deflection of D are shown in Figs. 4.15(b) and (c), respectively. Therefore, if the actual vertical deflection at

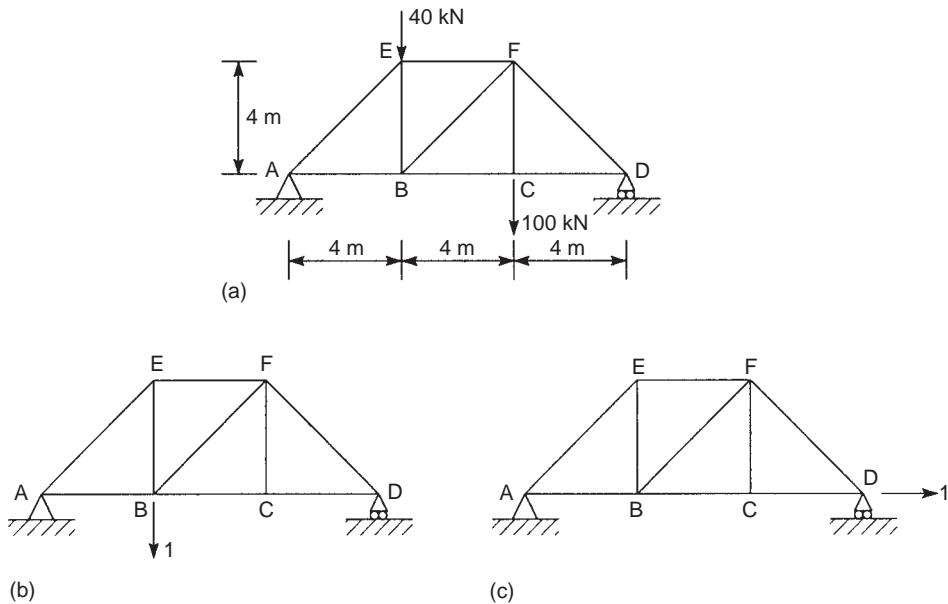


FIGURE 4.15 Deflection of a Truss Using the Unit Load Method

B is  $\delta_{B,v}$  and the horizontal deflection at D is  $\delta_{D,h}$ , the external virtual work done by the unit loads is  $1\delta_{B,v}$  and  $1\delta_{D,h}$ , respectively. The internal actual and virtual force systems constitute axial forces in all the members.

These axial forces are constant along the length of each member, so that for a truss comprising  $n$  members, Eq. (4.12) reduces to

$$W_{i,N} = \sum_{j=1}^n \frac{F_{A,j} F_{v,j} L_j}{E_j A_j} \quad (i)$$

in which  $F_{A,j}$  and  $F_{v,j}$  are the actual and virtual forces in the  $j$ th member, which has a length  $L_j$ , an area of cross-section  $A_j$ , and a Young's modulus  $E_j$ .

Since the forces  $F_{v,j}$  are due to a unit load, we write Eq. (i) in the form

$$W_{i,N} = \sum_{j=1}^n \frac{F_{A,j} F_{1,j} L_j}{E_j A_j} \quad (ii)$$

Also, in this particular example, the area of cross-section,  $A$ , and Young's modulus,  $E$ , are the same for all members, so that it is sufficient to calculate  $\sum_{j=1}^n F_{A,j} F_{1,j} L_j$  and then divide by  $EA$  to obtain  $W_{i,N}$ .

The forces in the members, whether actual or virtual, may be calculated by the method of joints.<sup>1</sup> Note that the support reactions corresponding to the three sets of applied loads (one actual and two virtual) must be calculated before the internal force systems can be determined. However, in Fig. 4.15(c), it is clear from inspection that  $F_{1,AB} = F_{1,BC} = F_{1,CD} = +1$ , while the forces in all other members are zero. The calculations are presented in Table 4.1; note that positive signs indicate tension and negative signs compression.

**Table 4.1 Example 4.9**

Member	$L$ (m)	$F_A$ (kN)	$F_{1,B}$	$F_{1,D}$	$F_A F_{1,B} L$ (kN m)	$F_A F_{1,D} L$ (kN m)
AE	5.7	-84.9	-0.94	0	+451.4	0
AB	4.0	+60.0	+0.67	+1.0	+160.8	+240.0
EF	4.0	-60.0	-0.67	0	+160.8	0
EB	4.0	+20.0	+0.67	0	+53.6	0
BF	5.7	-28.3	+0.47	0	-75.2	0
BC	4.0	+80.0	+0.33	+1.0	+105.6	+320.0
CD	4.0	+80.0	+0.33	+1.0	+105.6	+320.0
CF	4.0	+100.0	0	0	0	0
DF	5.7	-113.1	-0.47	0	+301.0	0
					$\sum = +1263.6$	$\sum = +880.0$

Equating internal and external virtual work done (Eq. (4.23)), we have

$$1\delta_{B,v} = \frac{1263.6 \times 10^6}{200,000 \times 1,800}$$

from which

$$\delta_{B,v} = 3.51 \text{ mm}$$

and

$$1\delta_{D,h} = \frac{880 \times 10^6}{200,000 \times 1,800}$$

which gives

$$\delta_{D,h} = 2.44 \text{ mm}$$

Both deflections are positive, which indicates that the deflections are in the directions of the applied unit loads. Note that in the preceding, it is unnecessary to specify units for the unit load, since the unit load appears, in effect, on both sides of the virtual work equation (the internal  $F_1$  forces are directly proportional to the unit load).

### Example 4.10

Determine the components of the deflection of the point C in the frame shown in Fig. 4.16(a); consider the effect of bending only.

The horizontal and vertical components of the deflection of C may be found by applying unit loads in turn at C, as shown in Figs. 4.16(b) and (c). The internal work done by this virtual force system—that is, the unit loads acting through real displacements—is given by Eq. (4.20), in which, for AB,

$$M_A = WL - (wx^2/2), \quad M_v = 1L(\text{horiz.}), \quad M_v = -1x \text{ (vert.)}$$

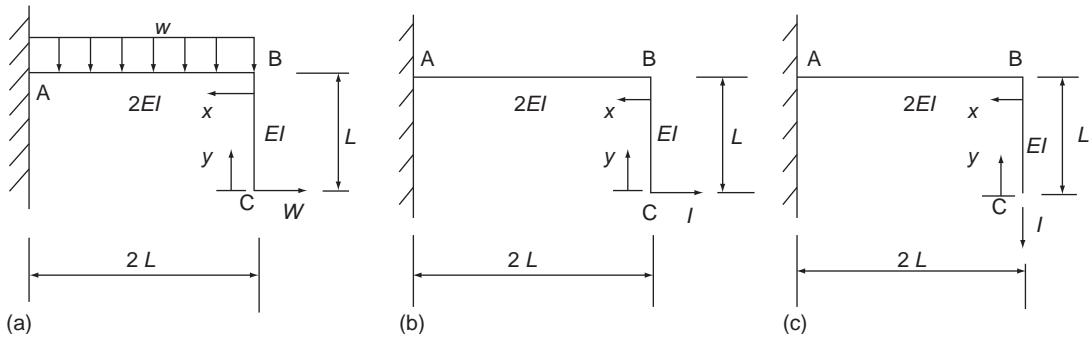


FIGURE 4.16 Frame of Example 4.10

for BC,

$$M_A = Wy, \quad M_v = 1y \text{ (horiz.)}, \quad M_v = 0 \text{ (vert.)}$$

Considering the horizontal deflection first, the total internal virtual work is

$$W_i = \int_0^L (Wy^2/EI) dy + \int_0^{2L} [WL - (wx^2/2)](L/2EI) dx$$

that is,

$$W_i = (W/EI)[y^3/3]_0^L + (L/2EI)[WLx - (wx^3/6)]_0^{2L}$$

which gives

$$W_i = 2L^3(2W - wL)/3EI \quad (i)$$

The virtual external work done by the unit load is  $1\delta_{C,H}$ , where  $\delta_{C,H}$  is the horizontal component of the deflection of C. Equating this to the internal virtual work given by Eq. (i) gives

$$\delta_{C,H} = 2L^3(2W - wL)/3EI \quad (ii)$$

Now, considering the vertical component of deflection,

$$W_i = (1/2EI) \int_0^{2L} [WL - (wx^2/2)](-x)dx$$

Note that, for BC,  $M_v = 0$ . Integrating this expression and substituting the limits gives

$$W_i = L^3(-W + wL)/EI \quad (iii)$$

The external virtual work done is  $1\delta_{C,V}$ , where  $\delta_{C,V}$  is the vertical component of the deflection of C. Equating the internal and external virtual work gives

$$\delta_{C,V} = L^3(-W + wL)/EI \quad (iv)$$

Note that the components of deflection can be either positive or negative, depending on the relative magnitudes of W and w. A positive value indicates a deflection in the direction of the applied unit load; a negative one indicates a deflection in the opposite direction to the applied unit load.

**Example 4.11**

A cantilever beam AB takes the form of a quadrant of a circle of radius,  $R$ , and is positioned on a horizontal plane. If the beam carries a vertically downward load,  $W$ , at its free end and its bending and torsional stiffnesses are  $EI$  and  $GJ$ , respectively, calculate the vertical component of the deflection at its free end.

A plan view of the beam is shown in Fig. 4.17. To determine the vertical displacement of B, we apply a virtual unit load at B vertically downward (i.e., into the plane of the page).

At a section of the beam where the radius at the section makes an angle,  $\alpha$ , with the radius through B,

$$\begin{aligned} M_A &= Wp = WR \sin\alpha, & M_v &= 1R \sin\alpha, \\ T_A &= W(R - R \cos\alpha), & T_v &= 1(R - R \cos\alpha) \end{aligned}$$

The total internal virtual work done is given by the summation of Eqs. (4.20) and (4.22), that is,

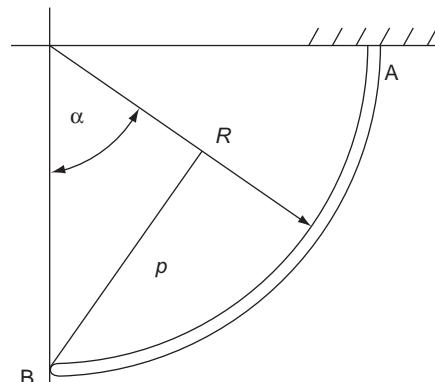
$$W_i = (1/EI) \int_0^{\pi/2} WR^2 \sin^2\alpha R d\alpha + (1/GJ) \int_0^{\pi/2} WR^2(1 - \cos\alpha)^2 R d\alpha \quad (\text{i})$$

Integrating and substituting the limits in Eq. (i) gives

$$W_i = WR^3 \{(\pi/4EI) + (1/GJ)[(3\pi/4) - 2]\} \quad (\text{ii})$$

The external virtual work done by the unit load is  $1\delta_B$ , so that equating with Eq. (ii), we obtain

$$\delta_B = WR^3 \{(\pi/4EI) + (1/GJ)[(3\pi/4) - 2]\}$$



**FIGURE 4.17 Cantilever Beam of Example 4.11**

**Reference**

- [1] Megson THG. Structural and stress analysis. 4th ed. Oxford: Elsevier; 2019.

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## PROBLEMS

**P.4.1** Use the principle of virtual work to determine the support reactions in the beam ABCD shown in Fig. P.4.1.

Answer:  $R_A = 1.25W$ ,  $R_D = 1.75W$

**P.4.2** Find the support reactions in the beam ABC shown in Fig. P.4.2 using the principle of virtual work.

Answer:  $R_A = (W + 2wL)/4$ ,  $R_C = (3W + 2wL)/4$

**P.4.2 MATLAB** Use the Symbolic Math Toolbox in MATLAB to repeat Problem P.4.2, assuming that the vertical force  $W$  in Fig. P.4.2 is located  $3L/5$  from A.

Answer:  $R_A = 2W/5 + wL/2$ ,  $R_C = 3W/5 + wL/2$

**P.4.3** Determine the reactions at the built-in end of the cantilever beam ABC shown in Fig. P.4.3 using the principle of virtual work.

Answer:  $R_A = 3W$ ,  $M_A = 2.5WL$

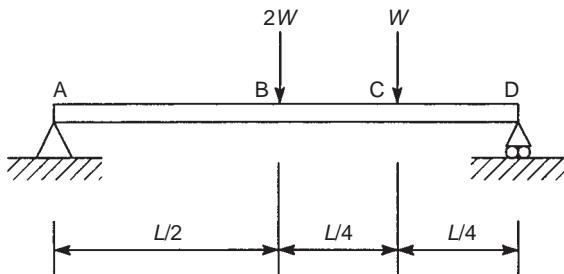


FIGURE P.4.1

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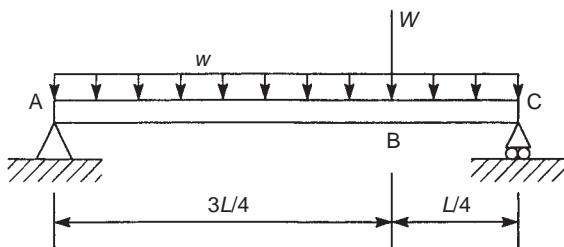


FIGURE P.4.2

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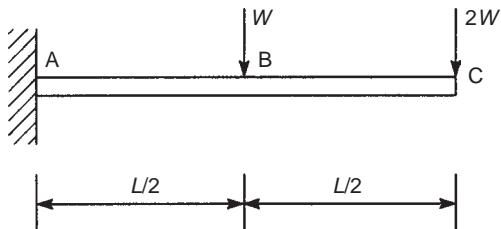


FIGURE P.4.3

**P.4.4** Find the bending moment at the three-quarter-span point in the beam shown in Fig. P.4.4. Use the principle of virtual work.

Answer:  $3wL^2/32$

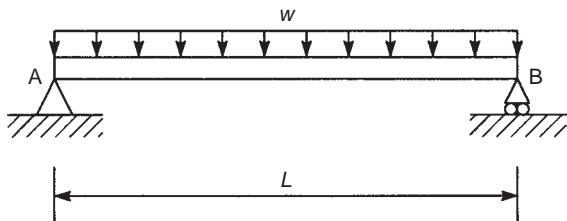


FIGURE P.4.4

**P.4.5** Calculate the forces in the members FG, GD, and CD of the truss shown in Fig. P.4.5 using the principle of virtual work. All horizontal and vertical members are 1 m long.

Answer: FG = +20 kN, GD = +28.3 kN, CD = -20 kN

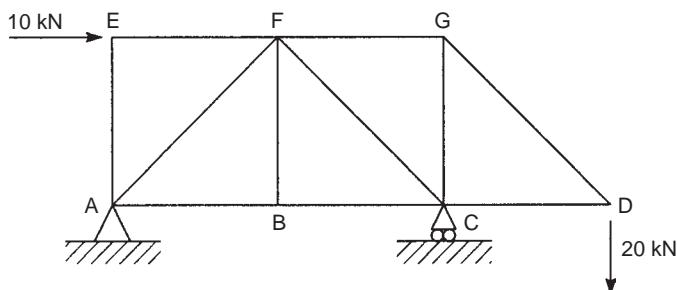


FIGURE P.4.5

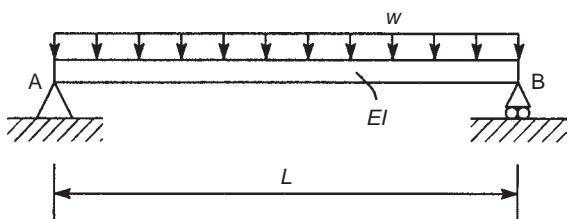


FIGURE P.4.6

**P.4.6** Use the principle of virtual work to calculate the vertical displacements at the quarter- and mid-span points in the beam shown in Fig. P.4.6.

Answer:  $19wL^4/2048EI$ ,  $5wL^4/384EI$  (both downward)

**P.4.6 MATLAB** Use the Symbolic Math Toolbox in MATLAB to repeat Problem P.4.6. Calculate the vertical displacements at increments of  $L/8$  along the beam.

Answer:

Distance from A	Vertical Displacement
0	0
$L/8$	$497L^4w/98,304EI$
$L/4$	$19L^4w/2,048EI$
$3L/8$	$395L^4w/32,768EI$
$L/2$	$5L^4w/384EI$
$5L/8$	$395L^4w/32,768EI$
$3L/5$	$19L^4w/2,048EI$
$7L/8$	$497L^4w/9,8304EI$
$L$	0

**P.4.7** The frame shown in Fig. P.4.7 consists of a cranked beam simply supported at A and F and reinforced by a tie bar pinned to the beam at B and E. The beam carries a uniformly distributed load of intensity,  $w$ , over the outer parts AB and EF. Considering the effects of bending and axial load only determines the axial force in the tie bar and the bending moments at B and C. The bending stiffness of the beam is  $EI$  and its cross-sectional area is  $3A$ , while the corresponding values for the tie bar are  $EI$  and  $A$ .

Answer: Force in tie bar is  $9wL^3/8[L^2 + (4I/A)]$ ,  $M(\text{at B}) = wL^2/2$  (counterclockwise),  $M(\text{at C}) = (TL - wL^2)/2$  (clockwise).

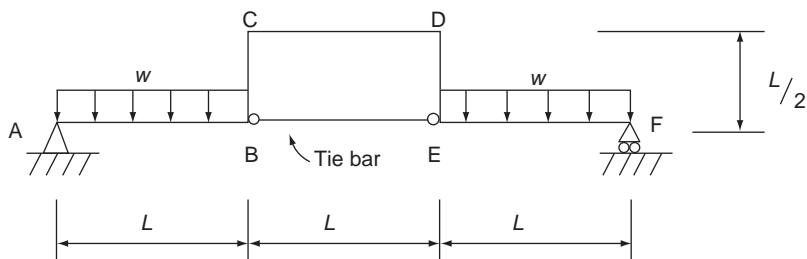


FIGURE P.4.7

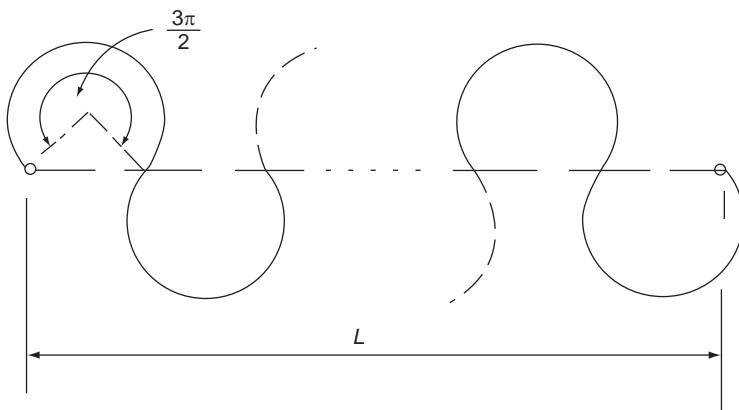


FIGURE P.4.8

**P.4.8** The flat tension spring shown in Fig. P.4.8 consists of a length of wire of circular cross-section having a diameter,  $d$ , and Young's modulus,  $E$ . The spring consists of  $n$  open loops each of which subtends an angle of  $3\pi/2$  radians at its center; the length between the ends of the spring is  $L$ . Considering bending and axial strains only calculate the stiffness of the spring.

$$\text{Answer: } (\sqrt{2})\pi Ed^2/L \{ [48L^2(\pi + 1)/n^2d^2] + (3\pi - 2) \}$$

**P.4.9** The circular fuselage frame shown in Fig. P.4.9 has a cut-out at the bottom and is loaded as shown. The member AB is pinned to the frame at A and B. If the second moment of area of the cross-section of the frame is  $416,000 \text{ mm}^4$  and it has a Young's modulus of  $267,000 \text{ N/mm}^2$  while the area of cross-section of the member AB is  $130 \text{ mm}^2$  and its Young's modulus is  $44,800 \text{ N/mm}^2$ , calculate the axial force in the member AB.

$$\text{Answer: } 15.8 \text{ kN}$$

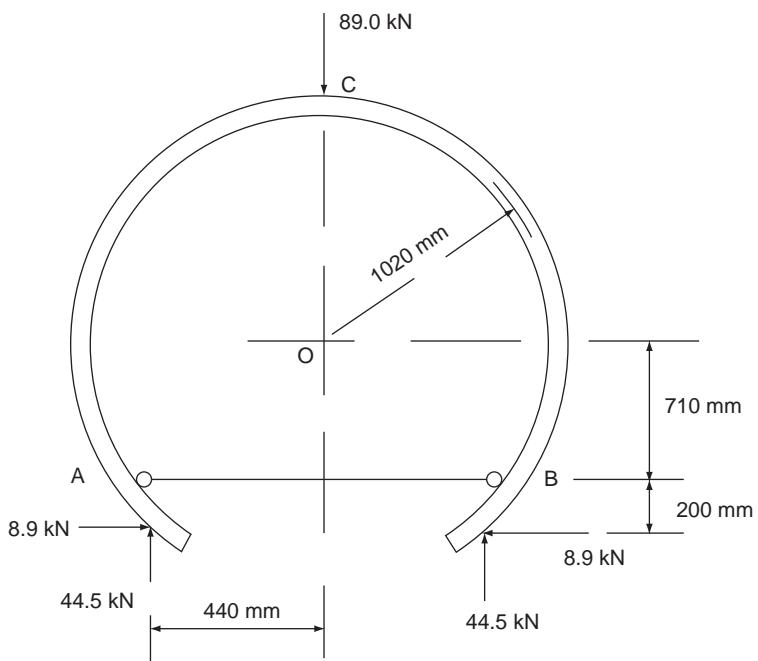


FIGURE P.4.9

# Energy methods

# 5

In [Chapter 2](#), we saw that the elasticity method of structural analysis embodies the determination of stresses and displacements by employing equations of equilibrium and compatibility in conjunction with the relevant force–displacement or stress–strain relationships. In addition, in [Chapter 4](#), we investigated the use of virtual work in calculating forces, reactions, and displacements in structural systems. A powerful alternative but equally fundamental approach is the use of energy methods. These, while providing exact solutions for many structural problems, find their greatest use in the rapid approximate solution of problems for which exact solutions do not exist. Also, many structures that are statically indeterminate—that is, they cannot be analyzed by the application of the equations of statical equilibrium alone—may be conveniently analyzed using an energy approach. Further, energy methods provide comparatively simple solutions for deflection problems not readily solved by more elementary means.

Generally, as we shall see, modern analysis<sup>1</sup> uses the methods of *total complementary energy* and *total potential energy*. Either method may be employed to solve a particular problem, although as a general rule, deflections are more easily found using complementary energy and forces by potential energy.

Although energy methods are applicable to a wide range of structural problems and may even be used as indirect methods of forming equations of equilibrium or compatibility,<sup>1,2</sup> we shall be concerned in this chapter with the solution of deflection problems and the analysis of statically indeterminate structures. We also include some methods restricted to the solution of linear systems: that is, the *unit load method*, the *principle of superposition*, and the *reciprocal theorem*.

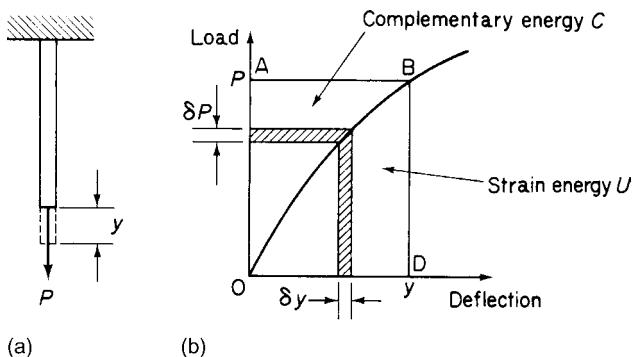
## 5.1 STRAIN ENERGY AND COMPLEMENTARY ENERGY

[Figure 5.1\(a\)](#) shows a structural member subjected to a steadily increasing load  $P$ . As the member extends, the load  $P$  does work, and from the law of conservation of energy, this work is stored in the member as *strain energy*. A typical load–deflection curve for a member possessing nonlinear elastic characteristics is shown in [Fig. 5.1\(b\)](#). The strain energy  $U$  produced by a load  $P$  and corresponding extension  $y$  is then

$$U = \int_0^y P \, dy \quad (5.1)$$

and is clearly represented by the area OBD under the load–deflection curve. Engesser (1889) called the area OBA above the curve the *complementary energy*  $C$ , and from [Fig. 5.1\(b\)](#),

$$C = \int_0^P y \, dP \quad (5.2)$$



**FIGURE 5.1 (a) Strain Energy of a Member Subjected to Simple Tension; (b) Load–Deflection Curve for a Nonlinearly Elastic Member**

Complementary energy, as opposed to strain energy, has no physical meaning, being purely a convenient mathematical quantity. However, it is possible to show that complementary energy obeys the law of conservation of energy in the type of situation usually arising in engineering structures, so that its use as an energy method is valid.

Differentiation of Eqs. (5.1) and (5.2) with respect to  $y$  and  $P$ , respectively, gives

$$\frac{dU}{dy} = P, \quad \frac{dC}{dP} = y$$

Bearing these relationships in mind, we can now consider the interchangeability of strain and complementary energy. Suppose that the curve of Fig. 5.1(b) is represented by the function

$$P = by^n$$

where the coefficient  $b$  and exponent  $n$  are constants. Then,

$$dP = nb y^{n-1} dy$$

and since

$$y = \left(\frac{P}{b}\right)^{1/n}$$

$$dy = \frac{1}{n} \left(\frac{P}{b}\right)^{(1/n)-1} dP$$

Substituting in Eqs. (5.1) and (5.2),

$$U = \int_0^y P dy = \frac{1}{n} \int_0^P \left(\frac{P}{b}\right)^{1/n} dP$$

$$C = \int_0^P y dP = n \int_0^y b y^n dy$$

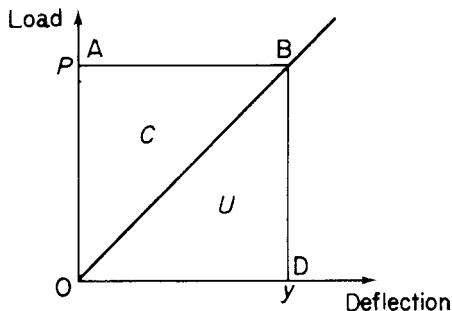


FIGURE 5.2 Load-Deflection Curve for a Linearly Elastic Member

Hence,

$$\frac{dU}{dy} = P, \quad \frac{dU}{dP} = \frac{1}{n} \left( \frac{P}{b} \right)^{1/n} = \frac{1}{n} y \quad (5.3)$$

$$\frac{dC}{dP} = y, \quad \frac{dC}{dy} = bny^n = nP \quad (5.4)$$

When  $n = 1$ ,

$$\left. \begin{aligned} \frac{dU}{dy} &= \frac{dC}{dy} = P \\ \frac{dU}{dP} &= \frac{dC}{dP} = y \end{aligned} \right\} \quad (5.5)$$

and the strain and complementary energies are completely interchangeable. Such a condition is found in a linearly elastic member; its related load-deflection curve being that shown in Fig. 5.2. Clearly, area OBD ( $U$ ) is equal to area OBA ( $C$ ).

We see that the latter of Eqs. (5.5) is in the form of what is commonly known as Castigliano's first theorem, in which the differential of the strain energy  $U$  of a structure with respect to a load is equated to the deflection of the load. To be mathematically correct, however, the differential of the complementary energy  $C$  is what should be equated to deflection (compare Eqs. (5.3) and (5.4)).

## 5.2 PRINCIPLE OF THE STATIONARY VALUE OF THE TOTAL COMPLEMENTARY ENERGY

Consider an elastic system in equilibrium supporting forces  $P_1, P_2, \dots, P_n$ , which produce real corresponding displacements  $\Delta_1, \Delta_2, \dots, \Delta_n$ . If we impose virtual forces  $\delta P_1, \delta P_2, \dots, \delta P_n$  on the system acting through the real displacements, then the total virtual work done by the system is (see Chapter 4)

$$-\int_{\text{vol}} y \, dP + \sum_{r=1}^n \Delta_r \delta P_r$$

The first term in this expression is the negative virtual work done by the particles in the elastic body, while the second term represents the virtual work of the externally applied virtual forces. From the principle of virtual work,

$$-\int_{\text{vol}} y \, dP + \sum_{r=1}^n \Delta_r \delta P_r = 0 \quad (5.6)$$

Comparing Eq. (5.6) with Eq. (5.2), we see that each term represents an increment in complementary energy: the first, of the internal forces; the second, of the external loads. Equation (5.6) may therefore be rewritten

$$\delta(C_i + C_e) = 0 \quad (5.7)$$

where

$$C_i = \int_{\text{vol}} \int_0^P y \, dP \quad \text{and} \quad C_e = - \sum_{r=1}^n \Delta_r P_r \quad (5.8)$$

We now call the quantity  $(C_i + C_e)$  the *total complementary energy*  $C$  of the system.

The displacements specified in Eq. (5.6) are real displacements of a continuous elastic body; they therefore obey the condition of compatibility of displacement, so that Eqs. (5.6) and (5.7) are equations of geometrical compatibility. The *principle of the stationary value of the total complementary energy* may then be stated as follows:

*For an elastic body in equilibrium under the action of applied forces, the true internal forces (or stresses) and reactions are those for which the total complementary energy has a stationary value.*

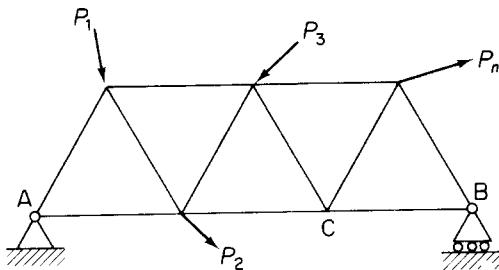
In other words, the true internal forces (or stresses) and reactions are those that satisfy the condition of compatibility of displacement. This property of the total complementary energy of an elastic system is particularly useful in the solution of statically indeterminate structures, in which an infinite number of stress distributions and reactive forces may be found to satisfy the requirements of equilibrium.

### 5.3 APPLICATION TO DEFLECTION PROBLEMS

Generally, deflection problems are most readily solved by the complementary energy approach, although for linearly elastic systems, there is no difference between the methods of complementary and potential energy, since, as we have seen, complementary and strain energy then become completely interchangeable. We illustrate the method by reference to the deflections of frames and beams, which may or may not possess linear elasticity.

Let us suppose that we must find the deflection  $\Delta_2$  of the load  $P_2$  in the simple pin-jointed framework consisting, say, of  $k$  members and supporting loads  $P_1, P_2, \dots, P_n$ , as shown in Fig. 5.3. From Eqs. (5.8), the total complementary energy of the framework is given by

$$C = \sum_{i=1}^k \int_0^{F_i} \lambda_i \, dF_i - \sum_{r=1}^n \Delta_r P_r \quad (5.9)$$



**FIGURE 5.3 Determination of the Deflection of a Point on a Framework by the Method of Complementary Energy**

where  $\lambda_i$  is the extension of the  $i$ th member,  $F_i$  is the force in the  $i$ th member, and  $\Delta_r$  is the corresponding displacement of the  $r$ th load  $P_r$ . From the principle of the stationary value of the total complementary energy,

$$\frac{\partial C}{\partial P_2} = \sum_{i=1}^k \lambda_i \frac{\partial F_i}{\partial P_2} - \Delta_2 = 0 \quad (5.10)$$

from which

$$\Delta_2 = \sum_{i=1}^k \lambda_i \frac{\partial F_i}{\partial P_2} \quad (5.11)$$

Equation (5.10) is seen to be identical to the principle of virtual forces in which virtual forces  $\delta F$  and  $\delta P$  act through real displacements  $\lambda$  and  $\Delta$ . Clearly, the partial derivatives with respect to  $P_2$  of the constant loads  $P_1, P_2, \dots, P_n$  vanish, leaving the required deflection  $\Delta_2$  as the unknown. At this stage, before  $\Delta_2$  can be evaluated, the load-displacement characteristics of the members must be known. For linear elasticity (see Eq. (1.40)),

$$\lambda_i = \frac{F_i L_i}{A_i E_i}$$

where  $L_i$ ,  $A_i$ , and  $E_i$  are, respectively, the length, cross-sectional area, and modulus of elasticity of the  $i$ th member. On the other hand, if the load-displacement relationship is of a nonlinear form, say,

$$F_i = b(\lambda_i)^c$$

in which  $b$  and  $c$  are known, then Eq. (5.11) becomes

$$\Delta_2 = \sum_{i=1}^k \left( \frac{F_i}{b} \right)^{1/c} \frac{\partial F_i}{\partial P_2}$$

The computation of  $\Delta_2$  is best accomplished in tabular form, but before the procedure is illustrated by an example, some aspects of the solution merit discussion.

We note that the support reactions do not appear in Eq. (5.9). This convenient absence derives from the fact that the displacements  $\Delta_1, \Delta_2, \dots, \Delta_n$  are the real displacements of the frame and fulfill the conditions of geometrical compatibility and boundary restraint. The complementary energy of the

reaction at A and the vertical reaction at B is therefore zero, since both their corresponding displacements are zero. If we examine Eq. (5.11), we note that  $\lambda_i$  is the extension of the  $i$ th member of the framework due to the applied loads  $P_1, P_2, \dots, P_n$ . Therefore, the loads  $F_i$  in the substitution for  $\lambda_i$  in Eq. (5.11) are those corresponding to the loads  $P_1, P_2, \dots, P_n$ . The term  $\partial F_i / \partial P_2$  in Eq. (5.11) represents the rate of change of  $F_i$  with  $P_2$  and is calculated by applying the load  $P_2$  to the *unloaded* frame and determining the corresponding member loads in terms of  $P_2$ . This procedure indicates a method for obtaining the displacement of either a point on the frame in a direction not coincident with the line of action of a load or, in fact, a point such as C that carries no load at all. We place at the point and in the required direction a *fictitious* or *dummy* load, say  $P_f$ , the original loads being removed. The loads in the members due to  $P_f$  are then calculated and  $\partial F_i / \partial P_f$  obtained for each member. Substitution in Eq. (5.11) produces the required deflection.

It must be pointed out that it is not absolutely necessary to remove the actual loads during the application of  $P_f$ . The force in each member is then calculated in terms of the actual loading and  $P_f$ .  $F_i$  follows by substituting  $P_f = 0$ , and  $\partial F_i / \partial P_f$  is found by differentiation with respect to  $P_f$ . Obviously, the two approaches yield the same expressions for  $F_i$  and  $\partial F_i / \partial P_f$ , although the latter is arithmetically clumsier.

### Example 5.1

Calculate the vertical deflection of the point B and the horizontal movement of D in the pin-jointed framework shown in Fig. 5.4(a). All members of the framework are linearly elastic and have cross-sectional areas of  $1800 \text{ mm}^2$ . The value of E for the material of the members is  $200,000 \text{ N/mm}^2$ .

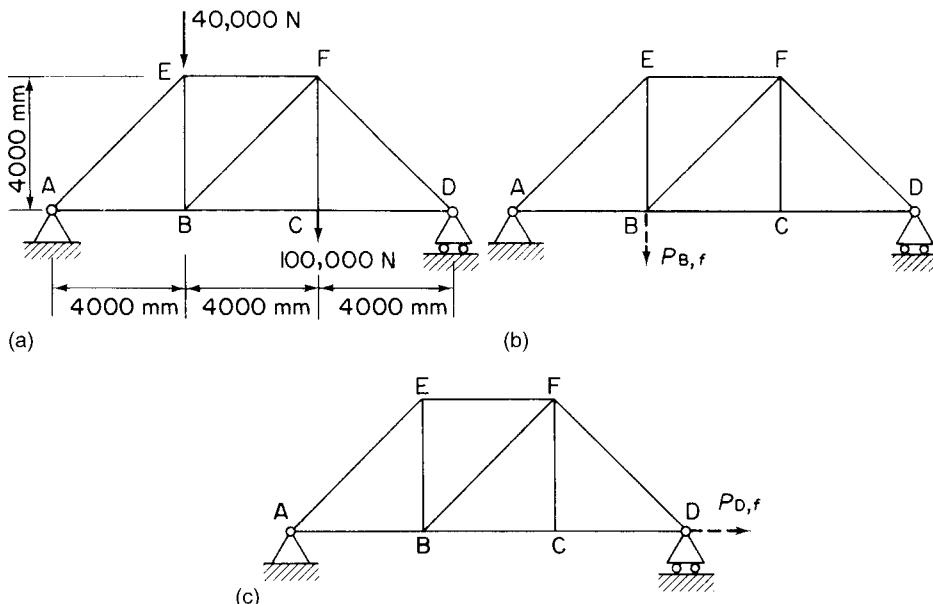


FIGURE 5.4 (a) Actual Loading of Framework; (b) Determination of Vertical Deflection of B; (c) Determination of Horizontal Deflection of D

**Table 5.1 Example 5.1**

① Member	② $L$ (mm)	③ $F$ (N)	④ $F_{B,f}$ (N)	⑤ $\partial F_{B,f}/\partial P_{B,f}$	⑥ $F_{D,f}$ (N)	⑦ $\partial F_{D,f}/\partial P_{D,f}$	⑧ $\times 10^6$ $FL\partial F_{B,f}/\partial P_{B,f}$	⑨ $\times 10^6$ $FL\partial F_{D,f}/\partial P_{D,f}$
AE	$4,000\sqrt{2}$	$-60,000\sqrt{2}$	$-2\sqrt{2}P_{B,f}/3$	$-2\sqrt{2}/3$	0	0	$320\sqrt{2}$	0
EF	4,000	-60,000	$-2P_{B,f}/3$	$-2/3$	0	0	160	0
FD	$4,000\sqrt{2}$	$-80,000\sqrt{2}$	$-\sqrt{2}P_{B,f}/3$	$-\sqrt{2}/3$	0	0	$640\sqrt{2}/3$	0
DC	4,000	80,000	$P_{B,f}/3$	$1/3$	$P_{D,f}$	1	$320/3$	320
CB	4,000	80,000	$P_{B,f}/3$	$1/3$	$P_{D,f}$	1	$320/3$	320
BA	4,000	60,000	$2P_{B,f}/3$	$2/3$	$P_{D,f}$	1	$480/3$	240
EB	4,000	20,000	$2P_{B,f}/3$	$2/3$	0	0	$160/3$	0
FB	$4,000\sqrt{2}$	$-20,000\sqrt{2}$	$\sqrt{2}P_{B,f}/3$	$\sqrt{2}/3$	0	0	$-160\sqrt{2}/3$	0
FC	4,000	100,000	0	0	0	0	0	0
							$\sum = 1268$	$\sum = 880$

The members of the framework are linearly elastic, so that Eq. (5.11) may be written

$$\Delta = \sum_{i=1}^k \frac{F_i L_i}{A_i E_i} \frac{\partial F_i}{\partial P} \quad (\text{i})$$

or, since each member has the same cross-sectional area and modulus of elasticity,

$$\Delta = \frac{1}{AE} \sum_{i=1}^k F_i L_i \frac{\partial F_i}{\partial P} \quad (\text{ii})$$

The solution is completed in Table 5.1, in which  $F$  are the member forces due to the actual loading of Fig. 5.4(a),  $F_{B,f}$  are the member forces due to the fictitious load  $P_{B,f}$  in Fig. 5.4(b), and  $F_{D,f}$  are the forces in the members produced by the fictitious load  $P_{D,f}$  in Fig. 5.4(c). We take tensile forces as positive and compressive forces as negative.

The vertical deflection of B is

$$\Delta_{B,v} = \frac{1,268 \times 10^6}{1,800 \times 200,000} = 3.52 \text{ mm}$$

and the horizontal movement of D is

$$\Delta_{D,h} = \frac{880 \times 10^6}{1,800 \times 200,000} = 2.44 \text{ mm}$$

which agree with the virtual work solution (Example 4.9).

The positive values of  $\Delta_{B,v}$  and  $\Delta_{D,h}$  indicate that the deflections are in the directions of  $P_{B,f}$  and  $P_{D,f}$ .

The analysis of beam deflection problems by complementary energy is similar to that of pin-jointed frameworks, except that we assume initially that displacements are caused primarily by bending action. Shear force effects are discussed later in the chapter.

**Example 5.2**

Determine the deflection of the free end of the tip-loaded cantilever beam shown in Fig. 5.5; the bending stiffness of the beam is  $EI$ .

From a comparison with Eq. (5.2) and Fig. 5.1(b), the complementary energy of the element  $\delta z$  in Fig. 5.5 as the bending moment changes by  $\delta M$  from the section  $z$  to the section  $z + \delta z$  and which produces the change  $\delta\theta$  in the slope of the beam is

$$\delta C = \delta\theta \delta M$$

Therefore, for the complete beam under the load  $P$ ,

$$C = \int_L^M d\theta dM - P\Delta_v \quad (\text{i})$$

From the principle of the stationary value of the total complementary energy,

$$\frac{\partial C}{\partial P} = \int_L^M d\theta \frac{dM}{dP} - \Delta_v = 0$$

or

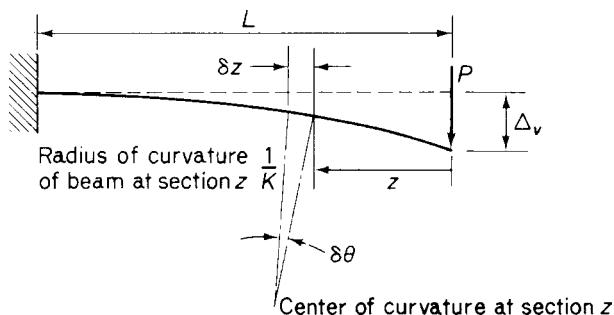
$$\Delta_v = \int_L^M d\theta \frac{dM}{dP} \quad (\text{ii})$$

Equation (ii) is applicable to either a nonlinear or linearly elastic beam. To proceed further, therefore, we require the load–displacement ( $M-\theta$ ) and bending moment–load ( $M-P$ ) relationships. It is immaterial for the purposes of this illustrative problem whether the system is linear or nonlinear, since the mechanics of the solution are the same in either case. We choose therefore a linear  $M-\theta$  relationship, as this is the case in the majority of the problems we consider. Hence, from Fig. 5.5,

$$\delta\theta = K\delta z$$

or

$$d\theta = \frac{M}{EI} dz \quad \left( \frac{1}{K} = \frac{EI}{M} \text{ from simple beam theory (see Eq. (16.7))} \right)$$



**FIGURE 5.5 Beam Deflection by the Method of Complementary Energy**

where the product *modulus of elasticity*  $\times$  *second moment of area of the beam cross-section* is known as the *bending or flexural rigidity* of the beam. Also,

$$M = Pz$$

so that

$$\frac{dM}{dP} = z$$

Substitution for  $d\theta$ ,  $M$ , and  $dM/dP$  in Eq. (ii) gives

$$\Delta_v = \int_0^L \frac{Pz^2}{EI} dz$$

or

$$\Delta_v = \frac{PL^3}{3EI}$$

The fictitious load method of the framework example may be employed in the solution of beam deflection problems where we require deflections at positions on the beam other than concentrated load points.

### Example 5.3

Determine the deflection of the tip of the cantilever beam shown in Fig. 5.6; the bending stiffness of the beam is  $EI$ .

First, we apply a fictitious load  $P_f$  at the point where the deflection is required. The total complementary energy of the system is then

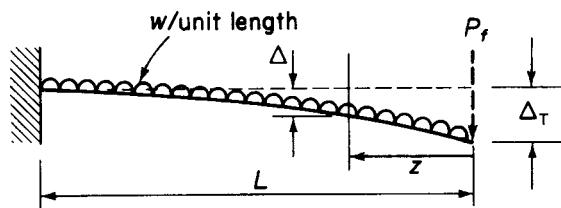
$$C = \int_L \int_0^M d\theta dM - \Delta_T P_f - \int_0^L \Delta w dz$$

where the symbols take their previous meanings and  $\Delta$  is the vertical deflection of any point on the beam. Then,

$$\frac{\partial C}{\partial P_f} = \int_0^L d\theta \frac{\partial M}{\partial P_f} - \Delta_T = 0 \quad (i)$$

As before,

$$d\theta = \frac{M}{EI} dz$$



**FIGURE 5.6 Deflection of a Uniformly Loaded Cantilever by the Method of Complementary Energy**

but

$$M = P_f z + \frac{wz^2}{2} \quad (P_f = 0)$$

Hence,

$$\frac{\partial M}{\partial P_f} = z$$

Substituting in Eq. (i) for  $d\theta$ ,  $M$ , and  $\partial M/\partial P_f$  and remembering that  $P_f = 0$ , we have

$$\Delta_T = \int_0^L \frac{wz^3}{2EI} dz$$

giving

$$\Delta_T = \frac{wL^4}{8EI}$$

Note that here, unlike the method for the solution of the pin-jointed framework, the fictitious load is applied to the loaded beam. However, no arithmetical advantage is gained by the former approach, although the result obviously is the same, since  $M$  would equal  $wz^2/2$  and  $\partial M/\partial P_f$  would have the value  $z$ . ■

### Example 5.4

Calculate the vertical displacements of the quarter- and mid-span points B and C of the simply supported beam of length  $L$  and flexural rigidity  $EI$  loaded, as shown in Fig. 5.7.

The total complementary energy  $C$  of the system including the fictitious loads  $P_{B,f}$  and  $P_{C,f}$  is

$$C = \int_L \int_0^M d\theta dM - P_{B,f} \Delta_B - P_{C,f} \Delta_C - \int_0^L \Delta w dz \quad (i)$$

Hence,

$$\frac{\partial C}{\partial P_{B,f}} = \int_L d\theta \frac{\partial M}{\partial P_{B,f}} - \Delta_B = 0 \quad (ii)$$

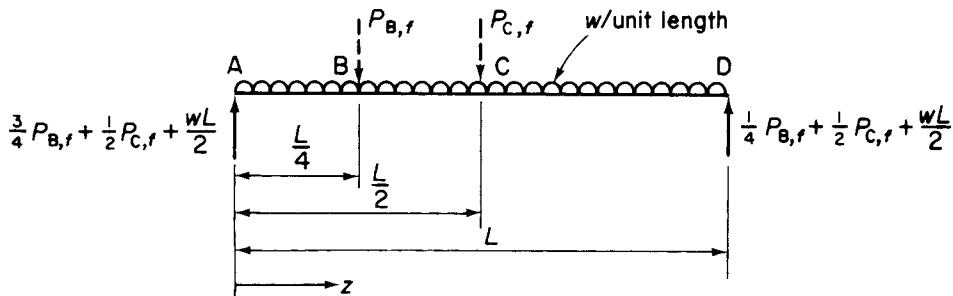


FIGURE 5.7 Deflection of a Simply Supported Beam by the Method of Complementary Energy

and

$$\frac{\partial C}{\partial P_{C,f}} = \int_L d\theta \frac{\partial M}{\partial P_{C,f}} - \Delta_C = 0 \quad (\text{iii})$$

Assuming a linearly elastic beam, Eqs. (ii) and (iii) become

$$\Delta_B = \frac{1}{EI} \int_0^L M \frac{\partial M}{\partial P_{B,f}} dz \quad (\text{iv})$$

$$\Delta_C = \frac{1}{EI} \int_0^L M \frac{\partial M}{\partial P_{C,f}} dz \quad (\text{v})$$

From A to B,

$$M = \left( \frac{3}{4} P_{B,f} + \frac{1}{2} P_{C,f} + \frac{wL}{2} \right) z - \frac{wz^2}{2}$$

so that

$$\frac{\partial M}{\partial P_{B,f}} = \frac{3}{4} z, \quad \frac{\partial M}{\partial P_{C,f}} = \frac{1}{2} z$$

From B to C,

$$M = \left( \frac{3}{4} P_{B,f} + \frac{1}{2} P_{C,f} + \frac{wL}{2} \right) z - \frac{wz^2}{2} - P_{B,f} \left( z - \frac{L}{4} \right)$$

giving

$$\frac{\partial M}{\partial P_{B,f}} = \frac{1}{4}(L-z), \quad \frac{\partial M}{\partial P_{C,f}} = \frac{1}{2}z$$

From C to D,

$$M = \left( \frac{1}{4} P_{B,f} + \frac{1}{2} P_{C,f} + \frac{wL}{2} \right) (L-z) - \frac{w}{2} (L-z)^2$$

so that

$$\frac{\partial M}{\partial P_{B,f}} = \frac{1}{4}(L-z), \quad \frac{\partial M}{\partial P_{C,f}} = \frac{1}{2}(L-z)$$

Substituting these values in Eqs. (iv) and (v) and remembering that  $P_{B,f} = P_{C,f} = 0$ , we have, from Eq. (iv),

$$\Delta_B = \frac{1}{EI} \left\{ \int_0^{L/4} \left( \frac{wLz}{2} - \frac{wz^2}{2} \right) \frac{3}{4} z dz + \int_{L/4}^{L/2} \left( \frac{wLz}{2} - \frac{wz^2}{2} \right) \frac{1}{4}(L-z) dz + \int_{L/2}^L \left( \frac{wLz}{2} - \frac{wz^2}{2} \right) \frac{1}{4}(L-z) dz \right\}$$

from which

$$\Delta_B = \frac{57wL^4}{6,144EI}$$

Similarly,

$$\Delta_C = \frac{5wL^4}{384EI}$$

**Example 5.5**

Use the principle of the stationary value of the total complementary energy of a system to calculate the horizontal displacement of the point C in the frame of [Example 4.10](#).

Referring to [Fig. 4.16](#), we apply a horizontal fictitious load,  $P_{C,f}$ , at C. The total complementary energy of the system, including the fictitious load, is given by

$$C = \int_L^M d\theta \, dM - P_{C,f} \Delta_{C,h} - \int_0^{2L} \Delta w \, dx - W \Delta_{C,h} \quad (\text{i})$$

where  $\Delta_{C,h}$  is the actual horizontal displacement of C. Then,

$$\partial C / \partial P_{C,f} = \int_L^M d\theta (\partial M / \partial P_{C,f}) - \Delta_{C,h} = 0 \quad (\text{ii})$$

Therefore, from Eq. (ii),

$$\Delta_{C,h} = (1/2EI) \int_0^{2L} M (\partial M / \partial P_{C,f}) dx + (1/EI) \int_0^L M (\partial M / \partial P_{C,f}) dy \quad (\text{iii})$$

In AB,

$$M = (W + P_{C,f})L - (wx^2/2)$$

so that

$$\partial M / \partial P_{C,f} = L$$

In BC,

$$M = (W + P_{C,f})y$$

and

$$\partial M / \partial P_{C,f} = y$$

Substituting these expressions in Eq. (iii) and remembering that  $P_{C,f} = 0$ , we have

$$\Delta_{C,h} = (1/2EI) \int_0^{2L} [WL - (wx^2/2)]L \, dx + (1/EI) \int_0^L Wy^2 \, dy \quad (\text{iv})$$

Integrating Eq. (iv) and substituting the limits gives

$$\Delta_{C,h} = (2L^3/3EI)(2W - wL)$$

which is the solution produced in [Example 4.10](#).

**Example 5.6**

Using the principle of the stationary value of the total complementary energy of a system, calculate the vertical displacement of the point B in the cantilever beam of [Ex. 4.11](#).

In this example, we need to consider the torsional contribution to the total complementary energy in addition to that due to bending. By comparison with Eq. (i) of Example 5.2, we see that the internal complementary energy of a bar subjected to torsion is given by

$$C_{i,T} = \int_L^T d\gamma \, dT \quad (i)$$

where  $\gamma$  is the angle of twist at any section of the bar.

Suppose that an imaginary load,  $P_{B,f}$ , is applied vertically at B (i.e., into the plane of the paper) in the cantilever beam of Example 4.11. The total complementary energy of the beam is then given by

$$C = \int_L^M d\theta \, dM + \int_L^T d\gamma \, dT - P_{B,f} \Delta_B \quad (ii)$$

where  $\Delta_B$  is the vertical deflection of B. Then,

$$\partial C / \partial P_{B,f} = \int_L^M d\theta (dM/dP_{B,f}) + \int_L^T d\gamma (dT/dP_{B,f}) - \Delta_B \quad (iii)$$

The first term on the right-hand side of Eq. (iii) is replaced as in Example 5.2. The second term is replaced in a similar manner using Eq. (3.12). Then,

$$\Delta_B = (1/EI) \int_0^L M (dM/dP_{B,f}) \, dz + (1/GJ) \int_0^L T (dT/dP_{B,f}) \, dz \quad (iv)$$

Referring to Fig. 4.17,

$$M = (W + P_{B,f})p = (W + P_{B,f})R \sin\alpha$$

Then,

$$(dM/dP_{B,f}) = R \sin\alpha$$

Also,

$$T = (W + P_{B,f})(R - R \cos\alpha)$$

And

$$(dT/dP_{B,f}) = R(1 - \cos\alpha)$$

Substituting these expressions in Eq. (iv) and remembering that  $P_{B,f} = 0$  gives

$$\Delta_B = (1/EI) \int_0^{\pi/2} WR^2 \sin^2\alpha R \, d\alpha + (1/GJ) \int_0^{\pi/2} WR^2(1 - \cos\alpha)^2 R \, d\alpha \quad (v)$$

The right-hand side of Eq. (v) is identical to the expression for the internal work done in Example 4.11. Therefore,

$$\Delta_B = WR^3 \{(\pi/4EI) + (1/GJ)[(3\pi/4) - 2]\}$$

as before. ■

The fictitious load method of determining deflections may be streamlined for linearly elastic systems and is then termed the *unit load method*; this we discuss later in the chapter.

## 5.4 APPLICATION TO THE SOLUTION OF STATICALLY INDETERMINATE SYSTEMS

In a statically determinate structure, the internal forces are determined uniquely by simple statical equilibrium considerations. This is not the case for a statically indeterminate system, in which, as already noted, an infinite number of internal force or stress distributions may be found to satisfy the conditions of equilibrium. The true force system is, as demonstrated in [Section 5.2](#), the one satisfying the conditions of compatibility of displacement of the elastic structure or, alternatively, that for which the total complementary energy has a stationary value. We apply the principle to a variety of statically indeterminate structures, beginning with the relatively simple singly redundant pin-jointed frame of [Example 5.7](#).

### Example 5.7

Determine the forces in the members of the pin-jointed framework shown in [Fig. 5.8](#). Each member has the same value of the product  $AE$ .

The first step is to choose the redundant member. In this example, no advantage is gained by the choice of any particular member, although in some cases careful selection can result in a decrease in the amount of arithmetical labor. Taking  $BD$  as the redundant member, we assume that it sustains a tensile force  $R$  due to the external loading. The total complementary energy of the framework is, with the notation of [Eq. \(5.9\)](#),

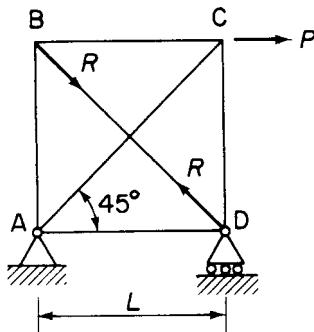
$$C = \sum_{i=1}^k \int_0^{F_i} \lambda_i dF_i - P\Delta$$

Hence,

$$\frac{\partial C}{\partial R} = \sum_{i=1}^k \lambda_i \frac{\partial F_i}{\partial R} = 0 \quad (\text{i})$$

or, assuming linear elasticity,

$$\frac{1}{AE} \sum_{i=1}^k F_i L_i \frac{\partial F_i}{\partial R} = 0 \quad (\text{ii})$$



**FIGURE 5.8** Analysis of a Statically Indeterminate Framework by the Method of Complementary Energy

**Table 5.2 Example 5.7**

<b>① Member</b>	<b>② Length</b>	<b>③ <math>F</math></b>	<b>④ <math>\partial F / \partial R</math></b>	<b>⑤ <math>FL\partial F / \partial R</math></b>
AB	$L$	$-R/\sqrt{2}$	$-1/\sqrt{2}$	$RL/2$
BC	$L$	$-R/\sqrt{2}$	$-1/\sqrt{2}$	$RL/2$
CD	$L$	$-(P + R/\sqrt{2})$	$-1/\sqrt{2}$	$L(P + R/\sqrt{2})/\sqrt{2}$
DA	$L$	$-R/\sqrt{2}$	$-1/\sqrt{2}$	$RL/2$
AC	$\sqrt{2}L$	$\sqrt{2}P + R$	1	$L(2P + \sqrt{2}R)$
BD	$\sqrt{2}L$	$R$	1	$\sqrt{2}RL$
				$\sum = 4.83RL + 2.707PL$

The solution is now completed in **Table 5.2**, where, as in **Table 5.1**, positive signs indicate tension.

Hence, from **Eq. (ii)**,

$$4.83RL + 2.707PL = 0$$

or

$$R = -0.56P$$

Substitution for  $R$  in column ③ of **Table 5.2** gives the force in each member. Having determined the forces in the members, the deflection of any point on the framework may be found by the method described in **Section 5.3**.

Unlike the statically determinate type, statically indeterminate frameworks may be subjected to self-straining. Therefore, internal forces are present before external loads are applied. Such a situation may be caused by a local temperature change or by an initial lack of fit of a member. Suppose that the member BD of the framework of **Fig. 5.8** is short by a known amount  $\Delta_R$  when the framework is assembled but is forced to fit. The load  $R$  in BD has suffered a displacement  $\Delta_R$  in addition to that caused by the change in length of BD produced by the load  $P$ . The total complementary energy is then

$$C = \sum_{i=1}^k \int_0^{F_i} \lambda_i dF_i - P\Delta - R\Delta_R$$

and

$$\frac{\partial C}{\partial R} = \sum_{i=1}^k \lambda_i \frac{\partial F_i}{\partial R} - \Delta_R = 0$$

or

$$\Delta_R = \frac{1}{AE} \sum_{i=1}^k F_i L_i \frac{\partial F_i}{\partial R} \quad (\text{iii})$$

Obviously, the summation term in **Eq. (iii)** has the same value as in the previous case, so that

$$\Delta_R = \frac{1}{AE} (4.83RL + 2.707PL)$$

which gives

$$R = -0.56P + \frac{AE}{4.83L} \Delta_R$$

Hence, the forces in the members due to both applied loads and an initial lack of fit.

Some care should be given to the sign of the lack of fit  $\Delta_R$ . We note here that the member BD is short by an amount  $\Delta_R$ , so that the assumption of a positive sign for  $\Delta_R$  is compatible with the tensile force  $R$ . If BD were initially too long, then the total complementary energy of the system would be written

$$C = \sum_{i=1}^k \int_0^{F_i} \lambda_i dF_i - P\Delta - R(-\Delta_R)$$

giving

$$-\Delta_R = \frac{1}{AE} \sum_{i=1}^k F_i L_i \frac{\partial F_i}{\partial R}$$

### Example 5.8

Calculate the loads in the members of the singly redundant pin-jointed framework shown in Fig. 5.9. The members AC and BD are  $30 \text{ mm}^2$  in cross-section, and all other members are  $20 \text{ mm}^2$  in cross-section. The members AD, BC, and DC are each  $800 \text{ mm}$  long.  $E = 200,000 \text{ N/mm}^2$ .

From the geometry of the framework  $\hat{A}B\hat{D} = \hat{C}B\hat{D} = 30^\circ$ ; therefore,  $BD = AC = 800\sqrt{3} \text{ mm}$ . Choosing CD as the redundant member and proceeding from Eq. (ii) of Example 5.7, we have

$$\frac{1}{E} \sum_{i=1}^k \frac{F_i L_i}{A_i} \frac{\partial F_i}{\partial R} = 0 \quad (\text{i})$$

From Table 5.3, we have

$$\sum_{i=1}^k \frac{F_i L_i}{A_i} \frac{\partial F_i}{\partial R} = -268 + 129.2R = 0$$

Hence,  $R = 2.1 \text{ N}$  and the forces in the members are tabulated in column ⑦ of Table 5.3.

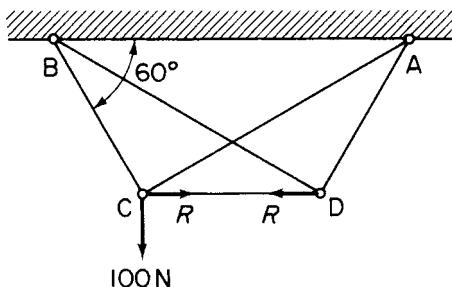


FIGURE 5.9 Framework of Example 5.8

**Table 5.3 Example 5.8 (tension positive)**

<b>① Member</b>	<b>② <math>L</math> (mm)</b>	<b>③ <math>A</math> (mm<math>^2</math>)</b>	<b>④ <math>F</math> (N)</b>	<b>⑤ <math>\partial F / \partial R</math></b>	<b>⑥ <math>(FL/A) \partial F / \partial R</math></b>	<b>⑦ Force (N)</b>
AC	$800\sqrt{3}$	30	$50 - \sqrt{3}R/2$	$-\sqrt{3}/2$	$-2000 + 20\sqrt{3}R$	48.2
CB	800	20	$86.6 + R/2$	$1/2$	$1732 + 10R$	87.6
BD	$800\sqrt{3}$	30	$-\sqrt{3}R/2$	$-\sqrt{3}/2$	$20\sqrt{3}R$	-1.8
CD	800	20	$R$	1	$40R$	2.1
AD	800	20	$R/2$	$1/2$	$10R$	1.0
					$\sum = -268 + 129.2R$	

**Example 5.8 MATLAB**

Repeat Example 5.8 using MATLAB and columns 1–4 of Table 5.3.

Values for  $R$  and the member forces, rounded to the first decimal place, are obtained through the following MATLAB file:

```
% Declare any needed variables
syms F L A E R
E = 200000;

% Define L, A, and F using columns 2-4 of Table 5.3
L = [800*sqrt(3); % L_AC
      800; % L_CB
      800*sqrt(3); % L_BD
      800; % L_CD
      800]; % L_AD
A = sym([30; % A_AC
          20; % A_CB
          30; % A_BD
          20; % A_CD
          20]); % A_AD
F = [50-sqrt(3)*R/2; % F_AC
      86.6+R/2; % F_CB
      -sqrt(3)*R/2; % F_BD
      R; % F_CD
      R/2]; % F_AD

% Use Eq. (5.16) to solve for R
eqI = F.*L.*diff(F,R)./(A.*E); % Eq. (5.16)
R_val = solve(sum(eqI),R);

% Substitute R_val into F to calculate the member forces
Force = subs(F,R,R_val);
```

```
% Output R and the member forces, rounded to the first decimal place, to the Command
Window
R = round(double(R_val)*10)/10;
Force = round(double(Force)*10)/10;

disp(['R =' num2str(R) ' N'])
disp(['Force = [' num2str(Force(1)), ' , ' num2str(Force(2)), ' , ' num2str(Force(3)),
' , ' num2str(Force(4)), ' , ' num2str(Force(5))'] N'])
```

The Command Window outputs resulting from this MATLAB file are as follows:

```
R = 2.1 N
Force = [48.2, 87.6, -1.8, 2.1, 1] N
```

### Example 5.9

A plane, pin-jointed framework consists of six bars forming a rectangle ABCD 4000 mm  $\times$  3000 mm with two diagonals, as shown in Fig. 5.10. The cross-sectional area of each bar is 200 mm<sup>2</sup> and the frame is unstressed when the temperature of each member is the same. Due to local conditions, the temperature of one of the 3000 mm members is raised by 30°C. Calculate the resulting forces in all the members if the coefficient of linear expansion  $\alpha$  of the bars is  $7 \times 10^{-6}/^\circ\text{C}$  and  $E = 200,000 \text{ N/mm}^2$ .

Suppose that BC is the heated member, then the increase in length of BC =  $3000 \times 30 \times 7 \times 10^{-6} = 0.63$  mm. Note that this is an extension of BC resulting in a compressive force in BC; the extension is therefore negative. From Eq. (iii) of Example 5.7,

$$-0.63 = \frac{1}{200 \times 200,000} \sum_{i=1}^k F_i L_i \frac{\partial F_i}{\partial R} \quad (\text{i})$$

Substitution from the summation of column ⑤ in Table 5.4 into Eq. (i) gives

$$R = \frac{-0.63 \times 200 \times 200,000}{48,000} = -525 \text{ N}$$

Column ⑥ of Table 5.4 is now completed for the force in each member.

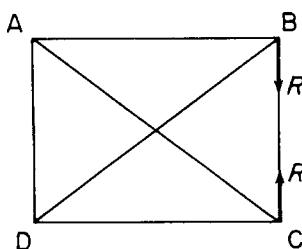


FIGURE 5.10 Framework of Example 5.9

**Table 5.4 Example 5.9 (tension positive)**

<b>① Member</b>	<b>② <math>L</math> (mm)</b>	<b>③ <math>F(N)</math></b>	<b>④ <math>\partial F/\partial R</math></b>	<b>⑤ <math>FL\partial F/\partial R</math></b>	<b>⑥ Force (N)</b>
AB	4,000	$4R/3$	$4/3$	$64,000R/9$	-700
BC	3,000	$R$	1	$3,000R$	-525
CD	4,000	$4R/3$	$4/3$	$64,000R/9$	-700
DA	3,000	$R$	1	$3,000R$	-525
AC	5,000	$-5R/3$	$-5/3$	$125,000R/9$	875
DB	5,000	$-5R/3$	$-5/3$	$125,000R/9$	875
$\sum = 48,000R$					

So far, our analysis has been limited to singly redundant frameworks, although the same procedure may be adopted to solve a multi-redundant framework of, say,  $m$  redundancies. Therefore, instead of a single equation of the type (i) in [Example 5.7](#), we would have  $m$  simultaneous equations

$$\frac{\partial C}{\partial R_j} = \sum_{i=1}^k \lambda_i \frac{\partial F_i}{\partial R_j} = 0 \quad (j = 1, 2, \dots, m)$$

from which the  $m$  unknowns  $R_1, R_2, \dots, R_m$  are obtained. The forces  $F$  in the members follow, being expressed initially in terms of the applied loads and  $R_1, R_2, \dots, R_m$ .

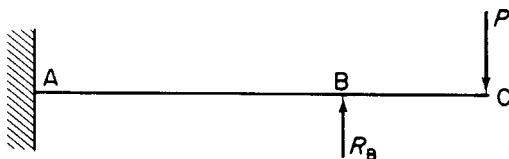
Other types of statically indeterminate structure are solved by the application of total complementary energy with equal facility. The propped cantilever of [Fig. 5.11](#) is an example of a singly redundant beam structure for which total complementary energy readily yields a solution.

The total complementary energy of the system is, with the notation of Eq. (i) of [Example 5.2](#),

$$C = \int_L \int_0^M d\theta \, dM - P\Delta_C - R_B\Delta_B$$

where  $\Delta_C$  and  $\Delta_B$  are the deflections at C and B, respectively. Usually, in problems of this type,  $\Delta_B$  is either zero for a rigid support or a known amount (sometimes in terms of  $R_B$ ) for a sinking support. Hence, for a stationary value of  $C$ ,

$$\frac{\partial C}{\partial R_B} = \int_L d\theta \frac{\partial M}{\partial R_B} - \Delta_B = 0$$

**FIGURE 5.11 Analysis of a Propped Cantilever by the Method of Complementary Energy**

from which equation  $R_B$  may be found;  $R_B$  being contained in the expression for the bending moment  $M$ .

Obviously, the same procedure is applicable to a beam having a multiredundant support system, for example, a continuous beam supporting a series of loads  $P_1, P_2, \dots, P_n$ . The total complementary energy of such a beam is given by

$$C = \int_L \int_0^M d\theta dM - \sum_{j=1}^m R_j \Delta_j - \sum_{r=1}^n P_r \Delta_r$$

where  $R_j$  and  $\Delta_j$  are the reaction and known deflection (at least in terms of  $R_j$ ) of the  $j$ th support point in a total of  $m$  supports. The stationary value of  $C$  gives

$$\frac{\partial C}{\partial R_j} = \int_L d\theta \frac{\partial M}{\partial R_j} - \Delta_j = 0 \quad (j = 1, 2, \dots, m)$$

producing  $m$  simultaneous equations for the  $m$  unknown reactions.

The intention here is not to suggest that continuous beams are best or most readily solved by the energy method; the moment distribution method produces a more rapid solution, especially for beams in which the degree of redundancy is large. Instead the purpose is to demonstrate the versatility and power of energy methods in their ready solution of a wide range of structural problems. A complete investigation of this versatility is impossible here due to restrictions of space; in fact, whole books have been devoted to this topic. We therefore limit our analysis to problems peculiar to the field of aircraft structures, with which we are primarily concerned. The remaining portion of this section is therefore concerned with the solution of frames and rings possessing varying degrees of redundancy.

The frameworks we considered in the earlier part of this section and in [Section 5.3](#) comprised members capable of resisting direct forces only. Of a more general type are composite frameworks, in which some or all of the members resist bending and shear loads in addition to direct loads. It is usual, however, except for the thin-walled structures in Part B of this book, to ignore deflections produced by shear forces. We, therefore, only consider bending and direct force contributions to the internal complementary energy of such structures. The method of analysis is illustrated in [Ex. 5.10](#).

### Example 5.9 MATLAB

Repeat [Example 5.9](#) using MATLAB and columns 1–3 of [Table 5.4](#).

Values for  $R$  and the member forces are obtained through the following MATLAB file:

```
% Declare any needed variables
syms delta_R alpha F L A E R T
A = 200;
alpha = 7*10^(-6);
E = 200000;
T = 30;
```

```
% Define L and F using columns 2-3 of Table 5.4
L = [4000; % L_AB
      3000; % L_BC
      4000; % L_CD
      3000; % L_DA
      5000; % L_AC
      5000]; % L_DB
F = [4*R/3; % F_AB
      R; % F_BC
      4*R/3; % F_CD
      R; % F_DA
     -5*R/3; % F_AC
     -5*R/3]; % F_DB

% Use Eq. (5.17) to solve for R assuming BC is the heated member
delta_R = L(2)*T*alpha;
eqI = F.*L.*diff(F,R)./(A*E); % Eq. (5.17)
R_val = solve(sum(eqI)+delta_R,R);

% Substitute R_val into F to calculate the member forces
Force = subs(F,R,R_val);

% Output R and the member forces to the Command Window
R = round(double(R_val));
Force = round(double(Force));

disp(['R=' num2str(R) 'N'])
disp(['Force = [' num2str(Force(1)), 'num2str(Force(2))', 'num2str(Force(3))',
      'num2str(Force(4))', 'num2str(Force(5))', 'num2str(Force(6))] N'])
```

The Command Window output resulting from this MATLAB file is as follows:

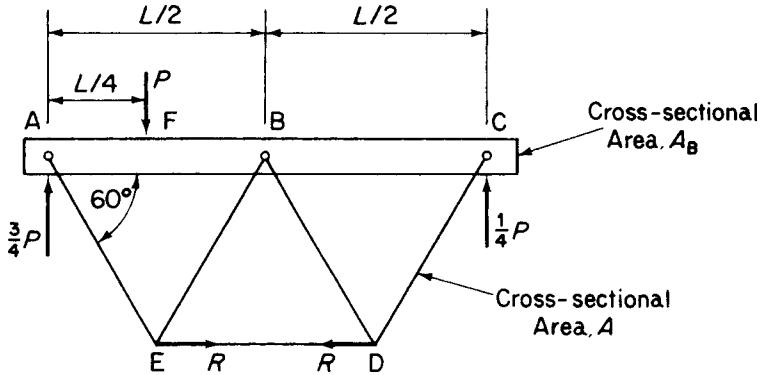
```
R = -525 N
Force = [-700, -525, -700, -525, 875, 875] N
```

### Example 5.10

The simply supported beam ABC shown in Fig. 5.12 is stiffened by an arrangement of pin-jointed bars capable of sustaining axial loads only. If the cross-sectional area of the beam is  $A_B$  and that of the bars is  $A$ , calculate the forces in the members of the framework, assuming that displacements are caused by bending and direct force action only.

We observe that if the beam were capable of supporting only direct loads, then the structure would be a relatively simple statically determinate pin-jointed framework. Since the beam resists bending moments (we are ignoring shear effects), the system is statically indeterminate with a single redundancy, the bending moment at any section of the beam. The total complementary energy of the framework is given, with the notation previously developed, by

$$C = \int_{ABC} \int_0^M d\theta \, dM + \sum_{i=1}^k \lambda_i \, dF_i - P\Delta \quad (i)$$



**FIGURE 5.12** Analysis of a Trussed Beam by the Method of Complementary Energy

If we suppose that the tensile load in the member ED is  $R$ , then, for  $C$  to have a stationary value,

$$\frac{\partial C}{\partial R} = \int_{ABC} d\theta \frac{\partial M}{\partial R} + \sum_{i=1}^k \lambda_i \frac{\partial F_i}{\partial R} = 0 \quad (\text{ii})$$

At this point, we assume the appropriate load–displacement relationships; again we take the system to be linear so that Eq. (ii) becomes

$$\int_0^L \frac{M}{EI} \frac{\partial M}{\partial R} dz + \sum_{i=1}^k \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial R} = 0 \quad (\text{iii})$$

The two terms in Eq. (iii) may be evaluated separately, bearing in mind that only the beam ABC contributes to the first term, while the complete structure contributes to the second. Evaluating the summation term by a tabular process, we have Table 5.5.

Summation of column ⑥ in Table 5.5 gives

$$\sum_{i=1}^k \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial R} = \frac{R L}{4E} \left( \frac{1}{A_B} + \frac{10}{A} \right) \quad (\text{iv})$$

**Table 5.5** Example 5.10 (tension positive)

<b>① Member</b>	<b>② Length</b>	<b>③ Area</b>	<b>④ F</b>	<b>⑤ <math>\partial F / \partial R</math></b>	<b>⑥ <math>(F/A) \partial F / \partial R</math></b>
AB	$L/2$	$A_B$	$-R/2$	$-1/2$	$R/4A_B$
BC	$L/2$	$A_B$	$-R/2$	$-1/2$	$R/4A_B$
CD	$L/2$	$A$	$R$	$1$	$R/A$
DE	$L/2$	$A$	$R$	$1$	$R/A$
BD	$L/2$	$A$	$-R$	$-1$	$R/A$
EB	$L/2$	$A$	$-R$	$-1$	$R/A$
AE	$L/2$	$A$	$R$	$1$	$R/A$

The bending moment at any section of the beam between A and F is

$$M = \frac{3}{4}Pz - \frac{\sqrt{3}}{2}Rz; \quad \text{hence, } \frac{\partial M}{\partial R} = -\frac{\sqrt{3}}{2}z$$

between F and B is

$$M = \frac{P}{4}(L-z) - \frac{\sqrt{3}}{2}R(L-z); \quad \text{hence, } \frac{\partial M}{\partial R} = -\frac{\sqrt{3}}{2}(L-z)$$

and between B and C is

$$M = \frac{P}{4}(L-z) - \frac{\sqrt{3}}{2}R(L-z); \quad \text{hence, } \frac{\partial M}{\partial R} = -\frac{\sqrt{3}}{2}(L-z)$$

Therefore,

$$\begin{aligned} \int_0^L \frac{M}{EI} \frac{\partial M}{\partial R} dz &= \frac{1}{EI} \left\{ \int_0^{L/4} -\left( \frac{3}{4}Pz - \frac{\sqrt{3}}{2}Rz \right) \frac{\sqrt{3}}{2}z dz \right. \\ &\quad + \int_{L/4}^{L/2} \left[ \frac{P}{4}(L-z) - \frac{\sqrt{3}}{2}R(L-z) \right] \left( -\frac{\sqrt{3}}{2}z \right) dz \\ &\quad \left. + \int_{L/2}^L -\left[ \frac{P}{4}(L-z) - \frac{\sqrt{3}}{2}R(L-z) \right] \frac{\sqrt{3}}{2}(L-z) dz \right\} \end{aligned}$$

giving

$$\int_0^L \frac{M}{EI} \frac{\partial M}{\partial R} dz = \frac{-11\sqrt{3}PL^3}{768EI} + \frac{RL^3}{16EI} \quad (\text{v})$$

Substituting from Eqs. (iv) and (v) into Eq. (iii),

$$-\frac{11\sqrt{3}PL^3}{768EI} + \frac{RL^3}{16EI} + \frac{RL}{4E} \left( \frac{A + 10A_B}{A_B A} \right) = 0$$

from which

$$R = \frac{11\sqrt{3}PL^2A_B A}{48[L^2A_B A + 4I(A + 10A_B)]}$$

hence, the forces in each member of the framework. The deflection  $\Delta$  of the load  $P$  or any point on the framework may be obtained by the method of Section 5.3. For example, the stationary value of the total complementary energy of Eq. (i) gives  $\Delta$ ; that is,

$$\frac{\partial C}{\partial P} = \int_{ABC} d\theta \frac{\partial M}{\partial R} + \sum_{i=1}^k \lambda_i \frac{\partial F_i}{\partial P} - \Delta = 0$$

Although braced beams are still found in modern light aircraft in the form of braced wing structures, a much more common structural component is the ring frame. The role of this particular component is

discussed in detail in [Chapter 12](#); it is therefore sufficient for the moment to say that ring frames form the basic shape of semi-monocoque fuselages reacting to shear loads from the fuselage skins, point loads from wing spar attachments, and distributed loads from floor beams. Usually, a ring is two-dimensional, supporting loads applied in its own plane. Our analysis is limited to the two-dimensional case.

### Example 5.11

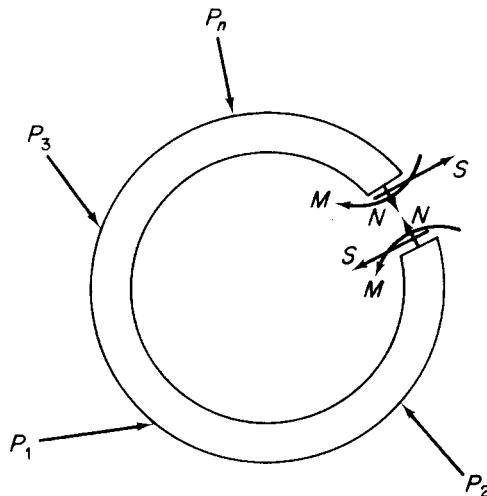
Determine the bending moment distribution in the two-dimensional ring shown in [Fig. 5.13](#); the bending stiffness of the ring is  $EI$ .

A two-dimensional ring has redundancies of direct load, bending moment, and shear at any section, as shown in [Fig. 5.13](#). However, in some special cases of loading, the number of redundancies may be reduced. For example, on a plane of symmetry, the shear loads and sometimes the normal or direct loads are zero, while on a plane of anti-symmetry the direct loads and bending moments are zero. Let us consider the simple case of the doubly symmetrical ring shown in [Fig. 5.14\(a\)](#). At a section in the vertical plane of symmetry, the internal shear and direct loads vanish, leaving one redundancy, the bending moment  $M_A$  ([Fig. 5.14\(b\)](#)). Note that, in the horizontal plane of symmetry, the internal shears are zero but the direct loads have a value  $P/2$ . The total complementary energy of the system (again ignoring shear strains) is

$$C = \int_{\text{ring}} \int_0^M d\theta dM - 2 \left( \frac{P}{2} \Delta \right)$$

taking the bending moment as positive when it increases the curvature of the ring. In this expression for  $C$ ,  $\Delta$  is the displacement of the top, A, of the ring relative to the bottom, B. Assigning a stationary value to  $C$ , we have

$$\frac{\partial C}{\partial M_A} = \int_{\text{ring}} d\theta \frac{\partial M}{\partial M_A} = 0$$



**FIGURE 5.13 Internal Force System in a Two-Dimensional Ring**

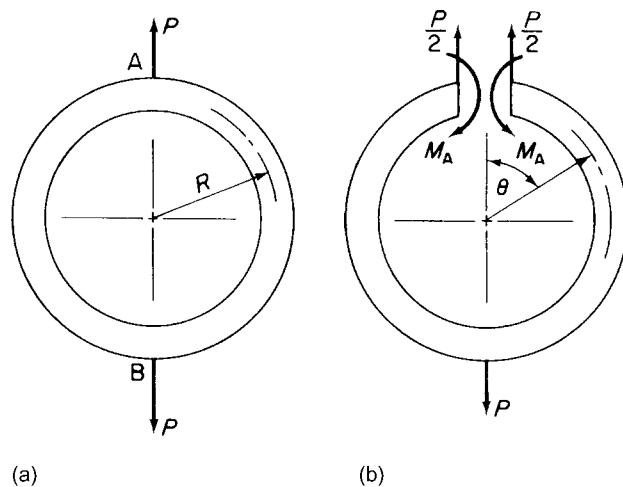


FIGURE 5.14 Doubly Symmetric Ring

or assuming linear elasticity and considering, from symmetry, half the ring,

$$\int_0^{\pi R} \frac{M}{EI} \frac{\partial M}{\partial M_A} ds = 0$$

Therefore, since

$$M = M_A - \frac{P}{2}R \sin\theta, \quad \frac{\partial M}{\partial M_A} = 1$$

and we have

$$\int_0^{\pi} \left( M_A - \frac{P}{2}R \sin\theta \right) R d\theta = 0$$

or

$$\left[ M_A \theta + \frac{P}{2}R \cos\theta \right]_0^\pi = 0$$

from which

$$M_A = \frac{PR}{\pi}$$

The bending moment distribution is then

$$M = PR \left( \frac{1}{\pi} - \frac{\sin\theta}{2} \right)$$

and is shown diagrammatically in Fig. 5.15.

We shall now consider a more representative aircraft structural problem.

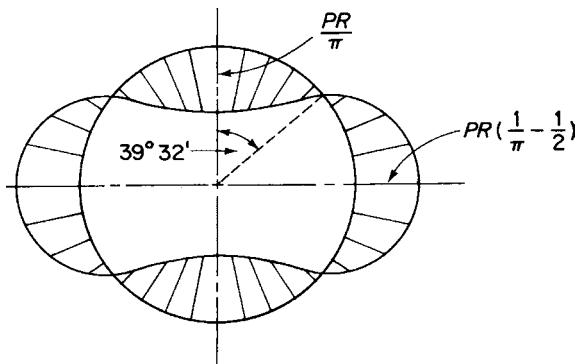


FIGURE 5.15 Distribution of Bending Moment in a Doubly Symmetric Ring

### Example 5.12

The circular fuselage frame shown in Fig. 5.16(a) supports a load,  $P$ , which is reacted to by a shear flow,  $q$  (i.e., a shear force per unit length, see Chapter 17), distributed around the circumference of the frame from the fuselage skin. If the bending stiffness of the frame is  $EI$ , calculate the distribution of bending moment round the frame.

The value and direction of this shear flow are quoted here but are derived from theory established in Section 17.3. From our previous remarks on the effect of symmetry, we observe that there is no shear force at the section A on the vertical plane of symmetry. The unknowns are therefore the bending moment  $M_A$  and normal force  $N_A$ . We proceed, as in the previous example, by writing down the total complementary energy  $C$  of the system. Then, neglecting shear strains,

$$C = \int_{\text{ring}} \int_0^M d\theta dM - P\Delta \quad (\text{i})$$

in which  $\Delta$  is the deflection of the point of application of  $P$  relative to the top of the frame. Note that  $M_A$  and  $N_A$  do not contribute to the complementary energy of the system, since, by symmetry, the rotation and horizontal displacements at A are zero. From the principle of the stationary value of the total complementary energy,

$$\frac{\partial C}{\partial M_A} = \int_{\text{ring}} d\theta \frac{\partial M}{\partial M_A} = 0 \quad (\text{ii})$$

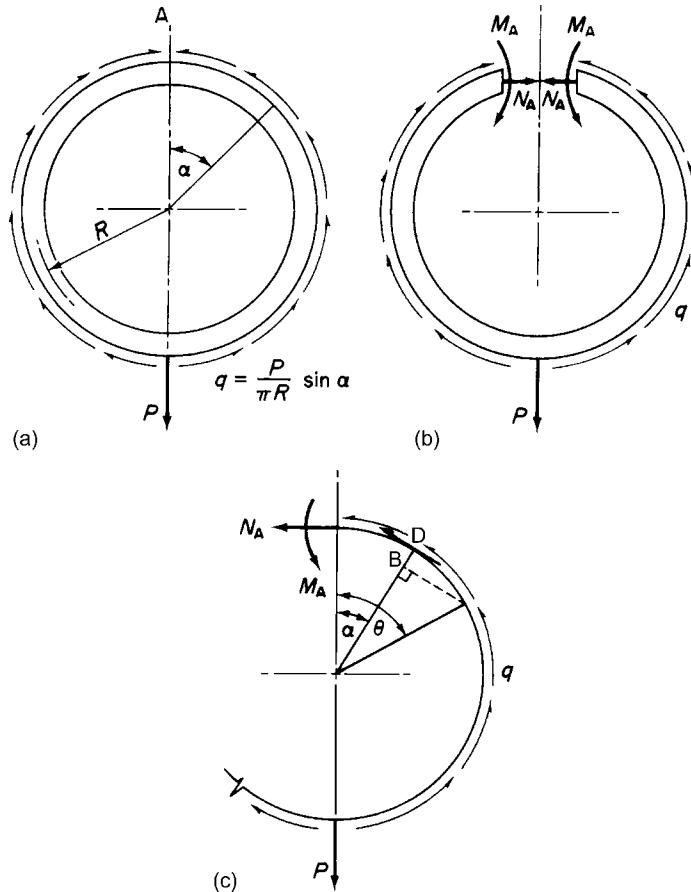
and

$$\frac{\partial C}{\partial N_A} = \int_{\text{ring}} d\theta \frac{\partial M}{\partial N_A} = 0 \quad (\text{iii})$$

The bending moment at a radial section inclined at an angle  $\theta$  to the vertical diameter is, from Fig. 5.16(c),

$$M = M_A + N_A R(1 - \cos\theta) + \int_0^\theta q BDR d\alpha$$

Note that the shear force on the element  $\delta s$  at D is  $q\delta s$  or  $qR\delta\theta$ . Then,



**FIGURE 5.16 Determination of Bending Moment Distribution in a Shear and Direct Loaded Ring**

$$M = M_A + N_A R (1 - \cos\theta) + \int_0^\theta \frac{P}{\pi R} \sin \alpha [R - R \cos(\theta - \alpha)] R \, d\alpha$$

The integration in the above may be carried out as follows.

$$\int_0^{\theta} \sin \alpha [R - R \cos(\theta - \alpha)] R d\alpha = R^2 \int_0^{\theta} (\sin \alpha - \cos \theta \cos \alpha \sin \alpha - \sin \theta \sin^2 \alpha) d\alpha$$

$$= R^2 \left[ -\cos \alpha + \cos \theta \frac{\cos 2\alpha}{4} - \frac{\sin \theta}{2} \left( \alpha - \frac{\sin 2\alpha}{2} \right) \right]_0^{\theta}$$

which simplifies to

$$R^2 \left( 1 - \cos \theta - \frac{\theta}{2} \sin \theta \right)$$

Then

$$M = M_A + N_A R(1 - \cos\theta) + \frac{PR}{\pi}(1 - \cos\theta - \frac{1}{2}\theta \sin\theta) \quad (\text{iv})$$

Hence,

$$\frac{\partial M}{\partial M_A} = 1, \quad \frac{\partial M}{\partial N_A} = R(1 - \cos\theta) \quad (\text{v})$$

Assuming that the fuselage frame is linearly elastic, we have, from Eqs. (ii) and (iii),

$$2 \int_0^\pi \frac{M}{EI} \frac{\partial M}{\partial M_A} R d\theta = 2 \int_0^\pi \frac{M}{EI} \frac{\partial M}{\partial N_A} R d\theta = 0 \quad (\text{vi})$$

Substituting in the first of Eqs. (vi) for  $M$  from Eq. (iv) and  $\partial M/\partial M_A$  from the first of Eqs. (v), we have

$$2 \int_0^\pi \frac{M}{EI} \frac{\partial M}{\partial M_A} R d\theta = 0 = \int_0^\pi \left[ M_A + N_A R(1 - \cos\theta) + \frac{PR}{\pi} \left( 1 - \cos\theta - \frac{\theta}{2} \sin\theta \right) \right] R d\theta$$

that is,

$$0 = \left[ M_A \theta + N_A R(\theta - \sin\theta) + \frac{PR}{\pi} \left( \theta - \sin\theta + \frac{\theta}{2} \cos\theta - \frac{\sin\theta}{2} \right) \right]_0^\pi$$

from which

$$-\frac{PR}{2\pi} = M_A + N_A R \quad (\text{vii})$$

Note that the term  $(\theta/2)\sin\theta$  in the above was integrated by parts.

Similarly, substituting in the second of Eqs. (vi) for  $M$  from Eq. (iv) and  $\partial M/\partial M_A$  from the second of Eqs. (v) gives

$$-\frac{7PR}{8\pi} = M_A + \frac{3}{2}N_A R \quad (\text{viii})$$

Then, subtracting Eq. (viii) from Eq. (vi),

$$-\frac{PR}{2\pi} + \frac{7PR}{8\pi} = -\frac{N_A}{2}R$$

from which

$$N_A = -\frac{3P}{4\pi}$$

Now substituting in Eq. (vii) (or Eq. (viii)) for  $N_A$ ,

$$MA = \frac{PR}{4\pi}$$

Alternatively, Eqs. (vii) and (viii) may be written in matrix form as follows:

$$\frac{PR}{\pi} \begin{Bmatrix} -1/2 \\ -7/8 \end{Bmatrix} = \begin{bmatrix} 1 & R \\ 1 & 3R/2 \end{bmatrix} \begin{Bmatrix} M_A \\ N_A \end{Bmatrix} \quad (\text{ix})$$

so that

$$\begin{Bmatrix} M_A \\ N_A \end{Bmatrix} = \frac{PR}{\pi} \begin{bmatrix} 1 & R \\ 1 & 3R/2 \end{bmatrix}^{-1} \begin{Bmatrix} -1/2 \\ -7/8 \end{Bmatrix}$$

or

$$\begin{Bmatrix} M_A \\ N_A \end{Bmatrix} = \frac{PR}{\pi} \begin{bmatrix} 3 & -2 \\ -2/R & 2/R \end{bmatrix} \begin{Bmatrix} -1/2 \\ -7/8 \end{Bmatrix}$$

which gives

$$M_A = \frac{PR}{4\pi}, \quad N_A = \frac{-3P}{4\pi}$$

as before.

The bending moment distribution follows from Eq. (iv) and is

$$M = \frac{PR}{4\pi} (2 - \cos\theta - 2\theta \sin\theta) \quad (\text{x})$$

The solution of Eq. (ix) involves the inversion of the matrix

$$\begin{bmatrix} 1 & R \\ 1 & 3R/2 \end{bmatrix}$$

which may be carried out using any of the standard methods detailed in texts on matrix analysis. In this example, Eqs. (vii) and (viii) are clearly most easily solved directly; however, the matrix approach illustrates the technique and serves as a useful introduction to the more detailed discussion in Chapter 6.

### Example 5.13

A two-cell fuselage has circular frames with a rigidly attached straight member across the middle. The bending stiffness of the lower half of the frame is  $2EI$ , while that of the upper half and also the straight member is  $EI$ .

Calculate the distribution of the bending moment in each part of the frame for the loading system shown in Fig. 5.17(a). Illustrate your answer by means of a sketch and show clearly the bending moment carried by each part of the frame at the junction with the straight member. Deformations due only to bending strains need be taken into account.

The loading is antisymmetrical, so that there are no bending moments or normal forces on the plane of antisymmetry; there remain three shear loads  $S_A$ ,  $S_D$ , and  $S_C$ , as shown in Fig. 5.17(b). The total complementary energy of the half-frame is then (neglecting shear strains)

$$C = \int_{\text{half-frame}} \int_0^M d\theta dM - M_0 \alpha_B - \frac{M_0}{r} \Delta_B \quad (\text{i})$$

where  $\alpha_B$  and  $\Delta_B$  are the rotation and deflection of the frame at B caused by the applied moment  $M_0$  and concentrated load  $M_0/r$ , respectively. From antisymmetry, there is no deflection at A, D, or C so that  $S_A$ ,  $S_C$ , and  $S_D$  make no contribution to the total complementary energy. In addition, overall equilibrium of the half-frame gives

$$S_A + S_D + S_C = \frac{M_0}{r} \quad (\text{ii})$$

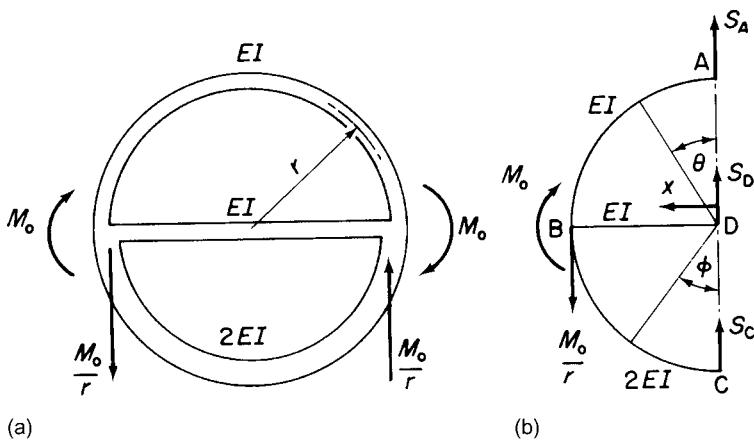


FIGURE 5.17 Determination of Bending Moment Distribution in an Antisymmetrical Fuselage Frame

Assigning stationary values to the total complementary energy and considering the half-frame only, we have

$$\frac{\partial C}{\partial S_A} = \int_{\text{half-frame}} d\theta \frac{\partial M}{\partial S_A} = 0$$

and

$$\frac{\partial C}{\partial S_D} = \int_{\text{half-frame}} d\theta \frac{\partial M}{\partial S_D} = 0$$

or, assuming linear elasticity,

$$\int_{\text{half-frame}} \frac{M}{EI} \frac{\partial M}{\partial S_A} ds = \int_{\text{half-frame}} \frac{M}{EI} \frac{\partial M}{\partial S_D} ds = 0 \quad (\text{iii})$$

In AB,

$$M = -S_A r \sin\theta \quad \text{and} \quad \frac{\partial M}{\partial S_A} = -r \sin\theta, \quad \frac{\partial M}{\partial S_D} = 0$$

In DB,

$$M = S_D x \quad \text{and} \quad \frac{\partial M}{\partial S_A} = 0, \quad \frac{\partial M}{\partial S_D} = x$$

In CB, where, from Eq. (ii),  $S_C = \frac{M_0}{r} - S_A - S_D$ .

$$M = S_C r \sin\phi = \left( \frac{M_0}{r} - S_A - S_D \right) r \sin\phi$$

Thus,

$$\frac{\partial M}{\partial S_A} = -r \sin\phi \quad \text{and} \quad \frac{\partial M}{\partial S_D} = -r \sin\phi$$

Substituting these expressions in Eq. (iii) and integrating as in Example 5.12, we have

$$3.365S_A + S_C = M_0/r \quad (\text{iv})$$

$$S_A + 2.178S_C = M_0/r \quad (\text{v})$$

which, with Eq. (ii), enable  $S_A$ ,  $S_D$ , and  $S_C$  to be found. In matrix form, these equations are written

$$\begin{Bmatrix} M_0/r \\ M_0/r \\ M_0/r \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3.365 & 0 & 1 \\ 1 & 0 & 2.178 \end{bmatrix} \begin{Bmatrix} S_A \\ S_D \\ S_C \end{Bmatrix} \quad (\text{vi})$$

from which we obtain

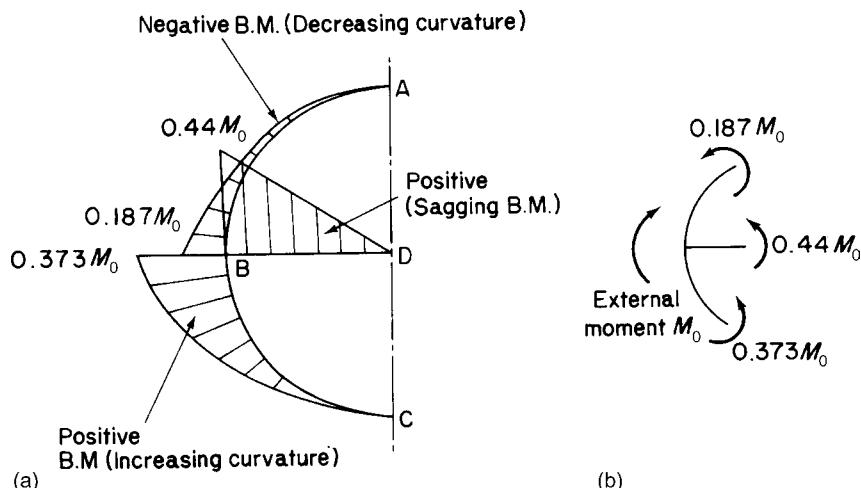
$$\begin{Bmatrix} S_A \\ S_D \\ S_C \end{Bmatrix} = \begin{bmatrix} 0 & 0.345 & -0.159 \\ 1 & -0.187 & -0.373 \\ 0 & -0.159 & 0.532 \end{bmatrix} \begin{Bmatrix} M_0/r \\ M_0/r \\ M_0/r \end{Bmatrix} \quad (\text{vii})$$

which give

$$S_A = 0.187 M_0/r, \quad S_D = 0.44 M_0/r, \quad S_C = 0.373 M_0/r$$

Again, the square matrix of Eq. (vi) has been inverted to produce Eq. (vii). Alternatively, Eqs. (iv) and (v) may be solved simultaneously.

The bending moment distribution with directions of bending moment is shown in Fig. 5.18. ■



**FIGURE 5.18 Distribution of Bending Moment in the Frame of Example 5.13**

So far in this chapter we considered the application of the principle of the stationary value of the total complementary energy of elastic systems in the analysis of various types of structure. Although the majority of the examples used to illustrate the method are of linearly elastic systems, it was pointed out that generally they may be used with equal facility for the solution of nonlinear systems.

In fact, the question of whether a structure possesses linear or nonlinear characteristics arises only after the initial step of writing down expressions for the total potential or complementary energies. However, a great number of structures are linearly elastic and possess unique properties that enable solutions, in some cases, to be more easily obtained. The remainder of this chapter is devoted to these methods.

## 5.5 UNIT LOAD METHOD

In [Section 5.3](#), we discussed the *dummy* or *fictitious* load method of obtaining deflections of structures. For a linearly elastic structure, the method may be streamlined as follows.

Consider the framework of [Fig. 5.3](#), in which we require, say, to find the vertical deflection of the point C. Following the procedure of [Section 5.3](#), we would place a vertical dummy load  $P_f$  at C and write down the total complementary energy of the framework (see [Eq. \(5.9\)](#)):

$$C = \sum_{i=1}^k \int_0^{F_i} \lambda_i dF_i - \sum_{r=1}^n \Delta_r P_r$$

For a stationary value of  $C$ ,

$$\frac{\partial C}{\partial P_f} = \sum_{i=1}^k \lambda_i \frac{\partial F_i}{\partial P_f} - \Delta_C = 0 \quad (5.12)$$

from which

$$\Delta_C = \sum_{i=1}^k \lambda_i \frac{\partial F_i}{\partial P_f} \quad (5.13)$$

as before. If, instead of the arbitrary dummy load  $P_f$ , we had placed a unit load at C, then the load in the  $i$ th linearly elastic member would be

$$F_i = \frac{\partial F_i}{\partial P_f} 1$$

Therefore, the term  $\partial F_i / \partial P_f$  in [Eq. \(5.13\)](#) is equal to the load in the  $i$ th member due to a unit load at C, and [Eq. \(5.13\)](#) may be written

$$\Delta_C = \sum_{i=1}^k \frac{F_{i,0} F_{i,1} L_i}{A_i E_i} \quad (5.14)$$

where  $F_{i,0}$  is the force in the  $i$ th member due to the actual loading and  $F_{i,1}$  is the force in the  $i$ th member due to a unit load placed at the position and in the direction of the required deflection. Thus, in [Example 5.1](#), columns ④ and ⑥ in [Table 5.1](#) would be eliminated, leaving column ⑤ as  $F_{B,1}$  and column ⑦ as  $F_{D,1}$ . Obviously column ③ is  $F_0$ .

Similar expressions for deflection due to bending and torsion of linear structures follow from the well-known relationships between bending and rotation and torsion and rotation. Hence, for a member of length  $L$  and flexural and torsional rigidities  $EI$  and  $GJ$ , respectively,

$$\Delta_{B,M} = \int_L \frac{M_0 M_1}{EI} dz, \quad \Delta_T = \int_L \frac{T_0 T_1}{GJ} dz \quad (5.15)$$

where  $M_0$  is the bending moment at any section produced by the actual loading and  $M_1$  is the bending moment at any section due to a unit load applied at the position and in the direction of the required deflection; similarly for torsion.

Generally, shear deflections of slender beams are ignored but may be calculated when required for particular cases. Of greater interest in aircraft structures is the calculation of the deflections produced by the large shear stresses experienced by thin-walled sections. This problem is discussed in [Chapter 20](#).

**Example 5.14**

A steel rod of uniform circular cross-section is bent as shown in Fig. 5.19, AB and BC being horizontal and CD vertical. The arms AB, BC, and CD are of equal length. The rod is encastré at A and the other end D is free. A uniformly distributed load covers the length BC. Find the components of the displacement of the free end D in terms of  $EI$  and  $GJ$ .

Since the cross-sectional area  $A$  and modulus of elasticity  $E$  are not given, we assume that displacements due to axial distortion are to be ignored. We place, in turn, unit loads in the assumed positive directions of the axes  $xyz$ .

First, consider the displacement in the direction parallel to the  $x$  axis. From Eqs. (5.15),

$$\Delta_x = \int_L \frac{M_0 M_1}{EI} ds + \int_L \frac{T_0 T_1}{GJ} ds$$

Employing a tabular procedure,

Plane	$M_0$			$M_1$			$T_0$			$T_1$		
	$xy$	$xz$	$yz$	$xy$	$xz$	$yz$	$xy$	$xz$	$yz$	$xy$	$xz$	$yz$
CD	0	0	0	y	0	0	0	0	0	0	0	0
CB	0	0	$-wz^2/2$	0	z	0	0	0	0	l	0	0
BA	$-wlx$	0	0	l	l	0	0	0	$wl^2/2$	0	0	0

Hence, substituting for  $M_0$ ,  $M_1$ , etc. in the expression for  $\Delta_x$  and noting that the second integral involving torsion disappears,

$$\Delta_x = \int_0^l -\frac{wl^2 x}{EI} dx$$

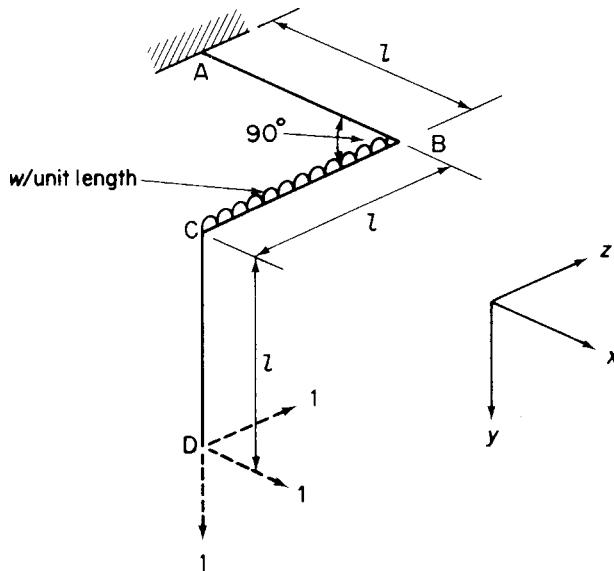


FIGURE 5.19 Deflection of a Bent Rod

or

$$\Delta_x = -\frac{wl^4}{2EI}$$

Similarly,

$$\Delta_y = wl^4 \left( \frac{11}{24EI} + \frac{1}{2GJ} \right)$$

$$\Delta_z = wl^4 \left( \frac{1}{6EI} + \frac{1}{2GJ} \right)$$

## 5.6 FLEXIBILITY METHOD

An alternative approach to the solution of statically indeterminate beams and frames is to release the structure—that is, remove redundant members or supports—until the structure becomes statically determinate. The displacement of some point in the released structure is then determined by, say, the unit load method. The actual loads on the structure are removed and unknown forces applied to the points where the structure has been released; the displacement at the point produced by these unknown forces must, from compatibility, be the same as that in the released structure. The unknown forces are then obtained; this approach is known as *the flexibility method*.

### Example 5.15

Determine the forces in the members of the truss shown in Fig. 5.20(a); the cross-sectional area  $A$  and Young's modulus  $E$  are the same for all members.

The truss in Fig. 5.20(a) is clearly externally statically determinate but has a degree of internal statical indeterminacy equal to 1. We therefore release the truss so that it becomes statically determinate by “cutting” one of the members, say BD, as shown in Fig. 5.20(b). Due to the actual loads ( $P$  in this case), the cut ends of the member BD

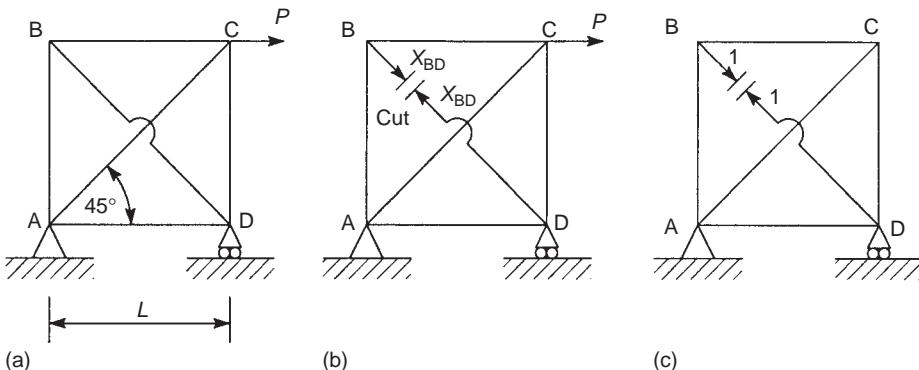


FIGURE 5.20 Analysis of a Statically Indeterminate Truss

separate or come together, depending on whether the force in the member (before it was cut) is tensile or compressive; we shall assume that it was tensile.

We are assuming that the truss is linearly elastic, so that the relative displacement of the cut ends of the member BD (in effect, the movement of B and D away from or toward each other along the diagonal BD) may be found using, say, the unit load method. Thus, we determine the forces  $F_{a,j}$ , in the members produced by the actual loads. We then apply equal and opposite unit loads to the cut ends of the member BD, as shown in Fig 5.20(c), and calculate the forces,  $F_{1,j}$  in the members. The displacement of B relative to D,  $\Delta_{BD}$  (see Eq. (ii) in Ex. 4.9), is given by

$$\Delta_{BD} = \sum_{j=1}^n \frac{F_{a,j} F_{1,j} L_j}{AE}$$

The forces,  $F_{a,j}$ , are the forces in the members of the released truss due to the actual loads and are not, therefore, the actual forces in the members of the complete truss. We therefore redesignate the forces in the members of the released truss as  $F_{0,j}$ . The expression for  $\Delta_{BD}$  becomes

$$\Delta_{BD} = \sum_{j=1}^n \frac{F_{0,j} F_{1,j} L_j}{AE} \quad (\text{i})$$

In the actual structure, this displacement is prevented by the force,  $X_{BD}$ , in the redundant member BD. If, therefore, we calculate the displacement,  $a_{BD}$ , in the direction of BD produced by a unit value of  $X_{BD}$ , the displacement due to  $X_{BD}$  is  $X_{BD}a_{BD}$ . Clearly, from compatibility,

$$\Delta_{BD} + X_{BD}a_{BD} = 0 \quad (\text{ii})$$

from which  $X_{BD}$  is found;  $a_{BD}$  is a *flexibility coefficient*. Having determined  $X_{BD}$ , the actual forces in the members of the complete truss may be calculated by, say, the method of joints or the method of sections.

In Eq. (ii),  $a_{BD}$  is the displacement of the released truss in the direction of BD produced by a unit load. Therefore, in using the unit load method to calculate this displacement, the actual member forces ( $F_{1,j}$ ) and the member forces produced by the unit load ( $F_{1,j}$ ) are the same. Therefore, from Eq. (i),

$$a_{BD} = \sum_{j=1}^n \frac{F_{1,j}^2 L_j}{AE} \quad (\text{iii})$$

The solution is completed in Table 5.6.

**Table 5.6 Example 5.15**

Member	$L_j$ (m)	$F_{0,j}$	$F_{1,j}$	$F_{0,j} F_{1,j} L_j$	$F_{1,j}^2 L_j$	$F_{a,j}$
AB	$L$	0	-0.71	0	$0.5L$	+0.40P
BC	$L$	0	-0.71	0	$0.5L$	+0.40P
CD	$L$	$-P$	-0.71	$0.71PL$	$0.5L$	-0.60P
BD	$1.41L$	—	1.0	—	$1.41L$	-0.56P
AC	$1.41L$	$1.41P$	1.0	$2.0PL$	$1.41L$	+0.85P
AD	$L$	0	-0.71	0	$0.5L$	+0.40P
				$\sum = 2.71PL$	$\sum = 4.82L$	

From Table 5.6,

$$\Delta_{BD} = \frac{2.71PL}{AE}, \quad a_{BD} = \frac{4.82L}{AE}$$

Substituting these values in Eq. (i), we have

$$\frac{2.71PL}{AE} + X_{BD} \frac{4.82L}{AE} = 0$$

from which

$$X_{BD} = -0.56P$$

(i.e., compression). The actual forces,  $F_{aj}$ , in the members of the complete truss of Fig. 5.20(a) are now calculated using the method of joints and are listed in the final column of Table 5.6.

We note in the preceding that  $\Delta_{BD}$  is positive, which means that  $\Delta_{BD}$  is in the direction of the unit loads; that is, B approaches D and the diagonal BD in the released structure decreases in length. Therefore, in the complete structure, the member BD, which prevents this shortening, must be in compression as shown; also  $a_{BD}$  will always be positive, since it contains the term  $F_{1,j}^2$ . Finally, we note that the cut member BD is included in the calculation of the displacements in the released structure, since its deformation, under a unit load, contributes to  $a_{BD}$ .

### Example 5.16

Calculate the forces in the members of the truss shown in Fig. 5.21(a). All members have the same cross-sectional area  $A$  and Young's modulus  $E$ .

By inspection we see that the truss is both internally and externally statically indeterminate, since it would remain stable and in equilibrium if one of the diagonals, AD or BD, and the support at C were removed; the degree of indeterminacy is therefore 2. Unlike the truss in Example 5.15, we could not remove *any* member, since if BC or CD were removed, the outer half of the truss would become a mechanism while the portion ABDE would remain statically indeterminate. Therefore, we select AD and the support at C as the releases, giving the statically determinate truss shown in Fig. 5.21(b); we designate the force in the member AD as  $X_1$  and the vertical reaction at C as  $R_2$ .

In this case, we shall have two compatibility conditions, one for the diagonal AD and one for the support at C. We therefore need to investigate three loading cases: one in which the actual loads are applied to the released statically determinate truss in Fig. 5.21(b); a second in which unit loads are applied to the cut member AD (Fig. 5.21(c)); and a third in which a unit load is applied at C in the direction of  $R_2$  (Fig. 5.21(d)). By comparison with the previous example, the compatibility conditions are

$$\Delta_{AD} + a_{11}X_1 + a_{12}R_2 = 0 \quad (i)$$

$$v_C + a_{21}X_1 + a_{22}R_2 = 0 \quad (ii)$$

in which  $\Delta_{AD}$  and  $v_C$  are, respectively, the change in length of the diagonal AD and the vertical displacement of C due to the actual loads acting on the released truss, while  $a_{11}$ ,  $a_{12}$ , and so forth, are flexibility coefficients, which we have previously defined. The calculations are similar to those carried out in Example 5.15 and are shown in Table 5.7. From Table 5.7,

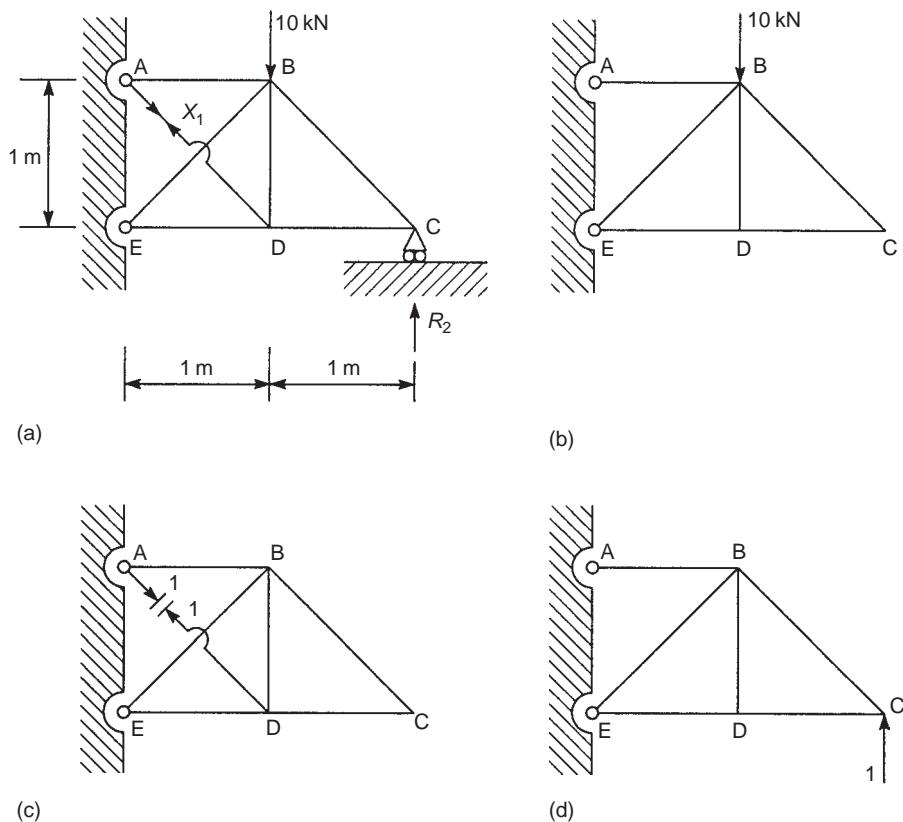


FIGURE 5.21 Statically Indeterminate Truss of Example 5.16

Table 5.7 Example 5.16

Member	$L_j$	$F_{0j}$	$F_{1j}$ ( $X_1$ )	$F_{1j}$ ( $R_2$ )	$F_{0j}F_{1j}$ ( $X_1$ ) $L_j$	$F_{0j}F_{1j}$ ( $R_2$ ) $L_j$	$F_{1^2j}$ ( $X_1$ ) $L_j$	$F_{1^2j}$ ( $R_2$ ) $L_j$	$F_{1,j}(X_1)$ $F_{1,j}(R_2)L_j$	$F_{aj}$
AB	1	10.0	-0.71	-2.0	-7.1	-20.0	0.5	4.0	1.41	0.67
BC	1.41	0	0	-1.41	0	0	0	2.81	0	-4.45
CD	1	0	0	1.0	0	0	0	1.0	0	3.15
DE	1	0	-0.71	1.0	0	0	0.5	1.0	-0.71	0.12
AD	1.41	0	1.0	0	0	0	1.41	0	0	4.28
BE	1.41	-14.14	1.0	1.41	-20.0	-28.11	1.41	2.81	2.0	-5.4
BD	1	0	-0.71	0	0	0	0.5	0	0	-3.03
					$\sum = -27.1$	$\sum = -48.11$	$\sum = 4.32$	$\sum = 11.62$	$\sum = 2.7$	

$$\Delta_{AD} = \sum_{j=1}^n \frac{F_{0,j} F_{1,j}(X_1) L_j}{AE} = \frac{-27.1}{AE} \quad (\text{i.e., AD increases in length})$$

$$v_C = \sum_{j=1}^n \frac{F_{0,j} F_{1,j}(R_2) L_j}{AE} = \frac{-48.11}{AE} \quad (\text{i.e., C is displaced downward})$$

$$a_{11} = \sum_{j=1}^n \frac{F_{1,j}^2(X_1) L_j}{AE} = \frac{4.32}{AE}$$

$$a_{22} = \sum_{j=1}^n \frac{F_{1,j}^2(R_2) L_j}{AE} = \frac{11.62}{AE}$$

$$a_{12} = a_{21} = \sum_{j=1}^n \frac{F_{1,j}(X_1) F_{1,j}(R_2) L_j}{AE} = \frac{2.7}{AE}$$

Substituting in Eqs. (i) and (ii) and multiplying through by  $AE$ , we have

$$-27.1 + 4.32X_1 + 2.7R_2 = 0 \quad (\text{iii})$$

$$-48.11 + 2.7X_1 + 11.62R_2 = 0 \quad (\text{iv})$$

Solving Eqs. (iii) and (iv), we obtain

$$X_1 = 4.28\text{kN}, R_2 = 3.15\text{kN}$$

The actual forces,  $F_{a,j}$ , in the members of the complete truss are now calculated by the method of joints and are listed in the final column of Table 5.7.

### 5.6.1 Self-straining trusses

Statically indeterminate trusses, unlike the statically determinate type, may be subjected to self-straining, in which internal forces are present before external loads are applied. Such a situation may be caused by a local temperature change or by an initial lack of fit of a member. In cases such as these, the term on the right-hand side of the compatibility equations, Eq. (ii) in Example 5.15 and Eqs. (i) and (ii) in Example 5.16, would not be zero.

#### Example 5.17

The truss shown in Fig. 5.22(a) is unstressed when the temperature of each member is the same, but due to local conditions, the temperature in the member BC is increased by  $30^\circ\text{C}$ . If the cross-sectional area of each member is  $200 \text{ mm}^2$  and the coefficient of linear expansion of the members is  $7 \times 10^{-6}/^\circ\text{C}$ , calculate the resulting forces in the members; Young's modulus  $E = 200,000 \text{ N/mm}^2$ .

Due to the temperature rise, the increase in length of the member BC is  $3 \times 10^3 \times 30 \times 7 \times 10^{-6} = 0.63 \text{ mm}$ . The truss has a degree of internal statical indeterminacy equal to 1 (by inspection). We therefore release the truss by cutting the member BC, which has experienced the temperature rise, as shown in Fig. 5.22(b); we suppose that the force in BC is  $X_1$ . Since there are no external loads on the truss,  $\Delta_{BC}$  is zero and the compatibility condition becomes

$$a_{11}X_1 = -0.63 \quad (\text{i})$$

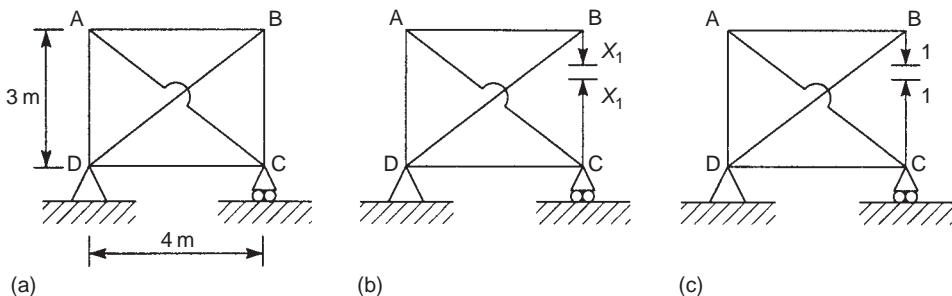


FIGURE 5.22 Self-Straining Due to a Temperature Change

Table 5.8 Example 5.17

Member	$L_j$ (mm)	$F_{1,j}$	$F_{1,j}^2 L_j$	$F_{aj}$ (N)
AB	4,000	1.33	7,111.1	-700
BC	3,000	1.0	3,000.0	-525
CD	4,000	1.33	7,111.1	-700
DA	3,000	1.0	3,000.0	-525
AC	5,000	-1.67	13,888.9	875
DB	5,000	-1.67	13,888.9	875
$\sum = 48,000.0$				

in which, as before,

$$a_{11} = \sum_{j=1}^n \frac{F_{1,j}^2 L_j}{AE}$$

Note that the extension of BC is negative, since it is opposite in direction to  $X_1$ . The solution is now completed in Table 5.8. Hence,

$$a_{11} = \frac{48,000}{200 \times 200,000} = 1.2 \times 10^{-3}$$

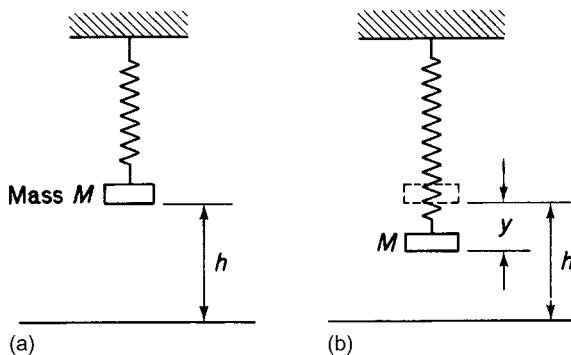
Then, from Eq. (i),

$$X_1 = -525 \text{ N}$$

The forces,  $F_{aj}$ , in the members of the complete truss are given in the final column of Table 5.8. Compare the preceding with the solution of Example 5.9.

## 5.7 TOTAL POTENTIAL ENERGY

In the spring-mass system shown in its unstrained position in Fig. 5.23(a), we normally define the *potential energy* of the mass as the product of its weight,  $Mg$ , and its height,  $h$ , above some arbitrarily fixed datum. In other words, it possesses energy by virtue of its position. After deflection to an



**FIGURE 5.23 (a) Potential Energy of a Spring–Mass System; (b) Loss in Potential Energy Due to a Change in Position**

equilibrium state (Fig. 5.23(b)), the mass has lost an amount of potential energy equal to  $Mgy$ . Therefore, we may associate deflection with a loss of potential energy. Alternatively, we may argue that the gravitational force acting on the mass does work during its displacement, resulting in a loss of energy. Applying this reasoning to the elastic system of Fig. 5.1(a) and assuming that the potential energy of the system is zero in the unloaded state, the *loss* of potential energy of the load  $P$  as it produces a deflection  $y$  is  $Py$ . Thus, the potential energy  $V$  of  $P$  in the deflected equilibrium state is given by

$$V = -Py$$

We now define the *total potential energy* (TPE) of a system in its deflected equilibrium state as the sum of its internal or strain energy and the potential energy of the applied external forces. Hence, for the single member-force configuration of Fig. 5.1(a),

$$\text{TPE} = U + V = \int_0^y P \, dy - Py$$

For a general system consisting of loads  $P_1, P_2, \dots, P_n$  producing *corresponding displacements* (i.e., displacements in the directions of the loads),  $\Delta_1, \Delta_2, \dots, \Delta_n$  the potential energy of all the loads is

$$V = \sum_{r=1}^n V_r = \sum_{r=1}^n (-P_r \Delta_r)$$

and the total potential energy of the system is given by

$$\text{TPE} = U + V = U + \sum_{r=1}^n (-P_r \Delta_r) \quad (5.16)$$

## 5.8 PRINCIPLE OF THE STATIONARY VALUE OF THE TOTAL POTENTIAL ENERGY

Let us now consider an elastic body in equilibrium under a series of external loads,  $P_1, P_2, \dots, P_n$ , and suppose that we impose small virtual displacements  $\delta\Delta_1, \delta\Delta_2, \dots, \delta\Delta_n$  in the directions of the loads. The virtual work done by the loads is

$$\sum_{r=1}^n P_r \delta \Delta_r$$

This work is accompanied by an increment of strain energy  $\delta U$  in the elastic body, since by specifying virtual displacements of the loads we automatically impose virtual displacements on the particles of the body itself, as the body is continuous and is assumed to remain so. This increment in strain energy may be regarded as negative virtual work done by the particles so that the total work done during the virtual displacement is

$$-\delta U + \sum_{r=1}^n P_r \delta \Delta_r$$

The body is in equilibrium under the applied loads, so that by the principle of virtual work, this expression must be equal to zero. Hence,

$$\delta U - \sum_{r=1}^n P_r \delta \Delta_r = 0 \quad (5.17)$$

The loads  $P_r$  remain constant during the virtual displacement; therefore, Eq. (5.17) may be written

$$\delta U - \delta \sum_{r=1}^n P_r \Delta_r = 0$$

or, from Eq. (5.16),

$$\delta(U + V) = 0 \quad (5.18)$$

Thus, the total potential energy of an elastic system has a stationary value for all small displacements if the system is in equilibrium. It may also be shown that if the stationary value is a minimum, the equilibrium is stable. A qualitative demonstration of this fact is sufficient for our purposes, although mathematical proofs exist.<sup>1</sup> In Fig. 5.24, the positions A, B, and C of a particle correspond to different equilibrium states. The total potential energy of the particle in each of its three positions is proportional to its height  $h$  above some arbitrary datum, since we are considering a single particle for which the

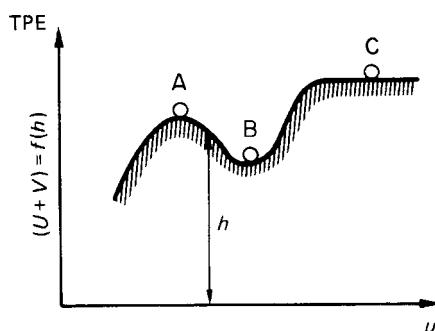


FIGURE 5.24 States of Equilibrium of a Particle

strain energy is zero. Clearly, at each position, the first-order variation,  $\partial(U + V)/\partial u$ , is zero (indicating equilibrium), but only at B, where the total potential energy is a minimum, is the equilibrium stable. At A and C, we have unstable and neutral equilibrium, respectively.

To summarize, the principle of the stationary value of the total potential energy may be stated as follows:

*The total potential energy of an elastic system has a stationary value for all small displacements when the system is in equilibrium; further, the equilibrium is stable if the stationary value is a minimum.*

This principle is often used in the approximate analysis of structures where an exact analysis does not exist. We illustrate the application of the principle in [Example 5.18](#), where we suppose that the displaced form of the beam is unknown and must be assumed; this approach is called the *Rayleigh–Ritz* method.

### Example 5.18

Determine the deflection of the mid-span point of the linearly elastic, simply supported beam shown in [Fig. 5.25](#); the flexural rigidity of the beam is  $EI$ .

The assumed displaced shape of the beam must satisfy the boundary conditions for the beam. Generally, trigonometric or polynomial functions have been found to be the most convenient, where, however, the simpler the function, the less accurate is the solution. Let us suppose that the displaced shape of the beam is given by

$$v = v_B \sin \frac{\pi z}{L} \quad (\text{i})$$

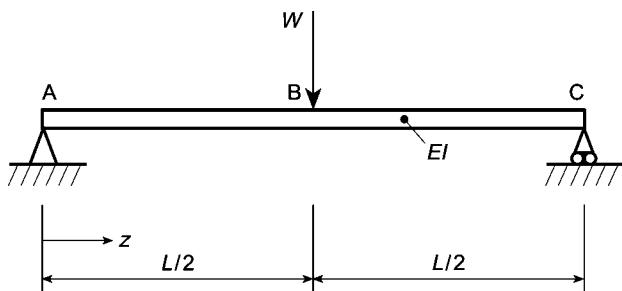
in which  $v_B$  is the displacement at the mid-span point. From [Eq. \(i\)](#), we see that  $v = 0$  when  $z = 0$  and  $z = L$  and that  $v = v_B$  when  $z = L/2$ . Also  $dv/dz = 0$  when  $z = L/2$ , so that the displacement function satisfies the boundary conditions of the beam.

The strain energy,  $U$ , due to bending of the beam, is given by<sup>3</sup>

$$U = \int_L \frac{M^2}{2EI} dz \quad (\text{ii})$$

Also,

$$M = -EI \frac{d^2 v}{dz^2} \quad (\text{see Chapter 16}) \quad (\text{iii})$$



**FIGURE 5.25 Approximate Determination of Beam Deflection Using Total Potential Energy**

From Eq. (i),

$$\frac{dv}{dz} = v_B \frac{\pi}{L} \cos \frac{\pi z}{L}$$

and

$$\frac{d^2v}{dz^2} = -v_B \frac{\pi^2}{L^2} \sin \frac{\pi z}{L}$$

Substituting in Eq. (iii),

$$M = EI v_B \frac{\pi^2}{L^2} \sin \frac{\pi z}{L}$$

so that, from Eq. (ii),

$$U = \frac{EI}{2} \int_0^L \frac{v_B^2 \pi^4}{L^4} \sin^2 \frac{\pi z}{L} dz$$

that is,

$$U = \frac{EI v_B^2 \pi^4}{2 L^4} \int_0^L \frac{1}{2} \left( 1 - \cos \frac{2\pi z}{L} \right) dz$$

giving

$$U = \frac{\pi^4 E I v_B^2}{4 L^3}$$

The total potential energy of the beam is then given by

$$TPE = U + V = \frac{\pi^4 E l v_B^2}{4 L^3} - W v_B$$

Then, from the principle of the stationary value of the total potential energy,

$$\frac{\partial(U + V)}{\partial v_B} = \frac{\pi^4 E l v_B}{2 L^3} - W = 0$$

from which

$$v_B = \frac{2 W L^3}{\pi^4 E I} = 0.02053 \frac{W L^3}{E I} \quad (\text{iv})$$

The exact expression for the mid-span displacement is<sup>3</sup>

$$v_B = \frac{W L^3}{48 E I} = 0.02083 \frac{W L^3}{E I} \quad (\text{v})$$

Comparing the exact (Eq. (v)) and approximate results (Eq. (iv)), we see that the difference is less than 2%. Further, the approximate displacement is less than the exact displacement, since, by assuming a displaced shape, we have, in effect, forced the beam into taking that shape by imposing restraint; the beam is therefore stiffer.

## 5.9 PRINCIPLE OF SUPERPOSITION

An extremely useful principle employed in the analysis of linearly elastic structures is that of superposition. The principle states that if the displacements at all points in an elastic body are proportional to the forces producing them—that is, the body is linearly elastic—the effect on such a body of a number of forces is the sum of the effects of the forces applied separately. We make immediate use of the principle in the derivation of the reciprocal theorem in the following section.

## 5.10 RECIPROCAL THEOREM

The reciprocal theorem is an exceptionally powerful method of analysis of linearly elastic structures and is accredited, in turn, to Maxwell, Betti, and Rayleigh. However, before we establish the theorem, we consider a useful property of linearly elastic systems resulting from the principle of superposition. This principle enables us to express the deflection of any point in a structure in terms of a constant coefficient and the applied loads. For example, a load  $P_1$  applied at a point 1 in a linearly elastic body produces a deflection  $\Delta_1$  at the point given by

$$\Delta_1 = a_{11}P_1$$

in which the *influence* or *flexibility* coefficient  $a_{11}$  is defined as the deflection at the point 1 in the direction of  $P_1$ , produced by a unit load at the point 1 applied in the direction of  $P_1$ . Clearly, if the body supports a system of loads such as those shown in Fig. 5.26, each of the loads  $P_1, P_2, \dots, P_n$  contributes to the deflection at the point 1. Thus, the *corresponding deflection*  $\Delta_1$  at the point 1 (i.e., the total deflection in the direction of  $P_1$  produced by all the loads) is

$$\Delta_1 = a_{11}P_1 + a_{12}P_2 + \cdots + a_{1n}P_n$$

where  $a_{12}$  is the deflection at the point 1 in the direction of  $P_1$  produced by a unit load at the point 2 in the direction of the load  $P_2$ , and so on. The corresponding deflections at the points of application of the complete system of loads are

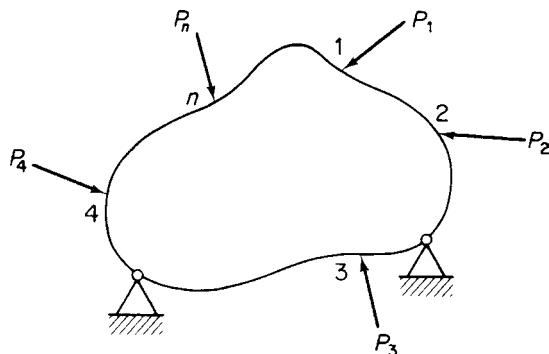


FIGURE 5.26 Linearly Elastic Body Subjected to Loads  $P_1, P_2, P_3, \dots, P_n$

$$\left. \begin{aligned} \Delta_1 &= a_{11}P_1 + a_{12}P_2 + a_{13}P_3 + \cdots + a_{1n}P_n \\ \Delta_2 &= a_{21}P_1 + a_{22}P_2 + a_{23}P_3 + \cdots + a_{2n}P_n \\ \Delta_3 &= a_{31}P_1 + a_{32}P_2 + a_{33}P_3 + \cdots + a_{3n}P_n \\ &\vdots \\ \Delta_n &= a_{n1}P_1 + a_{n2}P_2 + a_{n3}P_3 + \cdots + a_{nn}P_n \end{aligned} \right\} \quad (5.19)$$

or, in matrix form,

$$\left\{ \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \vdots \\ \Delta_n \end{array} \right\} = \left[ \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{array} \right] \left\{ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \end{array} \right\}$$

which may be written in shorthand matrix notation as

$$\{\Delta\} = [A]\{P\}$$

Suppose now that an elastic body is subjected to a gradually applied force  $P_1$  at a point 1 and then, while  $P_1$  remains in position, a force  $P_2$  is gradually applied at another point 2. The total strain energy  $U$  of the body is given by

$$U_1 = \frac{P_1}{2}(a_{11}P_1) + \frac{P_2}{2}(a_{22}P_2) + P_1(a_{12}P_2) \quad (5.20)$$

The third term on the right-hand side of Eq. (5.20) results from the additional work done by  $P_1$  as it is displaced through a further distance  $a_{12}P_2$  by the action of  $P_2$ . If we now remove the loads and apply  $P_2$  followed by  $P_1$ , we have

$$U_2 = \frac{P_2}{2}(a_{22}P_2) + \frac{P_1}{2}(a_{11}P_1) + P_2(a_{21}P_1) \quad (5.21)$$

By the principle of superposition, the strain energy stored is independent of the order in which the loads are applied. Hence,

$$U_1 = U_2$$

and it follows that

$$a_{12} = a_{21} \quad (5.22)$$

Thus, in its simplest form, the reciprocal theorem states the following:

*The deflection at point 1 in a given direction due to a unit load at point 2 in a second direction is equal to the deflection at point 2 in the second direction due to a unit load at point 1 in the first direction.*

In a similar manner, we derive the relationship between moments and rotations:

*The rotation at a point 1 due to a unit moment at a point 2 is equal to the rotation at point 2 produced by a unit moment at point 1.*

Finally, we have

*The rotation at point 1 due to a unit load at point 2 is numerically equal to the deflection at point 2 in the direction of the unit load due to a unit moment at point 1.*

**Example 5.19**

A cantilever 800 mm long with a prop 500 mm from the wall deflects in accordance with the following observations when a point load of 40 N is applied to its end:

Distance (mm)	0	100	200	300	400	500	600	700	800
Deflection (mm)	0	-0.3	-1.4	-2.5	-1.9	0	2.3	4.8	10.6

What is the angular rotation of the beam at the prop due to a 30 N load applied 200 mm from the wall, together with a 10 N load applied 350 mm from the wall?

The initial deflected shape of the cantilever is plotted as shown in Fig. 5.27(a) and the deflections at D and E produced by the 40 N load determined. The solution then proceeds as follows.

Deflection at D due to 40 N load at C = -1.4 mm.

Hence, from the reciprocal theorem, the deflection at C due to a 40 N load at D = -1.4 mm.

It follows that the deflection at C due to a 30 N load at D =  $-3/4 \times 1.4 = -1.05$  mm.

Similarly, the deflection at C due to a 10 N load at E =  $-1/4 \times 2.4 = -0.6$  mm.

Therefore, the total deflection at C, produced by the 30 and 10 N loads acting simultaneously (Fig. 5.27(b)), is  $-1.05 - 0.6 = -1.65$  mm, from which the angular rotation of the beam at B,  $\theta_B$ , is given by

$$\theta_B = \tan^{-1} \frac{1.65}{300} = \tan^{-1} 0.0055$$

or

$$\theta_B = 0^\circ 19'$$

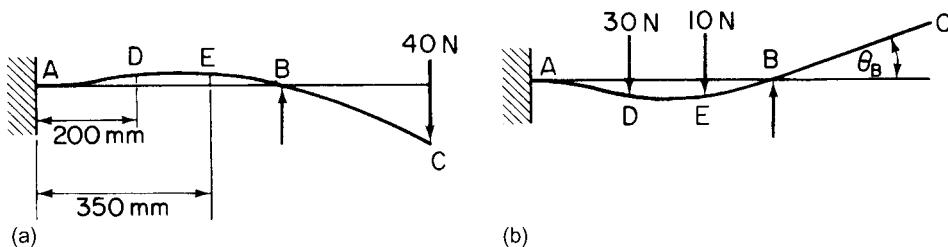


FIGURE 5.27 (a) Given Deflected Shape of Propped Cantilever; (b) Determination of the Deflection of C

**Example 5.20**

An elastic member is pinned to a drawing board at its ends A and B. When a moment  $M$  is applied at A, A rotates  $\theta_A$ , B rotates  $\theta_B$ , and the center deflects  $\delta_1$ . The same moment  $M$  applied to B rotates B,  $\theta_C$ , and deflects the center through  $\delta_2$ . Find the moment induced at A when a load  $W$  is applied to the center in the direction of the measured deflections, both A and B being restrained against rotation.

The three load conditions and the relevant displacements are shown in Fig. 5.28. From Figs. 5.28(a) and (b), the rotation at A due to  $M$  at B is, from the reciprocal theorem, equal to the rotation at B due to  $M$  at A. Hence,

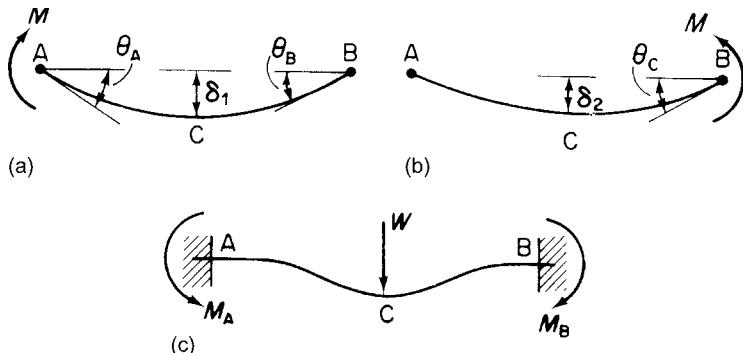


FIGURE 5.28 Model Analysis of a Fixed Beam

$$\theta_{A(b)} = \theta_B$$

It follows that the rotation at A due to  $M_B$  at B is

$$\theta_{A(c),1} = (M_B/M)\theta_B \quad (\text{i})$$

Also the rotation at A due to unit load at C is equal to the deflection at C due to unit moment at A. Therefore,

$$\frac{\theta_{A(c),2}}{W} = \frac{\delta_1}{M}$$

or

$$\theta_{A(c),2} = \frac{W}{M}\delta_1 \quad (\text{ii})$$

where  $\theta_{A(c),2}$  is the rotation at A due to  $W$  at C. Finally, the rotation at A due to  $M_A$  at A is, from Fig. 5.28(a) and (c),

$$\theta_{A(c),3} = \frac{M_A}{M}\theta_A \quad (\text{iii})$$

The total rotation at A produced by  $M_A$  at A,  $W$  at C, and  $M_B$  at B is, from Eqs. (i), (ii), and (iii),

$$\theta_{A(c),1} + \theta_{A(c),2} + \theta_{A(c),3} = \frac{M_B}{M}\theta_B + \frac{W}{M}\delta_1 + \frac{M_A}{M}\theta_A = 0 \quad (\text{iv})$$

since the end A is restrained from rotation. Similarly, the rotation at B is given by

$$\frac{M_B}{M}\theta_C + \frac{W}{M}\delta_2 + \frac{M_A}{M}\theta_B = 0 \quad (\text{v})$$

Then Eq. (iv)  $\times \theta_C -$  Eq. (v)  $\times \theta_B$  gives

$$\frac{W}{M}(\delta_1\theta_C - \delta_2\theta_B) + \frac{M_A}{M}(\theta_A\theta_C - \theta_B^2) = 0$$

from which

$$M_A = W \left( \frac{\delta_2\theta_B - \delta_1\theta_C}{\theta_A\theta_C - \theta_B^2} \right)$$

The fact that the arbitrary moment  $M$  does not appear in the expression for the restraining moment at A (similarly it does not appear in  $M_B$ ), produced by the load  $W$ , indicates an extremely useful application of the reciprocal theorem, namely the model analysis of statically indeterminate structures. For example, the fixed beam of Fig. 5.28(c) could be a full-scale bridge girder. It is then only necessary to construct a model, say of Perspex, having the same flexural rigidity  $EI$  as the full-scale beam and measure rotations and displacements produced by an arbitrary moment  $M$  to obtain fixing moments in the full-scale beam supporting a full-scale load.

## 5.11 TEMPERATURE EFFECTS

A uniform temperature applied across a beam section produces an expansion of the beam, as shown in Fig. 5.29, provided there are no constraints. However, a linear temperature gradient across the beam section causes the upper fibers of the beam to expand more than the lower ones, producing a bending strain, as shown in Fig. 5.30, without the associated bending stresses, again provided no constraints are present.

Consider an element of the beam of depth  $h$  and length  $\delta z$  subjected to a linear temperature gradient over its depth, as shown in Fig. 5.31(a). The upper surface of the element increases in length to  $\delta z(1 + \alpha t)$  (see Section 1.15.1), where  $\alpha$  is the coefficient of linear expansion of the material of the beam. Thus, from Fig. 5.31(b),

$$\frac{R}{\delta z} = \frac{R + h}{\delta z(1 + \alpha t)}$$

giving

$$R = h/\alpha t \quad (5.23)$$

Also,

$$\delta\theta = \delta z/R$$

so that, from Eq. (5.23),

$$\delta\theta = \frac{\delta z\alpha t}{h} \quad (5.24)$$

We may now apply the principle of the stationary value of the total complementary energy in conjunction with the unit load method to determine the deflection  $\Delta_{Te}$ , due to the temperature of any point of the beam shown in Fig. 5.30. We have seen that this principle is equivalent to the application of the principle of virtual work, where virtual forces act through real displacements. Therefore, we may specify

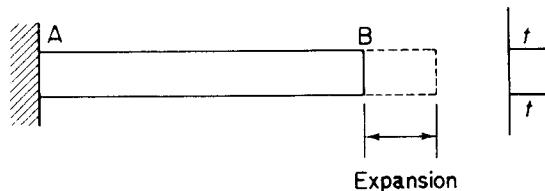


FIGURE 5.29 Expansion of Beam Due to Uniform Temperature

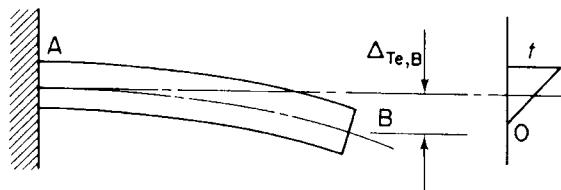


FIGURE 5.30 Bending of Beam Due to Linear Temperature Gradient

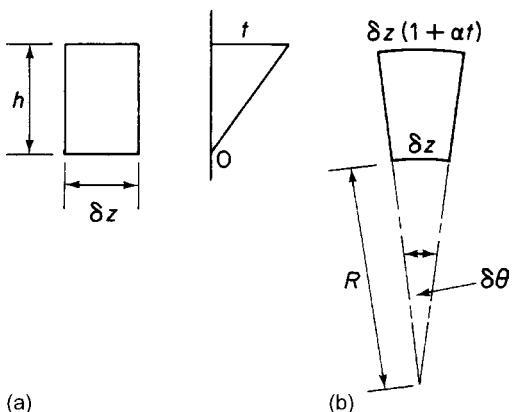


FIGURE 5.31 (a) Linear Temperature Gradient Applied to Beam Element; (b) Bending of Beam Element Due to Temperature Gradient

that the displacements are those produced by the temperature gradient, while the virtual force system is the unit load. Thus, the deflection  $\Delta_{Te,B}$  of the tip of the beam is found by writing down the increment in total complementary energy caused by the application of a virtual unit load at B and equating the resulting expression to zero (see Eqs. (5.7) and Eq. (i) of Example 5.2). Thus,

$$\delta C = \int_L M_1 d\theta - 1 \Delta_{Te,B} = 0$$

or

$$\Delta_{Te,B} = \int_L M_1 d\theta \quad (5.25)$$

where  $M_1$  is the bending moment at any section due to the unit load. Substituting for  $d\theta$  from Eq. (5.24), we have

$$\Delta_{Te,B} = \int_L M_1 \frac{\alpha t}{h} dz \quad (5.26)$$

where  $t$  can vary arbitrarily along the span of the beam but only linearly with depth. For a beam supporting some form of external loading, the total deflection is given by the superposition of the temperature deflection from Eq. (5.26) and the bending deflection from Eq. (5.15); thus,

$$\Delta = \int_L M_1 \left( \frac{M_0}{EI} + \frac{\alpha t}{h} \right) dz \quad (5.27)$$

### Example 5.21

Determine the deflection of the tip of the cantilever in Fig. 5.32 with the temperature gradient shown.

Applying a unit load vertically downward at B,  $M_1 = 1 \times z$ . Also the temperature  $t$  at a section  $z$  is  $t_0(l-z)/l$ . Substituting in Eq. (5.26) gives

$$\Delta_{Te,B} = \int_0^l z \frac{\alpha}{h} \frac{t_0}{l} (l-z) dz \quad (i)$$

Integrating Eq. (i) gives

$$\Delta_{Te,B} = \frac{\alpha t_0 l^2}{6h} \quad (\text{i.e., downward})$$

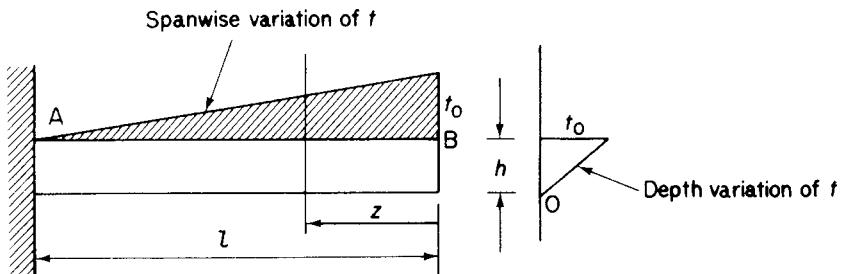


FIGURE 5.32 Beam of Example 5.21

### References

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### Further reading

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## PROBLEMS

**P.5.1.** Find the magnitude and the direction of the movement of the joint C of the plane pin-jointed frame loaded as shown in Fig. P.5.1. The value of  $L/AE$  for each member is  $1/20 \text{ mm/N}$ .

Answer: 5.24 mm at  $14.7^\circ$  to left of vertical

**P.5.2.** A rigid triangular plate is suspended from a horizontal plane by three vertical wires attached to its corners. The wires are each 1 mm diameter, 1440 mm long, with a modulus of elasticity of  $196,000 \text{ N/mm}^2$ . The ratio of the lengths of the sides of the plate is 3:4:5. Calculate the deflection at the point of application due to a 100 N load placed at a point equidistant from the three sides of the plate.

Answer: 0.33 mm

**P.5.3.** The pin-jointed space frame shown in Fig. P.5.3 is attached to rigid supports at points 0, 4, 5, and 9 and is loaded by a force  $P$  in the  $x$  direction and a force  $3P$  in the negative  $y$  direction at the point 7. Find the rotation of member 27 about the  $z$  axis due to this loading. Note that the plane frames 01234 and 56789 are identical. All members have the same cross-sectional area  $A$  and Young's modulus  $E$ .

Answer:  $382P/9AE$

**P.5.4.** A horizontal beam is of uniform material throughout but has a second moment of area of  $I$  for the central half of the span  $L$  and  $I/2$  for each section in both outer quarters of the span. The beam carries a single central concentrated load  $P$ .

- Derive a formula for the central deflection of the beam, due to  $P$ , when simply supported at each end of the span.
- If both ends of the span are encastré, determine the magnitude of the fixed end moments.

Answer:  $3PL^3/128EI$ ,  $5PL/48$  (hogging)

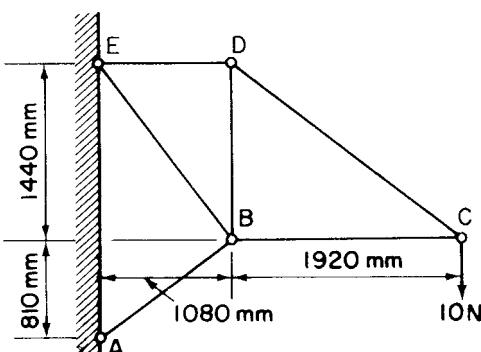


FIGURE P.5.1

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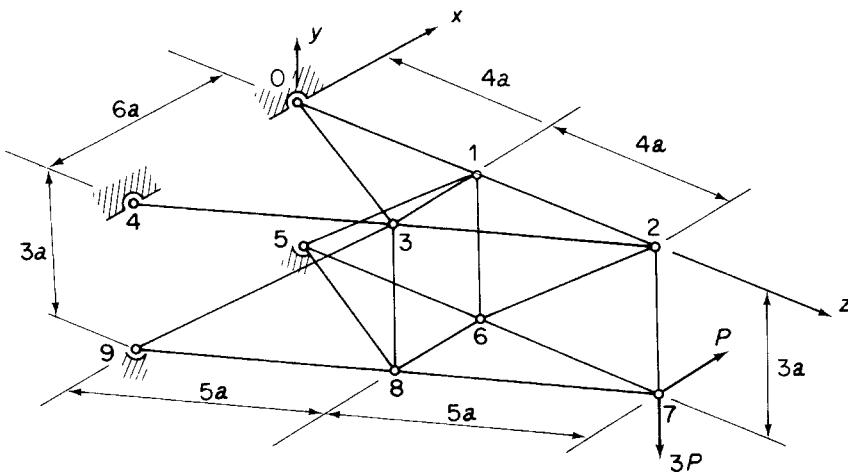


FIGURE P.5.3

**P.5.5.** The tubular steel post shown in Fig. P.5.5 supports a load of 250 N at the free end C. The outside diameter of the tube is 100 mm and the wall thickness is 3 mm. Neglecting the weight of the tube, find the horizontal deflection at C. The modulus of elasticity is 206,000 N/mm<sup>2</sup>.

Answer: 53.3 mm

**P.5.6.** A simply supported beam AB of span  $L$  and uniform section carries a distributed load of intensity varying from zero at A to  $w_0$ /unit length at B according to the law

$$w = \frac{2w_0z}{L} \left(1 - \frac{z}{2L}\right)$$

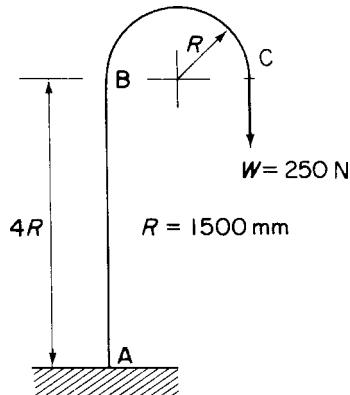


FIGURE P.5.5

per unit length. If the deflected shape of the beam is given approximately by the expression

$$v = a_1 \sin \frac{\pi z}{L} + a_2 \sin \frac{2\pi z}{L}$$

evaluate the coefficients  $a_1$  and  $a_2$  and find the deflection of the beam at mid-span.

Answer:  $a_1 = 2w_0L^4(\pi^2 + 4)/EI\pi^7$ ,  $a_2 = -w_0L^4/16EI\pi^5$ ,  $0.00918 w_0L^4/EI$ .

**P.5.7.** A uniform simply supported beam, span  $L$ , carries a distributed loading that varies according to a parabolic law across the span. The load intensity is zero at both ends of the beam and  $w_0$  at its mid-point. The loading is normal to a principal axis of the beam cross-section, and the relevant flexural rigidity is  $EI$ . Assuming that the deflected shape of the beam can be represented by the series

$$v = \sum_{i=1}^{\infty} a_i \sin \frac{i\pi z}{L}$$

find the coefficients  $a_i$  and the deflection at the mid-span of the beam using only the first term in this series.

Answer:  $a_i = 32w_0L^4/EI\pi^7 i^7$  ( $i$  odd),  $w_0L^4/94.4EI$ .

**P.5.8.** Figure P.5.8 shows a plane pin-jointed framework pinned to a rigid foundation. All its members are made of the same material and have equal cross-sectional area  $A$ , except member 12, which has area  $A\sqrt{2}$ . Under some system of loading, member 14 carries a tensile stress of  $0.7 \text{ N/mm}^2$ .

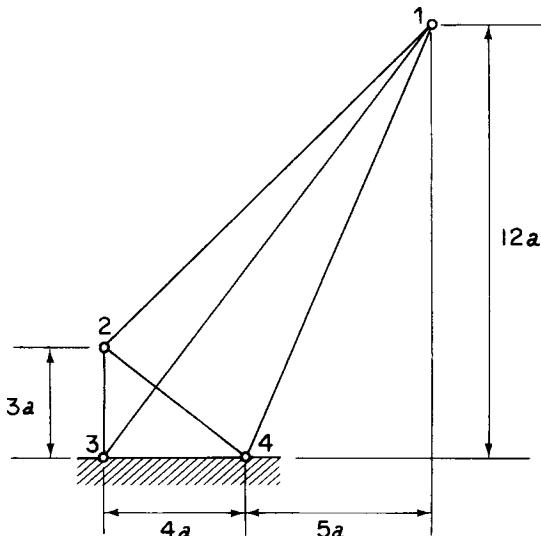


FIGURE P.5.8

Calculate the change in temperature that, if applied to member 14 only, reduces the stress in that member to zero. Take the coefficient of linear expansion as  $\alpha = 24 \times 10^{-6}/^\circ\text{C}$  and Young's modulus  $E = 70,000 \text{ N/mm}^2$ .

Answer:  $5.6^\circ\text{C}$

**P.5.8. MATLAB** Use the Symbolic Math Toolbox in MATLAB to repeat Problem P.5.8, assuming that member 14 carries tensile stresses of 0.5, 0.7, 0.9, and  $1.1 \text{ N/mm}^2$ .

- Answer:
- (i)  $4.0^\circ\text{C}$
  - (ii)  $5.6^\circ\text{C}$
  - (iii)  $7.2^\circ\text{C}$
  - (iv)  $8.8^\circ\text{C}$

**P.5.9.** The plane, pin-jointed rectangular framework shown in Fig. P.5.9(a) has one member (24) that is loosely attached at joint 2, so that relative movement between the end of the member and the joint may occur when the framework is loaded. This movement is a maximum of 0.25 mm and takes place only in the direction 24. Figure P.5.9(b) shows joint 2 in detail when the framework is not loaded. Find the value of the load  $P$  at which member 24 just becomes an effective part of the structure and also the loads in all the members when  $P$  is 10,000 N. All bars are of the same material ( $E = 70,000 \text{ N/mm}^2$ ) and have a cross-sectional area of  $300 \text{ mm}^2$ .

Answer:  $P = 294 \text{ N}$ ,  $F_{12} = 2481.6 \text{ N}(T)$ ,  $F_{23} = 1861.2 \text{ N}(T)$ ,  $F_{34} = 2481.6 \text{ N}(T)$ ,  
 $F_{41} = 5638.9 \text{ N}(C)$ ,  $F_{13} = 9398.1 \text{ N}(T)$ ,  $F_{24} = 3102.0 \text{ N}(C)$ .

**P.5.10.** The plane frame ABCD of Fig. P.5.10 consists of three straight members with rigid joints at B and C, freely hinged to rigid supports at A and D. The flexural rigidity of AB and CD is twice that of BC. A distributed load is applied to AB, varying linearly in intensity from zero at A to  $w$  per unit length

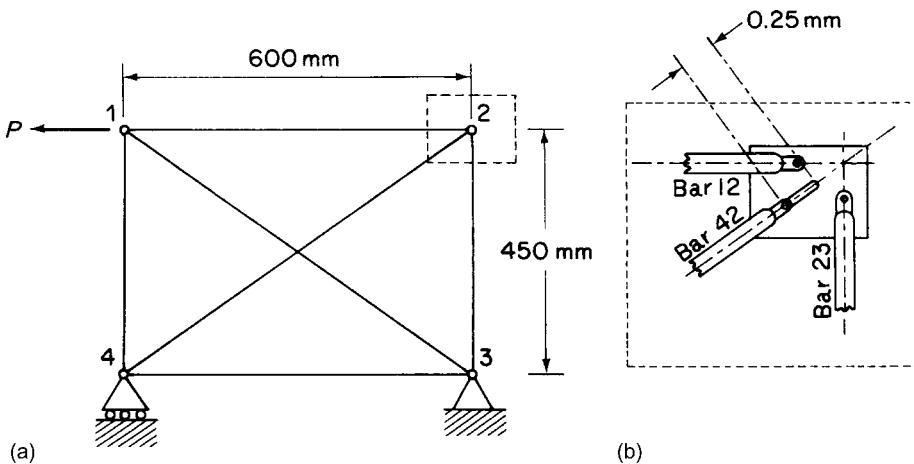


FIGURE P.5.9

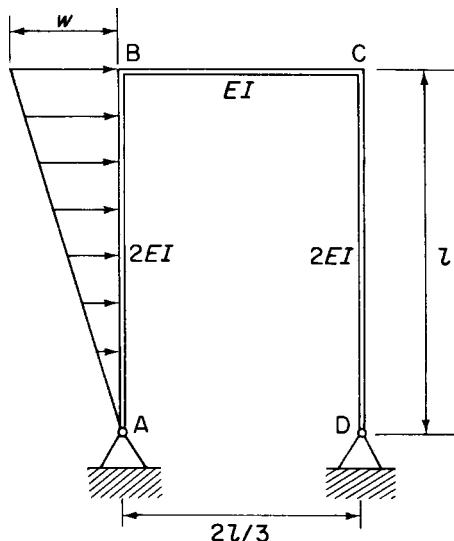


FIGURE P.5.10

at B. Determine the distribution of bending moment in the frame, illustrating your results with a sketch showing the principal values.

Answer:  $M_B = 7wl^2/45$ ,  $M_C = 8wl^2/45$ ,

Cubic distribution on AB, linear on BC and CD

**P.5.11.** A bracket BAC is composed of a circular tube AB, whose second moment of area is  $1.5I$ , and a beam AC, whose second moment of area is  $I$  and which has negligible resistance to torsion. The two members are rigidly connected together at A and built into a rigid abutment at B and C, as shown in Fig. P.5.11. A load  $P$  is applied at A in a direction normal to the plane of the figure. Determine the fraction of the load supported at C. Both members are of the same material, for which  $G = 0.38E$ .

Answer:  $0.72P$ .

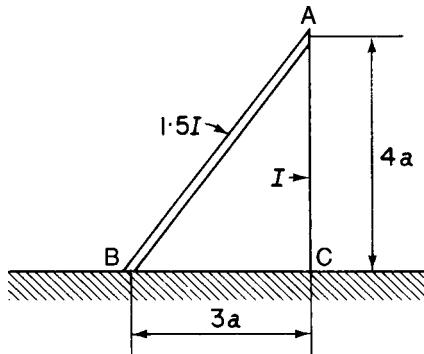


FIGURE P.5.11

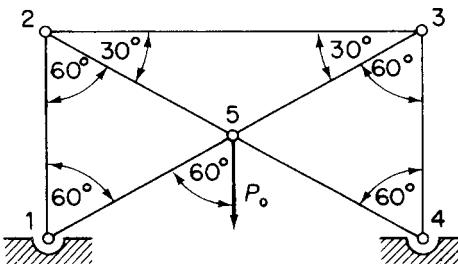


FIGURE P.5.12

**P.5.12.** In the plane pin-jointed framework shown in Fig. P.5.12, bars 25, 35, 15, and 45 are linearly elastic with modulus of elasticity  $E$ . The remaining three bars obey a nonlinear elastic stress-strain law given by

$$\varepsilon = \frac{\tau}{E} \left[ 1 + \left( \frac{\tau}{\tau_0} \right)^n \right]$$

where  $\tau$  is the stress corresponding to strain  $\varepsilon$ . Each of bars 15, 45, and 23 has a cross-sectional area  $A$ , and each of the remainder has an area of  $A/\sqrt{3}$ . The length of member 12 is equal to the length of member 34 =  $2L$ . If a vertical load  $P_0$  is applied at joint 5, as shown, show that the force in the member 23, that is,  $F_{23}$ , is given by the equation

$$\alpha^n x^{n+1} + 3.5x + 0.8 = 0$$

where

$$x = F_{23}/P_0 \quad \text{and} \quad \alpha = P_0/A\tau_0$$

**P.5.13.** Figure P.5.13 shows a plan view of two beams, AB 9150 mm long and DE 6100 mm long. The simply supported beam AB carries a vertical load of 100,000 N applied at F, a distance one-third of the span from B. This beam is supported at C on the encastré beam DE. The beams are of uniform cross-section and have the same second moment of area  $83.5 \times 10^6 \text{ mm}^4$ .  $E = 200,000 \text{ N/mm}^2$ . Calculate the deflection of C.

Answer: 5.6 mm

**P.5.14.** The plane structure shown in Fig. P.5.14 consists of a uniform continuous beam ABC pinned to a fixture at A and supported by a framework of pin-jointed members. All members other than ABC have the same cross-sectional area  $A$ . For ABC, the area is  $4A$  and the second moment of area for bending is  $Aa^2/16$ . The material is the same throughout. Find (in terms of  $w$ ,  $A$ ,  $a$ , and Young's modulus  $E$ ) the vertical displacement of point D under the vertical loading shown. Ignore shearing strains in the beam ABC.

Answer:  $30,232 wa^2/3AE$

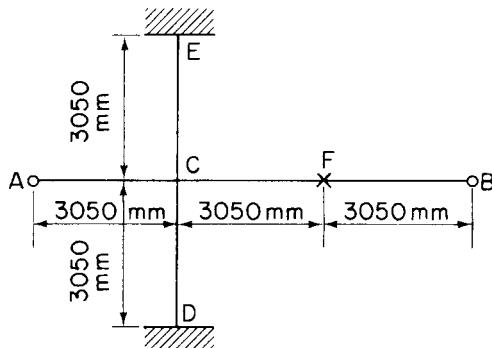


FIGURE P.5.13

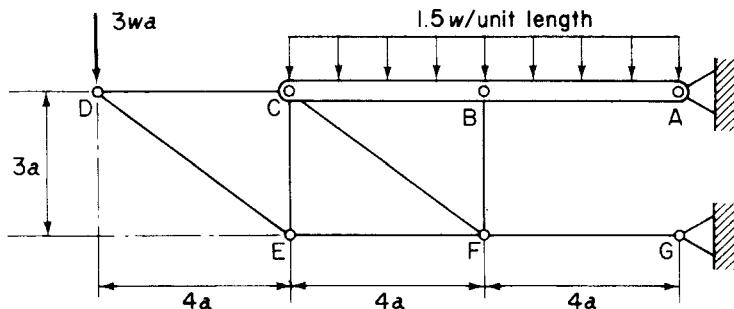


FIGURE P.5.14

**P.5.15.** The fuselage frame shown in Fig. P.5.15 consists of two parts, ACB and ADB, with frictionless pin joints at A and B. The bending stiffness is constant in each part, with value  $EI$  for ACB and  $xEI$  for ADB. Find  $x$  so that the maximum bending moment in ADB is one-half of that in ACB. Assume that the deflections are due to bending strains only.

Answer: 0.092

**P.5.16.** A transverse frame in a circular section fuel tank is of radius  $r$  and constant bending stiffness  $EI$ . The loading on the frame consists of the hydrostatic pressure due to the fuel and the vertical support reaction  $P$ , which is equal to the weight of fuel carried by the frame, shown in Fig. P.5.16. Taking into account only strains due to bending, calculate the distribution of bending moment around the frame in terms of the force  $P$ , the frame radius  $r$ , and the angle  $\theta$ .

Answer:  $M = Pr(0.160 - 0.080 \cos\theta - 0.159\theta \sin\theta)$

**P.5.17.** The frame shown in Fig. P.5.17 consists of a semicircular arc, center B, radius  $a$ , of constant flexural rigidity  $EI$  jointed rigidly to a beam of constant flexural rigidity  $2EI$ . The frame is subjected to an outward loading, as shown, arising from an internal pressure  $p_0$ . Find the bending moment at points

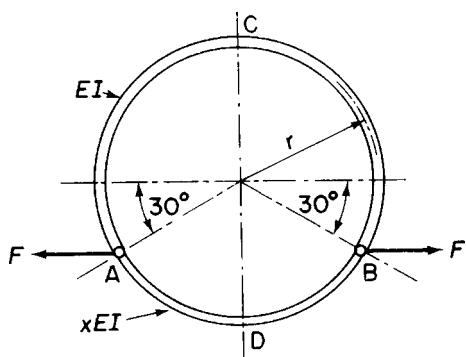


FIGURE P.5.15

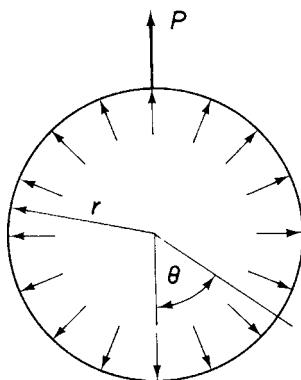


FIGURE P.5.16

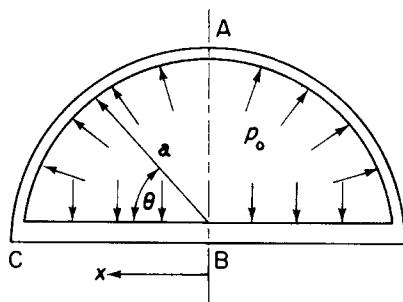


FIGURE P.5.17

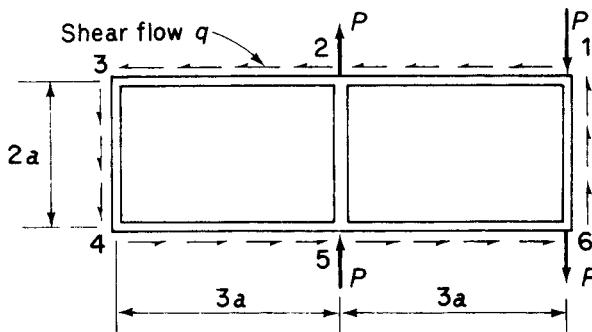


FIGURE P.5.18

A, B, and C and locate any points of contraflexure. A is the mid-point of the arc. Neglect deformations of the frame due to shear and normal forces.

Answer:  $M_A = -0.057p_0a^2$ ,  $M_B = -0.292p_0a^2$ ,  $M_C = 0.208p_0a^2$

Points of contraflexure: in AC, at  $51.7^\circ$  from horizontal; in BC,  $0.764a$  from B

**P.5.18.** The rectangular frame shown in Fig. P.5.18 consists of two horizontal members 123 and 456 rigidly joined to three vertical members 16, 25, and 34. All five members have the same bending stiffness  $EI$ . The frame is loaded in its own plane by a system of point loads  $P$ , which are balanced by a constant shear flow  $q$  around the outside. Determine the distribution of the bending moment in the frame and sketch the bending moment diagram. In the analysis, take into account only bending deformations.

Answer: Shears only at mid-points of vertical members. On the lower half of the frame,  $S_{43} = 0.27P$  to right,  $S_{52} = 0.69P$  to left,  $S_{61} = 1.08P$  to left; the bending moment diagram follows.

**P.5.19.** A circular fuselage frame, shown in Fig. P.5.19, of radius  $r$  and constant bending stiffness  $EI$ , has a straight floor beam of length  $r\sqrt{2}$ , bending stiffness  $EI$ , rigidly fixed to the frame at either end. The frame is loaded by a couple  $T$  applied at its lowest point and a constant equilibrating shear flow  $q$  around its periphery. Determine the distribution of the bending moment in the frame, illustrating your answer by means of a sketch. In the analysis, deformations due to shear and end load may be considered negligible. The depth of the frame cross-section in comparison with the radius  $r$  may also be neglected.

Answer:  $M_{14} = T(0.29 \sin\theta - 0.16\theta)$ ,  $M_{24} = 0.30Tx/r$ ,  $M_{43} = T(0.59 \sin\theta - 0.16\theta)$ .

**P.5.20.** A thin-walled member BCD is rigidly built in at D and simply supported at the same level at C, as shown in Fig. P.5.20. Find the horizontal deflection at B due to the horizontal force  $F$ . Full account must be taken of deformations due to shear and direct strains, as well as to bending. The member is of uniform cross-section, of area  $A$ , relevant second moment of area in bending  $I = Ar^2/400$ , and "reduced" effective area in shearing  $A' = A/4$ . Poisson's ratio for the material is  $\nu = 1/3$ . Give the answer in terms of  $F$ ,  $r$ ,  $A$ , and Young's modulus  $E$ .

Answer:  $448 Fr/EA$ .

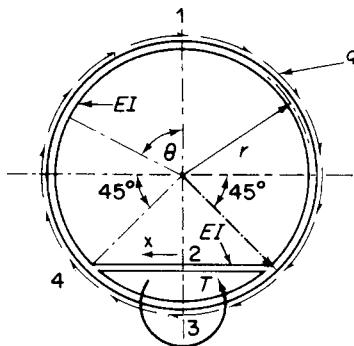


FIGURE P.5.19

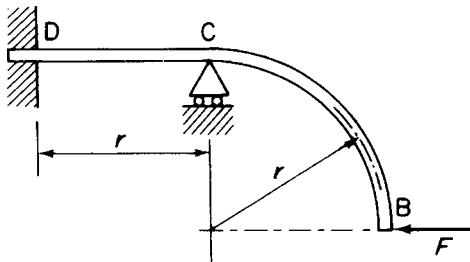


FIGURE P.5.20

**P.5.21.** Figure P.5.21 shows two cantilevers, the end of one being vertically above the other and connected to it by a spring AB. Initially, the system is unstrained. A weight  $W$  placed at A causes a vertical deflection at A of  $\delta_1$  and a vertical deflection at B of  $\delta_2$ . When the spring is removed, the weight  $W$  at A causes a deflection at A of  $\delta_3$ . Find the extension of the spring when it is replaced and the weight  $W$  is transferred to B.

Answer:  $\delta_2 (\delta_1 - \delta_2)/(\delta_3 - \delta_1)$

**P.5.22.** A beam 2400 mm long is supported at two points A and B, which are 1440 mm apart; point A is 360 mm from the left-hand end of the beam and point B is 600 mm from the right-hand end; the value of  $EI$  for the beam is  $240 \times 10^8$  N mm $^2$ . Find the slope at the supports due to a load of 2000 N applied at

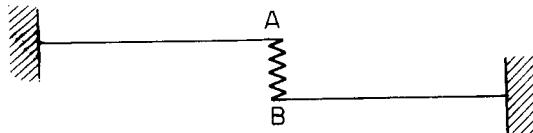


FIGURE P.5.21

the mid-point of AB. Use the reciprocal theorem in conjunction with this result to find the deflection at the mid-point of AB due to loads of 3000 N applied at each of the extreme ends of the beam.

Answer: 0.011, 15.8 mm

**P.5.22. MATLAB** Use the Symbolic Math Toolbox in MATLAB to repeat Problem P.5.22. In addition, use the reciprocal theorem to calculate the displacement at the mid-point of AB due to the following combinations of loads at C and D:

	(i)	(ii)	(iii)	(iv)	(v)
C:	2000 N	2500 N	3000 N	3500 N	4000 N
D:	4000 N	3500 N	3000 N	2500 N	2000 N

Answer: 0.011 rad

(i)	(ii)	(iii)	(iv)	(v)
17.16 mm	16.5 mm	15.84 mm	15.18 mm	14.52 mm

**P.5.23.** Figure P.5.23 shows a cantilever beam subjected to linearly varying temperature gradients along its length and through its depth. Calculate the deflection at the free end of the beam.

Answer:  $2\alpha t_0 L^2 / 3h$

**P.5.24.** Figure P.5.24 shows a frame pinned to its support at A and B. The frame center-line is a circular arc, and the section is uniform, of bending stiffness  $EI$  and depth  $d$ . Find an expression for the maximum stress produced by a uniform temperature gradient through the depth, the temperatures on the outer and inner surfaces being, respectively, raised and lowered by amount  $T$ . The points A and B are unaltered in position.

Answer:  $1.30ET\alpha$

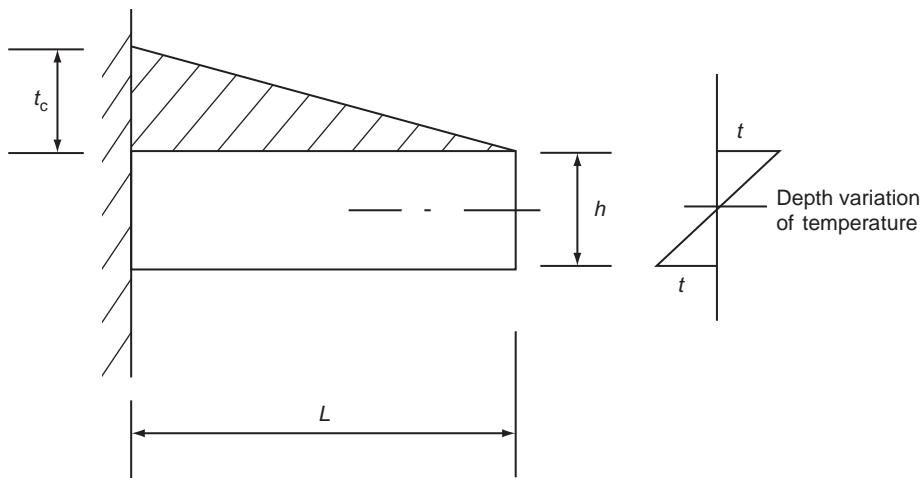


FIGURE P.5.23

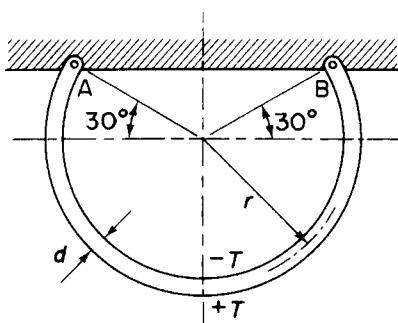


FIGURE P.5.24

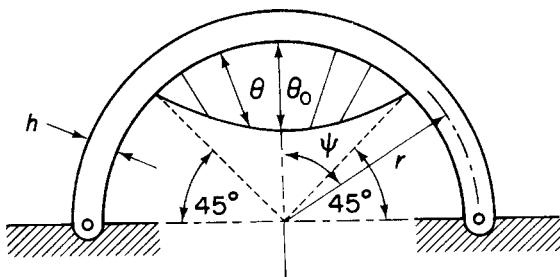


FIGURE P.5.25

**P.5.25.** A uniform, semicircular fuselage frame is pin-jointed to a rigid portion of the structure and is subjected to a given temperature distribution on the inside, as shown in Fig. P.5.25. The temperature falls linearly across the section of the frame to zero on the outer surface. Find the values of the reactions at the pin-joints and show that the distribution of the bending moment in the frame is

$$M = \frac{0.59 EI\alpha\theta_0 \cos \psi}{h}$$

given that

(a) The temperature distribution is

$$\begin{aligned} \theta &= \theta_0 \cos^2 \psi, & \text{for } -\pi/4 < \Psi < \pi/4 \\ \theta &= 0, & \text{for } -\pi/4 > \Psi > \pi/4 \end{aligned}$$

(b) Only bending deformations are to be taken into account:

$\alpha$  = coefficient of linear expansion of frame material

$EI$  = bending rigidity of frame

$h$  = depth of cross-section

$r$  = mean radius of frame.

# Matrix methods

# 6

Actual aircraft structures consist of numerous components generally arranged in an irregular manner. These components are usually continuous and therefore, theoretically, possess an infinite number of degrees of freedom and redundancies. Analysis is possible only if the actual structure is replaced by an idealized approximation or model. This procedure is discussed to some extent in Chapter 20, where we note that the greater the simplification introduced by the idealization, the less complex but more inaccurate becomes the analysis. In aircraft design, where structural weight is of paramount importance, an accurate knowledge of component loads and stresses is essential, so that at some stage in the design these must be calculated as accurately as possible. This accuracy is achieved only by considering an idealized structure that closely represents the actual structure. Standard methods of structural analysis are inadequate for coping with the necessary degree of complexity in such idealized structures. This situation led, in the late 1940s and early 1950s, to the development of *matrix methods* of analysis and at the same time to the emergence of high-speed, electronic, digital computers. Conveniently, matrix methods are ideally suited for expressing structural theory and expressing that theory in a form suitable for numerical solution by computer.

A structural problem may be formulated in either of two ways. One approach proceeds with the displacements of the structure as the unknowns, the internal forces then follow from the determination of these displacements; in the alternative approach, forces are treated as being initially unknown. In the language of matrix methods, these two approaches are known as the *stiffness* (or *displacement*) method and the *flexibility* (or *force*) method, respectively. The most widely used of these two methods is the stiffness method, and for this reason we shall concentrate on this particular approach. Argyris and Kelsey,<sup>1</sup> however, showed that complete duality exists between the two methods, in that the form of the governing equations is the same whether they are expressed in terms of displacements or forces.

Generally, actual structures must be idealized to some extent before they become amenable to analysis. Examples of some simple idealizations and their effect on structural analysis are presented in Chapter 20 for aircraft structures. Outside the realms of aeronautical engineering, the representation of a truss girder by a pin-jointed framework is a well-known example of the idealization of what are known as “skeletal” structures. Such structures are assumed to consist of a number of elements joined at points called *nodes*. The behavior of each element may be determined by basic methods of structural analysis, and hence the behavior of the complete structure is obtained by superposition. Operations such as this are easily carried out by matrix methods, as we shall see later in this chapter.

A more difficult type of structure to idealize is the continuum structure; in this category are dams, plates, shells, and, obviously, aircraft fuselage and wing skins. A method, extending the matrix technique for skeletal structures, of representing continua by any desired number of elements connected at their nodes was developed by Clough et al.<sup>2</sup> at the Boeing Aircraft Company and the University of

California at Berkeley. The elements may be of any desired shape, but the simplest, used in plane stress problems, are the triangular and quadrilateral elements. We shall discuss the *finite element method*, as it is known, in greater detail later.

Initially, we shall develop the matrix stiffness method of solution for simple skeletal and beam structures. The fundamentals of matrix algebra are assumed.

## 6.1 NOTATION

Generally, we shall consider structures subjected to forces,  $F_{x,1}, F_{y,1}, F_{z,1}, F_{x,2}, F_{y,2}, F_{z,2}, \dots, F_{x,n}, F_{y,n}, F_{z,n}$ , at nodes 1, 2, ...,  $n$ , at which the displacements are  $u_1, v_1, w_1, u_2, v_2, w_2, \dots, u_n, v_n, w_n$ . The numerical suffixes specify nodes, while the algebraic suffixes relate the direction of the forces to an arbitrary set of axes,  $x, y, z$ . Nodal displacements  $u, v, w$  represent displacements in the positive directions of the  $x, y$ , and  $z$  axes, respectively. The forces and nodal displacements are written as column matrices (alternatively known as *column vectors*)

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{z,1} \\ F_{x,2} \\ F_{y,2} \\ F_{z,2} \\ \vdots \\ F_{x,n} \\ F_{y,n} \\ F_{z,n} \end{Bmatrix} \quad \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}$$

which, once established for a particular problem, may be abbreviated to

$$\{F\} \quad \{\delta\}$$

The generalized force system  $\{F\}$  can contain moments  $M$  and torques  $T$  in addition to direct forces, in which case  $\{\delta\}$  includes rotations  $\theta$ . Therefore, in referring simply to a nodal force system, we imply the possible presence of direct forces, moments, and torques, while the corresponding nodal displacements can be translations and rotations.

For a complete structure, the nodal forces and nodal displacements are related through a *stiffness matrix*  $[K]$ . We shall see that, in general,

$$\{F\} = [K]\{\delta\} \tag{6.1}$$

where  $[K]$  is a symmetric matrix of the form

$$[K] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \tag{6.2}$$

The element  $k_{ij}$  (that is, the element located on row  $i$  and in column  $j$ ) is known as the *stiffness influence coefficient* (note  $k_{ij} = k_{ji}$ ). Once the stiffness matrix  $[K]$  has been formed, the complete solution to a problem follows from routine numerical calculations that are carried out, in most practical cases, by computer.

## 6.2 STIFFNESS MATRIX FOR AN ELASTIC SPRING

The formation of the stiffness matrix  $[K]$  is the most crucial step in the matrix solution of any structural problem. We shall show in the subsequent work how the stiffness matrix for a complete structure may be built up from a consideration of the stiffness of its individual elements. First, however, we shall investigate the formation of  $[K]$  for a simple spring element, which exhibits many of the characteristics of an actual structural member.

The spring of stiffness  $k$  shown in Fig. 6.1 is aligned with the  $x$  axis and supports forces  $F_{x,1}$  and  $F_{x,2}$  at its nodes 1 and 2, where the displacements are  $u_1$  and  $u_2$ . We build up the stiffness matrix for this simple case by examining different states of nodal displacement. First, we assume that node 2 is prevented from moving, such that  $u_1 = u_1$  and  $u_2 = 0$ . Hence,

$$F_{x,1} = ku_1$$

and, from equilibrium, we see that

$$F_{x,2} = -F_{x,1} = -ku_1 \quad (6.3)$$

which indicates that  $F_{x,2}$  has become a reactive force in the opposite direction to  $F_{x,1}$ . Second, we take the reverse case, where  $u_1 = 0$  and  $u_2 = u_2$ , and obtain

$$F_{x,2} = ku_2 = -F_{x,1} \quad (6.4)$$

By superposition of these two conditions, we obtain relationships between the applied forces and the nodal displacements for the state when  $u_1 = u_1$  and  $u_2 = u_2$ . Thus,

$$\left. \begin{aligned} F_{x,1} &= ku_1 - ku_2 \\ F_{x,2} &= -ku_1 + ku_2 \end{aligned} \right\} \quad (6.5)$$

Writing Eq. (6.5) in matrix form, we have

$$\left\{ \begin{aligned} F_{x,1} \\ F_{x,2} \end{aligned} \right\} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \left\{ \begin{aligned} u_1 \\ u_2 \end{aligned} \right\} \quad (6.6)$$

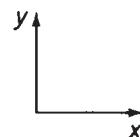
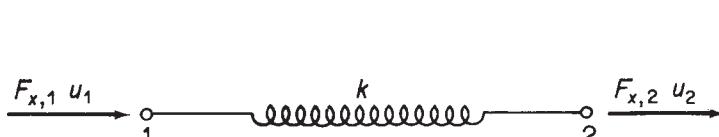


FIGURE 6.1 Determination of Stiffness Matrix for a Single Spring

and by comparison with Eq. (6.1), we see that the stiffness matrix for this spring element is

$$[K] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad (6.7)$$

which is a symmetric matrix of order  $2 \times 2$ .

### 6.3 STIFFNESS MATRIX FOR TWO ELASTIC SPRINGS IN LINE

Bearing in mind the results of the previous section, we shall now proceed, initially by a similar process, to obtain the stiffness matrix of the composite two-spring system shown in Fig. 6.2. The notation and sign convention for the forces and nodal displacements are identical to those specified in Section 6.1.

First, let us suppose that  $u_1 = u_1$  and  $u_2 = u_3 = 0$ . By comparison with the single spring case, we have

$$F_{x,1} = k_a u_1 = -F_{x,2} \quad (6.8)$$

but, in addition,  $F_{x,3} = 0$ , since  $u_2 = u_3 = 0$ .

Second, we put  $u_1 = u_3 = 0$  and  $u_2 = u_2$ . Clearly, in this case, the movement of node 2 takes place against the combined spring stiffnesses  $k_a$  and  $k_b$ . Hence,

$$\left. \begin{aligned} F_{x,2} &= (k_a + k_b)u_2 \\ F_{x,1} &= -k_a u_2, F_{x,3} = -k_b u_2 \end{aligned} \right\} \quad (6.9)$$

Thus, the reactive force  $F_{x,1}$  ( $= -k_a u_2$ ) is not directly affected by the fact that node 2 is connected to node 3 but is determined solely by the displacement of node 2. Similar conclusions are drawn for the reactive force  $F_{x,3}$ .

Finally, we set  $u_1 = u_2 = 0, u_3 = u_3$  and obtain

$$\left. \begin{aligned} F_{x,3} &= k_b u_3 = -F_{x,2} \\ F_{x,1} &= 0 \end{aligned} \right\} \quad (6.10)$$

Superimposing these three displacement states, we have, for the condition  $u_1 = u_1, u_2 = u_2, u_3 = u_3$ ,

$$\left. \begin{aligned} F_{x,1} &= k_a u_1 - k_a u_2 \\ F_{x,2} &= -k_a u_1 + (k_a + k_b)u_2 - k_b u_3 \\ F_{x,3} &= -k_b u_2 + k_b u_3 \end{aligned} \right\} \quad (6.11)$$

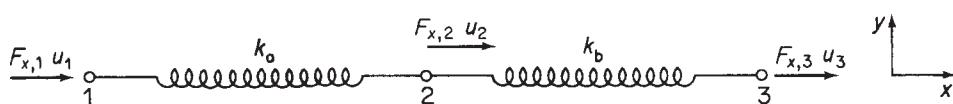


FIGURE 6.2 Stiffness Matrix for a Two-Spring System

Writing Eqs. (6.11) in matrix form gives

$$\begin{Bmatrix} F_{x,1} \\ F_{x,2} \\ F_{x,3} \end{Bmatrix} = \begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a + k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad (6.12)$$

Comparison of Eqs. (6.12) with Eq. (6.1) shows that the stiffness matrix  $[K]$  of this two-spring system is

$$[K] = \begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a + k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \quad (6.13)$$

Equation (6.13) is a symmetric matrix of order  $3 \times 3$ .

It is important to note that the order of a stiffness matrix may be predicted from a knowledge of the number of nodal forces and displacements. For example, Eq. (6.7) is a  $2 \times 2$  matrix connecting *two* nodal forces with *two* nodal displacements; Eq. (6.13) is a  $3 \times 3$  matrix relating *three* nodal forces to *three* nodal displacements. We deduce that a stiffness matrix for a structure in which  $n$  nodal forces relate to  $n$  nodal displacements is of order  $n \times n$ . The order of the stiffness matrix does not, however, bear a direct relation to the number of nodes in a structure, since it is possible for more than one force to be acting at any one node.

So far we have built up the stiffness matrices for the single- and two-spring assemblies by considering various states of displacement in each case. Such a process clearly becomes tedious for more complex assemblies involving a large number of springs, so that a shorter, alternative procedure is desirable. From our remarks in the preceding paragraph and by reference to Eq. (6.2), we could have deduced at the outset of the analysis that the stiffness matrix for the two-spring assembly would be of the form

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \quad (6.14)$$

The element  $k_{11}$  of this matrix relates the force at node 1 to the displacement at node 1 and so on. Hence, remembering the stiffness matrix for the single spring (Eq. (6.7)), we may write down the stiffness matrix for an elastic element connecting nodes 1 and 2 in a structure as

$$[K_{12}] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \quad (6.15)$$

and for the element connecting nodes 2 and 3 as

$$[K_{23}] = \begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix} \quad (6.16)$$

In our two-spring system, the stiffness of the spring joining nodes 1 and 2 is  $k_a$  and that of the spring joining nodes 2 and 3 is  $k_b$ . Therefore, by comparison with Eq. (6.7), we rewrite Eqs. (6.15) and (6.16) as

$$[K_{12}] = \begin{bmatrix} k_a & -k_a \\ -k_a & k_a \end{bmatrix}, \quad [K_{23}] = \begin{bmatrix} k_b & -k_b \\ -k_b & k_b \end{bmatrix} \quad (6.17)$$

Substituting in Eq. (6.14) gives

$$[K] = \begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a + k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix}$$

which is identical to Eq. (6.13). We see that only the  $k_{22}$  term (linking the force at node 2 to the displacement at node 2) receives contributions from both springs. This results from the fact that node 2 is directly connected to both nodes 1 and 3, while nodes 1 and 3 are each joined directly only to node 2. Also, the elements  $k_{13}$  and  $k_{31}$  of  $[K]$  are zero, since nodes 1 and 3 are not directly connected and are therefore not affected by each other's displacement.

The formation of a stiffness matrix for a complete structure thus becomes a relatively simple matter of the superposition of individual or element stiffness matrices. The procedure may be summarized as follows: terms of the form  $k_{ii}$  on the main diagonal consist of the sum of the stiffnesses of all the structural elements meeting at node  $i$  while off-diagonal terms of the form  $k_{ij}$  consist of the sum of the stiffnesses of all the elements connecting node  $i$  to node  $j$ .

An examination of the stiffness matrix reveals that it possesses certain properties. For example, the sum of the elements in any column is zero, indicating that the conditions of equilibrium are satisfied. Also, the nonzero terms are concentrated near the leading diagonal, while all the terms in the leading diagonal are positive; the latter property derives from the physical behavior of any actual structure in which positive nodal forces produce positive nodal displacements.

Further inspection of Eq. (6.13) shows that its determinant vanishes. As a result, the stiffness matrix  $[K]$  is singular and its inverse does not exist. We shall see that this means that the associated set of simultaneous equations for the unknown nodal displacements cannot be solved, for the simple reason that we have placed no limitation on any of the displacements  $u_1$ ,  $u_2$ , or  $u_3$ . Thus, the application of external loads results in the system moving as a rigid body. Sufficient boundary conditions must therefore be specified to enable the system to remain stable under load. In this particular problem, we shall demonstrate the solution procedure by assuming that node 1 is fixed; that is,  $u_1 = 0$ .

The first step is to rewrite Eq. (6.13) in partitioned form as

$$\begin{Bmatrix} F_{x,1} \\ F_{x,2} \\ F_{x,3} \end{Bmatrix} = \begin{bmatrix} k_a & \vdots & -k_a & 0 \\ \dots & & \dots & \dots \\ -k_a & \vdots & k_a + k_b & -k_b \\ 0 & \vdots & -k_b & k_b \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix} \quad (6.18)$$

In Eq. (6.18),  $F_{x,1}$  is the unknown reaction at node 1,  $u_1$  and  $u_2$  are unknown nodal displacements, while  $F_{x,2}$  and  $F_{x,3}$  are known applied loads. Expanding Eq. (6.18) by matrix multiplication, we obtain

$$\{F_{x,1}\} = [-k_a \quad 0] \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \quad \begin{Bmatrix} F_{x,2} \\ F_{x,3} \end{Bmatrix} = \begin{bmatrix} k_a + k_b & -k_b \\ -k_b & k_b \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \quad (6.19)$$

Inversion of the second of Eqs. (6.19) gives  $u_2$  and  $u_3$  in terms of  $F_{x,2}$  and  $F_{x,3}$ . Substitution of these values in the first equation then yields  $F_{x,1}$ .

Thus,

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} k_a + k_b & -k_b \\ -k_b & k_b \end{bmatrix}^{-1} \begin{Bmatrix} F_{x,2} \\ F_{x,3} \end{Bmatrix}$$

or

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1/k_a & 1/k_a \\ 1/k_a & 1/k_b + 1/k_a \end{bmatrix} \begin{Bmatrix} F_{x,2} \\ F_{x,3} \end{Bmatrix}$$

Hence,

$$\{F_{x,1}\} = [-k_a \quad 0] \begin{bmatrix} 1/k_a & 1/k_a \\ 1/k_a & 1/k_b + 1/k_a \end{bmatrix} \begin{Bmatrix} F_{x,2} \\ F_{x,3} \end{Bmatrix}$$

which gives

$$F_{x,1} = -F_{x,2} - F_{x,3}$$

as would be expected from equilibrium considerations. In problems where reactions are not required, equations relating known applied forces to unknown nodal displacements may be obtained by deleting the rows and columns of  $[K]$  corresponding to zero displacements. This procedure eliminates the necessity of rearranging rows and columns in the original stiffness matrix when the fixed nodes are not conveniently grouped together.

Finally, the internal forces in the springs may be determined from the force–displacement relationship of each spring. Thus, if  $S_a$  is the force in the spring joining nodes 1 and 2, then

$$S_a = k_a(u_2 - u_1)$$

Similarly for the spring between nodes 2 and 3,

$$S_b = k_b(u_3 - u_2)$$

## 6.4 MATRIX ANALYSIS OF PIN-JOINTED FRAMEWORKS

The formation of stiffness matrices for pin-jointed frameworks and the subsequent determination of nodal displacements follow a similar pattern to that described for a spring assembly. A member in such a framework is assumed to be capable of carrying axial forces only and obeys a unique force–deformation relationship given by (see Eq. (1.40))

$$F = \frac{AE}{L} \delta$$

where  $F$  is the force in the member,  $\delta$  its change in length,  $A$  its cross-sectional area,  $L$  its unstrained length, and  $E$  its modulus of elasticity. This expression is seen to be equivalent to the spring–displacement relationships of Eqs. (6.3) and (6.4), so that we immediately write down the stiffness matrix for a member by replacing  $k$  by  $AE/L$  in Eq. (6.7):

$$[K] = \begin{bmatrix} AE/L & -AE/L \\ -AE/L & AE/L \end{bmatrix}$$

or

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (6.20)$$

so that, for a member aligned with the  $x$  axis, joining nodes  $i$  and  $j$  subjected to nodal forces  $F_{x,i}$  and  $F_{x,j}$ , we have

$$\begin{Bmatrix} F_{x,i} \\ F_{x,j} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \quad (6.21)$$

The solution proceeds in a similar manner to that given in the previous section for a spring or spring assembly. However, some modification is necessary, since frameworks consist of members set at various angles to one another. Figure 6.3 shows a member of a framework inclined at an angle  $\theta$  to a set of arbitrary reference axes  $x, y$ . We shall refer every member of the framework to this *global coordinate system*, as it is known, when we consider the complete structure, but we shall use a *member* or *local* coordinate system  $\bar{x}, \bar{y}$  when considering individual members. Nodal forces and displacements referred to local coordinates are written as  $\bar{F}$ ,  $\bar{u}$ , etc., so that Eq. (6.21) becomes, in terms of local coordinates,

$$\begin{Bmatrix} \bar{F}_{x,i} \\ \bar{F}_{x,j} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_i \\ \bar{u}_j \end{Bmatrix} \quad (6.22)$$

where the element stiffness matrix is written  $[\bar{K}_{ij}]$ .

In Fig. 6.3, external forces  $\bar{F}_{x,i}$  and  $\bar{F}_{x,j}$  are applied to nodes  $i$  and  $j$ . Note that  $\bar{F}_{y,i}$  and  $\bar{F}_{y,j}$  do not exist, since the member can only support axial forces. However,  $\bar{F}_{x,i}$  and  $\bar{F}_{x,j}$  have components  $F_{x,i}$ ,  $F_{y,i}$  and  $F_{x,j}$ ,  $F_{y,j}$ , respectively, so that whereas only two force components appear for the member in terms of local coordinates, four components are present when global coordinates are used. Therefore, if we are to transfer from local to global coordinates, Eq. (6.22) must be expanded to an order consistent with the use of global coordinates; that is,

$$\begin{Bmatrix} \bar{F}_{x,i} \\ \bar{F}_{y,i} \\ \bar{F}_{x,j} \\ \bar{F}_{y,j} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{u}_j \\ \bar{v}_j \end{Bmatrix} \quad (6.23)$$

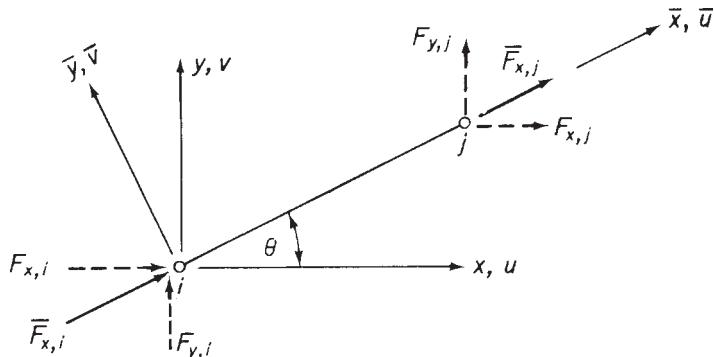


FIGURE 6.3 Local and Global Coordinate Systems for a Member of a Plane Pin-Jointed Framework

Equation (6.23) does not change the basic relationship between  $\overline{F}_{x,i}$ ,  $\overline{F}_{x,j}$  and  $\overline{u}_i$ ,  $\overline{u}_j$  as defined in Eq. (6.22).

From Fig. 6.3, we see that

$$\begin{aligned}\overline{F}_{x,i} &= F_{x,i} \cos\theta + F_{y,i} \sin\theta \\ \overline{F}_{y,i} &= -F_{x,i} \sin\theta + F_{y,i} \cos\theta\end{aligned}$$

and

$$\begin{aligned}\overline{F}_{x,j} &= F_{x,j} \cos\theta + F_{y,j} \sin\theta \\ \overline{F}_{y,j} &= -F_{x,j} \sin\theta + F_{y,j} \cos\theta\end{aligned}$$

Writing  $\lambda$  for  $\cos\theta$  and  $\mu$  for  $\sin\theta$ , we express the preceding equations in matrix form as

$$\begin{Bmatrix} \overline{F}_{x,i} \\ \overline{F}_{y,i} \\ \overline{F}_{x,j} \\ \overline{F}_{y,j} \end{Bmatrix} = \begin{bmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \end{bmatrix} \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \end{Bmatrix} \quad (6.24)$$

or, in abbreviated form,

$$\{\overline{F}\} = [T]\{F\} \quad (6.25)$$

where  $[T]$  is known as the *transformation matrix*. A similar relationship exists between the sets of nodal displacements. Thus, again using our shorthand notation,

$$\{\overline{\delta}\} = [T]\{\delta\} \quad (6.26)$$

Substituting now for  $\{\overline{F}\}$  and  $\{\overline{\delta}\}$  in Eq. (6.23) from Eqs. (6.25) and (6.26), we have

$$[T]\{F\} = [\overline{K}_{ij}][T]\{\delta\}$$

Hence,

$$\{F\} = [T^{-1}][\overline{K}_{ij}][T]\{\delta\} \quad (6.27)$$

It may be shown that the inverse of the transformation matrix is its transpose; that is,

$$[T^{-1}] = [T]^T$$

Therefore, we rewrite Eq. (6.27) as

$$\{F\} = [T]^T[\overline{K}_{ij}][T]\{\delta\} \quad (6.28)$$

The nodal force system referred to global coordinates,  $\{F\}$ , is related to the corresponding nodal displacements by

$$\{F\} = [K_{ij}]\{\delta\} \quad (6.29)$$

where  $[K_{ij}]$  is the member stiffness matrix referred to global coordinates. Comparison of Eqs. (6.28) and (6.29) shows that

$$[K_{ij}] = [T]^T [\bar{K}_{ij}] [T]$$

Initially the product  $[\bar{K}_{ij}] [T]$  is calculated where  $[\bar{K}_{ij}]$  is defined in Eq. (6.23) and  $[T]$  is defined in Eq. (6.24). Then,

$$[\bar{K}_{ij}] [T] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \end{bmatrix} = \begin{bmatrix} \lambda & \mu & -\lambda & -\mu \\ 0 & 0 & 0 & 0 \\ -\lambda & -\mu & \lambda & \mu \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then,

$$[T]^T [\bar{K}_{ij}] [T] = \begin{bmatrix} \lambda & -\mu & 0 & 0 \\ \mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -\mu \\ 0 & 0 & \mu & \lambda \end{bmatrix} \begin{bmatrix} \lambda & \mu & -\lambda & -\mu \\ 0 & 0 & 0 & 0 \\ -\lambda & -\mu & \lambda & \mu \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which

$$[K_{ij}] = \frac{AE}{L} \begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \quad (6.30)$$

By evaluating  $\lambda (= \cos\theta)$  and  $\mu (= \sin\theta)$  for each member and substituting in Eq. (6.30), we obtain the stiffness matrix, referred to global coordinates, for each member of the framework.

In Section 6.3, we determined the internal force in a spring from the nodal displacements. Applying similar reasoning to the framework member, we may write down an expression for the internal force  $S_{ij}$  in terms of the local coordinates:

$$S_{ij} = \frac{AE}{L} (\bar{u}_j - \bar{u}_i) \quad (6.31)$$

Now,

$$\bar{u}_j = \lambda u_j + \mu v_j$$

$$\bar{u}_i = \lambda u_i + \mu v_i$$

Hence,

$$\bar{u}_j - \bar{u}_i = \lambda(u_j - u_i) + \mu(v_j - v_i)$$

Substituting in Eq. (6.31) and rewriting in matrix form, we have

$$S_{ij} = \frac{AE}{L} \begin{bmatrix} \lambda & \mu \\ ij \end{bmatrix} \left\{ \begin{array}{l} u_j - u_i \\ v_j - v_i \end{array} \right\} \quad (6.32)$$

**Example 6.1**

Determine the horizontal and vertical components of the deflection of node 2 and the forces in the members of the pin-jointed framework shown in Fig. 6.4. The product  $AE$  is constant for all members.

We see in this problem that nodes 1 and 3 are pinned to a fixed foundation and are therefore not displaced. Hence, with the global coordinate system shown,

$$u_1 = v_1 = u_3 = v_3 = 0$$

The external forces are applied at node 2 such that  $F_{x,2} = 0$ ,  $F_{y,2} = -W$ ; the nodal forces at 1 and 3 are then unknown reactions.

The first step in the solution is to assemble the stiffness matrix for the complete framework by writing down the member stiffness matrices referred to the global coordinate system using Eq. (6.30). The direction cosines  $\lambda$  and  $\mu$  take different values for each of the three members; therefore, remembering that the angle  $\theta$  is measured counterclockwise from the positive direction of the  $x$  axis, we have Table 6.1.

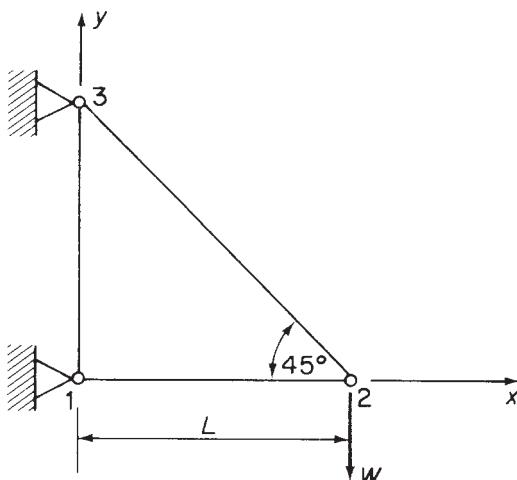


FIGURE 6.4 Pin-Jointed Framework of Example 6.1

**Table 6.1** Example 6.1

Member	$\theta$	$\lambda$	$\mu$
1–2	0	1	0
1–3	90	0	1
2–3	135	$-1/\sqrt{2}$	$1/\sqrt{2}$

The member stiffness matrices are therefore

$$\begin{aligned} [K_{12}] &= \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & [K_{13}] &= \frac{AE}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\ [K_{23}] &= \frac{AE}{\sqrt{2}L} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned} \quad (\text{i})$$

The next stage is to add the member stiffness matrices to obtain the stiffness matrix for the complete framework. Since there are six possible nodal forces producing six possible nodal displacements, the complete stiffness matrix is of the order  $6 \times 6$ . Although the addition is not difficult in this simple problem, care must be taken when solving more complex structures to ensure that the matrix elements are placed in the correct position in the complete stiffness matrix. This may be achieved by expanding each member stiffness matrix to the order of the complete stiffness matrix by inserting appropriate rows and columns of zeros. Such a method is, however, time and space consuming. An alternative procedure is suggested here. The complete stiffness matrix is of the form shown in Eq. (ii):

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \end{Bmatrix} = \begin{bmatrix} [k_{11}] & [k_{12}] & [k_{13}] \\ [k_{21}] & [k_{22}] & [k_{23}] \\ [k_{31}] & [k_{32}] & [k_{33}] \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (\text{ii})$$

The complete stiffness matrix has been divided into a number of submatrices in which  $[k_{11}]$  is a  $2 \times 2$  matrix relating the nodal forces  $F_{x,1}, F_{y,1}$  to the nodal displacements  $u_1, v_1$ , and so on. It is a simple matter to divide each member stiffness matrix into submatrices of the form  $[k_{11}]$ , as shown in Eqs. (iii). All that remains is to insert each submatrix into its correct position in Eq. (ii), adding the matrix elements where they overlap; for example, the  $[k_{11}]$  submatrix in Eq. (ii) receives contributions from  $[K_{12}]$  and  $[K_{13}]$ . The complete stiffness matrix is then of the form shown in Eq. (iv). It is sometimes helpful, when considering the stiffness matrix separately, to write the nodal displacement above the appropriate column (see Eq. (iv)). We note that  $[K]$  is symmetrical, all the diagonal terms are positive, and the sum of each row and column is zero:

$$[K_{12}] = \frac{AE}{L} \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ k_{11} & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ k_{12} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ k_{21} & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ k_{22} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$[K_{13}] = \frac{AE}{L} \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ k_{11} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & k_{13} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ k_{31} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & k_{33} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

(iii)

$$[K_{23}] = \frac{AE}{\sqrt{2}L} \begin{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ k_{22} & \frac{1}{2} \end{bmatrix} & \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ k_{23} & -\frac{1}{2} \end{bmatrix} \\ \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ k_{32} & -\frac{1}{2} \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ k_{33} & \frac{1}{2} \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 + \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & -1 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 1 + \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{cases} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 \\ u_3 = 0 \\ v_3 = 0 \end{cases}$$

(iv)

If we now delete rows and columns in the stiffness matrix corresponding to zero displacements, we obtain the unknown nodal displacements  $u_2$  and  $v_2$  in terms of the applied loads  $F_{x,2}$  ( $= 0$ ) and  $F_{y,2}$  ( $= -W$ ):

$$\begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 + \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} \quad (\text{v})$$

Inverting Eq. (v) gives

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{L}{AE} \begin{bmatrix} 1 & 1 \\ 1 & 1 + 2\sqrt{2} \end{bmatrix} \begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} \quad (\text{vi})$$

from which

$$u_2 = \frac{L}{AE} (F_{x,2} + F_{y,2}) = -\frac{WL}{AE} \quad (\text{vii})$$

$$v_2 = \frac{L}{AE} [F_{x,2} + (1 + 2\sqrt{2})F_{y,2}] = -\frac{WL}{AE} (1 + 2\sqrt{2}) \quad (\text{viii})$$

The reactions at nodes 1 and 3 are now obtained by substituting for  $u_2$  and  $v_2$  from Eq. (vi) into Eq. (iv):

$$\begin{aligned} \begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,3} \\ F_{y,3} \end{Bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 + 2\sqrt{2} \end{bmatrix} \begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} \\ &= \begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} \end{aligned}$$

giving

$$F_{x,1} = -F_{x,2} - F_{y,2} = W$$

$$F_{y,1} = 0$$

$$F_{x,3} = F_{y,2} = -W$$

$$F_{y,3} = W$$

Finally, the forces in the members are found from Eqs. (6.32), (vii), and (viii):

$$S_{12} = \frac{AE}{L} [1 \quad 0] \begin{Bmatrix} u_2 - u_1 \\ v_2 - v_1 \end{Bmatrix} = -W \quad (\text{compression})$$

$$S_{13} = \frac{AE}{L} [0 \quad 1] \begin{Bmatrix} u_3 - u_1 \\ v_3 - v_1 \end{Bmatrix} = -0 \quad (\text{as expected})$$

$$S_{23} = \frac{AE}{\sqrt{2}L} \left[ -\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right] \begin{Bmatrix} u_3 - u_2 \\ v_3 - v_2 \end{Bmatrix} = \sqrt{2}W \quad (\text{tension})$$

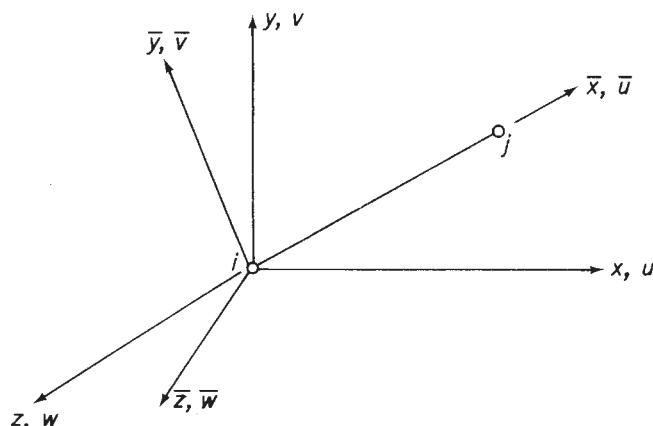
## 6.5 APPLICATION TO STATICALLY INDETERMINATE FRAMEWORKS

The matrix method of solution described in the previous sections for spring and pin-jointed framework assemblies is completely general and therefore applicable to any structural problem. We observe that at no stage in [Example 6.1](#) did the question of the degree of indeterminacy of the framework arise. It follows that problems involving statically indeterminate frameworks (and other structures) are solved in an identical manner to that presented in [Example 6.1](#), the stiffness matrices for the redundant members being included in the complete stiffness matrix as before.

## 6.6 MATRIX ANALYSIS OF SPACE FRAMES

The procedure for the matrix analysis of space frames is similar to that for plane pin-jointed frameworks. The main difference lies in the transformation of the member stiffness matrices from local to global coordinates, since, as we see from [Fig. 6.5](#), axial nodal forces  $\bar{F}_{x,i}$  and  $\bar{F}_{x,j}$  each now has three global components  $F_{x,i}, F_{y,i}, F_{z,i}$  and  $F_{x,j}, F_{y,j}, F_{z,j}$ , respectively. The member stiffness matrix referred to global coordinates is therefore of the order of  $6 \times 6$  so that  $[K_{ij}]$  of [Eq. \(6.22\)](#) must be expanded to the same order to allow for this. Hence,

$$[\bar{K}_{ij}] = \frac{AE}{L} \begin{bmatrix} \bar{u}_i & \bar{v}_i & \bar{w}_i & \bar{u}_j & \bar{v}_j & \bar{w}_j \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.33)$$



**FIGURE 6.5 Local and Global Coordinate Systems for a Member in a Pin-Jointed Space Frame**

In Fig. 6.5, the member  $ij$  is of length  $L$ , cross-sectional area  $A$ , and modulus of elasticity  $E$ . Global and local coordinate systems are designated as for the two-dimensional case. Further, we suppose that

$$\begin{aligned}\theta_{x\bar{x}} &= \text{angle between } x \text{ and } \bar{x} \\ \theta_{x\bar{y}} &= \text{angle between } x \text{ and } \bar{y} \\ &\vdots \\ \theta_{z\bar{y}} &= \text{angle between } z \text{ and } \bar{y} \\ &\vdots\end{aligned}$$

Therefore, nodal forces referred to the two systems of axes are related as follows:

$$\left. \begin{aligned}\bar{F}_x &= F_x \cos\theta_{x\bar{x}} + F_y \cos\theta_{x\bar{y}} + F_z \cos\theta_{x\bar{z}} \\ \bar{F}_y &= F_x \cos\theta_{y\bar{x}} + F_y \cos\theta_{y\bar{y}} + F_z \cos\theta_{y\bar{z}} \\ \bar{F}_z &= F_x \cos\theta_{z\bar{x}} + F_y \cos\theta_{z\bar{y}} + F_z \cos\theta_{z\bar{z}}\end{aligned}\right\} \quad (6.34)$$

Writing

$$\left. \begin{aligned}\lambda_{\bar{x}} &= \cos\theta_{x\bar{x}}, & \lambda_{\bar{y}} &= \cos\theta_{x\bar{y}}, & \lambda_{\bar{z}} &= \cos\theta_{x\bar{z}} \\ \mu_{\bar{x}} &= \cos\theta_{y\bar{x}}, & \mu_{\bar{y}} &= \cos\theta_{y\bar{y}}, & \mu_{\bar{z}} &= \cos\theta_{y\bar{z}} \\ v_{\bar{x}} &= \cos\theta_{z\bar{x}}, & v_{\bar{y}} &= \cos\theta_{z\bar{y}}, & v_{\bar{z}} &= \cos\theta_{z\bar{z}}\end{aligned}\right\} \quad (6.35)$$

we may express Eq. (6.34) for nodes  $i$  and  $j$  in matrix form as

$$\left\{ \begin{array}{c} \bar{F}_{x,i} \\ \bar{F}_{y,i} \\ \bar{F}_{z,i} \\ \bar{F}_{x,j} \\ \bar{F}_{y,j} \\ \bar{F}_{z,j} \end{array} \right\} = \left[ \begin{array}{cccccc} \lambda_{\bar{x}} & \mu_{\bar{x}} & v_{\bar{x}} & 0 & 0 & 0 \\ \lambda_{\bar{y}} & \mu_{\bar{y}} & v_{\bar{y}} & 0 & 0 & 0 \\ \lambda_{\bar{z}} & \mu_{\bar{z}} & v_{\bar{z}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\bar{x}} & \mu_{\bar{x}} & v_{\bar{x}} \\ 0 & 0 & 0 & \lambda_{\bar{y}} & \mu_{\bar{y}} & v_{\bar{y}} \\ 0 & 0 & 0 & \lambda_{\bar{z}} & \mu_{\bar{z}} & v_{\bar{z}} \end{array} \right] \left\{ \begin{array}{c} F_{x,i} \\ F_{y,i} \\ F_{z,i} \\ F_{x,j} \\ F_{y,j} \\ F_{z,j} \end{array} \right\} \quad (6.36)$$

or in abbreviated form,

$$\{\bar{F}\} = [T]\{F\}$$

The derivation of  $[K_{ij}]$  for a member of a space frame proceeds on identical lines to that for the plane frame member. Thus, as before,

$$[K_{ij}] = [T]^T [\bar{K}_{ij}] [T]$$

Substituting for  $[T]$  and  $[\bar{K}_{ij}]$  from Eqs. (6.36) and (6.33) gives

$$[K_{ij}] = \frac{AE}{L} \begin{bmatrix} \lambda_{\bar{x}}^2 & \lambda_{\bar{x}}\mu_{\bar{x}} & \lambda_{\bar{x}}v_{\bar{x}} & -\lambda_{\bar{x}}^2 & -\lambda_{\bar{x}}\mu_{\bar{x}} & -\lambda_{\bar{x}}v_{\bar{x}} \\ \lambda_{\bar{x}}\mu_{\bar{x}} & \mu_{\bar{x}}^2 & \mu_{\bar{x}}v_{\bar{x}} & -\lambda_{\bar{x}}\mu_{\bar{x}} & -\mu_{\bar{x}}^2 & -\mu_{\bar{x}}v_{\bar{x}} \\ \lambda_{\bar{x}}v_{\bar{x}} & \mu_{\bar{x}}v_{\bar{x}} & v_{\bar{x}}^2 & -\lambda_{\bar{x}}v_{\bar{x}} & -\mu_{\bar{x}}v_{\bar{x}} & -v_{\bar{x}}^2 \\ -\lambda_{\bar{x}}^2 & -\lambda_{\bar{x}}\mu_{\bar{x}} & -\lambda_{\bar{x}}v_{\bar{x}} & \lambda_{\bar{x}}^2 & \lambda_{\bar{x}}\mu_{\bar{x}} & \lambda_{\bar{x}}v_{\bar{x}} \\ -\lambda_{\bar{x}}\mu_{\bar{x}} & -\mu_{\bar{x}}^2 & -\mu_{\bar{x}}v_{\bar{x}} & \lambda_{\bar{x}}\mu_{\bar{x}} & \mu_{\bar{x}}^2 & \mu_{\bar{x}}v_{\bar{x}} \\ -\lambda_{\bar{x}}v_{\bar{x}} & -\mu_{\bar{x}}v_{\bar{x}} & -v_{\bar{x}}^2 & \lambda_{\bar{x}}v_{\bar{x}} & \mu_{\bar{x}}v_{\bar{x}} & v_{\bar{x}}^2 \end{bmatrix} \quad (6.37)$$

All the suffixes in Eq. (6.37) are  $\bar{x}$ , so that we may rewrite the equation in simpler form, namely

$$[K_{ij}] = \frac{AE}{L} \begin{bmatrix} \lambda^2 & & & \vdots & & & & \text{SYM} \\ \lambda\mu & \mu^2 & & \vdots & & & & \\ \lambda\nu & \mu\nu & \nu^2 & \vdots & & & & \\ \dots & \\ -\lambda^2 & -\lambda\mu & -\lambda\nu & \vdots & \lambda^2 & & & \\ -\lambda\mu & -\mu^2 & -\mu\nu & \vdots & \lambda\mu & \mu^2 & & \\ -\lambda\nu & -\mu\nu & -\nu^2 & \vdots & \lambda\nu & \mu\nu & \nu^2 & \end{bmatrix} \quad (6.38)$$

where  $\lambda$ ,  $\mu$ , and  $\nu$  are the direction cosines between the  $x$ ,  $y$ ,  $z$ , and  $\bar{x}$  axes, respectively.

The complete stiffness matrix for a space frame is assembled from the member stiffness matrices in a similar manner to that for the plane frame and the solution completed as before.

## 6.7 STIFFNESS MATRIX FOR A UNIFORM BEAM

Our discussion so far has been restricted to structures comprising members capable of resisting axial loads only. Many structures, however, consist of beam assemblies in which the individual members resist shear and bending forces in addition to axial loads. We shall now derive the stiffness matrix for a uniform beam and consider the solution of rigid jointed frameworks formed by an assembly of beams, or beam elements, as they are sometimes called.

Figure 6.6 shows a uniform beam  $ij$  of flexural rigidity  $EI$  and length  $L$  subjected to nodal forces  $F_{y,i}, F_{y,j}$  and nodal moments  $M_i, M_j$  in the  $xy$  plane. The beam suffers nodal displacements and rotations  $v_i, v_j$  and  $\theta_i, \theta_j$ . We do not include axial forces here, since their effects have already been determined in our investigation of pin-jointed frameworks.

The stiffness matrix  $[K_{ij}]$  may be built up by considering various deflected states for the beam and superimposing the results, as we did initially for the spring assemblies of Figs 6.1 and 6.2 or,

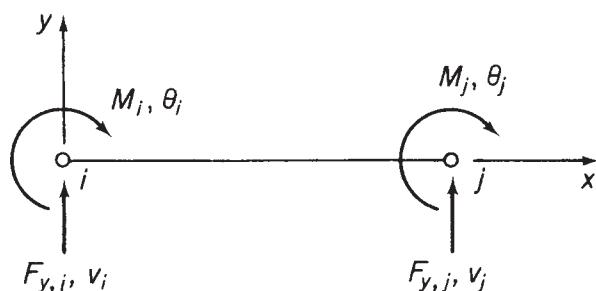


FIGURE 6.6 Forces and Moments on a Beam-Element

alternatively, it may be written down directly from the well-known beam slope–deflection equations.<sup>3</sup> We shall adopt the latter procedure. From slope–deflection theory, we have

$$M_i = -\frac{6EI}{L^2}v_i + \frac{4EI}{L}\theta_i + \frac{6EI}{L^2}v_j + \frac{2EI}{L}\theta_j \quad (6.39)$$

and

$$M_j = -\frac{6EI}{L^2}v_i + \frac{2EI}{L}\theta_i + \frac{6EI}{L^2}v_j + \frac{4EI\theta_j}{L} \quad (6.40)$$

Also, considering vertical equilibrium, we obtain

$$F_{y,i} + F_{y,j} = 0 \quad (6.41)$$

and, from moment equilibrium about node  $j$ , we have

$$F_{y,i}L + M_i + M_j = 0 \quad (6.42)$$

Hence, the solution of Eqs. (6.39)–(6.42) gives

$$-F_{y,i} = F_{y,j} = -\frac{12EI}{L^3}v_i + \frac{6EI}{L^2}\theta_i + \frac{12EI}{L^3}v_j + \frac{6EI}{L^2}\theta_j \quad (6.43)$$

Expressing Eqs. (6.39), (6.40), and (6.43) in matrix form yields

$$\begin{Bmatrix} F_{y,i} \\ M_i \\ F_{y,j} \\ M_j \end{Bmatrix} = EI \begin{bmatrix} 12/L^3 & -6/L^2 & -12/L^3 & -6/L^2 \\ -6/L^2 & 4/L & 6/L^2 & 2/L \\ -12/L^3 & 6/L^2 & 12/L^3 & 6/L^2 \\ -6/L^2 & 2/L & 6/L^2 & 4/L \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} \quad (6.44)$$

which is of the form

$$\{F\} = [K_{ij}]\{\delta\}$$

where  $[K_{ij}]$  is the stiffness matrix for the beam.

It is possible to write Eq. (6.44) in an alternative form such that the elements of  $[K_{ij}]$  are pure numbers:

$$\begin{Bmatrix} F_{y,i} \\ M_i/L \\ F_{y,j} \\ M_j/L \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i L \\ v_j \\ \theta_j L \end{Bmatrix}$$

This form of Eq. (6.44) is particularly useful in numerical calculations for an assemblage of beams in which  $EI/L^3$  is constant.

Equation (6.44) is derived for a beam whose axis is aligned with the  $x$  axis so that the stiffness matrix defined by Eq. (6.44) is actually  $[\bar{K}_{ij}]$ , the stiffness matrix referred to a local coordinate

system. If the beam is positioned in the  $xy$  plane with its axis arbitrarily inclined to the  $x$  axis, then the  $x$  and  $y$  axes form a global coordinate system, and it becomes necessary to transform Eq. (6.44) to allow for this. The procedure is similar to that for the pin-jointed framework member of Section 6.4, in that  $[\bar{K}_{ij}]$  must be expanded to allow for the fact that nodal displacements  $\bar{u}_i$  and  $\bar{u}_j$ , which are irrelevant for the beam in local coordinates, have components  $u_i$ ,  $v_i$  and  $u_j$ ,  $v_j$  in global coordinates:

$$[\bar{K}_{ij}] = EI \begin{bmatrix} u_i & v_i & \theta_i & u_j & v_j & \theta_j \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12/L^3 & -6/L^2 & 0 & -12/L^3 & -6/L^2 \\ 0 & -6/L^2 & 4/L & 0 & 6/L^2 & 2/L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12/L^3 & 6/L^2 & 0 & 12/L^3 & 6/L^2 \\ 0 & -6/L^2 & 2/L & 0 & 6/L^2 & 4/L \end{bmatrix} \quad (6.45)$$

We may deduce the transformation matrix  $[T]$  from Eq. (6.24) if we remember that, although  $u$  and  $v$  transform in exactly the same way as in the case of a pin-jointed member, the rotations  $\theta$  remain the same in either local or global coordinates.

Hence,

$$[T] = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.46)$$

where  $\lambda$  and  $\mu$  have previously been defined. Thus, since

$$[K_{ij}] = [T]^T [\bar{K}_{ij}] [T] \quad (\text{see Section 6.4})$$

we have, from Eqs. (6.45) and (6.46),

$$[K_{ij}] = EI \begin{bmatrix} 12\mu^2/L^3 & & & & & \\ -12\lambda\mu/L^3 & 12\lambda^2/L^3 & & & & \\ 6\mu/L^2 & -6\lambda/L^2 & 4/L & & & \\ -12\mu^2/L^3 & 12\lambda\mu/L^3 & -6\mu/L^2 & 12\mu^2/L^3 & & \\ 12\lambda\mu/L^3 & -12\lambda^2/L^3 & 6\lambda/L^2 & -12\lambda\mu/L^3 & 12\lambda^2/L^3 & \\ 6\mu/L^2 & -6\lambda/L^2 & 2/L & 6\mu/L^2 & 6\lambda/L^2 & 4\lambda/L \end{bmatrix} \quad \text{SYM} \quad (6.47)$$

Again, the stiffness matrix for the complete structure is assembled from the member stiffness matrices, the boundary conditions are applied, and the resulting set of equations solved for the unknown nodal displacements and forces.

The internal shear forces and bending moments in a beam may be obtained in terms of the calculated nodal displacements. For a beam joining nodes  $i$  and  $j$ , we shall have obtained the unknown values of  $v_i$ ,  $\theta_i$  and  $v_j$ ,  $\theta_j$ . The nodal forces  $F_{y,i}$  and  $M_i$  are then obtained from Eq. (6.44) if the beam is aligned with the  $x$  axis. Hence,

$$\left. \begin{aligned} F_{y,i} &= EI \left( \frac{12}{L^3} v_i - \frac{6}{L^2} \theta_i - \frac{12}{L^3} v_j - \frac{6}{L^2} \theta_j \right) \\ M_i &= EI \left( -\frac{6}{L^2} v_i + \frac{4}{L} \theta_i + \frac{6}{L^2} v_j + \frac{2\theta_j}{L} \right) \end{aligned} \right\} \quad (6.48)$$

Similar expressions are obtained for the forces at node  $j$ . From Fig. 6.6, we see that the shear force  $S_y$  and bending moment  $M$  in the beam are given by

$$\left. \begin{aligned} S_y &= F_{y,i} \\ M &= F_{y,i}x + M_i \end{aligned} \right\} \quad (6.49)$$

Substituting Eq. (6.48) into Eq. (6.49) and expressing in matrix form yields

$$\left. \begin{aligned} \begin{Bmatrix} S_y \\ M \end{Bmatrix} &= EI \begin{bmatrix} \frac{12}{L^3} & -\frac{6}{L^2} & -\frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{12}{L^3}x - \frac{6}{L^2} & -\frac{6}{L^2}x + \frac{4}{L} & \frac{12}{L^3}x + \frac{6}{L^2} & -\frac{6}{L^2}x + \frac{2}{L} \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} \end{aligned} \right\} \quad (6.50)$$

The matrix analysis of the beam in Fig. 6.6 is based on the condition that no external forces are applied between the nodes. Obviously, in a practical case, a beam supports a variety of loads along its length, and therefore such beams must be idealized into a number of *beam-elements* for which this condition holds. The idealization is accomplished by merely specifying nodes at points along the beam such that any element lying between adjacent nodes carries, at the most, a uniform shear and a linearly varying bending moment. For example, the beam of Fig. 6.7 is idealized into beam-elements 1–2, 2–3, and 3–4, for which the unknown nodal displacements are  $v_2$ ,  $\theta_2$ ,  $v_3$ ,  $\theta_3$ ,  $v_4$  and  $\theta_4$  ( $v_1 = \theta_1 = v_3 = 0$ ).

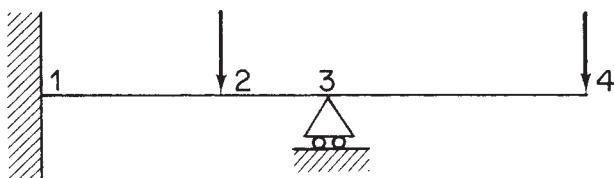
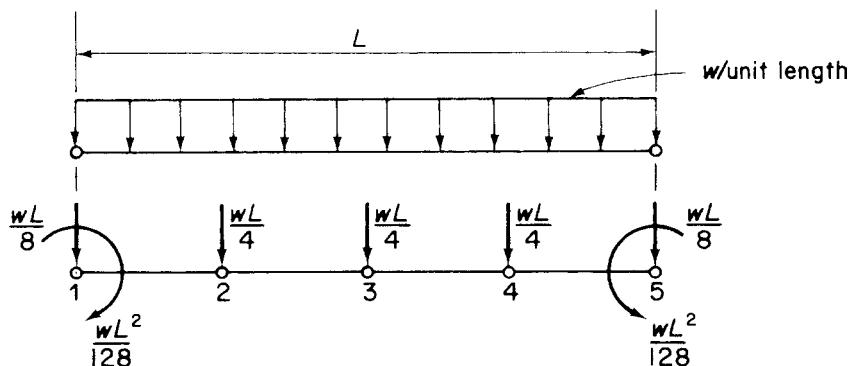


FIGURE 6.7 Idealization of a Beam Into Beam-Elements

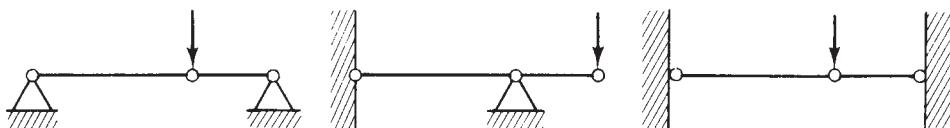
Beams supporting distributed loads require special treatment, in that the distributed load is replaced by a series of statically equivalent point loads at a selected number of nodes. Clearly, the greater the number of nodes chosen, the more accurate but more complicated and therefore time consuming is the analysis. [Figure 6.8](#) shows a typical idealization of a beam supporting a uniformly distributed load. Details of the analysis of such beams may be found in Martin.<sup>4</sup>

Many simple beam problems may be idealized into a combination of two beam-elements and three nodes. A few examples of such beams are shown in [Fig. 6.9](#). If we therefore assemble a stiffness matrix for the general case of a two beam-element system, we may use it to solve a variety of problems simply by inserting the appropriate loading and support conditions. Consider the assemblage of two beam-elements shown in [Fig. 6.10](#). The stiffness matrices for the beam-elements 1–2 and 2–3 are obtained from [Eq. \(6.44\)](#):

$$[K_{12}] = EI_a \begin{bmatrix} v_1 & \theta_1 \\ 12/L_a^3 & -6/L_a^2 \\ -6/L_a^2 & 4/L_a \\ -12/L_a^3 & 6/L_a^2 \\ -6/L_a^2 & 2/L_a \end{bmatrix} \begin{bmatrix} v_2 & \theta_2 \\ -12/L_a^3 & -6/L_a^2 \\ 6/L_a^2 & 2/L_a \\ 12/L_a^3 & 6/L_a^2 \\ 6/L_a^2 & 4/L_a \end{bmatrix} \quad (6.51)$$



**FIGURE 6.8 Idealization of a Beam Supporting a Uniformly Distributed Load**



**FIGURE 6.9 Idealization of Beams Into Beam-Elements**

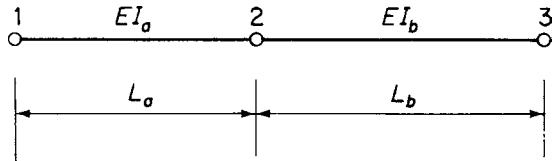


FIGURE 6.10 Assemblage of Two Beam-Elements

$$[K_{23}] = EI_b \begin{bmatrix} \frac{v_2}{12/L_b^3} & \frac{\theta_2}{-6/L_b^2} & \frac{v_3}{-12/L_b^3} & \frac{\theta_3}{-6/L_b^2} \\ -6/L_b^2 & k_{22} & 4/L_b & k_{23} \\ -12/L_b^3 & 6/L_b^2 & 12/L_b^3 & 6/L_b^2 \\ -6/L_b^2 & 2/L_b & 6/L_b^2 & 4/L_b \end{bmatrix} \quad (6.52)$$

The complete stiffness matrix is formed by superimposing  $[K_{12}]$  and  $[K_{23}]$  as described in Example 6.1. Hence,

$$[K] = E \begin{bmatrix} \frac{12I_a}{L_a^3} & -\frac{6I_a}{L_a^2} & -\frac{12I_a}{L_a^3} & -\frac{6I_a}{L_a^2} & 0 & 0 \\ -\frac{6I_a}{L_a^2} & \frac{4I_a}{L_a} & \frac{6I_a}{L_a^2} & \frac{2I_a}{L_a} & 0 & 0 \\ -\frac{12I_a}{L_a^3} & \frac{6I_a}{L_a^2} & 12\left(\frac{I_a}{L_a^3} + \frac{I_b}{L_b^3}\right) & 6\left(\frac{I_a}{L_a^2} - \frac{I_b}{L_b^2}\right) & -\frac{12I_b}{L_b^3} & -\frac{6I_b}{L_b^2} \\ 0 & -\frac{6I_a}{L_a^2} & 6\left(\frac{I_a}{L_a^2} - \frac{I_b}{L_b^2}\right) & 4\left(\frac{I_a}{L_a} + \frac{I_b}{L_b}\right) & \frac{6I_b}{L_b^2} & \frac{2I_b}{L_b} \\ 0 & 0 & -\frac{12I_b}{L_b^3} & \frac{6I_b}{L_b^2} & \frac{12I_b}{L_b^3} & \frac{6I_b}{L_b^2} \\ 0 & 0 & -\frac{6I_b}{L_b^2} & \frac{2I_b}{L_b} & \frac{6I_b}{L_b^2} & \frac{4I_b}{L_b} \end{bmatrix} \quad (6.53)$$

### Example 6.2

Determine the unknown nodal displacements and forces in the beam shown in Fig. 6.11. The beam is of uniform section throughout.

The beam may be idealized into two beam-elements, 1–2 and 2–3. From Fig. 6.11, we see that  $v_1 = v_3 = 0$ ,  $F_{y,2} = -W$ ,  $M_2 = +M$ . Therefore, eliminating rows and columns corresponding to zero displacements from Eq. (6.53), we obtain

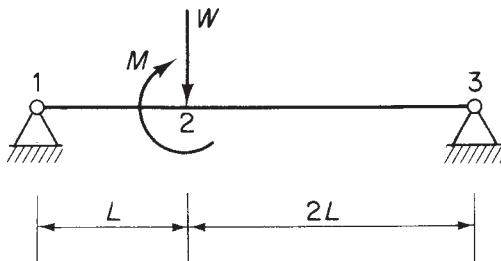


FIGURE 6.11 Beam of Example 6.2

$$\begin{Bmatrix} F_{y,2} = -W \\ M_2 = M \\ M_1 = 0 \\ M_3 = 0 \end{Bmatrix} = EI \begin{bmatrix} 27/2L^3 & 9/2L^2 & 6/L^2 & -3/2L^2 \\ 9/2L^2 & 6/L & 2/L & 1/L \\ 6/L^2 & 2/L & 4/L & 0 \\ -3/2L^2 & 1/L & 0 & 2/L \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_1 \\ \theta_3 \end{Bmatrix} \quad (i)$$

Equation (i) may be written such that the elements of  $[K]$  are pure numbers:

$$\begin{Bmatrix} F_{y,2} = -W \\ M_2/L = M/L \\ M_1/L = 0 \\ M_3/L = 0 \end{Bmatrix} = \frac{EI}{2L^3} \begin{bmatrix} 27 & 9 & 12 & -3 \\ 9 & 12 & 4 & 2 \\ 12 & 4 & 8 & 0 \\ -3 & 2 & 0 & 4 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2L \\ \theta_1L \\ \theta_3L \end{Bmatrix} \quad (ii)$$

Expanding Eq. (ii) by matrix multiplication, we have

$$\begin{Bmatrix} -W \\ M/L \end{Bmatrix} = \frac{EI}{2L^3} \left( \begin{bmatrix} 27 & 9 \\ 9 & 12 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2L \end{Bmatrix} + \begin{bmatrix} 12 & -3 \\ 4 & 2 \end{bmatrix} \begin{Bmatrix} \theta_1L \\ \theta_3L \end{Bmatrix} \right) \quad (iii)$$

and

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \frac{EI}{2L^3} \left( \begin{bmatrix} 12 & 4 \\ -3 & 2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2L \end{Bmatrix} + \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \theta_1L \\ \theta_3L \end{Bmatrix} \right) \quad (iv)$$

Equation (iv) gives

$$\begin{Bmatrix} \theta_1L \\ \theta_3L \end{Bmatrix} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2L \end{Bmatrix} \quad (v)$$

Substituting Eq. (v) in Eq. (iii), we obtain

$$\begin{Bmatrix} v_2 \\ \theta_2L \end{Bmatrix} = \frac{L^3}{9EI} \begin{bmatrix} -4 & -2 \\ -2 & 3 \end{bmatrix} \begin{Bmatrix} -W \\ M/L \end{Bmatrix} \quad (vi)$$

from which the unknown displacements at node 2 are

$$v_2 = -\frac{4WL^3}{9EI} - \frac{2ML^2}{9EI}$$

$$\theta_2 = \frac{2WL^2}{9EI} + \frac{1ML}{3EI}$$

In addition, from Eq. (v), we find that

$$\theta_1 = \frac{5WL^2}{9EI} + \frac{1}{6} \frac{ML}{EI}$$

$$\theta_3 = -\frac{4WL^2}{9EI} - \frac{1}{3} \frac{ML}{EI}$$

Note that the solution has been obtained by inverting two  $2 \times 2$  matrices rather than the  $4 \times 4$  matrix of Eq. (ii). This simplification has been brought about by the fact that  $M_1 = M_3 = 0$ .

The internal shear forces and bending moments can now be found using Eq. (6.50). For the beam-element 1–2, we have

$$S_{y,12} = EI \left( \frac{12}{L^3} v_1 - \frac{6}{L^2} \theta_1 - \frac{12}{L^3} v_2 - \frac{6}{L^2} \theta_2 \right)$$

or

$$S_{y,12} = \frac{2}{3} W - \frac{1}{3} \frac{M}{L}$$

and

$$M_{12} = EI \left[ \left( \frac{12}{L^3} x - \frac{6}{L^2} \right) v_1 + \left( -\frac{6}{L^2} x + \frac{4}{L} \right) \theta_1 + \left( -\frac{12}{L^3} x + \frac{6}{L^2} \right) v_2 + \left( -\frac{6}{L^2} x + \frac{2}{L} \right) \theta_2 \right]$$

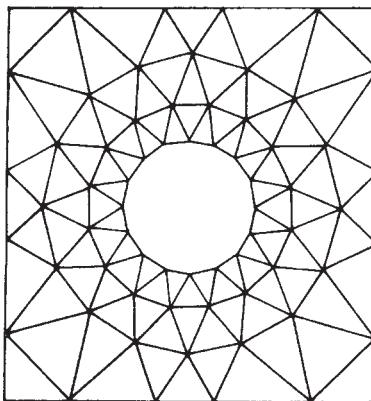
which reduces to

$$M_{12} = \left( \frac{2}{3} W - \frac{1}{3} \frac{M}{L} \right) x$$

■

## 6.8 FINITE ELEMENT METHOD FOR CONTINUUM STRUCTURES

In the previous sections, we have discussed the matrix method of solution of structures composed of elements connected only at nodal points. For skeletal structures consisting of arrangements of beams, these nodal points fall naturally at joints and at positions of concentrated loading. Continuum structures, such as flat plates, aircraft skins, shells, etc., possess no such natural subdivisions and must therefore be artificially idealized into a number of elements before matrix methods can be used. These *finite elements*, as they are known, may be two- or three-dimensional, but the most commonly used are two-dimensional triangular and quadrilateral shaped elements. The idealization may be carried out in any number of different ways, depending on such factors as the type of problem, the accuracy of the solution required, and the time and money available. For example, a *coarse* idealization involving a small number of large elements provides a comparatively rapid but very approximate solution, while a *fine* idealization of small elements produces more accurate results but takes longer and consequently costs more. Frequently, *graded meshes* are used, in which small elements are placed in regions where high stress concentrations are expected, for example,



**FIGURE 6.12 Finite Element Idealization of a Flat Plate With a Central Hole**

around cut-outs and loading points. The principle is illustrated in Fig. 6.12, where a graded system of triangular elements is used to examine the stress concentration around a circular hole in a flat plate.

Although the elements are connected at an infinite number of points around their boundaries, it is assumed that they are interconnected only at their corners or nodes. Thus, compatibility of displacement is ensured only at the nodal points. However, in the finite element method, a displacement pattern is chosen for each element that may satisfy some, if not all, of the compatibility requirements along the sides of adjacent elements.

Since we are employing matrix methods of solution, we are concerned initially with the determination of nodal forces and displacements. Therefore, the system of loads on the structure must be replaced by an equivalent system of nodal forces. Where these loads are concentrated, the elements are chosen such that a node occurs at the point of application of the load. In the case of distributed loads, equivalent nodal concentrated loads must be calculated.<sup>4</sup>

The solution procedure is identical in outline to that described in the previous sections for skeletal structures; the differences lie in the idealization of the structure into finite elements and the calculation of the stiffness matrix for each element. The latter procedure, which in general terms is applicable to all finite elements, may be specified in a number of distinct steps. We shall illustrate the method by establishing the stiffness matrix for the simple one-dimensional beam-element of Fig. 6.6, for which we have already derived the stiffness matrix using slope-deflection.

### 6.8.1 Stiffness matrix for a beam-element

The first step is to choose a suitable coordinate and node numbering system for the element and define its nodal displacement vector  $\{\delta^e\}$  and nodal load vector  $\{F^e\}$ . Use is made here of the superscript  $e$  to denote element vectors, since, in general, a finite element possesses more than two nodes. Again, we are not concerned with axial or shear displacements, so that for the beam-element of Fig. 6.6, we have

$$\{\delta^e\} = \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}, \{F^e\} = \begin{Bmatrix} F_{y,i} \\ M_i \\ F_{y,j} \\ M_j \end{Bmatrix}$$

Since each of these vectors contains four terms, the element stiffness matrix  $[K^e]$  is of order of  $4 \times 4$ .

In the second step, we select a displacement function that uniquely defines the displacement of all points in the beam–element in terms of the nodal displacements. This displacement function may be taken as a polynomial that must include four arbitrary constants, corresponding to the four nodal degrees of freedom of the element:

$$v(x) = \alpha_1 + \alpha_2x + \alpha_3x^2 + \alpha_4x^3 \quad (6.54)$$

Equation (6.54) is of the same form as that derived from elementary bending theory for a beam subjected to concentrated loads and moments and may be written in matrix form as

$$\{v(x)\} = [1 \ x \ x^2 \ x^3] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

or in abbreviated form as

$$\{v(x)\} = [f(x)]\{\alpha\} \quad (6.55)$$

The rotation  $\theta$  at any section of the beam–element is given by  $\partial v/\partial x$ ; therefore,

$$\theta = \alpha_2 + 2\alpha_3x + 3\alpha_4x^2 \quad (6.56)$$

From Eqs. (6.54) and (6.56), we can write expressions for the nodal displacements  $v_i$ ,  $\theta_i$  and  $v_j$ ,  $\theta_j$  at  $x = 0$  and  $x = L$ , respectively. Hence,

$$\left. \begin{array}{l} v_i = \alpha_1 \\ \theta_i = \alpha_2 \\ v_j = \alpha_1 + \alpha_2L + \alpha_3L^2 + \alpha_4L^3 \\ \theta_j = \alpha_2 + 2\alpha_3L + 3\alpha_4L^2 \end{array} \right\} \quad (6.57)$$

Writing Eqs. (6.57) in matrix form gives

$$\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad (6.58)$$

or

$$\{\delta^e\} = [A]\{\alpha\} \quad (6.59)$$

The third step follows directly from Eqs. (6.58) and (6.55), in that we express the displacement at any point in the beam–element in terms of the nodal displacements. Using Eq. (6.59), we obtain

$$\{\alpha\} = [A^{-1}]\{\delta^e\} \quad (6.60)$$

Substituting in Eq. (6.55) gives

$$\{v(x)\} = [f(x)][A^{-1}]\{\delta^e\} \quad (6.61)$$

where  $[A^{-1}]$  is obtained by inverting  $[A]$  in Eq. (6.58) and may be shown to be given by

$$[A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/L^2 & -2/L & 3/L^2 & -1/L \\ 2/L^3 & 1/L^2 & -2/L^3 & 1/L^2 \end{bmatrix} \quad (6.62)$$

In step four, we relate the strain  $\{\varepsilon(x)\}$  at any point  $x$  in the element to the displacement  $\{v(x)\}$  and hence to the nodal displacements  $\{\delta^e\}$ . Since we are concerned here with bending deformations only, we may represent the strain by the curvature  $\partial^2 v / \partial x^2$ . Hence, from Eq. (6.54),

$$\frac{\partial^2 v}{\partial x^2} = 2\alpha_3 + 6\alpha_4 x \quad (6.63)$$

or in matrix form,

$$\{\varepsilon\} = [0 \ 0 \ 2 \ 6x] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad (6.64)$$

which we write as

$$\{\varepsilon\} = [C]\{\alpha\} \quad (6.65)$$

Substituting for  $\{\alpha\}$  in Eq. (6.65) from Eq. (6.60), we have

$$\{\varepsilon\} = [C][A^{-1}]\{\delta^e\} \quad (6.66)$$

Step five relates the internal stresses in the element to the strain  $\{\varepsilon\}$ , and hence, using Eq. (6.66), to the nodal displacements  $\{\delta^e\}$ . In our beam-element, the stress distribution at any section depends entirely on the value of the bending moment  $M$  at that section. Therefore, we may represent a “state of stress”  $\{\sigma\}$  at any section by the bending moment  $M$ , which, from simple beam theory, is given by

$$M = EI \frac{\partial^2 v}{\partial x^2} \quad (\text{see Eqs. (16.31)})$$

or

$$\{\sigma\} = [EI]\{\varepsilon\} \quad (6.67)$$

which we write as

$$\{\sigma\} = [D]\{\varepsilon\} \quad (6.68)$$

The matrix  $[D]$  in Eq. (6.68) is the “elasticity” matrix relating “stress” and “strain.” In this case,  $[D]$  consists of a single term, the flexural rigidity  $EI$  of the beam. Generally, however,  $[D]$  is of a higher order. If we now substitute for  $\{\varepsilon\}$  in Eq. (6.68) from Eq. (6.66), we obtain the “stress” in terms of the nodal displacements; that is,

$$\{\sigma\} = [D][C][A^{-1}]\{\delta^e\} \quad (6.69)$$

The element stiffness matrix is finally obtained in step six, in which we replace the internal “stresses”  $\{\sigma\}$  by a statically equivalent nodal load system  $\{F^e\}$ , thereby relating nodal loads to nodal displacements (from Eq. (6.69)) and defining the element stiffness matrix  $[K^e]$ . This is achieved by employing

the principle of the stationary value of the total potential energy of the beam (see Section 5.8), which comprises the internal strain energy  $U$  and the potential energy  $V$  of the nodal loads. Thus,

$$U + V = \frac{1}{2} \int_{\text{vol}} \{\varepsilon\}^T \{\sigma\} d(\text{vol}) - \{\delta^e\}^T \{F^e\} \quad (6.70)$$

Substituting in Eq. (6.70) for  $\{\varepsilon\}$  from Eq. (6.66) and  $\{\sigma\}$  from Eq. (6.69), we have

$$U + V = \frac{1}{2} \int_{\text{vol}} \{\delta^e\}^T [A^{-1}]^T [C]^T [D] [C] [A^{-1}] \{\delta^e\} d(\text{vol}) - \{\delta^e\}^T \{F^e\} \quad (6.71)$$

The total potential energy of the beam has a stationary value with respect to the nodal displacements  $\{\delta^e\}^T$ ; hence, from Eq. (6.71),

$$\frac{\partial(U + V)}{\partial(\delta^e)} = \int_{\text{vol}} [A^{-1}]^T [C]^T [D] [C] [A^{-1}] \{\delta^e\} d(\text{vol}) - \{F^e\} = 0 \quad (6.72)$$

from which

$$\{F^e\} = \left[ \int_{\text{vol}} [C]^T [A^{-1}]^T [D] [C] [A^{-1}] d(\text{vol}) \right] \{\delta^e\} \quad (6.73)$$

or, writing  $[C][A^{-1}]$  as  $[B]$ , we obtain

$$\{F^e\} = \left[ \int_{\text{vol}} [B]^T [D] [B] d(\text{vol}) \right] \{\delta^e\} \quad (6.74)$$

from which the element stiffness matrix is clearly

$$\{K^e\} = \left[ \int_{\text{vol}} [B]^T [D] [B] d(\text{vol}) \right] \quad (6.75)$$

From Eqs. (6.62) and (6.64) we have

$$[B] = [C][A^{-1}] = [0 \ 0 \ 2 \ 6x] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/L^2 & -2/L & 3/L^2 & -1/L \\ 2/L^3 & 1/L^2 & -2/L^3 & 1/L^2 \end{bmatrix}$$

or

$$[B]^T = \begin{bmatrix} -\frac{6}{L^2} + \frac{12x}{L^3} \\ -\frac{4}{L} + \frac{6x}{L^2} \\ \frac{6}{L^2} - \frac{12x}{L^3} \\ -\frac{2}{L} + \frac{6x}{L^2} \end{bmatrix} \quad (6.76)$$

Hence,

$$[K^e] = \int_0^L \begin{bmatrix} -\frac{6}{L^2} + \frac{12x}{L^3} \\ -\frac{4}{L} + \frac{6x}{L^2} \\ \frac{6}{L^2} - \frac{12x}{L^3} \\ -\frac{2}{L} + \frac{6x}{L^2} \end{bmatrix} [EI] \begin{bmatrix} -\frac{6}{L^2} + \frac{12x}{L^3} & -\frac{4}{L} + \frac{6x}{L^2} & \frac{6}{L^2} - \frac{12x}{L^3} & -\frac{2}{L} + \frac{6x}{L^2} \end{bmatrix} dx$$

which gives

$$[K^e] = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \quad (6.77)$$

Equation (6.77) is identical to the stiffness matrix (see Eq. (6.44)) for the uniform beam of Fig. 6.6.

Finally, in step seven, we relate the internal “stresses,”  $\{\sigma\}$ , in the element to the nodal displacements  $\{\delta^e\}$ . This has in fact been achieved to some extent in Eq. (6.69), namely

$$\{\sigma\} = [D][C][A^{-1}]\{\delta^e\}$$

or, from the preceding,

$$\{\sigma\} = [D][B]\{\delta^e\} \quad (6.78)$$

Equation (6.78) is usually written

$$\{\sigma\} = [H]\{\delta^e\} \quad (6.79)$$

in which  $[H] = [D][B]$  is the stress–displacement matrix. For this particular beam–element,  $[D] = EI$  and  $[B]$  is defined in Eq. (6.76). Thus,

$$[H] = [EI] \begin{bmatrix} -\frac{6}{L^2} + \frac{12x}{L^3} & -\frac{4}{L} + \frac{6x}{L^2} & \frac{6}{L^2} - \frac{12x}{L^3} & -\frac{2}{L} + \frac{6x}{L^2} \end{bmatrix} \quad (6.80)$$

## 6.8.2 Stiffness matrix for a triangular finite element

Triangular finite elements are used in the solution of plane stress and plane strain problems. Their advantage over other shaped elements lies in their ability to represent irregular shapes and boundaries with relative simplicity.

In the derivation of the stiffness matrix, we shall adopt the step-by-step procedure of the previous example. Initially, therefore, we choose a suitable coordinate and node numbering system for the element and define its nodal displacement and nodal force vectors. Figure 6.13 shows a triangular element referred to axes Oxy and having nodes  $i$ ,  $j$ , and  $k$  lettered counterclockwise. It may be shown that the inverse of the  $[A]$  matrix for a triangular element contains terms giving the actual area of the element; this area is positive if the preceding node lettering or numbering system is adopted. The element

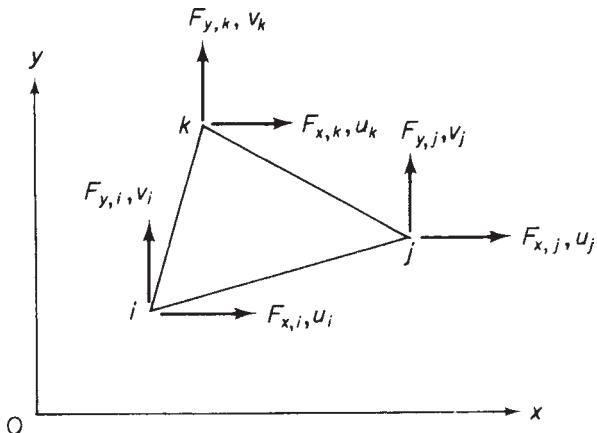


FIGURE 6.13 Triangular Element for Plane Elasticity Problems

is to be used for plane elasticity problems and therefore has two degrees of freedom per node, giving a total of six degrees of freedom for the element, which results in a  $6 \times 6$  element stiffness matrix  $[K^e]$ . The nodal forces and displacements are shown, and the complete displacement and force vectors are

$$\{\delta^e\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix} \quad \{F^e\} = \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \\ F_{x,k} \\ F_{y,k} \end{Bmatrix} \quad (6.81)$$

We now select a displacement function that must satisfy the boundary conditions of the element, that is, the condition that each node possesses two degrees of freedom. Generally, for computational purposes, a polynomial is preferable to, say, a trigonometric series, since the terms in a polynomial can be calculated much more rapidly by a digital computer. Furthermore, the total number of degrees of freedom is six, so that only six coefficients in the polynomial can be obtained. Suppose that the displacement function is

$$\left. \begin{aligned} u(x, y) &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x, y) &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \right\} \quad (6.82)$$

The constant terms,  $\alpha_1$  and  $\alpha_4$ , are required to represent any in-plane rigid body motion, that is, motion without strain, while the linear terms enable states of constant strain to be specified; Eqs. (6.82) ensure compatibility of displacement along the edges of adjacent elements. Writing Eqs. (6.82) in matrix form gives

$$\left. \begin{aligned} u(x, y) \\ v(x, y) \end{aligned} \right\} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \left\{ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{array} \right\} \quad (6.83)$$

Comparing Eq. (6.83) with Eq. (6.55), we see that it is of the form

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = [f(x, y)]\{\alpha\} \quad (6.84)$$

Substituting values of displacement and coordinates at each node in Eq. (6.84), we have, for node  $i$ ,

$$\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \end{bmatrix} \{\alpha\}$$

Similar expressions are obtained for nodes  $j$  and  $k$ , so that, for the complete element, we obtain

$$\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 1 & x_j & y_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 1 & x_k & y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \quad (6.85)$$

From Eq. (6.81) and by comparison with Eqs. (6.58) and (6.59), we see that Eq. (6.85) takes the form

$$\{\delta^e\} = [A]\{\alpha\}$$

Hence (step 3), we obtain

$$\{\alpha\} = [A^{-1}]\{\delta^e\} \quad (\text{compare with Eq. (6.60)})$$

The inversion of  $[A]$ , defined in Eq. (6.85), may be achieved algebraically, as illustrated in Example 6.3. Alternatively, the inversion may be carried out numerically for a particular element by computer. Substituting for  $\{\alpha\}$  from the preceding into Eq. (6.84) gives

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = [f(x, y)][A^{-1}]\{\delta^e\} \quad (\text{compare with Eq. (6.61)}) \quad (6.86)$$

The strains in the element are

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (6.87)$$

From Eqs. (1.18) and (1.20), we see that

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (6.88)$$

Substituting for  $u$  and  $v$  in Eqs. (6.88) from Eqs. (6.82) gives

$$\begin{aligned} \varepsilon_x &= \alpha_2 \\ \varepsilon_y &= \alpha_6 \\ \gamma_{xy} &= \alpha_3 + \alpha_5 \end{aligned}$$

or in matrix form,

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \quad (6.89)$$

which is of the form

$$\{\varepsilon\} = [C]\{\alpha\} \quad (\text{see Eqs. (6.64) and (6.65)})$$

Substituting for  $\{\alpha\} (= [A^{-1}] \{\delta^e\})$ , we obtain

$$\{\varepsilon\} = [C][A^{-1}]\{\delta^e\} \quad (\text{compare with Eq. (6.66)})$$

or

$$\{\varepsilon\} = [B]\{\delta^e\} \quad (\text{see Eq. 6.76})$$

where  $[C]$  is defined in [Eq. \(6.89\)](#).

In step five, we relate the internal stresses  $\{\sigma\}$  to the strain  $\{\varepsilon\}$ , and hence, using step four, to the nodal displacements  $\{\delta^e\}$ . For plane stress problems,

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (6.90)$$

and

$$\left. \begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} \\ \varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy} \end{aligned} \right\} \quad (\text{see Chapter 1})$$

Thus, in matrix form,

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (6.91)$$

It may be shown that (see [Chapter 1](#))

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (6.92)$$

which has the form of [Eq. \(6.68\)](#); that is,

$$\{\sigma\} = [D]\{\varepsilon\}$$

Substituting for  $\{\varepsilon\}$  in terms of the nodal displacements  $\{\delta^e\}$ , we obtain

$$\{\sigma\} = [D][B]\{\delta^e\} \quad (\text{see Eq. (6.69)})$$

In the case of plane strain, the elasticity matrix  $[D]$  takes a different form to that defined in Eq. (6.92). For this type of problem,

$$\begin{aligned}\varepsilon_x &= \frac{\sigma_x}{E} - \frac{v\sigma_y}{E} - \frac{v\sigma_z}{E} \\ \varepsilon_y &= \frac{\sigma_y}{E} - \frac{v\sigma_x}{E} - \frac{v\sigma_z}{E} \\ \varepsilon_z &= \frac{\sigma_z}{E} - \frac{v\sigma_x}{E} - \frac{v\sigma_y}{E} = 0 \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} = \frac{2(1+v)}{E} \tau_{xy}\end{aligned}$$

Eliminating  $\sigma_z$  and solving for  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$ , gives (see Eqs. (1.43)–(1.45))

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E(1-v)}{(1+v)(1-2v)} \begin{bmatrix} 1 & \frac{v}{1-v} & 0 \\ \frac{v}{1-v} & 1 & 0 \\ 0 & 0 & \frac{(1-2v)}{2(1-v)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (6.93)$$

which again takes the form

$$\{\sigma\} = [D]\{\varepsilon\}$$

Step six, in which the internal stresses  $\{\sigma\}$  are replaced by the statically equivalent nodal forces  $\{F^e\}$ , proceeds in an identical manner to that described for the beam-element. Thus,

$$\{F^e\} = \left[ \int_{\text{vol}} [B]^T [D] [B] d(\text{vol}) \right] \{\delta^e\}$$

as in Eq. (6.74), whence

$$\{K^e\} = \left[ \int_{\text{vol}} [B]^T [D] [B] d(\text{vol}) \right]$$

In this expression  $[B] = [C][A^{-1}]$ , where  $[A]$  is defined in Eq. (6.85) and  $[C]$  in Eq. (6.89). The elasticity matrix  $[D]$  is defined in Eq. (6.92) for plane stress problems or in Eq. (6.93) for plane strain problems. We note that the  $[C]$ ,  $[A]$  (therefore  $[B]$ ), and  $[D]$  matrices contain only constant terms and may therefore be taken outside the integration in the expression for  $[K^e]$ , leaving only  $\int d(\text{vol})$ , which is simply the area  $A$  of the triangle times its thickness  $t$ . Thus,

$$[K^e] = [[B]]^T [D] [B] A t \quad (6.94)$$

Finally, the element stresses follow from Eq. (6.79); that is,

$$\{\sigma\} = [H]\{\delta^e\}$$

where  $[H] = [D][B]$  and  $[D]$  and  $[B]$  have previously been defined. It is usually found convenient to plot the stresses at the centroid of the element.

Of all the finite elements in use, the triangular element is probably the most versatile. It may be used to solve a variety of problems ranging from two-dimensional flat plate structures to three-dimensional folded plates and shells. For three-dimensional applications, the element stiffness matrix [ $K^e$ ] is transformed from an in-plane  $xy$  coordinate system to a three-dimensional system of global coordinates by the use of a transformation matrix similar to those developed for the matrix analysis of skeletal structures. In addition to the above, triangular elements may be adapted for use in plate flexure problems and for the analysis of bodies of revolution.

### Example 6.3

A constant strain triangular element has corners 1(0, 0), 2(4, 0), and 3(2, 2) referred to a Cartesian Oxy axes system and is 1 unit thick. If the elasticity matrix [ $D$ ] has elements  $D_{11} = D_{22} = a$ ,  $D_{12} = D_{21} = b$ ,  $D_{13} = D_{23} = D_{31} = D_{32} = 0$  and  $D_{33} = c$ , derive the stiffness matrix for the element.

From Eq. (6.82),

$$u_1 = \alpha_1 + \alpha_2(0) + \alpha_3(0)$$

that is,

$$u_1 = \alpha_1 \quad (i)$$

$$u_2 = \alpha_1 + \alpha_2(4) + \alpha_3(0)$$

that is,

$$u_2 = \alpha_1 + 4\alpha_2 \quad (ii)$$

$$u_3 = \alpha_1 + \alpha_2(2) + \alpha_3(2)$$

that is,

$$u_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \quad (iii)$$

From Eq. (i),

$$\alpha_1 = u_1 \quad (iv)$$

and, from Eqs (ii) and (iv),

$$\alpha_2 = \frac{u_2 - u_1}{4} \quad (v)$$

Then, from Eqs (iii) to (v),

$$\alpha_3 = \frac{2u_3 - u_1 - u_2}{4} \quad (vi)$$

Substituting for  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  in the first of Eqs. (6.82) gives

$$u = u_1 + \left(\frac{u_2 - u_1}{4}\right)x + \left(\frac{2u_3 - u_1 - u_2}{4}\right)y$$

or

$$u = \left(1 - \frac{x}{4} - \frac{y}{4}\right)u_1 + \left(\frac{x}{4} - \frac{y}{4}\right)u_2 + \frac{y}{2}u_3 \quad (vii)$$

Similarly,

$$v = \left(1 - \frac{x}{4} - \frac{y}{4}\right)v_1 + \left(\frac{x}{4} - \frac{y}{4}\right)v_2 + \frac{y}{2}v_3 \quad (\text{viii})$$

Now, from Eq. (6.88),

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} = -\frac{u_1}{4} + \frac{u_2}{4} \\ \varepsilon_y &= \frac{\partial v}{\partial y} = -\frac{v_1}{4} - \frac{v_2}{4} + \frac{v_3}{2}\end{aligned}$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{u_1}{4} - \frac{u_2}{4} - \frac{v_1}{4} + \frac{v_2}{4}$$

Hence, since, from Eq. (6.66),  $\{\varepsilon\} = [B] \{\delta^e\}$

$$[B]\{\delta^e\} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -1 & -1 & 1 & 2 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (\text{ix})$$

Also, the elements of the  $[D]$  matrix are given, i.e.,

$$[D] = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

Hence,

$$[D][B] = \frac{1}{4} \begin{bmatrix} -a & -b & a & -b & 0 & 2b \\ -b & -a & b & -a & 0 & 2a \\ -c & -c & -c & c & 2c & 0 \end{bmatrix}$$

and

$$[B]^T [D] [B] = \frac{1}{16} \begin{bmatrix} a+c & b+c & -a+c & b-c & -2c & -2b \\ b+c & a+c & -b+c & a-c & -2c & -2a \\ -a+c & -b+c & a+c & -b-c & -2c & 2b \\ b-c & a-c & -b-c & a+c & 2c & -2a \\ -2c & -2c & -2c & 2c & 4c & 0 \\ -2b & -2a & 2b & -2a & 0 & 4a \end{bmatrix}$$

Then, from Eq. (6.94), in which the volume  $At$  of the element is  $2 \times 1 \times 2 \times 1 = 4$  units,

$$[K^e] = \frac{1}{4} \begin{bmatrix} a+c & b+c & -a+c & b-c & -2c & -2b \\ b+c & a+c & -b+c & a-c & -2c & -2a \\ -a+c & -b+c & a+c & -b-c & -2c & 2b \\ b-c & a-c & -b-c & a+c & 2c & -2a \\ -2c & -2c & -2c & 2c & 4c & 0 \\ -2b & -2a & 2b & -2a & 0 & 4a \end{bmatrix}$$

■

### 6.8.3 Stiffness matrix for a quadrilateral element

Quadrilateral elements are frequently used in combination with triangular elements to build up particular geometrical shapes.

Figure 6.14 shows a quadrilateral element referred to axes Oxy and having corner nodes,  $i, j, k$ , and  $l$ ; the nodal forces and displacements are also shown, and the displacement and force vectors are

$$\{\delta^e\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ u_l \\ v_l \end{Bmatrix}, \{F^e\} = \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \\ F_{x,k} \\ F_{y,k} \\ F_{x,l} \\ F_{y,l} \end{Bmatrix} \quad (6.95)$$

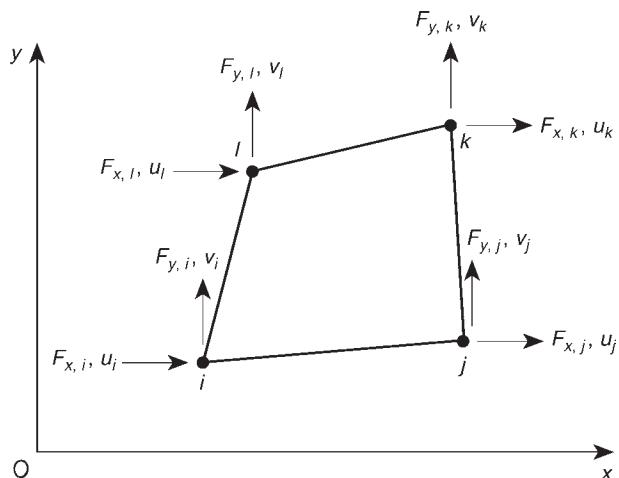


FIGURE 6.14 Quadrilateral Element Subjected to Nodal In-Plane Forces and Displacements

As in the case of the triangular element, we select a displacement function that satisfies the total of eight degrees of freedom of the nodes of the element; again this displacement function is in the form of a polynomial with a maximum of eight coefficients:

$$\left. \begin{aligned} u(x, y) &= \alpha_1 + \alpha_2x + \alpha_3y + \alpha_4xy \\ v(x, y) &= \alpha_5 + \alpha_6x + \alpha_7y + \alpha_8xy \end{aligned} \right\} \quad (6.96)$$

The constant terms,  $\alpha_1$  and  $\alpha_5$ , are required, as before, to represent the in-plane rigid body motion of the element, while the two pairs of linear terms enable states of constant strain to be represented throughout the element. Further, the inclusion of the  $xy$  terms results in both the  $u(x, y)$  and  $v(x, y)$  displacements having the same algebraic form, so that the element behaves in exactly the same way in the  $x$  direction as it does in the  $y$  direction.

Writing Eqs. (6.96) in matrix form gives

$$\left. \begin{aligned} u(x, y) \\ v(x, y) \end{aligned} \right\} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \left. \begin{aligned} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{aligned} \right\} \quad (6.97)$$

or

$$\left. \begin{aligned} u(x, y) \\ v(x, y) \end{aligned} \right\} = [f(x, y)]\{\alpha\} \quad (6.98)$$

Now, substituting the coordinates and values of displacement at each node, we obtain

$$\left. \begin{aligned} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ u_l \\ v_l \end{aligned} \right\} = \begin{bmatrix} 1 & x_i & y_i & x_i y_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_i & y_i & x_i y_i \\ 1 & x_j & y_j & x_j y_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_j & y_j & x_j y_j \\ 1 & x_k & y_k & x_k y_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_k & y_k & x_k y_k \\ 1 & x_l & y_l & x_l y_l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_l & y_l & x_l y_l \end{bmatrix} \left. \begin{aligned} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{aligned} \right\} \quad (6.99)$$

which is of the form

$$\{\delta^e\} = [A]\{\alpha\}$$

Then,

$$\{\alpha\} = [A^{-1}]\{\delta^e\} \quad (6.100)$$

The inversion of  $[A]$  is illustrated in Example 6.4 but, as in the case of the triangular element, is most easily carried out by means of a computer. The remaining analysis is identical to that for the triangular element except that the  $\{\varepsilon\}$ - $\{\alpha\}$  relationship (see Eq. (6.89)) becomes

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \quad (6.101)$$

**Example 6.4**

A rectangular element used in a plane stress analysis has corners whose coordinates (in meters), referred to an Oxy axes system, are 1(-2, -1), 2(2, -1), 3(2, 1), and 4(-2, 1); the displacements (also in meters) of the corners are

$$\begin{aligned} u_1 &= 0.001, & u_2 &= 0.003, & u_3 &= -0.003, & u_4 &= 0 \\ v_1 &= -0.004, & v_2 &= -0.002, & v_3 &= 0.001, & v_4 &= 0.001 \end{aligned}$$

If Young's modulus  $E = 200,000$  N/mm<sup>2</sup> and Poisson's ratio  $\nu = 0.3$ , calculate the stresses at the center of the element.

From the first of Eqs. (6.96),

$$u_1 = \alpha_1 - 2\alpha_2 - \alpha_3 + 2\alpha_4 = 0.001 \quad (\text{i})$$

$$u_2 = \alpha_1 + 2\alpha_2 - \alpha_3 - 2\alpha_4 = 0.003 \quad (\text{ii})$$

$$u_3 = \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = -0.003 \quad (\text{iii})$$

$$u_4 = \alpha_1 - 2\alpha_2 + \alpha_3 - 2\alpha_4 = 0 \quad (\text{iv})$$

Subtracting Eq. (ii) from Eq. (i),

$$\alpha_2 - \alpha_4 = 0.0005 \quad (\text{v})$$

Now, subtracting Eq. (iv) from Eq. (iii),

$$\alpha_2 + \alpha_4 = -0.00075 \quad (\text{vi})$$

Then, subtracting Eq. (vi) from Eq. (v),

$$\alpha_4 = -0.000625 \quad (\text{vii})$$

whence, from either Eq. (v) or (vi),

$$\alpha_2 = -0.000125 \quad (\text{viii})$$

Adding Eqs. (i) and (ii),

$$\alpha_1 - \alpha_3 = 0.002 \quad (\text{ix})$$

Adding Eqs. (iii) and (iv),

$$\alpha_1 + \alpha_3 = -0.0015 \quad (\text{x})$$

Then, adding Eqs. (ix) and (x),

$$\alpha_1 = 0.00025 \quad (\text{xi})$$

and, from either Eq. (ix) or (x),

$$\alpha_3 = -0.00175 \quad (\text{xi})$$

The second of Eqs. (6.96) is used to determine  $\alpha_5, \alpha_6, \alpha_7, \alpha_8$  in an identical manner to the preceding. Thus,

$$\begin{aligned}\alpha_5 &= -0.001 \\ \alpha_6 &= 0.00025 \\ \alpha_7 &= 0.002 \\ \alpha_8 &= -0.00025\end{aligned}$$

Now, substituting for  $\alpha_1, \alpha_2, \dots, \alpha_8$  in Eqs. (6.96),

$$u_i = 0.00025 - 0.000125x - 0.00175y - 0.000625xy$$

and

$$v_i = -0.001 + 0.00025x + 0.002y - 0.00025xy$$

Then, from Eqs. (6.88),

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} = -0.000125 - 0.000625y \\ \varepsilon_y &= \frac{\partial v}{\partial y} = 0.002 - 0.00025x \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -0.0015 - 0.000625x - 0.00025y\end{aligned}$$

Therefore, at the center of the element ( $x = 0, y = 0$ ),

$$\begin{aligned}\varepsilon_x &= -0.000125 \\ \varepsilon_y &= 0.002 \\ \gamma_{xy} &= -0.0015\end{aligned}$$

so that, from Eqs. (6.92),

$$\sigma_x = \frac{E}{1 - v^2}(\varepsilon_x + v\varepsilon_y) = \frac{200,000}{1 - 0.3^2}[-0.000125 + (0.3 \times 0.002)]$$

that is,

$$\sigma_x = 104.4 \text{ N/mm}^2$$

$$\sigma_y = \frac{E}{1 - v^2}(\varepsilon_y + v\varepsilon_x) = \frac{200,000}{1 - 0.3^2}[0.002 + (0.3 \times 0.000125)]$$

that is,

$$\sigma_y = 431.3 \text{ N/mm}^2$$

and

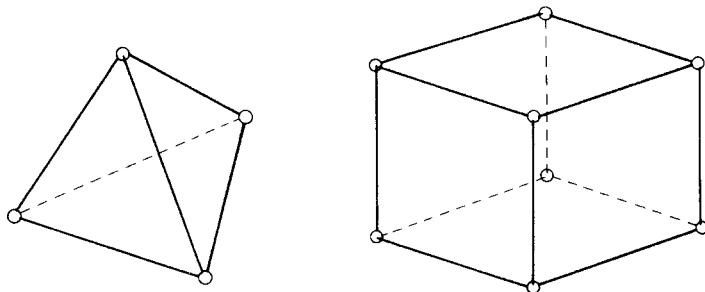
$$\tau_{xy} = \frac{E}{1 - v^2} \times \frac{1}{2}(1 - v)\gamma_{xy} = \frac{E}{2(1 + v)}\gamma_{xy}$$

Thus,

$$\tau_{xy} = \frac{200,000}{2(1+0.3)} \times (-0.0015)$$

that is,

$$\tau_{xy} = -115.4 \text{ N/mm}^2$$



**FIGURE 6.15 Tetrahedron and Rectangular Prism Finite Elements for Three-Dimensional Problems**

The application of the finite element method to three-dimensional solid bodies is a straightforward extension of the analysis of two-dimensional structures. The basic three-dimensional elements are the tetrahedron and the rectangular prism, both shown in Fig. 6.15. The tetrahedron has four nodes, each possessing 3 degrees of freedom, a total of 12 degrees of freedom for the element, while the prism has eight nodes and therefore a total of 24 degrees of freedom. Displacement functions for each element require polynomials in  $x$ ,  $y$ , and  $z$ ; for the tetrahedron, the displacement function is of the first degree with 12 constant coefficients, while that for the prism may be of a higher order to accommodate the 24 degrees of freedom. A development in the solution of three-dimensional problems has been the introduction of curvilinear coordinates. This enables the tetrahedron and prism to be distorted into arbitrary shapes that are better suited for fitting actual boundaries. For more detailed discussions of the finite element method, reference should be made to the work of Jenkins,<sup>5</sup> Zienkiewicz and Cheung,<sup>6</sup> and to the many research papers published on the method.

New elements and new applications of the finite element method are still being developed, some of which lie outside the field of structural analysis. These fields include soil mechanics, heat transfer, fluid and seepage flow, magnetism, and electricity.

#### Example 6.4 MATLAB

Repeat Example 6.4 using MATLAB. Use matrix methods when possible.

Values for the stresses in the element, rounded to the first decimal place, are obtained through the following MATLAB file:

```
% Declare any needed variables
E = 200000; % Youngs Modulus
```

```

v_p = 0.3; % Poisson's ratio
x = [-2 2 2 -2]'; % x-location of corners
y = [-1 -1 1 1]'; % y-location of corners
u = [0.001 0.003 -0.003 0]; % x-displacement of corners
v = [-0.004 -0.002 0.001 0.001]; % y-displacement of corners

% Calculate alpha values using Eqs (6.99) and (6.100)
A = zeros(8,8);
delta = zeros(8,1);
for i=1:1:4
    A((i-1)*2+1,:) = [1 x(i) y(i) x(i)*y(i) 0 0 0 0];
    A((i-1)*2+2,:) = [0 0 0 1 x(i) y(i) x(i)*y(i)];
    delta((i-1)*2+1) = u(i);
    delta((i-1)*2+2) = v(i);
end
alpha = A\delta;

```

% Calculate the strains at the center ( $x=0, y=0$ ) of the element using Eq. (6.101)

```

x = 0;
y = 0;
B = [0 1 0 y 0 0 0 0;
      0 0 0 0 0 1 x;
      0 0 1 x 0 1 0 y];
strain = B*alpha;

```

% Calculate the stresses using Eq. (6.92)

```

C = [1 v_p 0;
      v_p 1 0;
      0 0 (1-v_p)/2];
sig = E/(1-v_p^2)*(C*strain);

```

% Output stresses rounded to the first decimal to the Command Window

```

disp(['sig_x =' num2str(round(sig(1)*10)/10) 'N/mm^2'])
disp(['sig_y =' num2str(round(sig(2)*10)/10) 'N/mm^2'])
disp(['tau_xy =' num2str(round(sig(3)*10)/10) 'N/mm^2'])

```

The Command Window outputs resulting from this MATLAB file are as follows:

```

sig_x = 104.4 N/mm^2
sig_y = 431.3 N/mm^2
tau_xy = -115.4 N/mm^2

```

## References

- [1] Argyris JH, Kelsey S. Energy theorems and structural analysis. London: Butterworth Scientific Publications; 1960.
- [2] Clough RW, Turner MJ, Martin HC, Topp LJ. Stiffness and deflection analysis of complex structures. J Aero Sciences 1956;23(9).

- [3] Megson THG. Structural and stress analysis. 4th ed. Oxford: Elsevier; 2019.
- [4] Martin HC. Introduction to matrix methods of structural analysis. New York: McGraw-Hill; 1966.
- [5] Jenkins WM. Matrix and digital computer methods in structural analysis. London: McGraw-Hill; 1969.
- [6] Zienkiewicz OC, Cheung YK. The finite element method in structural and continuum mechanics. London: McGraw-Hill; 1967.

## Further reading

Zienkiewicz OC, Holister GS. Stress analysis. London: John Wiley and Sons; 1965.

## PROBLEMS

**P.6.1.** Figure P.6.1 shows a square symmetrical pin-jointed truss 1234, pinned to rigid supports at 2 and 4 and loaded with a vertical load at 1. The axial rigidity  $EA$  is the same for all members. Use the stiffness method to find the displacements at nodes 1 and 3 and solve for all the internal member forces and support reactions.

$$\text{Answer: } v_1 = -PL/\sqrt{2}AE, \quad v_3 = -0.293PL/AE, \quad S_{12} = P/2 = S_{14},$$

$$S_{23} = -0.207P = S_{43}, \quad S_{13} = 0.293P$$

$$F_{x,2} = -F_{x,4} = 0.207P, \quad F_{y,2} = F_{y,4} = P/2.$$

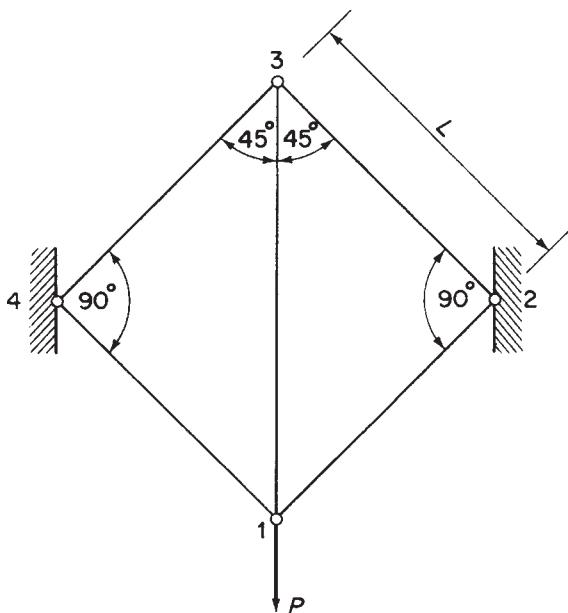


FIGURE P.6.1

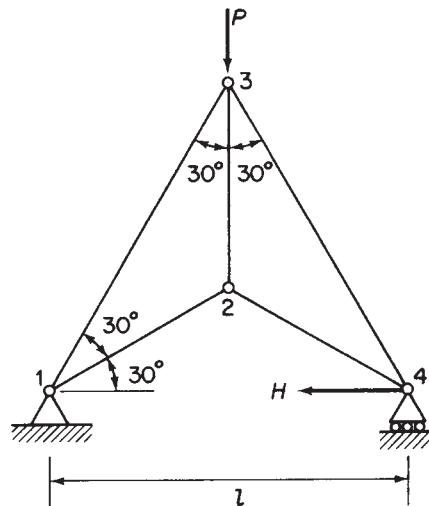


FIGURE P.6.2

**P.6.2.** Use the stiffness method to find the ratio  $H/P$  for which the displacement of node 4 of the plane pin-jointed frame shown loaded in Fig. P.6.2 is zero and, for that case, give the displacements of nodes 2 and 3. All members have equal axial rigidity  $EA$ .

$$\text{Answer: } H/P = 0.449, \quad v_2 = -4Pl/(9 + 2\sqrt{3})AE, \\ v_3 = -6Pl/(9 + 2\sqrt{3})AE.$$

**P.6.3.** Form the matrices required to solve completely the plane truss shown in Fig. P.6.3 and determine the force in member 24. All members have equal axial rigidity.

$$\text{Answer: } S_{24} = 0.$$

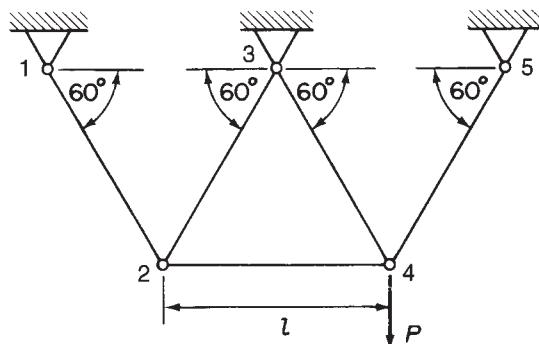


FIGURE P.6.3

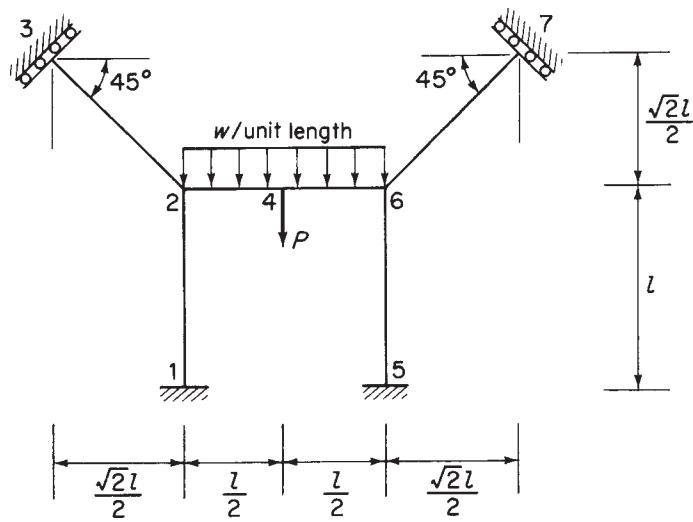


FIGURE P.6.4

**P.6.4.** The symmetrical plane rigid jointed frame 1234567, shown in Fig. P.6.4, is fixed to rigid supports at 1 and 5 and supported by rollers inclined at  $45^\circ$  to the horizontal at nodes 3 and 7. It carries a vertical point load  $P$  at node 4 and a uniformly distributed load  $w$  per unit length on the span 26. Assuming the same flexural rigidity  $EI$  for all members, set up the stiffness equations that, when solved, give the nodal displacements of the frame. Explain how the member forces can be obtained.

**P.6.5.** The frame shown in Fig. P.6.5 has the planes  $xz$  and  $yz$  as planes of symmetry. Note that node 8 is vertically above node 1. The nodal coordinates of one-quarter of the frame are given in Table P.6.5(i). In this structure, the deformation of each member is due to a single effect, this being axial, bending, or

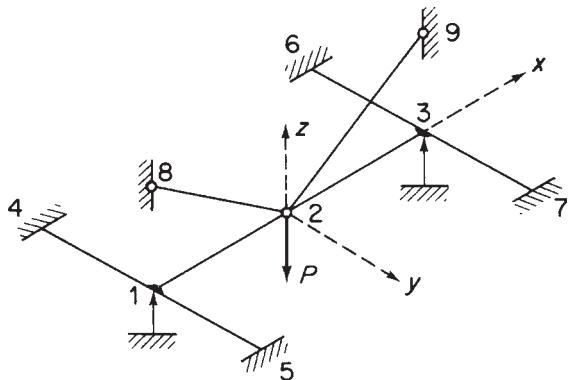


FIGURE P.6.5

**Table P.6.5(i)** Nodal Coordinates

Node	<i>x</i>	<i>y</i>	<i>z</i>
2	0	0	0
3	<i>L</i>	0	0
7	<i>L</i>	0.8 <i>L</i>	0
9	<i>L</i>	0	<i>L</i>

**Table P.6.5(ii)** Mode of Deformation

Member	Effect		
	Axial	Bending	Torsional
23	—	<i>EI</i>	—
37	—	—	<i>GJ</i> = 0.8 <i>EI</i>
29	$EA = 6\sqrt{2}\frac{EI}{L^2}$	—	—

torsional. The mode of deformation of each member is given in [Table P.6.5\(ii\)](#), together with the relevant rigidity. Use the *direct stiffness* method to find all the displacements and hence calculate the forces in all the members. For member 123, plot the shear force and bending moment diagrams. Briefly outline the sequence of operations in a typical computer program suitable for linear frame analysis.

Answer:  $S_{29} = S_{28} = \sqrt{2}P/6$  (tension)  
 $M_3 = -M_1 = PL/9$  (hogging),  $M_2 = 2PL/9$  (sagging)  
 $SF_{12} = -SF_{23} = P/3$

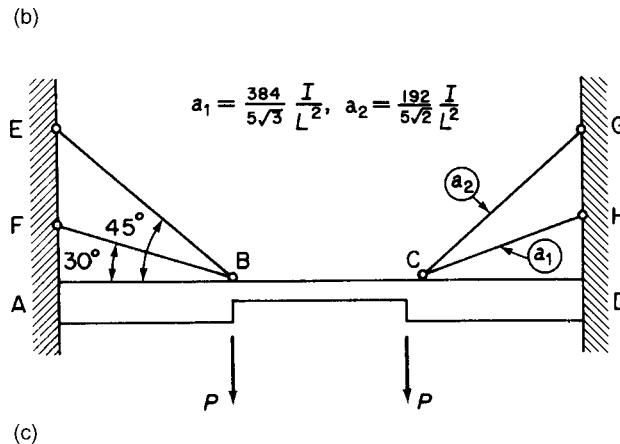
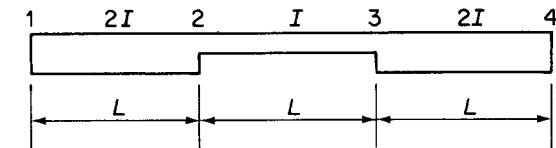
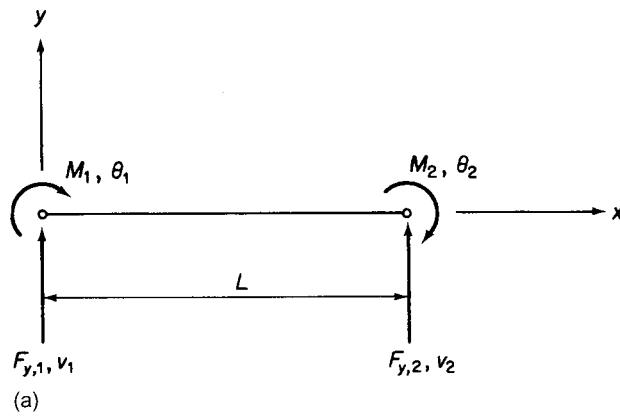
Twisting moment in 37,  $PL/18$  (counterclockwise).

**P.6.6.** Given that the force–displacement (stiffness) relationship for the beam–element shown in [Fig. P.6.6\(a\)](#) may be expressed in the following form:

$$\begin{Bmatrix} F_{y,1} \\ M_1/L \\ F_{y,2} \\ M_2/L \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 L \\ v_2 \\ \theta_2 L \end{Bmatrix}$$

Obtain the force–displacement (stiffness) relationship for the variable section beam ([Fig. P.6.6\(b\)](#)), composed of elements 12, 23, and 34. Such a beam is loaded and supported symmetrically, as shown in [Fig. P.6.6\(c\)](#). Both ends are rigidly fixed, and the ties FB, CH have a cross-section area  $a_1$  and the ties EB, CG a cross-section area  $a_2$ . Calculate the deflections under the loads, the forces in the ties, and all other information necessary for sketching the bending moment and shear force diagrams for the beam. Neglect axial effects in the beam. The ties are made from the same material as the beam.

Answer:  $v_B = v_C = -5PL^3/144EI, \theta_B = -\theta_C = PL^2/24EI,$   
 $S_1 = 2P/3, S_2 = \sqrt{2}P/3,$   
 $F_{y,A} = P/3, M_A = -PL/4.$



**FIGURE P.6.6**

**P.6.7.** The symmetrical rigid jointed grillage shown in Fig. P.6.7 is encastré at 6, 7, 8, and 9 and rests on simple supports at 1, 2, 4, and 5. It is loaded with a vertical point load  $P$  at 3. Use the stiffness method to find the displacements of the structure and calculate the support reactions and the forces in all the members. Plot the bending moment diagram for 123. All members have the same section properties and  $GJ = 0.8EI$ .

Answer:  $F_{y,1} = F_{y,5} = -P/16$

$$F_{y,2} = F_{y,4} = 9P/16$$

$$M_{21} = M_{45} = -Pl/16$$

$$M_{21} = M_{43} = -Pl/18 \text{ (hogging)}$$

$M_{23} = M_{43} = -Fl/12$  (logging) Twisting moment in 02, 82, 74, and 94 is  $Fl/90$ .

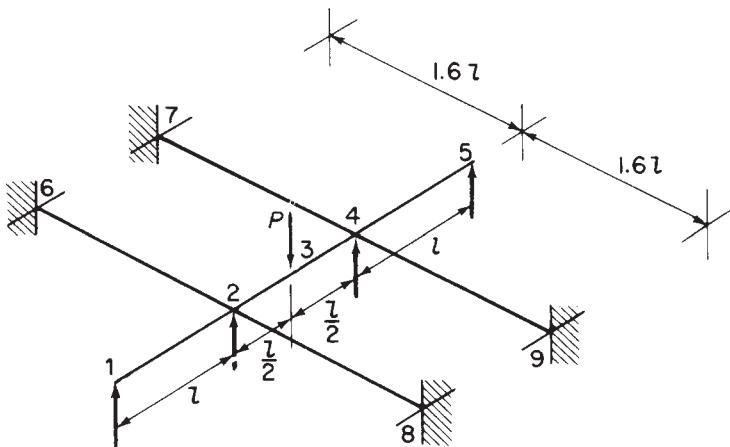


FIGURE P.6.7

**P.6.8.** It is required to form the stiffness matrix of a triangular element 123 with coordinates  $(0, 0)$ ,  $(a, 0)$ , and  $(0, a)$ , respectively, to be used for “plane stress” problems.

- (a) Form the  $[B]$  matrix.
- (b) Obtain the stiffness matrix  $[K^e]$ .

Why, in general, is a finite element solution not an exact solution?

**P.6.9.** It is required to form the stiffness matrix of a triangular element 123 for use in stress analysis problems. The coordinates of the element are  $(1, 1)$ ,  $(2, 1)$ , and  $(2, 2)$ , respectively.

- (a) Assume a suitable displacement field explaining the reasons for your choice.
- (b) Form the  $[B]$  matrix.
- (c) Form the matrix that gives, when multiplied by the element nodal displacements, the stresses in the element. Assume a general  $[D]$  matrix.

**P.6.10.** It is required to form the stiffness matrix for a rectangular element of side  $2a \times 2b$  and thickness  $t$  for use in “plane stress” problems.

- (a) Assume a suitable displacement field.
- (b) Form the  $[C]$  matrix.
- (c) Obtain  $\int_{\text{vol}} [C]^T [D] [C] dV$ .

Note that the stiffness matrix may be expressed as

$$[K^e] = [A^{-1}]^T \left[ \int_{\text{vol}} [C]^T [D] [C] dV \right] [A^{-1}]$$

**P.6.11.** A square element 1234, whose corners have coordinates  $x, y$  (in meters) of  $(-1, -1)$ ,  $(1, -1)$ ,  $(1, 1)$ , and  $(-1, 1)$ , respectively, was used in a plane stress finite element analysis. The following nodal displacements (mm) were obtained:

$$\begin{aligned} u_1 &= 0.1 & u_2 &= 0.3 & u_3 &= 0.6 & u_4 &= 0.1 \\ v_1 &= 0.1 & v_2 &= 0.3 & v_3 &= 0.7 & v_4 &= 0.5 \end{aligned}$$

If Young's modulus  $E = 200,000 \text{ N/mm}^2$  and Poisson's ratio  $\nu = 0.3$ , calculate the stresses at the center of the element.

$$\text{Answer: } \sigma_x = 51.65 \text{ N/mm}^2, \quad \sigma_y = 55.49 \text{ N/mm}^2, \quad \tau_{xy} = 13.46 \text{ N/mm}^2$$

**P.6.12.** A rectangular element used in plane stress analysis has corners whose coordinates in meters referred to an Oxy axes system are 1(-2, -1), 2(2, -1), 3(2, 1), 4(-2, 1). The displacements of the corners (in meters) are

$$\begin{aligned} u_1 &= 0.001 & u_2 &= 0.003 & u_3 &= -0.003 & u_4 &= 0 \\ v_1 &= -0.004 & v_2 &= -0.002 & v_3 &= 0.001 & v_4 &= 0.001 \end{aligned}$$

If Young's modulus is 200,000 N/mm<sup>2</sup> and Poisson's ratio is 0.3, calculate the strains at the center of the element.

$$\text{Answer: } \varepsilon_x = -0.000125, \quad \varepsilon_y = 0.002, \quad \gamma_{xy} = -0.0015.$$

**P.6.12 MATLAB** Use MATLAB to repeat Problem P.6.12. In addition, calculate the strains at the following (x,y) locations in the element:

$$\begin{array}{lll} (-1, 0.5) & (0, 0.5) & (1, 0.5) \\ (-1, 0) & (0, 0) & (1, 0) \\ (-1, -0.5) & (0, -0.5) & (1, -0.5) \end{array}$$

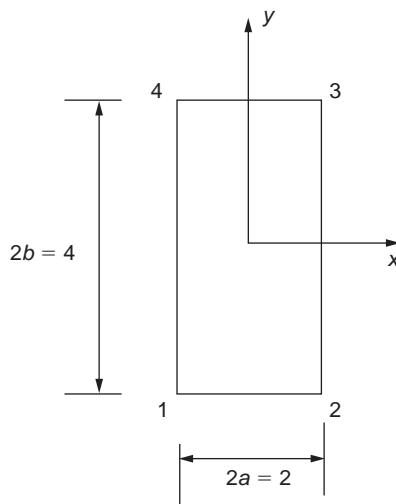
- Answer:
- (i)  $(x, y) = (-1, 0.5)$ ,  $\varepsilon_x = -0.0004375$ ,  $\varepsilon_y = 0.0023$ ,  $\gamma_{xy} = -0.0010$
  - (ii)  $(x, y) = (0, 0.5)$ ,  $\varepsilon_x = -0.0004375$ ,  $\varepsilon_y = 0.0020$ ,  $\gamma_{xy} = -0.0016$
  - (iii)  $(x, y) = (1, 0.5)$ ,  $\varepsilon_x = -0.0004375$ ,  $\varepsilon_y = 0.0018$ ,  $\gamma_{xy} = -0.0023$
  - (iv)  $(x, y) = (-1, 0)$ ,  $\varepsilon_x = -0.000125$ ,  $\varepsilon_y = 0.0023$ ,  $\gamma_{xy} = -0.000875$
  - (v)  $(x, y) = (0, 0)$ ,  $\varepsilon_x = -0.000125$ ,  $\varepsilon_y = 0.0020$ ,  $\gamma_{xy} = -0.0015$
  - (vi)  $(x, y) = (1, 0)$ ,  $\varepsilon_x = -0.000125$ ,  $\varepsilon_y = 0.0018$ ,  $\gamma_{xy} = -0.0021$
  - (vii)  $(x, y) = (-1, -0.5)$ ,  $\varepsilon_x = 0.0001875$ ,  $\varepsilon_y = 0.0023$ ,  $\gamma_{xy} = -0.00075$
  - (viii)  $(x, y) = (0, -0.5)$ ,  $\varepsilon_x = 0.0001875$ ,  $\varepsilon_y = 0.002$ ,  $\gamma_{xy} = -0.0014$
  - (ix)  $(x, y) = (1, -0.5)$ ,  $\varepsilon_x = 0.0001875$ ,  $\varepsilon_y = 0.0018$ ,  $\gamma_{xy} = -0.002$

**P.6.13.** A constant strain triangular element has corners 1(0,0), 2(4,0), and 3(2,2) and is 1 unit thick. If the elasticity matrix [D] has elements  $D_{11} = D_{22} = a$ ,  $D_{12} = D_{21} = b$ ,  $D_{13} = D_{23} = D_{31} = D_{32} = 0$  and  $D_{33} = c$ , derive the stiffness matrix for the element.

Answer:

$$[K^e] = \frac{1}{4} \begin{bmatrix} a+c & & & & & \\ b+c & a+c & & & & \\ -a+c & -b+c & a+c & & & \\ b-c & a-c & -b-c & a+c & & \\ -2c & -2c & -2c & 2c & 4c & \\ -2b & -2a & 2b & -2a & 0 & 4a \end{bmatrix}$$

**P.6.14.** The following interpolation formula is suggested as a displacement function for deriving the stiffness of a plane stress rectangular element of uniform thickness  $t$  shown in Fig. P.6.14.

**FIGURE P.6.14**

$$u = \frac{1}{4ab} [(a-x)(b-y)u_1 + (a+x)(b-y)u_2 + (a+x)(b+y)u_3 + (a-x)(b+y)u_4]$$

Form the strain matrix and obtain the stiffness coefficients  $K_{11}$  and  $K_{12}$  in terms of the material constants  $c$ ,  $d$ , and  $e$  defined next. In the elasticity matrix  $[D]$ ,

$$D_{11} = D_{22} = c \quad D_{12} = d \quad D_{33} = e \quad \text{and} \quad D_{13} = D_{23} = 0$$

$$\text{Answer: } K_{11} = t(4c + e)/6, \quad K_{12} = t(d + e)/4$$

# Bending of thin plates

# 7

Generally, we define a thin plate as a sheet of material whose thickness is small compared with its other dimensions but which is capable of resisting bending in addition to membrane forces. Such a plate forms a basic part of an aircraft structure, being, for example, the area of stressed skin bounded by adjacent stringers and ribs in a wing structure or by adjacent stringers and frames in a fuselage.

In this chapter, we investigate the effect of a variety of loading and support conditions on the small deflection of rectangular plates. Two approaches are presented: an “exact” theory based on the solution of a differential equation; and an energy method relying on the principle of the stationary value of the total potential energy of the plate and its applied loading. The latter theory is used in Chapter 9 to determine buckling loads for unstiffened and stiffened panels.

## 7.1 PURE BENDING OF THIN PLATES

The thin rectangular plate of Fig. 7.1 is subjected to pure bending moments of intensity  $M_x$  and  $M_y$  per unit length uniformly distributed along its edges. The former bending moment is applied along the edges parallel to the  $y$  axis, the latter along the edges parallel to the  $x$  axis. We assume that these bending moments are positive when they produce compression at the upper surface of the plate and tension at the lower.

If we further assume that the displacement of the plate in a direction parallel to the  $z$  axis is small compared with its thickness  $t$  and sections that are plane before bending remain plane after bending, then, as in the case of simple beam theory, the middle plane of the plate does not deform during the bending and is therefore a *neutral plane*. We take the neutral plane as the reference plane for our system of axes.

Let us consider an element of the plate of side  $\delta x \delta y$  and having a depth equal to the thickness  $t$  of the plate, as shown in Fig. 7.2(a). Suppose that the radii of curvature of the neutral plane  $n$  are  $\rho_x$  and  $\rho_y$  in the  $xz$  and  $yz$  planes, respectively (Fig. 7.2(b)). Positive curvature of the plate corresponds to the positive bending moments that produce displacements in the positive direction of the  $z$  or downward axis. Again, as in simple beam theory, the direct strains  $\epsilon_x$  and  $\epsilon_y$ , corresponding to direct stresses  $\sigma_x$  and  $\sigma_y$  of an elemental lamina of thickness  $\delta z$  a distance  $z$  below the neutral plane are given by

$$\epsilon_x = \frac{z}{\rho_x}, \quad \epsilon_y = \frac{z}{\rho_y} \quad (\text{see Eq. (16.1)}) \quad (7.1)$$

Referring to Eqs. (1.52), we have

$$\epsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y), \quad \epsilon_y = \frac{1}{E}(\sigma_y - v\sigma_x) \quad (7.2)$$

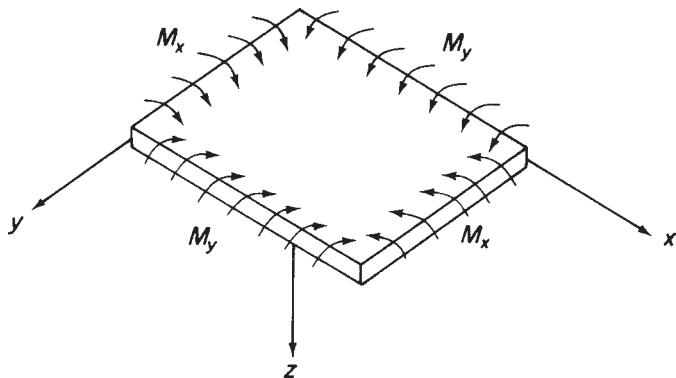


FIGURE 7.1 Plate Subjected to Pure Bending

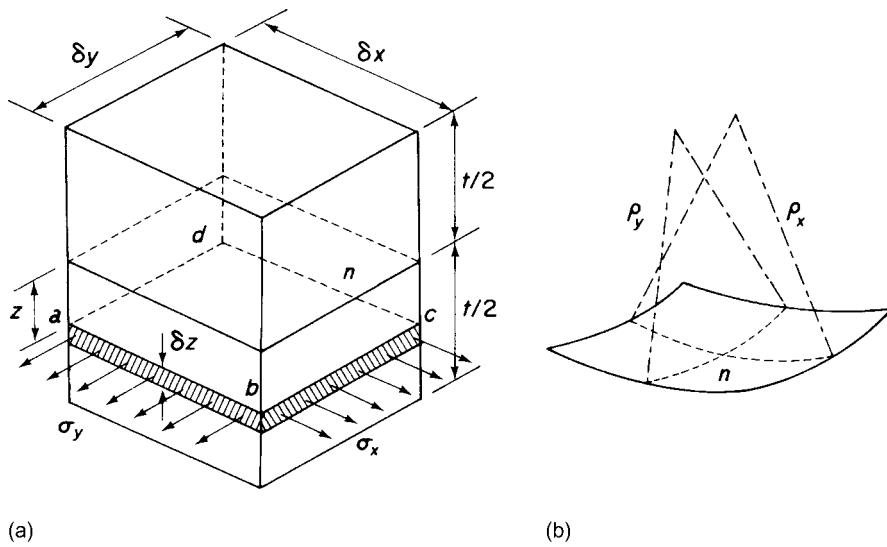


FIGURE 7.2 (a) Direct Stress on Lamina of Plate Element; (b) Radii of Curvature of Neutral Plane

Then,

$$E \frac{z}{\rho_x} = \sigma_z - v\sigma_y$$

and

$$E \frac{z}{\rho_y} = \sigma_y - v\sigma_x$$

Therefore,

$$E \frac{z}{\rho_x} + vE \frac{z}{\rho_y} = \sigma_x(1 - v^2)$$

from which

$$\left. \begin{aligned} \sigma_x &= \frac{Ez}{1-v^2} \left( \frac{1}{\rho_x} + \frac{v}{\rho_y} \right) \\ \sigma_y &= \frac{Ez}{1-v^2} \left( \frac{1}{\rho_y} + \frac{v}{\rho_x} \right) \end{aligned} \right\} \quad (7.3)$$

Similarly,

As would be expected from our assumption of plane sections remaining plane, the direct stresses vary linearly across the thickness of the plate, their magnitudes depending on the curvatures (i.e., bending moments) of the plate. The internal direct stress distribution on each vertical surface of the element must be in equilibrium with the applied bending moments. Thus,

$$M_x \delta y = \int_{-t/2}^{t/2} \sigma_x z \delta y dz$$

and

$$M_y \delta x = \int_{-t/2}^{t/2} \sigma_y z \delta x dz$$

Substituting for  $\sigma_x$  and  $\sigma_y$  from Eqs. (7.3) gives

$$\begin{aligned} M_x &= \int_{-t/2}^{t/2} \frac{Ez^2}{1-v^2} \left( \frac{1}{\rho_x} + \frac{v}{\rho_y} \right) dz \\ M_y &= \int_{-t/2}^{t/2} \frac{Ez^2}{1-v^2} \left( \frac{1}{\rho_y} + \frac{v}{\rho_x} \right) dz \end{aligned}$$

Let

$$D = \int_{-t/2}^{t/2} \frac{Ez^2}{1-v^2} dz = \frac{Et^3}{12(1-v^2)} \quad (7.4)$$

Then,

$$M_x = D \left( \frac{1}{\rho_x} + \frac{v}{\rho_y} \right) \quad (7.5)$$

$$M_y = D \left( \frac{1}{\rho_y} + \frac{v}{\rho_x} \right) \quad (7.6)$$

in which  $D$  is known as the *flexural rigidity* of the plate.

If  $w$  is the deflection of any point on the plate in the  $z$  direction, then we may relate  $w$  to the curvature of the plate in the same manner as the well-known mathematical expression for beam curvature:

$$\frac{1}{\rho_x} = -\frac{\partial^2 w}{\partial x^2} \quad \frac{1}{\rho_y} = -\frac{\partial^2 w}{\partial y^2}$$

the negative signs resulting from the fact that the centers of curvature occur above the plate, in which region  $z$  is negative. Equations (7.5) and (7.6) then become

$$M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right) \quad (7.7)$$

$$M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + v \frac{\partial^2 w}{\partial x^2} \right) \quad (7.8)$$

Equations (7.7) and (7.8) define the deflected shape of the plate provided that  $M_x$  and  $M_y$  are known. If either  $M_x$  or  $M_y$  is zero, then

$$\frac{\partial^2 w}{\partial x^2} = -v \frac{\partial^2 w}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 w}{\partial y^2} = -v \frac{\partial^2 w}{\partial x^2}$$

and the plate has curvatures of opposite signs. The case of  $M_y = 0$  is illustrated in Fig. 7.3. A surface possessing two curvatures of opposite sign is known as an *anticlastic surface*, as opposed to a *synclastic surface*, which has curvatures of the same sign. Further, if  $M_x = M_y = M$ , then from Eqs. (7.5) and (7.6),

$$\frac{1}{\rho_x} = \frac{1}{\rho_y} = \frac{1}{\rho}$$

Therefore, the deformed shape of the plate is spherical and of curvature

$$\frac{1}{\rho} = \frac{M}{D(1+v)} \quad (7.9)$$

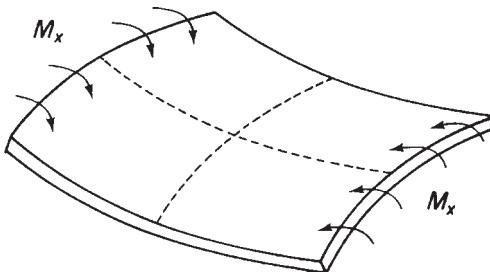


FIGURE 7.3 Anticlastic Bending

## 7.2 PLATES SUBJECTED TO BENDING AND TWISTING

In general, the bending moments applied to the plate will not be in planes perpendicular to its edges. Such bending moments, however, may be resolved in the normal manner into tangential and perpendicular components, as shown in Fig. 7.4. The perpendicular components are seen to be  $M_x$  and  $M_y$  as before, while the tangential components  $M_{xy}$  and  $M_{yx}$  (again, these are moments per unit length) produce twisting of the plate about axes parallel to the  $x$  and  $y$  axes. The system of suffixes and the sign convention for these twisting moments must be clearly understood to avoid confusion.  $M_{xy}$  is a twisting

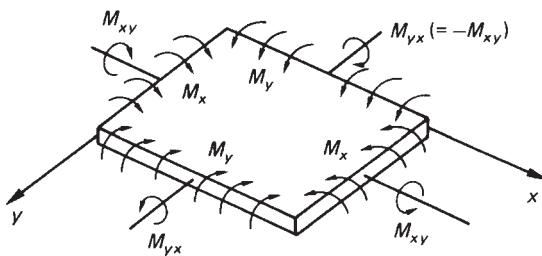


FIGURE 7.4 Plate Subjected to Bending and twisting

moment intensity in a vertical  $x$  plane parallel to the  $y$  axis, while  $M_{yx}$  is a twisting moment intensity in a vertical  $y$  plane parallel to the  $x$  axis. Note that the first suffix gives the direction of the axis of the twisting moment. We also define positive twisting moments as being clockwise when viewed along their axes in directions parallel to the positive directions of the corresponding  $x$  or  $y$  axis. In Fig. 7.4, therefore, all moment intensities are positive.

Since the twisting moments are tangential moments or torques, they are resisted by a system of horizontal shear stresses  $\tau_{xy}$ , as shown in Fig. 7.6. From a consideration of complementary shear stresses (see Fig. 7.6),  $M_{xy} = -M_{yx}$ , so that we may represent a general moment application to the plate in terms of  $M_x$ ,  $M_y$ , and  $M_{xy}$  as shown in Fig. 7.5(a). These moments produce tangential and normal moments,  $M_t$  and  $M_n$ , on an arbitrarily chosen diagonal plane FD. We may express these moment intensities (in an analogous fashion to the complex stress systems of Section 1.6) in terms of  $M_x$ ,  $M_y$ , and  $M_{xy}$ . Thus, for equilibrium of the triangular element ABC of Fig. 7.5(b) in a plane perpendicular to AC,

$$M_n AC = M_x AB \cos\alpha + M_y BC \sin\alpha - M_{xy} AB \sin\alpha - M_{xy} BC \cos\alpha$$

giving

$$M_n = M_x \cos^2\alpha + M_y \sin^2\alpha - M_{xy} \sin 2\alpha \quad (7.10)$$

Similarly, for equilibrium in a plane parallel to CA,

$$M_t AC = M_x AB \sin\alpha - M_y BC \cos\alpha + M_{xy} AB \cos\alpha - M_{xy} BC \sin\alpha$$

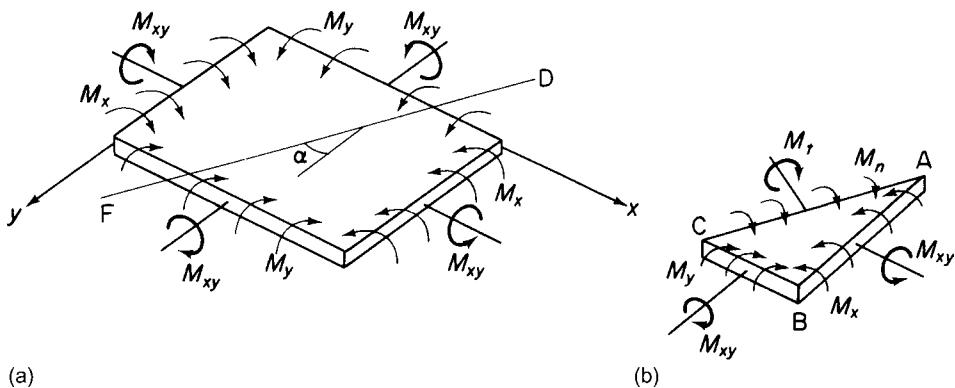
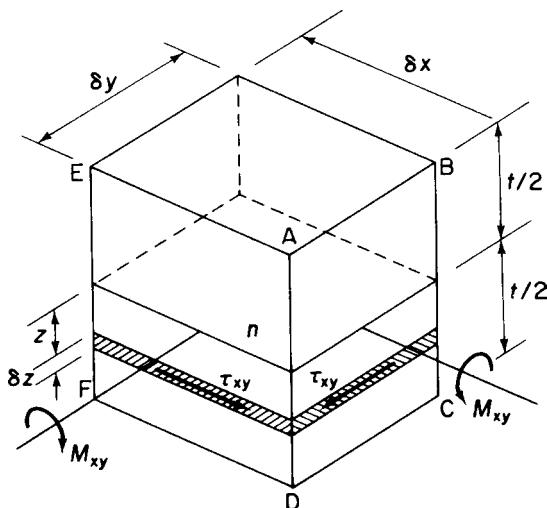


FIGURE 7.5 (a) Plate Subjected to Bending and twisting; (b) Tangential and Normal Moments on an Arbitrary Plane

FIGURE 7.6 Complementary Shear Stresses Due to Twisting Moments  $M_{xy}$ 

or

$$M_t = \frac{(M_x - M_y)}{2} \sin 2\alpha + M_{xy} \cos 2\alpha \quad (7.11)$$

(compare Eqs. (7.10) and (7.11) with Eqs. (1.8) and (1.9)). We observe, from Eq. (7.11), that there are two values of  $\alpha$ , differing by  $90^\circ$  and given by

$$\tan 2\alpha = -\frac{2M_{xy}}{M_x - M_y}$$

for which  $M_t = 0$ , leaving normal moments of intensity  $M_n$  on two mutually perpendicular planes. These moments are termed *principal moments* and their corresponding curvatures *principal curvatures*. For a plate subjected to pure bending and twisting in which  $M_x$ ,  $M_y$ , and  $M_{xy}$  are invariable throughout the plate, the principal moments are the algebraically greatest and least moments in the plate. It follows that there are no shear stresses on these planes and that the corresponding direct stresses, for a given value of  $z$  and moment intensity, are the algebraically greatest and least values of direct stress in the plate.

Let us now return to the loaded plate of Fig. 7.5(a). We have established, in Eqs. (7.7) and (7.8), the relationships between the bending moment intensities  $M_x$  and  $M_y$  and the deflection  $w$  of the plate. The next step is to relate the twisting moment  $M_{xy}$  to  $w$ . From the principle of superposition, we may consider  $M_{xy}$  acting separately from  $M_x$  and  $M_y$ . As stated previously,  $M_{xy}$  is resisted by a system of horizontal complementary shear stresses on the vertical faces of sections taken throughout the thickness of the plate parallel to the  $x$  and  $y$  axes. Consider an element of the plate formed by such sections, as shown in Fig. 7.6. The complementary shear stresses on a lamina of the element a distance  $z$  below the neutral plane are, in accordance with the sign convention of Section 1.2,  $\tau_{xy}$ . Therefore, on the face ABCD,

$$M_{xy}\delta y = - \int_{-t/2}^{t/2} \tau_{xy}\delta yz \, dz$$

and, on the face ADFE,

$$M_{xy}\delta x = - \int_{-t/2}^{t/2} \tau_{xy}\delta xz \, dz$$

giving

$$M_{xy} = - \int_{-t/2}^{t/2} \tau_{xy}z \, dz$$

or, in terms of the shear strain  $\gamma_{xy}$  and modulus of rigidity  $G$  (see Eqs. (1.51)),

$$M_{xy} = -G \int_{-t/2}^{t/2} \gamma_{xy}z \, dz \quad (7.12)$$

Referring to Eqs. (1.20), the shear strain  $\gamma_{xy}$  is given by

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

We require, of course, to express  $\gamma_{xy}$  in terms of the deflection  $w$  of the plate; this may be accomplished as follows. An element taken through the thickness of the plate suffers rotations equal to  $\partial w/\partial x$  and  $\partial w/\partial y$  in the  $xz$  and  $yz$  planes, respectively. Considering the rotation of such an element in the  $xz$  plane, as shown in Fig. 7.7, we see that the displacement  $u$  in the  $x$  direction of a point a distance  $z$  below the neutral plane is

$$u = -\frac{\partial w}{\partial x} z$$

Similarly, the displacement  $v$  in the  $y$  direction is

$$v = -\frac{\partial w}{\partial y} z$$

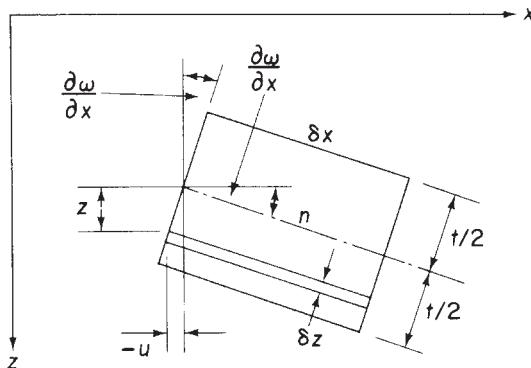


FIGURE 7.7 Determination of Shear Strain  $\gamma_{xy}$

Hence, substituting for  $u$  and  $v$  in the expression for  $\gamma_{xy}$ , we have

$$\gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y} \quad (7.13)$$

whence from Eq. (7.12),

$$M_{xy} = G \int_{-t/2}^{t/2} 2z^2 \frac{\partial^2 w}{\partial x \partial y} dz$$

or

$$M_{xy} = \frac{Gt^3}{6} \frac{\partial^2 w}{\partial x \partial y}$$

Replacing  $G$  by the expression  $E/2(1+v)$  established in Eq. (1.50) gives

$$M_{xy} = \frac{Et^3}{12(1+v)} \frac{\partial^2 w}{\partial x \partial y}$$

Multiplying the numerator and denominator of this equation by the factor  $(1-v)$  yields

$$M_{xy} = D(1-v) \frac{\partial^2 w}{\partial x \partial y} \quad (7.14)$$

Equations (7.7), (7.8), and (7.14) relate the bending and twisting moments to the plate deflection and are analogous to the bending moment-curvature relationship for a simple beam.

### 7.3 PLATES SUBJECTED TO A DISTRIBUTED TRANSVERSE LOAD

The relationships between bending and twisting moments and plate deflection are now employed in establishing the general differential equation for the solution of a thin rectangular plate, supporting a distributed transverse load of intensity  $q$  per unit area (see Fig. 7.8). The distributed load may,

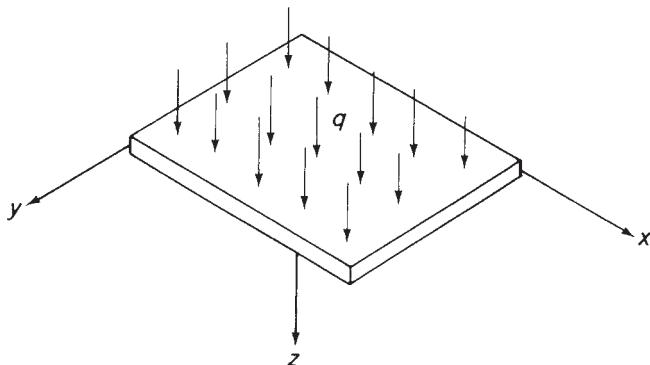
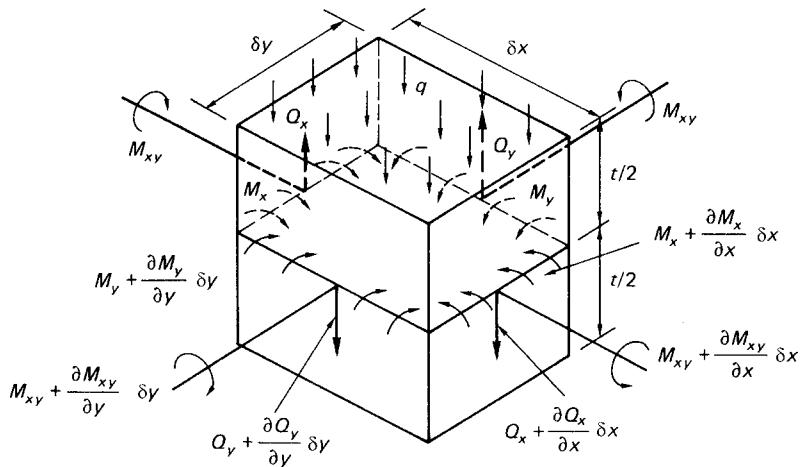


FIGURE 7.8 Plate Supporting a Distributed Transverse Load



**FIGURE 7.9 Plate Element Subjected to Bending, Twisting, and Transverse Loads**

in general, vary over the surface of the plate and is therefore a function of  $x$  and  $y$ . We assume, as in the preceding analysis, that the middle plane of the plate is the neutral plane and that the plate deforms such that plane sections remain plane after bending. This latter assumption introduces an apparent inconsistency in the theory. For plane sections to remain plane, the shear strains  $\gamma_{xz}$  and  $\gamma_{yz}$  must be zero. However, the transverse load produces transverse shear forces (and therefore stresses), as shown in Fig. 7.9. We therefore assume that, although  $\gamma_{xz} = \tau_{xz}/G$  and  $\gamma_{yz} = \tau_{yz}/G$  are negligible, the corresponding shear forces are of the same order of magnitude as the applied load  $q$  and the moments  $M_x$ ,  $M_y$ , and  $M_{xy}$ . This assumption is analogous to that made in slender beam theory, in which shear strains are ignored.

The element of plate shown in Fig. 7.9 supports bending and twisting moments as previously described and, in addition, vertical shear forces  $Q_x$  and  $Q_y$  per unit length on faces perpendicular to the  $x$  and  $y$  axes, respectively. The variation of shear stresses  $\tau_{xz}$  and  $\tau_{yz}$  along the small edges  $\delta x$ ,  $\delta y$  of the element is neglected and the resultant shear forces  $Q_x\delta y$  and  $Q_y\delta x$  are assumed to act through the centroid of the faces of the element. From the previous sections,

$$M_x = \int_{-t/2}^{t/2} \sigma_x z dz, \quad M_y = \int_{-t/2}^{t/2} \sigma_y z dz \quad M_{xy} = (-M_{yx}) = - \int_{-t/2}^{t/2} \tau_{xyz} dz$$

In a similar fashion,

$$Q_x = \int_{-t/2}^{t/2} \tau_{xz} dz, \quad Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz \quad (7.15)$$

For equilibrium of the element parallel to Oz and assuming that the weight of the plate is included in  $q$ ,

$$\left( Q_x + \frac{\partial Q_x}{\partial x} \delta x \right) \delta y - Q_x \delta y + \left( Q_y + \frac{\partial Q_y}{\partial y} \delta y \right) \delta x - Q_y \delta x + q \delta x \delta y = 0$$

or, after simplification,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \quad (7.16)$$

Taking moments about the  $x$  axis,

$$\begin{aligned} M_{xy}\delta y - \left( M_{xy} + \frac{\partial M_{xy}}{\partial x} \delta x \right) \delta y - M_y \delta x + \left( M_y + \frac{\partial M_y}{\partial y} \delta y \right) \delta x \\ - \left( Q_y + \frac{\partial Q_y}{\partial y} \delta y \right) \delta x \delta y + Q_x \frac{\delta y^2}{2} - \left( Q_x + \frac{\partial Q_x}{\partial x} \delta x \right) \frac{\delta y^2}{2} - q \delta x \frac{\delta y^2}{2} = 0 \end{aligned}$$

Simplifying this equation and neglecting small quantities of a higher order than those retained gives

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0 \quad (7.17)$$

Similarly, taking moments about the  $y$  axis, we have

$$\frac{\partial M_{xy}}{\partial y} - \frac{\partial M_x}{\partial x} + Q_x = 0 \quad (7.18)$$

Substituting in Eq. (7.16) for  $Q_x$  and  $Q_y$  from Eqs. (7.18) and (7.17), we obtain

$$\frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial y \partial x} = -q$$

or

$$\frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \quad (7.19)$$

Replacing  $M_x$ ,  $M_{xy}$ , and  $M_y$  in Eq. (7.19) from Eqs. (7.7), (7.14), and (7.8) gives

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \quad (7.20)$$

This equation may also be written

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{q}{D}$$

or

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 w = \frac{q}{D}$$

The operator  $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$  is the well-known Laplace operator in two dimensions and is sometimes written as  $\nabla^2$ . Therefore,

$$(\nabla^2)^2 w = \frac{q}{D}$$

Generally, the transverse distributed load  $q$  is a function of  $x$  and  $y$ , so that the determination of the deflected form of the plate reduces to obtaining a solution to Eq. (7.20), which satisfies the known boundary conditions of the problem. The bending and twisting moments follow from Eqs. (7.7), (7.8), and (7.14), and the shear forces per unit length  $Q_x$  and  $Q_y$  are found from Eqs. (7.17) and (7.18) by substitution for  $M_x$ ,  $M_y$ , and  $M_{xy}$  in terms of the deflection  $w$  of the plate; thus,

$$Q_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = -D \frac{\partial}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right) \quad (7.21)$$

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = -D \frac{\partial}{\partial y} \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right) \quad (7.22)$$

Direct and shear stresses are then calculated from the relevant expressions relating them to  $M_x$ ,  $M_y$ ,  $M_{xy}$ ,  $Q_x$ , and  $Q_y$ .

Before discussing the solution to Eq. (7.20) for particular cases, we shall establish boundary conditions for various types of edge support.

### 7.3.1 The simply supported edge

Let us suppose that the edge  $x = 0$  of the thin plate shown in Fig. 7.10 is free to rotate but not to deflect. The edge is then said to be *simply supported*. The bending moment along this edge must be zero and also the deflection  $w = 0$ . Thus,

$$(w)_{x=0} = 0 \quad \text{and} \quad (M_x)_{x=0} = -D \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right)_{x=0} = 0$$

The condition that  $w = 0$  along the edge  $x = 0$  also means that

$$\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} = 0$$

along this edge. These boundary conditions therefore reduce to

$$(w)_{x=0} = 0, \quad \left( \frac{\partial^2 w}{\partial x^2} \right)_{x=0} = 0 \quad (7.23)$$

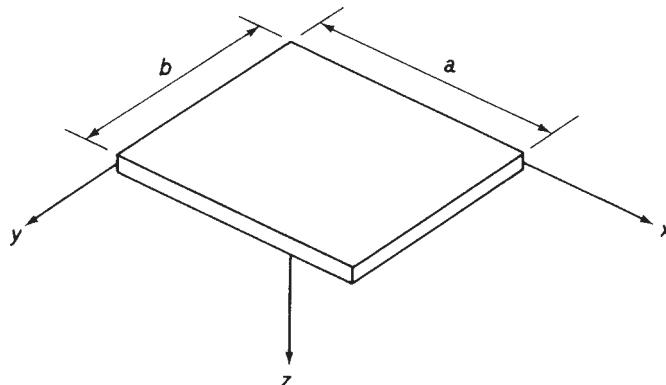


FIGURE 7.10 Plate of Dimensions  $a \times b$

### 7.3.2 The built-in edge

If the edge  $x = 0$  is built in or firmly clamped so that it can neither rotate nor deflect, then, in addition to  $w$ , the slope of the middle plane of the plate normal to this edge must be zero. That is,

$$(w)_{x=0} = 0, \quad \left(\frac{\partial w}{\partial x}\right)_{x=0} = 0 \quad (7.24)$$

### 7.3.3 The free edge

Along a free edge, there are no bending moments, twisting moments, or vertical shearing forces, so that, if  $x = 0$  is the free edge, then

$$(M_x)_{x=0} = 0, \quad (M_{xy})_{x=0} = 0, \quad (Q_x)_{x=0} = 0$$

giving, in this instance, three boundary conditions. However, Kirchhoff (1850) showed that only two boundary conditions are necessary to obtain a solution to Eq. (7.20), and that the reduction is obtained by replacing the two requirements of zero twisting moment and zero shear force by a single equivalent condition. Thomson and Tait (1883) gave a physical explanation of how this reduction may be effected. They pointed out that the horizontal force system equilibrating the twisting moment  $M_{xy}$  may be replaced along the edge of the plate by a vertical force system.

Consider two adjacent elements  $\delta y_1$  and  $\delta y_2$  along the edge of the thin plate of Fig. 7.11. The twisting moment  $M_{xy}\delta y_1$  on the element  $\delta y_1$  may be replaced by forces  $M_{xy}$  a distance  $\delta y_1$  apart. Note that  $M_{xy}$  being a twisting moment per unit length, has the dimensions of force. The twisting moment on the adjacent element  $\delta y_2$  is  $[M_{xy} + (\partial M_{xy}/\partial y)\delta y]\delta y_2$ . Again, this may be replaced by forces  $M_{xy} + (\partial M_{xy}/\partial y)\delta y$ . At the common surface of the two adjacent elements there is now a resultant force  $(\partial M_{xy}/\partial y)\delta y$  or a vertical force per unit length of  $\partial M_{xy}/\partial y$ . For the sign convention for  $Q_x$  shown in

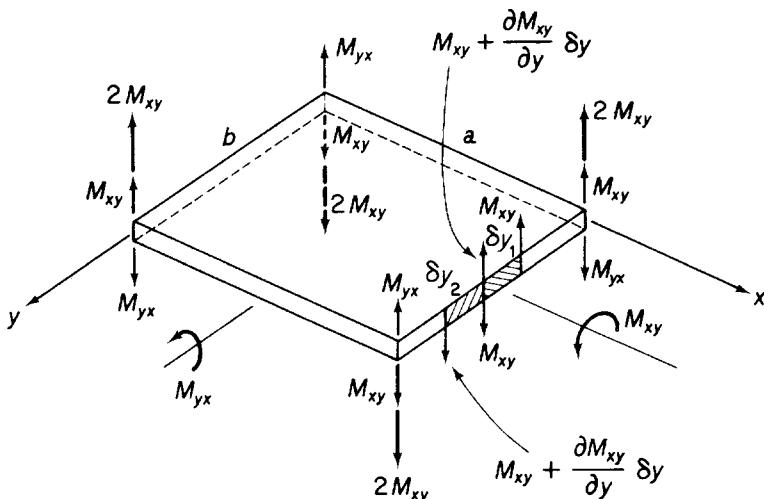


FIGURE 7.11 Equivalent Vertical Force System

[Fig. 7.9](#), we have a statically equivalent vertical force per unit length of  $(Q_x - \partial M_{xy}/\partial y)$ . The separate conditions for a free edge of  $(M_{xy})_{x=0} = 0$  and  $(Q_x)_{x=0} = 0$  are therefore replaced by the equivalent condition

$$\left( Q_x - \frac{\partial M_{xy}}{\partial y} \right)_{x=0} = 0$$

or, in terms of deflection,

$$\left[ \frac{\partial^3 w}{\partial x^3} + (2 - v) \frac{\partial^3 w}{\partial x \partial y^2} \right]_{x=0} = 0 \quad (7.25)$$

Also, for the bending moment along the free edge to be zero,

$$(M_x)_{x=0} = \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right)_{x=0} = 0 \quad (7.26)$$

The replacement of the twisting moment  $M_{xy}$  along the edges  $x = 0$  and  $x = a$  of a thin plate by a vertical force distribution results in leftover concentrated forces at the corners of  $M_{xy}$ , as shown in [Fig. 7.11](#). By the same argument, concentrated forces  $M_{yx}$  are produced by the replacement of the twisting moment  $M_{yx}$ . Since  $M_{xy} = -M_{yx}$ , the resultant forces  $2M_{xy}$  act at each corner as shown and must be provided by external supports if the corners of the plate are not to move. The directions of these forces are easily obtained if the deflected shape of the plate is known. For example, a thin plate simply supported along all four edges and uniformly loaded has  $\partial w/\partial x$  positive and numerically increasing, with increasing  $y$  near the corner  $x = 0, y = 0$ . Hence,  $\partial^2 w/\partial x \partial y$  is positive at this point and, from [Eq. \(7.14\)](#), we see that  $M_{xy}$  is positive and  $M_{yx}$  is negative; the resultant force  $2M_{xy}$  is therefore downward. From symmetry, the force at each remaining corner is also  $2M_{xy}$  downward, so that the tendency is for the corners of the plate to rise.

Having discussed various types of boundary conditions, we shall proceed to obtain the solution for the relatively simple case of a thin rectangular plate of dimensions  $a \times b$ , simply supported along each of its four edges and carrying a distributed load  $q(x, y)$ . We have shown that the deflected form of the plate must satisfy the differential equation

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x, y)}{D} \quad (\text{see Eq. (7.20)})$$

with the boundary conditions

$$(w)_{x=0,a} = 0, \quad \left( \frac{\partial^2 w}{\partial x^2} \right)_{x=0,a} = 0$$

$$(w)_{y=0,b} = 0, \quad \left( \frac{\partial^2 w}{\partial y^2} \right)_{y=0,b} = 0$$

Navier (1820) showed that these conditions are satisfied by representing the deflection  $w$  as an infinite trigonometrical or Fourier series:

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (7.27)$$

in which  $m$  represents the number of half waves in the  $x$  direction and  $n$  the corresponding number in the  $y$  direction. Further,  $A_{mn}$  are unknown coefficients which must satisfy the preceding differential equation and may be determined as follows.

We may also represent the load  $q(x, y)$  by a Fourier series:

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (7.28)$$

A particular coefficient  $a_{m'n'}$  is calculated by first multiplying both sides of Eq. (7.28) by  $\sin(m'\pi x/a) \sin(n'\pi y/b)$  and integrating with respect to  $x$  from 0 to  $a$  and with respect to  $y$  from 0 to  $b$ . Thus,

$$\begin{aligned} & \int_0^a \int_0^b q(x, y) \sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b} \, dx \, dy \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^a \int_0^b a_{mn} \sin \frac{m\pi x}{a} \sin \frac{m'\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{n'\pi y}{b} \, dx \, dy \\ &= \frac{ab}{4} a_{m'n'} \end{aligned}$$

The integrands in the above are evaluated as follows. If  $m \neq m'$  then, using the trigonometrical relationship  $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$$\begin{aligned} \int_0^a \sin \frac{m\pi x}{a} \sin \frac{m'\pi x}{a} \, dx &= \frac{1}{2} \left[ \int_0^a \cos \frac{\pi}{a} (m - m')x \, dx - \int_0^a \cos \frac{\pi}{a} (m + m')x \, dx \right] \\ &= \frac{1}{2} \left[ \frac{a}{\pi(m - m')} \sin \frac{\pi}{a} (m - m')x - \frac{a}{\pi(m + m')} \sin \frac{\pi}{a} (m + m')x \right]_0^a \\ &= 0 \quad (\text{Note, } m \text{ and } m' \text{ are integers}) \end{aligned}$$

For the case when  $m = m'$ ,

$$\int_0^a \sin \frac{m\pi x}{a} \sin \frac{m\pi x}{a} \, dx = \int_0^a \sin^2 \frac{m\pi x}{a} \, dx = \int_0^a \frac{1}{2} \left( 1 - \cos \frac{2m\pi x}{a} \right) \, dx = \frac{a}{2}$$

Similarly,

$$\begin{aligned} \int_0^b \sin \frac{n\pi y}{b} \sin \frac{n\pi y}{b} \, dy &= 0 \quad \text{when } n \neq n' \quad \text{and} \\ &= \frac{b}{2} \quad \text{when } n = n' \end{aligned}$$

It follows that

$$a_{m'n'} = \frac{4}{ab} \int_0^a \int_0^b q(x, y) \sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b} \, dx \, dy \quad (7.29)$$

Substituting now for  $w$  and  $q(x, y)$  from Eqs. (7.27) and (7.28) into the differential equation for  $w$ , we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ A_{mn} \left[ \left( \frac{m\pi}{a} \right)^4 + 2 \left( \frac{m\pi}{a} \right)^2 \left( \frac{n\pi}{b} \right)^2 + \left( \frac{n\pi}{b} \right)^4 \right] - \frac{a_{mn}}{D} \right\} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = 0$$

This equation is valid for all values of  $x$  and  $y$ , so that

$$A_{mn} \left[ \left( \frac{m\pi}{a} \right)^4 + 2 \left( \frac{m\pi}{a} \right)^2 \left( \frac{n\pi}{b} \right)^2 + \left( \frac{n\pi}{b} \right)^4 \right] - \frac{a_{mn}}{D} = 0$$

or in alternative form,

$$A_{mn} \pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - \frac{a_{mn}}{D} = 0$$

giving

$$A_{mn} = \frac{1}{\pi^4 D} \frac{a_{mn}}{\left[ (m^2/a^2) + (n^2/b^2) \right]^2}$$

Hence,

$$w = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn}}{\left[ (m^2/a^2) + (n^2/b^2) \right]^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (7.30)$$

in which  $a_{mn}$  is obtained from Eq. (7.29). Equation (7.30) is the general solution for a thin rectangular plate under a transverse load  $q(x, y)$ .

### Example 7.1

A thin rectangular plate  $a \times b$  is simply supported along its edges and carries a uniformly distributed load of intensity  $q_0$ . Determine the deflected form of the plate and the distribution of bending moment.

Since  $q(x, y) = q_0$ , we find from Eq. (7.29) that

$$a_{mn} = \frac{4q_0}{ab} \int_0^a \int_0^b \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy = \frac{16q_0}{\pi^2 mn}$$

where  $m$  and  $n$  are odd integers. For  $m$  or  $n$  even,  $a_{mn} = 0$ . Hence, from Eq. (7.30),

$$w = \frac{16q_0}{\pi^6 D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\sin(m\pi x/a) \sin(n\pi y/b)}{mn[(m^2/a^2) + (n^2/b^2)]^2} \quad (i)$$

The maximum deflection occurs at the center of the plate, where  $x = a/2$ ,  $y = b/2$ . Thus,

$$w_{\max} = \frac{16q_0}{\pi^6 D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\sin(m\pi/2) \sin(n\pi/2)}{mn[(m^2/a^2) + (n^2/b^2)]^2} \quad (ii)$$

This series is found to converge rapidly, the first few terms giving a satisfactory answer. For a square plate, taking  $\nu = 0.3$ , summation of the first four terms of the series gives

$$w_{\max} = 0.0443q_0 \frac{a^4}{Et^3}$$

Substitution for  $w$  from Eq. (i) into the expressions for bending moment, Eqs. (7.7) and (7.8), yields

$$M_x = \frac{16q_0}{\pi^4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{[(m^2/a^2) + \nu(n^2/b^2)]}{mn[(m^2/a^2) + (n^2/b^2)]^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (\text{iii})$$

$$M_y = \frac{16q_0}{\pi^4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{[\nu(m^2/a^2) + (n^2/b^2)]}{mn[(m^2/a^2) + (n^2/b^2)]^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (\text{iv})$$

Maximum values occur at the center of the plate. For a square plate  $a = b$ , and the first five terms give

$$M_{x,\max} = M_{y,\max} = 0.0479q_0a^2$$

Comparing Eqs. (7.3) with Eqs. (7.5) and (7.6), we observe that

$$\sigma_x = \frac{12M_{xz}}{t^3}, \quad \sigma_y = \frac{12M_{yz}}{t^3}$$

Again the maximum values of these stresses occur at the center of the plate and on the top and bottom surfaces at  $z = \pm t/2$ , so that

$$\sigma_{x,\max} = \frac{6M_x}{t^2}, \quad \sigma_{y,\max} = \frac{6M_y}{t^2}$$

For the square plate,

$$\sigma_{x,\max} = \sigma_{y,\max} = 0.287q_0 \frac{a^2}{t^2}$$

The twisting moment and shear stress distributions follow in a similar manner.

The infinite series (Eq. (7.27)) assumed for the deflected shape of a plate gives an exact solution for displacements and stresses. However, a more rapid, but approximate, solution may be obtained by assuming a displacement function in the form of a polynomial. The polynomial must, of course, satisfy the governing differential equation (Eq. (7.20)) and the boundary conditions of the specific problem. The “guessed” form of the deflected shape of a plate is the basis for the energy method of solution described in Section 7.6.

### Example 7.2

Show that the deflection function

$$w = A(x^2y^2 - bx^2y - axy^2 + abxy)$$

is valid for a rectangular plate of sides  $a$  and  $b$ , built in on all four edges and subjected to a uniformly distributed load of intensity  $q$ . If the material of the plate has a Young’s modulus  $E$  and is of thickness  $t$ , determine the distributions of bending moment along the edges of the plate.

Differentiating the deflection function gives

$$\frac{\partial^4 w}{\partial x^4} = 0, \quad \frac{\partial^4 w}{\partial y^4} = 0, \quad \frac{\partial^4 w}{\partial x^2 \partial y^2} = 4A$$

Substituting in Eq. (7.20), we have

$$0 + 2 \times 4A + 0 = \text{constant} = \frac{q}{D}$$

The deflection function is therefore valid and

$$A = \frac{q}{8D}$$

The bending moment distributions are given by Eqs. (7.7) and (7.8); that is,

$$M_x = -\frac{q}{4}[y^2 - by + v(x^2 - ax)] \quad (\text{i})$$

$$M_y = -\frac{q}{4}[x^2 - ax + v(y^2 - by)] \quad (\text{ii})$$

For the edges  $x = 0$  and  $x = a$ ,

$$M_x = -\frac{q}{4}(y^2 - by), \quad M_y = -\frac{vq}{4}(y^2 - by)$$

For the edges  $y = 0$  and  $y = b$ ,

$$M_x = -\frac{vq}{4}(x^2 - ax), \quad M_y = -\frac{q}{4}(x^2 - ax)$$

## Example 7.2 MATLAB

Repeat Example 7.2 using the Symbolic Math Toolbox in MATLAB.

Expressions for  $M_x$  and  $M_y$  along the edges of the plate are obtained through the following MATLAB file:

```
% Declare any needed variables
syms w A x y b a q M_x M_y v D

% Define the given deflection function
w = A*(x^2*y^2 - b*x^2*y - a*x*y^2 + a*b*x*y);

% Check Eq. (7.20)
check = diff(w,x,4) + 2*diff(diff(w,x,2),y,2) + diff(w,y,4) - q/D;
if diff(check,x) == sym(0) && diff(check,y) == sym(0) && check ~= -q/D
    disp('The deflection function is valid')
    disp(' ')
else
    disp('The deflection function is invalid')
end

% Solve check for the constant A and substitute back into w
A_val = solve(check,A);
w = subs(w,A,A_val);
```

```
% Calculate the bending moment distributions using Eqs (7.7) and (7.8)
eqI = -D*(diff(w,x,2) + v*diff(w,y,2)); % M_x
eqII = -D*(diff(w,y,2) + v*diff(w,x,2)); % M_y

% Output expressions of M_x, M_y for the edge x=0 to the Command Window
disp('For the edge x = 0:')
disp(['M_x =' char(simplify(subs(eqI,x,0)))] )
disp(['M_y =' char(simplify(subs(eqII,x,0)))] )
disp('')

% Output expressions of M_x, M_y for the edge x=a to the Command Window
disp('For the edge x = a:')
disp(['M_x =' char(simplify(subs(eqI,x,a)))] )
disp(['M_y =' char(simplify(subs(eqII,x,a)))] )
disp('')

% Output expressions of M_x, M_y for the edge y=0 to the Command Window
disp('For the edge y = 0:')
disp(['M_x =' char(simplify(subs(eqI,y,0)))] )
disp(['M_y =' char(simplify(subs(eqII,y,0)))] )
disp('')

% Output expressions of M_x, M_y for the edge y=b to the Command Window
disp('For the edge y = b:')
disp(['M_x =' char(simplify(subs(eqI,y,b)))] )
disp(['M_y =' char(simplify(subs(eqII,y,b)))] )
disp('')

else
    disp('The deflection function does not satisfy Equation (7.20)')
    disp('')
end
```

The Command Window outputs resulting from this MATLAB file are as follows. The deflection function is valid.

For the edge  $x = 0$ :

$$\begin{aligned} M_x &= (q*y*(b - y))/4 \\ M_y &= (q*v*y*(b - y))/4 \end{aligned}$$

For the edge  $x = a$ :

$$\begin{aligned} M_x &= (q*y*(b - y))/4 \\ M_y &= (q*v*y*(b - y))/4 \end{aligned}$$

For the edge  $y = 0$ :

$$\begin{aligned} M_x &= (q*v*x*(a - x))/4 \\ M_y &= (q*x*(a - x))/4 \end{aligned}$$

For the edge  $y = b$ :

$$\begin{aligned} M_x &= (q*v*x*(a - x))/4 \\ M_y &= (q*x*(a - x))/4 \end{aligned}$$

## 7.4 COMBINED BENDING AND IN-PLANE LOADING OF A THIN RECTANGULAR PLATE

So far our discussion has been limited to small deflections of thin plates produced by different forms of transverse loading. In these cases, we assumed that the middle or neutral plane of the plate remained unstressed. Additional in-plane tensile, compressive, or shear loads produce stresses in the middle plane, and these, if of sufficient magnitude, affect the bending of the plate. Where the in-plane stresses are small compared with the critical buckling stresses, it is sufficient to consider the two systems separately; the total stresses are then obtained by superposition. On the other hand, if the in-plane stresses are not small, then their effect on the bending of the plate must be considered.

The elevation and plan of a small element  $\delta x \delta y$  of the middle plane of a thin deflected plate are shown in Fig. 7.12. Direct and shear forces per unit length produced by the in-plane loads are given the notation  $N_x$ ,  $N_y$ , and  $N_{xy}$  and are assumed to be acting in positive senses in the directions shown. Since there are no resultant forces in the  $x$  or  $y$  directions from the transverse loads (see Fig. 7.9), we need only include the in-plane loads shown in Fig. 7.12 when considering the equilibrium of the element in these directions. For equilibrium parallel to  $Ox$ ,

$$\left( N_x + \frac{\partial N_x}{\partial x} \delta x \right) \delta y \cos \left( \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \delta x \right) - N_x \delta y \cos \frac{\partial w}{\partial x} + \left( N_{yx} + \frac{\partial N_{yx}}{\partial y} \delta y \right) \delta x - N_{yx} \delta x = 0$$

For small deflections,  $\partial w / \partial x$  and  $(\partial^2 w / \partial x^2) \delta x$  are small, and the cosines of these angles are therefore approximately equal to one. The equilibrium equation thus simplifies to

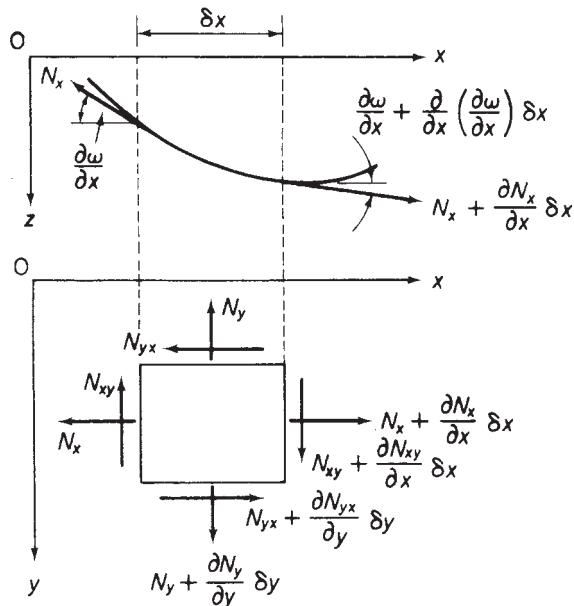


FIGURE 7.12 In-Plane Forces on Plate Element

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0 \quad (7.31)$$

Similarly, for equilibrium in the  $y$  direction, we have

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad (7.32)$$

Note that the components of the in-plane shear loads per unit length are, to a first order of approximation, the value of the shear load multiplied by the projection of the element on the relevant axis.

The determination of the contribution of the shear loads to the equilibrium of the element in the  $z$  direction is complicated by the fact that the element possesses curvature in both the  $xz$  and  $yz$  planes. Therefore, from Fig. 7.13, the component in the  $z$  direction due to the  $N_{xy}$  shear loads only is

$$\left( N_{xy} + \frac{\partial N_{xy}}{\partial x} \delta x \right) \delta y \left( \frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x \partial y} \delta x \right) - N_{xy} \delta y \frac{\partial w}{\partial y} \quad \left( \text{Note: } \sin \frac{\partial w}{\partial y} = \frac{\partial w}{\partial y}, \text{ etc.} \right)$$

or

$$N_{xy} \frac{\partial^2 w}{\partial x \partial y} \delta x \delta y + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} \delta x \delta y$$

neglecting terms of a lower order. Similarly, the contribution of  $N_{yx}$  is

$$N_{yx} \frac{\partial^2 w}{\partial x \partial y} \delta x \delta y + \frac{\partial N_{yx}}{\partial y} \frac{\partial w}{\partial x} \delta x \delta y$$

The components arising from the direct forces per unit length are readily obtained from Fig. 7.12, namely

$$\left( N_x + \frac{\partial N_x}{\partial x} \delta x \right) \delta y \left( \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \delta x \right) - N_x \delta y \frac{\partial w}{\partial x}$$

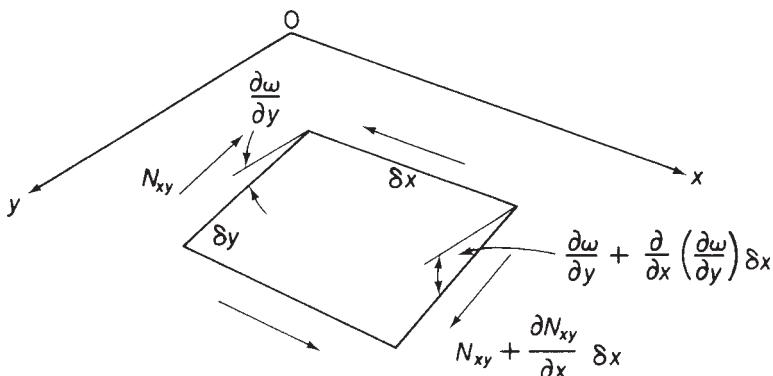


FIGURE 7.13 Component of Shear Loads in the  $z$  Direction

or

$$N_x \frac{\partial^2 w}{\partial x^2} \delta x \delta y + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} \delta x \delta y$$

and, similarly,

$$N_y \frac{\partial^2 w}{\partial y^2} \delta x \delta y + \frac{\partial N_y}{\partial y} \frac{\partial w}{\partial y} \delta x \delta y$$

The total force in the  $z$  direction is found from the summation of these expressions and is

$$\begin{aligned} & N_x \frac{\partial^2 w}{\partial x^2} \delta x \delta y + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} \delta x \delta y + N_y \frac{\partial^2 w}{\partial y^2} \delta x \delta y + \frac{\partial N_y}{\partial y} \frac{\partial w}{\partial y} \delta x \delta y \\ & + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} \delta x \delta y + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \delta x \delta y + \frac{\partial N_{xy}}{\partial y} \frac{\partial w}{\partial x} \delta x \delta y \end{aligned}$$

in which  $N_{yx}$  is equal to and is replaced by  $N_{xy}$ . Using Eqs. (7.31) and (7.32), we reduce this expression to

$$\left( N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \delta x \delta y$$

Since the in-plane forces do not produce moments along the edges of the element, Eqs. (7.17) and (7.18) remain unaffected. Further, Eq. (7.16) may be modified simply by the addition of the preceding vertical component of the in-plane loads to  $q \delta x \delta y$ . Therefore, the governing differential equation for a thin plate supporting transverse and in-plane loads is, from Eq. (7.20),

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} \left( q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \quad (7.33)$$

### Example 7.3

Determine the deflected form of the thin rectangular plate of Example 7.1 if, in addition to a uniformly distributed transverse load of intensity  $q_0$ , it supports an in-plane tensile force  $N_x$  per unit length.

The uniform transverse load may be expressed as a Fourier series (see Eq. (7.28) and Example 7.1); that is,

$$q = \frac{16q_0}{\pi^2} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Equation (7.33) then becomes, on substituting for  $q$ ,

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{N_x}{D} \frac{\partial^2 w}{\partial x^2} = \frac{16q_0}{\pi^2 D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (i)$$

The appropriate boundary conditions are

$$w = \frac{\partial^2 w}{\partial x^2} = 0, \text{ at } x = 0 \text{ and } a$$

$$w = \frac{\partial^2 w}{\partial y^2} = 0, \text{ at } y = 0 \text{ and } b$$

These conditions may be satisfied by the assumption of a deflected form of the plate given by

$$\begin{aligned} w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \frac{\partial^4 w}{\partial x^4} &= \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} A_{mn} \frac{m^4 \pi^4}{a^4} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \frac{\partial^4 w}{\partial x^2 \partial y^2} &= \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} A_{mn} \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \frac{n^2 \pi^2}{b^2} \sin \frac{n\pi y}{b} \\ \frac{\partial^4 w}{\partial y^4} &= \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \frac{n^4 \pi^4}{b^4} \sin \frac{n\pi y}{b} \end{aligned}$$

Substituting these expressions in Eq. (i) gives

$$\begin{aligned} &\sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left[ \frac{m^4 \pi^4}{a^4} + 2 \left( \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right)^2 + \frac{n^4 \pi^4}{b^4} + \frac{N_x m^2 \pi^2}{D a^2} \right] \\ &= \frac{16q_0}{\pi^2 D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned}$$

from which

$$A_{mn} = \frac{16q_0}{\pi^6 D m n \left[ \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 + \frac{N_x m^2}{\pi^2 D a^2} \right]}, \text{ for odd } m \text{ and } n$$

$$A_{mn} = 0, \text{ for even } m \text{ and } n$$

Therefore,

$$w = \frac{16q_0}{\pi^6 D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{mn \left[ \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 + \frac{N_x m^2}{\pi^2 D a^2} \right]} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (\text{ii})$$

Comparing Eq. (ii) with Eq. (i) of Example 7.1, we see that, as a physical inspection would indicate, the presence of a tensile in-plane force decreases deflection. Conversely, a compressive in-plane force would increase the deflection.



## 7.5 BENDING OF THIN PLATES HAVING A SMALL INITIAL CURVATURE

Suppose that a thin plate has an initial curvature so that the deflection of any point in its middle plane is  $w_0$ . We assume that  $w_0$  is small compared with the thickness of the plate. The application of transverse and in-plane loads causes the plate to deflect a further amount  $w_1$ , so that the total deflection is then  $w = w_0 + w_1$ . However, in the derivation of Eq. (7.33), we note that the left-hand side is obtained from expressions for bending moments, which themselves depend on the change of curvature. We therefore use the deflection  $w_1$  on the left-hand side, not  $w$ . The effect on bending of the in-plane forces depends on the total deflection  $w$  so that we write Eq. (7.33):

$$\begin{aligned} & \frac{\partial^4 w_1}{\partial x^4} + 2 \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^4} \\ &= \frac{1}{D} \left[ q + N_x \frac{\partial^2 (w_0 + w_1)}{\partial x^2} + N_y \frac{\partial^2 (w_0 + w_1)}{\partial y^2} + 2N_{xy} \frac{\partial^2 (w_0 + w_1)}{\partial x \partial y} \right] \end{aligned} \quad (7.34)$$

The effect of an initial curvature on deflection is therefore equivalent to the application of a transverse load of intensity:

$$N_x \frac{\partial^2 w_0}{\partial x^2} + N_y \frac{\partial^2 w_0}{\partial y^2} + 2N_{xy} \frac{\partial^2 w_0}{\partial x \partial y}$$

Thus, in-plane loads alone produce bending, provided there is an initial curvature.

Assuming that the initial form of the deflected plate is

$$w_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (7.35)$$

then, by substitution in Eq. (7.34), we find that, if  $N_x$  is compressive and  $N_y = N_{xy} = 0$ ,

$$w_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (7.36)$$

where

$$B_{mn} = \frac{A_{mn} N_x}{(\pi^2 D/a^2)[m + (n^2 a^2/b^2)]^2 - N_x}$$

We shall return to the consideration of initially curved plates in the discussion of the experimental determination of buckling loads of flat plates in Chapter 9.

## 7.6 ENERGY METHOD FOR THE BENDING OF THIN PLATES

Two types of solution are obtainable for thin-plate bending problems by the application of the principle of the stationary value of the total potential energy of the plate and its external loading. The first, in which the form of the deflected shape of the plate is known, produces an exact solution; the second, the Rayleigh–Ritz method, assumes an approximate deflected shape in the form of a series having a finite

number of terms chosen to satisfy the boundary conditions of the problem and also to give the kind of deflection pattern expected.

In Chapter 5, we saw that the total potential energy of a structural system comprised the internal or strain energy of the structural member plus the potential energy of the applied loading. We now proceed to derive expressions for these quantities for the loading cases considered in the preceding sections.

### 7.6.1 Strain energy produced by bending and twisting

In thin-plate analysis, we are concerned with deflections normal to the loaded surface of the plate. These, as in the case of slender beams, are assumed to be primarily due to bending action, so that the effects of shear strain and shortening or stretching of the middle plane of the plate are ignored. Therefore, it is sufficient for us to calculate the strain energy produced by bending and twisting only, as this will be applicable, for the reason of the previous assumption, to all loading cases. It must be remembered that we are neglecting the contributions of only shear and direct strains on the deflection of the plate; the stresses producing them must not be ignored.

Consider the element  $\delta x \times \delta y$  of a thin plate  $a \times b$  shown in elevation in the  $xz$  plane in Fig. 7.14(a). Bending moments  $M_x$  per unit length applied to its  $\delta y$  edge produce a change in slope between its ends equal to  $(\partial^2 w / \partial x^2) \delta x$ . However, since we regard the moments  $M_x$  as positive in the sense shown, then this change in slope, or relative rotation, of the ends of the element is negative, as the slope decreases with increasing  $x$ . The bending strain energy due to  $M_x$  is then

$$\frac{1}{2} M_x \delta y \left( -\frac{\partial^2 w}{\partial x^2} \delta x \right)$$

Similarly, in the  $yz$  plane, the contribution of  $M_y$  to the bending strain energy is

$$\frac{1}{2} M_y \delta x \left( -\frac{\partial^2 w}{\partial y^2} \delta y \right)$$

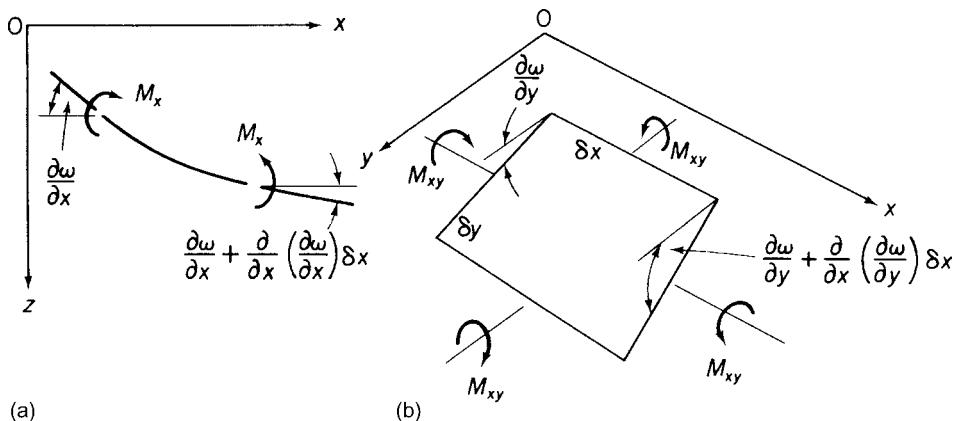


FIGURE 7.14 (a) Strain Energy of an Element Due to Bending; (b) Strain Energy Due to Twisting

The strain energy due to the twisting moment per unit length,  $M_{xy}$ , applied to the  $\delta y$  edges of the element, is obtained from Fig. 7.14(b). The relative rotation of the  $\delta y$  edges is  $(\partial^2 w / \partial x \partial y) \delta x$ , so that the corresponding strain energy is

$$\frac{1}{2} M_{xy} \delta y \frac{\partial^2 w}{\partial x \partial y} \delta x$$

Finally, the contribution of the twisting moment  $M_{xy}$  on the  $\delta x$  edges is, in a similar fashion,

$$\frac{1}{2} M_{xy} \delta x \frac{\partial^2 w}{\partial x \partial y} \delta y$$

The total strain energy of the element from bending and twisting is thus

$$\frac{1}{2} \left( -M_x \frac{\partial^2 w}{\partial x^2} - M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \delta x \delta y$$

Substitution for  $M_x$ ,  $M_y$ , and  $M_{xy}$  from Eqs. (7.7), (7.8), and (7.14) gives the total strain energy of the element as

$$\frac{D}{2} \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \delta x \delta y$$

which on rearranging becomes

$$\frac{D}{2} \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} \delta x \delta y$$

Hence, the total strain energy  $U$  of the rectangular plate  $a \times b$  is

$$U = \frac{D}{2} \int_0^a \int_0^b \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy \quad (7.37)$$

Note that if the plate is subject to pure bending only, then  $M_{xy} = 0$  and, from Eq. (7.14),  $\partial^2 w / \partial x \partial y = 0$ , so that Eq. (7.37) simplifies to

$$U = \frac{D}{2} \int_0^a \int_0^b \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] dx dy \quad (7.38)$$

## 7.6.2 Potential energy of a transverse load

An element  $\delta x \times \delta y$  of the transversely loaded plate of Fig. 7.8 supports a load  $q \delta x \delta y$ . If the displacement of the element normal to the plate is  $w$ , then the potential energy  $\delta V$  of the load on the element referred to the undeflected plate position is

$$\delta V = -wq \delta x \delta y \quad (\text{see Section 5.7})$$

Therefore, the potential energy  $V$  of the total load on the plate is given by

$$V = - \int_0^a \int_0^b wq dx dy \quad (7.39)$$

### 7.6.3 Potential energy of in-plane loads

We may consider each load  $N_x$ ,  $N_y$ , and  $N_{xy}$  in turn, then use the principle of superposition to determine the potential energy of the loading system when they act simultaneously. Consider an elemental strip of width  $\delta y$  along the length  $a$  of the plate in Fig. 7.15(a). The compressive load on this strip is  $N_x \delta y$ , and due to the bending of the plate, the horizontal length of the strip decreases by an amount  $\lambda$ , as shown in Fig. 7.15(b). The potential energy  $\delta V_x$  of the load  $N_x \delta y$ , referred to the undeflected position of the plate as the datum, is then

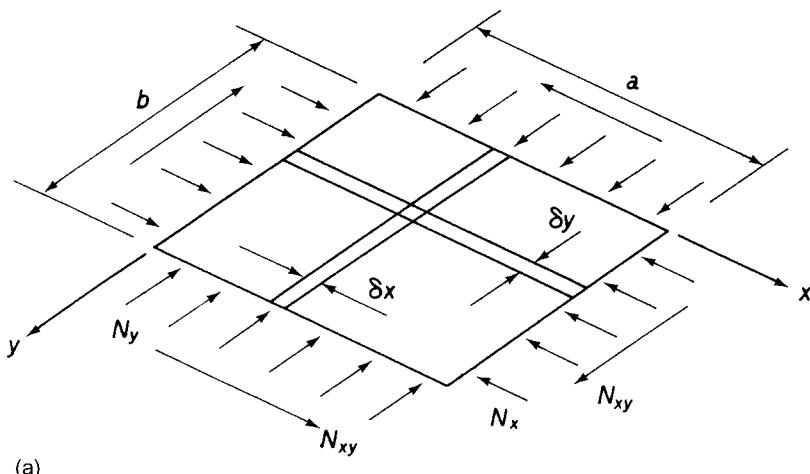
$$\delta V_x = -N_x \lambda \delta y \quad (7.40)$$

From Fig. 7.15(b), the length of a small element  $\delta a$  of the strip is

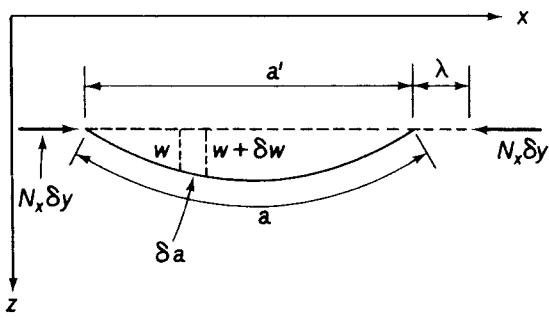
$$\delta a = (\delta x^2 + \delta w^2)^{\frac{1}{2}}$$

and, since  $\partial w / \partial x$  is small,

$$\delta a \approx \delta x \left[ 1 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]$$



(a)



(b)

FIGURE 7.15 (a) In-Plane Loads on a Plate; (b) Shortening of an Element Due to Bending

Hence,

$$a = \int_0^{a'} \left[ 1 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] dx$$

giving

$$a = a' + \int_0^{a'} \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 dx$$

and

$$\lambda = a - a' = \int_0^{a'} \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 dx$$

Since

$$\int_0^{a'} \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 dx \text{ only differs from } \int_0^a \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 dx$$

by a term of negligible order, we write

$$\lambda = \int_0^a \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 dx \quad (7.41)$$

The potential energy  $V_x$  of the  $N_x$  loading follows from Eqs. (7.40) and (7.41), thus

$$V_x = -\frac{1}{2} \int_0^a \int_0^b N_x \left( \frac{\partial w}{\partial x} \right)^2 dx dy \quad (7.42)$$

Similarly,

$$V_y = -\frac{1}{2} \int_0^a \int_0^b N_y \left( \frac{\partial w}{\partial y} \right)^2 dx dy \quad (7.43)$$

The potential energy of the in-plane shear load  $N_{xy}$  may be found by considering the work done by  $N_{xy}$  during the shear distortion corresponding to the deflection  $w$  of an element. This shear strain is the reduction in the right angle  $C_2AB_1$  to the angle  $C_1AB_1$  of the element in Fig. 7.16 or, rotating  $C_2A$  with respect to  $AB_1$  to  $AD$  in the plane  $C_1AB_1$ , the angle  $DAC_1$ . The displacement  $C_1C_2$  is equal to  $(\partial w / \partial y) \delta y$ , and the angle  $DC_2C_1$  is  $\partial w / \partial x$ . Thus,  $C_1D$  is equal to

$$\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \delta y$$

and the angle  $DAC_1$ , representing the shear strain corresponding to the bending displacement  $w$ , is

$$\frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$$

so that the work done on the element by the shear force  $N_{xy} \delta y$  is

$$\frac{1}{2} N_{xy} \delta x \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$$

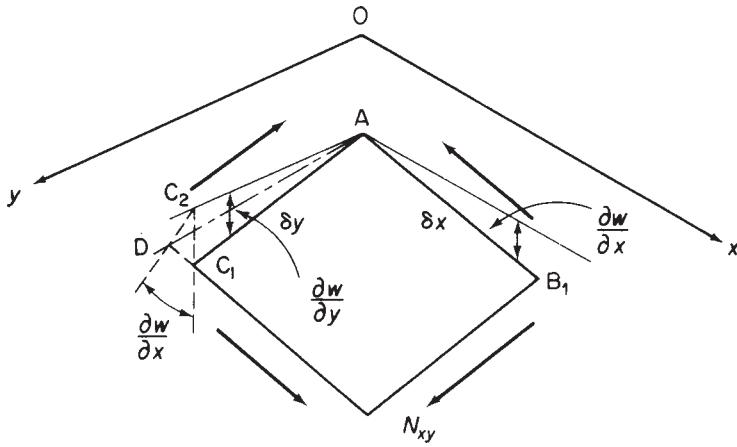


FIGURE 7.16 Calculation of Shear Strain Corresponding to Bending Deflection

Similarly, the work done by the shear force  $N_{xy}\delta y$  is

$$\frac{1}{2}N_{xy}\delta y \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$$

and the total work done, taken over the complete plate, is

$$\frac{1}{2} \int_0^a \int_b^b 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} dx dy$$

It follows immediately that the potential energy of the  $N_{xy}$  loads is

$$V_{xy} = -\frac{1}{2} \int_0^a \int_0^b 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} dx dy \quad (7.44)$$

and, for the complete in-plane loading system, we have, from Eqs. (7.42), (7.43), and (7.44), a potential energy of

$$V = -\frac{1}{2} \int_0^a \int_0^b \left[ N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy \quad (7.45)$$

We are now in a position to solve a wide range of thin-plate problems, provided that the deflections are small, obtaining exact solutions if the deflected form is known or approximate solutions if the deflected shape has to be guessed.

Considering the rectangular plate of Section 7.3, simply supported along all four edges and subjected to a uniformly distributed transverse load of intensity  $q_0$ , we know that its deflected shape is given by Eq. (7.27), namely

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

The total potential energy of the plate is, from Eqs. (7.37) and (7.39),

$$U + V = \int_0^a \int_0^b \left\{ \frac{D}{2} \left[ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-v) \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \right] - w q_0 \right\} dx dy \quad (7.46)$$

Substituting in Eq. (7.46) for  $w$  and realizing that “cross-product” terms integrate to zero, we have

$$\begin{aligned} U + V &= \int_0^a \int_0^b \left\{ \frac{D}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left[ \pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \right. \right. \\ &\quad \left. \left. - 2(1-v) \frac{m^2 n^2 \pi^4}{a^2 b^2} \left( \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} - \cos^2 \frac{m\pi x}{a} \cos^2 \frac{n\pi y}{b} \right) \right] \right. \\ &\quad \left. - q_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right\} dx dy \end{aligned}$$

The term multiplied by  $2(1-v)$  integrates to zero, and the mean value of  $\sin^2$  or  $\cos^2$  over a complete number of half waves is  $\frac{1}{2}$ , thus integration of the above expression yields

$$U + V = \frac{D}{2} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} A_{mn}^2 \frac{\pi^4 ab}{4} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - q_0 \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} A_{mn} \frac{4ab}{\pi^2 mn} \quad (7.47)$$

From the principle of the stationary value of the total potential energy, we have

$$\frac{\partial(U + V)}{\partial A_{mn}} = \frac{D}{2} 2A_{mn} \frac{\pi^4 ab}{4} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - q_0 \frac{4ab}{\pi^2 mn} = 0$$

so that

$$A_{mn} = \frac{16q_0}{\pi^6 D m n [(m^2/a^2) + (n^2/b^2)]^2}$$

giving a deflected form

$$w = \frac{16q_0}{\pi^6 D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\sin(m\pi x/a) \sin(n\pi y/b)}{mn[(m^2/a^2) + (n^2/b^2)]^2}$$

which is the result obtained in Eq. (i) of Example 7.1.

This solution is exact, since we know the true deflected shape of the plate in the form of an infinite series for  $w$ . Frequently, the appropriate infinite series is not known, so that only an approximate solution may be obtained. The method of solution, known as the *Rayleigh-Ritz* method, involves the selection of a series for  $w$  containing a finite number of functions of  $x$  and  $y$ . These functions are chosen to satisfy the boundary conditions of the problem as far as possible and also to give the type of deflection pattern expected. Naturally, the more representative the guessed functions are, the more accurate the solution becomes.

Suppose that the guessed series for  $w$  in a particular problem contains three functions of  $x$  and  $y$ . Thus,

$$w = A_1 f_1(x, y) + A_2 f_2(x, y) + A_3 f_3(x, y)$$

where  $A_1$ ,  $A_2$ , and  $A_3$  are unknown coefficients. We now substitute for  $w$  in the appropriate expression for the total potential energy of the system and assign stationary values with respect to  $A_1$ ,  $A_2$ , and  $A_3$  in turn. Thus,

$$\frac{\partial(U + V)}{\partial A_1} = 0, \quad \frac{\partial(U + V)}{\partial A_2} = 0, \quad \frac{\partial(U + V)}{\partial A_3} = 0$$

giving three equations that are solved for  $A_1$ ,  $A_2$ , and  $A_3$ .

### Example 7.4

A rectangular plate  $a \times b$  is simply supported along each edge and carries a uniformly distributed load of intensity  $q_0$ . Assuming a deflected shape given by

$$w = A_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

determine the value of the coefficient  $A_{11}$  and hence find the maximum value of deflection.

The expression satisfies the boundary conditions of zero deflection and zero curvature (i.e., zero bending moment) along each edge of the plate. Substituting for  $w$  in Eq. (7.46), we have

$$\begin{aligned} U + V &= \int_0^a \int_0^b \left[ \frac{DA_{11}^2}{2} \left\{ \frac{\pi^4}{(a^2 b^2)^2} (a^2 + b^2)^2 \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} - 2(1 - v) \right. \right. \\ &\quad \times \left. \left. \left[ \frac{\pi^4}{a^2 b^2} \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} - \frac{\pi^4}{a^2 b^2} \cos^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b} \right] \right\} \right. \\ &\quad \left. - q_0 A_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right] dx dy \end{aligned}$$

from which

$$U + V = \frac{DA_{11}^2}{2} \frac{\pi^4}{4a^3 b^3} (a^2 + b^2)^2 - q_0 A_{11} \frac{4ab}{\pi^2}$$

so that

$$\frac{\partial(U + V)}{\partial A_{11}} = \frac{DA_{11} \pi^4}{4a^3 b^3} (a^2 + b^2)^2 - q_0 \frac{4ab}{\pi^2} = 0$$

and

$$A_{11} = \frac{16q_0 a^4 b^4}{\pi^6 D (a^2 + b^2)^2}$$

giving

$$w = \frac{16q_0 a^4 b^4}{\pi^6 D (a^2 + b^2)^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

At the center of the plate,  $w$  is a maximum and

$$w_{\max} = \frac{16q_0 a^4 b^4}{\pi^6 D (a^2 + b^2)^2}$$

For a square plate and assuming  $v = 0.3$ ,

$$w_{\max} = 0.0455q_0 \frac{a^4}{Et^3}$$

which compares favorably with the result of [Example 7.1](#).

In this chapter, we have dealt exclusively with small deflections of thin plates. For a plate subjected to large deflections, the middle plane is stretched due to *bending*, so that [Eq. \(7.33\)](#) requires modification. The relevant theory is outside the scope of this book but may be found in a variety of references.

#### **Example 7.4 MATLAB**

Repeat Example 7.4 using the Symbolic Math Toolbox in MATLAB.

Expressions for  $A_{11}$  and the maximum deflection are obtained through the following MATLAB file:

```
% Declare any needed variables
syms w A_11 x y b a q_0 UV v D

% Define the given deflection function
w = A_11*sin(pi*x/a)*sin(pi*y/b);

% Evaluate Eq. (7.46) to calculate U+V (UV)
w_xx = diff(w,x,2);
w_yy = diff(w,y,2);
w_xy = diff(diff(w,x),y);

UV = int(int((D/2*((w_xx+w_yy)^2 - 2*(1-v)*(w_xx*w_yy-w_xy^2)) - w*q_0),y,0,b,
'IgnoreAnalyticConstraints',true),x,0,a);

% Differentiate UV with respect to A_11, set equal to 0, and solve for A_11
A_11val = solve(diff(UV,A_11),A_11);

% Substitute A_11 back into w
w = subs(w,A_11,A_11val);

% Due to the boundary conditions and the form of w, there will only be one maximum value.
% Therefore, w_max will occur where the gradient of w is 0.
x_max = solve(diff(w,x),x,'IgnoreAnalyticConstraints',true);
y_max = solve(diff(w,y),y,'IgnoreAnalyticConstraints',true);

% Substitute x_max and y_max into w
w_max = subs(subs(w,x,x_max),y,y_max);

% Output expressions for A_11 and w_max to the Command Window
disp(['A_11 =' char(A_11val)])
disp(['w_max =' char(w_max)])
```

The Command Window outputs resulting from this MATLAB file are as follows:

```
A_11 = (16*a^4*b^4*q_0)/(D*pi^6*(a^2 + b^2)^2)
w_max = (16*a^4*b^4*q_0)/(D*pi^6*(a^2 + b^2)^2)
```

## Further reading

- Jaeger JC. Elementary theory of elastic plates. New York: Pergamon Press; 1964.  
 Timoshenko SP, Woinowsky-Krieger S. Theory of plates and shells. 2nd ed. New York: McGraw-Hill; 1959.  
 Timoshenko SP, Gere JM. Theory of elastic stability. 2nd ed. New York: McGraw-Hill; 1961.  
 Wang CT. Applied elasticity. New York: McGraw-Hill; 1953.

## PROBLEMS

**P.7.1.** A plate 10 mm thick is subjected to bending moments  $M_x$  equal to 10 Nm/mm and  $M_y$  equal to 5 Nm/mm. Calculate the maximum direct stresses in the plate.

Answer:  $\sigma_{x,\max} = \pm 600\text{N/mm}^2$ ,  $\sigma_{y,\max} = \pm 300\text{N/mm}^2$ .

**P.7.2.** For the plate and loading of problem P.7.1, find the maximum twisting moment per unit length in the plate and the direction of the planes on which this occurs.

Answer: 2.5 Nm/mm at  $45^\circ$  to the  $x$  and  $y$  axes.

**P.7.3.** The plate of the previous two problems is subjected to a twisting moment of 5 Nm/mm along each edge in addition to the bending moments of  $M_x = 10 \text{ Nm/mm}$  and  $M_y = 5 \text{ Nm/mm}$ . Determine the principal moments in the plate, the planes on which they act, and the corresponding principal stresses.

Answer: 13.1 Nm/mm, 1.9 Nm/mm,  $\alpha = -31.7^\circ$ ,  $\alpha = +58.3^\circ$ ,  $\pm 786 \text{ N/mm}^2$ ,  $\pm 114 \text{ N/mm}^2$ .

**P.7.3 MATLAB** Use MATLAB to repeat Problem P.7.3 for the following combinations of  $M_x$ ,  $M_y$ , and  $M_{xy}$ . Do not calculate the principal stresses.

	(i)	(ii)	(iii)	(iv)	(v)
$M_x$	6	8	10	12	14
$M_x$	7	6	5	4	3
$M_x$	3	4	5	6	7

Answer: (i)  $M_I = 9.5 \text{ Nm/mm}$ ,  $M_{II} = 3.5 \text{ Nm/mm}$ ,  $\alpha = -49.7^\circ$  or  $40.3^\circ$   
 (ii)  $M_I = 11.1 \text{ Nm/mm}$ ,  $M_{II} = 2.9 \text{ Nm/mm}$ ,  $\alpha = -38^\circ$  or  $52^\circ$   
 (iii)  $M_I = 13.1 \text{ Nm/mm}$ ,  $M_{II} = 1.9 \text{ Nm/mm}$ ,  $\alpha = -31.7^\circ$  or  $58.3^\circ$   
 (iv)  $M_I = 15.2 \text{ Nm/mm}$ ,  $M_{II} = 0.8 \text{ Nm/mm}$ ,  $\alpha = -28.2^\circ$  or  $61.8^\circ$   
 (v)  $M_I = 17.4 \text{ Nm/mm}$ ,  $M_{II} = 0.4 \text{ Nm/mm}$ ,  $\alpha = -25.9^\circ$  or  $64.1^\circ$

**P.7.4.** A thin rectangular plate of length  $a$  and width  $2a$  is simply supported along the edges  $x = 0$ ,  $x = a$ ,  $y = -a$ , and  $y = +a$ . The plate has a flexural rigidity  $D$ , a Poisson's ratio of 0.3, and carries a load distribution given by  $q(x, y) = q_0 \sin(\pi x/a)$ . If the deflection of the plate is represented by the expression

$$w = \frac{qa^4}{D\pi^4} \left( 1 + A \cosh \frac{\pi y}{a} + B \frac{\pi y}{a} \sinh \frac{\pi y}{a} \right) \sin \frac{\pi x}{a}$$

determine the values of the constants  $A$  and  $B$ .

Answer:  $A = -0.2213$ ,  $B = 0.0431$ .

**P.7.5.** A thin, elastic square plate of side  $a$  is simply supported on all four sides and supports a uniformly distributed load  $q$ . If the origin of axes coincides with the center of the plate, show that the deflection of the plate can be represented by the expression

$$w = \frac{q}{96(1-v)D} [2(x^4 + y^4) - 3a^2(1-v)(x^2 + y^2) - 12vx^2y^2 + A]$$

where  $D$  is the flexural rigidity,  $v$  is Poisson's ratio, and  $A$  is a constant. Calculate the value of  $A$  and hence the central deflection of the plate.

Answer:  $A = a^4(5 - 3v)/4$ , Cen. def. =  $qa^4(5 - 3v)/384D(1 - v)$

**P.7.6.** The deflection of a square plate of side  $a$  that supports a lateral load represented by the function  $q(x, y)$  is given by

$$w(x, y) = w_0 \cos \frac{\pi x}{a} \cos \frac{3\pi y}{a}$$

where  $x$  and  $y$  are referred to axes whose origin coincides with the center of the plate and  $w_0$  is the deflection at the center. If the flexural rigidity of the plate is  $D$  and Poisson's ratio is  $v$ , determine the loading function  $q$ , the support conditions of the plate, the reactions at the plate corners, and the bending moments at the center of the plate.

Answer:  $q(x, y) = w_0 D 100 \frac{\pi^4}{a^4} \cos \frac{\pi x}{a} \cos \frac{3\pi y}{a}$

The plate is simply supported on all edges. Reactions:  $-6w_0 D \left(\frac{\pi}{a}\right)^2 (1 - v)$

$$M_x = w_0 D \left(\frac{\pi}{a}\right)^2 (1 + 9v), \quad M_y = w_0 D \left(\frac{\pi}{a}\right)^2 (9 + v)$$

**P.7.7.** A simply supported square plate  $a \times a$  carries a distributed load according to the formula

$$q(x, y) = q_0 \frac{x}{a}$$

where  $q_0$  is its intensity at the edge  $x = a$ . Determine the deflected shape of the plate.

Answer:  $w = \frac{8q_0 a^4}{\pi^6 D} \sum_{m=1,2,3}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{m+1}}{mn(m^2+n^2)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}$

**P.7.8.** An elliptic plate of major and minor axes  $2a$  and  $2b$  and of small thickness  $t$  is clamped along its boundary and is subjected to a uniform pressure difference  $p$  between the two faces. Show that the usual differential equation for normal displacements of a thin flat plate subject to lateral loading is satisfied by the solution

$$w = w_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2$$

where  $w_0$  is the deflection at the center, which is taken as the origin.

Determine  $w_0$  in terms of  $p$  and the relevant material properties of the plate and hence expressions for the greatest stresses due to bending at the center and at the ends of the minor axis.

$$\text{Answer: } w_0 = \frac{3p(1 - v^2)}{2Et^3 \left( \frac{3}{a^4} + \frac{2}{a^2 b^2} + \frac{3}{b^4} \right)}$$

Center,

$$\sigma_{x,\max} = \frac{\pm 3pa^2b^2(b^2 + va^2)}{t^2(3b^4 + 2a^2b^2 + 3a^4)}, \quad \sigma_{y,\max} = \frac{\pm 3pa^2b^2(a^2 + vb^2)}{t^2(3b^4 + 2a^2b^2 + 3a^4)}$$

Ends of minor axis,

$$\sigma_{x,\max} = \frac{\pm 6pa^4b^2}{t^2(3b^4 + 2a^2b^2 + 3a^4)}, \quad \sigma_{y,\max} = \frac{\pm 6pb^4a^2}{t^2(3b^4 + 2a^2b^2 + 3a^4)}$$

**P.7.9.** Use the energy method to determine the deflected shape of a rectangular plate  $a \times b$ , simply supported along each edge and carrying a concentrated load  $W$  at a position  $(\xi, \eta)$  referred to axes through a corner of the plate. The deflected shape of the plate can be represented by the series

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\text{Answer: } A_{mn} = \frac{4W \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}}{\pi^4 Dab [(m^2/a^2) + (n^2/b^2)]^2}$$

**P.7.10.** If, in addition to the point load  $W$ , the plate of problem P.7.9 supports an in-plane compressive load of  $N_x$  per unit length on the edges  $x = 0$  and  $x = a$ , calculate the resulting deflected shape.

$$\text{Answer: } A_{mn} = \frac{4W \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}}{abD\pi^4 \left[ \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - \frac{m^2 N_x}{\pi^2 a^2 D} \right]}$$

**P.7.11.** A square plate of side  $a$  is simply supported along all four sides and is subjected to a transverse, uniformly distributed load of intensity  $q_0$ . It is proposed to determine the deflected shape of the plate by the Rayleigh–Ritz method employing a “guessed” form for the deflection of

$$w = A_{11} \left( 1 - \frac{4x^2}{a^2} \right) \left( 1 - \frac{4y^2}{a^2} \right)$$

in which the origin is taken at the center of the plate.

Comment on the degree to which the boundary conditions are satisfied and find the central deflection assuming  $v = 0.3$ .

$$\text{Answer: } 0.0389 q_0 a^4 / Et^3$$

**P.7.11 MATLAB** Use the Symbolic Math Toolbox in MATLAB to repeat Problem P.7.11. In addition, calculate the deflections at the following  $(x,y)$  locations in the plate.

$$(a/4, a/4), (0, 0), (a/3, a/3)$$

- Answer:
- (i)  $(x, y) = (a/4, a/4), w = 0.0218a^4q_0/Et^3$
  - (ii)  $(x, y) = (0, 0), w = 0.0388a^4q_0/Et^3$
  - (iii)  $(x, y) = (a/3, a/3), w = 0.012a^4q_0/Et^3$

**P.7.12.** A rectangular plate  $a \times b$ , simply supported along each edge, possesses a small initial curvature in its unloaded state given by

$$w_0 = A_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

Determine, using the energy method, its final deflected shape when it is subjected to a compressive load  $N_x$  per unit length along the edges  $x = 0, x = a$ .

Answer:

$$w = \frac{A_{11}}{\left[ 1 - \frac{N_x a^2}{\pi^2 D} \left/ \left( 1 + \frac{a^2}{b^2} \right)^2 \right. \right]} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

# Columns

# 8

A large proportion of an aircraft's structure consists of thin webs stiffened by slender longerons or stringers. Both are susceptible to failure by buckling at a buckling stress or critical stress, which is frequently below the limit of proportionality and seldom appreciably above the yield stress of the material. Clearly, for this type of structure, buckling is the most critical mode of failure, so that the prediction of buckling loads of columns, thin plates, and stiffened panels is extremely important in aircraft design. In this chapter, we consider the buckling failure of all these structural elements and also the flexural-torsional failure of thin-walled open tubes of low torsional rigidity.

Two types of structural instability arise: primary and secondary. The former involves the complete element, there being no change in cross-sectional area while the wavelength of the buckle is of the same order as the length of the element. Generally, solid and thick-walled columns experience this type of failure. In the latter mode, changes in cross-sectional area occur and the wavelength of the buckle is of the order of the cross-sectional dimensions of the element. Thin-walled columns and stiffened plates may fail in this manner.

## 8.1 EULER BUCKLING OF COLUMNS

The first significant contribution to the theory of the buckling of columns was made as early as 1744 by Euler. His classical approach is still valid, and likely to remain so, for slender columns possessing a variety of end restraints. Our initial discussion is therefore a presentation of the Euler theory for the small elastic deflection of perfect columns. However, we investigate first the nature of buckling and the difference between theory and practice.

It is common experience that if an increasing axial compressive load is applied to a slender column, there is a value of the load at which the column will suddenly bow or buckle in some unpredetermined direction. This load is patently the buckling load of the column or something very close to the buckling load. Clearly, this displacement implies a degree of asymmetry in the plane of the buckle caused by geometrical or material imperfections of the column and its load. However, in our theoretical stipulation of a perfect column in which the load is applied precisely along the perfectly straight centroidal axis, there is perfect symmetry, so that, theoretically, there can be no sudden bowing or buckling. We therefore require a precise definition of the buckling load that may be used in our analysis of the perfect column.

If the perfect column of Fig. 8.1 is subjected to a compressive load  $P$ , only shortening of the column occurs, no matter what the value of  $P$ . However, if the column is displaced a small amount by a lateral load  $F$ , then, at values of  $P$  below the critical or buckling load,  $P_{CR}$ , removal of  $F$  results in a return of the column to its undisturbed position, indicating a state of stable equilibrium. At the critical load, the displacement does not disappear and, in fact, the column remains in any displaced position as long as

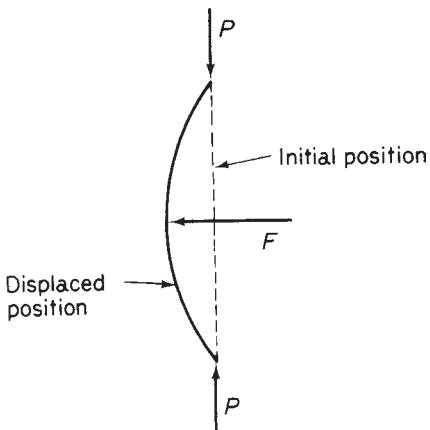


FIGURE 8.1 Definition of Buckling Load for a Perfect Column

the displacement is small. Thus, the buckling load  $P_{CR}$  is associated with a state of *neutral equilibrium*. For  $P > P_{CR}$ , enforced lateral displacements increase and the column is unstable.

Consider the pin-ended column AB of Fig. 8.2. We assume that it is in the displaced state of neutral equilibrium associated with buckling, so that the compressive load  $P$  has attained the critical value  $P_{CR}$ . Simple bending theory (see Chapter 16) gives

$$EI \frac{d^2v}{dz^2} = -M$$

or

$$EI \frac{d^2v}{dz^2} = -P_{CR}v \quad (8.1)$$

so that the differential equation of bending of the column is

$$\frac{d^2v}{dz^2} + \frac{P_{CR}}{EI}v = 0 \quad (8.2)$$

The well-known solution of Eq. (8.2) is

$$v = A \cos \mu z + B \sin \mu z \quad (8.3)$$

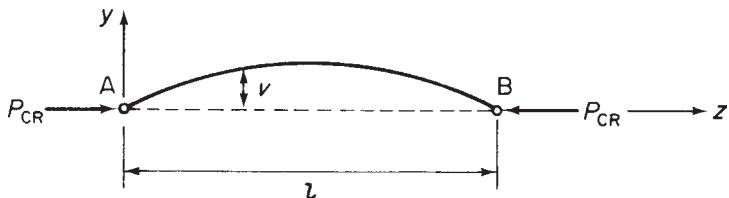


FIGURE 8.2 Determination of Buckling Load for a Pin-Ended Column

where  $\mu^2 = P_{CR}/EI$  and  $A$  and  $B$  are unknown constants. The boundary conditions for this particular case are  $v = 0$  at  $z = 0$  and  $l$ . Therefore,  $A = 0$  and

$$B \sin \mu l = 0$$

For a nontrivial solution (i.e.,  $v \neq 0$ ),

$$\sin \mu l = 0 \text{ or } \mu l = n\pi, \text{ where } n = 1, 2, 3, \dots$$

giving

$$\frac{P_{CR}l^2}{EI} = n^2\pi^2$$

or

$$P_{CR} = \frac{n^2\pi^2EI}{l^2} \quad (8.4)$$

Note that Eq. (8.3) cannot be solved for  $v$  no matter how many of the available boundary conditions are inserted. This is to be expected, since the neutral state of equilibrium means that  $v$  is indeterminate.

The smallest value of buckling load, in other words the smallest value of  $P$  that can maintain the column in a neutral equilibrium state, is obtained by substituting  $n = 1$  in Eq. (8.4). Hence,

$$P_{CR} = \frac{\pi^2EI}{l^2} \quad (8.5)$$

Other values of  $P_{CR}$  corresponding to  $n = 2, 3, \dots$ , are

$$P_{CR} = \frac{4\pi^2EI}{l^2}, \frac{9\pi^2EI}{l^2}, \dots$$

These higher values of buckling load cause more complex modes of buckling such as those shown in Fig. 8.3. The different shapes may be produced by applying external restraints to a very slender column at the points of contraflexure to prevent lateral movement. If no restraints are provided, then these forms of buckling are unstable and have little practical meaning.

The critical stress,  $\sigma_{CR}$ , corresponding to  $P_{CR}$ , is, from Eq. (8.5),

$$\sigma_{CR} = \frac{\pi^2E}{(l/r)^2} \quad (8.6)$$

where  $r$  is the radius of gyration of the cross-sectional area of the column ( $I = Ar^2$ ). The term  $l/r$  is known as the *slenderness ratio* of the column. For a column that is not doubly symmetrical,  $r$  is the

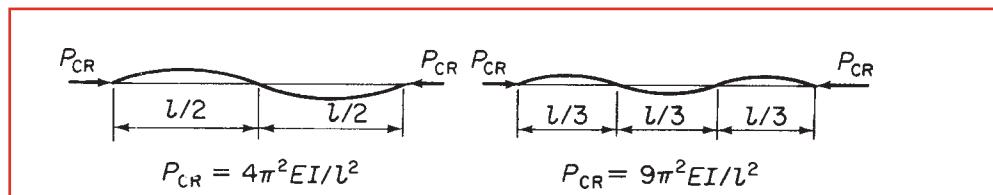


FIGURE 8.3 Buckling Loads for Different Buckling Modes of a Pin-Ended Column

least radius of gyration of the cross-section, since the column bends about an axis about which the flexural rigidity  $EI$  is least. Alternatively, if buckling is prevented in all but one plane, then  $EI$  is the flexural rigidity in that plane.

Equations (8.5) and (8.6) may be written in the form

$$P_{CR} = \frac{\pi^2 EI}{l_e^2} \quad (8.7)$$

and

$$\sigma_{CR} = \frac{\pi^2 E}{(l_e/r)^2} \quad (8.8)$$

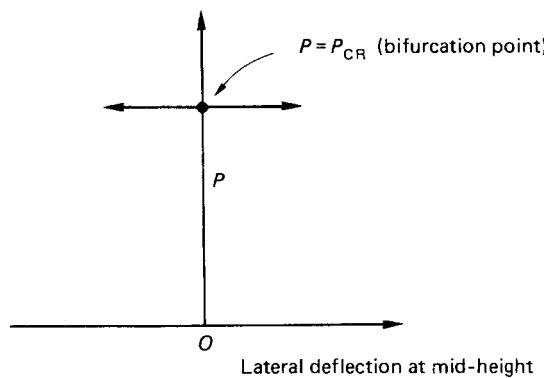
where  $l_e$  is the *effective length* of the column. This is the length of a *pin-ended column* that has the same critical load as that of a column of length  $l$  but with different end conditions. The determination of critical load and stress is carried out in an identical manner to that for the pin-ended column, except that the boundary conditions are different in each case. Table 8.1 gives the solution in terms of effective length for columns having a variety of end conditions. In addition, the boundary conditions referred to the coordinate axes of Fig. 8.2 are quoted. The last case in Table 8.1 involves the solution of a transcendental equation; this is most readily accomplished by a graphical method.

Let us now examine the buckling of the perfect pin-ended column of Fig. 8.2 in greater detail. We showed, in Eq. (8.4), that the column buckles at *discrete* values of axial load and that associated with each value of buckling load is a particular buckling mode (Fig. 8.3). These discrete values of buckling load are called *eigenvalues*, their associated functions (in this case  $v = B \sin n\pi z/l$ ) are called *eigenfunctions*, and the problem itself is called an *eigenvalue problem*.

Further, suppose that the lateral load  $F$  in Fig. 8.1 is removed. Since the column is perfectly straight, homogeneous, and loaded exactly along its axis, it suffers only axial compression as  $P$  is increased. This situation, theoretically, continues until yielding of the material of the column occurs. However, as we have seen, for values of  $P$  below  $P_{CR}$ , the column is in stable equilibrium, whereas for  $P > P_{CR}$ , the column is unstable. A plot of load against lateral deflection at mid-height therefore has the form shown in Fig. 8.4, where, at the point  $P = P_{CR}$ , it is theoretically possible for the column to take one of three deflection paths. Thus, if the column remains undisturbed, the deflection at mid-height continues to be zero but unstable (i.e., the trivial solution of Eq. (8.3),  $v = 0$ ), or if disturbed, the column buckles in either of two lateral directions; the point at which this possible branching occurs is called a *bifurcation point*; further bifurcation points occur at the higher values of  $P_{CR}(4\pi^2 EI/l^2, 9\pi^2 EI/l^2, \dots)$ .

**Table 8.1** Column Length Solutions

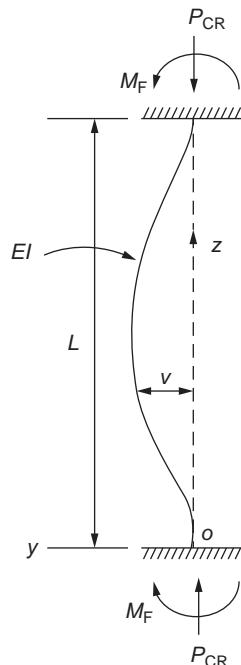
Ends	$l_e/l$	Boundary conditions
Both pinned	1.0	$v = 0$ at $z = 0$ and $l$
Both fixed	0.5	$v = 0$ at $z = 0$ and $z = l$ , $dv/dz = 0$ at $z = l$
One fixed, the other free	2.0	$v = 0$ and $dv/dz = 0$ at $z = 0$
One fixed, the other pinned	0.6998	$dv/dz = 0$ at $z = 0$ , $v = 0$ at $z = l$ and $z = 0$



**FIGURE 8.4 Behavior of a Perfect Pin-Ended Column**

### Example 8.1

A column of length  $L$  and flexural rigidity  $EI$  is firmly built in at each end as shown in Fig. 8.5. Determine the smallest value of axial load that will cause the column to buckle.



**FIGURE 8.5 Buckling of a Slender Column with Fixed Ends**

Since the ends of the column are built in fixing moments  $M_F$  are induced as shown. Then, at any section Z, the bending moment  $M$  is given by

$$M = P_{CR} v - M_F$$

By comparison with Eq. (8.1),

$$EI \frac{d^2v}{dz^2} = -P_{CR} v + M_F$$

Rearranging,

$$\frac{d^2v}{dz^2} + \frac{P_{CR}}{EI} v = \frac{M_F}{EI} \quad (i)$$

The solution of Eq. (i) is of standard form and is

$$v = A \cos \mu z + B \sin \mu z + \frac{M_F}{P_{CR}} \quad (ii)$$

$$\text{where } \mu^2 = \frac{P_{CR}}{EI}.$$

The boundary conditions are; when  $z = 0, v = 0$  and when  $z = L, v = 0$ . From the first of these,  $A = -M_F/P_{CR}$ , and from the second,

$$0 = \frac{-M_F}{P_{CR}} \cos \mu L + B \sin \mu L + \frac{M_F}{P_{CR}}$$

which gives

$$B = \frac{-M_F}{P_{CR}} \frac{(1 - \cos \mu L)}{\sin \mu L}$$

Then, Eq. (ii) becomes

$$v = \frac{-M_F}{P_{CR}} \left[ \cos \mu z + \frac{(1 - \cos \mu L)}{\sin \mu L} \sin \mu z - 1 \right] \quad (iii)$$

Note that  $v$  is indeterminate, since  $M_F$  cannot be found. Also, since  $dv/dz = 0$  when  $z = L$ , we have, from Eq. (iii),

$$0 = 1 - \cos \mu L$$

so that

$$\cos \mu L = 1$$

and

$$\mu L = n\pi \quad \text{where } n = 0, 2, 4, \dots$$

For a nontrivial solution, i.e.,  $n \neq 0$ , and taking the smallest value of buckling load ( $n = 2$ ),

$$P_{CR} = \frac{4\pi^2 EI}{L^2} \quad (iv)$$

**Example 8.2**

A uniform column of length  $L$  and flexural stiffness  $EI$  is simply supported at its ends and has an additional elastic support at mid-span. This support is such that, if a lateral displacement  $v_c$  occurs at this point, a restoring force  $kv_c$  is generated at the point. Derive an equation giving the buckling load of the column. If the buckling load is  $4\pi^2EI/L^2$ , find the value of  $k$ . Also, if the elastic support is infinitely stiff, show that the buckling load is given by the equation  $\tan \lambda L/2 = \lambda L/2$ , where  $\lambda = \sqrt{P/EI}$ .

The column is shown in its displaced position in Fig. 8.6. The bending moment at any section in the left-hand half of the column is given by

$$M = Pv - \frac{kv_c}{2}z$$

so that, by comparison with Eq. (8.1),

$$EI \frac{d^2v}{dz^2} = -Pv + \frac{kv_c}{2}z$$

giving

$$\frac{d^2v}{dz^2} + \lambda^2 v = \frac{kv_c}{2EI}z \quad (i)$$

The solution of Eq. (i) is of standard form and is

$$v = A \cos \lambda z + B \sin \lambda z + \frac{kv_c}{2P}z$$

The constants  $A$  and  $B$  are found using the boundary conditions of the column, which are  $v = 0$  when  $z = 0$ ,  $v = v_c$ , when  $z = L/2$  and  $(dv/dz) = 0$  when  $z = L/2$ .

From the first of these,  $A = 0$ , while from the second,

$$B = v_c(1 - kL/4P)/\sin(\lambda L/2)$$

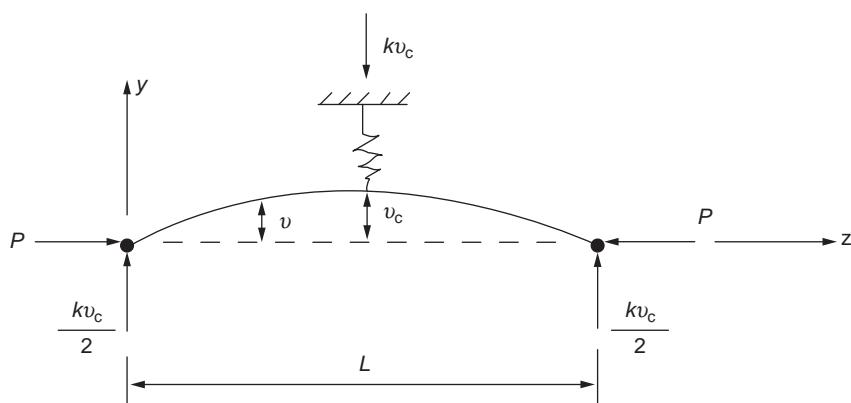


FIGURE 8.6 Column of Example 8.2

The third boundary condition gives, since  $v_c \neq 0$ , the required equation; that is,

$$\left(1 - \frac{kL}{4P}\right) \cos \frac{\lambda L}{2} + \frac{k}{2P\lambda} \sin \frac{\lambda L}{2} = 0$$

Rearranging,

$$P = \frac{kL}{4} \left(1 - \frac{\tan(\lambda L/2)}{\lambda L/2}\right)$$

If  $P$  (buckling load) =  $4\pi^2 EI/L^2$ , then  $\lambda L/2 = \pi$ , so that  $k = 4P/L$ . Finally, if  $k \rightarrow \infty$ ,

$$\tan \frac{\lambda L}{2} = \frac{\lambda L}{2} \quad (\text{ii})$$

Note that Eq. (ii) is the transcendental equation that would be derived when determining the buckling load of a column of length  $L/2$ , built in at one end, and pinned at the other.

### Example 8.2 MATLAB

Repeat the derivation of the column buckling load in Example 8.2 using MATLAB.

The expression for the column buckling load ( $P$ ) is obtained through the following MATLAB file:

```
% Declare any needed variables
syms M P v k v_c z EI lambda L A B C PI
P_buck = EI/(L^2)*4*PI^2;
lambda_sq = P_buck/EI;

% Define the bending moment equation at any section in the column
M = P*v - k*v_c*z/2;

% Substituting M into Eq. (8.1) and solving the second order differential
% equation results in the following general solution for v
C = solve(EI*diff(C,2)+subs(M,v,C),C); %Equation 8.1
v = A*cos(lambda*z) + B*sin(lambda*z) + C; % General solution of Eq.(8.1)

% Check boundary conditions to solve for A and B and P
% Boundary Condition #1: v = 0 when z = 0
A_val = solve(subs(v,z,0),A);
v = subs(v,A,A_val);

% Boundary Condition #2: v = v_c when z = L/2
B_val = solve(subs(v,z,L/2)-v_c,B);
v = subs(v,B,B_val);

% Boundary Condition #3: v_z = 0 when z = L/2
P = solve(subs(diff(v,z),z,L/2),P);
```

```
% Output a simplified expression of P to the Command Window
disp(['P =' char(simplify(P))])
```

The Command Window output resulting from this MATLAB file is as follows:

```
P = (L*k)/4 - (k*tan((L*lambda)/2))/(2*lambda)
```

## 8.2 INELASTIC BUCKLING

We have shown that the critical stress, Eq. (8.8), depends only on the elastic modulus of the material of the column and the slenderness ratio  $l/r$ . For a given material, the critical stress increases as the slenderness ratio decreases, that is, as the column becomes shorter and thicker. A point is reached when the critical stress is greater than the yield stress of the material, so that Eq. (8.8) is no longer applicable. For mild steel, this point occurs at a slenderness ratio of approximately 100, as shown in Fig. 8.7. We therefore require some alternative means of predicting column behavior at low values of the slenderness ratio.

It was assumed in the derivation of Eq. (8.8) that the stresses in the column remain within the elastic range of the material, so that the modulus of elasticity  $E (= d\sigma/d\varepsilon)$  was constant. Above the elastic limit,  $d\sigma/d\varepsilon$  depends upon the value of stress and whether the stress is increasing or decreasing. Thus, in Fig. 8.8, the elastic modulus at the point A is the tangent modulus  $E_t$  if the stress is increasing but  $E$  if the stress is decreasing.

Consider a column having a plane of symmetry and subjected to a compressive load  $P$  such that the direct stress in the column  $P/A$  is above the elastic limit. If the column is given a small deflection,  $v$ , in its plane of symmetry, then the stress on the concave side increases while the stress on the convex side decreases. Thus, in the cross-section of the column shown in Fig. 8.9(a) the compressive stress

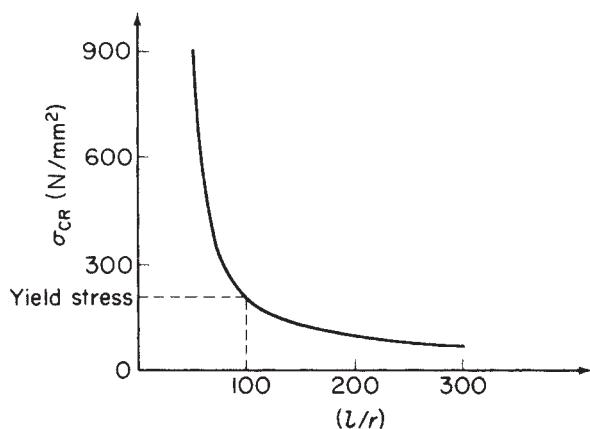


FIGURE 8.7 Critical Stress-Slenderness Ratio for a Column

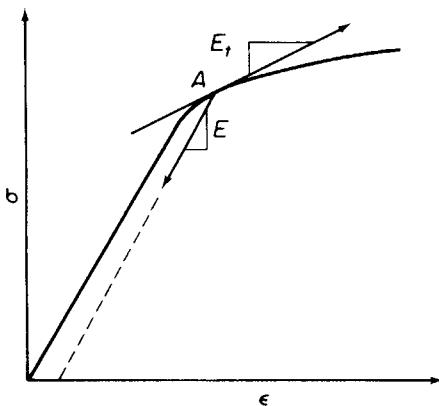


FIGURE 8.8 Elastic Moduli for a Material Stressed Above the Elastic Limit

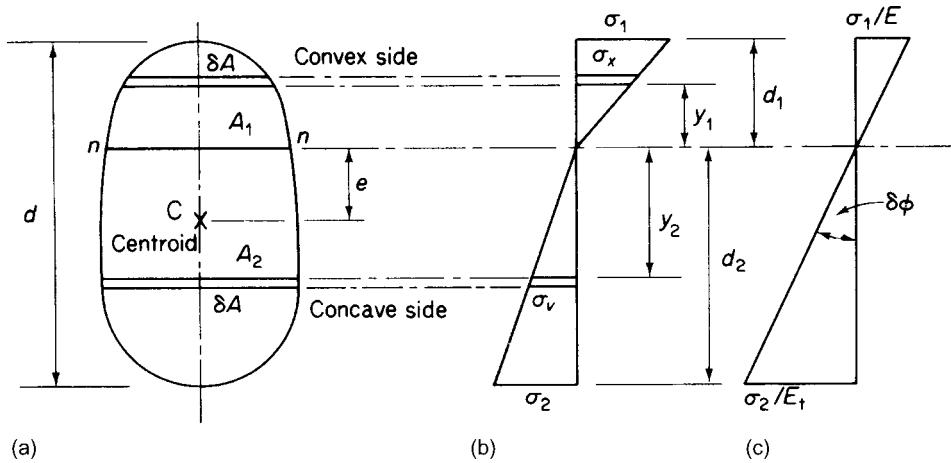


FIGURE 8.9 Determination of Reduced Elastic Modulus

decreases in the area  $A_1$  and increases in the area  $A_2$ , while the stress on the line  $nn$  is unchanged. Since these changes take place outside the elastic limit of the material, we see, from our remarks in the previous paragraph, that the modulus of elasticity of the material in the area  $A_1$  is  $E$  while that in  $A_2$  is  $E_t$ . The homogeneous column now behaves as if it were nonhomogeneous, with the result that the stress distribution is changed to the form shown in Fig. 8.9(b); the linearity of the distribution follows from an assumption that plane sections remain plane.

As the axial load is unchanged by the disturbance,

$$\int_0^{d_1} \sigma_x dA = \int_0^{d_2} \sigma_v dA \quad (8.9)$$

Also,  $P$  is applied through the centroid of each end section a distance  $e$  from  $nn$ , so that

$$\int_0^{d_1} \sigma_x(y_1 + e) dA + \int_0^{d_2} \sigma_v(y_2 - e) dA = -Pv \quad (8.10)$$

From Fig. 8.9(b),

$$\sigma_x = \frac{\sigma_1}{d_1} y_1, \quad \sigma_v = \frac{\sigma_2}{d_2} y_2 \quad (8.11)$$

The angle between two close, initially parallel sections of the column is equal to the change in slope  $d^2v/dz^2$  of the column between the two sections. This, in turn, must be equal to the angle  $\delta\phi$  in the strain diagram of Fig. 8.9(c). Hence,

$$\frac{d^2v}{dz^2} = \frac{\sigma_1}{Ed_1} = \frac{\sigma_2}{E_t d_2} \quad (8.12)$$

and Eq. (8.9) becomes, from Eqs. (8.11) and (8.12),

$$E \frac{d^2v}{dz^2} \int_0^{d_1} y_1 dA - E_t \frac{d^2v}{dz^2} \int_0^{d_2} y_2 dA = 0 \quad (8.13)$$

Further, in a similar manner, from Eq. (8.10),

$$\frac{d^2v}{dz^2} \left( E \int_0^{d_1} y_1^2 dA + E_t \int_0^{d_2} y_2^2 dA \right) + e \frac{d^2v}{dz^2} \left( E \int_0^{d_1} y_1 dA - E_t \int_0^{d_2} y_2 dA \right) = -Pv \quad (8.14)$$

The second term on the left-hand side of Eq. (8.14) is zero, from Eq. (8.13). Therefore, we have

$$\boxed{\frac{d^2v}{dz^2} (EI_1 + E_t I_2) = -Pv} \quad (8.15)$$

in which

$$I_1 = \int_0^{d_1} y_1^2 dA \quad \text{and} \quad I_2 = \int_0^{d_2} y_2^2 dA$$

the second moments of area about  $nn$  of the convex and concave sides of the column, respectively. Putting

$$E_r I = EI_1 + E_t I_2$$

or

$$\boxed{E_r = E \frac{I_1}{I} + E_t \frac{I_2}{I}} \quad (8.16)$$

where  $E_r$ , known as the *reduced modulus*, gives

$$E_r \frac{d^2v}{dz^2} + Pv = 0$$

Comparing this with Eq. (8.2), we see that, if  $P$  is the critical load  $P_{\text{CR}}$ , then

$$P_{\text{CR}} = \frac{\pi^2 E_r I}{l_e^2} \quad (8.17)$$

and

$$\sigma_{\text{CR}} = \frac{\pi^2 E_r}{(l_e/r)^2} \quad (8.18)$$

This method for predicting critical loads and stresses outside the elastic range is known as the *reduced modulus theory*. From Eq. (8.13), we have

$$E \int_0^{d_1} y_1 \, dA - E_t \int_0^{d_2} y_2 \, dA = 0 \quad (8.19)$$

which, together with the relationship  $d = d_1 + d_2$ , enables the position of  $nn$  to be found.

It is possible that the axial load  $P$  is increased at the time of the lateral disturbance of the column such that no strain reversal occurs on its convex side. The compressive stress therefore increases over the complete section so that the tangent modulus applies over the whole cross-section. The analysis is then the same as that for column buckling within the elastic limit except that  $E_t$  is substituted for  $E$ . Hence, the *tangent modulus theory* gives

$$P_{\text{CR}} = \frac{\pi^2 E_t I}{l_e^2} \quad (8.20)$$

and

$$\sigma_{\text{CR}} = \frac{\pi^2 E_t}{(l_e/r)^2} \quad (8.21)$$

By a similar argument, a reduction in  $P$  could result in a decrease in stress over the whole cross-section. The elastic modulus applies in this case and the critical load and stress are given by the standard Euler theory, namely Eqs. (8.7) and (8.8).

In Eq. (8.16),  $I_1$  and  $I_2$  are together greater than  $I$  while  $E$  is greater than  $E_t$ . It follows that the reduced modulus  $E_r$  is greater than the tangent modulus  $E_t$ . Consequently, buckling loads predicted by the reduced modulus theory are greater than buckling loads derived from the tangent modulus theory, so that, although we specified theoretical loading situations where the different theories apply, there remains the difficulty of deciding which should be used for design purposes.

Extensive experiments carried out on aluminum alloy columns by the aircraft industry in the 1940s showed that the actual buckling load was approximately equal to the tangent modulus load. Shanley (1947) explained that, for columns with small imperfections, an increase of axial load and bending occur simultaneously. He then showed analytically that, after the tangent modulus load is reached, the strain on the concave side of the column increases rapidly while that on the convex side decreases slowly. The large deflection corresponding to the rapid strain increase on the concave side, which occurs soon after the tangent modulus load is passed, means that it is possible to exceed the tangent

modulus load by only a small amount. It follows that the buckling load of columns is given most accurately for practical purposes by the tangent modulus theory.

Empirical formulae have been used extensively to predict buckling loads, although in view of the close agreement between experiment and the tangent modulus theory, they would appear unnecessary. Several formulae are in use; for example, the *Rankine*, *Straight-line*, and *Johnson's parabolic* formulae are given in many books on elastic stability.<sup>1</sup>

### Example 8.3

A short column of rectangular cross-section, width  $b$  and depth  $d$ , is fabricated from material having a Young's modulus  $E$  and a tangent modulus  $E_t$ . Determine, in terms of  $E$  and  $E_t$ , an expression for the reduced modulus  $E_r$ .

For a rectangular section of width  $b$ , Eq. (8.19) becomes

$$E \int_0^{d_1} b y_1 dy - E_t \int_0^{d_2} b y_2 dy = 0$$

Integrating and substituting the limits gives

$$Ed_1^2 - E_t d_2^2 = 0 \quad (\text{i})$$

Substituting in Eq. (i) for  $d_2 (=d - d_1)$ , we obtain

$$d_1^2(E - E_t) + 2E_t d d_1 - E_t d^2 = 0$$

Using the standard formula for the solution of quadratic equations, we obtain

$$d_1 = \frac{-2E_t d \pm \sqrt{4E_t^2 d^2 + 4(E - E_t)E_t d^2}}{2(E - E_t)}$$

Simplifying,

$$d_1 = d \frac{(-E_t \pm \sqrt{EE_t})}{(E - E_t)}$$

Since  $d_1$  cannot be negative and since  $E > E_t$ , the negative square root term is neglected so that

$$d_1 = d \frac{(-E_t + \sqrt{EE_t})}{(E - E_t)}$$

This expression is most easily simplified by dividing the denominator by the numerator using the normal rules of algebraic division. Then

$$d_1 = \frac{d}{1 + \sqrt{(E/E_t)}} \quad (\text{ii})$$

It follows that

$$d_2 = \frac{d}{1 + \sqrt{(E_t/E)}} \quad (\text{iii})$$

Now, in Eq. (8.16),

$$I = \frac{bd^3}{12}, I_1 = \frac{bd_1^3}{3}, I_2 = \frac{bd_2^3}{3} \quad (\text{see Section 16.4})$$

Substituting for  $d_1$  and  $d_2$  in these expressions from Eqs. (ii) and (iii), respectively, and then in Eq. (8.16) gives

$$E_r = \frac{4E}{\left(1 + \sqrt{E/E_t}\right)^3} + \frac{4E_t}{\left(1 + \sqrt{E_t/E}\right)^3}$$

that is,

$$E_r = \frac{4E(E_t)^{3/2}}{\left(\sqrt{E_t} + \sqrt{E}\right)^3} + \frac{4E_t(E)^{3/2}}{\left(\sqrt{E} + \sqrt{E_t}\right)^3}$$

which simplifies to

$$E_r = \frac{4EE_t}{\left(\sqrt{E_t} + \sqrt{E}\right)^2} \quad (\text{iv})$$

Note that in the case of elastic failure occurring across the complete cross-section of the column then, in Eq. (iv),  $E = E_t$  and  $E_r = E_t$ . If no elastic failure occurs, then  $E_t = E$  and  $E_r = E$ .

### 8.3 EFFECT OF INITIAL IMPERFECTIONS

Obviously, it is impossible in practice to obtain a perfectly straight homogeneous column and to ensure that it is exactly axially loaded. An actual column may be bent with some eccentricity of load. Such imperfections influence to a large degree the behavior of the column, which, unlike the perfect column, begins to bend immediately the axial load is applied.

Let us suppose that a column, initially bent, is subjected to an increasing axial load  $P$ , as shown in Fig. 8.10. In this case, the bending moment at any point is proportional to the change in curvature of the column from its initial bent position. Thus,

$$EI \frac{d^2v}{dz^2} - EI \frac{d^2v_0}{dz^2} = -Pv \quad (8.22)$$

which, on rearranging, becomes

$$\frac{d^2v}{dz^2} + \lambda^2 v = \frac{d^2v_0}{dz^2} \quad (8.23)$$

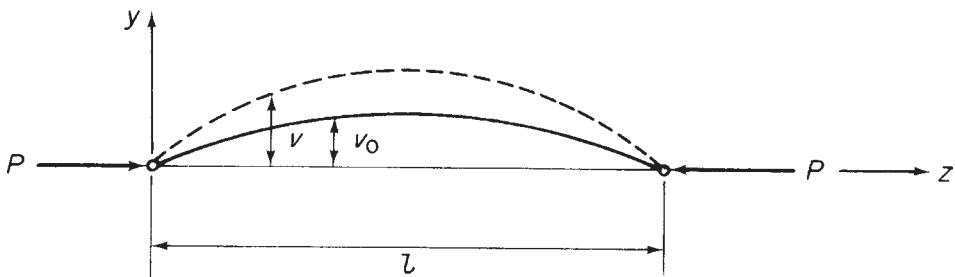


FIGURE 8.10 Initially Bent Column

where  $\lambda^2 = P/EI$ . The final deflected shape,  $v$ , of the column depends upon the form of its unloaded shape,  $v_0$ . Assuming that

$$v_0 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi z}{l} \quad (8.24)$$

and substituting in Eq. (8.23), we have

$$\frac{d^2v}{dz^2} + \lambda^2 v = -\frac{\pi^2}{l^2} \sum_{n=1}^{\infty} n^2 A_n \sin \frac{n\pi z}{l}$$

The general solution to this equation is

$$v = B \cos \lambda z + D \sin \lambda z + \sum_{n=1}^{\infty} \frac{n^2 A_n}{n^2 - \alpha} \sin \frac{n\pi z}{l}$$

where  $B$  and  $D$  are constants of integration and  $\alpha = \lambda^2 l^2 / \pi^2$ . The boundary conditions are  $v = 0$  at  $z = 0$  and  $l$ , giving  $B = D = 0$ , from which

$$v = \sum_{n=1}^{\infty} \frac{n^2 A_n}{n^2 - \alpha} \sin \frac{n\pi z}{l} \quad (8.25)$$

Note that, in contrast to the perfect column, we are able to obtain a nontrivial solution for deflection. This is to be expected, since the column is in stable equilibrium in its bent position at all values of  $P$ .

An alternative form for  $\alpha$  is

$$\alpha = \frac{Pl^2}{\pi^2 EI} = \frac{P}{P_{CR}}$$

(see Eq. (8.5)). Thus,  $\alpha$  is always less than 1 and approaches unity when  $P$  approaches  $P_{CR}$ , so that the first term in Eq. (8.25) usually dominates the series. A good approximation, therefore, for deflection when the axial load is in the region of the critical load is

$$v = \frac{A_1}{1 - \alpha} \sin \frac{\pi z}{l} \quad (8.26)$$

or, at the center of the column, where  $z = l/2$ ,

$$v = \frac{A_1}{1 - P/P_{CR}} \quad (8.27)$$

in which  $A_1$  is seen to be the initial central deflection. If central deflections  $\delta (= v - A_1)$  are measured from the initially bowed position of the column, then from Eq. (8.27), we obtain

$$\frac{A_1}{1 - P/P_{CR}} - A_1 = \delta$$

which gives, on rearranging,

$$\delta = P_{CR} \frac{\delta}{P} - A_1 \quad (8.28)$$

and we see that a graph of  $\delta$  plotted against  $\delta/P$  has a slope, in the region of the critical load, equal to  $P_{CR}$  and an intercept equal to the initial central deflection. This is the well-known *Southwell plot* for the experimental determination of the elastic buckling load of an imperfect column.

Timoshenko and Gere<sup>1</sup> also showed that Eq. (8.27) may be used for a perfectly straight column with small eccentricities of column load.

### Example 8.4

The pin-jointed column shown in Fig. 8.11 carries a compressive load  $P$  applied eccentrically at a distance  $e$  from the axis of the column. Determine the maximum bending moment in the column.

The bending moment at any section of the column is given by

$$M = P(e + v)$$

Then, by comparison with Eq. (8.1),

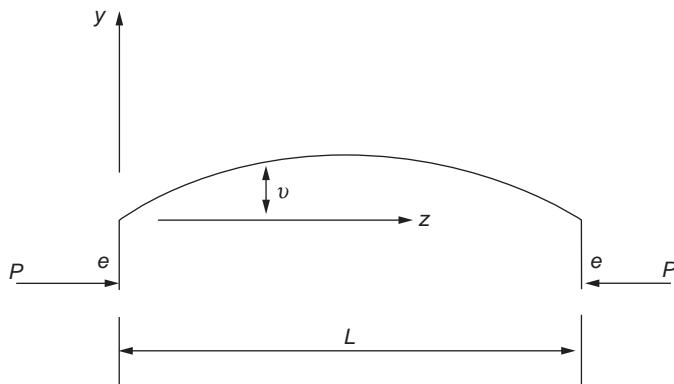
$$EI \frac{d^2v}{dz^2} = -P(e + v)$$

giving

$$\frac{d^2v}{dz^2} + \mu^2 v = -\frac{Pe}{EI} \quad (\mu^2 = P/EI) \quad (i)$$

The solution of Eq. (i) is of standard form and is

$$v = A \cos \mu z + B \sin \mu z - e$$



**FIGURE 8.11 Eccentrically Loaded Column of Example 8.4.**

The boundary conditions are  $v = 0$  when  $z = 0$  and  $(dv/dz) = 0$  when  $z = L/2$ . From the first of these,  $A = e$ , while from the second,

$$B = e \tan \frac{\mu L}{2}$$

The equation for the deflected shape of the column is then

$$v = e \cos \mu z + e \tan \frac{\mu L}{2} \sin \mu z - e$$

or

$$v = \frac{e}{\cos(\mu L/2)} \left( \cos \mu z \cos \frac{\mu L}{2} + \sin \mu z \sin \frac{\mu L}{2} \right) - e$$

which simplifies to

$$v = e \left[ \frac{\cos \mu(z - L/2)}{\cos \mu L/2} - 1 \right]$$

The maximum value of  $v$  occurs at mid-span, where  $z = L/2$ ; that is,

$$v_{\max} = e \left( \sec \frac{\mu L}{2} - 1 \right)$$

The maximum bending moment is given by

$$M(\max) = Pe + Pv_{\max}$$

so that

$$M(\max) = Pe \sec \frac{\mu L}{2}$$

## 8.4 STABILITY OF BEAMS UNDER TRANSVERSE AND AXIAL LOADS

Stresses and deflections in a linearly elastic beam subjected to transverse loads, as predicted by simple beam theory, are directly proportional to the applied loads. This relationship is valid if the deflections are small, such that the slight change in geometry produced in the loaded beam has an insignificant effect on the loads themselves. This situation changes drastically when axial loads act simultaneously with the transverse loads. The internal moments, shear forces, stresses, and deflections then become dependent upon the magnitude of the deflections as well as the magnitude of the external loads. They are also sensitive, as we observed in the previous section, to beam imperfections, such as initial curvature and eccentricity of axial load. Beams supporting both axial and transverse loads are sometimes known as *beam-columns* or simply as *transversely loaded columns*.

We consider first the case of a pin-ended beam carrying a uniformly distributed load of intensity  $w$  per unit length and an axial load  $P$ , as shown in Fig. 8.12. The bending moment at any section of the beam is

$$M = Pv + \frac{wlz}{2} - \frac{wz^2}{2} = -EI \frac{d^2v}{dz^2}$$

giving

$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = \frac{w}{2EI}(z^2 - lz) \quad (8.29)$$

The standard solution to Eq. (8.29) is

$$v = A \cos \lambda z + B \sin \lambda z + \frac{w}{2P} \left( z^2 - lz - \frac{2}{\lambda^2} \right)$$

where  $A$  and  $B$  are unknown constants and  $\lambda^2 = P/EI$ . Substituting the boundary conditions  $v = 0$  at  $z = 0$  and  $l$  gives

$$A = \frac{w}{\lambda^2 P}, \quad B = \frac{w}{\lambda^2 P \sin \lambda l} (l - \cos \lambda l)$$

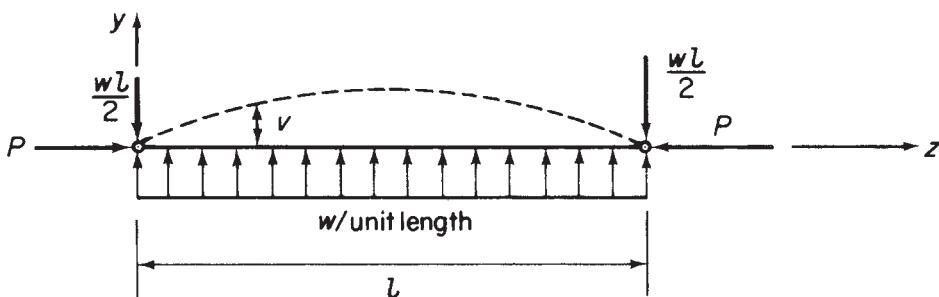


FIGURE 8.12 Bending of a Uniformly Loaded Beam-Column

so that the deflection is determinate for any value of  $w$  and  $P$  and is given by

$$v = \frac{w}{\lambda^2 P} \left[ \cos \lambda z + \left( \frac{1 - \cos \lambda l}{\sin \lambda l} \right) \sin \lambda z \right] + \frac{w}{2P} \left( z^2 - lz - \frac{2}{\lambda^2} \right) \quad (8.30)$$

In beam-columns, as in beams, we are primarily interested in maximum values of stress and deflection. For this particular case, the maximum deflection occurs at the center of the beam and is, after some transformation of Eq. (8.30),

$$v_{\max} = \frac{w}{\lambda^2 P} \left( \sec \frac{\lambda l}{2} - 1 \right) - \frac{wl^2}{8P} \quad (8.31)$$

The corresponding maximum bending moment is

$$M_{\max} = -Pv_{\max} - \frac{wl^2}{8}$$

or, from Eq. (8.31),

$$M_{\max} = \frac{w}{\lambda^2} \left( 1 - \sec \frac{\lambda l}{2} \right) \quad (8.32)$$

We may rewrite Eq. (8.32) in terms of the Euler buckling load  $P_{CR} = \pi^2 EI/l^2$  for a pin-ended column, hence

$$M_{\max} = \frac{wl^2 P_{CR}}{\pi^2 P} \left( 1 - \sec \frac{\pi}{2} \sqrt{\frac{P}{P_{CR}}} \right) \quad (8.33)$$

As  $P$  approaches  $P_{CR}$ , the bending moment (and deflection) becomes infinite. However, this theory is based on the assumption of small deflections (otherwise,  $d^2v/dz^2$  is not a close approximation for curvature), so that such a deduction is invalid. The indication is, though, that large deflections are produced by the presence of a compressive axial load no matter how small the transverse load might be.

Let us consider now the beam-column of Fig. 8.13, with hinged ends carrying a concentrated load  $W$  at a distance  $a$  from the right-hand support. For

$$z \leq l - a, \quad EI \frac{d^2v}{dz^2} = -M = -Pv - \frac{Waz}{l} \quad (8.34)$$

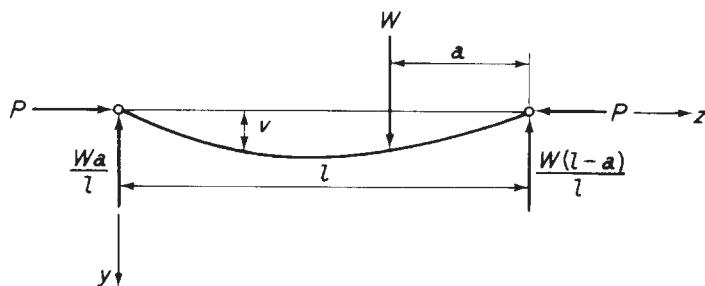


FIGURE 8.13 Beam-Column Supporting A Point Load

and, for

$$z \geq l - a, EI \frac{d^2v}{dz^2} = -M = -Pv - \frac{W}{l}(l - a)(l - z) \quad (8.35)$$

Writing

$$\lambda^2 = \frac{P}{EI}$$

Eq. (8.34) becomes

$$\frac{d^2v}{dz^2} + \lambda^2 v = -\frac{Wa}{Ell} z$$

the general solution of which is

$$v = A \cos \lambda z + B \sin \lambda z - \frac{Wa}{Pl} z \quad (8.36)$$

Similarly, the general solution of Eq. (8.35) is

$$v = C \cos \lambda z + D \sin \lambda z - \frac{W}{Pl}(l - a)(l - z) \quad (8.37)$$

where  $A$ ,  $B$ ,  $C$ , and  $D$  are constants, which are found from the boundary conditions as follows.

When  $z = 0$ ,  $v = 0$ , therefore, from Eq. (8.36),  $A = 0$ . At  $z = l$ ,  $v = 0$ , giving, from Eq. (8.37),  $C = -D \tan \lambda l$ . At the point of application of the load, the deflection and slope of the beam given by Eqs (8.36) and (8.37) must be the same. Hence, equating deflections,

$$B \sin \lambda(l - a) - \frac{Wa}{Pl}(l - a) = D[\sin \lambda(l - a) - \tan \lambda l \cos \lambda(l - a)] - \frac{Wa}{Pl}(l - a)$$

that is,

$$B = D \left[ 1 - \frac{\tan \lambda l}{\tan \lambda(l - a)} \right]$$

and, equating slopes,

$$B \lambda \cos \lambda(l - a) - \frac{Wa}{Pl} = D \lambda [\cos \lambda(l - a) + \tan \lambda l \sin \lambda(l - a)] + \frac{W}{Pl}(l - a)$$

that is,

$$B = D \lambda \left[ 1 + \frac{\tan \lambda l \sin \lambda(l - a)}{\cos \lambda(l - a)} \right] + \frac{Wl}{Pl \cos \lambda(l - a)}$$

Now equating the two expressions for  $B$  and expressing all trigonometrical terms in the form of sines and cosines,

$$D \left[ 1 - \frac{\sin \lambda l \cos \lambda(l - a)}{\cos \lambda l \sin \lambda(l - a)} - 1 + \frac{\sin \lambda l \sin \lambda(l - a)}{\cos \lambda l \cos \lambda(l - a)} \right] = \frac{W}{P \lambda \cos \lambda(l - a)}$$

or

$$D \frac{\sin \lambda l}{\cos \lambda l} \left[ \frac{\cos \lambda(l-a)}{\sin \lambda(l-a)} + \frac{\sin \lambda(l-a)}{\cos \lambda(l-a)} \right] = \frac{W}{P \lambda \cos \lambda(l-a)}$$

Since  $\cos^2 \lambda(l-a) + \sin^2 \lambda(l-a) = 1$ , the above reduces to

$$\frac{-D \sin \lambda l}{\cos \lambda l \sin \lambda(l-a)} = \frac{W}{P \lambda}$$

which gives

$$D = -\frac{W \sin \lambda(l-a) \cos \lambda l}{P \lambda \sin \lambda l}$$

Now substituting for  $D$  in the first of the expressions for  $B$  have

$$B = \frac{W \sin \lambda a}{P \lambda \sin \lambda l}$$

The deflected shape of the column is then

$$v = \frac{W \sin \lambda a}{P \lambda \sin \lambda l} \sin \lambda z - \frac{Wa}{Pl} z, \quad \text{for } z \leq l-a \quad (8.38)$$

$$v = \frac{W \sin \lambda(l-a)}{P \lambda \sin \lambda l} \sin \lambda(l-z) - \frac{W}{Pl}(l-a)(l-z), \quad \text{for } z \geq l-a \quad (8.39)$$

These equations for the beam-column deflection enable the bending moment and resulting bending stresses to be found at all sections.

A particular case arises when the load is applied at the center of the span. The deflection curve is then symmetrical with a maximum deflection under the load of

$$v_{\max} = \frac{W}{2P\lambda} \tan \frac{\lambda l}{2} - \frac{Wl}{4P}$$

Finally, we consider a beam-column subjected to end moments  $M_A$  and  $M_B$  in addition to an axial load  $P$  (Fig. 8.14). The deflected form of the beam-column may be found by using the principle of superposition and the results of the previous case. First, we imagine that  $M_B$  acts alone with the axial load  $P$ . If we assume that the point load  $W$  moves towards  $B$  and simultaneously increases so that the product  $Wa = \text{constant} = M_B$ , then, in the limit as  $a$  tends to zero, we have the moment  $M_B$  applied at  $B$ .

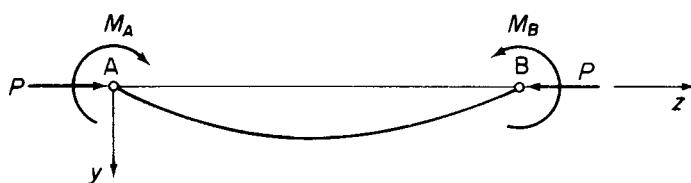


FIGURE 8.14 Beam-Column Supporting End Moments

The deflection curve is then obtained from Eq. (8.38) by substituting  $\lambda a$  for  $\sin \lambda a$  (since  $\lambda a$  is now very small) and  $M_B$  for  $Wa$ :

$$v = \frac{M_B}{P} \left( \frac{\sin \lambda z}{\sin \lambda l} - \frac{z}{l} \right) \quad (8.40)$$

In a similar way, we find the deflection curve corresponding to  $M_A$  acting alone. Suppose that  $W$  moves toward A such that the product  $W(l - a) = \text{constant} = M_A$ . Then, as  $(l - a)$  tends to zero, we have  $\sin \lambda(l - a) = \lambda(l - a)$ , and Eq. (8.39) becomes

$$v = \frac{M_A}{P} \left[ \frac{\sin \lambda(l - z)}{\sin \lambda l} - \frac{(l - z)}{l} \right] \quad (8.41)$$

The effect of the two moments acting simultaneously is obtained by superposition of the results of Eqs. (8.40) and (8.41). Hence, for the beam-column of Fig. 8.14,

$$v = \frac{M_B}{P} \left( \frac{\sin \lambda z}{\sin \lambda l} - \frac{z}{l} \right) + \frac{M_A}{P} \left[ \frac{\sin \lambda(l - z)}{\sin \lambda l} - \frac{(l - z)}{l} \right] \quad (8.42)$$

Equation (8.42) is also the deflected form of a beam-column supporting eccentrically applied end loads at A and B. For example, if  $e_A$  and  $e_B$  are the eccentricities of  $P$  at the ends A and B, respectively, then  $M_A = Pe_A$ ,  $M_B = Pe_B$ , giving a deflected form of

$$v = e_B \left( \frac{\sin \lambda z}{\sin \lambda l} - \frac{z}{l} \right) + e_A \left[ \frac{\sin \lambda(l - z)}{\sin \lambda l} - \frac{(l - z)}{l} \right] \quad (8.43)$$

Other beam-column configurations featuring a variety of end conditions and loading regimes may be analyzed by a similar procedure.

## 8.5 ENERGY METHOD FOR THE CALCULATION OF BUCKLING LOADS IN COLUMNS

The fact that the total potential energy of an elastic body possesses a stationary value in an equilibrium state may be used to investigate the neutral equilibrium of a buckled column. In particular, the energy method is extremely useful when the deflected form of the buckled column is unknown and has to be guessed.

First, we consider the pin-ended column shown in its buckled position in Fig. 8.15. The internal or strain energy  $U$  of the column is assumed to be produced by bending action alone and is given by the well-known expression

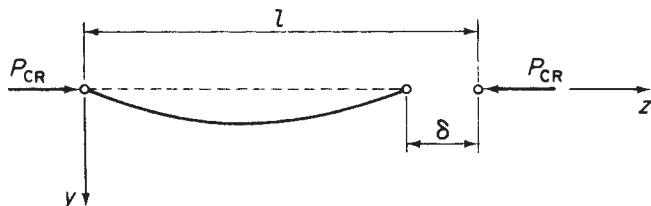


FIGURE 8.15 Shortening of a Column Due to Buckling

$$U = \int_0^l \frac{M^2}{2EI} dz \quad (8.44)$$

or alternatively, since  $EI d^2v/dz^2 = -M$ ,

$$U = \frac{EI}{2} \int_0^l \left( \frac{d^2v}{dz^2} \right)^2 dz \quad (8.45)$$

The potential energy  $V$  of the buckling load  $P_{CR}$ , referred to the straight position of the column as the datum, is then

$$V = -P_{CR}\delta$$

where  $\delta$  is the axial movement of  $P_{CR}$  caused by the bending of the column from its initially straight position. By reference to Fig. 7.16(b) and Eq. (7.41), we see that

$$\delta = \frac{1}{2} \int_0^l \left( \frac{dv}{dz} \right)^2 dz$$

giving

$$V = -\frac{P_{CR}}{2} \int_0^l \left( \frac{dv}{dz} \right)^2 dz \quad (8.46)$$

The total potential energy of the column in the neutral equilibrium of its buckled state is therefore

$$U + V = \int_0^l \frac{M^2}{2EI} dz - \frac{P_{CR}}{2} \int_0^l \left( \frac{dv}{dz} \right)^2 dz \quad (8.47)$$

or, using the alternative form of  $U$  from Eq. (8.45),

$$U + V = \frac{EI}{2} \int_0^l \left( \frac{d^2v}{dz^2} \right)^2 dz - \frac{P_{CR}}{2} \int_0^l \left( \frac{dv}{dz} \right)^2 dz \quad (8.48)$$

We saw in Chapter 7 that exact solutions of plate bending problems are obtainable by energy methods when the deflected shape of the plate is known. An identical situation exists in the determination of critical loads for column and thin-plate buckling modes. For the pin-ended column under discussion, a deflected form of

$$v = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi z}{l} \quad (8.49)$$

satisfies the boundary conditions of

$$(v)_{z=0} = (v)_{z=l} = 0, \quad \left( \frac{d^2v}{dz^2} \right)_{z=0} = \left( \frac{d^2v}{dz^2} \right)_{z=l} = 0$$

and is capable, within the limits for which it is valid and if suitable values for the constant coefficients  $A_n$  are chosen, of representing any continuous curve. We are therefore in a position to find  $P_{CR}$  exactly. Substituting Eq. (8.49) into Eq. (8.48) gives

$$U + V = \frac{EI}{2} \int_0^l \left( \frac{\pi}{l} \right)^4 \left( \sum_{n=1}^{\infty} n^2 A_n \sin \frac{n\pi z}{l} \right)^2 dz - \frac{P_{CR}}{2} \int_0^l \left( \frac{\pi}{l} \right)^2 \left( \sum_{n=1}^{\infty} n A_n \cos \frac{n\pi z}{l} \right)^2 dz \quad (8.50)$$

The product terms in both integrals of Eq. (8.50) disappear on integration, leaving only integrated values of the squared terms. Thus,

$$U + V = \frac{\pi^4 EI}{4l^3} \sum_{n=1}^{\infty} n^4 A_n^2 - \frac{\pi^2 P_{\text{CR}}}{4l} \sum_{n=1}^{\infty} n^2 A_n^2 \quad (8.51)$$

Assigning a stationary value to the total potential energy of Eq. (8.51) with respect to each coefficient  $A_n$  in turn, then taking  $A_n$  as being typical, we have

$$\frac{\partial(U + V)}{\partial A_n} = \frac{\pi^4 EI n^4 A_n}{2l^3} - \frac{\pi^2 P_{\text{CR}} n^2 A_n}{2l} = 0$$

from which

$$P_{\text{CR}} = \frac{\pi^2 EI n^2}{l^2}$$

as before.

We see that each term in Eq. (8.49) represents a particular deflected shape with a corresponding critical load. Hence, the first term represents the deflection of the column shown in Fig. 8.15, with  $P_{\text{CR}} = \pi^2 EI/l^2$ . The second and third terms correspond to the shapes shown in Fig. 8.3, having critical loads of  $4\pi^2 EI/l^2$  and  $9\pi^2 EI/l^2$  and so on. Clearly, the column must be constrained to buckle into these more complex forms. In other words, the column is being forced into an unnatural shape, is consequently stiffer, and offers greater resistance to buckling, as we observe from the higher values of critical load. Such buckling modes, as stated in Section 8.1, are unstable and are generally of academic interest only.

If the deflected shape of the column is known, it is immaterial which of Eqs. (8.47) or (8.48) is used for the total potential energy. However, when only an approximate solution is possible, Eq. (8.47) is preferable, since the integral involving bending moment depends upon the accuracy of the assumed form of  $v$ , whereas the corresponding term in Eq. (8.48) depends upon the accuracy of  $d^2v/dz^2$ . Generally, for an assumed deflection curve,  $v$  is obtained much more accurately than  $d^2v/dz^2$ .

Suppose that the deflection curve of a particular column is unknown or extremely complicated. We then assume a reasonable shape that satisfies, as far as possible, the end conditions of the column and the pattern of the deflected shape (Rayleigh–Ritz method). Generally, the assumed shape is in the form of a finite series involving a series of unknown constants and assumed functions of  $z$ . Let us suppose that  $v$  is given by

$$v = A_1 f_1(z) + A_2 f_2(z) + A_3 f_3(z)$$

Substitution in Eq. (8.47) results in an expression for total potential energy in terms of the critical load and the coefficients  $A_1$ ,  $A_2$ , and  $A_3$  as the unknowns. Assigning stationary values to the total potential energy with respect to  $A_1$ ,  $A_2$ , and  $A_3$  in turn produces three simultaneous equations from which the ratios  $A_1/A_2$ ,  $A_1/A_3$ , and the critical load are determined. Absolute values of the coefficients are unobtainable, since the deflections of the column in its buckled state of neutral equilibrium are indeterminate.

As a simple illustration, consider the column shown in its buckled state in Fig. 8.16. An approximate shape may be deduced from the deflected shape of a tip-loaded cantilever. Thus,

$$v = \frac{v_0 z^2}{2l^3} (3l - z) \quad (\text{see Example 16.5})$$

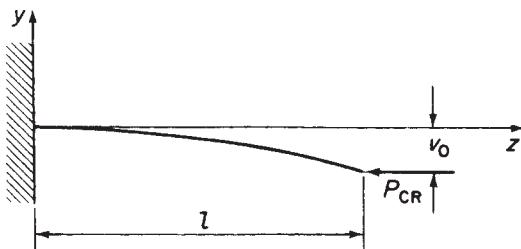


FIGURE 8.16 Buckling Load for a Built-in Column by the Energy Method

This expression satisfies the end conditions of deflection, that is,  $v = 0$  at  $z = 0$  and  $v = v_0$  at  $z = l$ . In addition, it satisfies the conditions that the slope of the column is zero at the built-in end and that the bending moment, that is,  $d^2v/dz^2$ , is zero at the free end. The bending moment at any section is  $M = P_{CR} (v_0 - v)$ , so that substitution for  $M$  and  $v$  in Eq. (8.47) gives

$$U + V = \frac{P_{CR}^2 v_0^2}{2EI} \int_0^l \left(1 - \frac{3z^2}{2l^2} + \frac{z^3}{2l^3}\right)^2 dz - \frac{P_{CR}}{2} \int_0^l \left(\frac{3v_0}{2l^3}\right)^3 z^2 (2l - z)^2 dz$$

Integrating and substituting the limits, we have

$$U + V = \frac{17}{35} \frac{P_{CR}^2 v_0^2 l}{2EI} - \frac{3}{5} P_{CR} \frac{v_0^2}{l} = 0$$

Hence,

$$\frac{\partial(U + V)}{\partial v_0} = \frac{17}{35} \frac{P_{CR}^2 v_0 l}{EI} - \frac{6P_{CR} v_0}{5l} = 0$$

from which

$$P_{CR} = \frac{42EI}{17l^2} = 2.471 \frac{EI}{l^2}$$

This value of the critical load compares with the exact value (see Table 8.1) of  $\pi^2 EI / 4l^2 = 2.467 EI / l^2$ ; the error, in this case, is seen to be extremely small. Approximate values of the critical load obtained by the energy method are always greater than the correct values. The explanation lies in the fact that an assumed deflected shape implies the application of constraints to force the column to take up an artificial shape. This, as we have seen, has the effect of stiffening the column with a consequent increase in critical load.

It will be observed that the solution for this example may be obtained by simply equating the increase in internal energy ( $U$ ) to the work done by the external critical load ( $-V$ ). This is always the case when the assumed deflected shape contains a single unknown coefficient, such as  $v_0$  in the above example.

## 8.6 FLEXURAL-TORSIONAL BUCKLING OF THIN-WALLED COLUMNS

It is recommended that the reading of this section be delayed until after [Chapter 28](#) has been studied.

In some instances, thin-walled columns of open cross-section do not buckle in bending as predicted by the Euler theory but twist without bending or bend and twist simultaneously, producing flexural-torsional buckling. The solution to this type of problem relies on the theory presented in [Chapter 28](#) for the torsion of open-section beams subjected to warping (axial) restraint. Initially, however, we shall establish a useful analogy between the bending of a beam and the behavior of a pin-ended column.

The bending equation for a simply supported beam carrying a uniformly distributed load of intensity  $w_y$  and having  $C_x$  and  $C_y$  as principal centroidal axes is

$$EI_{xx} \frac{d^4 v}{dz^4} = w_y \quad (\text{see Chapter 16}) \quad (8.52)$$

Also, the equation for the buckling of a pin-ended column about the  $C_x$  axis is (see [Eq. \(8.1\)](#))

$$EI_{xx} \frac{d^2 v}{dz^2} = -P_{CR} v \quad (8.53)$$

Differentiating [Eq. \(8.53\)](#) twice with respect to  $z$  gives

$$EI_{xx} \frac{d^4 v}{dz^4} = -P_{CR} \frac{d^2 v}{dz^2} \quad (8.54)$$

Comparing [Eqs. \(8.52\)](#) and [\(8.54\)](#), we see that the behavior of the column may be obtained by considering it as a simply supported beam carrying a uniformly distributed load of intensity  $w_y$  given by

$$w_y = -P_{CR} \frac{d^2 v}{dz^2} \quad (8.55)$$

Similarly, for buckling about the  $C_y$  axis,

$$w_x = -P_{CR} \frac{d^2 u}{dz^2} \quad (8.56)$$

Consider now a thin-walled column having the cross-section shown in [Fig. 8.17](#) and suppose that the centroidal axes  $C_{xy}$  are principal axes (see [Chapter 16](#));  $S(x_S, y_S)$  is the shear center of the column (see [Chapter 17](#)) and its cross-sectional area is  $A$ . Due to the flexural-torsional buckling produced, say, by a compressive axial load  $P$ , the cross-section suffers translations  $u$  and  $v$  parallel to  $C_x$  and  $C_y$ , respectively, and a rotation  $\theta$ , positive counterclockwise, about the shear center  $S$ . Thus, due to translation,  $C$  and  $S$  move to  $C'$  and  $S'$  and, due to rotation about  $S'$ ,  $C'$  moves to  $C''$ . The total movement of  $C$ ,  $u_C$ , in the  $x$  direction is given by

$$u_C = u + C'D = u + C'C'' \sin \alpha \quad (S' \hat{C}' C'' \simeq 90^\circ)$$

But

$$C'C'' = C'S'\theta = CS\theta$$

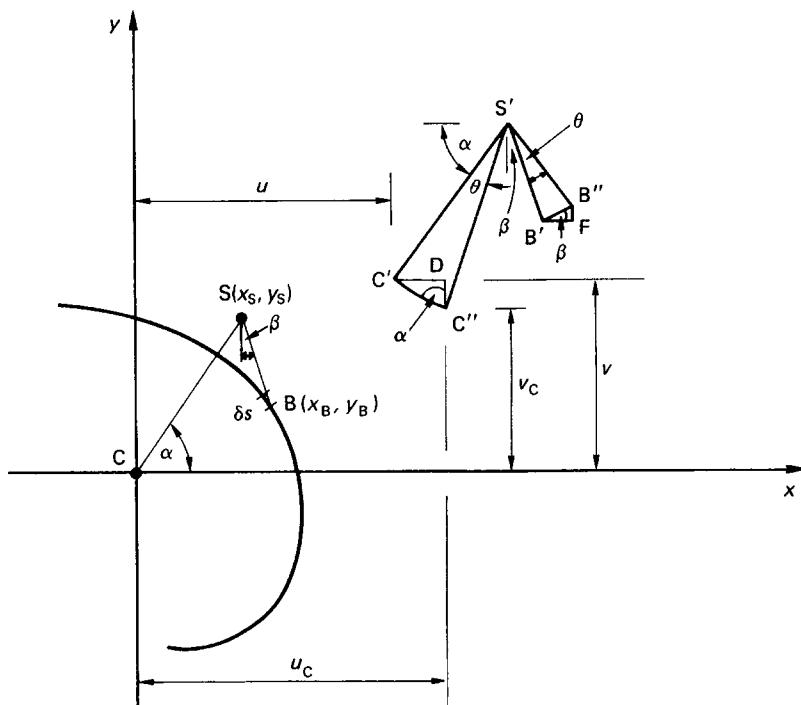


FIGURE 8.17 Flexural-Torsional Buckling of a Thin-Walled Column

Hence,

$$u_C = u + \theta CS \sin \alpha = u + y_S \theta \quad (8.57)$$

Also, the total movement of C in the y direction is

$$v_C = v - DC'' = v - C'C'' \cos \alpha = v - \theta CS \cos \alpha$$

so that

$$v_C = v - x_S \theta \quad (8.58)$$

Since, at this particular cross-section of the column, the centroidal axis has been displaced, the axial load  $P$  produces bending moments about the displaced  $x$  and  $y$  axes given, respectively, by

$$M_x = Pv_C = P(v - x_S \theta) \quad (8.59)$$

and

$$M_y = Pu_C = P(u + y_S \theta) \quad (8.60)$$

From simple beam theory (Chapter 16),

$$EI_{xx} \frac{d^2v}{dz^2} = -M_x = -P(v - x_S \theta) \quad (8.61)$$

and

$$EI_{yy} \frac{d^2u}{dz^2} = -M_y = -P(u + y_S \theta) \quad (8.62)$$

where  $I_{xx}$  and  $I_{yy}$  are the second moments of area of the cross-section of the column about the principal centroidal axes,  $E$  is Young's modulus for the material of the column, and  $z$  is measured along the centroidal longitudinal axis.

The axial load  $P$  on the column at any cross-section is distributed as a uniform direct stress  $\sigma$ . Thus, the direct load on any element of length  $\delta s$  at a point  $B(x_B, y_B)$  is  $\sigma t \delta s$  acting in a direction parallel to the longitudinal axis of the column. In a similar manner to the movement of  $C$  to  $C''$ , the point  $B$  is displaced to  $B''$ . The horizontal movement of  $B$  in the  $x$  direction is then

$$u_B = u + B'F = u + B'B'' \cos \beta$$

But

$$B'B'' = S'B'\theta = SB\theta$$

Hence,

$$u_B = u + \theta SB \cos \beta$$

or

$$u_B = u + (y_S - y_B)\theta \quad (8.63)$$

Similarly, the movement of  $B$  in the  $y$  direction is

$$v_B = v - (x_S - x_B)\theta \quad (8.64)$$

Therefore, from Eqs. (8.63) and (8.64) and referring to Eqs. (8.55) and (8.56), we see that the compressive load on the element  $\delta s$  at  $B$ ,  $\sigma t \delta s$ , is equivalent to lateral loads

$$-\sigma t \delta s \frac{d^2}{dz^2} [u + (y_S - y_B)\theta] \text{ in the } x \text{ direction}$$

and

$$-\sigma t \delta s \frac{d^2}{dz^2} [v - (x_S - x_B)\theta] \text{ in the } y \text{ direction}$$

The lines of action of these equivalent lateral loads do not pass through the displaced position  $S'$  of the shear center and therefore produce a torque about  $S'$  leading to the rotation  $\theta$ . Suppose that the element  $\delta s$  at  $B$  is of unit length in the longitudinal  $z$  direction. The torque per unit length of the column  $\delta T(z)$  acting on the element at  $B$  is then given by

$$\begin{aligned}\delta T(z) = & -\sigma t \delta s \frac{d^2}{dz^2} [u + (y_S - y_B)\theta](y_S - y_B) \\ & + \sigma t \delta s \frac{d^2}{dz^2} [v - (x_S - x_B)\theta](x_S - x_B)\end{aligned}\quad (8.65)$$

Integrating Eq. (8.65) over the complete cross-section of the column gives the torque per unit length acting on the column; that is,

$$\begin{aligned}T(z) = & - \int_{\text{Sect}} \sigma t \frac{d^2 u}{dz^2} (y_S - y_B) ds - \int_{\text{Sect}} \sigma t (y_S - y_B)^2 \frac{d^2 \theta}{dz^2} ds \\ & + \int_{\text{Sect}} \sigma t \frac{d^2 v}{dz^2} (x_S - x_B) ds - \int_{\text{Sect}} \sigma t (x_S - x_B)^2 \frac{d^2 \theta}{dz^2} ds\end{aligned}\quad (8.66)$$

Expanding Eq. (8.66) and noting that  $\sigma$  is constant over the cross-section, we obtain

$$\begin{aligned}T(z) = & -\sigma \frac{d^2 u}{dz^2} y_S \int_{\text{Sect}} t ds + \sigma \frac{d^2 u}{dz^2} \int_{\text{Sect}} t y_B ds - \sigma \frac{d^2 \theta}{dz^2} y_S^2 \int_{\text{Sect}} t ds \\ & + \sigma \frac{d^2 \theta}{dz^2} 2y_S \int_{\text{Sect}} t y_B ds - \sigma \frac{d^2 \theta}{dz^2} \int_{\text{Sect}} t y_B^2 ds + \sigma \frac{d^2 v}{dz^2} x_S \int_{\text{Sect}} t ds \\ & - \sigma \frac{d^2 v}{dz^2} \int_{\text{Sect}} t x_B ds - \sigma \frac{d^2 \theta}{dz^2} x_S^2 \int_{\text{Sect}} t ds + \sigma \frac{d^2 \theta}{dz^2} 2x_S \int_{\text{Sect}} t x_B ds \\ & - \sigma \frac{d^2 \theta}{dz^2} \int_{\text{Sect}} t x_B^2 ds\end{aligned}\quad (8.67)$$

In Eq. (8.67)  $\int_{\text{Sect}} t ds = A$ , the cross-sectional area of the column so that  $\sigma \int_{\text{Sect}} t ds = P$  the applied axial load. Further,  $\int_{\text{Sect}} t x_B ds$  and  $\int_{\text{Sect}} t y_B ds$  are the first moments of area of the column cross-section about the centroid of area C and are therefore zero. Also  $\int_{\text{Sect}} t y_B^2 ds$  and  $\int_{\text{Sect}} t x_B^2 ds$  are the second moments of area  $I_{xx}$  and  $I_{yy}$  of the column cross-section about the x and y axes, respectively (see Section 16.4).

Equation (8.67) may then be rewritten:

$$T(z) = P \left( x_S \frac{d^2 v}{dz^2} - y_S \frac{d^2 u}{dz^2} \right) - \frac{P}{A} \frac{d^2 \theta}{dz^2} (A y_S^2 + I_{xx} + A x_S^2 + I_{yy}) \quad (8.68)$$

In Eq. (8.68), the term  $I_{xx} + I_{yy} + A(x_S^2 + y_S^2)$  is the polar second moment of area  $I_0$  of the column about the shear center S. Thus, Eq. (8.68) becomes

$$T(z) = P \left( x_S \frac{d^2 v}{dz^2} - y_S \frac{d^2 u}{dz^2} \right) - I_0 \frac{P}{A} \frac{d^2 \theta}{dz^2} \quad (8.69)$$

Substituting for  $T(z)$  from Eq. (8.69) in Eq. (28.11), the general equation for the torsion of a thin-walled beam, we have

$$EI \frac{d^4 \theta}{dz^4} - \left( GJ - I_0 \frac{P}{A} \right) \frac{d^2 \theta}{dz^2} - P x_S \frac{d^2 v}{dz^2} + P y_S \frac{d^2 u}{dz^2} = 0 \quad (8.70)$$

Equations (8.61), (8.62), and (8.70) form three simultaneous equations which may be solved to determine the flexural-torsional buckling loads.

As an example, consider the case of a column of length  $L$  in which the ends are restrained against rotation about the  $z$  axis and against deflection in the  $x$  and  $y$  directions; the ends are also free to rotate about the  $x$  and  $y$  axes and are free to warp. Thus,  $u = v = \theta = 0$  at  $z = 0$  and  $z = L$ . Also, since the column is free to rotate about the  $x$  and  $y$  axes at its ends,  $M_x = M_y = 0$  at  $z = 0$  and  $z = L$ , and from Eqs. (8.61) and (8.62),

$$\frac{d^2v}{dz^2} = \frac{d^2u}{dz^2} = 0 \text{ at } z = 0 \text{ and } z = L$$

Further, the ends of the column are free to warp so that

$$\frac{d^2\theta}{dz^2} = 0 \text{ at } z = 0 \text{ and } z = L \quad (\text{see Eq. (28.1)})$$

An assumed buckled shape given by

$$u = A_1 \sin \frac{\pi z}{L}, \quad v = A_2 \sin \frac{\pi z}{L} \quad \theta = A_3 \sin \frac{\pi z}{L} \quad (8.71)$$

in which  $A_1$ ,  $A_2$ , and  $A_3$  are unknown constants, satisfies the preceding boundary conditions. Substituting for  $u$ ,  $v$ , and  $\theta$  from Eqs. (8.71) into Eqs. (8.61), (8.62), and (8.70), we have

$$\left. \begin{aligned} &\left( P - \frac{\pi^2 EI_{xx}}{L^2} \right) A_2 - Px_S A_3 = 0 \\ &\left( P - \frac{\pi^2 EI_{yy}}{L^2} \right) A_1 - Py_S A_3 = 0 \\ &Py_S A_1 - Px_S A_2 - \left( \frac{\pi^2 E\Gamma}{L^2} + GJ - \frac{I_0}{A} P \right) A_3 = 0 \end{aligned} \right\} \quad (8.72)$$

For nonzero values of  $A_1$ ,  $A_2$ , and  $A_3$ , the determinant of Eqs. (8.72) must equal zero, that is,

$$\begin{vmatrix} 0 & P - \pi^2 EI_{xx}/L^2 & -Px_S \\ P - \pi^2 EI_{yy}/L^2 & 0 & Py_S \\ Py_S & -Px_S & I_0 P/A - \pi^2 E\Gamma/L^2 - GJ \end{vmatrix} = 0 \quad (8.73)$$

The roots of the cubic equation formed by the expansion of the determinant give the critical loads for the flexural-torsional buckling of the column; clearly, the lowest value is significant.

In the case where the shear center of the column and the centroid of area coincide, that is, the column has a doubly symmetrical cross-section,  $x_S = y_S = 0$  and Eqs. (8.61), (8.62), and (8.70) reduce, respectively, to

$$EI_{xx} \frac{d^2v}{dz^2} = -Pv \quad (8.74)$$

$$EI_{yy} \frac{d^2u}{dz^2} = -Pu \quad (8.75)$$

$$EI \frac{d^4\theta}{dz^4} - \left( GJ - I_0 \frac{P}{A} \right) \frac{d^2\theta}{dz^2} = 0 \quad (8.76)$$

Equations (8.74), (8.75), and (8.76), unlike Eqs. (8.61), (8.62), and (8.70), are uncoupled and provide three separate values of buckling load. Thus, Eqs. (8.74) and (8.75) give values for the Euler buckling loads about the  $x$  and  $y$  axes, respectively, while Eq. (8.76) gives the axial load that produces pure torsional buckling; clearly the buckling load of the column is the lowest of these values. For the column whose buckled shape is defined by Eqs. (8.71), substitution for  $v$ ,  $u$ , and  $\theta$  in Eqs. (8.74), (8.75), and (8.76), respectively, gives

$$P_{CR(x)} = \frac{\pi^2 EI_{xx}}{L^2} \quad P_{CR(y)} = \frac{\pi^2 EI_{yy}}{L^2} \quad P_{CR(\theta)} = \frac{A}{I_0} \left( GJ + \frac{\pi^2 EI}{L^2} \right) \quad (8.77)$$

### Example 8.5

A thin-walled pin-ended column is 2 m long and has the cross-section shown in Fig. 8.18. If the ends of the column are free to warp, determine the lowest value of axial load that causes buckling, and specify the buckling mode. Take  $E = 75,000 \text{ N/mm}^2$  and  $G = 21,000 \text{ N/mm}^2$ .

Since the cross-section of the column is doubly symmetrical, the shear center coincides with the centroid of area and  $x_S = y_S = 0$ ; Eqs. (8.74), (8.75), and (8.76) therefore apply. Further, the boundary conditions are those

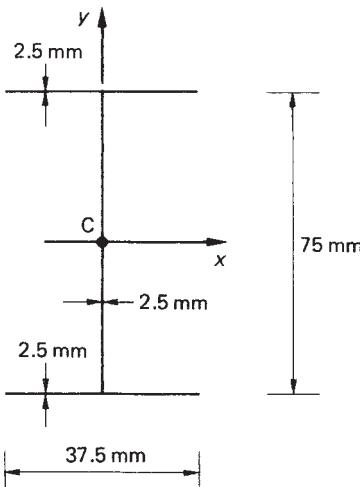


FIGURE 8.18 Column Section of Example 8.5

of the column whose buckled shape is defined by Eqs. (8.71), so that the buckling load of the column is the lowest of the three values given by Eqs. (8.77).

The cross-sectional area  $A$  of the column is

$$A = 2.5(2 \times 37.5 + 75) = 375 \text{ mm}^2$$

The second moments of area of the cross-section about the centroidal axes Cxy are (see Chapter 16), respectively,

$$\begin{aligned} I_{xx} &= 2 \times 37.5 \times 2.5 \times 37.5^2 + 2.5 \times 75^3/12 = 3.52 \times 10^5 \text{ mm}^4 \\ I_{yy} &= 2 \times 2.5 \times 37.5^3/12 = 0.22 \times 10^5 \text{ mm}^4 \end{aligned}$$

The polar second moment of area  $I_0$  is

$$I_0 = I_{xx} + I_{yy} + A(x_S^2 + y_S^2) \quad (\text{see the derivation of Eq. (8.69)})$$

that is,

$$I_0 = 3.52 \times 10^5 + 0.22 \times 10^5 = 3.74 \times 10^5 \text{ mm}^4$$

The torsion constant  $J$  is obtained using Eq. (18.11), which gives

$$J = 2 \times 37.5 \times 2.5^3/3 + 75 \times 2.5^3/3 = 781.3 \text{ mm}^4$$

Finally,  $\Gamma$  is found using the method of Section 28.2 and is

$$\Gamma = 2.5 \times 37.5^3 \times 75^2/24 = 30.9 \times 10^6 \text{ mm}^6$$

Substituting these values in Eqs. (8.77), we obtain

$$P_{CR(xx)} = 6.5 \times 10^4 \text{ N}, \quad P_{CR(yy)} = 0.41 \times 10^4 \text{ N}, \quad P_{CR(\theta)} = 2.22 \times 10^4 \text{ N}$$

The column therefore buckles in bending about the Cy axis when subjected to an axial load of  $0.41 \times 10^4 \text{ N}$ .

Equation (8.73) for the column whose buckled shape is defined by Eqs. (8.71) may be rewritten in terms of the three separate buckling loads given by Eqs. (8.77):

$$\begin{vmatrix} 0 & P - P_{CR(xx)} & -Px_S \\ P - P_{CR(yy)} & 0 & Py_S \\ Py_S & -Px_S & I_0(P - P_{CR(\theta)})/A \end{vmatrix} = 0 \quad (8.78)$$

If the column has, say, Cx as an axis of symmetry, then the shear center lies on this axis and  $y_S = 0$ . Equation (8.78) thereby reduces to

$$\begin{vmatrix} P - P_{CR(xx)} & -Px_S \\ -Px_S & I_0(P - P_{CR(\theta)})/A \end{vmatrix} = 0 \quad (8.79)$$

The roots of the quadratic equation formed by expanding Eq. (8.79) are the values of axial load that produce flexural-torsional buckling about the longitudinal and x axes. If  $P_{CR(yy)}$  is less than the smallest of these roots, the column will buckle in pure bending about the y axis.

**Example 8.6**

A column of length 1 m has the cross-section shown in Fig. 8.19. If the ends of the column are pinned and free to warp, calculate its buckling load;  $E = 70,000 \text{ N/mm}^2$ ,  $G = 30,000 \text{ N/mm}^2$ .

In this case, the shear center S is positioned on the Cx axis, so that  $y_S = 0$ , and Eq. (8.79) applies. The distance  $\bar{x}$  of the centroid of area C from the web of the section is found by taking first moments of area about the web:

$$2(100 + 100 + 100)\bar{x} = 2 \times 2 \times 100 \times 50$$

which gives

$$\bar{x} = 33.3 \text{ mm}$$

The position of the shear center S is found using the method of Example 17.1; this gives  $x_S = -76.2 \text{ mm}$ . The remaining section properties are found by the methods specified in Example 8.5 and follow:

$$\begin{aligned} A &= 600 \text{ mm}^2, & I_{xx} &= 1.17 \times 10^6 \text{ mm}^4, & I_{yy} &= 0.67 \times 10^6 \text{ mm}^4, \\ I_0 &= 5.35 \times 10^6 \text{ mm}^4, & J &= 800 \text{ mm}^4, & \Gamma &= 2488 \times 10^6 \text{ mm}^6 \end{aligned}$$

From Eq. (8.77),

$$P_{\text{CR}(yy)} = 4.63 \times 10^5 \text{ N}, \quad P_{\text{CR}(xx)} = 8.08 \times 10^5 \text{ N}, \quad P_{\text{CR}(0)} = 1.97 \times 10^5 \text{ N}$$

Expanding Eq. (8.79),

$$(P - P_{\text{CR}(xx)})(P - P_{\text{CR}(0)})I_0/A - P^2 x_S^2 = 0 \quad (\text{i})$$

Rearranging Eq. (i),

$$P^2(1 - Ax_S^2/I_0) - P(P_{\text{CR}(xx)} + P_{\text{CR}(0)}) + P_{\text{CR}(xx)}P_{\text{CR}(0)} = 0 \quad (\text{ii})$$

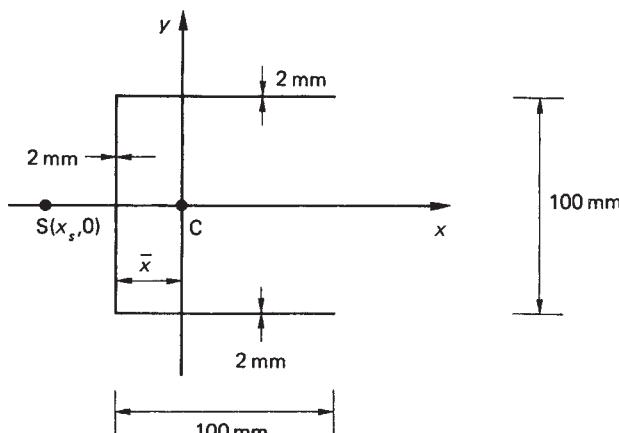


FIGURE 8.19 Column Section of Example 8.6

Substituting the values of the constant terms in Eq. (ii), we obtain

$$P^2 - 29.13 \times 10^5 P + 46.14 \times 10^{10} = 0 \quad (\text{iii})$$

The roots of Eq. (iii) give two values of critical load, the lowest of which is

$$P = 1.68 \times 10^5 \text{ N}$$

It can be seen that this value of flexural-torsional buckling load is lower than any of the uncoupled buckling loads  $P_{\text{CR}(xx)}$ ,  $P_{\text{CR}(yy)}$ , or  $P_{\text{CR}(\theta)}$ ; the reduction is due to the interaction of the bending and torsional buckling modes.

### Example 8.6 MATLAB

Repeat Example 8.6 using the MATLAB and the calculated section properties.

The value of the critical buckling load is obtained through the following MATLAB file:

```
% Declare any needed variables
syms P
L = 1000;
t = 2;
E = 70000;
G = 30000;
y_s = 0;
x_s = -76.2;
A = 600;
I_xx = 1.17e6;
I_yy = 0.67e6;
I_0 = 5.32e6;
J = 800;
T = 2488e6;

% Evaluate Eq.(8.77)
P_CRYy = pi^2*E*I_yy/L^2;
P_CRxx = pi^2*E*I_xx/L^2;
P_CRTtheta = A*(G*J + pi^2*E*T/L^2)/I_0;

% Substitute results of Eq. (8.77) into Eq.(8.79)
eq_I = det([P-P_CRYy -P*x_s; -P*x_s I_0*(P-P_CRTtheta)/A]);

% Solve eq_I for the critical buckling load (P)
P = solve(eq_I,P);

% Output the minimum value of P to the Command Window
disp(['P =' num2str(min(double(P))) 'N'])
```

The Command Window output resulting from this MATLAB file is as follows:

$P = 167785.096 \text{ N}$

**Example 8.7**

A thin-walled column has the cross-section shown in Fig. 8.20, is of length  $L$ , and is subjected to an axial load through its shear center S. If the ends of the column are prevented from warping and twisting, determine the value of direct stress when failure occurs due to torsional buckling.

The torsion bending constant  $\Gamma$  is found using the method described in Section 28.2. The position of the shear center is given but is obvious by inspection. The swept area  $2\Gamma A_{R,0}$  is determined as a function of  $s$ , and its distribution is shown in Fig. 8.21. The center of gravity of the “wire” is found by taking moments about the  $s$  axis.

Then,

$$2A'_R 5td = td \left( \frac{d^2}{2} + \frac{5d^2}{4} + \frac{3d^2}{2} + \frac{5d^2}{4} + \frac{d^2}{2} \right)$$

which gives

$$2A'_R = d^2$$

The torsion bending constant is then the “moment of inertia” of the “wire” and is

$$\Gamma = 2td \frac{1}{3}(d^2)^2 + \frac{td}{3} \left( \frac{d^2}{2} \right)^2 \times 2 + td \left( \frac{d^2}{2} \right)^2$$

from which

$$\Gamma = \frac{13}{12}td^5$$

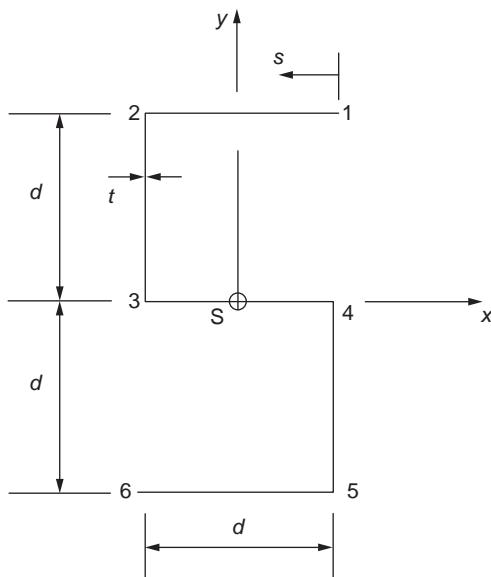


FIGURE 8.20 Section of Column of Example 8.7

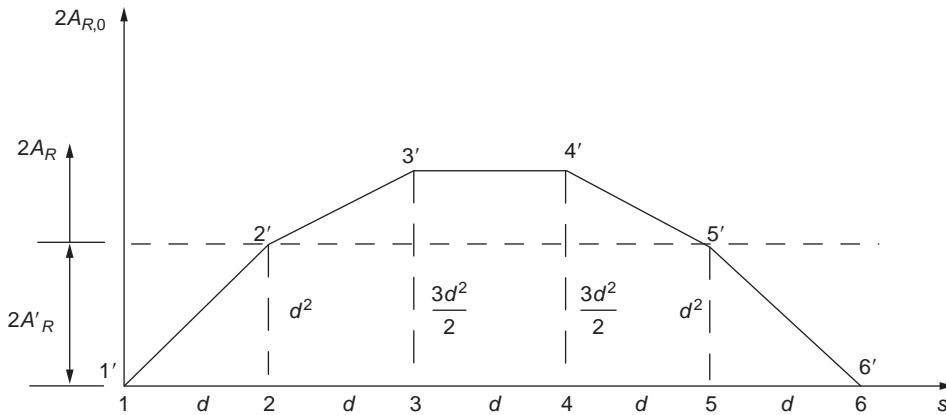


FIGURE 8.21 Determination of Torsion Bending Constant for Column Section of Example 8.7

Also, the torsion constant  $J$  is given by (see Section 3.4)

$$J = \sum \frac{st^3}{3} = \frac{5dt^3}{3}$$

The shear center of the section and the centroid of area coincide, so that the torsional buckling load is given by Eq. (8.76). Rewriting this equation,

$$\frac{d^4\theta}{dz^4} + \mu^2 \frac{d^2\theta}{dz^2} = 0 \quad (i)$$

where

$$\mu^2 = (\sigma I_0 - GJ)/EI, \quad (\sigma = P/A)$$

The solution of Eq. (i) is

$$\theta = A \cos \mu z + B \sin \mu z + Cz + D \quad (ii)$$

The boundary conditions are  $\theta = 0$  when  $z = 0$  and  $z = L$ , and since the warping is suppressed at the ends of the beam,

$$\frac{d\theta}{dz} = 0, \quad \text{when } z = 0 \text{ and } z = L \quad (\text{see Eq.(18.19)})$$

Putting  $\theta = 0$  at  $z = 0$  in Eq. (ii),

$$0 = A + D$$

or

$$A = -D$$

Also,

$$\frac{d\theta}{dz} = -\mu A \sin \mu z + \mu B \cos \mu z + C$$

and, since  $(d\theta/dz) = 0$  at  $z = 0$ ,

$$C = -\mu B$$

When  $z = L$ ,  $\theta = 0$ , so that, from Eq. (ii),

$$0 = A \cos \mu L + B \sin \mu L + CL + D$$

which may be rewritten

$$0 = B(\sin \mu L - \mu L) + A(\cos \mu L - 1) \quad (\text{iii})$$

Then, for  $(d\theta/dz) = 0$  at  $z = L$ ,

$$0 = \mu B \cos \mu L - \mu A \sin \mu L - \mu B$$

or

$$0 = B(\cos \mu L - 1) - A \sin \mu L \quad (\text{iv})$$

Eliminating  $A$  from Eqs. (iii) and (iv),

$$0 = B[2(1 - \cos \mu L) - \mu L \sin \mu L] \quad (\text{v})$$

Then, since  $C = -\mu B$ ,

$$0 = -C[2(1 - \cos \mu L) - \mu L \sin \mu L] \quad (\text{vi})$$

or

$$B = -C$$

But  $B = -C/\mu$ , so that to satisfy both equations,  $B = C = 0$  and

$$\theta = A \cos \mu z - A = A(\cos \mu z - 1) \quad (\text{vii})$$

Since  $\theta = 0$  at  $z = l$ ,

$$\cos \mu L = 1$$

or

$$\mu L = 2n\pi$$

Therefore,

$$\mu^2 L^2 = 4n^2 \pi^2$$

or

$$\frac{\sigma I_0 - GJ}{E\Gamma} = \frac{4n^2 \pi^2}{L^2}$$

The lowest value of torsional buckling load corresponds to  $n = 1$ , so that, rearranging the preceding,

$$\sigma = \frac{1}{I_0} \left( GJ + \frac{4\pi^2 EI\Gamma}{L^2} \right) \quad (\text{viii})$$

The polar second moment of area  $I_0$  is given by

$$I_0 = I_{xx} + I_{yy} \quad (\text{see Megson}^2)$$

that is,

$$I_0 = 2\left(td d^2 + \frac{td^3}{3}\right) + \frac{3td^3}{12} + 2td \frac{d^2}{4}$$

which gives

$$I_0 = \frac{41}{12}td^3$$

Substituting for  $I_0$ ,  $J$ , and  $\Gamma$  in Eq. (viii),

$$\sigma = \frac{4}{41d^2} \left( 5Gt^2 + \frac{13\pi^2 Ed^4}{L^2} \right)$$



## References

- [1] Timoshenko SP, Gere JM. Theory of elastic stability. 2nd ed. New York: McGraw-Hill; 1961.
- [2] Megson THG. Structural and stress analysis. 4th ed. Oxford: Elsevier; 2019.

## PROBLEMS

**P.8.1.** A strut of length  $L$  and flexural rigidity  $EI$  is subjected to an axial load  $P$ . Find the critical value of  $P$  when the strut has end conditions:

- (i) One end fixed, the other free.
- (ii) One end fixed, the other pinned.

Answer: (i)  $\pi^2 EI / 4L^2$ , (ii)  $2.05\pi^2 EI / L^2$ .

**P.8.2.** If the column of Example 8.1 has a spring support at mid-span identical to that in Example 8.2, derive an expression giving the buckling load of the column.

Answer:  $P = \frac{k}{2} \left( \frac{L}{2} - \frac{1}{\lambda} \sin \frac{\lambda L}{2} \right) / \left[ 1 - \frac{14}{\lambda^2 L^2} \left( 1 - \cos \frac{\lambda L}{2} \right) \right]$

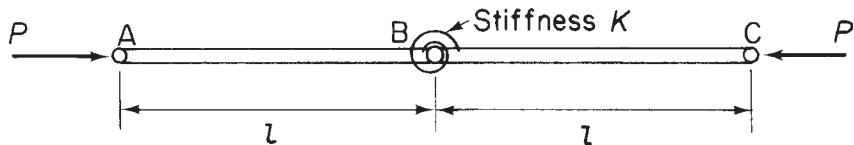


FIGURE P.8.3

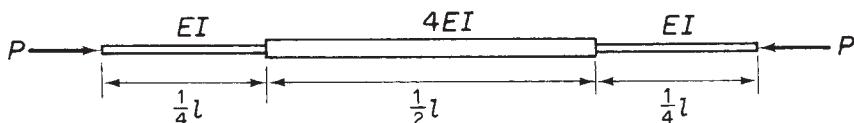


FIGURE P.8.4

**P.8.3.** The system shown in Fig. P.8.3 consists of two bars AB and BC, each of bending stiffness  $EI$  elastically hinged together at B by a spring of stiffness  $K$  (i.e., bending moment applied by spring =  $K \times$  change in slope across B). Regarding A and C as simple pin joints, obtain an equation for the first buckling load of the system. What are the lowest buckling loads when (a)  $K \rightarrow \infty$ , (b)  $EI \rightarrow \infty$ ? Note that B is free to move vertically.

Answer:  $\mu K / \tan \mu l$ .

**P.8.4.** A pin-ended column of length  $l$  and constant flexural stiffness  $EI$  is reinforced to give a flexural stiffness  $4EI$  over its central half (see Fig. P.8.4). Considering symmetric modes of buckling only, obtain the equation whose roots yield the flexural buckling loads and solve for the lowest buckling load.

Answer:  $\tan \mu l / 8 = 1/\sqrt{2}$ ,  $P = 24.2EI/l^2$

**P.8.5.** A uniform column of length  $l$  and bending stiffness  $EI$  is built in at one end and free at the other and has been designed so that its lowest flexural buckling load is  $P$  (see Fig. P.8.5). Subsequently it has to carry an increased load, and for this it is provided with a lateral spring at the free end. Determine the necessary spring stiffness  $k$  so that the buckling load becomes  $4P$ .

Answer:  $k = 4P\mu / (\mu l - \tan \mu l)$ .

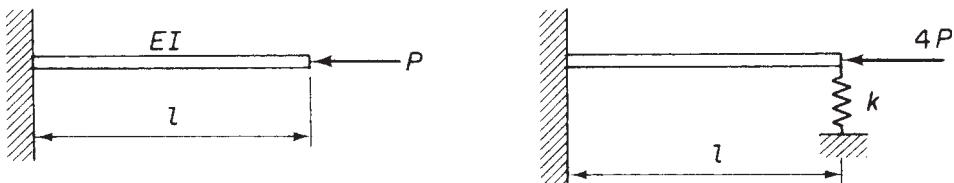


FIGURE P.8.5

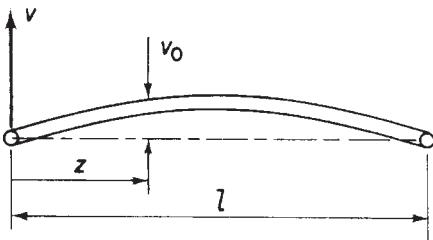


FIGURE P.8.6

**P.8.6.** A uniform, pin-ended column of length  $l$  and bending stiffness  $EI$  has an initial curvature such that the lateral displacement at any point between the column and the straight line joining its ends is given by

$$v_0 = a \frac{4z}{l^2} (l - z) \quad (\text{see Fig. P.8.6})$$

Show that the maximum bending moment due to a compressive end load  $P$  is given by

$$M_{\max} = -\frac{8aP}{(\lambda l)^2} \left( \sec \frac{\lambda l}{2} - 1 \right)$$

where

$$\lambda^2 = P/EI$$

**P.8.7.** The uniform pin-ended column shown in Fig. P.8.7 is bent at the center so that its eccentricity there is  $\delta$ . If the two halves of the column are otherwise straight and have a flexural stiffness  $EI$ , find the value of the maximum bending moment when the column carries a compression load  $P$ .

Answer:  $-P \frac{2\delta}{l} \sqrt{\frac{EI}{P}} \tan \sqrt{\frac{P}{EI}} \frac{l}{2}$

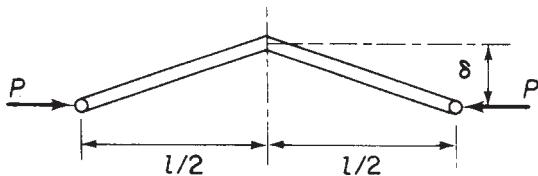


FIGURE P.8.7

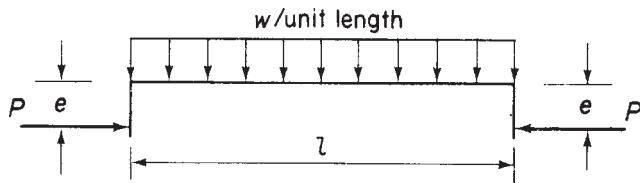


FIGURE P.8.8

**P.8.8.** A straight uniform column of length  $l$  and bending stiffness  $EI$  is subjected to uniform lateral loading  $w/\text{unit length}$ . The end attachments do not restrict rotation of the column ends. The longitudinal compressive force  $P$  has eccentricity  $e$  from the centroids of the end sections and is placed so as to oppose the bending effect of the lateral loading, as shown in Fig. P.8.8. The eccentricity  $e$  can be varied and is to be adjusted to the value, which, for given values of  $P$  and  $w$ , will result in the least maximum bending moment on the column. Show that

$$e = (w/P\mu^2) \tan^2 \mu l / 4$$

where

$$\mu^2 = P/EI$$

Deduce the end moment that gives the optimum condition when  $P$  tends to zero.

Answer:  $wl^2/16$ .

**P.8.9.** The relation between stress  $\sigma$  and strain  $\varepsilon$  in compression for a certain material is

$$10.5 \times 10^6 \varepsilon = \sigma + 21,000 \left( \frac{\sigma}{49,000} \right)^{16}$$

Assuming the tangent modulus equation to be valid for a uniform strut of this material, plot the graph of  $\sigma_b$  against  $l/r$ , where  $\sigma_b$  is the flexural buckling stress,  $l$  the equivalent pin-ended length, and  $r$  the least radius of gyration of the cross-section. Estimate the flexural buckling load for a tubular strut of this material, of 1.5 units outside diameter and 0.08 units wall thickness with effective length 20 units.

Answer: 14,454 force units

**P.8.10.** A short column has a doubly symmetrical I-section and is fabricated from material for which Young's modulus is  $E$  and the tangent modulus is  $E_t$ ; its flange width is  $b$  and its overall depth is  $d$ . Assuming that the direct stresses are concentrated in the flanges at distances  $\pm d/2$  from the horizontal axis of symmetry obtain an expression for the reduced section modulus  $E_r$  in terms of  $E$  and  $E_t$ .

Answer:  $E_r = 2E/[1 + (E_t/E)]$ .

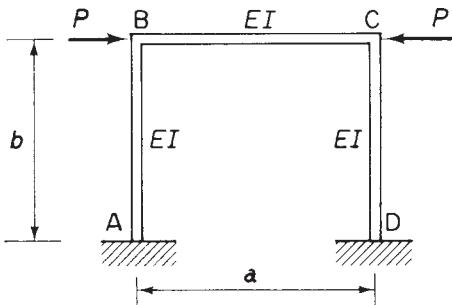


FIGURE P.8.11

**P.8.11.** A rectangular portal frame ABCD is rigidly fixed to a foundation at A and D and is subjected to a compression load  $P$  applied at each end of the horizontal member BC (see Fig. P.8.11). If all the members have the same bending stiffness  $EI$ , show that the buckling loads for modes that are symmetrical about the vertical center line are given by the transcendental equation,

$$\frac{\lambda a}{2} = -\frac{1}{2} \left( \frac{a}{b} \right) \tan \left( \frac{\lambda a}{2} \right)$$

where

$$\lambda^2 = P/EI$$

**P.8.12.** A compression member (Fig. P.8.12) is made of circular section tube, diameter  $d$ , thickness  $t$ . The member is not perfectly straight when unloaded, having a slightly bowed shape that may be represented by the expression

$$v = \delta \sin \left( \frac{\pi z}{l} \right)$$

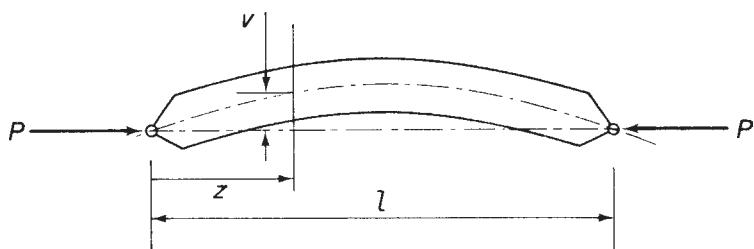


FIGURE P.8.12

Show that when the load  $P$  is applied, the maximum stress in the member can be expressed as

$$\sigma_{\max} = \frac{P}{\pi dt} \left[ 1 + \frac{1 - \alpha}{1 - \alpha} \frac{4\delta}{d} \right]$$

where

$$\alpha = P/P_e, \quad P_e = \pi^2 EI/l^2$$

Assume  $t$  is small compared with  $d$ , so that the following relationships are applicable:

$$\text{Cross-sectional area of tube} = \pi dt$$

$$\text{Second moment of area of tube} = \pi d^3 t/8$$

**P.8.13.** Figure P.8.13 illustrates an idealized representation of part of an aircraft control circuit. A uniform, straight bar of length  $a$  and flexural stiffness  $EI$  is built in at the end A and hinged at B to a link BC, of length  $b$ , whose other end C is pinned, so that it is free to slide along the line ABC between smooth, rigid guides. A, B, and C are initially in a straight line, and the system carries a compression force  $P$ , as shown. Assuming that the link BC has a sufficiently high flexural stiffness to prevent its buckling as a pin-ended strut, show, by setting up and solving the differential equation for flexure of AB, that buckling of the system, of the type illustrated in Fig. P.8.13, occurs when  $P$  has such a value that

$$\tan \lambda a = \lambda(a + b)$$

where

$$\lambda^2 = P/EI$$

**P.8.14.** A pin-ended column of length  $l$  has its central portion reinforced, the second moment of its area being  $I_2$ , while that of the end portions, each of length  $a$ , is  $I_1$ . Use the energy method to determine the critical load of the column, assuming that its centerline deflects into the parabola  $v = kz(l - z)$  and taking the more accurate of the two expressions for the bending moment. In the case where  $I_2 = 1.6I_1$  and  $a = 0.2 l$ , find the percentage increase in strength due to the reinforcement, and compare it with the percentage increase in weight on the basis that the radius of gyration of the section is not altered.

Answer:  $P_{CR} = 14.96EI_1/l^2$ , 52 percent, 36 percent

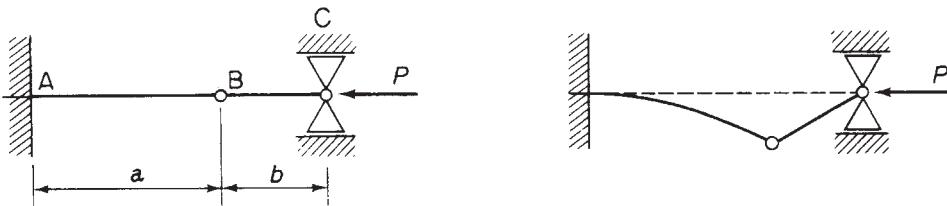


FIGURE P.8.13

**P.8.14 MATLAB** Use the MATLAB to repeat Problem P.8.14 for the following relations of  $I_1$  and  $I_2$ .

	(i)	(ii)	(iii)	(iv)	(v)
$l_2$	$1.4 l_1$	$1.5 l_1$	$1.6 l_1$	$1.7 l_1$	$1.8 l_1$

- Answer:
- (i)  $P_{CR} = 13.38EI_1/l^2, 36\%, 24\%$
  - (ii)  $P_{CR} = 14.18EI_1/l^2, 44\%, 30\%$
  - (iii)  $P_{CR} = 14.96EI_1/l^2, 52\%, 36\%$
  - (iv)  $P_{CR} = 15.72EI_1/l^2, 59\%, 42\%$
  - (v)  $P_{CR} = 16.47EI_1/l^2, 67\%, 48\%$

**P.8.15.** A tubular column of length  $l$  is tapered in wall thickness so that the area and the second moment of area of its cross-section decrease uniformly from  $A_1$  and  $I_1$  at its center to  $0.2A_1$  and  $0.2I_1$  at its ends. Assuming a deflected center-line of parabolic form and taking the more correct form for the bending moment, use the energy method to estimate its critical load when tested between pin-centers, in terms of the preceding data and Young's modulus  $E$ . Hence, show that the saving in weight by using such a column instead of one having the same radius of gyration and constant thickness is about 15 percent.

Answer:  $7.01EI_1/l^2$

**P.8.16.** A uniform column (Fig. P.8.16), of length  $l$  and bending stiffness  $EI$ , is rigidly built in at the end  $z = 0$  and simply supported at the end  $z = l$ . The column is also attached to an elastic foundation of constant stiffness  $k$ /unit length. Representing the deflected shape of the column by a polynomial,

$$v = \sum_{n=0}^p a_n \eta^n, \quad \text{where } \eta = z/l$$

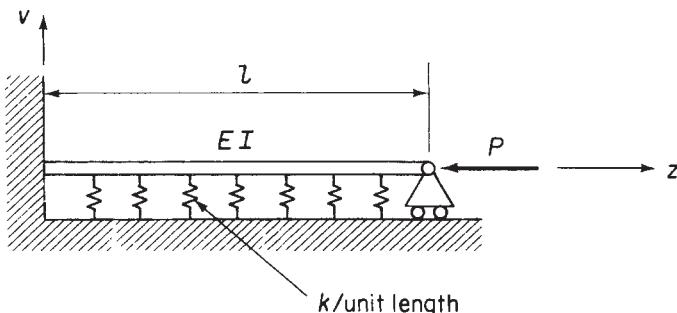


FIGURE P.8.16

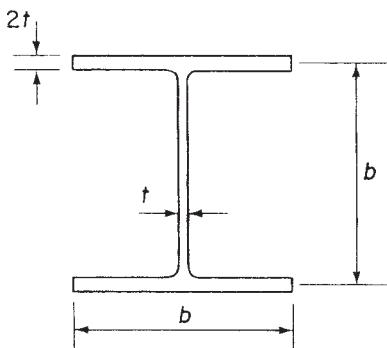


FIGURE P.8.17

determine the form of this function by choosing a minimum number of terms  $p$  such that all the kinematic (geometric) and static boundary conditions are satisfied, allowing for one arbitrary constant only. Using the result thus obtained, find an approximation to the lowest flexural buckling load  $P_{\text{CR}}$  by the Rayleigh–Ritz method.

Answer:  $P_{\text{CR}} = 21.05EI/l^2 + 0.09kl^2$

**P.8.17.** Figure P.8.17 shows the doubly symmetrical cross-section of a thin-walled column with rigidly fixed ends. Find an expression, in terms of the section dimensions and Poisson's ratio, for the column length for which the purely flexural and the purely torsional modes of instability occur at the same axial load. In which mode does failure occur if the length is less than the value found? The possibility of local instability is to be ignored.

Answer:  $l = (2\pi b^2/t) \sqrt{(1+\nu)/255}$ , torsion

**P.8.18.** A column of length  $2l$  with the doubly symmetric cross-section, shown in Fig. P.8.18, is compressed between the parallel platens of a testing machine that fully prevents twisting and warping of the ends. Using the data that follows, determine the average compressive stress at which the column first buckles in torsion:

$$l = 500 \text{ mm}, \quad b = 25.0 \text{ mm}, \quad t = 2.5 \text{ mm}, \quad E = 70,000 \text{ N/mm}^2, \quad E/G = 2.6$$

Answer:  $\sigma_{\text{CR}} = 282 \text{ N/mm}^2$ .

**P.8.19.** A pin-ended column of length 1.0 m has the cross-section shown in Fig. P.8.19. If the ends of the column are free to warp, determine the lowest value of axial load that will cause the column to buckle and specify the mode. Take  $E = 70,000 \text{ N/mm}^2$  and  $G = 25,000 \text{ N/mm}^2$ .

Answer: 5,527 N. The column buckles in bending about an axis in the plane of its web.

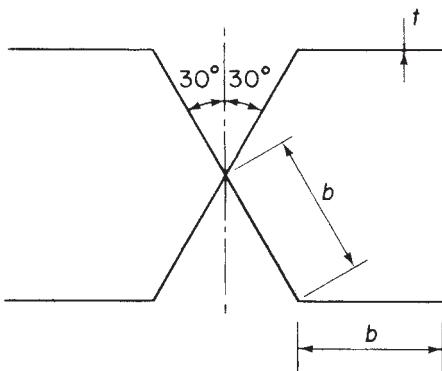


FIGURE P.8.18

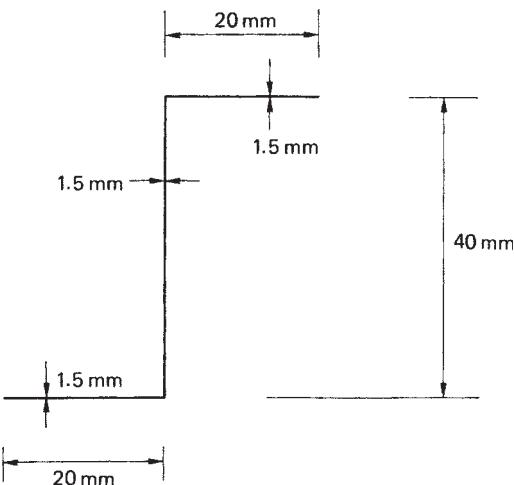


FIGURE P.8.19

**P.8.19. MATLAB** Use MATLAB to repeat Problem P.8.19 for the following column lengths.

$$L \quad \begin{matrix} \text{(i)} \\ .5\text{ m} \end{matrix} \quad \begin{matrix} \text{(ii)} \\ 1\text{ m} \end{matrix} \quad \begin{matrix} \text{(iii)} \\ 1.5\text{ m} \end{matrix} \quad \begin{matrix} \text{(iv)} \\ 2\text{ m} \end{matrix} \quad \begin{matrix} \text{(v)} \\ 2.5\text{ m} \end{matrix} \quad \begin{matrix} \text{(vi)} \\ 3\text{ m} \end{matrix}$$

- Answer:
- (i) 22108 N
  - (ii) 5527 N
  - (iii) 2456 N
  - (iv) 1382 N
  - (v) 884 N
  - (vi) 614 N

The column buckles in bending about the  $y$  axis (plane of the web) for all selected values of  $L$ .

**P.8.20.** A pin-ended column of height 3.0 m has a circular cross-section of diameter 80 mm, wall thickness 2.0 mm, and is converted to an open section by a narrow longitudinal slit; the ends of the column are free to warp. Determine the values of axial load that would cause the column to buckle in (a) pure bending and (b) pure torsion. Hence, determine the value of the flexural-torsional buckling load. Take  $E = 70,000 \text{ N/mm}^2$  and  $G = 22,000 \text{ N/mm}^2$ . Note: the position of the shear center of the column section may be found using the method described in Chapter 17.

Answer: (a)  $3.09 \times 10^4 \text{ N}$ , (b)  $1.78 \times 10^4 \text{ N}$ ,  $1.19 \times 10^4 \text{ N}$

# Thin plates

# 9

We shall see in [Chapter 12](#), when we examine the structural components of aircraft and spacecraft, that they consist mainly of thin plates stiffened by arrangements of ribs and stringers. Thin plates under relatively small compressive loads are prone to buckle and so must be stiffened to prevent this. The determination of buckling loads for thin plates in isolation is relatively straightforward, but when stiffened by ribs and stringers, the problem becomes complex and frequently relies on an empirical solution. In fact, the stiffeners may buckle before the plate and, depending on their geometry, may buckle as a column or suffer local buckling of, say, a flange.

In this chapter, we present the theory for the determination of buckling loads of flat plates and examine some of the different empirical approaches various researchers have suggested. In addition, we investigate the particular case of flat plates that, when reinforced by horizontal flanges and vertical stiffeners, form the spars of aircraft wing structures; these are known as *tension field beams*.

## 9.1 BUCKLING OF THIN PLATES

A thin plate may buckle in a variety of modes, depending upon its dimensions, the loading, and the method of support. Usually, however, buckling loads are much lower than those likely to cause failure in the material of the plate. The simplest form of buckling arises when compressive loads are applied to simply supported opposite edges and the unloaded edges are free, as shown in [Fig. 9.1](#). A thin plate in this configuration behaves in exactly the same way as a pin-ended column, so that the critical load is that predicted by the [Euler theory](#). Once this critical load is reached, the plate is incapable of supporting any further load. This is not the case, however, when the unloaded edges are supported against displacement out of the  $xy$  plane. Buckling, for such plates, takes the form of a bulging displacement of the central region of the plate while the parts adjacent to the supported edges remain straight as shown in [Fig. 9.2](#). These parts enable the plate to resist higher loads—an important factor in aircraft design.

At this stage, we are not concerned with this postbuckling behavior but rather with the prediction of the critical load that causes the initial bulging of the central area of the plate. For the analysis, we may conveniently employ the method of total potential energy, since we already, in [Chapter 7](#), derived expressions for strain and potential energy corresponding to various load and support configurations. In these expressions, we assumed that the displacement of the plate comprises bending deflections only and that these are small in comparison with the thickness of the plate. These restrictions therefore apply in the subsequent theory.

First, we consider the relatively simple case of the thin plate of [Fig. 9.1](#), loaded as shown, but simply supported along all four edges. We saw in [Chapter 7](#) that its true deflected shape may be represented by the infinite double trigonometrical series

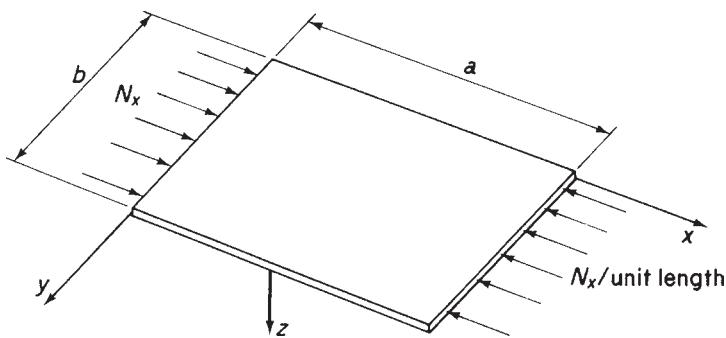


FIGURE 9.1 Buckling of a Thin Flat Plate

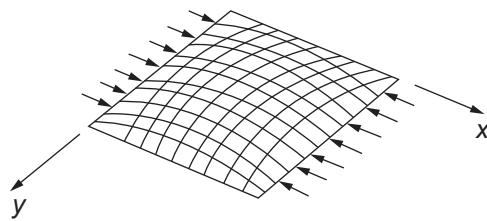


FIGURE 9.2 Buckling of a Plate Supported on All Four Edges

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Also, the total potential energy of the plate is, from Eqs. (7.37) and (7.45),

$$U + V = \frac{1}{2} \int_0^a \int_0^b \left[ D \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} - N_x \left( \frac{\partial w}{\partial x} \right)^2 \right] dx dy \quad (9.1)$$

The integration of Eq. (9.1) on substituting for  $w$  is similar to those integrations carried out in Chapter 7. Thus, by comparison with Eq. (7.47),

$$U + V = \frac{\pi^4 abD}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) - \frac{\pi^2 b}{8a} N_x \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 A_{mn}^2 \quad (9.2)$$

The total potential energy of the plate has a stationary value in the neutral equilibrium of its buckled state (i.e.,  $N_x = N_{x,CR}$ ). Therefore, differentiating Eq. (9.2) with respect to each unknown coefficient  $A_{mn}$ , we have

$$\frac{\partial(U + V)}{\partial A_{mn}} = \frac{\pi^4 abD}{4} A_{mn} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - \frac{\pi^2 b}{4a} N_{x,CR} m^2 A_{mn} = 0$$

and, for a nontrivial solution,

$$N_{x,\text{CR}} = \pi^2 a^2 D \frac{1}{m^2} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \quad (9.3)$$

Exactly the same result may have been deduced from Eq. (ii) of [Example 7.3](#), where the displacement  $w$  becomes infinite for a negative (compressive) value of  $N_x$  equal to that of [Eq. \(9.3\)](#).

We observe, from [Eq. \(9.3\)](#), that each term in the infinite series for displacement corresponds, as in the case of a column, to a different value of critical load (note, the problem is an eigenvalue problem). The lowest value of critical load evolves from some critical combination of integers  $m$  and  $n$ , that is, the number of half-waves in the  $x$  and  $y$  directions, and the plate dimensions. Clearly  $n = 1$  gives a minimum value, so that no matter what the values of  $m$ ,  $a$ , and  $b$ , the plate buckles into a half sine wave in the  $y$  direction. Thus, we may write [Eq. \(9.3\)](#) as

$$N_{x,\text{CR}} = \pi^2 a^2 D \frac{1}{m^2} \left( \frac{m^2}{a^2} + \frac{1}{b^2} \right)^2$$

or

$$N_{x,\text{CR}} = \frac{k\pi^2 D}{b^2} \quad (9.4)$$

where the plate *buckling coefficient*  $k$  is given by the minimum value of

$$k = \left( \frac{mb}{a} + \frac{a}{mb} \right)^2 \quad (9.5)$$

for a given value of  $a/b$ . To determine the minimum value of  $k$  for a given value of  $a/b$ , we plot  $k$  as a function of  $a/b$  for different values of  $m$ , as shown by the dotted curves in [Fig. 9.3](#). The minimum value of  $k$  is obtained from the lower envelope of the curves shown solid in the figure.

It can be seen that  $m$  varies with the ratio  $a/b$  and that  $k$  and the buckling load are a minimum when  $k = 4$  at values of  $a/b = 1, 2, 3, \dots$ . As  $a/b$  becomes large,  $k$  approaches 4, so that long narrow plates tend to buckle into a series of squares as shown in [Fig. 9.4](#).

The transition from one buckling mode to the next may be found by equating values of  $k$  for the  $m$  and  $m + 1$  curves. Hence,

$$\frac{mb}{a} + \frac{a}{mb} = \frac{(m+1)b}{a} + \frac{a}{(m+1)b}$$

Then,

$$\frac{mb(m+1)}{a} + \frac{a(m+1)}{mb} = (m+1)^2 \frac{b}{a} + \frac{a}{b}$$

from which

$$\frac{a}{b} = (m+1) \left( \frac{mb}{a} + \frac{a}{mb} - \frac{mb}{a} - \frac{b}{a} \right)$$

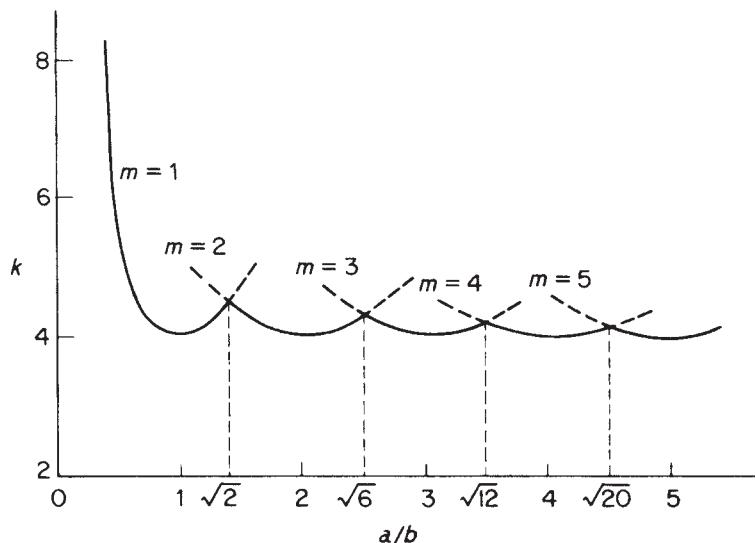
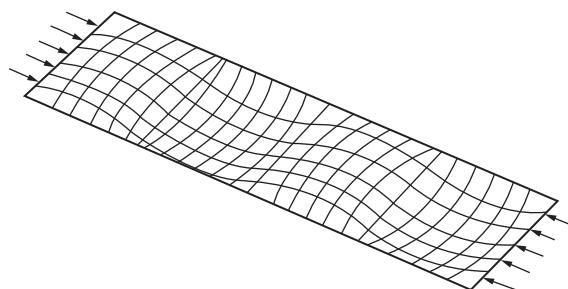
FIGURE 9.3 Buckling Coefficient  $k$  for Simply Supported Plates

FIGURE 9.4 Buckling of a Long, Narrow Rectangular Plate

so that

$$\frac{a}{b} = \frac{a}{b} \left( 1 - \frac{mb}{a} + \frac{a}{mb} - \frac{b}{a} \right)$$

Then,

$$0 = \frac{b}{a} \left( -m + \frac{a^2}{mb^2} - 1 \right)$$

giving

$$\frac{a}{b} = \sqrt{m(m+1)}$$

Substituting  $m = 1$ , we have  $a/b = \sqrt{2} = 1.414$ , and for  $m = 2$ ,  $a/b = \sqrt{6} = 2.45$ , and so on.

For a given value of  $a/b$ , the critical stress,  $\sigma_{CR} = N_{x,CR}/t$ , is found from Eqs. (9.4) and (7.4); that is,

$$\boxed{\sigma_{CR} = \frac{k\pi^2 E}{12(1 - v^2)} \left(\frac{t}{b}\right)^2} \quad (9.6)$$

In general, the critical stress for a uniform rectangular plate, with various edge supports and loaded by constant or linearly varying in-plane direct forces ( $N_x, N_y$ ) or constant shear forces ( $N_{xy}$ ) along its edges, is given by Eq. (9.6). The value of  $k$  remains a function of  $a/b$  but depends also upon the type of loading and edge support. Solutions for such problems have been obtained by solving the appropriate differential equation or by using the approximate (Rayleigh–Ritz) energy method. Values of  $k$  for a variety of loading and support conditions are shown in Fig. 9.5. For a plate subjected to shear loads along its four edges,  $k$  becomes the *shear buckling coefficient* (Fig. 9.5(c)) and  $\sigma_{CR}$  in Eq. (9.6) becomes the critical shear stress,  $\tau_{CR}$ .

We see from Fig. 9.5 that  $k$  is very nearly constant for  $a/b > 3$ . This fact is particularly useful in aircraft structures where longitudinal stiffeners are used to divide the skin into narrow panels (having small values of  $b$ ), thereby increasing the buckling stress of the skin.

## 9.2 INELASTIC BUCKLING OF PLATES

For plates having small values of  $b/t$ , the critical stress may exceed the elastic limit of the material of the plate. In such a situation, Eq. (9.6) is no longer applicable, since, as we saw in the case of columns,  $E$  becomes dependent on stress, as does Poisson's ratio  $v$ . These effects are usually included in a plasticity correction factor  $\eta$ , so that Eq. (9.6) becomes

$$\boxed{\sigma_{CR} = \frac{\eta k\pi^2 E}{12(1 - v^2)} \left(\frac{t}{b}\right)^2} \quad (9.7)$$

where  $E$  and  $v$  are elastic values of Young's modulus and Poisson's ratio. In the linearly elastic region,  $\eta = 1$ , which means that Eq. (9.7) may be applied at all stress levels. The derivation of a general expression for  $\eta$  is outside the scope of this book, but one<sup>1</sup> giving good agreement with experiment is

$$\eta = \frac{1 - v_e^2 E_s}{1 - v_p^2 E} \left[ \frac{1}{2} + \frac{1}{2} \left( \frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s} \right)^{\frac{1}{2}} \right]$$

where  $E_t$  and  $E_s$  are the tangent modulus and secant modulus (stress/strain) of the plate in the inelastic region and  $v_e$  and  $v_p$  are Poisson's ratio in the elastic and inelastic ranges.

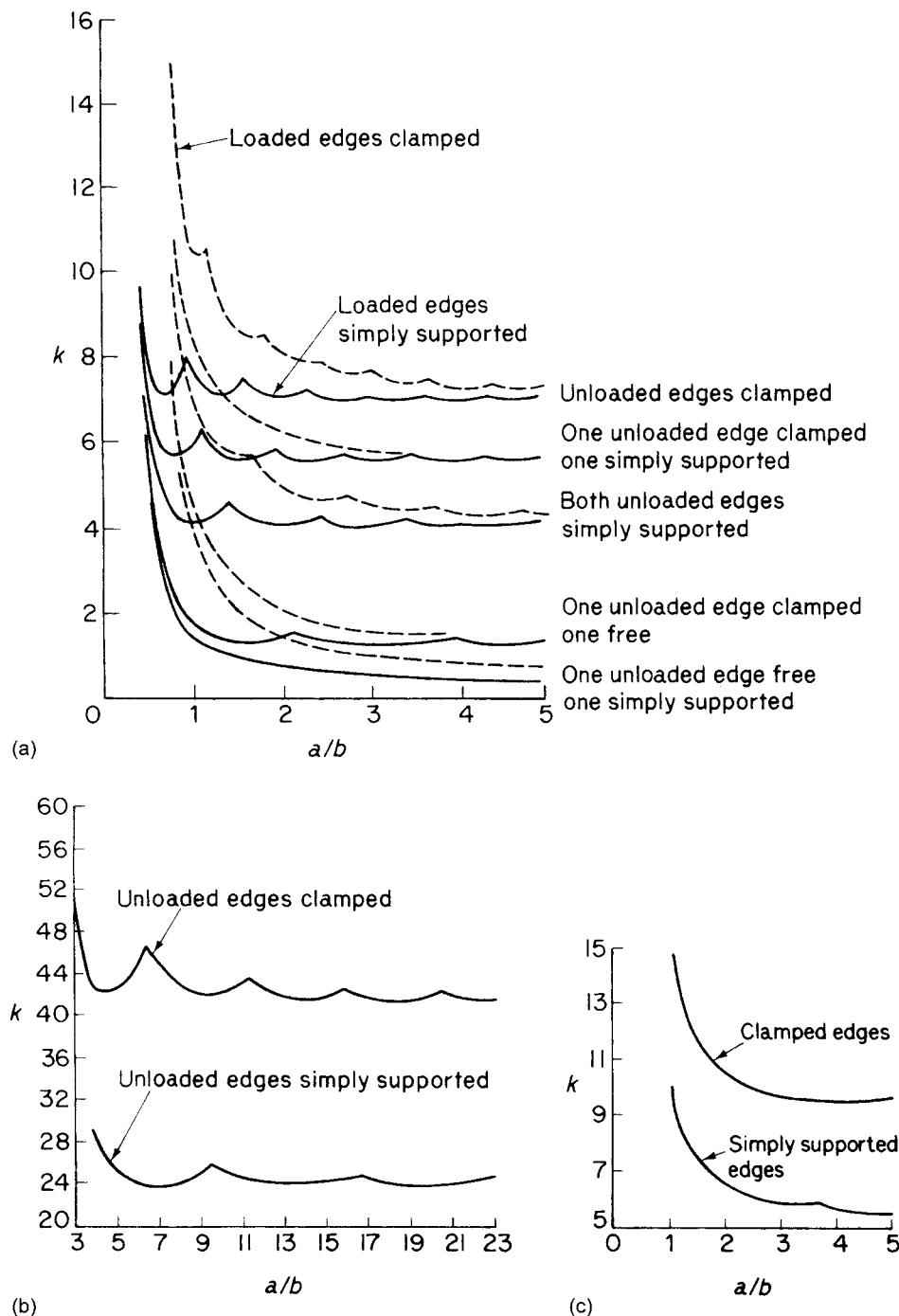


FIGURE 9.5 (a) Buckling Coefficients for Flat Plates in Compression; (b) Buckling Coefficients for Flat Plates in Bending; (c) Shear Buckling Coefficients for Flat Plates

### 9.3 EXPERIMENTAL DETERMINATION OF THE CRITICAL LOAD FOR A FLAT PLATE

In Section 8.3, we saw that the critical load for a column may be determined experimentally, without actually causing the column to buckle, by means of the Southwell plot. The critical load for an actual, rectangular thin plate is found in a similar manner.

The displacement of an initially curved plate from the zero load position was found, in Section 7.5, to be

$$w_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where

$$B_{mn} = \frac{A_{mn} N_x}{\frac{\pi^2 D}{a^2} \left( m + \frac{n^2 a^2}{mb^2} \right)^2 - N_x}$$

We see that the coefficients  $B_{mn}$  increase with an increase of compressive load intensity  $N_x$ . It follows that when  $N_x$  approaches the critical value,  $N_{x,CR}$ , the term in the series corresponding to the buckled shape of the plate becomes the most significant. For a square plate,  $n = 1$  and  $m = 1$  give a minimum value of critical load, so that at the center of the plate,

$$w_1 = \frac{A_{11} N_x}{N_{x,CR} - N_x}$$

or, rearranging,

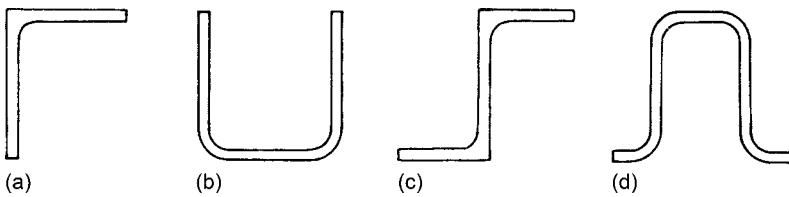
$$w_1 = N_{x,CR} \frac{w_1}{N_x} - A_{11}$$

Thus, a graph of  $w_1$  plotted against  $w_1/N_x$  has a slope, in the region of the critical load, equal to  $N_{x,CR}$ .

### 9.4 LOCAL INSTABILITY

We distinguished in the introductory remarks to Chapter 8 between primary and secondary (or local) instability. The latter form of buckling usually occurs in the flanges and webs of thin-walled columns having an effective slenderness ratio,  $l_e/r < 20$ . For  $l_e/r > 80$ , this type of column is susceptible to primary instability. In the intermediate range of  $l_e/r$  between 20 and 80, buckling occurs by a combination of both primary and secondary modes.

Thin-walled columns are encountered in aircraft structures in the shape of longitudinal stiffeners, which are normally fabricated by extrusion processes or by forming from a flat sheet. A variety of cross-sections are employed, although each is usually composed of flat plate elements arranged to form angle, channel, Z, or “top hat” sections, as shown in Fig. 9.6. We see that the plate elements fall into two distinct categories: flanges, which have a free unloaded edge; and webs, which are supported by the adjacent plate elements on both unloaded edges.



**FIGURE 9.6 (a) Extruded Angle; (b) Formed Channel; (c) Extruded Z; (d) Formed “Top Hat”**

In local instability, the flanges and webs buckle like plates, with a resulting change in the cross-section of the column. The wavelength of the buckle is of the order of the widths of the plate elements, and the corresponding critical stress is generally independent of the length of the column when the length is equal to or greater than three times the width of the largest plate element in the column cross-section.

Buckling occurs when the weakest plate element, usually a flange, reaches its critical stress, although in some cases all the elements reach their critical stresses simultaneously. When this occurs, the rotational restraint provided by adjacent elements to each other disappears and the elements behave as though they are simply supported along their common edges. These cases are the simplest to analyze and are found where the cross-section of the column is an equal-legged angle,  $T$ , cruciform, or a square tube of constant thickness. Values of local critical stress for columns possessing these types of section may be found using Eq. (9.7) and an appropriate value of  $k$ . For example,  $k$  for a cruciform section column is obtained from Fig. 9.5(a), for a plate that is simply supported on three sides with one edge free and has  $a/b > 3$ . Hence,  $k = 0.43$ , and if the section buckles elastically, the plasticity correction factor (see Eq. (9.7))  $\eta = 1$  and

$$\sigma_{CR} = 0.388E \left(\frac{t}{b}\right)^2, \quad (\nu = 0.3)$$

It must be appreciated that the calculation of local buckling stresses is generally complicated, with no particular method gaining universal acceptance, much of the information available being experimental. A detailed investigation of the topic is therefore beyond the scope of this book. When the individual plate elements do not reach their critical stresses simultaneously, an average compressive stress on the cross-section of the member may be found; this is called the *crippling* or *failure* stress  $\bar{\sigma}_f$  (see Section 9.6).

## 9.5 INSTABILITY OF STIFFENED PANELS

It is clear from Eq. (9.7) that plates having large values of  $b/t$  buckle at low values of critical stress. An effective method of reducing this parameter is to introduce stiffeners along the length of the plate thereby dividing a wide sheet into a number of smaller and more stable plates. Alternatively, the sheet may be divided into a series of wide short columns by stiffeners attached across its width. In the former type of structure, the longitudinal stiffeners carry part of the compressive load, while in the latter, all the load is supported by the plate. Frequently, both methods of stiffening are combined to form a grid-stiffened structure.

Stiffeners in earlier types of stiffened panel possessed a relatively high degree of strength compared with the thin skin, resulting in the skin buckling at a much lower stress level than the stiffeners. Such panels may be analyzed by assuming that the stiffeners provide simply supported edge conditions to a series of flat plates.

A more efficient structure is obtained by adjusting the stiffener sections so that buckling occurs in both stiffeners and skin at about the same stress. This is achieved by a construction involving closely spaced stiffeners of comparable thickness to the skin. Since their critical stresses are nearly the same, there is an appreciable interaction at buckling between skin and stiffeners so that the complete panel must be considered as a unit. However, caution must be exercised, since it is possible for the two simultaneous critical loads to interact and reduce the actual critical load of the structure<sup>2</sup> (see Example 8.5). Various modes of buckling are possible, including primary buckling, where the wavelength is of the order of the panel length, and local buckling, with wavelengths of the order of the width of the plate elements of the skin or stiffeners. A discussion of the various buckling modes of panels having Z-section stiffeners has been provided by Argyris and Dunne.<sup>3</sup>

The prediction of critical stresses for panels with a large number of longitudinal stiffeners is difficult and relies heavily on approximate (energy) and semi-empirical methods. Bleich<sup>4</sup> and Timoshenko (see Ref. 1, Chapter 8) give energy solutions for plates with one and two longitudinal stiffeners and also consider plates having a large number of stiffeners. Gerard and Becker<sup>5</sup> summarize much of the work on stiffened plates, and a large amount of theoretical and empirical data is presented by Argyris and Dunne in the *Handbook of Aeronautics*.<sup>3</sup>

For detailed work on stiffened panels, reference should be made to as much as possible of the preceding work. The literature is extensive, however, so that here we present a relatively simple approach suggested by Gerard.<sup>1</sup> Figure 9.7 represents a panel of width  $w$  stiffened by longitudinal members that may be flats (as shown), Z, I, channel, or “top hat” sections. It is possible for the panel to behave as a Euler column, its cross-section being that shown in Fig. 9.7. If the equivalent length of the panel acting as a column is  $l_e$ , then the Euler critical stress is

$$\sigma_{\text{CR,E}} = \frac{\pi^2 E}{(l_e/r)^2}$$

as in Eq. (8.8). In addition to the column buckling mode, individual plate elements constituting the panel cross-section may buckle as long plates. The buckling stress is then given by Eq. (9.7); that is,

$$\sigma_{\text{CR}} = \frac{\eta k \pi^2 E}{12(1 - v^2)} \left(\frac{t}{b}\right)^2$$

where the values of  $k$ ,  $t$ , and  $b$  depend upon the particular portion of the panel being investigated. For example, the portion of skin between stiffeners may buckle as a plate simply supported on all

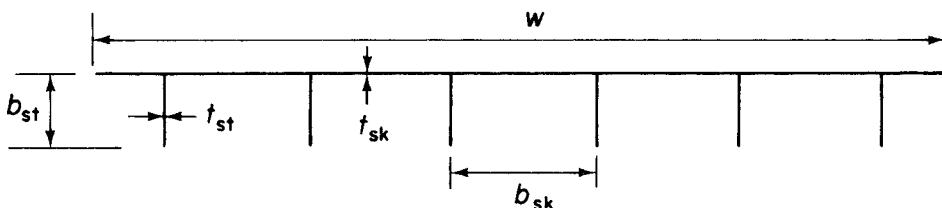


FIGURE 9.7 Stiffened Panel

four sides. Thus, for  $a/b > 3$ ,  $k = 4$  from Fig. 9.5(a), and, assuming that buckling takes place in the elastic range,

$$\sigma_{\text{CR}} = \frac{4\pi^2 E}{12(1 - v^2)} \left( \frac{t_{\text{sk}}}{b_{\text{sk}}} \right)^2$$

A further possibility is that the stiffeners may buckle as long plates simply supported on three sides with one edge free. Thus,

$$\sigma_{\text{CR}} = \frac{0.43\pi^2 E}{12(1 - v^2)} \left( \frac{t_{\text{st}}}{b_{\text{st}}} \right)^2$$

Clearly, the minimum value of these critical stresses is the critical stress for the panel taken as a whole.

The compressive load is applied to the panel over its complete cross-section. To relate this load to an applied compressive stress  $\sigma_A$  acting on each element of the cross-section, we divide the load per unit width, say  $N_x$ , by an equivalent skin thickness  $\bar{t}$ , hence,

$$\sigma_A = \frac{N_x}{\bar{t}}$$

where

$$\bar{t} = \frac{A_{\text{st}}}{b_{\text{sk}}} + t_{\text{sk}}$$

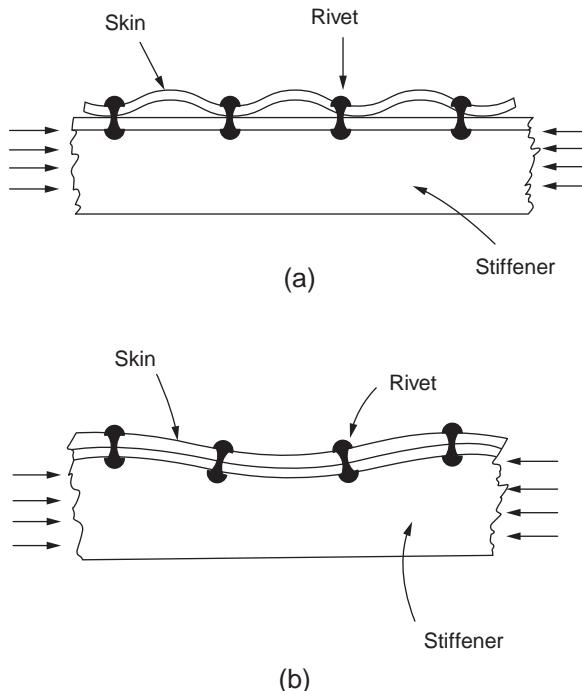
and  $A_{\text{st}}$  is the stiffener area.

The above remarks are concerned with the primary instability of stiffened panels. Values of local buckling stress have been determined by Boughan, Baab, and Gallaher for idealized web, Z, and T stiffened panels. The results are reproduced in Rivello<sup>6</sup> together with the assumed geometries.

Further types of instability found in stiffened panels occur where the stiffeners are riveted or spot-welded to the skin. Such structures may be susceptible to *interrivet buckling*, in which the skin buckles between rivets with a wavelength equal to the rivet pitch; or *wrinkling*, where the stiffener forms an elastic line support for the skin (see Fig. 9.8(a) and (b)). In the latter mode, the wavelength of the buckle is greater than the rivet pitch, and separation of skin and stiffener does not occur. Methods of estimating the appropriate critical stresses are given in Rivello<sup>6</sup> and the *Handbook of Aeronautics*.<sup>3</sup>

## 9.6 FAILURE STRESS IN PLATES AND STIFFENED PANELS

The previous discussion on plates and stiffened panels investigated the prediction of buckling stresses. However, as we have seen, plates retain some of their capacity to carry load, even though a portion of the plate has buckled. In fact, the ultimate load is not reached until the stress in the majority of the plate exceeds the elastic limit. The theoretical calculation of the ultimate stress is difficult, since non-linearity results from both large deflections and the inelastic stress-strain relationship.



**FIGURE 9.8 (a) Interrivet Buckling; (b) Wrinkling**

Gerard<sup>1</sup> proposes a semi-empirical solution for flat plates supported on all four edges. After elastic buckling occurs, theory and experiment indicate that the average compressive stress,  $\bar{\sigma}_a$ , in the plate and the unloaded edge stress,  $\sigma_e$ , are related by the following expression:

$$\frac{\bar{\sigma}_a}{\sigma_{CR}} = \alpha_1 \left( \frac{\sigma_e}{\sigma_{CR}} \right)^n \quad (9.8)$$

where

$$\sigma_{CR} = \frac{k\pi^2 E}{12(1 - v^2)} \left( \frac{t}{b} \right)^2$$

and  $\alpha_1$  is some unknown constant. Theoretical work by Stowell<sup>7</sup> and Mayers and Budiansky<sup>8</sup> shows that failure occurs when the stress along the unloaded edge is approximately equal to the compressive yield strength,  $\sigma_{cy}$ , of the material. Substituting  $\sigma_{cy}$  for  $\sigma_e$  in Eq. (9.8) and multiplying both sides by  $(\sigma_{CR}/\sigma_{cy})$  gives

$$\frac{\bar{\sigma}_a}{\sigma_{cy}} = \alpha_1 \frac{(\sigma_{cy})^n (\sigma_{CR})}{(\sigma_{CR})(\sigma_{cy})}$$

so that

$$\frac{\bar{\sigma}_f}{\sigma_{cy}} = \alpha_1 \left( \frac{\sigma_{CR}}{\sigma_{cy}} \right)^{1-n} \quad (9.9)$$

where the average compressive stress in the plate has become the average stress at failure  $\bar{\sigma}_f$ . Now substituting for  $\sigma_{CR}$  in Eq. (9.9) we have

$$\frac{\bar{\sigma}_f}{\sigma_{cy}} = \alpha_1 \left[ \frac{k\pi^2 E}{12(1 - v^2)} \left( \frac{t}{b} \right)^2 \right]^{1-n} \left( \frac{1}{\sigma_{cy}} \right)^{1-n}$$

which gives

$$\frac{\bar{\sigma}_f}{\sigma_{cy}} = \frac{\alpha_1 k^{1-n} \pi^{2(1-n)} E^{1-n}}{12(1 - v^2)^{1-n}} \left( \frac{t}{b} \right)^{2(1-n)} \left( \frac{1}{\sigma_{cy}} \right)^{1-n}$$

Putting

$$\frac{\alpha_1 \pi^{2(1-n)}}{[12(1 - v^2)]^{1-n}} = \alpha$$

yields

$$\frac{\bar{\sigma}_f}{\sigma_{cy}} = \alpha k^{1-n} \left[ \frac{t}{b} \left( \frac{E}{\sigma_{cy}} \right)^{\frac{1}{2}} \right]^{2(1-n)} \quad (9.10)$$

Having obtained the failure stress  $\bar{\sigma}_f$  for each wall of the cross-section, an average failure stress  $\bar{\sigma}_{f,a}$  may be calculated from

$$\bar{\sigma}_{f,a} = \frac{\sum_{i=1}^n \bar{\sigma}_{f,i} A_i}{\sum_{i=1}^n A_i}$$

in which  $A_i = bt_i$ . The failure load,  $P_f$ , for the cross-section is then given by

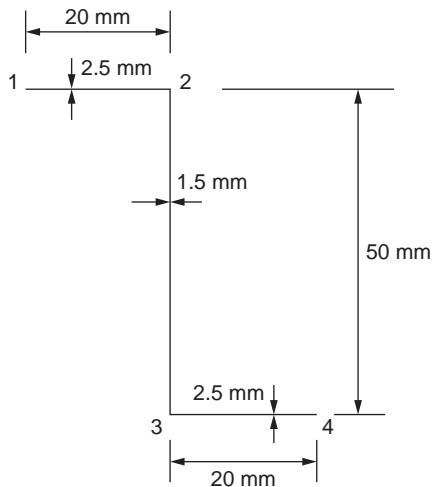
$$P_f = \bar{\sigma}_{f,a} A$$

where  $A$  is the total area of the cross-section.

### Example 9.1

Fig. 9.9 shows the cross-section of a long column. For the material of the column Young's modulus  $E = 70,000 \text{ N/mm}^2$ , Poisson's ratio  $v = 0.3$ , and the compressive yield strength is  $480 \text{ N/mm}^2$ . Calculate the failure stress for the column and hence the failure load.

Since the column is said to be long it may be assumed that for each wall of the column  $a/b > 3$ .



**FIGURE 9.9 Column Cross-Section of Example 9.1**

#### Flange 12 (and 34)

This may be regarded as a flat plate with one edge free and three edges simply supported. Then  $k=0.43$  (see Fig. 9.5(a)). Also, from Ref. 6,  $n=0.6$  and  $\alpha_1=0.8$  so that

$$\alpha = \frac{0.8 \times \pi^{0.8}}{\left[12(1 - 0.3^2)\right]^{0.4}} = 0.77$$

Then, from Eq. (9.10),

$$\frac{\bar{\sigma}_f}{\sigma_{cy}} = 0.77 \times 0.43^{0.4} \left[ \frac{2.5}{20} \left( \frac{70000}{480} \right)^{0.5} \right]^{0.8} = 0.76$$

and  $\bar{\sigma}_f = 0.76 \times 480 = 366.6 \text{ N/mm}^2$ .

#### Web 23

The web may be considered to be a flat plate simply supported on all four sides so that, from Fig. 9.5(a),  $k=4.0$ . Again, from Ref. 6,  $n=0.6$  and  $\alpha_1=0.8$  so that  $\alpha=0.77$  as for the flange 12. Then, from Eq. (9.10),

$$\frac{\bar{\sigma}_f}{\sigma_{cy}} = 0.77 \times 4.0^{0.4} \left[ \frac{1.5}{50} \left( \frac{70000}{480} \right)^{0.5} \right]^{0.8} = 0.6$$

which gives  $\bar{\sigma}_f = 0.6 \times 480 = 288.0 \text{ N/mm}^2$ .

The average failure stress is then

$$\bar{\sigma}_{f,a} = \frac{2 \times 366.6 \times 2.5 \times 20 + 288.0 \times 1.5 \times 50}{2 \times 2.5 \times 20 + 1.5 \times 50} = 332.9 \text{ N/mm}^2$$

The failure load follows and is

$$P_f = 332.9 \times 175 = 58,257.5 \text{ N}$$

Eq. (9.10) may be written in simplified form as

$$\frac{\bar{\sigma}_f}{\sigma_{cy}} = \beta \left[ \frac{t}{b} \left( \frac{E}{\sigma_{cy}} \right)^{\frac{1}{2}} \right]^m \quad (9.11)$$

where  $\beta = \alpha k^{m/2}$ . The constants  $\beta$  and  $m$  are determined by the best fit of Eq. (9.11) to test data.

Experiments on simply supported flat plates and square tubes of various aluminum and magnesium alloys and steel show that  $\beta = 1.42$  and  $m = 0.85$  fit the results within  $\pm 10$  percent up to the yield strength. Corresponding values for long clamped flat plates are  $\beta = 1.80$ ,  $m = 0.85$ .

Gerard<sup>9-12</sup> extended this method to the prediction of local failure stresses for the plate elements of thin-walled columns. Equation (9.11) becomes

$$\frac{\bar{\sigma}_f}{\sigma_{cy}} = \beta_g \left[ \left( \frac{gt^2}{A} \right) \left( \frac{E}{\sigma_{cy}} \right)^{\frac{1}{2}} \right]^m \quad (9.12)$$

where  $A$  is the cross-sectional area of the column,  $\beta_g$  and  $m$  are empirical constants, and  $g$  is the number of cuts required to reduce the cross-section to a series of flanged sections plus the number of flanges that would exist after the cuts are made. Examples of the determination of  $g$  are shown in Fig. 9.10.

The local failure stress in longitudinally stiffened panels was determined by Gerard<sup>10,12</sup> using a slightly modified form of Eqs. (9.11) and (9.12). Thus, for a section of the panel consisting of a stiffener and a width of skin equal to the stiffener spacing, eqn.

$$\frac{\bar{\sigma}_f}{\sigma_{cy}} = \beta_g \left[ \frac{gt_{sk}t_{st}}{A} \left( \frac{E}{\bar{\sigma}_{cy}} \right)^{\frac{1}{2}} \right]^m \quad (9.13)$$

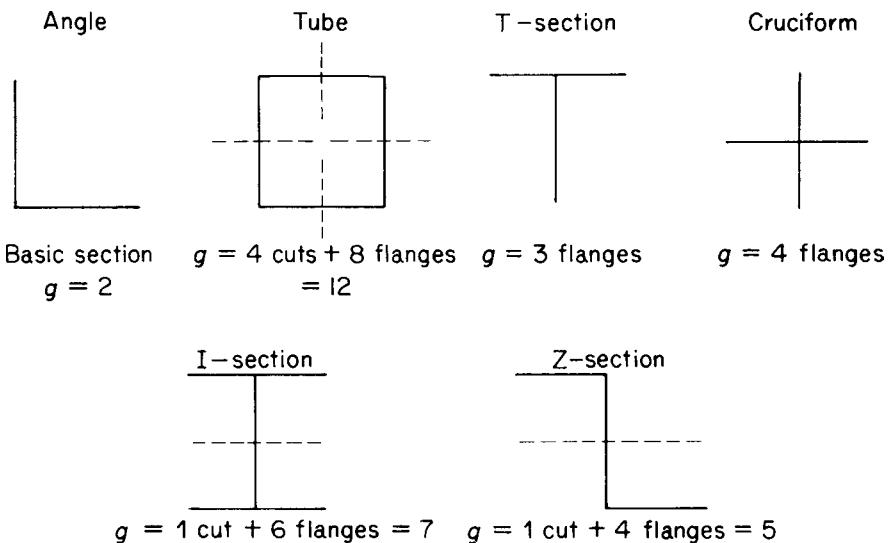
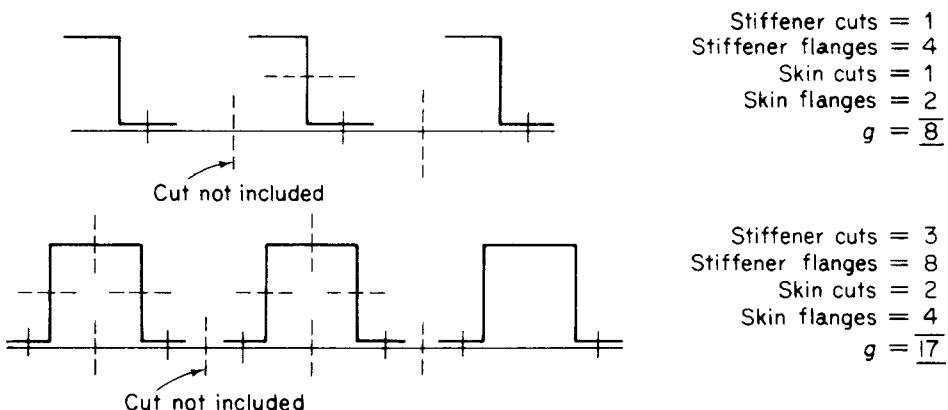


FIGURE 9.10 Determination of Empirical Constant  $g$



**FIGURE 9.11 Determination of  $g$  for Two Types of Stiffener–Skin Combination**

where  $t_{sk}$  and  $t_{st}$  are the skin and stiffener thicknesses, respectively. A weighted yield stress  $\bar{\sigma}_{cy}$  is used for a panel in which the material of the skin and stiffener have different yield stresses, thus,

$$\bar{\sigma}_{cy} = \frac{\sigma_{cy\ sk} + \bar{\sigma}_{cy\ st} [(\bar{t}/t_{st}) - 1]}{\bar{t}/t_{st}}$$

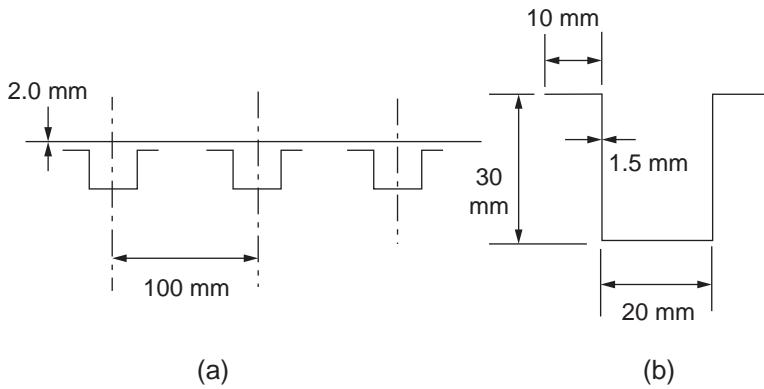
where  $\bar{t}$  is the average or equivalent skin thickness previously defined. The parameter  $g$  is obtained in a similar manner to that for a thin-walled column, except that the number of cuts in the skin and the number of equivalent flanges of the skin are included. A cut to the left of a stiffener is not counted, since it is regarded as belonging to the stiffener to the left of that cut. The calculation of  $g$  for two types of skin/stiffener combination is illustrated in Fig. 9.11. Equation (9.13) is applicable to either monolithic or built-up panels when, in the latter case, interrivet buckling and wrinkling stresses are greater than the local failure stress. Values of  $\beta_g$  and  $m$  are given in Rivello<sup>6</sup> where typically  $\beta_g = 0.56$  for a “top hat” section stiffener and  $m = 0.85$ .

The values of failure stress given by Eqs. (9.11), (9.12), and (9.13) are associated with local or secondary instability modes. Consequently, they apply when  $l/r \leq 20$ . In the intermediate range between the local and primary modes, failure occurs through a combination of both. At the moment, no theory satisfactorily predicts failure in this range, and we rely on test data and empirical methods. The NACA (now NASA) produced direct reading charts for the failure of “top hat,” Z, and Y section stiffened panels; a bibliography of the results is given by Gerard.<sup>10</sup>

It must be remembered that research into methods of predicting the instability and postbuckling strength of the thin-walled types of structure associated with aircraft construction is a continuous process. Modern developments include the use of the computer-based finite element technique (see Chapter 6) and the study of the sensitivity of thin-walled structures to imperfections produced during fabrication; much useful information and an extensive bibliography is contained in Murray.<sup>2</sup>

**Example 9.2**

Figure 9.12(a) shows part of the cross-section of the upper surface of a long wing panel in which the thin wing skin is stiffened by a series of “top hat” stiffeners whose cross-section is shown in Fig. 9.12(b). Both the skin and stiffeners are fabricated from aluminum alloy having a Young’s modulus of  $70,000 \text{ N/mm}^2$  although the compressive yield strength of the skin is  $450 \text{ N/mm}^2$  and that of the stiffeners is  $500 \text{ N/mm}^2$ . Take  $\beta_g = 0.56$  and  $m = 0.85$  and calculate the failure stress for the panel.



**FIGURE 9.12 Cross-Section of the Wing Panel of Example 9.2**

The area of cross-section of a stiffener is given by

$$A_{st} = 1.5 (2 \times 10 + 2 \times 30 + 20) = 150 \text{ mm}^2$$

and

$$A (\text{stiffener} + \text{skin}) = 150 + 100 \times 2.0 = 350 \text{ mm}^2$$

Then,

$$\bar{t} = \frac{150}{100} + 2.0 = 3.5 \text{ mm}$$

and

$$\bar{\sigma}_{cy} = \frac{450 + 500[(3.5/1.5) - 1]}{3.5/1.5} = 478.6 \text{ N/mm}^2$$

From Fig. 9.11,  $g = 17$ .

Then, from Eq. (9.13),

$$\bar{\sigma}_f = 478.6 \times 0.56 \left[ \frac{17 \times 2.0 \times 1.5}{350} \left( \frac{70,000}{478.6} \right)^{1/2} \right]^{0.85}$$

which gives

$$\bar{\sigma}_f = 433.8 \text{ N/mm}^2$$

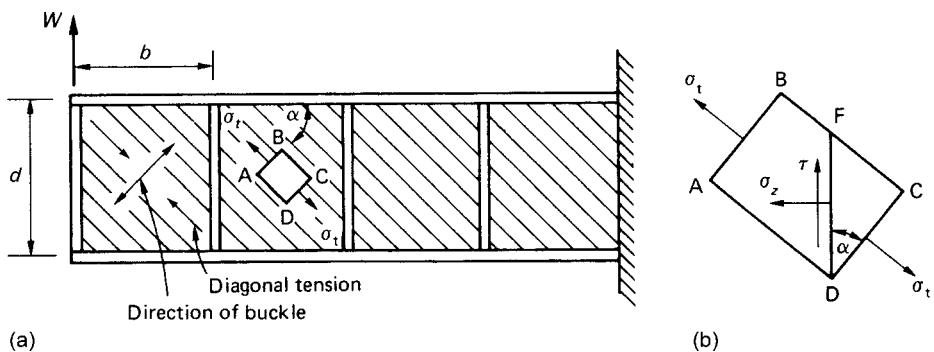


FIGURE 9.13 Diagonal Tension Field Beam

## 9.7 TENSION FIELD BEAMS

The spars of aircraft wings usually comprise an upper and a lower flange connected by thin, stiffened webs. These webs are often of such a thickness that they buckle under shear stresses at a fraction of their ultimate load. The form of the buckle is shown in Fig. 9.13(a), where the web of the beam buckles under the action of internal diagonal compressive stresses produced by shear, leaving a wrinkled web capable of supporting diagonal tension only in a direction perpendicular to that of the buckle; the beam is then said to be a *complete tension field beam*.

### 9.7.1 Complete diagonal tension

The theory presented here is due to H. Wagner.

[Em WEB leia-se ALMA](#)

The beam shown in Fig. 9.13(a) has concentrated flange areas having a depth  $d$  between their centroids and vertical stiffeners spaced uniformly along the length of the beam. It is assumed that the flanges resist the internal bending moment at any section of the beam while the web, of thickness  $t$ , resists the vertical shear force. The effect of this assumption is to produce a uniform shear stress distribution through the depth of the web (see Section 20.3) at any section. Therefore, at a section of the beam where the shear force is  $S$ , the shear stress  $\tau$  is given by

$$\tau = \frac{S}{td} \quad (9.14)$$

Consider now an element ABCD of the web in a panel of the beam, as shown in Fig. 9.13(a). The element is subjected to tensile stresses,  $\sigma_t$ , produced by the diagonal tension on the planes AB and CD; the angle of the diagonal tension is  $\alpha$ . On a vertical plane FD in the element, the shear stress is  $\tau$  and the direct stress  $\sigma_z$ . Now, considering the equilibrium of the element FCD (Fig. 9.13(b)) and resolving forces vertically, we have (see Section 1.6)

$$\sigma_t C D t \sin \alpha = \tau F D t$$

which gives

$$\sigma_t = \frac{\tau}{\sin \alpha \cos \alpha} = \frac{2\tau}{\sin 2\alpha} \quad (9.15)$$

or, substituting for  $\tau$  from Eq. (9.14) and noting that, in this case,  $S = W$  at all sections of the beam,

$$\sigma_t = \frac{2W}{td \sin 2\alpha} \quad (9.16)$$

Further, resolving forces horizontally for the element FCD,

$$\sigma_z FDt = \sigma_t CDt \cos \alpha$$

which gives

$$\sigma_z = \sigma_t \cos^2 \alpha$$

or, substituting for  $\sigma_t$  from Eq. (9.15),

$$\sigma_z = \frac{\tau}{\tan \alpha} \quad (9.17)$$

or, for this particular beam, from Eq. (9.14),

$$\sigma_z = \frac{W}{td \tan \alpha} \quad (9.18)$$

Since  $\tau$  and  $\sigma_t$  are constant through the depth of the beam, it follows that  $\sigma_z$  is constant through the depth of the beam.

The direct loads in the flanges are found by considering a length  $z$  of the beam, as shown in Fig. 9.14. On the plane  $mm'$ , direct and shear stresses  $\sigma_z$  and  $\tau$  are acting in the web, together with direct loads  $F_T$  and  $F_B$  in the top and bottom flanges, respectively.  $F_T$  and  $F_B$  are produced by a combination of the bending moment  $Wz$  at the section and the compressive action ( $\sigma_z$ ) of the diagonal tension. Taking moments about the bottom flange,

$$Wz = F_T d - \frac{\tau z w^2}{2}$$

Hence, substituting for  $\sigma_z$  from Eq. (9.18) and rearranging,

$$F_T = \frac{Wz}{d} + \frac{W}{2 \tan \alpha} \quad (9.19)$$

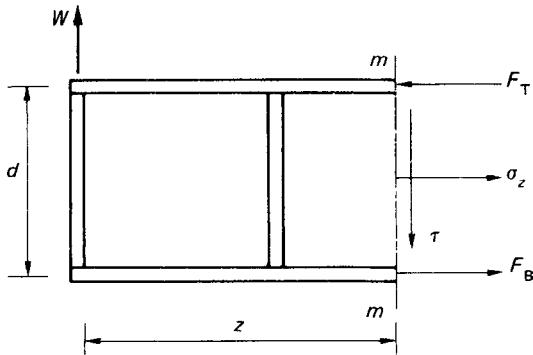
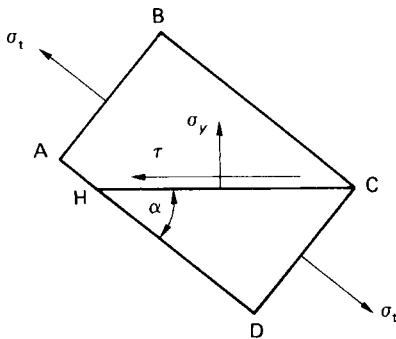


FIGURE 9.14 Determination of Flange Forces



**FIGURE 9.15 Stress System on a Horizontal Plane in the Beam Web**

Now, resolving forces horizontally,

$$F_B - F_T + \sigma_z t d = 0$$

which gives, on substituting for  $\sigma_z$  and  $F_T$  from Eqs. (9.18) and (9.19),

$$F_B = \frac{Wz}{d} - \frac{W}{2 \tan \alpha} \quad (9.20)$$

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The diagonal tension stress  $\sigma_t$  induces a direct stress  $\sigma_y$  on horizontal planes at any point in the web. Then, on a horizontal plane HC in the element ABCD of Fig. 9.13, there is a direct stress  $\sigma_y$  and a complementary shear stress  $\tau$ , as shown in Fig. 9.15.

From a consideration of the vertical equilibrium of the element HDC we have

$$\sigma_y H C t = \sigma_t C D t \sin \alpha$$

which gives

$$\sigma_y = \sigma_t \sin^2 \alpha$$

Substituting for  $\sigma_t$  from Eq. (9.15),

$$\sigma_y = \tau \tan \alpha \quad (9.21)$$

or, from Eq. (9.14), in which  $S = W$ ,

$$\sigma_y = \frac{W}{t d} \tan \alpha \quad (9.22)$$

The tensile stresses  $\sigma_y$  on horizontal planes in the web of the beam cause compression in the vertical stiffeners. Each stiffener may be assumed to support half of each adjacent panel in the beam, so that the compressive load  $P$  in a stiffener is given by

$$P = \sigma_y t b$$

which becomes, from Eq. (9.22),

$$P = \frac{Wb}{d} \tan \alpha \quad (9.23)$$

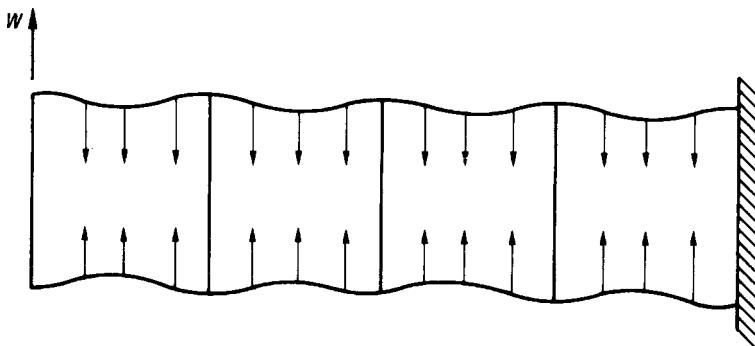


FIGURE 9.16 Bending of Flanges Due to Web Stress

If the load  $P$  is sufficiently high, the stiffeners buckle. Tests indicate that they buckle as columns of equivalent length:

$$\boxed{\begin{aligned} l_e &= d/\sqrt{4 - 2b/d}, & \text{for } b \leq 1.5d \\ l_e &= d, & \text{for } b \geq 1.5d \end{aligned}} \quad (9.24)$$

In addition to causing compression in the stiffeners, the direct stress  $\sigma_y$  produces bending of the beam flanges between the stiffeners, as shown in Fig. 9.16. Each flange acts as a continuous beam carrying a uniformly distributed load of intensity  $\sigma_y t$ . The maximum bending moment in a continuous beam with ends fixed against rotation occurs at a support and is  $wL^2/12$ , in which  $w$  is the load intensity and  $L$  the beam span. In this case, therefore, the maximum bending moment  $M_{\max}$  occurs at a stiffener and is given by

$$M_{\max} = \frac{\sigma_y tb^2}{12}$$

or, substituting for  $\sigma_y$  from Eq. (9.22),

$$M_{\max} = \frac{Wb^2 \tan \alpha}{12d} \quad (9.25)$$

Midway between the stiffeners this bending moment reduces to  $Wb^2 \tan \alpha / 24d$ .

The angle  $\alpha$  adjusts itself such that the total strain energy of the beam is a minimum. If it is assumed that the flanges and stiffeners are rigid, then the strain energy comprises the shear strain energy of the web only and  $\alpha = 45^\circ$ . In practice, both flanges and stiffeners deform, so that  $\alpha$  is somewhat less than  $45^\circ$ , usually of the order of  $40^\circ$  and, in the type of beam common to aircraft structures, rarely below  $38^\circ$ . For beams having all components made of the same material, the condition of minimum strain energy leads to various equivalent expressions for  $\alpha$ , one of which is

$$\tan^2 \alpha = \frac{\sigma_t + \sigma_F}{\sigma_t + \sigma_S} \quad (9.26)$$

in which  $\sigma_F$  and  $\sigma_S$  are the uniform direct compressive stresses induced by the diagonal tension in the flanges and stiffeners, respectively. Thus, from the second term on the right-hand side of either Eq. (9.19) or (9.20),

$$\sigma_F = \frac{W}{2A_F \tan \alpha} \quad (9.27)$$

in which  $A_F$  is the cross-sectional area of each flange. Also, from Eq. (9.23),

$$\sigma_S = \frac{Wb}{A_S d} \tan \alpha \quad (9.28)$$

where  $A_S$  is the cross-sectional area of a stiffener. Substitution of  $\sigma_t$  from Eq. (9.16) and  $\sigma_F$  and  $\sigma_S$  from Eqs. (9.27) and (9.28) into Eq. (9.26) produces an equation that may be solved for  $\alpha$ . An alternative expression for  $\alpha$ , again derived from a consideration of the total strain energy of the beam, is

$$\tan^4 \alpha = \frac{1 + td/2A_F}{1 + tb/A_S} \quad (9.29)$$

### Example 9.3

The beam shown in Fig. 9.17 is assumed to have a complete tension field web. If the cross-sectional areas of the flanges and stiffeners are, respectively,  $350 \text{ mm}^2$  and  $300 \text{ mm}^2$  and the elastic section modulus of each flange is  $750 \text{ mm}^3$ , determine the maximum stress in a flange and also whether or not the stiffeners buckle. The thickness of the web is 2 mm and the second moment of area of a stiffener about an axis in the plane of the web is  $2,000 \text{ mm}^4$ ;  $E = 70,000 \text{ N/mm}^2$ .

From Eq. (9.29),

$$\tan^4 \alpha = \frac{1 + 2 \times 400/(2 \times 350)}{1 + 2 \times 300/300} = 0.7143$$

so that

$$\alpha = 42.6^\circ$$

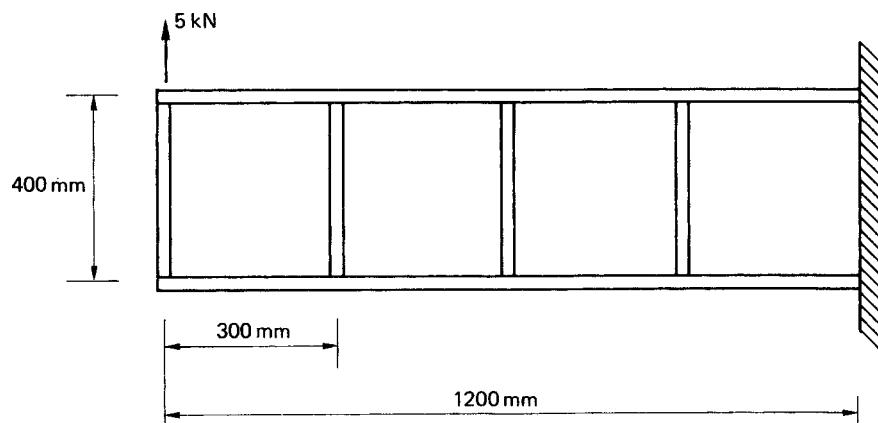


FIGURE 9.17 Beam of Example 9.3

The maximum flange stress occurs in the top flange at the built-in end, where the bending moment on the beam is greatest and the stresses due to bending and diagonal tension are additive. Therefore, from Eq. (9.19),

$$F_T = \frac{5 \times 1200}{400} + \frac{5}{2 \tan 42.6^\circ}$$

that is,

$$F_T = 17.7 \text{ kN}$$

Hence, the direct stress in the top flange produced by the externally applied bending moment and the diagonal tension is  $17.7 \times 10^3 / 350 = 50.6 \text{ N/mm}^2$ . In addition to this uniform compressive stress, local bending of the type shown in Fig. 9.16 occurs. The local bending moment in the top flange at the built-in end is found using Eq. (9.25):

$$M_{\max} = \frac{5 \times 10^3 \times 300^2 \tan 42.6^\circ}{12 \times 400} = 8.6 \times 10^4 \text{ Nmm}$$

The maximum compressive stress corresponding to this bending moment occurs at the lower extremity of the flange and is  $8.6 \times 10^4 / 750 = 114.9 \text{ N/mm}^2$ . Thus, the maximum stress in the flange occurs on the inside of the top flange at the built-in end of the beam, is compressive, and is equal to  $114.9 + 50.6 = 165.5 \text{ N/mm}^2$ .

The compressive load in a stiffener is obtained using Eq. (9.23):

$$P = \frac{5 \times 300 \tan 42.6^\circ}{400} = 3.4 \text{ kN}$$

Since, in this case,  $b < 1.5d$ , the equivalent length of a stiffener as a column is given by the first of Eqs. (9.24):

$$l_e = 400 / \sqrt{4 - 2 \times 300 / 400} = 253 \text{ mm}$$

From Eq. (8.7), the buckling load of a stiffener is then

$$P_{CR} = \frac{\pi^2 \times 70,000 \times 2,000}{253^2} = 22.0 \text{ kN}$$

Clearly, the stiffener does not buckle.

### Example 9.3: MATLAB

Repeat Example 9.3 using MATLAB.

The maximum stress in a flange and the determination whether or not the stiffener buckles are obtained through the following MATLAB file:

```
% Declare any needed variables
t = 2;
d = 400;
b = 300;
A_F = 350;
A_S = 300;
L = 1200;
```

```

E = 70000;
S = 750;
I = 2000;
W = 5;

% Solve Eq. (9.29) for the angle of diagonal tension (alpha)
alpha = atan(((1+t*d/(2*A_F))/(1+t*b/A_S))^(1/4));

% The maximum flange stress and bending moment will occur in the top of the
% flange at the built-in end where the stresses due to bending and diagonal
% tension are additive.

% Evaluate Eq. (9.19) in the top of the flange at the built-in end
z = L;
F_T = W*z/d + 5/(2*tan(alpha)); % Eq. (9.19)

% Calculate the direct stress in the top flange produced by the externally
% applied bending moment
sig_d = F_T/A_F;

% Calculate the local bending moment in the top flange using Eq. (9.25)
M_max = W*b^2*tan(alpha)/(12*d);

% Calculate the corresponding maximum compressive stress due to M_max
sig_M = M_max/S;

% Calculate the combined maximum stress in a flange
sig_max = sig_d + sig_M;

% Calculate the compressive load in a stiffener using Eq. (9.23)
P = W*b*tan(alpha)/d;

% Substitute P into Eq. (9.24) to calculate the equivalent length of a
% stiffener as a column
if b < 1.5*d
l_e = d/sqrt(4 - 2*b/d);
else
l_e = d;
end

% Calculate the buckling load of a stiffener using Eq. (8.7)
P_CR = pi^2*E*I/(l_e^2);

% Check if the stiffener will buckle
if P_CR > P
disp('The stiffener will not buckle')
else
disp('The stiffener will buckle')
end

% Output the combined max stress to the Command Window
disp(['sig_max =' char(vpa(sig_max*1000,4)) 'N/mm^2'])
disp(['P_CR =' num2str(round(P_CR/1000)) 'kN'])

```

The Command Window output resulting from this MATLAB file is as follows:

```
The stiffener will not buckle
sig_max = 165.5 N/mm^2
P_CR = 22 kN
```

In Eqs. (9.28) and (9.29,) it is implicitly assumed that a stiffener is fully effective in resisting axial load. This is the case if the centroid of area of the stiffener lies in the plane of the beam web. Such a situation arises when the stiffener consists of two members symmetrically arranged on opposite sides of the web. In the case where the web is stiffened by a single member attached to one side, the compressive load  $P$  is offset from the stiffener axis, thereby producing bending in addition to axial load. For a stiffener having its centroid a distance  $e$  from the center of the web, the combined bending and axial compressive stress,  $\sigma_c$ , at a distance  $e$  from the stiffener centroid, is

$$\sigma_c = \frac{P}{A_s} + \frac{Pe^2}{A_s r^2} \quad (\text{See Ref. [15]. Combined Bending and Direct Stress})$$

in which  $r$  is the radius of gyration of the stiffener cross-section about its neutral axis (note, second moment of area  $I = A r^2$ ). Then,

$$\sigma_c = \frac{P}{A_s} \left[ 1 + \left( \frac{e}{r} \right)^2 \right]$$

or

$$\sigma_c = \frac{P}{A_{S_e}}$$

where

$$A_{S_e} = \frac{A_s}{1 + (e/r)^2} \quad (9.30)$$

and is termed the *effective stiffener area*.

### 9.7.2 Incomplete diagonal tension

In modern aircraft structures, beams having extremely thin webs are rare. They retain, after buckling, some of their ability to support loads, so that even near failure, they are in a state of stress somewhere between that of pure diagonal tension and the pre-buckling stress. Such a beam is described as an *incomplete diagonal tension field beam* and may be analyzed by semi-empirical theory as follows.

It is assumed that the nominal web shear  $\tau (= S/t d)$  may be divided into a “true shear” component  $\tau_S$  and a diagonal tension component  $\tau_{DT}$  by writing

$$\tau_{DT} = k\tau, \quad \tau_S = (1 - k)\tau \quad (9.31)$$

where  $k$ , the *diagonal tension factor*, is a measure of the degree to which the diagonal tension is developed. A completely unbuckled web has  $k = 0$ , whereas  $k = 1$  for a web in complete diagonal tension. The value of  $k$  corresponding to a web having a critical shear stress  $\tau_{CR}$  is given by the empirical expression

$$k = \tanh\left(0.5 \log \frac{\tau}{\tau_{CR}}\right) \quad (9.32)$$

The ratio  $\tau/\tau_{CR}$  is known as the *loading ratio* or *buckling stress ratio*. The buckling stress  $\tau_{CR}$  may be calculated from the formula

$$\tau_{CR,elastic} = k_{ss}E \left(\frac{t}{b}\right)^2 \left[ R_d + \frac{1}{2}(R_b - R_d) \left(\frac{b}{d}\right)^3 \right] \quad (9.33)$$

where  $k_{ss}$  is the coefficient for a plate with simply supported edges and  $R_d$  and  $R_b$  are empirical restraint coefficients for the vertical and horizontal edges of the web panel, respectively. Graphs giving  $k_{ss}$ ,  $R_d$ , and  $R_b$  are reproduced in Kuhn.<sup>13</sup>

The stress equations (9.27) and (9.28) are modified in the light of these assumptions and may be rewritten in terms of the applied shear stress  $\tau$  as

$$\sigma_F = \frac{k\tau \cot \alpha}{(2A_F/tb) + 0.5(1-k)} \quad (9.34)$$

$$\sigma_S = \frac{k\tau \tan \alpha}{(A_S/tb) + 0.5(1-k)} \quad (9.35)$$

Further, the web stress  $\sigma_t$  given by Eq. (9.15) becomes two direct stresses:  $\sigma_1$  along the direction of  $\alpha$  given by

$$\sigma_1 = \frac{2k\tau}{\sin 2\alpha} + \tau(1-k) \sin 2\alpha \quad (9.36)$$

and  $\sigma_2$  perpendicular to this direction given by

$$\sigma_2 = -\tau(1-k) \sin 2\alpha \quad (9.37)$$

The secondary bending moment of Eq. (9.25) is multiplied by the factor  $k$ , while the effective lengths for the calculation of stiffener buckling loads become (see Eqs. (9.24))

$$\left. \begin{aligned} l_e &= d_s / \sqrt{1 + k^2(3 - 2b/d_s)}, & \text{for } b < 1.5d \\ l_e &= d_s, & \text{for } b > 1.5d \end{aligned} \right\}$$

where  $d_s$  is the actual stiffener depth, as opposed to the effective depth  $d$  of the web, taken between the web–flange connections, as shown in Fig. 9.18. We observe that Eqs. (9.34)–(9.37) are applicable to either incomplete or complete diagonal tension field beams, since, for the latter case,  $k = 1$ , giving the results of Eqs. (9.27), (9.28), and (9.15).

In some cases, beams taper along their lengths, in which case the flange loads are no longer horizontal but have vertical components that reduce the shear load carried by the web. Thus, in Fig. 9.19, where  $d$  is the depth of the beam at the section considered, we have, resolving forces vertically,

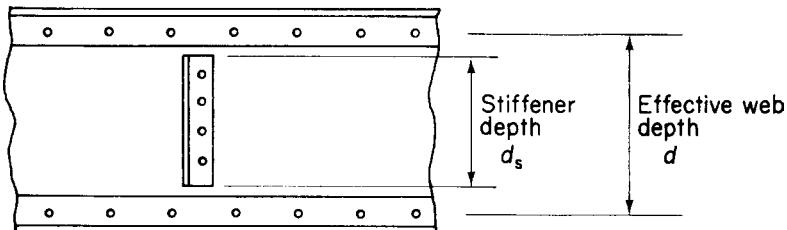


FIGURE 9.18 Calculation of Stiffener Buckling Load

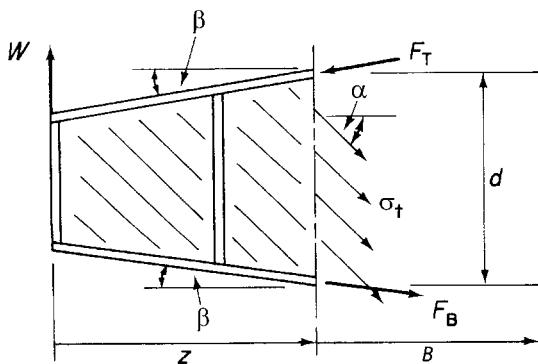


FIGURE 9.19 Effect of Taper on Diagonal Tension Field Beam Calculations

$$W - (F_T + F_B) \sin \beta - \sigma_t t(d \cos \alpha) \sin \alpha = 0 \quad (9.38)$$

For horizontal equilibrium,

$$(F_T - F_B) \cos \beta - \sigma_t t d \cos^2 \alpha = 0 \quad (9.39)$$

Taking moments about B,

$$Wz - F_T d \cos \beta + \frac{1}{2} \sigma_t t d^2 \cos^2 \alpha = 0 \quad (9.40)$$

Then, Eq. (9.38)  $\times \cos \beta$  – Eq.(9.39)  $\times \sin \beta$  gives

$$W \cos \beta - 2F_T \cos \beta \sin \beta - \sigma_t t d (\cos \alpha \sin \alpha \cos \beta - \cos^2 \alpha \cos \beta) = 0$$

Multiplying this equation by d and subtracting Eq. (9.40) multiplied by  $2\sin \beta$  gives

$$W \left( \frac{1 - 2z \tan \beta}{d} \right) - \sigma_t t d \left( \frac{\sin 2\alpha}{2} - \cos^2 \alpha \sin \beta + \cos^2 \alpha \sin \beta \right) = 0$$

whence

$$\sigma_t = \frac{2W}{td \sin 2\alpha} \left( \frac{1 - 2z \tan \beta}{d} \right) \quad (9.41)$$

Now substituting in Eq. (9.40) for  $\sigma_t$  and rearranging,

$$F_T = \frac{W}{d \cos \beta} \left[ z + \frac{d \cot \alpha}{2} \left( 1 - \frac{2z}{d} \tan \beta \right) \right] \quad (9.42)$$

Then, from either of Eq. (9.38) or (9.39),

$$F_B = \frac{W}{d \cos \beta} \left[ z - \frac{d \cot \alpha}{2} \left( 1 - \frac{2z}{d} \tan \beta \right) \right] \quad (9.43)$$

Equation (9.23) becomes

$$P = \frac{Wb}{d} \tan \alpha \left( 1 - \frac{2z}{d} \tan \beta \right) \quad (9.44)$$

Also, the shear force  $S$  at any section of the beam is, from Fig. 9.19,

$$S = W - (F_T + F_B) \sin \beta$$

or, substituting for  $F_T$  and  $F_B$  from Eqs. (9.42) and (9.43),

$$S = W \left( 1 - \frac{2z}{d} \tan \beta \right) \quad (9.45)$$

### 9.7.3 Postbuckling behavior

Sections 9.7.1 and 9.7.2 are concerned with beams in which the thin webs buckle to form tension fields; the beam flanges are then regarded as being subjected to bending action, as in Fig. 9.16. It is possible, if the beam flanges are relatively light, for failure due to yielding to occur in the beam flanges after the web has buckled, so that plastic hinges form and a failure mechanism of the type shown in Fig. 9.20 exists. This postbuckling behavior was investigated by Evans, Porter, and Rockey,<sup>14</sup> who developed a design method for beams subjected to bending and shear. Their method of analysis is presented here.

Suppose that the panel AXBZ in Fig. 9.20 collapses due to a shear load  $S$  and a bending moment  $M$ ; plastic hinges form at W, X, Y, and Z. In the initial stages of loading, the web remains perfectly flat until it reaches its critical stresses, that is,  $\tau_{cr}$  in shear and  $\sigma_{crb}$  in bending. The values of these stresses may be found approximately from

$$\left( \frac{\sigma_{mb}}{\sigma_{crb}} \right)^2 + \left( \frac{\tau_m}{\tau_{cr}} \right)^2 = 1 \quad (9.46)$$

where  $\sigma_{crb}$  is the critical value of bending stress with  $S = 0$ ,  $M \neq 0$ , and  $\tau_{cr}$  is the critical value of shear stress when  $S \neq 0$  and  $M = 0$ . Once the critical stress is reached, the web starts to buckle and cannot carry any increase in compressive stress, so that, as we have seen in Section 9.7.1, any additional load is carried by tension field action. It is assumed that the shear and bending stresses remain at their critical values  $\tau_m$  and  $\sigma_{mb}$  and that there are *additional* stresses  $\sigma_t$ , which are inclined at an angle  $\theta$  to the horizontal and which carry any increases in the applied load. At collapse – that is, at ultimate load conditions – the additional stress  $\sigma_t$  reaches its maximum value  $\sigma_{t(max)}$  and the panel is in the collapsed state shown in Fig. 9.20.

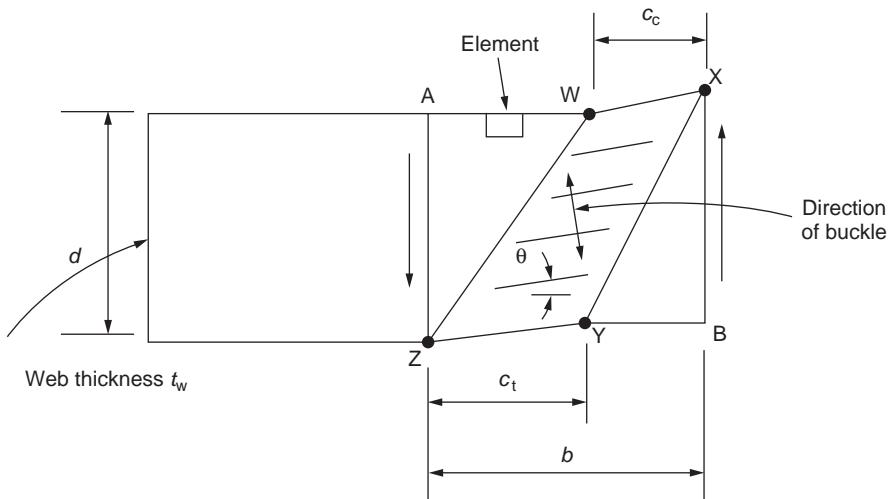


FIGURE 9.20 Collapse Mechanism of a Panel of a Tension Field Beam

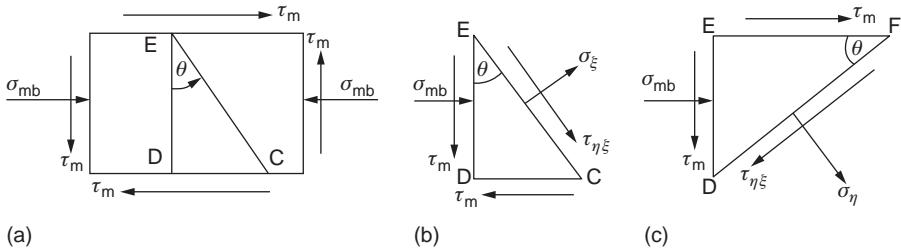


FIGURE 9.21 Determination of Stresses on Planes Parallel and Perpendicular to the Plane of the Buckle

Consider now the small rectangular element on the edge AW of the panel before collapse. The stresses acting on the element are shown in Fig. 9.21(a). The stresses on planes parallel to and perpendicular to the direction of the buckle may be found by considering the equilibrium of triangular elements within this rectangular element. Initially, we consider the triangular element CDE, which is subjected to the stress system shown in Fig. 9.21(b) and is in equilibrium under the action of the forces corresponding to these stresses. Note that the edge CE of the element is parallel to the direction of the buckle in the web.

For equilibrium of the element in a direction perpendicular to CE (see Section 1.6) and assuming unit thickness for the panel,

$$\sigma_\xi CE + \sigma_{mb} ED \cos \theta - \tau_m ED \sin \theta - \tau_m DC \cos \theta = 0$$

Dividing through by CE and rearranging, we have

$$\sigma_\xi = -\sigma_{mb} \cos^2 \theta + \tau_m \sin 2\theta \quad (9.47)$$

Similarly, by considering the equilibrium of the element in the direction EC, we have

$$\tau_{\eta\xi} = -\frac{\sigma_{mb}}{2} \sin 2\theta - \tau_m \cos 2\theta \quad (9.48)$$

Further, the direct stress  $\sigma_\eta$  on the plane FD (Fig. 9.21(c)), which is perpendicular to the plane of the buckle, is found from the equilibrium of the element FED. Then,

$$\sigma_\eta FD + \sigma_{mb} ED \sin \theta + \tau_m EF \sin \theta + \tau_m DE \cos \theta = 0$$

Dividing through by FD and rearranging gives

$$\sigma_\eta = -\sigma_{mb} \sin^2 \theta - \tau_m \sin 2\theta \quad (9.49)$$

Note that the shear stress on this plane forms a complementary shear stress system with  $\tau_{\eta\xi}$ .

The failure condition is reached by adding  $\sigma_{t(max)}$  to  $\sigma_\xi$  and using the von Mises theory of elastic failure<sup>15</sup>; that is,

$$\sigma_y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 + 3\tau^2 \quad (9.50)$$

where  $\sigma_y$  is the yield stress of the material,  $\sigma_1$  and  $\sigma_2$  are the direct stresses acting on two mutually perpendicular planes, and  $\tau$  is the shear stress acting on the same two planes. Hence, when the yield stress in the web is  $\sigma_{yw}$ , failure occurs when

$$\sigma_{yw}^2 = (\sigma_\xi + \sigma_{t(max)})^2 + \sigma_\eta^2 - \sigma_\eta (\sigma_\xi + \sigma_{t(max)}) + 3\tau_{\eta\xi}^2 \quad (9.51)$$

Equations (9.47), (9.48), (9.49), and (9.51) may be solved for  $\sigma_{t(max)}$ , which is then given by

$$\sigma_{t(max)} = -\frac{1}{2}A + \frac{1}{2}\left[A^2 - 4\left(\sigma_{mb}^2 + 3\tau_m^2 - \sigma_{yw}^2\right)\right]^{\frac{1}{2}} \quad (9.52)$$

where

$$A = 3\tau_m \sin 2\theta + \sigma_{mb} \sin^2 \theta - 2\sigma_{mb} \cos^2 \theta \quad (9.53)$$

These equations have been derived for a point on the edge of the panel but are applicable to any point within its boundary. Therefore, the resultant force  $F_w$ , corresponding to the tension field in the web, may be calculated and its line of action determined.

If the average stresses in the compression and tension flanges are  $\sigma_{cf}$  and  $\sigma_{tf}$  and the yield stress of the flanges is  $\sigma_{yf}$ , the reduced plastic moments in the flanges are<sup>15</sup>

$$M'_{pc} = M_{pc} \left[ 1 - \left( \frac{\sigma_{cf}}{\sigma_{yf}} \right)^2 \right] \quad (\text{compression flange}) \quad (9.54)$$

$$M'_{pt} = M_{pt} \left[ 1 - \left( \frac{\sigma_{tf}}{\sigma_{yf}} \right)^2 \right] \quad (\text{tension flange}) \quad (9.55)$$

The position of each plastic hinge may be found by considering the equilibrium of a length of flange and employing the principle of virtual work. In Fig. 9.22, the length WX of the upper flange of the beam is given a virtual displacement  $\phi$ . The work done by the shear force at X is equal to the energy absorbed by the plastic hinges at X and W and the work done *against* the tension field stress  $\sigma_{t(max)}$ . Suppose the average value of the tension field stress is  $\sigma_{tc}$ , that is, the stress at the mid-point of WX.

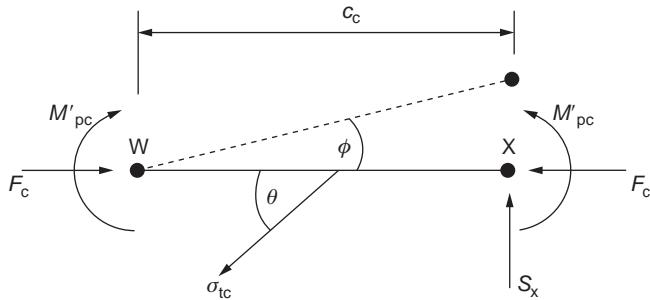


FIGURE 9.22 Determination of Plastic Hinge Position

Then,

$$S_x c_c \phi = 2M'_{pc} \phi + \sigma_{tc} t_w \sin^2 \theta \frac{c_c^2}{2} \phi$$

The minimum value of  $S_x$  is obtained by differentiating with respect to  $c_c$ ; that is,

$$\frac{dS_x}{dc_c} = -2 \frac{M'_{pc}}{c_c^2} + \sigma_{tc} t_w \frac{\sin^2 \theta}{2} = 0$$

which gives

$$c_c^2 = \frac{4M'_{pc}}{\sigma_{tc} t_w \sin^2 \theta} \quad (9.56)$$

Similarly, in the tension flange,

$$c_t^2 = \frac{4M'_{pt}}{\sigma_{tt} t_w \sin^2 \theta} \quad (9.57)$$

Clearly, for the plastic hinges to occur within a flange, both  $c_c$  and  $c_t$  must be less than  $b$ . Therefore, from Eq. (9.56),

$$M'_{pc} < \frac{t_w b^2 \sin^2 \theta}{4} \sigma_{tc} \quad (9.58)$$

where  $\sigma_{tc}$  is found from Eqs. (9.52) and (9.53) at the mid-point of WX.

The average axial stress in the compression flange between W and X is obtained by considering the equilibrium of half of the length of WX (Fig. 9.23).

Then,

$$F_c = \sigma_{cf} A_{cf} + \sigma_{tc} t_w \frac{c_c}{2} \sin \theta \cos \theta + \tau_m t_w \frac{c_c}{2}$$

from which

$$\sigma_{cf} = \frac{F_c - \frac{1}{2}(\sigma_{tc} \sin \theta \cos \theta + \tau_m)t_w c_c}{A_{cf}} \quad (9.59)$$

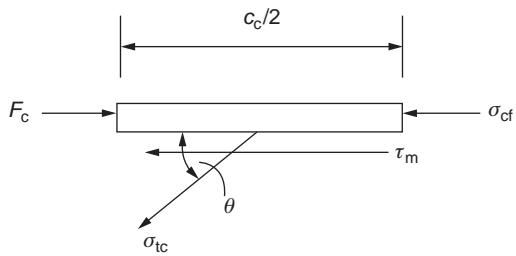


FIGURE 9.23 Determination of Flange Stress

where  $F_c$  is the force in the compression flange at W and  $A_{cf}$  is the cross-sectional area of the compression flange.

Similarly, for the tension flange,

$$\sigma_{tf} = \frac{F_t + \frac{1}{2}(\sigma_{tt} \sin \theta \cos \theta + \tau_m)t_w c_t}{A_{tf}} \quad (9.60)$$

The forces  $F_c$  and  $F_t$  are found by considering the equilibrium of the beam to the right of WY (Fig. 9.24). Then, resolving vertically and noting that  $S_{cr} = \tau_m t_w d$ ,

$$S_{ult} = F_w \sin \theta + \tau_m t_w d + \sum W_n \quad (9.61)$$

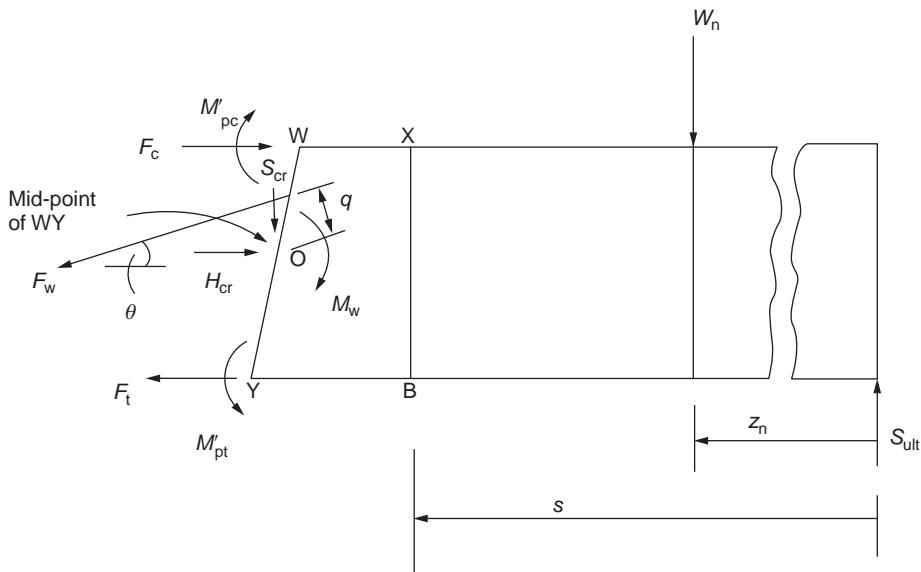


FIGURE 9.24 Determination of Flange Forces

Resolving horizontally and noting that  $H_{\text{cr}} = \tau_m t_w (b - c_c - c_t)$ ,

$$F_c - F_t = F_w \cos \theta - \tau_m t_w (b - c_c - c_t) \quad (9.62)$$

Taking moments about O, we have

$$F_c + F_t = \frac{2}{d} \left[ S_{\text{ult}} \left( s + \frac{b + c_c - c_t}{2} \right) + M'_{\text{pt}} - M'_{\text{pc}} + F_w q - M_w - \sum_n W_n z_n \right] \quad (9.63)$$

where  $W_1$  to  $W_n$  are external loads applied to the beam to the right of WY and  $M_w$  is the bending moment in the web when it has buckled and become a tension field: that is,

$$M_w = \frac{\sigma_{\text{mb}} b d^2}{b}$$

The flange forces are then

$$F_c = \frac{S_{\text{ult}}}{2d} (d \cot \theta + 2s + b + c_c - c_t) + \frac{1}{d} \left( M'_{\text{pt}} - M'_{\text{pc}} + F_w q - M_w - \sum_n W_n z_n \right) - \frac{1}{2} \tau_m t_w (d \cot \theta + b - c_c - c_t) \quad (9.64)$$

$$F_t = \frac{S_{\text{ult}}}{2d} (d \cot \theta + 2s + b + c_c - c_t) + \frac{1}{d} \left( M'_{\text{pt}} - M'_{\text{pc}} - F_w q - M_w - \sum_n W_n z_n \right) + \frac{1}{2} \tau_m t_w (d \cot \theta + b - c_c - c_t) \quad (9.65)$$

Evans, Porter, and Rockey adopted an iterative procedure for solving Eqs. (9.61)–(9.65), in which an initial value of  $\theta$  was assumed and  $\sigma_{\text{cf}}$  and  $\sigma_{\text{tf}}$  were taken to be zero. Then,  $c_c$  and  $c_t$  were calculated and approximate values of  $F_c$  and  $F_t$  found giving better estimates for  $\sigma_{\text{cf}}$  and  $\sigma_{\text{tf}}$ . The procedure was repeated until the required accuracy was obtained.

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## PROBLEMS

**P.9.1** A thin square plate of side  $a$  and thickness  $t$  is simply supported along each edge and has a slight initial curvature giving an initial deflected shape:

$$w_0 = \delta \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}$$

If the plate is subjected to a uniform compressive stress  $\sigma$  in the  $x$  direction (see Fig. P.9.1), find an expression for the *elastic* deflection  $w$  normal to the plate. Show also that the deflection at the mid-point of the plate can be presented in the form of a Southwell plot and illustrate your answer with a suitable sketch.

Answer:  $w = [\sigma t \delta / (4\pi^2 D/a^2 - \sigma t)] \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}$

**P.9.2** A uniform flat plate of thickness  $t$  has a width  $b$  in the  $y$  direction and length  $l$  in the  $x$  direction (see Fig. P.9.2). The edges parallel to the  $x$  axis are clamped and those parallel to the  $y$  axis are simply supported. A uniform compressive stress  $\sigma$  is applied in the  $x$  direction along the edges parallel to the  $y$  axis. Using an energy method, find an approximate expression for the

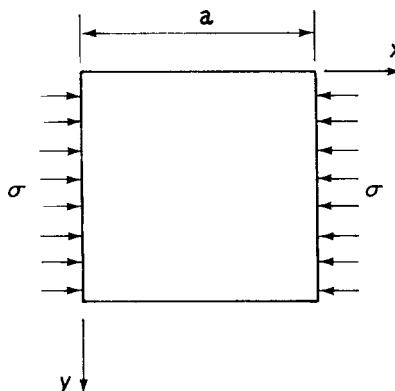


FIGURE P.9.1

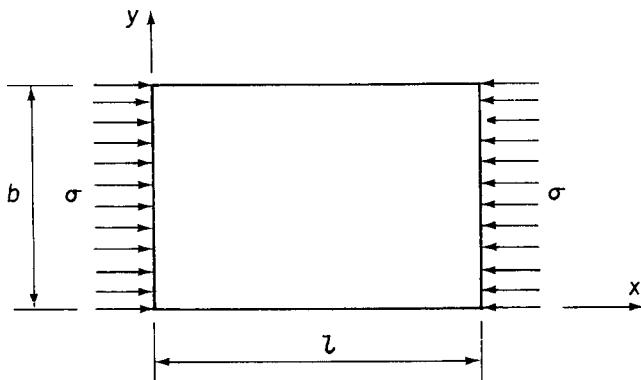


FIGURE P.9.2

magnitude of the stress  $\sigma$  that causes the plate to buckle, assuming that the deflected shape of the plate is given by

$$w = a_{11} \sin \frac{m\pi x}{l} \sin^2 \frac{\pi y}{b}$$

For this particular case,  $l = 2b$ , find the number of half waves  $m$  corresponding to the lowest critical stress, expressing the result to the nearest integer. Determine also the lowest critical stress.

Answer:  $m = 3$ ,  $\sigma_{CR} = [6E/(1 - v^2)](t/b)^2$

**P.9.3** A panel comprising a flat sheet and uniformly spaced Z-section stringers, a part of whose cross-section is shown in Fig. P.9.3, is to be investigated for strength under uniform compressive loads in a structure in which it is to be stabilized by frames a distance  $l$  apart,  $l$  being appreciably greater than the spacing  $b$ .

(a) State the modes of failure you would consider and how you would determine appropriate limiting stresses.

(b) Describe a suitable test to verify your calculations, giving particulars of the specimen, the manner of support, and the measurements you would take. The last should enable you to verify the assumptions made as well as to obtain the load supported.

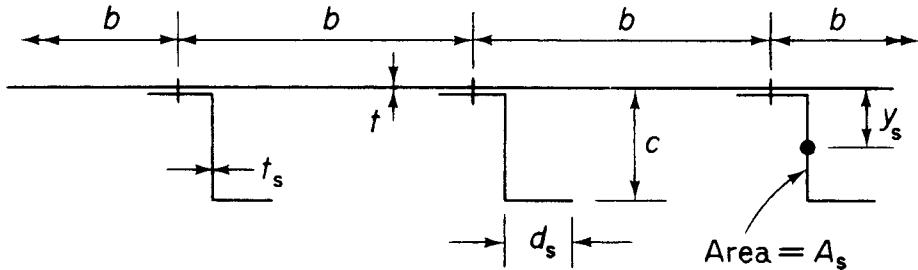


FIGURE P.9.3

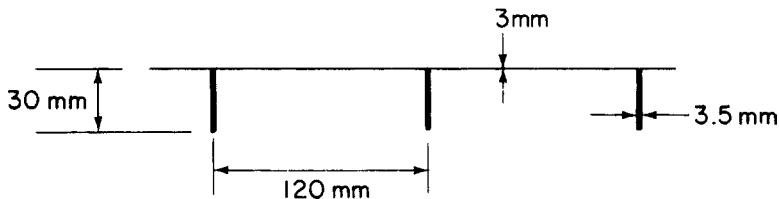


FIGURE P.9.4

**P.9.4** Part of a compression panel of internal construction is shown in Fig. P.9.4. The equivalent pin-center length of the panel is 500 mm. The material has a Young's modulus of 70,000 N/mm<sup>2</sup> and its elasticity may be taken as falling catastrophically when a compressive stress of 300 N/mm<sup>2</sup> is reached. Taking coefficients of 3.62 for buckling of a plate with simply supported sides and of 0.385 with one side simply supported and one free, determine (a) the load per millimeter width of panel when initial buckling may be expected and (b) the load per millimeter for ultimate failure. Treat the material as thin for calculating section constants and assume that, after initial buckling, the stress in the plate increases parabolically from its critical value in the center of sections.

Answer: 613.8 N/mm, 844.7 N/mm

**P.9.4 MATLAB** Use MATLAB to repeat part (a) of Problem P.9.4 for the following skin thicknesses ( $t_{sk}$ ):

	(i)	(ii)	(iii)	(iv)	(v)
$t_{sk}$	2 mm	2.5 mm	3 mm	3.5 mm	4 mm

- Answer:
- (i)  $N_{x,CR} = 202.4 \text{ N/mm}$
  - (ii)  $N_{x,CR} = 371.25 \text{ N/mm}$
  - (iii)  $N_{x,CR} = 613.8 \text{ N/mm}$
  - (iv)  $N_{x,CR} = 650.1 \text{ N/mm}$
  - (v)  $N_{x,CR} = 677.1 \text{ N/mm}$

**P.9.5** A long column has the cross-section shown in Fig. P.9.5. For the material of the column, Young's modulus  $E=70,000 \text{ N/mm}^2$ , Poisson's ratio  $\nu=0.3$ , and the compressive yield stress is 500 N/mm<sup>2</sup>. Calculate the failure load for the column.

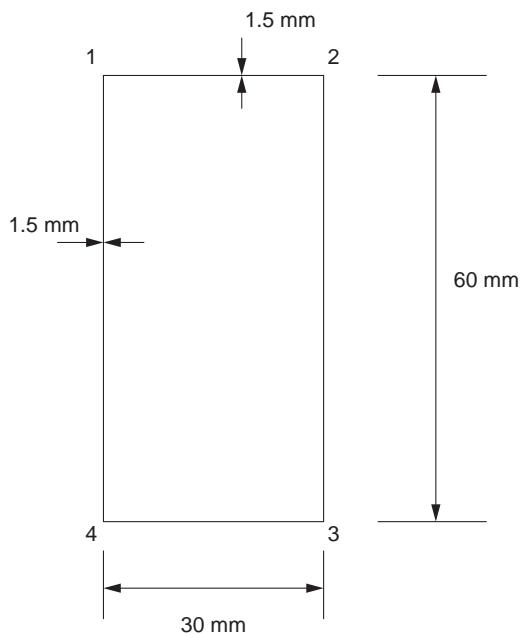
Answer: 75,105 N.

**P.9.6** Compare the value of critical load for the column of Fig. P.9.5 as predicted by Gerard (Eq. (9.12)) with that obtained in P.9.5.

Answer: 85,050 N, 13.2% greater.

**P.9.7** Examine the validity of the assumption in Example 9.2 that the web of the beam is in complete diagonal tension.

Answer: Assuming simply supported edges (the worst case), the critical shear load is 12.4 kN; the assumption is therefore invalid.

**FIGURE P.9.5**

**P.9.8** A simply supported beam has a span of 2.4 m and carries a central concentrated load of 10 kN. Each of the flanges of the beam has a cross-sectional area of  $300 \text{ mm}^2$  while that of the vertical web stiffeners is  $280 \text{ mm}^2$ . If the depth of the beam, measured between the centroids of area of the flanges, is 350 mm and the stiffeners are symmetrically arranged about the web and spaced at 300 mm intervals, determine the maximum axial load in a flange and the compressive load in a stiffener. It may be assumed that the beam web, of thickness 1.5 mm, is capable of resisting diagonal tension only.

Answer: 19.9 kN, 3.9 kN.

**P.9.9** The spar of an aircraft is to be designed as an incomplete diagonal tension beam, the flanges being parallel. The stiffener spacing will be 250 mm, the effective depth of web will be 750 mm, and the depth between web-to-flange attachments is 725 mm. The spar is to carry an ultimate shear force of 100,000 N. The maximum permissible shear stress is  $165 \text{ N/mm}^2$ , but it is also required that the shear stress should not exceed 15 times the critical shear stress for the web panel. Assuming  $\alpha$  to be  $40^\circ$  and using the following relationships

- (a) Select the smallest suitable web thickness from the following range of standard thicknesses.  
(Take Young's Modulus  $E$  as  $70,000 \text{ N/mm}^2$ .)

0.7 mm, 0.9 mm, 1.2 mm, 1.6 mm

- (b) Calculate the stiffener end load and the secondary bending moment in the flanges (assume stiffeners to be symmetrical about the web).

The shear stress buckling coefficient for the web may be calculated from the expression

$$K = 7.70 \left[ 1 + 0.75(b/d)^2 \right]$$

$b$  and  $d$  having their usual significance. The relationship between the diagonal tension factor and buckling stress ratio is

$\tau/\tau_{CR}$	5	7	9	11	13	15
$k$	0.37	0.40	0.42	0.48	0.51	0.53

Note that  $\alpha$  is the angle of diagonal tension measured from the spanwise axis of the beam, as in the usual notation.

Answer: 1.2 mm,  $130A_S/(1 + 0.0113A_S)$ , 238,910 N mm

**P.9.9 MATLAB** Use MATLAB to repeat Problem P.8.16 for the following stiffener spacings.

$b$	(i) 150 mm	(ii) 250 mm	(iii) 350 mm	(iv) 450 mm
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- Answer:
- (i) 0.9 mm,  $166A_S/(1 + 0.0247A_S)$ , 83 910 Nmm
  - (ii) 1.2 mm,  $130A_S/(1 + 0.0113A_S)$ , 238 910 Nmm
  - (iii) 1.2 mm,  $210A_S/(1 + 0.0101A_S)$ , 605 317 Nmm
  - (iv) 1.6 mm,  $106A_S/(1 + 0.00487A_S)$ , 811 829 Nmm

**P.9.10** The main compressive wing structure of an aircraft consists of stringers, having the section shown in Fig. P.9.10(b), bonded to a thin skin (Fig. P.9.10(a)). Find suitable values for the stringer spacing  $b$  and rib spacing  $L$  if local instability, skin buckling, and panel strut instability all occur at the same stress. Note that, in Fig. P.9.10(a), only two of several stringers are shown for diagrammatic clarity. Also, the thin skin should be treated as a flat plate, since the curvature is small. For a flat plate simply supported along two edges, assume a buckling coefficient of 3.62. Take  $E = 69,000 \text{ N/mm}^2$ .

Answer:  $b = 56.5 \text{ mm}$ ,  $L = 700 \text{ mm}$

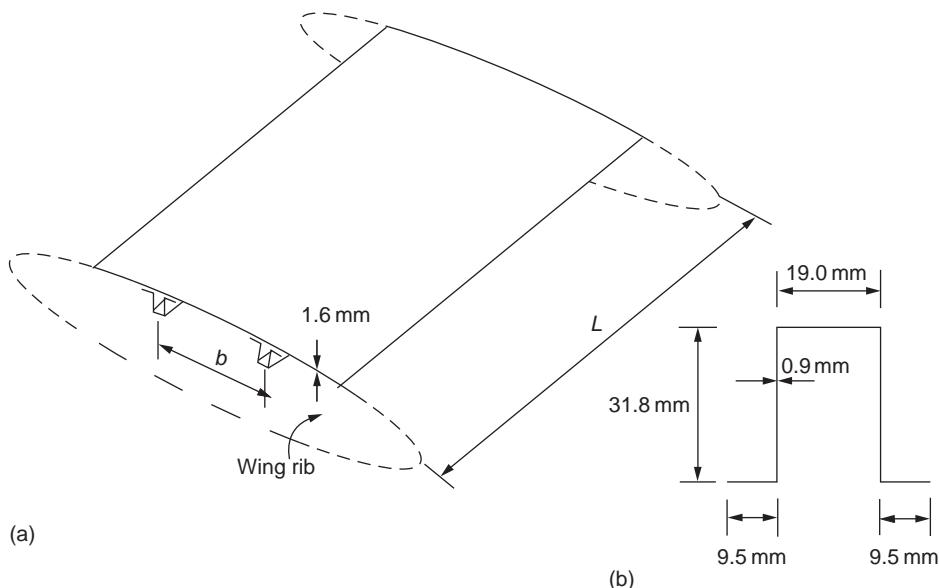


FIGURE P.9.10

# Structural vibration

# 10

Structures subjected to dynamic loading, particularly aircraft, vibrate or oscillate in a frequently complex manner. An aircraft, for example, possesses an infinite number of *natural* or *normal modes* of vibration. Simplifying assumptions, such as breaking down the structure into a number of concentrated masses connected by weightless beams (*lumped mass concept*), are made, but whatever method is employed, the natural modes and frequencies of vibration of a structure must be known before *flutter* speeds and frequencies can be found. We shall discuss flutter and other dynamic aeroelastic phenomena in Chapter 29, but for the moment, we shall concentrate on the calculation of the normal modes and frequencies of vibration of a variety of beam and mass systems.

## 10.1 OSCILLATION OF MASS-SPRING SYSTEMS

Let us suppose that the simple mass–spring system shown in Fig. 10.1 is displaced by a small amount  $x_0$  and suddenly released. The mass will oscillate about its initial position, and the equation of the resulting motion in the absence of damping forces is

$$m \frac{d^2x}{dt^2} = -kx$$

where  $k$  is the spring stiffness that produces a negative restoring force. Then,

$$\frac{d^2x}{dt^2} + \frac{kx}{m} = 0 \quad (10.1)$$

The solution of Eq. (10.1) is of standard form and is

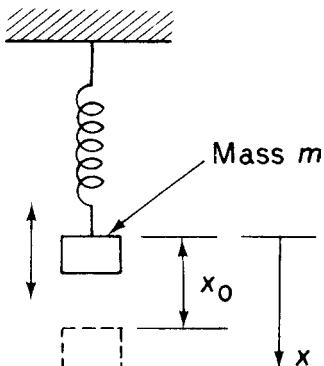
$$x = A \cos \omega t + B \sin \omega t$$

in which  $\omega^2 = k/m$ . Suppose we measure the time from the point when the mass passes through its initial position. Then, when  $t = 0$ ,  $x = 0$  so that  $A = 0$ . Also when  $x = \pm x_0$ ,  $dx/dt = 0$  (the mass will be instantaneously at rest), which gives

$$\frac{dx}{dt} = \omega B \cos \omega t = 0$$

For a nontrivial solution ( $B \neq 0$ )  $\cos \omega t = 0$  so that  $\omega t = \pi/2$  and, from the above,

$$x_0 = B \sin \frac{\pi}{2}$$



**FIGURE 10.1 Oscillation of a Mass-Spring System**

giving  $B = x_0$ . The solution of Eq. (10.1) is therefore

$$x = x_0 \sin \omega t$$

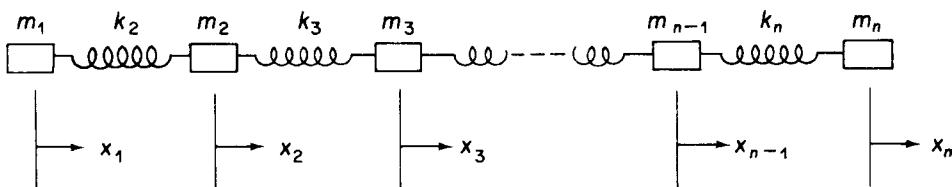
which may be written as

$$x = x_0 \sin (\omega t + \varepsilon) \quad (10.2)$$

in which  $\varepsilon$  is a phase angle that would allow for the time to be measured from a point other than the initial position of the mass. We now see that the mass  $m$  oscillates with simple harmonic motion of amplitude  $x_0$ . From Eq. (10.2) the time taken for the mass to travel from  $x = 0$  to  $x = x_0$  is given by  $\sin \omega t = 1$  or, as above,  $\omega t = \pi/2$ , that is,  $t = \pi/2\omega$ . One complete oscillation is  $4x_0$  so that the time for one complete oscillation is  $4(\pi/2\omega)$ . It follows that the *frequency* of the oscillation, that is, the number of oscillations/second is  $1/(2\pi/\omega)$  or  $\omega/2\pi$  *cycles/second*.

Both the frequency and periodic time are seen to depend upon the basic physical characteristics of the system, namely the spring stiffness and the magnitude of the mass. Therefore, although the amplitude of the oscillation may be changed by altering the size of the initial disturbance, its frequency is fixed. This frequency is the normal or natural frequency of the system, and the vertical simple harmonic motion of the mass is its normal mode of vibration.

Consider now the system of  $n$  masses connected by  $(n - 1)$  springs, as shown in Fig. 10.2. If we specify that motion takes place only in the direction of the spring axes, then the system has  $n$  degrees of freedom. It is therefore possible to set the system oscillating with simple harmonic motion in  $n$  ways. In each of these  $n$  modes of vibration, the masses oscillate in phase so that they all attain maximum



**FIGURE 10.2 Oscillation of an  $n$  Mass-Spring System**

amplitude at the same time and pass through their zero displacement positions at the same time. The set of amplitudes and the corresponding frequencies take up different values in each of the  $n$  modes. Again, these modes are termed *normal* or *natural modes of vibration* and the corresponding frequencies are called *normal* or *natural frequencies*.

The determination of normal modes and frequencies for a general spring–mass system involves the solution of a set of  $n$  simultaneous second-order differential equations of a type similar to Eq. (10.1). Associated with each solution are two arbitrary constants which determine the phase and amplitude of each mode of vibration. We can therefore relate the vibration of a system to a given set of initial conditions by assigning appropriate values to these constants.

A useful property of the normal modes of a system is their orthogonality, which is demonstrated by the provable fact that the product of the inertia forces in one mode and the displacements in another result in zero work done. In other words, displacements in one mode cannot be produced by inertia forces in another. It follows that the normal modes are independent of one another, so that the response of each mode to an externally applied force may be found without reference to the other modes. Therefore, by considering the response of each mode in turn and adding the resulting motions, we can find the response of the complete system to the applied loading. Another useful characteristic of normal modes is their “stationary property.” It can be shown that if an elastic system is forced to vibrate in a mode that is slightly different from a true normal mode, the frequency is only very slightly different from the corresponding natural frequency of the system. Reasonably accurate estimates of natural frequencies may therefore be made from “guessed” modes of displacement.

We shall proceed to illustrate the general method of solution by determining normal modes and frequencies of some simple beam–mass systems. Two approaches are possible: a *stiffness* or *displacement method*, in which spring or elastic forces are expressed in terms of stiffness parameters such as  $k$  in Eq. (10.1); and a *flexibility* or *force method*, in which elastic forces are expressed in terms of the flexibility  $\delta$  of the elastic system. In the latter approach,  $\delta$  is defined as the deflection due to unit force; the equation of motion of the spring–mass system of Fig. 10.1 then becomes

$$m\ddot{x} + \frac{x}{\delta} = 0 \quad (\text{where } \ddot{x} = d^2x/dt^2) \quad (10.3)$$

Again, the solution takes the form  $x = x_0 \sin(\omega t + \varepsilon)$ , but in this case,  $\omega^2 = 1/m\delta$ . Clearly, by our definitions of  $k$  and  $\delta$ , the product  $k\delta = 1$ . In problems involving rotational oscillations,  $m$  becomes the moment of inertia of the mass and  $\delta$  the rotation or displacement produced by unit moment.

Let us consider a spring–mass system having a finite number,  $n$ , degrees of freedom. The term *spring* is used here in a general sense, in that the  $n$  masses  $m_1, m_2, \dots, m_i, \dots, m_n$  may be connected by any form of elastic weightless member. Thus, if  $m_i$  is the mass at a point  $i$  where the displacement is  $x_i$  and  $\delta_{ij}$  is the displacement at the point  $i$  due to a unit load at a point  $j$  (note, from the reciprocal theorem,  $\delta_{ij} = \delta_{ji}$ ), the  $n$  equations of motion for the system are

$$\left. \begin{aligned} m_1\ddot{x}_1\delta_{11} + m_2\ddot{x}_2\delta_{12} + \dots + m_i\ddot{x}_i\delta_{1i} + \dots + m_n\ddot{x}_n\delta_{1n} + x_1 &= 0 \\ m_1\ddot{x}_1\delta_{21} + m_2\ddot{x}_2\delta_{22} + \dots + m_i\ddot{x}_i\delta_{2i} + \dots + m_n\ddot{x}_n\delta_{2n} + x_2 &= 0 \\ \dots & \\ m_1\ddot{x}_1\delta_{i1} + m_2\ddot{x}_2\delta_{i2} + \dots + m_i\ddot{x}_i\delta_{ii} + \dots + m_n\ddot{x}_n\delta_{in} + x_i &= 0 \\ \dots & \\ m_1\ddot{x}_1\delta_{n1} + m_2\ddot{x}_2\delta_{n2} + \dots + m_i\ddot{x}_i\delta_{ni} + \dots + m_n\ddot{x}_n\delta_{nn} + x_n &= 0 \end{aligned} \right\} \quad (10.4)$$

or

$$\sum_{j=1}^n m_j \ddot{x}_j \delta_{ij} x_j + x_i = 0, \quad (i = 1, 2, \dots, n) \quad (10.5)$$

Since each normal mode of the system oscillates with simple harmonic motion, the solution for the  $i$ th mode takes the form  $x = x_{0,i} \sin(\omega t + \varepsilon)$ , so that  $\ddot{x}_i = -\omega^2 x_{0,i} \sin(\omega t + \varepsilon) = -\omega^2 x_i$ . Equation (10.5) may therefore be written as

$$-\omega^2 \sum_{j=1}^n m_j \delta_{ij} x_j + x_i = 0, \quad (i = 1, 2, \dots, n) \quad (10.6)$$

For a nontrivial solution, that is,  $x_i \neq 0$ , the determinant of Eq. (10.6) must be zero. Hence,

$$\begin{vmatrix} (\omega^2 m_1 \delta_{11} - 1) & \omega^2 m_2 \delta_{12} & \dots & \omega^2 m_i \delta_{1i} & \dots & \omega^2 m_n \delta_{1n} \\ \omega^2 m_1 \delta_{21} & (\omega^2 m_2 \delta_{22} - 1) & \dots & \omega^2 m_i \delta_{2i} & \dots & \omega^2 m_n \delta_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \omega^2 m_1 \delta_{i1} & \omega^2 m_2 \delta_{i2} & & (\omega^2 m_i \delta_{ii} - 1) & & \omega^2 m_n \delta_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \omega^2 m_1 \delta_{n1} & \omega^2 m_2 \delta_{n2} & \dots & \omega^2 m_i \delta_{ni} & \dots & (\omega^2 m_n \delta_{nn} - 1) \end{vmatrix} = 0 \quad (10.7)$$

The solution of Eq. (10.7) gives the normal frequencies of vibration of the system. The corresponding modes may then be deduced, as we shall see in the following examples.

### Example 10.1

Determine the normal modes and frequencies of vibration of a weightless cantilever supporting masses  $m/3$  and  $m$  at points 1 and 2, as shown in Fig. 10.3. The flexural rigidity of the cantilever is  $EI$ .

The equations of motion of the system are, from Eqs. (10.4),

$$(m/3) \ddot{v}_1 \delta_{11} + m \ddot{v}_2 \delta_{12} + v_1 = 0 \quad (i)$$

$$(m/3) \ddot{v}_1 \delta_{21} + m \ddot{v}_2 \delta_{22} + v_2 = 0 \quad (ii)$$

where  $v_1$  and  $v_2$  are the vertical displacements of the masses at any instant of time. In this example, displacements are assumed to be caused by bending strains only; the flexibility coefficients  $\delta_{11}$ ,  $\delta_{22}$ , and  $\delta_{12}$  ( $= \delta_{21}$ ) may therefore be found by the unit load method described in Section 5.5. Then,

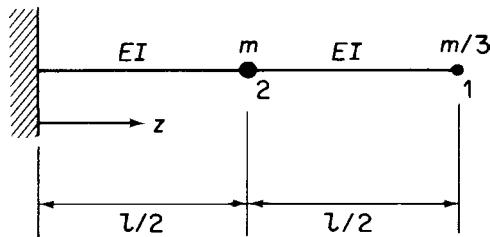


FIGURE 10.3 Mass–Beam System for Example 10.1

$$\delta_{ij} = \int_L \frac{M_i M_j}{EI} dz \quad (\text{iii})$$

where  $M_i$  is the bending moment at any section  $z$  due to a unit load at the point  $i$  and  $M_j$  is the bending moment at any section  $z$  produced by a unit load at the point  $j$ . Therefore, from Fig. 10.3,

$$\begin{aligned} M_1 &= 1(l-z), & 0 \leq z \leq l \\ M_2 &= 1(l/2-z), & 0 \leq z \leq l/2 \\ M_2 &= 0, & l/2 \leq z \leq l \end{aligned}$$

Hence,

$$\delta_{11} = \frac{1}{EI} \int_0^l M_1^2 dz = \frac{1}{EI} \int_0^l (l-z)^2 dz \quad (\text{iv})$$

$$\delta_{22} = \frac{1}{EI} \int_0^l M_2^2 dz = \frac{1}{EI} \int_0^{l/2} \left(\frac{l}{2}-z\right)^2 dz \quad (\text{v})$$

$$\delta_{12} = \delta_{21} = \frac{1}{EI} \int_0^l M_1 M_2 dz = \frac{1}{EI} \int_0^{l/2} (l-z) \left(\frac{l}{2}-z\right) dz \quad (\text{vi})$$

Integrating Eqs. (iv), (v), and (vi) and substituting limits, we obtain

$$\delta_{11} = \frac{l^3}{3EI}, \quad \delta_{22} = \frac{l^3}{24EI}, \quad \delta_{12} = \delta_{21} = \frac{5l^3}{48EI}$$

Each mass describes simple harmonic motion in the normal modes of oscillation, so that  $v_1 = v_{1,0} \sin(\omega t + \epsilon)$  and  $v_2 = v_{2,0} \sin(\omega t + \epsilon)$ . Hence,  $\ddot{v}_1 = -\omega^2 v_1$  and  $\ddot{v}_2 = -\omega^2 v_2$ . Substituting for  $\ddot{v}_1$ ,  $\ddot{v}_2$ ,  $\delta_{11}$ ,  $\delta_{22}$ , and  $\delta_{12}$  ( $= \delta_{21}$ ) in Eqs. (i) and (ii) and writing  $\lambda = ml^3/(3 \times 48EI)$ , we obtain

$$(1 - 16\lambda\omega^2)v_1 - 15\lambda\omega^2v_2 = 0 \quad (\text{vii})$$

$$5\lambda\omega^2v_1 - (1 - 6\lambda\omega^2)v_2 = 0 \quad (\text{viii})$$

For a nontrivial solution,

$$\begin{vmatrix} (1 - 16\lambda\omega^2) & -15\lambda\omega^2 \\ 5\lambda\omega^2 & -(1 - 6\lambda\omega^2) \end{vmatrix} = 0$$

Expanding this determinant, we have

$$-(1 - 16\lambda\omega^2)(1 - 6\lambda\omega^2) + 75(\lambda\omega^2)^2 = 0$$

or

$$21(\lambda\omega^2)^2 - 22\lambda\omega^2 + 1 = 0 \quad (\text{ix})$$

Inspection of Eq. (ix) shows that

$$\lambda\omega^2 = 1/21, \quad \text{or} \quad 1$$

Hence,

$$\omega^2 = \frac{3 \times 48EI}{21ml^3} \quad \text{or} \quad \frac{3 \times 48EI}{ml^3}$$

The normal or natural frequencies of vibration are therefore

$$f_1 = \frac{\omega_1}{2\pi} = \frac{2}{\pi} \sqrt{\frac{3EI}{7ml^3}}$$

$$f_2 = \frac{\omega_2}{2\pi} = \frac{6}{\pi} \sqrt{\frac{EI}{ml^3}}$$

The system is therefore capable of vibrating at two distinct frequencies. To determine the normal mode corresponding to each frequency, we first take the lower frequency  $f_1$  and substitute it in either Eq. (vii) or Eq. (viii). From Eq. (vii),

$$\frac{v_1}{v_2} = \frac{15\lambda\omega^2}{1 - 16\lambda\omega^2} = \frac{15 \times (1/21)}{1 - 16 \times (1/21)}$$

which is a positive quantity. Therefore, at the lowest natural frequency, the cantilever oscillates in such a way that the displacement of both masses has the same sign at the same instant of time. Such an oscillation takes the form shown in Fig. 10.4. Substituting the second natural frequency in Eq. (vii), we have

$$\frac{v_1}{v_2} = \frac{15\lambda\omega^2}{1 - 16\lambda\omega^2} = \frac{15}{1 - 16}$$

which is negative, so that the masses have displacements of opposite sign at any instant of time, as shown in Fig. 10.5.

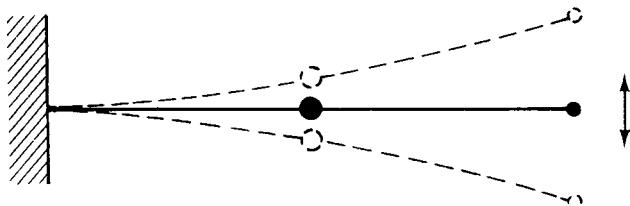


FIGURE 10.4 The First Natural Mode of the Mass–Beam System of Fig. 10.3

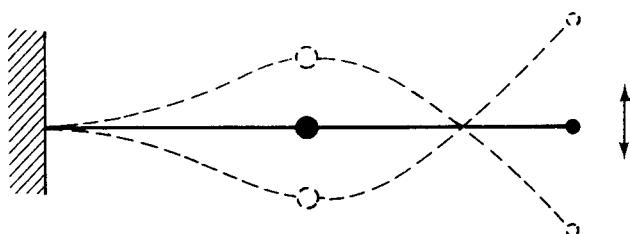


FIGURE 10.5 The Second Natural Mode of the Mass–Beam System of Fig. 10.3

**Example 10.2**

Find the lowest natural frequency of the weightless beam–mass system shown in Fig. 10.6. For the beam,  $GJ = (2/3)EI$ .

The equations of motion are

$$m\ddot{v}_1\delta_{11} + 4m\ddot{v}_2\delta_{12} + v_1 = 0 \quad (\text{i})$$

$$m\ddot{v}_1\delta_{21} + 4m\ddot{v}_2\delta_{22} + v_2 = 0 \quad (\text{ii})$$

In this problem, displacements are caused by bending and torsion, so that

$$\delta_{ij} = \int_L \frac{M_i M_j}{EI} ds + \int_L \frac{T_i T_j}{GJ} ds \quad (\text{iii})$$

From Fig. 10.6, we see that

$$\begin{aligned} M_1 &= 1x, & 0 \leq x \leq l \\ M_1 &= 1(2l - z), & 0 \leq z \leq 2l \\ M_2 &= 1(l - z), & 0 \leq z \leq l \\ M_2 &= 0, & l \leq z \leq 2l, \quad 0 \leq x \leq l \\ T_1 &= 1l, & 0 \leq z \leq 2l \\ T_1 &= 0, & 0 \leq x \leq l \\ T_2 &= 0, & 0 \leq z \leq 2l, \quad 0 \leq x \leq l \end{aligned}$$

Hence,

$$\delta_{11} = \int_0^l \frac{x^2}{EI} dx + \int_0^{2l} \frac{(2l - z)^2}{EI} dz + \int_0^{2l} \frac{l^2}{GJ} dz \quad (\text{iv})$$

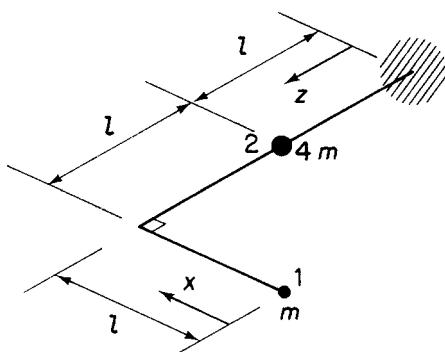


FIGURE 10.6 Mass–Beam System for Example 10.2

$$\delta_{22} = \int_0^l \frac{(l-z)^2}{EI} dz \quad (v)$$

$$\delta_{12} = \delta_{21} = \int_0^l \frac{(2l-z)(l-z)}{EI} dz \quad (vi)$$

from which we obtain

$$\delta_{11} = \frac{6l^3}{EI}, \quad \delta_{22} = \frac{l^3}{3EI}, \quad \delta_{12} = \delta_{21} = \frac{5l^3}{6EI}$$

Writing  $\lambda = ml^3/6EI$  and solving Eqs. (i) and (ii) in an identical manner to the solution of Eqs. (i) and (ii) in Example 10.1 results in a quadratic in  $\lambda\omega^2$ , namely

$$188(\lambda\omega^2)^2 - 44\lambda\omega^2 + 1 = 0 \quad (vii)$$

Solving Eq. (vii), we obtain

$$\lambda\omega^2 = \frac{44 \pm \sqrt{44^2 - 4 \times 188 \times 1}}{376}$$

which gives

$$\lambda\omega^2 = 0.21 \text{ or } 0.027$$

The lowest natural frequency therefore corresponds to  $\lambda\omega^2 = 0.027$  and is

$$\frac{1}{2\pi} \sqrt{\frac{0.162EI}{ml^3}}$$



### Example 10.3

Determine the natural frequencies of the system shown in Fig. 10.7 and sketch the normal modes. The flexural rigidity  $EI$  of the weightless beam is  $1.44 \times 10^6 \text{ Nm}^2$ ,  $l = 0.76 \text{ m}$ , the radius of gyration  $r$  of the mass  $m$  is  $0.152 \text{ m}$ , and its weight is  $1,435 \text{ N}$ .

In this problem, the mass possesses an inertia about its own center of gravity (its radius of gyration is not zero), which means that, in addition to translational displacements, it experiences rotation. The equations of motion are therefore

$$m\ddot{v}\delta_{11} + mr^2\ddot{\theta}\delta_{12} + v = 0 \quad (i)$$

$$m\ddot{v}\delta_{21} + mr^2\ddot{\theta}\delta_{22} + \theta = 0 \quad (ii)$$

where  $v$  is the vertical displacement of the mass at any instant of time and  $\theta$  is the rotation of the mass from its stationary position. Although the beam supports just one mass, it is subjected to two moment systems;  $M_1$  at any section  $z$  due to the weight of the mass and a constant moment  $M_2$  caused by the inertia couple of the mass as it rotates. Then,

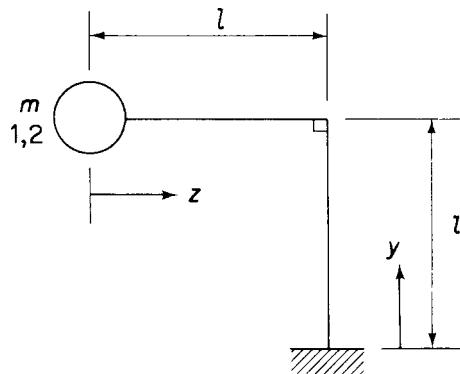


FIGURE 10.7 Mass–Beam System for Example 10.3

$$M_1 = 1z, \quad 0 \leq z \leq l$$

$$M_1 = 1l, \quad 0 \leq y \leq l$$

$$M_2 = 1, \quad 0 \leq z \leq l$$

$$M_2 = 1, \quad 0 \leq y \leq l$$

Hence,

$$\delta_{11} = \int_0^l \frac{z^2}{EI} dz + \int_0^l \frac{l^2}{EI} dy \quad (\text{iii})$$

$$\delta_{22} = \int_0^l \frac{dz}{EI} + \int_0^l \frac{dy}{EI} \quad (\text{iv})$$

$$\delta_{12} = \delta_{21} = \int_0^l \frac{zdz}{EI} + \int_0^l \frac{l}{EI} dy \quad (\text{v})$$

from which

$$\delta_{11} = \frac{4l^3}{3EI}, \quad \delta_{22} = \frac{2l}{EI}, \quad \delta_{12} = \delta_{21} = \frac{3l^2}{2EI}$$

Each mode oscillates with simple harmonic motion, so that

$$v = v_0 \sin(\omega t + \varepsilon), \quad \theta = \theta_0 \sin(\omega t + \varepsilon)$$

and

$$\ddot{v} = -\omega^2 v, \quad \ddot{\theta} = -\omega^2 \theta$$

Substituting in Eqs. (i) and (ii) gives

$$\left(1 - \omega^2 m \frac{4l^3}{3EI}\right) v - \omega^2 m r^2 \frac{3l^2}{2EI} \theta = 0 \quad (\text{vi})$$

$$-\omega^2 m \frac{3l^2}{2EI} v + \left(1 - \omega^2 \frac{2l}{EI}\right) \theta = 0 \quad (\text{vii})$$

Inserting the values of  $m$ ,  $r$ ,  $l$ , and  $EI$ , we have

$$\left(1 - \frac{1,435 \times 4 \times 0.76^3}{9.81 \times 3 \times 1.44 \times 10^6} \omega^2\right) v - \frac{1,435 \times 0.152^2 \times 3 \times 0.76^2}{9.81 \times 2 \times 1.44 \times 10^6} \omega^2 \theta = 0 \quad (\text{viii})$$

$$-\frac{1,435 \times 3 \times 0.76^2}{9.81 \times 2 \times 1.44 \times 10^6} \omega^2 v + \left(1 - \frac{1,435 \times 0.152^2 \times 2 \times 0.76}{9.81 \times 1.44 \times 10^6} \omega^2\right) \theta = 0 \quad (\text{ix})$$

or

$$(1 - 6 \times 10^{-5} \omega^2)v - 0.203 \times 10^{-5} \omega^2 \theta = 0 \quad (\text{x})$$

$$-8.8 \times 10^{-5} \omega^2 v + (1 - 0.36 \times 10^{-5} \omega^2) \theta = 0 \quad (\text{xi})$$

Solving Eqs. (x) and (xi) as before gives

$$\omega = 122 \text{ or } 1,300$$

from which the natural frequencies are

$$f_1 = \frac{61}{\pi}, \quad f_2 = \frac{650}{\pi}$$

From Eq. (x),

$$\frac{v}{\theta} = \frac{0.203 \times 10^{-5} \omega^2}{1 - 6 \times 10^{-5} \omega^2}$$

which is positive at the lowest natural frequency, corresponding to  $\omega = 122$ , and negative for  $\omega = 1,300$ . The modes of vibration are therefore as shown in Fig. 10.8.

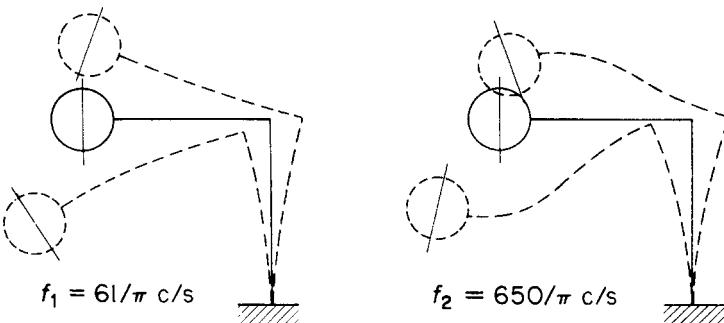


FIGURE 10.8 The First Two Natural Modes of Vibration of the Beam-Mass System of Fig. 10.7

## 10.2 OSCILLATION OF BEAMS

So far, we have restricted our discussion to weightless beams supporting concentrated, or otherwise, masses. We now investigate methods of determining normal modes and frequencies of vibration of beams possessing weight and therefore inertia. The equations of motion of such beams are derived on the assumption that vibration occurs in one of the principal planes of the beam and that the effects of rotary inertia and shear displacements may be neglected.

Figure 10.9(a) shows a uniform beam of cross-sectional area  $A$  vibrating in a principal plane about some axis Oz. The displacement of an element  $\delta z$  of the beam at any instant of time  $t$  is  $v$  and the moments and forces acting on the element are shown in Fig. 10.9(b). Taking moments about the vertical centerline of the element gives

$$S_y \frac{\partial z}{2} + M_x + \left( S_y + \frac{\partial S_y}{\partial z} \delta z \right) \frac{\partial z}{2} - \left( M_x + \frac{\partial M_x}{\partial z} \delta z \right) = 0$$

from which, neglecting second-order terms, we obtain

$$S_y = \frac{\partial M_x}{\partial z} \quad (10.8)$$

Considering the vertical equilibrium of the element,

$$\left( S_y + \frac{\partial S_y}{\partial z} \delta z \right) - S_y - \rho A \delta z \frac{\partial^2 v}{\partial t^2} = 0$$

so that

$$\frac{\partial S_y}{\partial z} = \rho A \frac{\partial^2 v}{\partial t^2} \quad (10.9)$$

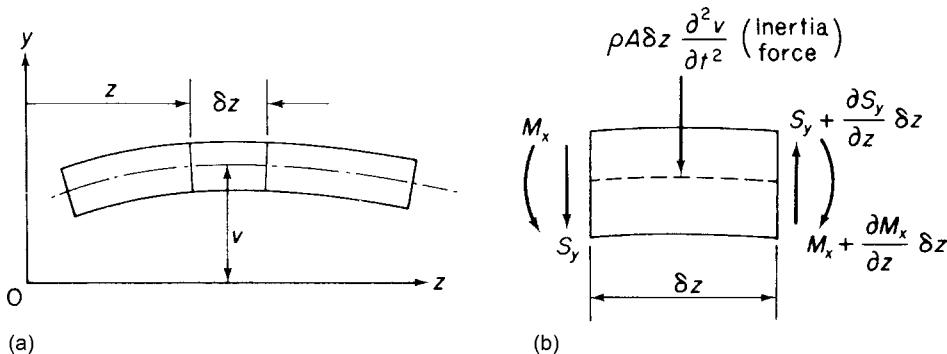


FIGURE 10.9 Vibration of a Beam Possessing Mass

From basic bending theory (Chapter 16),

$$M_x = -EI \frac{\partial^2 v}{\partial z^2} \quad (10.10)$$

It follows from Eqs. (10.8), (10.9), and (10.10) that

$$\frac{\partial^2}{\partial z^2} \left( -EI \frac{\partial^2 v}{\partial z^2} \right) = \rho A \frac{\partial^2 v}{\partial t^2} \quad (10.11)$$

Equation (10.11) is applicable to both uniform and nonuniform beams. In the latter case, the flexural rigidity,  $EI$ , and the mass per unit length,  $\rho A$ , are functions of  $z$ . For a beam of uniform section, Eq. (10.11) reduces to

$$EI \frac{\partial^4 v}{\partial z^4} + \rho A \frac{\partial^2 v}{\partial t^2} = 0 \quad (10.12)$$

In the normal modes of vibration, each element of the beam describes simple harmonic motion; thus,

$$v(z, t) = V(z) \sin(\omega t + \epsilon) \quad (10.13)$$

where  $V(z)$  is the amplitude of the vibration at any section  $z$ . Substituting for  $v$  from Eq. (10.13) in Eq. (10.12) yields

$$\frac{d^4 V}{dz^4} - \frac{\rho A \omega^2}{EI} V = 0 \quad (10.14)$$

Equation (10.14) is a fourth-order differential equation of standard form having the general solution

$$V = B \sin \lambda z + C \cos \lambda z + D \sinh \lambda z + F \cosh \lambda z \quad (10.15)$$

where

$$\lambda^4 = \frac{\rho A \omega^2}{EI}$$

and  $B$ ,  $C$ ,  $D$ , and  $F$  are unknown constants, which are determined from the boundary conditions of the beam. The ends of the beam may be

- (1) Simply supported or pinned, in which case the displacement and bending moment are zero, and therefore in terms of the function  $V(z)$ , we have  $V = 0$  and  $d^2V/dz^2 = 0$ ;
- (2) Fixed, giving zero displacement and slope, that is,  $V = 0$  and  $dV/dz = 0$ ;
- (3) Free, for which the bending moment and shear force are zero, hence,  $d^2V/dz^2 = 0$  and, from Eq. (10.8),  $d^3V/dz^3 = 0$ .

**Example 10.4**

Determine the first three normal modes of vibration and the corresponding natural frequencies of the uniform, simply supported beam shown in Fig. 10.10.

Since both ends of the beam are simply supported,  $V = 0$  and  $d^2V/dz^2 = 0$  at  $z = 0$  and  $z = L$ . From the first of these conditions and Eq. (10.15), we have

$$0 = C + F \quad (\text{i})$$

and, from the second,

$$0 = -\lambda^2 C + \lambda^2 F \quad (\text{ii})$$

Hence,  $C = F = 0$ . Applying these boundary conditions at  $z = L$  gives

$$0 = B \sin \lambda L + D \sinh \lambda L \quad (\text{iii})$$

and

$$0 = -\lambda^2 B \sin \lambda L + \lambda^2 D \sinh \lambda L \quad (\text{iv})$$

The only nontrivial solution ( $\lambda L \neq 0$ ) of Eqs. (iii) and (iv) is  $D = 0$  and  $\sin \lambda L = 0$ . It follows that

$$\lambda L = n\pi, \quad n = 1, 2, 3, \dots$$

Therefore,

$$\omega_n^2 = \left(\frac{n\pi}{L}\right)^4 \frac{EI}{\rho A}, \quad n = 1, 2, 3, \dots \quad (\text{v})$$

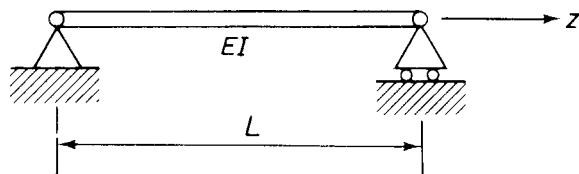
and the normal modes of vibration are given by

$$v(z, t) = B_n \sin \frac{n\pi z}{L} \sin (\omega_n t + \epsilon_n) \quad (\text{vi})$$

with natural frequencies

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \quad (\text{vii})$$

The first three normal modes of vibration are shown in Fig. 10.11.



**FIGURE 10.10 Beam of Example 10.4**

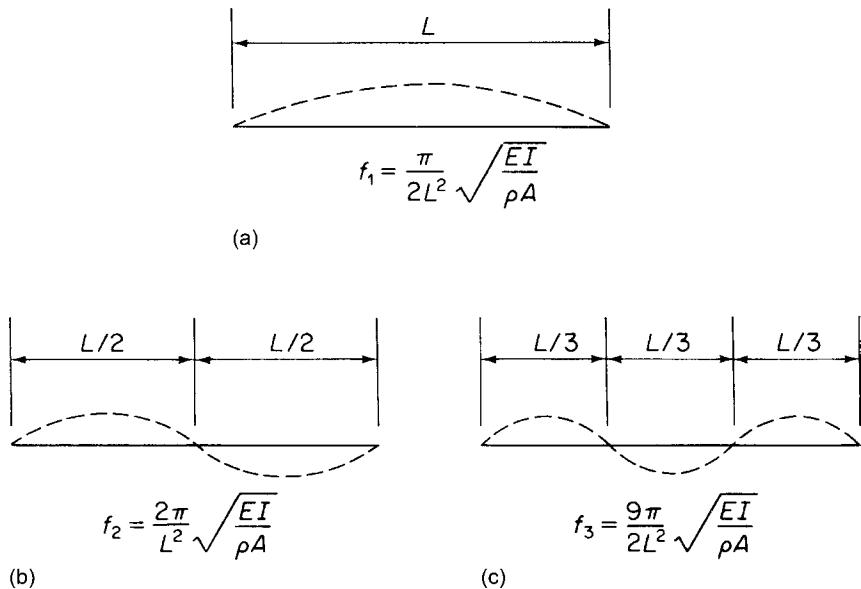


FIGURE 10.11 First Three Normal Modes of Vibration of the Beam of Example 10.4

**Example 10.5**

Find the first three normal modes and corresponding natural frequencies of the uniform cantilever beam shown in Fig. 10.12.

The boundary conditions in this problem are  $V = 0$ ,  $dV/dz = 0$  at  $z = 0$  and  $d^2V/dz^2 = 0$ ,  $d^3V/dz^3 = 0$  at  $z = L$ . Substituting these in turn in Eq. (10.15), we obtain

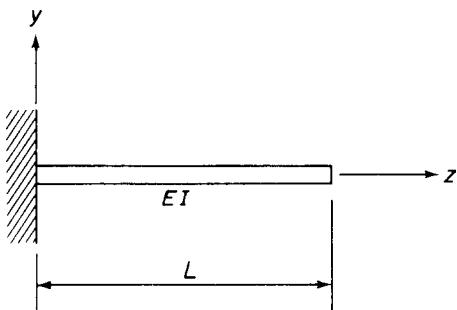


FIGURE 10.12 Cantilever Beam of Example 10.5

$$0 = C + F \quad (\text{i})$$

$$0 = \lambda B + \lambda D \quad (\text{ii})$$

$$0 = -\lambda^2 B \sin \lambda L - \lambda^2 C \cos \lambda L + \lambda^2 D \sinh \lambda L + \lambda^2 F \cosh \lambda L \quad (\text{iii})$$

$$0 = -\lambda^3 B \cos \lambda L + \lambda^3 C \sin \lambda L + \lambda^3 D \cosh \lambda L + \lambda^3 F \sinh \lambda L \quad (\text{iv})$$

From Eqs. (i) and (ii),  $C = -F$  and  $B = -D$ . Thus, replacing  $F$  and  $D$  in Eqs. (iii) and (iv), we obtain

$$B(-\sin \lambda L - \sinh \lambda L) + C(-\cos \lambda L - \cosh \lambda L) = 0 \quad (\text{v})$$

and

$$B(-\cos \lambda L - \cosh \lambda L) + C(\sin \lambda L - \sinh \lambda L) = 0 \quad (\text{vi})$$

Eliminating  $B$  and  $C$  from Eqs. (v) and (vi) gives

$$(-\sin \lambda L - \sinh \lambda L)(\sinh \lambda L - \sin \lambda L) + (\cos \lambda L + \cosh \lambda L)^2 = 0$$

Expanding this equation, and noting that  $\sin^2 \lambda L + \cos^2 \lambda L = 1$  and  $\cosh^2 \lambda L - \sinh^2 \lambda L = 1$  yields the frequency equation

$$\cos \lambda L \cosh \lambda L + 1 = 0 \quad (\text{vii})$$

Equation (vii) may be solved graphically or by Newton's method. The first three roots  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  are given by

$$\lambda_1 L = 1.875, \quad \lambda_2 L = 4.694, \quad \lambda_3 L = 7.855$$

from which are found the natural frequencies corresponding to the first three normal modes of vibration. The natural frequency of the  $r$ th mode ( $r \geq 4$ ) is obtained from the approximate relationship

$$\lambda_r L \approx (r - \frac{1}{2})\pi$$

and its shape in terms of a single arbitrary constant  $K_r$  is

$$V_r(z) = K_r [\cosh \lambda_r z - \cos \lambda_r z - k_r (\sinh \lambda_r z - \sin \lambda_r z)]$$

where

$$k_r = \frac{\cos \lambda_r L + \cosh \lambda_r L}{\sin \lambda_r L + \sinh \lambda_r L}, \quad r = 1, 2, 3, \dots$$

Figure 10.13 shows the first three normal mode shapes of the cantilever and their associated natural frequencies.

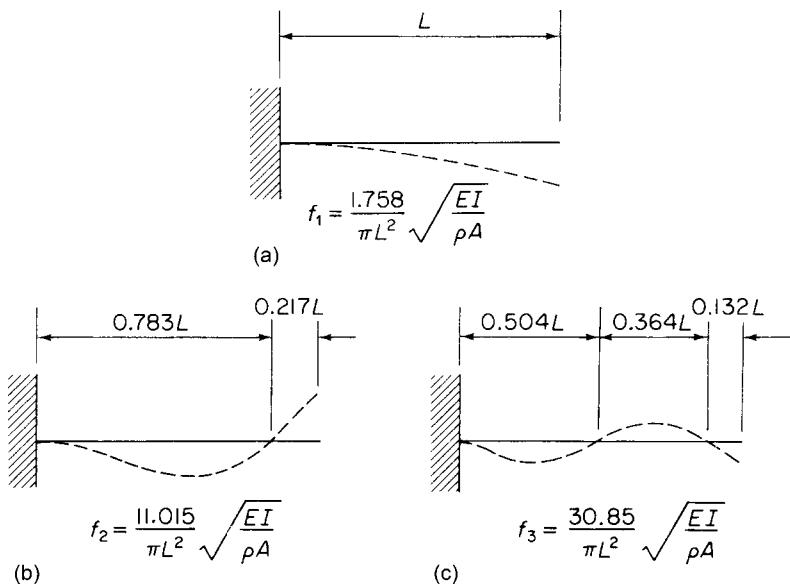


FIGURE 10.13 The First Three Normal Modes of Vibration of the Cantilever Beam of Example 10.5

### 10.3 APPROXIMATE METHODS FOR DETERMINING NATURAL FREQUENCIES

The determination of natural frequencies and normal mode shapes for beams of nonuniform section involves the solution of Eq. (10.11) and fulfillment of the appropriate boundary conditions. However, with the exception of a few special cases, such solutions do not exist, and the natural frequencies are obtained by approximate methods, such as the Rayleigh and Rayleigh–Ritz methods, which are presented here. Rayleigh's method is discussed first.

A beam vibrating in a normal or combination of normal modes possesses kinetic energy by virtue of its motion and strain energy as a result of its displacement from an initial unstrained condition. From the principle of conservation of energy, the sum of the kinetic and strain energies is constant with time. In computing the strain energy  $U$  of the beam, we assume that displacements are due to bending strains only so that

$$U = \int_L \frac{M^2}{2EI} dz \quad (\text{see Chapter 5}) \quad (10.16)$$

where

$$M = -EI \frac{\partial^2 v}{\partial z^2} \quad (\text{see Eq.(10.10)})$$

Substituting for  $v$  from Eq. (10.13) gives

$$M = -EI \frac{d^2V}{dz^2} \sin(\omega t + \varepsilon)$$

so that, from Eq. (10.16),

$$U = \frac{1}{2} \sin^2(\omega t + \varepsilon) \int_L EI \left( \frac{d^2V}{dz^2} \right)^2 dz \quad (10.17)$$

For a nonuniform beam, having a distributed mass  $\rho A(z)$  per unit length and carrying concentrated masses,  $m_1, m_2, m_3, \dots, m_n$  at distances  $z_1, z_2, z_3, \dots, z_n$  from the origin, the kinetic energy  $K_E$  may be written as

$$K_E = \frac{1}{2} \int_L \rho A(z) \left( \frac{\partial v}{\partial t} \right)^2 dz + \frac{1}{2} \sum_{r=1}^n m_r \left[ \left( \frac{\partial v}{\partial t} \right)_{z=z_r} \right]^2$$

Substituting for  $v(z)$  from Eq. (10.13), we have

$$K_E = \frac{1}{2} \omega^2 \cos^2(\omega t + \varepsilon) \left[ \int_L \rho A(z) V^2 dz + \sum_{r=1}^n m_r \{V(z_r)\}^2 \right] \quad (10.18)$$

Since  $K_E + U = \text{constant}$ , say  $C$ , then

$$\begin{aligned} & \frac{1}{2} \sin^2(\omega t + \varepsilon) \int_L EI \left( \frac{d^2V}{dz^2} \right)^2 dz + \frac{1}{2} \omega^2 \cos^2(\omega t + \varepsilon) \\ & \times \left[ \int_L \rho A(z) V^2 dz + \sum_{r=1}^n m_r \{V(z_r)\}^2 \right] = C \end{aligned} \quad (10.19)$$

Inspection of Eq. (10.19) shows that, when  $(\omega t + \varepsilon) = 0, \pi, 2\pi, \dots$ ,

$$\frac{1}{2} \omega^2 \left[ \int_L \rho A(z) V^2 dz + \sum_{r=1}^n m_r \{V(z_r)\}^2 \right] = C \quad (10.20)$$

and when

$$(\omega t + \varepsilon) = \pi/2, \quad 3\pi/2, \quad 5\pi/2, \dots,$$

then

$$\frac{1}{2} \int_L EI \left( \frac{d^2V}{dz^2} \right)^2 dz = C \quad (10.21)$$

In other words, the kinetic energy in the mean position is equal to the strain energy in the position of maximum displacement. From Eqs. (10.20) and (10.21),

$$\omega^2 = \frac{\int_L EI(d^2V/dz^2)^2 dz}{\int_L \rho A(z)V^2 dz + \sum_{r=1}^n m_r \{V(z_r)\}^2} \quad (10.22)$$

Equation (10.22) gives the exact value of natural frequency for a particular mode if  $V(z)$  is known. In the situation where a mode has to be “guessed,” Rayleigh’s principle states that if a mode is assumed that satisfies at least the slope and displacement conditions at the ends of the beam, then a good approximation to the true natural frequency is obtained. We have noted previously that if the assumed normal mode differs only slightly from the actual mode, then the stationary property of the normal modes ensures that the approximate natural frequency is only very slightly different from the true value. Furthermore, the approximate frequency is higher than the actual one, since the assumption of an approximate mode implies the presence of some constraints that force the beam to vibrate in a particular fashion; this has the effect of increasing the frequency.

The Rayleigh–Ritz method extends and improves the accuracy of the Rayleigh method by assuming a finite series for  $V(z)$ , namely

$$V(z) = \sum_{s=1}^n B_s V_s(z) \quad (10.23)$$

where each assumed function  $V_s(z)$  satisfies the slope and displacement conditions at the ends of the beam and the parameters  $B_s$  are arbitrary. Substitution of  $V(z)$  in Eq. (10.22) then gives approximate values for the natural frequencies. The parameters  $B_s$  are chosen to make these frequencies a minimum, thereby reducing the effects of the implied constraints. Having chosen a suitable series, the method of solution is to form a set of equations:

$$\frac{\partial \omega^2}{\partial B_s} = 0, \quad s = 1, 2, 3, \dots, n \quad (10.24)$$

Eliminating the parameter  $B_s$  leads to an  $n$ th-order determinant in  $\omega^2$ , whose roots give approximate values for the first  $n$  natural frequencies of the beam.

### Example 10.6

Determine the first natural frequency of a cantilever beam of length,  $L$ , flexural rigidity  $EI$ , and constant mass per unit length  $\rho A$ . The cantilever carries a mass  $2m$  at the tip, where  $m = \rho AL$ .

An exact solution to this problem may be found by solving Eq. (10.14) with the appropriate end conditions. Such a solution gives

$$\omega_1 = 1.1582 \sqrt{\frac{EI}{mL^3}}$$

and serves as a comparison for our approximate answer. As an assumed mode shape, we shall take the static deflection curve for a cantilever supporting a tip load, since, in this particular problem, the tip load  $2m$  is greater

than the mass  $\rho AL$  of the cantilever. If the reverse were true, we would assume the static deflection curve for a cantilever carrying a uniformly distributed load. Thus,

$$V(z) = a(3Lz^2 - z^3) \quad (\text{see Eq. (iv) of Example 16.5}) \quad (\text{i})$$

where the origin for  $z$  is taken at the built-in end and  $a$  is a constant term that includes the tip load and the flexural rigidity of the beam. From Eq. (i),

$$V(L) = 2aL^3 \quad \text{and} \quad \frac{d^2V}{dz^2} = 6a(L-z)$$

Substituting these values in Eq. (10.22), we obtain

$$\omega_1^2 = \frac{36EIa^2 \int_0^L (L-z)^2 dz}{\rho A a^2 \int_0^L (3L-z)^2 z^4 dz + 2m(2aL^3)^2} \quad (\text{ii})$$

Evaluating Eq. (ii) and expressing  $\rho A$  in terms of  $m$ , we obtain

$$\omega_1 = 1.1584 \sqrt{\frac{EI}{mL^3}} \quad (\text{iii})$$

which value is only 0.02 percent higher than the true value given previously. The estimation of higher natural frequencies requires the assumption of further, more complex shapes for  $V(z)$ . ■

It is clear from the previous elementary examples of normal mode and natural frequency calculation that the estimation of such modes and frequencies for a complete aircraft is a complex process. However, the aircraft designer is not restricted to calculation for the solution of such problems, although the advent of the digital computer has widened the scope and accuracy of this approach. Other possible methods are to obtain the natural frequencies and modes by direct measurement from the results of a *resonance test* on the actual aircraft or to carry out a similar test on a simplified scale model. Details of resonance tests are discussed in Section 29.4. Usually, a resonance test is impracticable, since the designer requires the information before the aircraft is built, although this type of test is carried out on the completed aircraft as a design check. The alternative of building a scale model has found favor for many years. Such models are usually designed to be as light as possible and to represent the stiffness characteristics of the full-scale aircraft. The inertia properties are simulated by a suitable distribution of added masses.

## PROBLEMS

**P.10.1.** Figure P.10.1 shows a massless beam ABCD of length  $3l$  and uniform bending stiffness  $EI$ , which carries concentrated masses  $2m$  and  $m$  at the points B and D, respectively. The beam is built in at end A and simply supported at C. In addition, a hinge at B allows only shear forces to be transmitted between sections AB and BCD. Calculate the natural frequencies of free, undamped oscillations of the system and determine the corresponding modes of vibration, illustrating your results by suitably dimensioned sketches.

Answer:  $\frac{1}{2\pi} \sqrt{\frac{3EI}{4ml^3}}, \quad \frac{1}{2\pi} \sqrt{\frac{3EI}{ml^3}}$

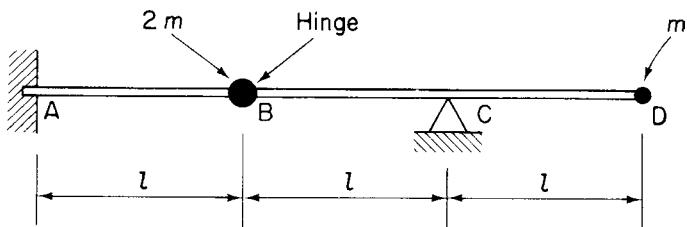


FIGURE P.10.1

**P.10.2.** Three massless beams 12, 23, and 24, each of length  $l$ , are rigidly joined together in one plane at the points 2, 12, and 23 being in the same straight line with 24 at right angles to them (see Fig. P.10.2). The bending stiffness of 12 is  $3EI$  while that of 23 and 24 is  $EI$ . The beams carry masses  $m$  and  $2m$  concentrated at the points 4 and 2, respectively. If the system is simply supported at 1 and 3, determine the natural frequencies of vibration in the plane of the figure.

$$\text{Answer: } \frac{1}{2\pi} \sqrt{\frac{2.13EI}{ml^3}}, \quad \frac{1}{2\pi} \sqrt{\frac{5.08EI}{ml^3}}$$

**P.10.3.** Two uniform circular tubes AB and BC are rigidly jointed at right angles at B and built in at A (Fig. P.10.3). The tubes themselves are massless but carry a mass of 20 kg at C, which has a polar radius of gyration of  $0.25a$  about an axis through its own center of gravity parallel to AB. Determine the natural frequencies and modes of vibration for small oscillations normal to the plane containing AB and BC. The tube has a mean diameter of 25 mm and wall thickness 1.25 mm. Assume that, for the material of the tube,  $E = 70,000 \text{ N/mm}^2$ ,  $G = 28,000 \text{ N/mm}^2$ , and  $a = 250 \text{ mm}$ .

Answer: 0.09 Hz, 0.62 Hz.

**P.10.4.** A uniform thin-walled cantilever tube, length  $L$ , circular cross-section of radius  $a$ , and thickness  $t$ , carries at its tip two equal masses  $m$ . One mass is attached to the tube axis while the other is mounted at the end of a light rigid bar at a distance of  $2a$  from the axis (see Fig. P.10.4). Neglecting the mass of the tube and assuming the stresses in the tube are given by basic bending theory and the Bredt–Batho theory of torsion, show that the frequencies  $\omega$  of the coupled torsion flexure oscillations which occur are given by

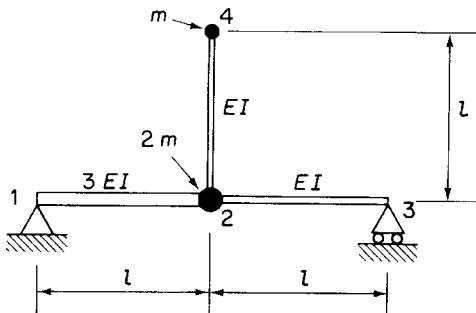


FIGURE P.10.2

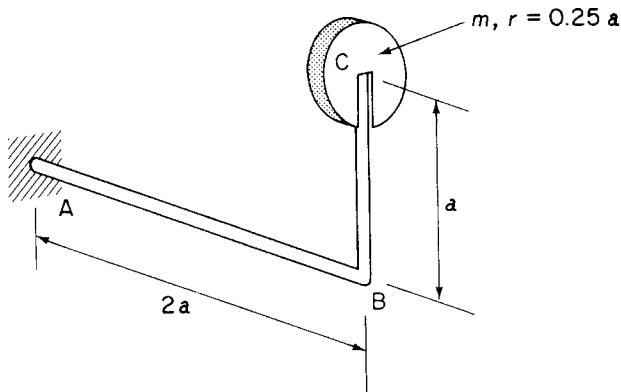


FIGURE P.10.3

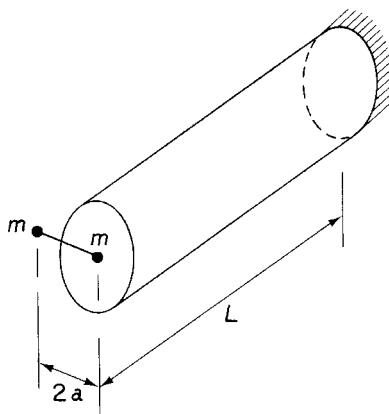


FIGURE P.10.4

$$\frac{1}{\omega^2} = \frac{mL^3}{3E\pi a^3 t} [1 + 2\lambda \pm (1 + 2\lambda + 2\lambda^2)^{\frac{1}{2}}]$$

where

$$\lambda = \frac{3E}{G} \frac{a^2}{L^2}$$

**P.10.5.** Figure P.10.5 shows the idealized cross-section of a single cell tube, with axis of symmetry  $xx$  and length 1,525 mm, in which the direct stresses due to bending are carried only in the four booms of the cross-section. The walls are assumed to carry only shear stresses. The tube is built in at the root and carries a weight of 4,450 N at its tip; the center of gravity of the weight coincides with the shear center of the tube cross-section. Assuming that the direct and shear stresses in the tube are given by basic bending theory, calculate the natural frequency of flexural vibrations of the weight in a vertical direction. The effect of the weight of the tube is to be neglected, and it should be noted that it is not necessary

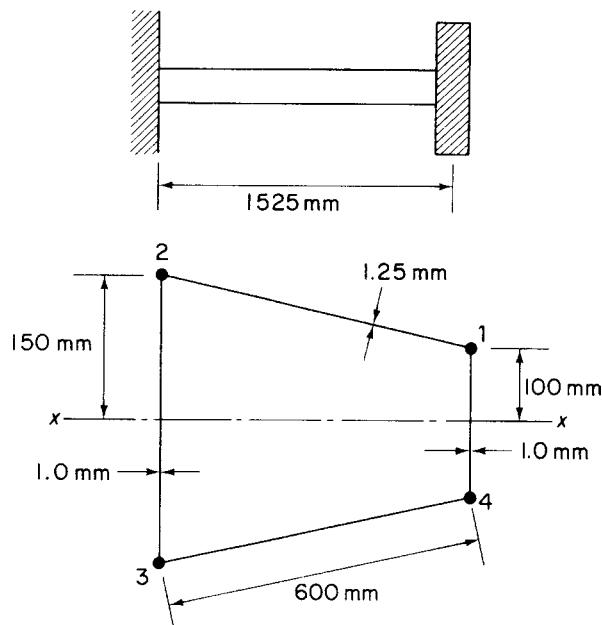


FIGURE P.10.5

to know the position of the shear center of the cross-section. The effect on the deflections of the shear strains in the tube walls must be included.

$$E = 70,000 \text{ N/mm}^2, \quad G = 26,500 \text{ N/mm}^2, \quad \text{boom areas } 970 \text{ mm}^2$$

Answer: 12.1 Hz.

**P.10.6.** A straight beam of length  $l$  is rigidly built in at its ends. For one-quarter of its length from each end, the bending stiffness is  $4EI$  and the mass/unit length is  $2m$ ; for the central half, the stiffness is  $EI$  and the mass  $m$  per unit length. In addition, the beam carries three mass concentrations,  $1/2 ml$  at  $1/4 l$  from each end and  $1/4 ml$  at the center, as shown in Fig. P.10.6. Use an energy method or other approximation to estimate the lowest frequency of natural flexural vibration. A first approximation solution will suffice, if it is accompanied by a brief explanation of a method of obtaining improved accuracy.

$$\text{Answer: } 3.7 \sqrt{\frac{EI}{ml^4}}.$$

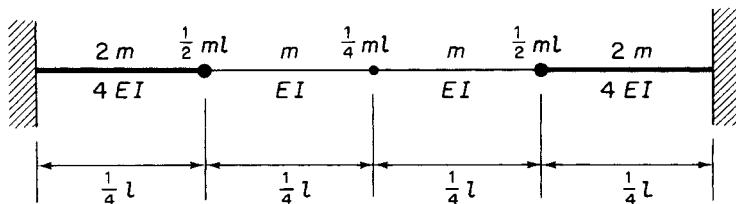


FIGURE P.10.6

# Materials

# 11

With the present chapter, we begin the purely aeronautical section of the book, where we consider structures peculiar to the field of aerospace engineering. These structures are typified by arrangements of thin, load-bearing skins, frames, and stiffeners, fabricated from lightweight, high-strength materials of which aluminum alloys are the most widely used examples.

As a preliminary to the analysis of the basic structural forms presented in subsequent chapters, we discuss the materials used in aerospace construction.

Several factors influence the selection of the structural material for an aircraft or space vehicle, but among these, strength allied to lightness is probably the most important. Other properties having varying, though sometimes critical, significance include: stiffness; toughness; resistance to corrosion; fatigue and the effects of environmental heating; ease of fabrication; availability and consistency of supply; and, not least important, cost.

The main groups of materials used in aircraft construction have been wood, steel, aluminum alloys with, more recently, titanium alloys, and fiber-reinforced composites. In the field of engine design, titanium alloys are used in the early stages of a compressor, while nickel-based alloys or steels are used for the hotter later stages. As we are concerned primarily with the materials involved in the construction of the airframe, discussion of materials used in engine manufacture falls outside the scope of this book.

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## 11.1 ALUMINUM ALLOYS

Pure aluminum is a relatively low-strength, extremely flexible metal with virtually no structural applications. However, when alloyed with other metals, its properties are improved significantly. For example, adding a few percent of copper and magnesium increases the yield and ultimate strengths by a factor of 4 to 6. Three groups of aluminum alloy have been used in the aerospace industry for many years and still play a major role in aircraft construction. In the first of these, aluminum is alloyed with copper, magnesium, manganese, silicon, and iron and has a typical composition of 4 percent copper, 0.5 percent magnesium, 0.5 percent manganese, 0.3 percent silicon, and 0.2 percent iron with the remainder being aluminum. In the wrought, heat-treated, naturally aged condition, this alloy possesses a 0.1 percent proof stress not less than  $230 \text{ N/mm}^2$ , a tensile strength not less than  $390 \text{ N/mm}^2$ , and an elongation at fracture of 15 percent. Artificial aging at a raised temperature of, for example,  $170^\circ\text{C}$  increases the proof stress to not less than  $370 \text{ N/mm}^2$  and the tensile strength to not less than  $460 \text{ N/mm}^2$  with an elongation of 8 percent.

The second group of alloys contain, in addition to the preceding, 1–2 percent of nickel, a higher content of magnesium, and possible variations in the amounts of copper, silicon, and iron. The most important property of these alloys is their retention of strength at high temperatures, which makes them particularly suitable for aero engine manufacture. A development of these alloys by Rolls-Royce and

High Duty Alloys Ltd. replaced some of the nickel by iron and reduced the copper content; these RR alloys, as they were called, were used for forgings and extrusions in aero engines and airframes.

The third group of alloys depends upon the inclusion of zinc and magnesium for their high strength and have a typical composition of 2.5 percent copper, 5 percent zinc, 3 percent magnesium, and up to 1 percent nickel with mechanical properties of 0.1 percent proof stress  $510 \text{ N/mm}^2$ , tensile strength  $585 \text{ N/mm}^2$ , and an elongation of 8 percent. In a modern development of this alloy, nickel has been eliminated and provision made for the addition of chromium and further amounts of manganese.

Alloys from each of these groups have been used extensively for airframes, skins, and other stressed components, the choice of alloy being influenced by factors such as strength (proof and ultimate stress), ductility, ease of manufacture (e.g., in extrusion and forging), resistance to corrosion and amenability to protective treatment, fatigue strength, freedom from liability to sudden cracking due to internal stresses, and resistance to fast crack propagation under load. Clearly, different types of aircraft or space vehicle have differing requirements. A military aircraft, for instance, having a relatively short life measured in hundreds of hours, does not call for the same degree of fatigue and corrosion resistance as a civil aircraft with a required life of 30,000 hours or more.

Unfortunately, as one particular property of aluminum alloys is improved, other desirable properties are sacrificed. For example, the extremely high static strength of the aluminum–zinc–magnesium alloys was accompanied for many years by a sudden liability to crack in an unloaded condition due to the retention of internal stresses in bars, forgings, and sheet after heat treatment. Although variations in composition eliminated this problem to a considerable extent, other deficiencies showed themselves. Early postwar passenger aircraft experienced large numbers of stress-corrosion failures of forgings and extrusions. The problem became so serious that, in 1953, it was decided to replace as many aluminum–zinc–manganese components as possible with the aluminum–4 percent copper Alloy L65 and to prohibit the use of forgings in zinc-bearing alloy in all future designs. However, improvements in the stress-corrosion resistance of the aluminum–zinc–magnesium alloys have resulted in recent years from British, American, and German research. Both British and American opinions agree on the benefits of including about 1 percent copper but disagree on the inclusion of chromium and manganese, while in Germany, the addition of silver has been found extremely beneficial. Improved control of casting techniques brought further improvements in resistance to stress corrosion. The development of aluminum–zinc–magnesium–copper alloys largely met the requirement for aluminum alloys possessing high strength, good fatigue crack growth resistance, and adequate toughness. Further development will concentrate on the production of materials possessing higher specific properties, bringing benefits in relation to weight saving rather than increasing strength and stiffness.

The first group of alloys possesses a lower static strength than the zinc-bearing alloys but are preferred for portions of the structure where fatigue considerations are of primary importance, such as the undersurfaces of wings, where tensile fatigue loads predominate. Experience has shown that the naturally aged version of these alloys has important advantages over the fully heat-treated forms in fatigue endurance and resistance to crack propagation. Furthermore, the inclusion of a higher percentage of magnesium was found, in America, to produce, in the naturally aged condition, mechanical properties between those of the normal naturally aged and artificially aged alloy. This alloy, designated 2024 (aluminum–copper alloys form the 2000 series), has the nominal composition of 4.5 percent copper, 1.5 percent magnesium, 0.6 percent manganese, with the remainder aluminum and appears to be a satisfactory compromise between the various important, but sometimes conflicting, mechanical properties.

Interest in aluminum–magnesium–silicon alloys has recently increased, although they have been in general use in the aerospace industry for decades. The reasons for this renewed interest are that they are potentially cheaper than aluminum–copper alloys and, being weldable, are capable of reducing manufacturing costs. In addition, variants, such as the ISO 6013 alloy, have improved property levels and, generally, possess a similar high fracture toughness and resistance to crack propagation as the 2000 series alloys.

Frequently, a particular form of an alloy is developed for a particular aircraft. An outstanding example of such a development was the use of Hiduminium RR58 as the basis for the main structural material, designated CM001, for Concorde. Hiduminium RR58 is a complex aluminum–copper–magnesium–nickel–iron alloy developed during World War II specifically for the manufacture of forged components in gas turbine aero engines. The chemical composition of the version used in Concorde, decided on the basis of elevated temperature, creep, fatigue, and tensile testing programs, has the detailed specification of materials listed in [Table 11.1](#). Generally, CM001 is found to possess better overall strength and fatigue characteristics over a wide range of temperatures than any of the other possible aluminum alloys.

**Table 11.1** Materials of CM001

	% Cu	% Mg	% Si	% Fe	% Ni	% Ti	% Al
Minimum	2.25	1.35	0.18	0.90	1.0	—	Remainder
Maximum	2.70	1.65	0.25	1.20	1.30	0.20	

The latest aluminum alloys to find general use in the aerospace industry are the aluminum–lithium alloys. Of these, the aluminum–lithium–copper–manganese alloy, 8090, developed in the United Kingdom, is extensively used in the main fuselage structure of GKN Westland Helicopters' design EH101; it has also been qualified for Eurofighter 2000 (now named the Typhoon) but has yet to be embodied. In the United States, the aluminum–lithium–copper alloy, 2095, has been used in the fuselage frames of the F16 as a replacement for 2124, resulting in a fivefold increase in fatigue life and a reduction in weight. Aluminum–lithium alloys can be successfully welded, possess a high fracture toughness, and exhibit a high resistance to crack propagation.

## 11.2 STEEL

The use of steel for the manufacture of thin-walled, box-section spars in the 1930s has been superseded by the aluminum alloys described in [Section 11.1](#). Clearly, its high specific gravity prevents steel's widespread use in aerospace construction, but it has retained some value as a material for castings for small components demanding high tensile strengths, high stiffness, and high resistance to wear. Such components include undercarriage pivot brackets, wing-root attachments, fasteners, and tracks.

Although the attainment of high and ultra-high tensile strengths presents no difficulty with steel, it is found that other properties are sacrificed and that it is difficult to manufacture into finished components. To overcome some of these difficulties, types of steel known as *maraging* steels were developed in 1961, from which carbon is either eliminated entirely or present only in very small amounts. Carbon, while producing the necessary hardening of conventional high-tensile steels, causes brittleness and

distortion; the latter is not easily rectifiable, as machining is difficult and cold forming impracticable. Welded fabrication is also almost impossible or very expensive. The hardening of maraging steels is achieved by the addition of other elements, such as nickel, cobalt, and molybdenum. A typical maraging steel has these elements present in the following proportions: nickel 17–19 percent, cobalt 8–9 percent, molybdenum 3–3.5 percent, with titanium 0.15–0.25 percent. The carbon content is a maximum of 0.03 percent, with traces of manganese, silicon, sulfur, phosphorus, aluminum, boron, calcium, and zirconium. Its 0.2 percent proof stress is nominally 1400 N/mm<sup>2</sup> and its modulus of elasticity 180,000 N/mm<sup>2</sup>.

The main advantages of maraging steels over conventional low-alloy steels are: higher fracture toughness and notched strength, simpler heat treatment, much lower volume change and distortion during hardening, very much simpler to weld, easier to machine, and better resistance to stress corrosion and hydrogen embrittlement. On the other hand, the material cost of maraging steels is three or more times greater than the cost of conventional steels, although this may be more than offset by the increased cost of fabricating a complex component from the latter steel.

Maraging steels have been used in aircraft arrester hooks, rocket motor cases, helicopter undercarriages, gears, ejector seats, and various structural forgings.

In addition to the preceding, steel in its stainless form has found applications primarily in the construction of super- and hypersonic experimental and research aircraft, where temperature effects are considerable. Stainless steel formed the primary structural material in the Bristol 188, built to investigate kinetic heating effects, and also in the American rocket aircraft, the X-15, capable of speeds of the order of Mach 5–6.

### 11.3 TITANIUM

The use of titanium alloys increased significantly in the 1980s, particularly in the construction of combat aircraft as opposed to transport aircraft. This increase continued in the 1990s to the stage where, for combat aircraft, the percentage of titanium alloy as a fraction of structural weight is of the same order as that of aluminum alloy. Titanium alloys possess high specific properties, have a good fatigue strength/tensile strength ratio with a distinct fatigue limit, and some retain considerable strength at temperatures up to 400–500°C. Generally, there is also a good resistance to corrosion and corrosion fatigue, although properties are adversely affected by exposure to temperature and stress in a salt environment. The latter poses particular problems in the engines of carrier-operated aircraft. Further disadvantages are a relatively high density, so that weight penalties are imposed if the alloy is extensively used, coupled with high primary and high fabrication costs, approximately seven times those of aluminum and steel.

In spite of this, titanium alloys were used in the airframe and engines of Concorde, while the Tornado wing carry-through box is fabricated from a weldable medium-strength titanium alloy. Titanium alloys are also used extensively in the F15 and F22 American fighter aircraft and are incorporated in the tail assembly of the Boeing 777 civil airliner. Other uses include forged components, such as flap and slat tracks and undercarriage parts.

New fabrication processes (e.g., superplastic forming combined with diffusion bonding) enable large and complex components to be produced, resulting in a reduction in production man-hours and weight. Typical savings are 30 percent in man-hours, 30 percent in weight, and 50 percent in cost

compared with conventional riveted titanium structures. It is predicted that the number of titanium components fabricated in this way for aircraft will increase significantly and include items such as access doors, sheet for areas of hot gas impingement, and so forth.

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## 11.4 POLYMERS

Polymers are generally not employed as structural materials, since they are of low strength and stiffness. Unlike metals, they do not possess a linear stress-strain relationship; in fact, the stiffness may change with the amount of strain. Also, their mechanical properties are temperature dependent, and at low temperatures they become brittle. There is a wide variety of polymers, but of this variety the following three are the main types.

*Elastomers.* For example, rubber, which is used for tires, seals, and liners, is flexible and exhibits a very large strain before failure occurs.

*Plastics.* Plain plastic materials have specific gravities of approximately unity and are therefore considerably heavier than wood, although of comparable strength. On the other hand, their specific gravities are less than half those of the aluminum alloys, so that they find uses as windows or lightly stressed parts whose dimensions are established by handling requirements rather than strength. They are also particularly useful as electrical insulators and as energy-absorbing shields for delicate instrumentation and even structures where severe vibration, such as in a rocket or space shuttle launch, occurs.

*Fibers.* Examples are natural fibers, synthetic fibers, and nylon; they are generally used to reinforce a low-strength polymer to form a composite structure.

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## 11.5 GLASS

The majority of modern aircraft have cabins pressurized for flight at high altitudes. Windscreens and windows are therefore subjected to loads normal to their midplanes. Glass is frequently the material employed for this purpose, in the form of plain or laminated plate or heat-strengthened plate. The types of plate glass used in aircraft have a modulus of elasticity between 70,000 and 75,000 N/mm<sup>2</sup> with a modulus of rupture in bending of 45 N/mm<sup>2</sup>. Heat-strengthened plate has a modulus of rupture of about four and a half times this figure.

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## 11.6 CERAMICS

Ceramics are not used as structural materials, as they are extremely brittle and will fracture at low values of strain. However, they are used as thermal protection for an actual structure that would be otherwise susceptible to mechanical degradation at high temperatures. They are also used in engine components and, in the form of tiles, provided the protective cover for the Space Shuttle. The degree of protection provided by the tiles on the leading edge of the wing of the Shuttle became clear when a piece of foam became detached from the fuel tank and damaged the ceramic skin of Space Shuttle *Columbia* in 2003 during launch. On reentry, the heat generated by air friction affected the resulting unprotected wing structure, causing the catastrophic break-up of *Columbia*.

## 11.7 COMPOSITE MATERIALS

Composite materials consist of strong fibers, such as glass or carbon, set in a polymer matrix, which is mechanically and chemically protective. The fibers may be continuous or discontinuous but possess a strength very much greater than that of the same bulk materials. For example, carbon fibers have a tensile strength of the order of 2,400 N/mm<sup>2</sup> and a modulus of elasticity of 400,000 N/mm<sup>2</sup>.

A sheet of fiber-reinforced material is anisotropic; that is, its properties depend on the direction of the fibers. Generally, therefore, in structural form, two or more sheets are sandwiched together to form a *lay-up* so that the fiber directions match those of the major loads.

In the early stages of the development of composite materials, glass fibers were used in a matrix of epoxy resin. This glass-reinforced plastic (GRP) was used for radomes and helicopter blades but found limited use in components of fixed-wing aircraft, due to its low stiffness. In the 1960s, new fibrous reinforcements were introduced; Kevlar, for example, is an aramid material with the same strength as glass but stiffer. Kevlar composites are tough but poor in compression and difficult to machine, so they were used in secondary structures. Another composite, using boron fiber and developed in the United States, was the first to possess sufficient strength and stiffness for primary structures.

These composites have now been replaced by carbon-fiber-reinforced plastics (CFRP), which have similar properties to boron composites but are very much cheaper. Typically, CFRP has a modulus of the order of three times that of GRP, one and a half times that of a Kevlar composite, and twice that of aluminum alloy. Its strength is three times that of aluminum alloy, approximately the same as that of GRP, and slightly less than that of Kevlar composites. CFRP does, however, suffer from some disadvantages. It is a brittle material and therefore does not yield plastically in regions of high stress concentration. Its strength is reduced by impact damage, which may not be visible, and the epoxy resin matrices can absorb moisture over a long period, which reduces its matrix-dependent properties, such as its compressive strength; this effect increases with increase of temperature. Further, the properties of CFRP are subject to more random variation than those of metals. All these factors must be allowed for in design. On the other hand, the stiffness of CFRP is much less affected than its strength by such conditions, and it is less prone to fatigue damage than metals are. It is estimated that replacing 40 percent of an aluminum alloy structure by CFRP would result in a 12 percent saving in total structural weight.

CFRP is included in the wing, tailplane, and forward fuselage of the latest Harrier development; is used in the Tornado taileron; and has been used to construct a complete Jaguar wing and engine bay door for testing purposes. The use of CFRP in the fabrication of helicopter blades led to significant increases in their service life, where fatigue resistance rather than stiffness is of primary importance. [Figure 11.1](#) shows the structural complexity of a Sea King helicopter rotor blade that incorporates CFRP, GRP, stainless steel, a honeycomb core, and foam filling. An additional advantage of the use of composites for helicopter rotor blades is that the molding techniques employed allow variations of cross-section along the span, resulting in substantial aerodynamic benefits. This approach is being employed in the fabrication of the main rotor blades of the GKN Westland Helicopters EH101.

A composite (fiberglass and aluminum) is used in the tail assembly of the Boeing 777, while the leading edge of the Airbus A310–300/A320 fin assembly is of conventional reinforced glass fiber construction, reinforced at the nose to withstand bird strikes. A complete composite airframe was produced for the Beechcraft Starship turboprop executive aircraft, which, however, was not a commercial success due to its canard configuration, causing drag and weight penalties.

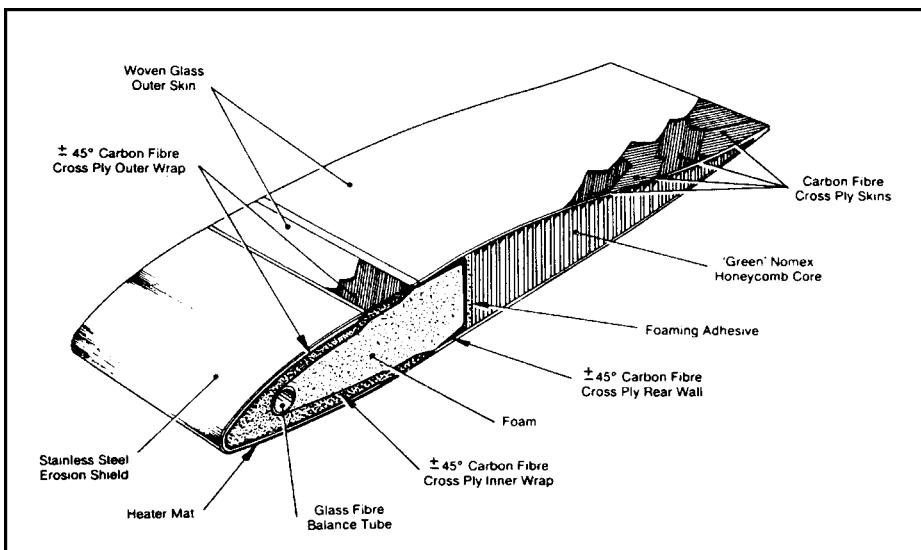


FIGURE 11.1 Sectional View of Helicopter Main Rotor Blade

(Courtesy Royal Aeronautical Society, Aerospace magazine)

The development of composite materials is continuing with research into the removal of strength-reducing flaws and local imperfections from carbon fibers. Other matrices, such as polyetheretherketone, which absorbs much less moisture than does epoxy resin, has an indefinite shelf life, and performs well under impact, are being developed; fabrication, however, requires much higher temperatures. Metal matrix composites, such as graphite–aluminum and boron–aluminum, are lightweight and retain their strength at higher temperatures than aluminum alloys but are expensive to produce. However, a fiber metal laminate (FML) is used as the upper fuselage skin of the Airbus A380 and as strap joint material in the Airbus A340 fuselage.

The use of composites in aircraft construction has increased steadily over the past decade. For example, composites in the Boeing 787 airliner comprise 50 percent of its structural weight, while a carbon-epoxy composite features significantly in the Boeing-McDonnell-Douglas F/A-18E jet fighter. Further, advanced composites are expected to feature more and more in the primary structure of both civil and military aircraft due to their potential for significant savings in weight.

We shall return to a more detailed analysis of the use of composites in aerospace structures in Chapter 26.

## 11.8 PROPERTIES OF MATERIALS

In Sections 11.1–11.7, we discussed the various materials used in aircraft construction and listed some of their properties. We now examine in more detail their behavior under load and also define different types of material.

### Ductility

A material is said to be *ductile* if it is capable of withstanding large strains under load before fracture occurs. These large strains are accompanied by a visible change in cross-sectional dimensions and therefore give warning of impending failure. Materials in this category include mild steel, aluminum and some of its alloys, copper, and polymers.

### Brittleness

A brittle material exhibits little deformation before fracture, the strain normally being below 5 percent. Brittle materials therefore may fail suddenly without visible warning. Included in this group are concrete, cast iron, high strength steel, timber, and ceramics.

### Elastic materials

A material is said to be *elastic* if deformations disappear completely on removal of the load. All known engineering materials are, in addition, *linearly elastic* within certain limits of stress, so that strain, within these limits, is directly proportional to stress.

### Plasticity

A material is perfectly *plastic* if no strain disappears after the removal of load. Ductile materials are *elastoplastic* and behave in an elastic manner until the *elastic limit* is reached, after which they behave plastically. When the stress is relieved, the elastic component of the strain is recovered but the plastic strain remains as a *permanent set*.

### Isotropic materials

In many materials, the elastic properties are the same in all directions at each point in the material, although they may vary from point to point; such a material is known as *isotropic*. An isotropic material having the same properties at all points is known as *homogeneous* (e.g., mild steel).

### Anisotropic materials

Materials having varying elastic properties in different directions are known as *anisotropic*.

### Orthotropic materials

Although a structural material may possess different elastic properties in different directions, this variation may be limited, as in the case of timber, which has just two values of Young's modulus, one in the direction of the grain and one perpendicular to the grain. A material whose elastic properties are limited to different values in three mutually perpendicular directions is known as *orthotropic*.

#### 11.8.1 Testing of engineering materials

The properties of engineering materials are determined mainly by the mechanical testing of specimens machined to prescribed sizes and shapes. The testing may be static or dynamic in nature, depending on the particular property being investigated. Possibly the most common mechanical static tests are tensile and compressive tests that are carried out on a wide range of materials. Ferrous and nonferrous metals are subjected to both forms of test, while compression tests are usually carried out on many nonmetallic materials. Other static tests include bending, shear, and hardness tests, while the toughness of a material, in other words, its ability to withstand shock loads, is determined by impact tests.

### Tensile tests

Tensile tests are normally carried out on metallic materials and, in addition, timber. Test pieces are machined from a batch of material, their dimensions being specified by Codes of Practice. They are commonly circular in cross-section, although flat test pieces having rectangular cross-sections are used when the batch of material is in the form of a plate. A typical test piece has the dimensions specified in Fig. 11.2. Usually, the diameter of a central portion of the test piece is fractionally less than that of the remainder to ensure that the test piece fractures between the gauge points.

Before the test begins, the mean diameter of the test piece is obtained by taking measurements at several sections using a micrometer screw gauge. Gauge points are punched at the required gauge length, the test piece is placed in the testing machine, and a suitable strain-measuring device, usually an extensometer, is attached to the test piece at the gauge points, so that the extension is measured over the given gauge length. Increments of load are applied and the corresponding extensions recorded. This procedure continues until yield occurs, when the extensometer is removed as a precaution against the damage caused if the test piece fractures unexpectedly. Subsequent extensions are measured by dividers placed in the gauge points until, ultimately, the test piece fractures. The final gauge length and the diameter of the test piece in the region of the fracture are measured so that the percentage elongation and percentage reduction in area may be calculated. These two parameters give a measure of the ductility of the material.

A stress-strain curve is drawn (see Figs 11.9 and 11.13), the stress normally being calculated on the basis of the original cross-sectional area of the test piece, that is, a *nominal stress* as opposed to an *actual stress* (which is based on the actual area of cross-section).

For ductile materials, there is a marked difference in the latter stages of the test, as a considerable reduction in cross-sectional area occurs between yield and fracture. From the stress-strain curve, the ultimate stress, the yield stress, and Young's modulus,  $E$ , are obtained.

There are a number of variations on the basic tensile test just described. Some of these depend upon the amount of additional information required and some upon the choice of equipment. There is a wide range of strain measuring devices to choose from, extending from different makes of mechanical extensometer, such as Huggenberger, Lindley, and Cambridge, to the electrical resistance strain gauge. The last normally is used on flat test pieces, one on each face to eliminate the effects of possible bending. At the same time, a strain gauge could be attached in a direction perpendicular to the direction of loading, so that lateral strains are measured. The ratio lateral strain/longitudinal strain is Poisson's ratio,  $\nu$ .

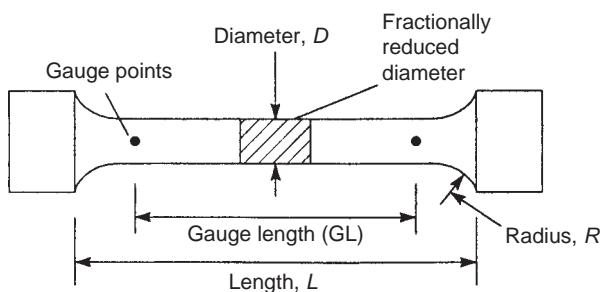


FIGURE 11.2 Standard Cylindrical Test Piece

Testing machines are usually driven hydraulically. More sophisticated versions employ load cells to record load and automatically plot load against extension or stress against strain on a pen recorder as the test proceeds – an advantage when investigating the distinctive behavior of mild steel at yield.

### **Compression tests**

A compression test is similar in operation to a tensile test, with the obvious difference that the load transmitted to the test piece is compressive rather than tensile. This is achieved by placing the test piece between the platens of the testing machine and reversing the direction of loading. Test pieces are normally cylindrical and are limited in length to eliminate the possibility of failure being caused by instability. Again, contractions are measured over a given gauge length by a suitable strain-measuring device.

Variations in test pieces occur when only the ultimate strength of the material in compression is required. For this purpose, concrete test pieces may take the form of cubes having edges approximately 10 cm long, while mild steel test pieces are still cylindrical in section but are of the order of 1 cm long.

### **Bending tests**

Many structural members are subjected primarily to bending moments. Bending tests are therefore carried out on simple beams constructed from the different materials to determine their behavior under this type of load.

Two forms of loading are employed, the choice depending upon the type specified in Codes of Practice for the particular material. In the first, a simply supported beam is subjected to a “two-point” loading system, as shown in Fig. 11.3(a). Two concentrated loads are applied symmetrically to the beam, producing zero shear force and constant bending moment in the central span of the beam (Figs. 11.3(b) and (c)). The condition of pure bending is therefore achieved in the central span.

The second form of loading system consists of a single concentrated load at midspan (Fig. 11.4(a)), which produces the shear force and bending moment diagrams shown in Figs. 11.4(b) and (c).

The loads may be applied manually by hanging weights on the beam or by a testing machine. Deflections are measured by a dial gauge placed underneath the beam. From the recorded results, a load-deflection diagram is plotted.

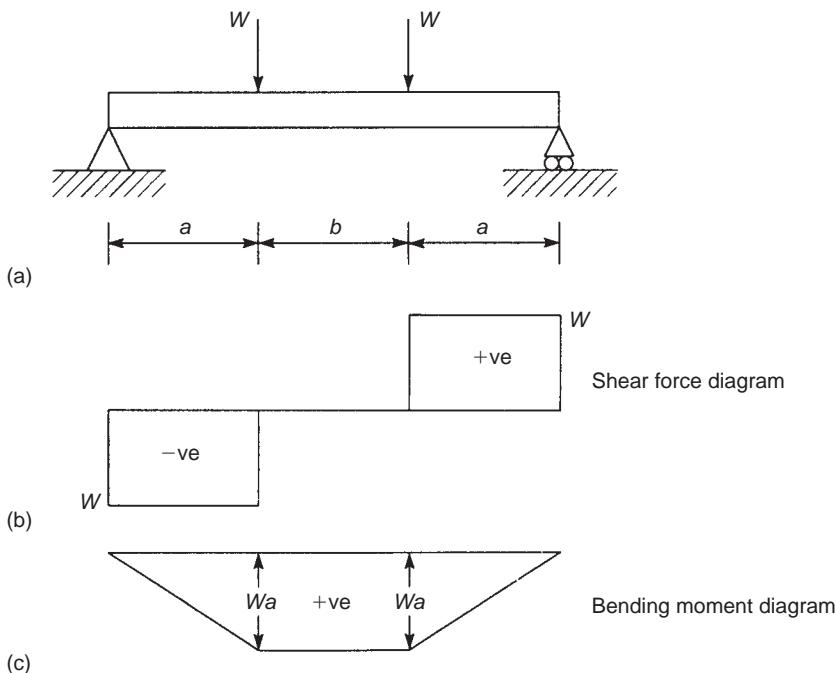
For most ductile materials, the test beams continue to deform without failure and fracture does not occur. Thus, plastic properties – for example, the ultimate strength in bending – cannot be determined for such materials. In the case of brittle materials, including cast iron, timber, and various plastics, failure does occur, so that plastic properties can be evaluated. For such materials, the ultimate strength in bending is defined by the *modulus of rupture*. This is taken to be the maximum direct stress in bending,  $\sigma_{x,u}$ , corresponding to the ultimate moment  $M_u$ , and is assumed to be related to  $M_u$  by the elastic relationship:

$$\sigma_{x,u} = \frac{M_u}{I} y_{\max} \quad (\text{see Eq. (16.9)})$$

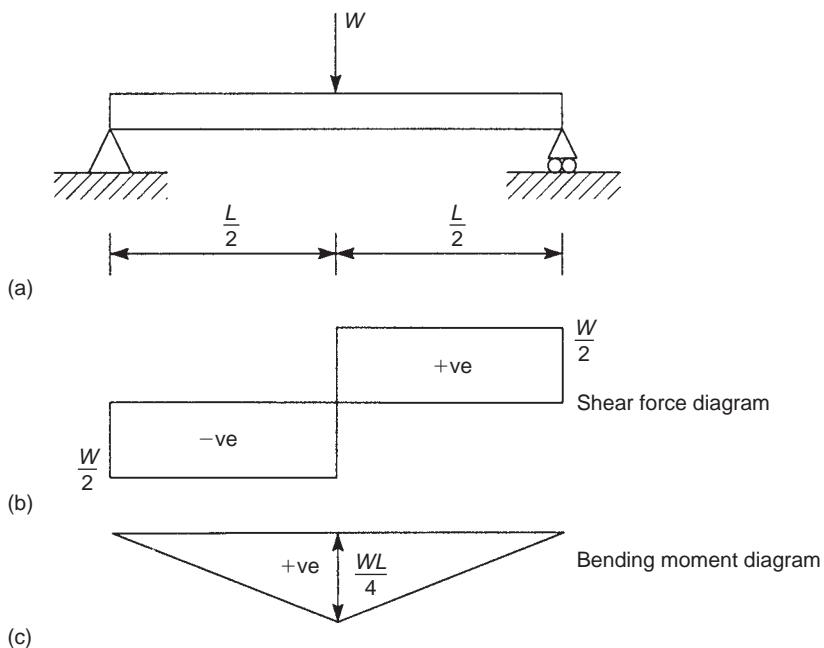
Other bending tests are designed to measure the ductility of a material and involve the bending of a bar around a pin. The angle of bending at which the bar starts to crack is then taken as an indication of its ductility.

### **Shear tests**

Two main types of shear test are used to determine the shear properties of materials. One type investigates the direct or transverse shear strength of a material and is used in connection with the shear strength of bolts, rivets, and beams. A typical arrangement is shown diagrammatically in Fig. 11.5,



**FIGURE 11.3 Bending Test on a Beam, "Two-Point" Load**



**FIGURE 11.4 Bending Test on a Beam, Single Load**

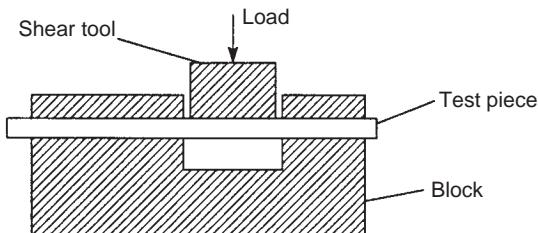


FIGURE 11.5 Shear Test

where the test piece is clamped to a block and the load is applied through the shear tool until failure occurs. In the arrangement shown the test piece is subjected to double shear, whereas if it is extended only partially across the gap in the block, it is subjected to single shear. In either case, the average shear strength is taken as the maximum load divided by the shear-resisting area.

The other type of shear test is used to evaluate the basic shear properties of a material, such as the shear modulus,  $G$ , the shear stress at yield, and the ultimate shear stress. In the usual form of test, a solid circular-section test piece is placed in a torsion machine and twisted by controlled increments of torque. The corresponding angles of twist are recorded and torque-twist diagrams plotted, from which the shear properties of the material are obtained. The method is similar to that used to determine the tensile properties of a material from a tensile test and uses relationships derived in [Chapter 3](#).

### **Hardness tests**

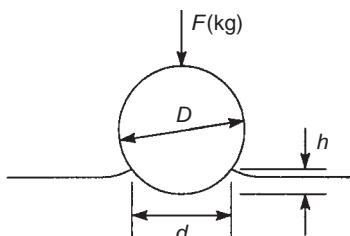
The machinability of a material and its resistance to scratching or penetration are determined by its "hardness." There also appears to be a connection between the hardness of some materials and their tensile strength, so that hardness tests may be used to determine the properties of a finished structural member where tensile and other tests would be impracticable. Hardness tests are also used to investigate the effects of heat treatment, hardening and tempering, and cold forming. Two types of hardness test are in common use: *indentation tests* and *scratch and abrasion tests*.

Indentation tests may be subdivided into two classes: static and dynamic. Of the static tests, the *Brinell* is the most common. In this, a hardened steel ball is pressed into the material under test by a static load acting for a fixed period of time. The load in kilograms divided by the spherical area of the indentation in square millimeters is called the *Brinell hardness number (BHN)*. In [Fig. 11.6](#), if  $D$  is the diameter of the ball,  $F$  the load in kilograms,  $h$  the depth of the indentation, and  $d$  the diameter of the indentation, then

$$BHN = \frac{F}{\pi D h} = \frac{2F}{\pi D [D - \sqrt{D^2 - d^2}]}$$

In practice, the hardness number of a given material is found to vary with  $F$  and  $D$ , so that for uniformity the test is standardized. For steel and hard materials,  $F = 3000$  kg and  $D = 10$  mm, while for soft materials,  $F = 500$  kg and  $D = 10$  mm; in addition, the load is usually applied for 15 s.

In the Brinell test, the dimensions of the indentation are measured by means of a microscope. To avoid this rather tedious procedure, direct reading machines have been devised, of which the *Rockwell* is typical. The indenting tool, again a hardened sphere, is first applied under a definite light load.



**FIGURE 11.6 Brinell Hardness Test**

The indenting tool is then replaced by a diamond cone with a rounded point, which is then applied under a specified indentation load. The difference between the depth of the indentation under the two loads is taken as a measure of the hardness of the material and is read directly from the scale.

A typical dynamic hardness test is performed by the *Shore Scleroscope*, which consists of a small hammer approximately 20 mm long and 6 mm in diameter fitted with a blunt, rounded, diamond point. The hammer is guided by a vertical glass tube and allowed to fall freely from a height of 25 cm onto the specimen, which it indents before rebounding. A certain proportion of the energy of the hammer is expended in forming the indentation, so that the height of the rebound, which depends upon the energy still possessed by the hammer, is taken as a measure of the hardness of the material.

A number of tests have been devised to measure the “scratch hardness” of materials. In one test, the smallest load in grams that, when applied to a diamond point, produces a scratch visible to the naked eye on a polished specimen of material is called its *hardness number*. In other tests, the magnitude of the load required to produce a definite width of scratch is taken as the measure of hardness. Abrasion tests, involving the shaking over a period of time of several specimens placed in a container, measure the resistance to wear of some materials. In some cases, there appears to be a connection between wear and hardness number, although the results show no level of consistency.

### **Impact tests**

It has been found that certain materials, particularly heat-treated steels, are susceptible to failure under shock loading, whereas an ordinary tensile test on the same material shows no abnormality. Impact tests measure the ability of materials to withstand shock loads and provide an indication of their *toughness*. Two main tests are in use, the *Izod* and the *Charpy*.

Both tests rely on a striker or weight attached to a pendulum. The pendulum is released from a fixed height, the weight strikes a notched test piece, and the angle through which the pendulum then swings is a measure of the toughness of the material. The arrangement for the Izod test is shown diagrammatically in Fig. 11.7(a). The specimen and the method of mounting are shown in detail in Fig. 11.7(b). The Charpy test is similar in operation except that the test piece is supported in a different manner, as shown in the plan view in Fig. 11.8.

### **11.8.2 Stress-strain curves**

We now examine in detail the properties of the different materials from the viewpoint of the results obtained from tensile and compression tests.

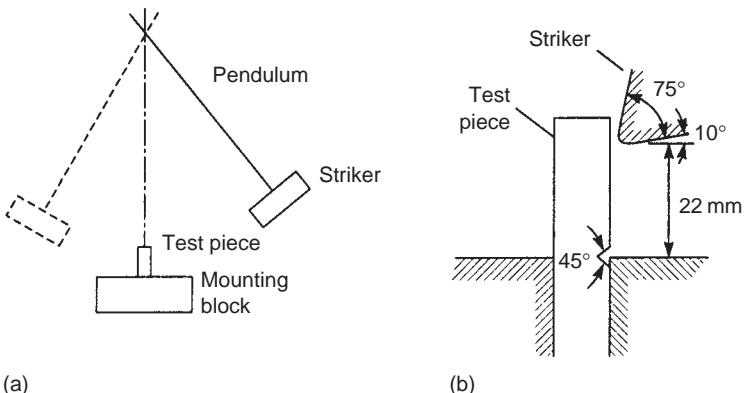


FIGURE 11.7 Izod Impact Test

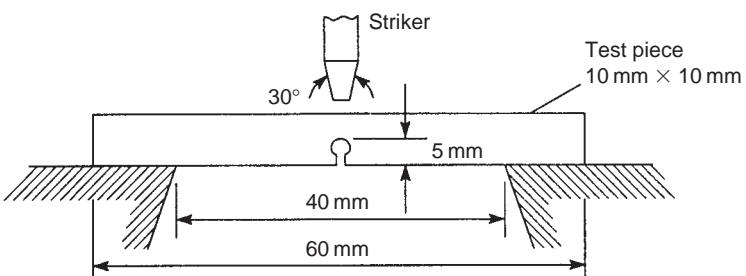
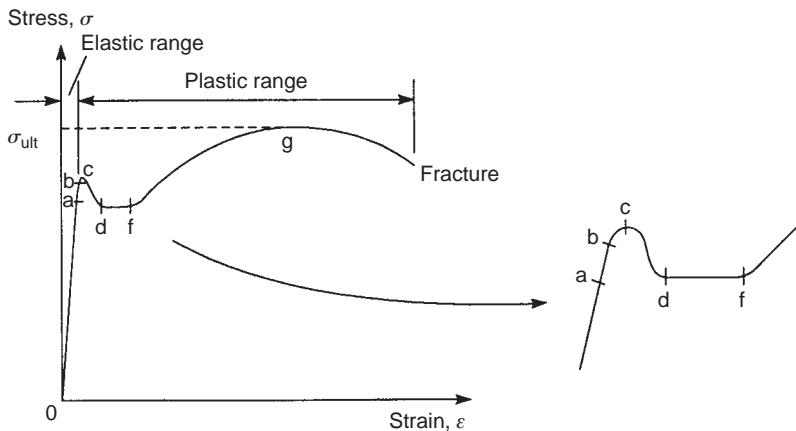


FIGURE 11.8 Charpy Impact Test

### **Low carbon steel (mild steel)**

A nominal stress-strain curve for mild steel, a ductile material, is shown in Fig. 11.9. From 0 to a, the stress-strain curve is linear, the material in this range obeying Hooke's law. Beyond a, the *limit of proportionality*, stress is no longer proportional to strain, and the stress-strain curve continues to b, the *elastic limit*, which is defined as the maximum stress that can be applied to a material without producing a permanent plastic deformation or *permanent set* when the load is removed. In other words, if the material is stressed beyond b and the load then removed, a residual strain exists at zero load. For many materials, it is impossible to detect a difference between the limit of proportionality and the elastic limit. From 0 to b, the material is said to be in the *elastic range*, while from b to fracture, the material is in the *plastic range*. The transition from the elastic to the plastic range may be explained by considering the arrangement of crystals in the material. As the load is applied, slipping occurs between the crystals aligned most closely to the direction of load. As the load is increased, more and more crystals slip with each equal load increment, until appreciable strain increments are produced and the plastic range is reached.

A further increase in stress from b results in the mild steel reaching its *upper yield point* at c, followed by a rapid fall in stress to its *lower yield point* at d. The existence of a lower yield point for mild

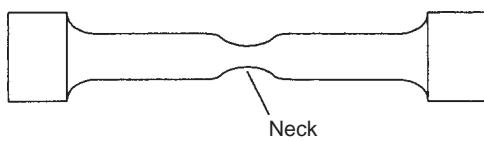


**FIGURE 11.9 Stress–Strain Curve for Mild Steel**

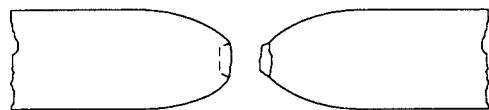
steel is a peculiarity of the tensile test, wherein the movement of the ends of the test piece produced by the testing machine does not proceed as rapidly as its plastic deformation; the load therefore decreases, as does the stress. From d to f, the strain increases at a roughly constant value of stress until *strain hardening* again causes an increase in stress. This increase in stress continues, accompanied by a large increase in strain to g, the *ultimate stress*,  $\sigma_{\text{ult}}$ , of the material. At this point, the test piece begins, visibly, to “neck” as shown in Fig. 11.10. The material in the test piece in the region of the “neck” is almost perfectly plastic at this stage, and from this point onward to fracture, there is a reduction in nominal stress.

For mild steel, yielding occurs at a stress of the order of  $300 \text{ N/mm}^2$ . At fracture, the strain (i.e., the elongation) is of the order of 30 percent. The gradient of the linear portion of the stress–strain curve gives a value for Young’s modulus in the region of  $200,000 \text{ N/mm}^2$ .

The characteristics of the fracture are worthy of examination. In a cylindrical test piece, the two halves of the fractured test piece have ends that form a “cup and cone” (Fig. 11.11). The actual failure planes in this case are inclined at approximately  $45^\circ$  to the axis of loading and coincide with planes of maximum shear stress. Similarly, if a flat tensile specimen of mild steel is polished then stressed, a pattern of fine lines appears on the polished surface at yield. These lines, which were first discovered by Lüder in 1854, intersect approximately at right angles and are inclined at  $45^\circ$  to the axis of the specimen, thereby coinciding with planes of maximum shear stress. These forms of yielding and fracture



**FIGURE 11.10 “Necking” of a Test Piece in the Plastic Range**



**FIGURE 11.11 “Cup-and-Cone” Failure of a Mild Steel Test Piece**

suggest that the crystalline structure of the steel is relatively weak in shear, with yielding taking the form of the sliding of one crystal plane over another rather than the tearing apart of two crystal planes.

The behavior of mild steel in compression is very similar to its behavior in tension, particularly in the elastic range. In the plastic range, it is not possible to obtain ultimate and fracture loads, since, due to compression, the area of cross-section increases as the load increases, producing a “barrelling” effect, as shown in Fig. 11.12. This increase in cross-sectional area tends to decrease the true stress, thereby increasing the load resistance. Ultimately, a flat disc is produced. For design purposes, the ultimate stresses of mild steel in tension and compression are assumed to be the same.

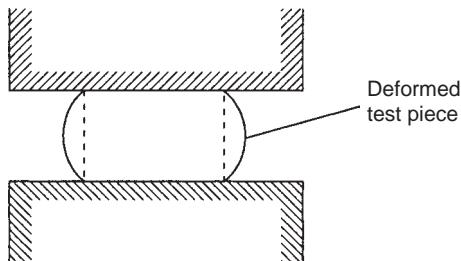
Higher grades of steel have greater strengths than mild steel but are not as ductile. They also possess the same Young’s modulus, so that the higher stresses are accompanied by higher strains.

### Aluminum

Aluminum and some of its alloys are also ductile materials, although their stress-strain curves do not have the distinct yield stress of mild steel. A typical stress-strain curve is shown in Fig. 11.13. The points *a* and *b* again mark the limit of proportionality and elastic limit, respectively, but are difficult to determine experimentally. Instead, a *proof stress* is defined, which is the stress required to produce a given permanent strain on removal of the load. In Fig. 11.13, a line drawn parallel to the linear portion of the stress-strain curve from a strain of 0.001 (i.e., a strain of 0.1 percent) intersects the stress-strain curve at the 0.1 percent proof stress. For elastic design this, or the 0.2 percent proof stress, is taken as the working stress.

Beyond the limit of proportionality, the material extends plastically, reaching its ultimate stress,  $\sigma_{ult}$ , at *d* before finally fracturing under a reduced nominal stress at *f*.

A feature of the fracture of aluminum alloy test pieces is the formation of a “double cup,” as shown in Fig. 11.14, implying that failure was initiated in the central portion of the test piece while the outer surfaces remained intact. Again, considerable “necking” occurs.



**FIGURE 11.12 “Barrelling” of a Mild Steel Test Piece in Compression**

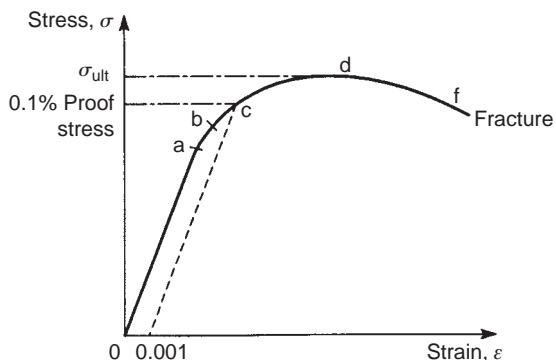


FIGURE 11.13 Stress–Strain Curve for Aluminum

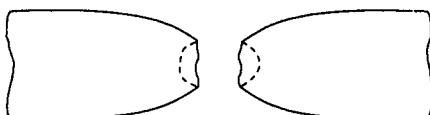


FIGURE 11.14 “Double-Cup” Failure of an Aluminum Alloy Test Piece

In compression tests on aluminum and its ductile alloys, similar difficulties are encountered to those experienced with mild steel. The stress–strain curve is very similar in the elastic range to that obtained in a tensile test, but the ultimate strength in compression cannot be determined; in design, its value is assumed to coincide with that in tension.

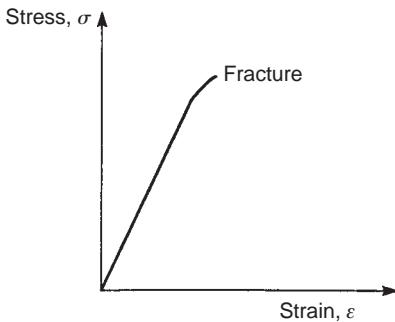
Aluminum and its alloys can suffer a form of corrosion, particularly in the salt-laden atmosphere of coastal regions. The surface becomes pitted and covered by a white furry deposit. This can be prevented by an electrolytic process called *anodizing*, which covers the surface with an inert coating. Aluminum alloys also corrode if they are placed in direct contact with other metals, such as steel. To prevent this, plastic is inserted between the possible areas of contact.

### ***Brittle materials***

These include cast iron, high-strength steel, concrete, timber, ceramics, and glass. The plastic range for brittle materials extends to only small values of strain. A typical stress–strain curve for a brittle material under tension is shown in Fig. 11.15. Little or no yielding occurs, and fracture takes place very shortly after the elastic limit is reached.

The fracture of a cylindrical test piece takes the form of a single failure plane approximately perpendicular to the direction of loading with no visible “necking” and an elongation of the order of 2–3 percent.

In compression, the stress–strain curve for a brittle material is very similar to that in tension, except that failure occurs at a much higher value of stress; for concrete, the ratio is of the order of 10:1. This is thought to be due to the presence of microscopic cracks in the material, giving rise to high stress concentrations, which are more likely to have a greater effect in reducing tensile strength than compressive strength.



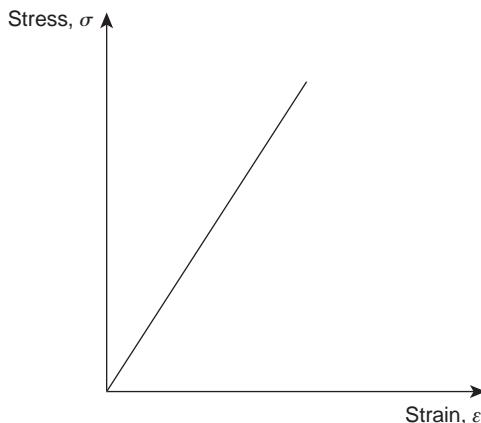
**FIGURE 11.15 Stress–Strain Curve for a Brittle Material**

### Composites

Fiber composites have stress–strain characteristics indicating that they are brittle materials (Fig. 11.16). There is little or no plasticity, and the modulus of elasticity is less than that of steel and aluminum alloy. However, the fibers themselves can have much higher values of strength and modulus of elasticity than the composite. For example, carbon fibers have a tensile strength of the order  $2,400 \text{ N/mm}^2$  and a modulus of elasticity of  $400,000 \text{ N/mm}^2$ .

Fiber composites are highly durable, require no maintenance, and can be used in hostile chemical and atmospheric environments; vinyls and epoxy resins provide the best resistance.

All the stress–strain curves described in the preceding discussion are those produced in tensile or compression tests in which the strain is applied at a negligible rate. A rapid strain application results in significant changes in the apparent properties of the materials, giving possible variations in yield stress of up to 100 percent.



**FIGURE 11.16 Stress–Strain Curve for a Fiber Composite**

### 11.8.3 Strain hardening

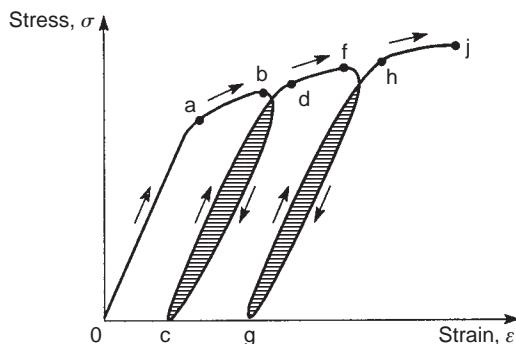
The stress-strain curve for a material is influenced by the *strain history*, or the loading and unloading of the material, within the plastic range. For example, in Fig. 11.17, a test piece is initially stressed in tension beyond the yield stress at a to a value at b. The material is then unloaded to c and reloaded to f, producing an increase in yield stress from the value at a to the value at d. Subsequent unloading to g and loading to j increases the yield stress still further to the value at h. This increase in strength resulting from the loading and unloading is known as *strain hardening*. It can be seen, from Fig. 11.17, that the stress-strain curve during the unloading and loading cycles form loops (the shaded areas in Fig. 11.17). These indicate that strain energy is lost during the cycle, the energy being dissipated in the form of heat produced by internal friction. This energy loss is known as *mechanical hysteresis* and the loops as *hysteresis loops*. Although the ultimate stress is increased by strain hardening, it is not influenced to the same extent as yield stress. The increase in strength produced by strain hardening is accompanied by decreases in toughness and ductility.

### 11.8.4 Creep and relaxation

We saw in Chapter 1 that a given load produces a calculable value of stress in a structural member and hence a corresponding value of strain once the full value of the load is transferred to the member. However, after this initial or ‘instantaneous’ stress and its corresponding value of strain have been attained, a great number of structural materials continue to deform slowly and progressively under load over a period of time. This behavior is known as *creep*. A typical creep curve is shown in Fig. 11.18.

Some materials, such as plastics and rubber, exhibit creep at room temperatures, but most structural materials require high temperatures or long-duration loading at moderate temperatures. In some “soft” metals, such as zinc and lead, creep occurs over a relatively short period of time, whereas materials such as concrete may be subject to creep over a period of years. Creep occurs in steel to a slight extent at normal temperatures but becomes very important at temperatures above 316°C.

Closely related to creep is *relaxation*. Whereas creep involves an increase in strain under constant stress, relaxation is the decrease in stress experienced over a period of time by a material subjected to a constant strain.



**FIGURE 11.17** Strain Hardening of a Material

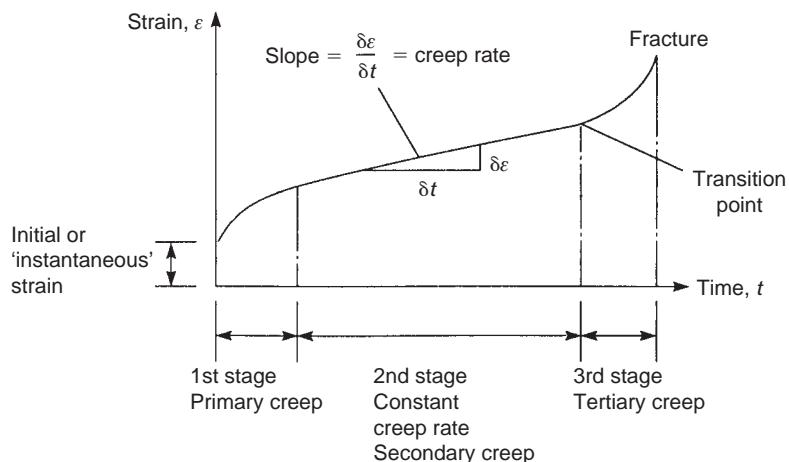


FIGURE 11.18 Typical Creep Curve

### 11.8.5 Fatigue

Structural members are frequently subjected to repetitive loading over a long period of time. For example, the members of a bridge structure suffer variations in loading possibly thousands of times a day as traffic moves over the bridge. In these circumstances, a structural member may fracture at a level of stress substantially below the ultimate stress for nonrepetitive static loads; this phenomenon is known as *fatigue*.

Fatigue cracks are most frequently initiated at sections in a structural member where changes in geometry, such as holes, notches, or sudden changes in section, cause *stress concentrations*. Designers seek to eliminate such areas by ensuring that rapid changes in section are as smooth as possible. At re-entrant corners, for example, fillets are provided, as shown in Fig. 11.19.

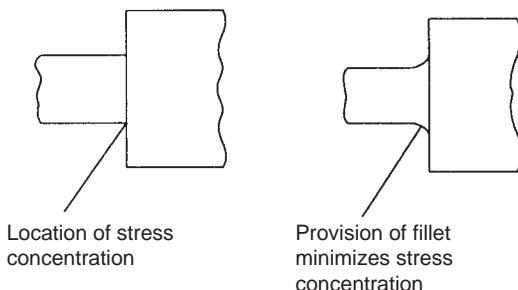
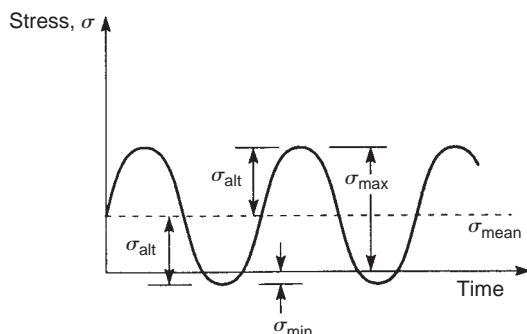


FIGURE 11.19 Stress Concentration Location

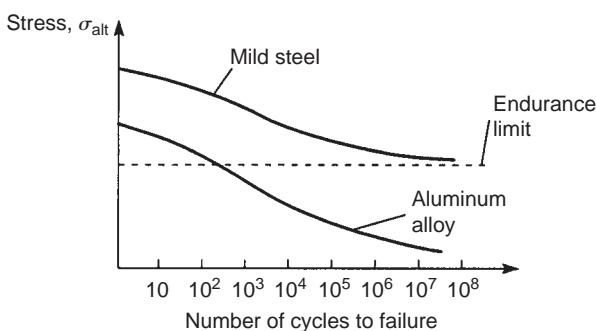
Other factors that affect the failure of a material under repetitive loading are the type of loading (fatigue is primarily a problem with repeated tensile stresses, probably because microscopic cracks can propagate more easily under tension), temperature, the material, surface finish (machine marks are potential crack propagators), corrosion, and residual stresses produced by welding.

Frequently in structural members, an alternating stress,  $\sigma_{\text{alt}}$ , is superimposed on a static or mean stress,  $\sigma_{\text{mean}}$ , as illustrated in Fig. 11.20. The value of  $\sigma_{\text{alt}}$  is the most important factor in determining the number of cycles of load that produce failure. The stress  $\sigma_{\text{alt}}$  that can be withstood for a specified number of cycles is called the *fatigue strength* of the material. Some materials, such as mild steel, possess a stress level that can be withstood for an indefinite number of cycles. This stress is known as the *endurance limit* of the material; no such limit has been found for aluminum and its alloys. Fatigue data are frequently presented in the form of an  $S$ - $n$  curve or stress–endurance curve, as shown in Fig. 11.21.

In many practical situations, the amplitude of the alternating stress varies and is frequently random in nature. The  $S$ - $n$  curve does not, therefore, apply directly, and an alternative means of predicting failure is required. *Miner's cumulative damage theory* suggests that failure occurs when



**FIGURE 11.20 Alternating Stress in Fatigue Loading**



**FIGURE 11.21 Stress–Endurance Curves**

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \dots + \frac{n_r}{N_r} = 1 \quad (11.1)$$

where  $n_1, n_2, \dots, n_r$  are the number of applications of stresses  $\sigma_{\text{alt}}$ ,  $\sigma_{\text{mean}}$  and  $N_1, N_2, \dots, N_r$  are the number of cycles to failure of stresses  $\sigma_{\text{alt}}, \sigma_{\text{mean}}$ .

We examine fatigue and its effect on aircraft design in much greater detail in [Chapter 15](#).

## PROBLEMS

**P.11.1** Describe a simple tensile test and show, with the aid of sketches, how measures of the ductility of the material of the specimen may be obtained. Sketch typical stress–strain curves for mild steel and an aluminum alloy showing their important features.

**P.11.2** A bar of metal 25 mm in diameter is tested on a length of 250 mm. In tension, the results in [Table P.11.2](#) were recorded. A torsion test gave the results also in [Table P.11.2](#). Represent these results in graphical form and determine Young's modulus,  $E$ ; the modulus of rigidity,  $G$ ; Poisson's ratio,  $v$ ; and the bulk modulus,  $K$ ; for the metal.

Answer:  $E \simeq 205,000 \text{ N/mm}^2$ ,  $G \simeq 80,700 \text{ N/mm}^2$ ,  $v \simeq 0.27$ ,  $K \simeq 148,600 \text{ N/mm}^2$

**P.11.2 MATLAB** Use MATLAB to repeat Problem P.11.2 for the following bar diameter values:

	(i)	(ii)	(iii)	(iv)	(v)
$d$	24.8 mm	24.9 mm	25 mm	25.1 mm	25.2 mm

Assume that the slopes of the tension and torsion data provided in [Table P.11.2](#) are 402.6 kN/mm and 12.38 kN m/rad, respectively.

**Table P.11.2** Problem 11.2

<b>Tension:</b>				
Load (kN)	10.4	31.2	52.0	72.8
Extension (mm)	0.036	0.089	0.140	0.191
<b>Torsion:</b>				
Torque (kNm)	0.051	0.152	0.253	0.354
Angle of twist (degrees)	0.24	0.71	1.175	1.642

- Answer: (i)  $E \approx 208,400 \text{ N/mm}^2$ ,  $G \approx 83,300 \text{ N/mm}^2$ ,  $v \approx 0.25$ ,  $K \approx 138,900 \text{ N/mm}^2$   
(ii)  $E \approx 206,700 \text{ N/mm}^2$ ,  $G \approx 82,000 \text{ N/mm}^2$ ,  $v \approx 0.26$ ,  $K \approx 143,500 \text{ N/mm}^2$   
(iii)  $E \approx 205,000 \text{ N/mm}^2$ ,  $G \approx 80,700 \text{ N/mm}^2$ ,  $v \approx 0.27$ ,  $K \approx 148,600 \text{ N/mm}^2$   
(iv)  $E \approx 203,400 \text{ N/mm}^2$ ,  $G \approx 79,400 \text{ N/mm}^2$ ,  $v \approx 0.28$ ,  $K \approx 154,100 \text{ N/mm}^2$   
(v)  $E \approx 201,800 \text{ N/mm}^2$ ,  $G \approx 78,200 \text{ N/mm}^2$ ,  $v \approx 0.29$ ,  $K \approx 160,200 \text{ N/mm}^2$

**P.11.3** The actual stress-strain curve for a particular material is given by  $\sigma = C\varepsilon^n$ , where  $C$  is a constant. Assuming that the material suffers no change in volume during plastic deformation, derive an expression for the nominal stress-strain curve and show that this has a maximum value when  $\varepsilon = n/(1-n)$ .

Answer:  $\sigma_{\text{nom}} = C\varepsilon^n / (1 + \varepsilon)$

**P.11.4** A structural member is to be subjected to a series of cyclic loads that produce different levels of alternating stress, as shown in [Table P.11.4](#). Determine whether or not a fatigue failure is probable.

Answer: Not probable ( $n_1/N_1 + n_2/N_2 + \dots = 0.39$ )

**Table P.11.4** Problem 11.4

Loading	Number of cycles	Number of cycles to failure
1	$10^4$	$5 \times 10^4$
2	$10^5$	$10^6$
3	$10^6$	$24 \times 10^7$
4	$10^7$	$12 \times 10^7$

# Structural components of aircraft and spacecraft

# 12

Aircraft are generally built up from the basic components of wings, fuselages, tail units, and control surfaces. There are variations in particular aircraft: for example, a delta wing aircraft would not necessarily possess a horizontal tail, although this is present in a canard configuration, such as that of the Eurofighter (Typhoon). Each component has one or more specific functions and must be designed to ensure that it can carry out these functions safely. In this chapter, we describe the various loads to which aircraft components are subjected, their function and fabrication, and the design of connections.

Spacecraft, apart from the Space Shuttle, which had a more or less conventional layout, consist generally of a long narrow tube containing the thrust structure, fuel tanks, and payload. We shall examine such structures in some detail, although a comprehensive study of spacecraft design is outside the scope of this book. For such studies reference should be made to any of the several texts available, typical of which is that by Wijker.<sup>1</sup>

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## 12.1 STRUCTURAL COMPONENTS: AIRCRAFT

Conventional aircraft usually consist of fuselage, wings, and tailplane. The fuselage contains crew and payload, the latter being passengers, cargo, or weapons plus fuel, depending on the type of aircraft and its function, the wings provide the lift and the tailplane is the main contributor to directional control. In addition, ailerons, elevators, and the rudder enable the pilot to maneuver the aircraft and maintain its stability in flight, while wing flaps provide the necessary increase of lift for take-off and landing.

### 12.1.1 Loads on structural components

The structure of an aircraft is required to support two distinct classes of load: the first, termed *ground loads*, includes all loads encountered by the aircraft during movement or transportation on the ground, such as taxiing and landing loads, towing and hoisting loads; while the second, *air loads*, comprises loads imposed on the structure during flight by maneuvers and gusts. In addition, aircraft designed for a particular role encounter loads peculiar to their sphere of operation. Carrier-borne aircraft, for instance, are subjected to catapult take-off and arrested landing loads: most large civil and practically all military aircraft have pressurized cabins for high-altitude flying; amphibious aircraft must be capable of landing on water and aircraft designed to fly at high speed at low altitude, such as the Tornado, require a structure of above average strength to withstand the effects of flight in extremely turbulent air.

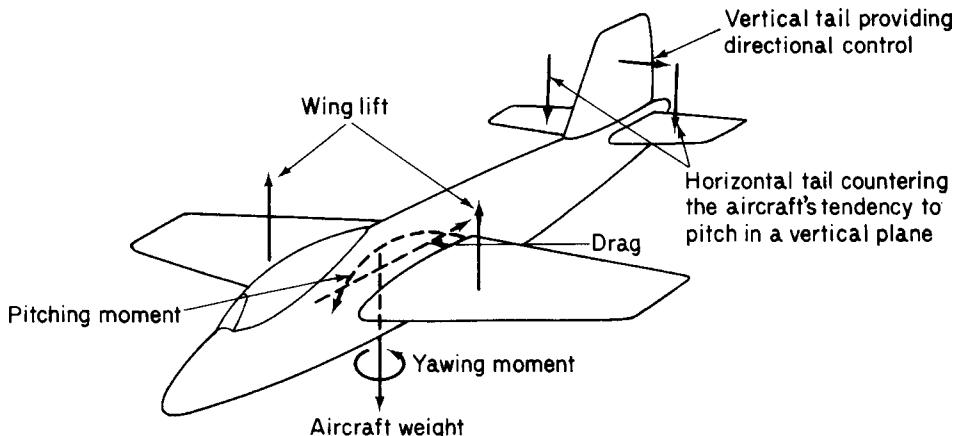
The two classes of loads may be further divided into *surface forces*, which act upon the surface of the structure, such as aerodynamic and hydrostatic pressure, and *body forces*, which act over the

volume of the structure and are produced by gravitational and inertial effects. Calculation of the distribution of aerodynamic pressure over the various surfaces of an aircraft's structure is presented in numerous texts on aerodynamics and therefore is not attempted here. However, we discuss the types of load induced by these various effects and their action on the different structural components.

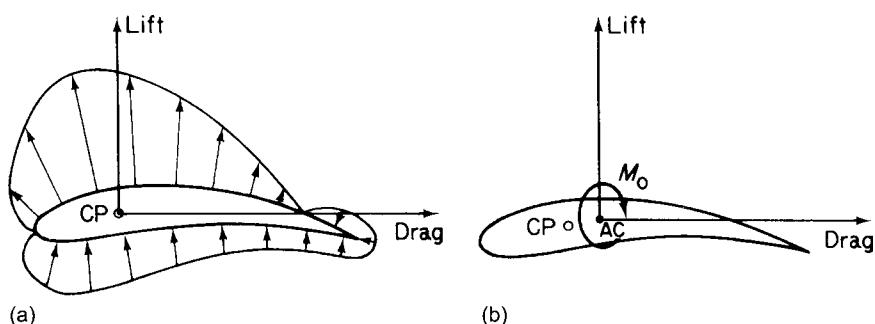
Basically, all air loads are the result of the pressure distribution over the surfaces of the skin produced by steady flight, maneuver, or gust conditions. Generally, these results cause direct loads, bending, shear, and torsion in all parts of the structure in addition to local, normal pressure loads imposed on the skin.

**Figure 12.1** shows typical aerodynamic force resultants experienced by an aircraft in steady flight.

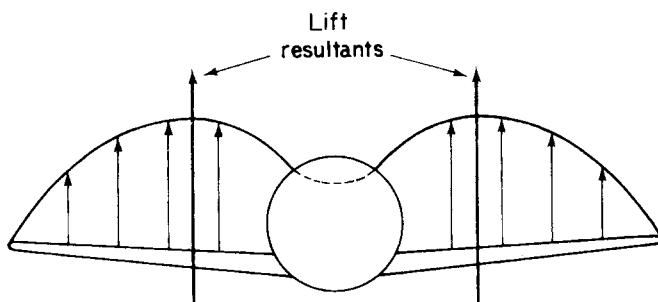
The force on an aerodynamic surface (wing, vertical or horizontal tail) results from a differential pressure distribution caused by incidence, camber, or a combination of both. Such a pressure distribution, shown in **Fig. 12.2(a)**, has vertical (lift) and horizontal (drag) resultants acting at a center of pressure (CP). (In practice, lift and drag are measured perpendicular and parallel to the flight path, respectively.) Clearly, the position of the CP changes as the pressure distribution varies with speed or wing incidence.



**FIGURE 12.1 Principal Aerodynamic Forces on an Aircraft During Flight**



**FIGURE 12.2 (a) Pressure Distribution Around an Aerofoil; (b) Transference of Lift and Drag Loads to the AC**



**FIGURE 12.3 Typical Lift Distribution for a Wing–Fuselage Combination**

However, there is, conveniently, a point in the aerofoil section about which the moment due to the lift and drag forces remains constant. We therefore replace the lift and drag forces acting at the CP by lift and drag forces acting at the aerodynamic center (AC) plus a constant moment  $M_0$ , as shown in Fig. 12.2(b). (Actually, at high Mach numbers, the position of the AC changes due to compressibility effects.)

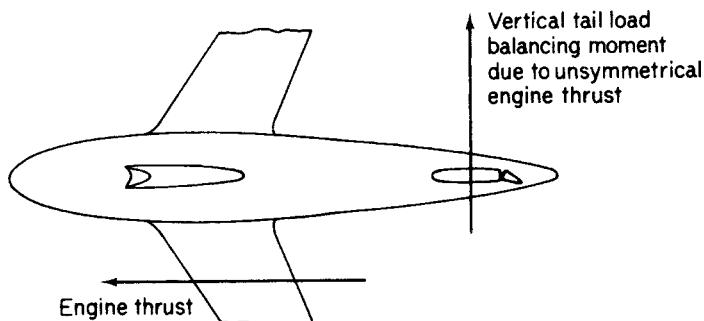
While the chordwise pressure distribution fixes the position of the resultant aerodynamic load in the wing cross-section, the spanwise distribution locates its position in relation, say, to the wing root. A typical distribution for a wing–fuselage combination is shown in Fig. 12.3. Similar distributions occur on horizontal and vertical tail surfaces.

We see therefore that wings, tailplane, and the fuselage are each subjected to direct, bending, shear, and torsional loads and must be designed to withstand critical combinations of these. Note that maneuvers and gusts do not introduce different loads but result only in changes of magnitude and position of the type of existing loads shown in Fig. 12.1. Over and above these basic in-flight loads, fuselages may be pressurized and thereby support hoop and longitudinal stresses (see Chapter 22); wings may carry weapons or extra fuel tanks with resulting additional aerodynamic and body forces contributing to the existing bending, shear, and torsion; and the thrust and weight of engines may affect either the fuselage or wings, depending on their relative positions.

Ground loads encountered in landing and taxiing subject the aircraft to concentrated shock loads through the undercarriage system. The majority of aircraft have their main undercarriage located in the wings, with a nosewheel or tail wheel in the vertical plane of symmetry. Clearly, the position of the main undercarriage should be such as to produce minimum loads on the wing structure compatible with the stability of the aircraft during ground maneuvers. This may be achieved by locating the undercarriage just forward of the flexural axis of the wing and as close to the wing root as possible. In this case, the shock landing load produces a given shear, minimum bending plus torsion, with the latter being reduced as far as practicable by offsetting the torque caused by the vertical load in the undercarriage leg by a torque in an opposite sense due to braking.

Other loads include engine thrust on the wings or fuselage, which acts in the plane of symmetry but may, in the case of engine failure, cause severe fuselage bending moments, as shown in Fig. 12.4; concentrated shock loads during a catapult launch; and hydrodynamic pressure on the fuselages or floats of seaplanes.

In Chapter 13, we examine in detail the calculation of ground and air loads for a variety of cases.



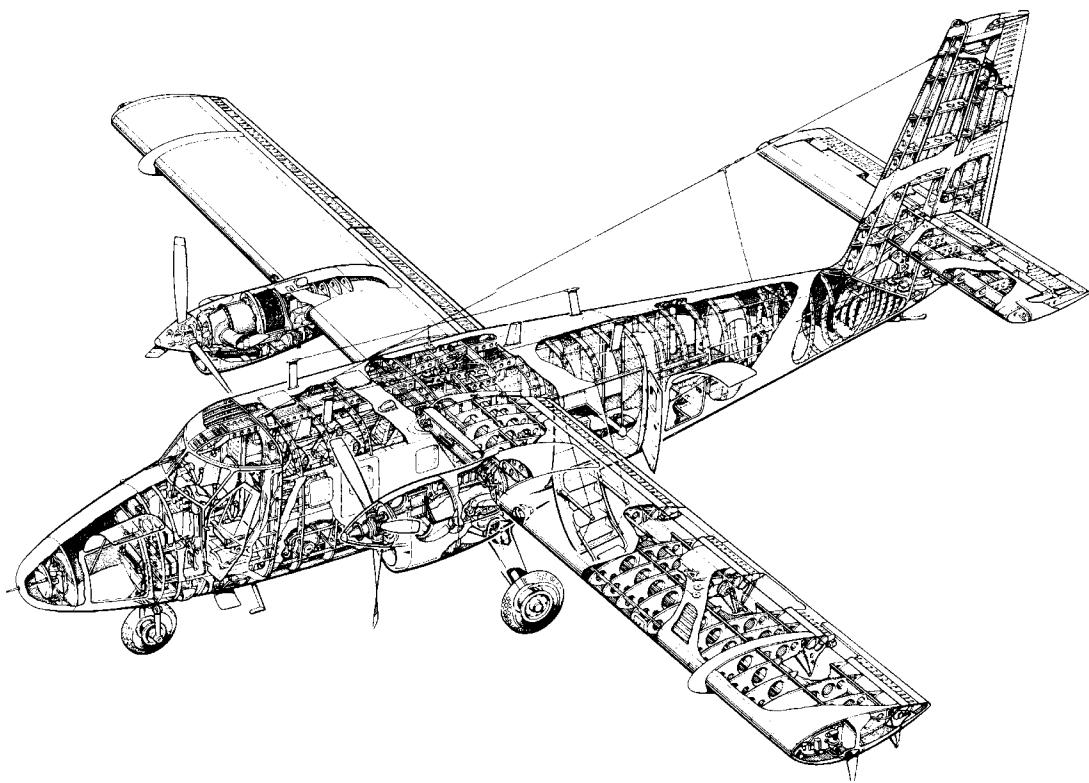
**FIGURE 12.4 Fuselage and Wing Bending Caused by an Unsymmetrical Engine Load**

### 12.1.2 Function of structural components

The basic functions of an aircraft's structure are to transmit and resist the applied loads and to provide an aerodynamic shape and to protect passengers, payload, systems, and the like from the environmental conditions encountered in flight. These requirements, in most aircraft, result in thin-shell structures where the outer surface or skin of the shell is usually supported by longitudinal stiffening members and transverse frames to enable it to resist bending, compressive, and torsional loads without buckling. Such structures are known as *semi-monocoque*, while thin shells that rely entirely on their skins for their capacity to resist loads are referred to as *monocoque*.

First, we consider wing sections, which, while performing the same function, can differ widely in their structural complexity, as can be seen by comparing Figs. 12.5 and 12.6. In Fig. 12.5, the wing of a small, light passenger aircraft, the De Havilland Canada Twin Otter, comprises a relatively simple arrangement of two spars, ribs, stringers, and skin, while the wing of the Harrier in Fig. 12.6 consists of numerous spars, ribs, and skin. However, no matter how complex the internal structural arrangement, the different components perform the same kind of function.

The shape of the cross-section is governed by aerodynamic considerations and clearly must be maintained for all combinations of load; this is one of the functions of the ribs. They also act with the skin in resisting the distributed aerodynamic pressure loads; they distribute concentrated loads (e.g., undercarriage and additional wing store loads) into the structure and redistribute stress around discontinuities, such as undercarriage wells, inspection panels, and fuel tanks, in the wing surface. Ribs increase the column buckling stress of the longitudinal stiffeners by providing end restraint and establishing their column length; in a similar manner, they increase the plate buckling stress of the skin panels. The dimensions of ribs are governed by their spanwise position in the wing and by the loads they are required to support. In the outer portions of the wing, where the cross-section may be relatively small if the wing is tapered and the loads are light, ribs act primarily as formers for the aerofoil shape. A light structure is sufficient for this purpose, whereas at sections closer to the wing root, where the ribs are required to absorb and transmit large concentrated applied loads, such as those from the undercarriage, engine thrust, and fuselage attachment point reactions, a much more rugged construction is necessary. Between these two extremes are ribs that support hinge reactions from ailerons, flaps, and other control surfaces, plus the many internal loads from fuel, armament, and systems installations.



**FIGURE 12.5** De Havilland Canada Twin Otter

(courtesy of De Havilland Aircraft of Canada Ltd.)

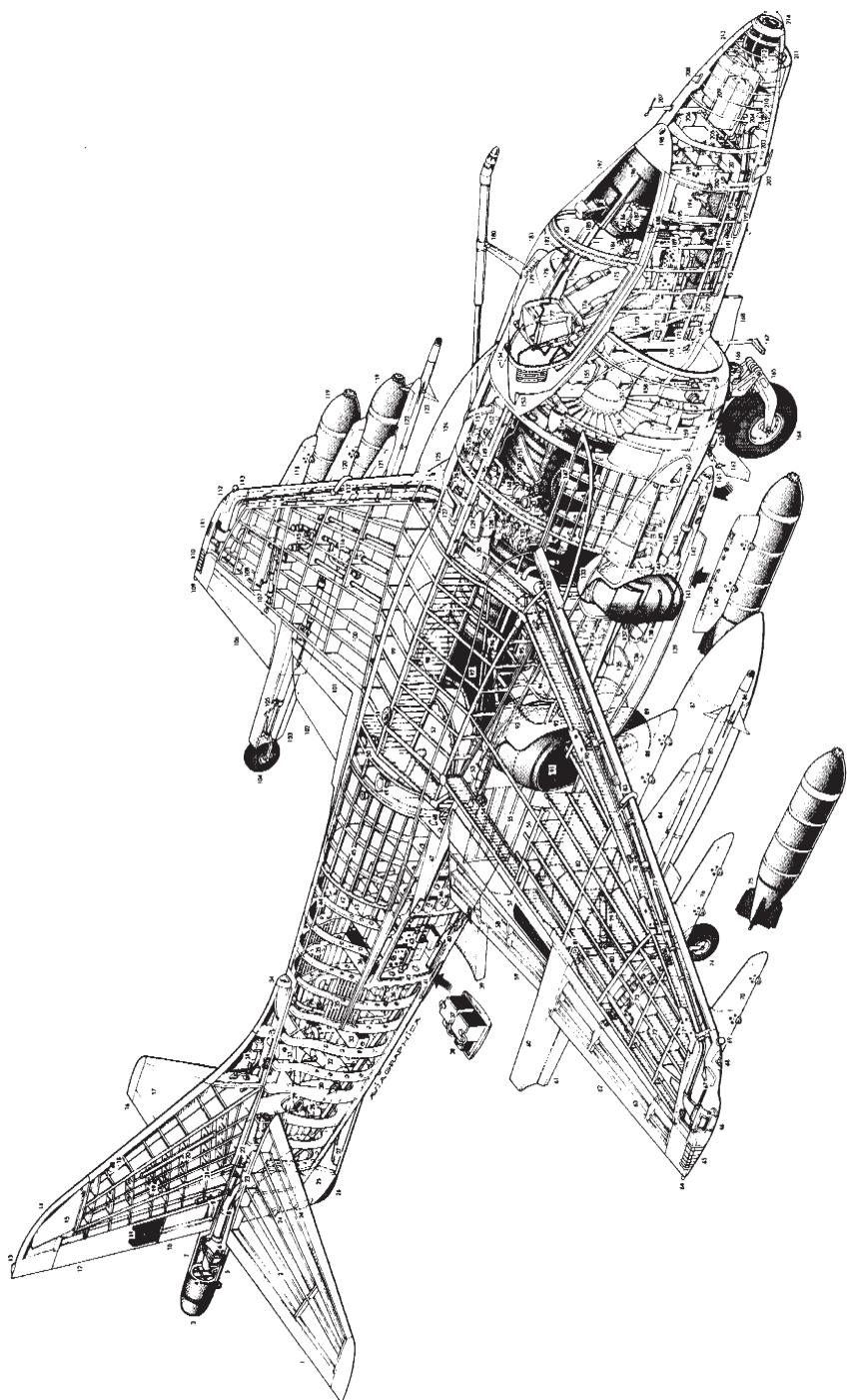


FIGURE 12.6 Harrier

1. Starboard all-moving tailplane;
2. Tailplane composite construction;
3. Tail radome;
4. Military equipment;
5. Tail pitch control air valve;
6. Yaw control air valves;
7. Tail "bullet" fairing;
8. Reaction control system air ducting;
9. Trim tab actuator;
10. Rudder trim tab;
11. Rudder composite construction;
12. Rudder;
13. Antenna;
14. Fin tip aerial fairing;
15. Upper broad band communications antenna;
16. Port tailplane;
17. Graphite epoxy tailplane skin;
18. Port side temperature probe;
19. MAD compensator;
20. Formation lighting strip;
21. Fin construction;
22. Fin attachment joint;
23. Tailplane pivot sealing plate;
24. Aerials;
25. Ventral fin;
26. Tail bumper;
27. Lower broad band communications antenna;
28. Tailplane hydraulic jack;
29. Heat exchanger air exhaust;
30. Aft fuselage frames;
31. Rudder hydraulic actuator;
32. Avionics equipment air conditioning plant;
33. Avionics equipment racks;
34. Heat exchanger ram air intake;
35. Electrical system circuit breaker panels, port and starboard;
36. Avionic equipment;
37. Chaff and flare dispensers;
38. Dispenser electronic control units;
39. Ventral airbrake;
40. Airbrake hydraulic jack;
41. Formation lighting strip;
42. Avionics bay access door, port and starboard;
43. Avionics equipment racks;
44. Fuselage frame and stringer construction;
45. Rear fuselage fuel tank;
46. Main undercarriage wheel bay;
47. Wing root fillet;
48. Wing spar-fuselage attachment joint;
49. Water filler cap;
50. Engine fire extinguisher bottle;
51. Anticollision light;
52. Water tank;
53. Flap hydraulic actuator;
54. Flap hinge fitting;
55. Nimonic fuselage heat shield;
56. Main undercarriage bay doors (closed after cycling of mainwheels);
57. Flap vane composite construction;
58. Flap composite construction;
59. Starboard slotted flap, lowered;
60. Outrigger wheel fairing;
61. Outrigger leg doors;
62. Starboard aileron;
63. Aileron composite construction;
64. Fuel jettison;
65. Formation lighting panel;
66. Roll control airvalve;
67. Wing tip fairing;
68. Starboard navigation light;
69. Radar warning aerial;
70. Outboard pylon;
71. Pylon attachment joint;
72. Graphite epoxy composite wing construction;
73. Aileron hydraulic actuator;
74. Starboard outrigger wheel;
75. BL755 600-lb. (272-kg) cluster bomb (CBU);
76. Intermediate pylon;
77. Reaction control air ducting;
78. Aileron control rod;
79. Outrigger hydraulic retraction jack;
80. Outrigger leg strut;
81. Leg pivot fixing;
82. Multspar wing construction;
83. Leading-edge wing fence;
84. Outrigger pylon;
85. Missile launch rail;
86. AIM-9L Sidewinder air-to-air missile;
87. External fuel tank, 300 U.S. gal (1 135 l);
88. Inboard pylon;
89. Aft retracting twin main wheels;
90. Inboard pylon attachment joint;
91. Rear (hot stream) swiveling exhaust nozzle;
92. Position of pressure refueling connection on port side;
93. Rear nozzle bearing;
94. Center fuselage flank fuel tank;
95. Hydraulic reservoir;
96. Nozzle bearing cooling air duct;
97. Engine exhaust divider duct;
98. Wing panel center rib;
99. Center section integral fuel tank;
100. Port wing integral fuel tank;
101. Flap vane;
102. Port slotted flap, lowered;
103. Outrigger wheel fairing;
104. Port outrigger wheel;
105. Torque scissor links;
106. Port aileron;
107. Aileron hydraulic actuator;
108. Aileron-air valve interconnection;
109. Fuel jettison;
110. Formation lighting panel;
111. Port roll control air valve;
112. Port navigation light;
113. Radar warning aerial;
114. Port wing reaction control air duct;
115. Fuel pumps;
116. Fuel system piping;
117. Port wing leading-edge fence;
118. Outboard pylon;
119. BL755 cluster bombs (maximum load, seven);
120. Intermediate pylon;
121. Port outrigger pylon;
122. Missile launch rail;
123. AIM-9L Sidewinder air-to-air missile;
124. Port leading-edge root extension (LERX);
125. Inboard pylon;
126. Hydraulic pumps;
127. APU intake;
128. Gas turbine starter–auxiliary power unit (APU);
129. Alternator cooling air exhaust;
130. APU exhaust;
131. Engine fuel control unit;
132. Engine bay venting ram air intake;
133. Rotary nozzle bearing;
134. Nozzle fairing construction;
135. Ammunition tank, 100 rounds;
136. Cartridge case collector box;
137. Ammunition feed chute;
138. Fuel vent;
139. Gun pack strake;
140. Fuselage centerline pylon;
141. Zero scarf forward (fan air) nozzle;
142. Ventral gun pack (two);
143. Aden 25-mm cannon;
144. Engine drain mast;
145. Hydraulic system ground connectors;
146. Forward fuselage flank fuel tank;
147. Engine electronic control units;
148. Engine accessory equipment gearbox;
149. Gearbox driven alternator;
150. Rolls-Royce Pegasus 11 Mk 105 vectored thrust turbofan;
151. Formation lighting strips;
152. Engine oil tank;
153. Bleed air spill duct;
154. Air conditioning intake scoops;
155. Cockpit air conditioning system heat exchanger;
156. Engine compressor-fan face;
157. Heat exchanger discharge to intake duct;
158. Nose undercarriage hydraulic retraction jack;
159. Intake blow-in doors;
160. Engine bay venting air scoop;
161. Cannon muzzle fairing;
162. Lift augmentation retractable cross-dam;
163. Cross-dam hydraulic jack;
164. Nosewheel;
165. Nosewheel forks;
166. Landing/taxiing lamp;
167. Retractable boarding step;
168. Nosewheel doors (closed after cycling of undercarriage);
169. Nosewheel door jack;
170. Boundary layer bleed air duct;
171. Nose undercarriage wheel bay;
172. Kick-in boarding steps;
173. Cockpit rear pressure bulkhead;
174. Starboard side console panel;
175. Martin-Baker Type 12 ejection seat;
176. Safety harness;
177. Ejection seat headrest;
178. Port engine air intake;
179. Probe hydraulic jack;
180. Retractable in-flight refueling probe (bolt-on pack);
181. Cockpit canopy cover;
182. Miniature detonating cord (MDC) canopy breaker;
183. Canopy frame;
184. Engine throttle and nozzle angle control levers;
185. Pilot's head-up display;
186. Instrument panel;
187. Moving map display;
188. Control column;
189. Central warning system panel;
190. Cockpit pressure floor;
191. Underfloor control runs;
192. Formation lighting strips;
193. Aileron trim actuator;
194. Rudder pedals;
195. Cockpit section composite construction;
196. Instrument panel shroud;
197. One-piece wraparound windscreen panel;
198. Ram air intake (cockpit fresh air);
199. Front pressure bulkhead;
200. Incidence vane;
201. Air data computer;
202. Pitot tube;
203. Lower IFF aerial;
204. Nose pitch control air valve;
205. Pitch trim control actuator;
206. Electrical system equipment;
207. Yaw vane;
208. Upper IFF aerial;
209. Avionic equipment;
210. ARBS heat exchanger;
211. MIRLS sensors;
212. Hughes Angle Rate Bombing System (ARBS);
213. Composite construction nose cone;
214. ARBS glazed aperture

(courtesy of Pilot Press Ltd.)

The primary function of the wing skin is to form an impermeable surface for supporting the aerodynamic pressure distribution from which the lifting capability of the wing is derived. These aerodynamic forces are transmitted in turn to the ribs and stringers by the skin through plate and membrane action. Resistance to shear and torsional loads is supplied by shear stresses developed in the skin and spar webs, while axial and bending loads are reacted by the combined action of skin and stringers.

Although the thin skin is efficient for resisting shear and tensile loads, it buckles under comparatively low compressive loads. Rather than increase the skin thickness and suffer a consequent weight penalty, stringers are attached to the skin and ribs, thereby dividing the skin into small panels and increasing the buckling and failing stresses. This stabilizing action of the stringers on the skin is, in fact, reciprocated to some extent, although the effect normal to the surface of the skin is minimal. Stringers rely chiefly on rib attachments for preventing column action in this direction. We noted in the previous paragraph the combined action of stringers and skin in resisting axial and bending loads.

The role of spar webs in developing shear stresses to resist shear and torsional loads was mentioned previously; they perform a secondary but significant function in stabilizing, with the skin, the spar flanges, or caps, which are therefore capable of supporting large compressive loads from axial and bending effects. In turn, spar webs exert a stabilizing influence on the skin in a similar manner to the stringers.

While the majority of these remarks have been directed toward wing structures, they apply, as can be seen by referring to Figs 12.5 and 12.6, to all the aerodynamic surfaces, namely wings, horizontal and vertical tails—except in the obvious cases of undercarriage loading—engine thrust, and so forth.

Fuselages, while of different shape to the aerodynamic surfaces, comprise members that perform functions similar to their counterparts in the wings and tailplane. However, there are differences in the generation of the various types of load. Aerodynamic forces on the fuselage skin are relatively low; on the other hand, the fuselage supports large concentrated loads, such as wing reactions, tailplane reactions, and undercarriage reactions, and it carries payloads of varying size and weight, which may cause large inertia forces. Furthermore, aircraft designed for high-altitude flight must withstand internal pressure. The shape of the fuselage cross-section is determined by operational requirements. For example, the most efficient sectional shape for a pressurized fuselage is circular or a combination of circular elements. Irrespective of shape, the basic fuselage structure is essentially a single-cell, thin-walled tube comprising skin, transverse frames, and stringers; transverse frames that extend completely across the fuselage are known as *bulkheads*. Three different types of fuselage are shown in Figs. 12.5–12.7. In Fig. 12.5, the fuselage is unpressurized, so that, in the passenger-carrying area, a more rectangular shape is employed to maximize space. The Harrier fuselage in Fig. 12.6 contains the engine, fuel tanks, and the like, so that its cross-sectional shape is, to some extent, predetermined, while in Fig. 12.7, the passenger-carrying fuselage of the British Aerospace 146 is pressurized and therefore circular in cross-section.

### 12.1.3 Fabrication of structural components

The introduction of all-metal, stressed skin aircraft resulted in methods and types of fabrication that remain in use to the present day. However, improvements in engine performance and advances in aerodynamics led to higher maximum lift, higher speeds, and therefore higher wing loadings, so that improved techniques of fabrication are necessary, particularly in the construction of wings. The increase in wing loading from about  $350 \text{ N/m}^2$  for 1917–1918 aircraft to around  $4800 \text{ N/m}^2$  for modern aircraft, coupled with a drop in the structural percentage of the total weight from 30–40 to 22–25 percent, gives some indication of the improvements in materials and structural design.

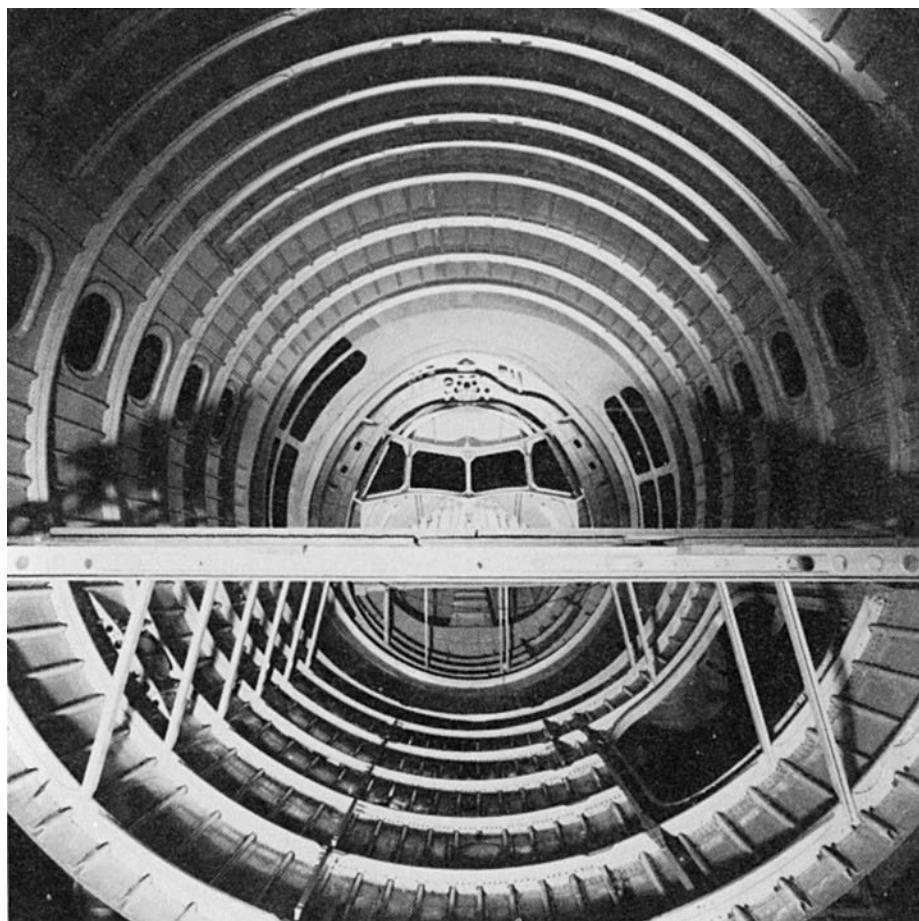
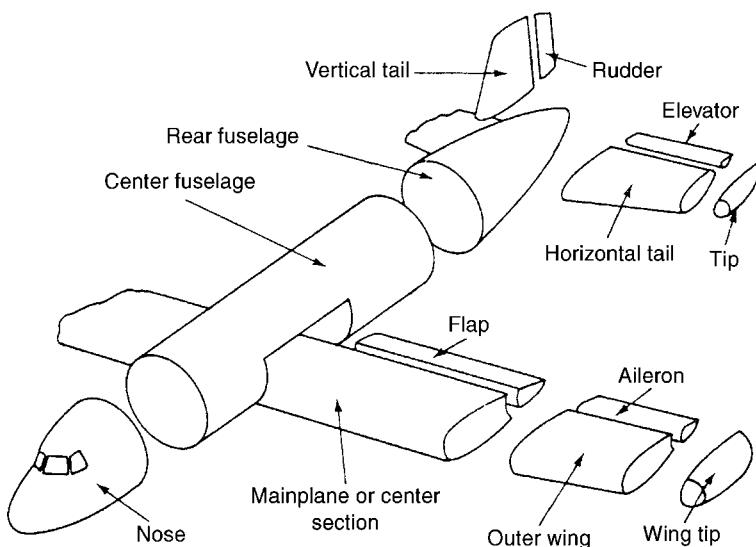


FIGURE 12.7 British Aerospace 146

(courtesy of British Aerospace)

For purposes of construction, aircraft are divided into a number of subassemblies. These are built in specially designed jigs, possibly in different parts of the factory or even different factories, before being forwarded to the final assembly shop. A typical breakdown into subassemblies of a medium-sized civil aircraft is shown in Fig. 12.8. Each subassembly relies on numerous minor assemblies such as spar webs, ribs, and frames, and these, in turn, are supplied with individual components from the detail workshop.

Although the wings (and tail surfaces) of fixed-wing aircraft generally consist of spars, ribs, skin, and stringers, methods of fabrication and assembly differ. The wing of the aircraft of Fig. 12.5 relies on fabrication techniques that have been employed for many years. In this form of construction, the spars comprise thin aluminum alloy webs and flanges, the latter being extruded or machined and bolted or riveted to the web. The ribs are formed in three parts from sheet metal by large presses and rubber dies and have flanges round their edges so that they can be riveted to the skin and spar webs; cutouts around



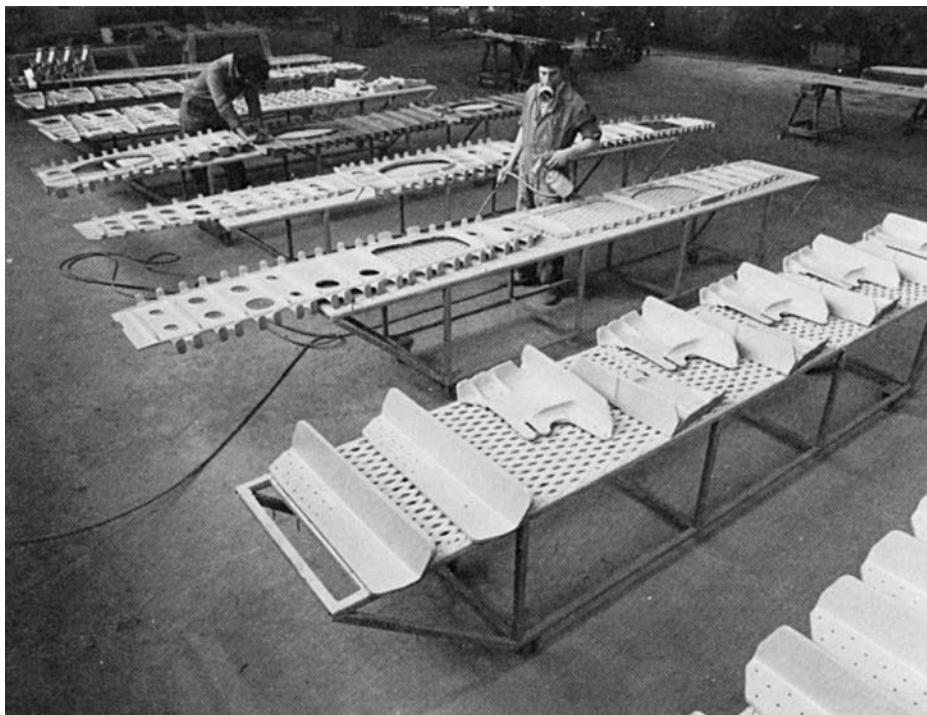
**FIGURE 12.8 Typical Subassembly Breakdown**

their edges allow the passage of spanwise stringers. Holes are cut in the ribs at positions of low stress for lightness and to accommodate control runs, fuel, and electrical systems.

Finally, the skin is riveted to the rib flanges and longitudinal stiffeners. Where the curvature of the skin is large – for example, at the leading edge – the aluminum alloy sheets are passed through “rolls” to preform them to the correct shape. A further, aerodynamic requirement is that forward chordwise sections of the wing be as smooth as possible, to delay transition from laminar to turbulent flow. Therefore, countersunk rivets are used in these positions as opposed to dome-headed rivets nearer the trailing edge.

The wing is attached to the fuselage through reinforced fuselage frames, frequently by bolts. In some aircraft, the wing spars are continuous through the fuselage, depending on the demands of space. In a high-wing aircraft (Fig. 12.5), deep spars passing through the fuselage would cause obstruction problems. In this case, a short third spar provides an additional attachment point. The ideal arrangement is obviously where continuity of the structure is maintained over the entire surface of the wing. In most practical cases, this is impossible, since cutouts in the wing surface are required for retracting undercarriages, bomb and gun bays, inspection panels, etc. The last are usually located on the undersurface of the wing and are fastened to stiffeners and rib flanges by screws, enabling them to resist direct and shear loads. Doors covering undercarriage wells and weapon bays are incapable of resisting wing stresses, so that provision must be made for transferring the loads from skin, flanges, and shear webs around the cutout. This may be achieved by inserting strong bulkheads or increasing the spar flange areas, although, no matter the method employed, increased cost and weight result.

The different structural requirements of aircraft designed for differing operational roles lead to a variety of wing constructions. For instance, high-speed aircraft require relatively thin wing sections, which support high wing loadings. To withstand the correspondingly high surface pressures and to obtain sufficient strength, much thicker skins are necessary. Wing panels are therefore frequently machined integrally with stringers from solid slabs of material, as are the wing ribs. Figure 12.9



**FIGURE 12.9 Wing Ribs for the European Airbus**

(courtesy of British Aerospace)

shows wing ribs for the European Airbus, in which web stiffeners, flanged lightness holes, and skin attachment lugs have been integrally machined from solid. This integral method of construction involves no new design principles and has the advantages of combining a high grade of surface finish, free from irregularities, with a more efficient use of material, since skin thicknesses are easily tapered to coincide with the spanwise decrease in bending stresses.

An alternative form of construction is the sandwich panel, which comprises a light honeycomb or corrugated metal core sandwiched between two outer skins of the stress-bearing sheet (see Fig. 12.10). The primary function of the core is to stabilize the outer skins, although it may be stress bearing as well. Sandwich panels are capable of developing high stresses, have smooth internal and external surfaces, and require small numbers of supporting rings or frames. They also possess a high resistance to fatigue from jet efflux. The uses of this method of construction include lightweight “planks” for cabin furniture, monolithic fairing shells generally having plastic facing skins, and the stiffening of flying control surfaces. Thus, for example, the ailerons and rudder of the British Aerospace Jaguar are fabricated from aluminum honeycomb, while fiberglass- and aluminum-faced honeycomb are used extensively in the wings and tail surfaces of the Boeing 747. Some problems, mainly disbonding and internal corrosion, have been encountered in service.

The general principles relating to wing construction are applicable to fuselages, with the exception that integral construction is not used in fuselages for obvious reasons. Figures 12.5, 12.6, and 12.7 show

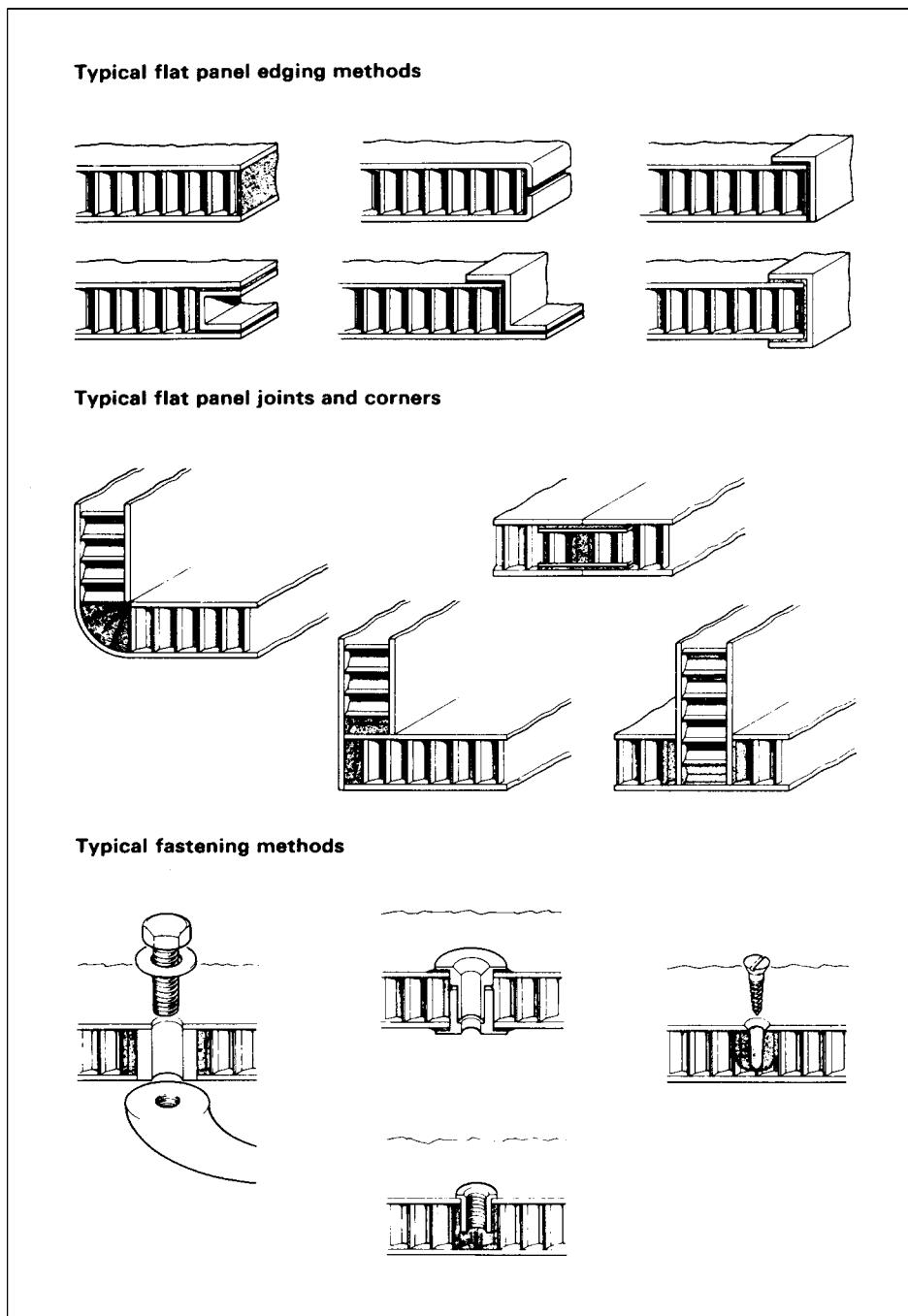


FIGURE 12.10 Sandwich Panels

(courtesy of Ciba-Geigy Plastics)

that the same basic method of construction is employed in aircraft having widely differing roles. Generally, the fuselage frames that support large concentrated floor loads or loads from wing or tailplane attachment points are heavier than lightly loaded frames and require stiffening, with additional provision for transmitting the concentrated load into the frame and the skin.

With the frames in position in the fuselage jig, stringers, passing through cutouts, are riveted to the frame flanges. Before the skin is riveted to the frames and stringers, other subsidiary frames, such as door and window frames, are riveted or bolted in position. The areas of the fuselage in the regions of these cutouts are reinforced by additional stringers, portions of frame, and increased skin thickness, to react to the high shear flows and direct stresses developed.

On completion, the various subassemblies are brought together for final assembly. Fuselage sections are usually bolted together through flanges around their peripheries, while wings and the tailplane are attached to pick-up points on the relevant fuselage frames. Wing spars on low-wing civil aircraft usually pass completely through the fuselage, simplifying wing design and the method of attachment. On smaller, military aircraft, engine installations frequently prevent this, so that wing spars are attached directly to and terminate at the fuselage frame. Clearly, at these positions, frame–stringer–skin structures require reinforcement.

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## 12.2 STRUCTURAL COMPONENTS: SPACECRAFT

A typical arrangement for the components of a spacecraft is shown diagrammatically in Fig. 12.11 where, in fact, the actual spacecraft is the payload and the complete structure is the launch vehicle. Variations in the structure occur, which depend upon the spacecraft mission that can vary from a planetary orbiter role to one involving telecommunications and navigation.

Generally, spacecraft, as the name implies, operate in space. However, the major loads sustained by the spacecraft structure are produced by the extremely high level of thrust required at launch. Two types of structure are used to transmit these loads through the spacecraft structure to the payload section, one of which has a truss or frame format while the other consists of a central cylindrical shell. In the former system the truss members are mainly struts that must be checked for stability as well as strength and stiffness, while in the latter case internal systems are attached at strong points directly to the cylindrical shell or by combinations of struts, platforms, and shear webs.

The thrust generated at the lower end of the launch vehicle is transferred to the payload section using one of the structural systems described above. However, a launch vehicle generally comprises multiple stages that may be placed in sequence one on top of the other or in parallel. We shall now examine in more detail the different components of the launch vehicle structure.

### ***Thrust structure***

The loads (thrust), which can be of the order of 15,000 kN produced by the propulsion system, are transferred into the main stages of the launch vehicle by means of the thrust structure. Since this has a similar external diameter to the launch vehicle, the loads must have a more concentrated form before they can be absorbed by the central truss or cylinder. This is achieved by inserting a connector in the form of a truncated cone as shown diagrammatically in Fig. 12.12. Generally, this will take the form of a stiffened shell.

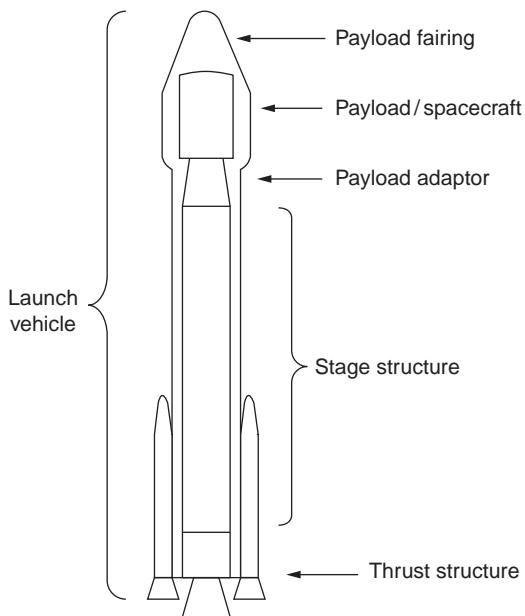


FIGURE 12.11 Diagrammatic Arrangement of Spacecraft Components

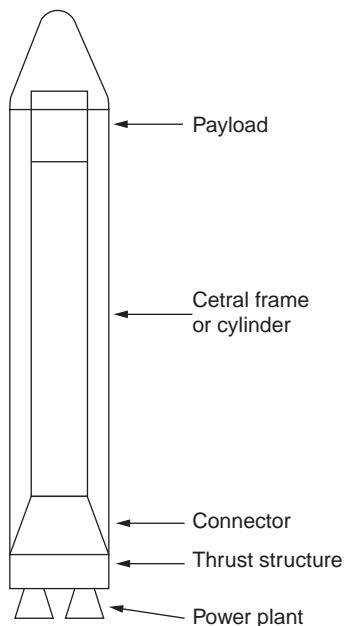


FIGURE 12.12 Thrust Structure Connector

### **Stage structures**

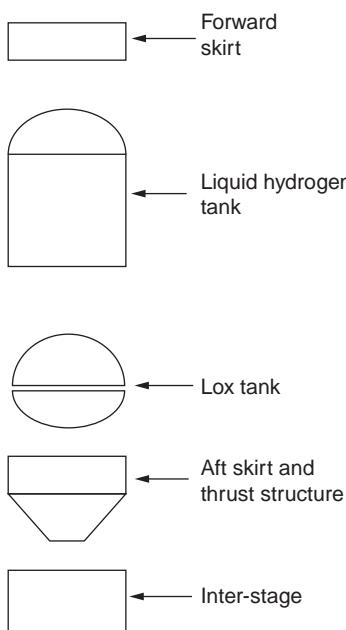
Launch vehicles, as illustrated in Fig. 12.11, contain multiple stages, each of which contains a propulsion system. These stages are placed in sequence with one on top of the other or in parallel. The main stage forms the core of the launch vehicle while booster stages, which are often placed in parallel, increase the thrust during the initial part of the launch (see Fig. 12.11). In some cases, the fuel tanks are part of the load-bearing structure where the different components (see Fig. 12.13) are connected to one another by interface structures. When the fuel tanks are not part of the primary structure, an external skin stiffened by longerons and circular ribs transfers the loads to other stages of the launch vehicle.

### **Adapters**

Adaptors are similar in function and geometry to the truncated cone, which transmits the loads from the thrust structure into the central primary structure. An adaptor is located in the upper part of the launch vehicle and transfers the loads from the stages to the spacecraft in the payload area under the fairing. Again, a stiffened shell design is used.

### **Payload fairings**

The function of the payload fairing is to protect the payload during the launch and ascent from the effects of aerodynamic heating and pressure. Once the launch vehicle is outside the atmosphere, the fairing is jettisoned thereby reducing structural weight.



**FIGURE 12.13 Exploded View of a Typical Spacecraft Structure**

### 12.2.1 Loads on structural components

Generally spacecraft loads are of a different type to those experienced by aircraft. They also fall into different categories, which include ground handling and transportation, launching, and those occurring when the spacecraft is in orbit; these, however, are usually very much smaller than those produced during launch. In addition to the loads generated by thrust during launch, significant vibrations occur, so the resonant frequency of the launch vehicle structure and the frequency of the vibrations must not coincide. Other forms of loading include internal pressure, thermal stress, and shock loads. We shall consider some simple loading cases in [Chapter 25](#).

## 12.3 CONNECTIONS

The fabrication of aircraft components generally involves the joining of one part of the component to another. For example, fuselage skins are connected to stringers and frames, while wing skins are connected to stringers and wing ribs unless, as in some military aircraft with high wing loadings, the stringers are machined integrally with the wing skin (see [Section 12.1.3](#)). With the advent of all-metal – that is, aluminum alloy – construction, riveted joints became the main form of connection with some welding – although aluminum alloys are difficult to weld – and in the modern era, some glued joints, which use epoxy resin.

### 12.3.1 Riveted connections

In general, riveted joints are stressed in complex ways, and an accurate analysis is very often difficult to achieve because of the discontinuities in the region of the joint. Fairly crude assumptions as to joint behavior are made, but, when combined with experience, safe designs are produced.

#### Simple lap joint

[Figure 12.14](#) shows two plates of thickness  $t$  connected by a single line of rivets; this type of joint, termed a *lap joint*, is one of the simplest used in construction.

Suppose that the plates carry edge loads of  $P/\text{unit width}$ , that the rivets are of diameter  $d$  and are spaced at a distance  $b$  apart, and that the distance from the line of rivets to the edge of each plate is  $a$ . Four possible modes of failure must be considered: **rivet shear**, **bearing pressure**, **plate failure in tension**, and **shear failure in a plate**.

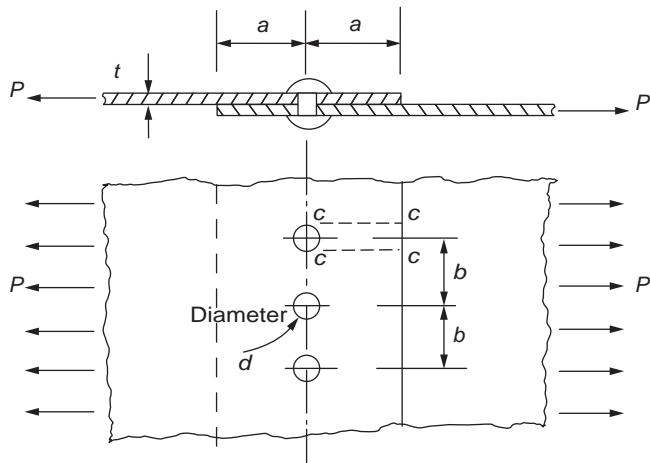
#### Rivet shear

The rivets may fail by shear across their diameter at the interface of the plates. Then, if the maximum shear stress the rivets will withstand is  $\tau_1$ , failure occurs when

$$Pb = \tau_1 \left( \frac{\pi d^2}{4} \right)$$

which gives

$$P = \frac{\pi d^2 \tau_1}{4b} \quad (12.1)$$



**FIGURE 12.14 Simple Riveted Lap Joint**

### Bearing pressure

Either the rivet or plate may fail due to bearing pressure. Suppose that  $p_b$  is this pressure then failure occurs when

$$\frac{Pb}{td} = p_b$$

so that

$$P = \frac{p_b td}{b} \quad (12.2)$$

### Plate failure in tension

The area of plate in tension along the line of rivets is reduced due to the presence of rivet holes. Therefore, if the ultimate tensile stress in the plate is  $\sigma_{ult}$ , failure occurs when

$$\frac{Pb}{t(b-d)} = \sigma_{ult}$$

from which

$$P = \frac{\sigma_{ult} t(b-d)}{b} \quad (12.3)$$

### Shear failure in a plate

Shearing of the plates may occur on the planes  $cc$ , resulting in the rivets being dragged out of the plate. If the maximum shear stress at failure of the material of the plates is  $\tau_2$ , then a failure of this type occurs when

$$Pb = 2at\tau_2$$

which gives

$$P = \frac{2at\tau_2}{b} \quad (12.4)$$

### Example 12.1

A joint in a fuselage skin is constructed by riveting the abutting skins between two straps, as shown in Fig. 12.15. The fuselage skins are 2.5 mm thick and the straps are each 1.2 mm thick; the rivets have a diameter of 4 mm. If the tensile stress in the fuselage skin must not exceed 125 N/mm<sup>2</sup> and the shear stress in the rivets is limited to 120 N/mm<sup>2</sup>, determine the maximum allowable rivet spacing such that the joint is equally strong in shear and tension.

A tensile failure in the plate occurs on the reduced plate cross-section along the rivet lines. This area is given by

$$A_p = (b - 4) \times 2.5 \text{ mm}^2$$

The failure load/unit width  $P_f$  is then given by

$$P_f b = (b - 4) \times 2.5 \times 125 \quad (\text{i})$$

The area of cross-section of each rivet is

$$A_r = \frac{\pi \times 4^2}{4} = 12.6 \text{ mm}^2$$

Since each rivet is in double shear (i.e., two failure shear planes), the area of cross-section in shear is

$$2 \times 12.6 = 25.2 \text{ mm}^2$$

Then, the failure load/unit width in shear is given by

$$P_f b = 25.2 \times 120 \quad (\text{ii})$$

For failure to occur simultaneously in shear and tension, that is, equating Eqs. (i) and (ii),

$$25.2 \times 120 = (b - 4) \times 2.5 \times 125$$

from which

$$b = 13.7 \text{ mm}$$

Say, a rivet spacing of 13 mm.

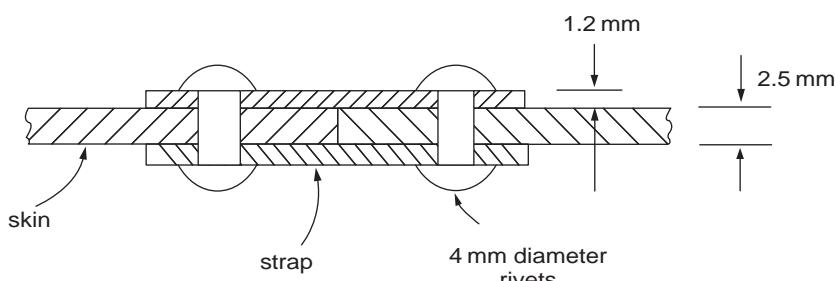


FIGURE 12.15 Joint of Example 12.1

### Joint efficiency

The efficiency of a joint or connection is measured by comparing the actual failure load with that which would apply if there were no rivet holes in the plate. Then, for the joint shown in Fig. 12.14, the joint efficiency  $\eta$  is given by

$$\eta = \frac{\sigma_{ult} t(b-d)/b}{\sigma_{ult} t} = \frac{b-d}{b} \quad (12.5)$$

### Group-riveted joints

Rivets may be grouped on each side of a joint such that the efficiency of the joint is a maximum. Suppose that two plates are connected as shown in Fig. 12.16 and that six rivets are required on each side. If it is assumed that each rivet is equally loaded, then the single rivet on the line aa takes one-sixth of the total load. The two rivets on the line bb then share two-sixths of the load, while the three rivets on the line cc share three-sixths of the load. On the line bb the area of cross-section of the plate is reduced by two rivet holes and that on the line cc by three rivet holes, so that, relatively, the joint is as strong at these sections as at aa. Therefore, a more efficient joint is obtained than if the rivets were arranged in, say, two parallel rows of three.

### Eccentrically loaded riveted joints

The bracketed connection shown in Fig. 12.17 carries a load  $P$  offset from the centroid of the rivet group. The rivet group is then subjected to a shear load  $P$  through its centroid and a moment or torque  $Pe$  about its centroid.

It is assumed that the shear load  $P$  is distributed equally among the rivets, causing a shear force in each rivet parallel to the line of action of  $P$ . The moment  $Pe$  is assumed to produce a shear force  $S$  in each rivet, where  $S$  acts in a direction perpendicular to the line joining a particular rivet to the centroid of the rivet group. Furthermore, the value of  $S$  is assumed to be proportional to the distance of the rivet from the centroid of the rivet group. Then,

$$Pe = \sum Sr$$

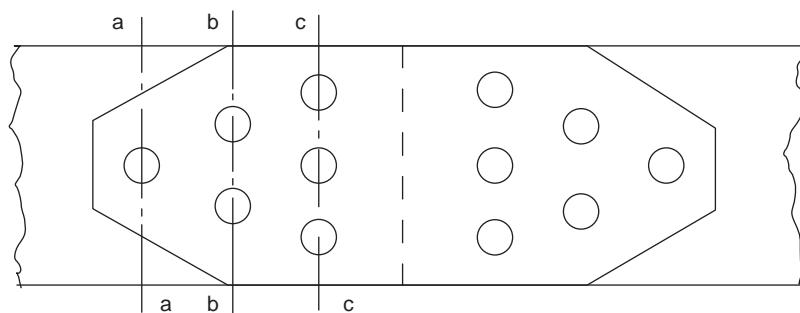
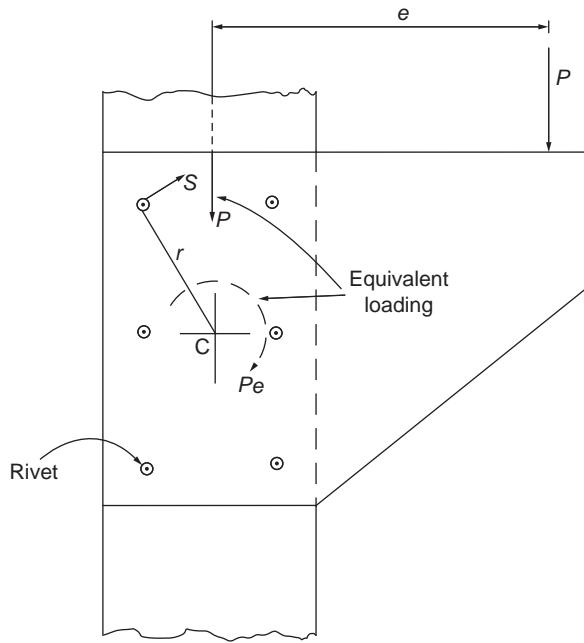


FIGURE 12.16 A Group-Riveted Joint



**FIGURE 12.17 Eccentrically Loaded Joint**

If  $S = kr$ , where  $k$  is a constant for all rivets, then

$$Pe = k \sum r^2$$

from which

$$k = Pe / \sum r^2$$

and

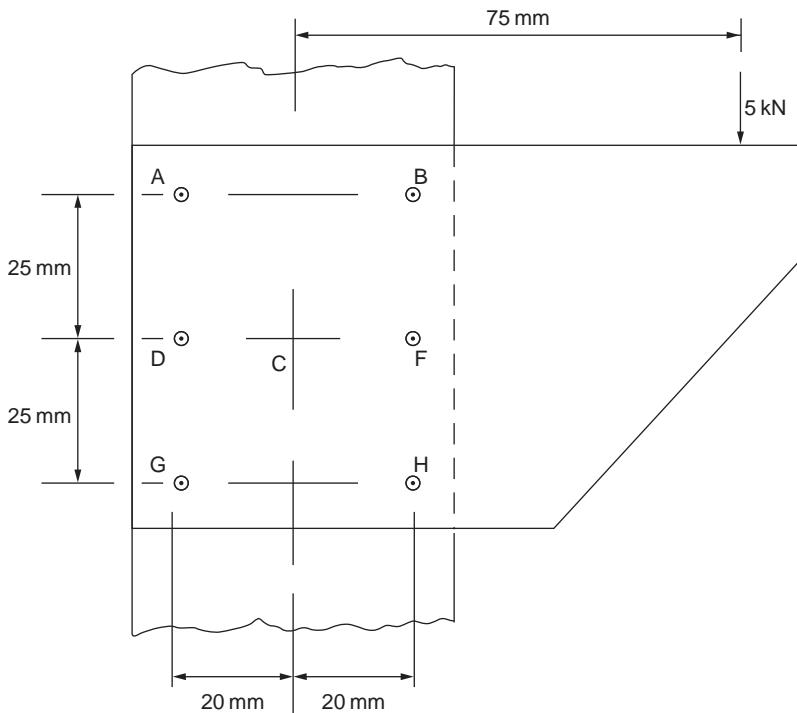
$$S = \frac{Pe}{\sum r^2} r \quad (12.6)$$

The resultant force on a rivet is then the vector sum of the forces due to  $P$  and  $Pe$ .

### Example 12.2

The bracket shown in Fig. 12.18 carries an offset load of 5 kN. Determine the resultant shear forces in the rivets A and B.

The vertical shear force on each rivet is  $5/6 = 0.83$  kN. The moment ( $Pe$ ) on the rivet group is  $5 \times 75 = 375$  kNm. The distance of rivet A (and B, G, and H) from the centroid C of the rivet group is given by



**FIGURE 12.18 Joint of Example 12.2**

$$r = (20^2 + 25^2)^{1/2} = (1025)^{1/2} = 32.02 \text{ mm}$$

The distance of D (and F) from C is 20 mm. Therefore,

$$\sum r^2 = 2 \times 400 + 4 \times 1025 = 4900$$

From Eq. (12.6), the shear forces on rivets A and B due to the moment are

$$S = \frac{375}{4900} \times 32.02 = 2.45 \text{ kN}$$

On rivet A, the force system due to  $P$  and  $Pe$  is that shown in Fig. 12.19(a), while that on B is shown in Fig. 12.19(b).

The resultant forces may then be calculated using the rules of vector addition or determined graphically using the parallelogram of forces.<sup>2</sup>

The design of riveted connections is carried out in the actual design of the rear fuselage of a single-engine trainer/semi-aerobatic aircraft in the Appendix.

### 12.3.2 Welded connections

Welded connections have the advantage that a weld is almost as strong as the parent material and, in effect, provides an integral structure.

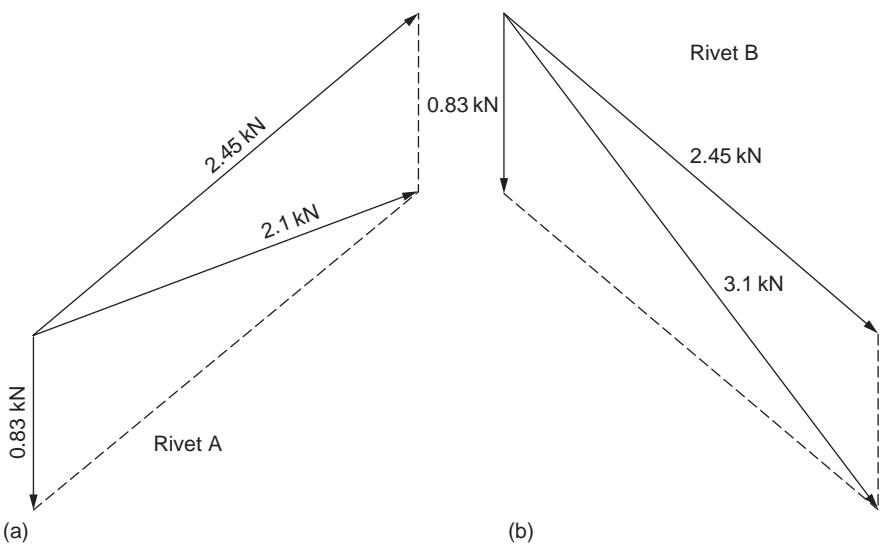


FIGURE 12.19 Force Diagrams for Rivets of Example 12.2

There are two major welding techniques employed in aerospace structures, *laser beam welding* and *friction stir welding*. In both cases heat is employed, but in laser beam welding heat is applied locally so that when the material reaches its melting temperature, then, only a small area is affected.

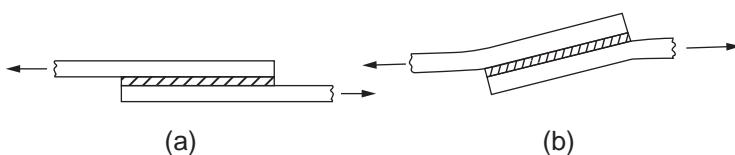
In friction stir welding, the material remains in its solid state and a cylindrical tool is rotated and forced under very high pressure into the sheet. The friction between the tool and the sheet generates sufficient heat to soften the material so that the resulting plasticized flow mixes. This method does not require additional welding material and can be applied to more alloys, including high-strength aeronautical alloys, than laser beam welding can.

Structures fabricated from composite materials can only be welded if they are thermoplastic. One technique, *resistance welding*, adds an electrically resistant material between the surfaces and applies an electric current through the material giving the required amount of heat to bond the parts, that is, the composite matrices, together.

### 12.3.3 Use of adhesives

An adhesive bonded joint is one considered to be only capable of resisting shear loads and even though such a joint does possess some tensile strength, this is usually ignored.

Generally, adhesive bonding produces joints of high durability and, in the absence of rivet holes and the accompanying stress concentrations, they possess fatigue lives greater than, say, riveted joints. Also, since the joint components are not weakened during the bonding process by, for example, the



**FIGURE 12.20 Distortion in an Overlap Joint**



**FIGURE 12.21 Scarf Joints in a Composite Structure**

generation of heat and the resulting plasticizing of the material, an adhesive joint is generally stronger than the individual components.

Two methods of bonding are currently in use, *hot bonding* and *cold bonding*. Of the two, hot bonding is usually preferred, since the process produces a higher quality of joint.

Adhesive bonded joints are produced under a carefully controlled repeatable procedure. However, the level of consistency of the quality of adhesive bonded joints is not sufficiently high for them to be used in primary structures without some reinforcement provided by, say, riveting.

In metallic structures, such as those comprised of aluminum components, the surfaces are pretreated by anodizing and priming. Anodizing is an electrolytic process that covers the surfaces with an inert coating.

The use of adhesive bonding for composite structural components is rather simpler than for metallic structures, since the adhesive requires curing as do the composites so that, if the curing is carried out simultaneously, an integral combination, in effect, results. However, when two sheets of material are bonded together in an overlap format as shown in Fig. 12.20(a), the joint will deform as shown in Fig. 12.20(b). This deformation causes bending stresses in the sheets and additional tensile stresses in the adhesive. In this situation, these tensile stresses must be calculated, as they clearly affect the joint strength.

This particular problem may overcome by employing a *scarf joint* in which the layered composition of the composite is utilized by curtailing each layer in the laminate at different positions; two examples are shown in Fig. 12.21.

## References

- [1] Wijker JJ. Spacecraft structures. Berlin: Springer Verlag; 2010.
- [2] Megson THG. Structural and stress analysis. 4th ed. Oxford: Elsevier; 2019.

## PROBLEMS

**P.12.1.** Examine possible uses of new materials in future aircraft manufacture.

**P.12.2.** Describe the main features of a stressed skin structure. Discuss the structural functions of the various components, with particular reference either to the fuselage or to the wing of a medium-sized transport aircraft.

**P.12.3.** Describe, with aid of sketches, a suitable structural layout for a spacecraft launch vehicle.

**P.12.4.** The primary structure of a launch vehicle consists of a cylindrical shell of diameter 1.5 m and height 7.5 m. If the thrust generated at liftoff is 15,000 kN, determine the required skin thickness of the shell. Check the stability of the shell assuming that one end is fixed and the other free. The maximum allowable stress in the shell is  $242 \text{ N/mm}^2$  and Young's modulus  $E = 72,000 \text{ N/mm}^2$ .

Answer: 13.15 mm. Shell is stable ( $P_{CR} = 54954 \text{ kN}$ ).

**P.12.5.** If the cylindrical shell of P.12.4 is to be replaced by 24 tubes of the same material and of outside diameter 75 mm, determine the required thickness of a tube and check its stability assuming both ends are fixed.

In the event of a tube being unstable, suggest a suitable stabilizing system.

Answer: 13.3 mm. Tubes are unstable ( $P_{CR} = 64.9 \text{ kN}$ ). Insert five intermediate, equally spaced stiffening rings ( $P_{CR} = 2328.5 \text{ kN}$ ).

**P.12.6.** The double riveted butt joint shown in Fig. P.12.6 connects two plates, which are each 2.5 mm thick, and the rivets have a diameter of 3 mm. If the failure strength of the rivets in shear is  $370 \text{ N/mm}^2$  and the ultimate tensile strength of the plate is  $465 \text{ N/mm}^2$ , determine the necessary rivet pitch if the joint is to be designed so that failure due to shear in the rivets and failure due to tension in the plate occur simultaneously. Calculate also the joint efficiency.

Answer: The rivet pitch is 12 mm, joint efficiency is 75 percent.

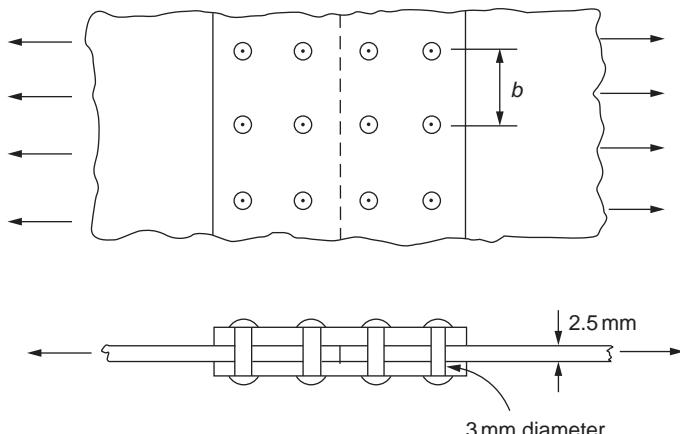


FIGURE P.12.6

**P.12.7.** The rivet group shown in Fig. P.12.7 connects two narrow lengths of plate, one of which carries a 15 kN load positioned as shown. If the ultimate shear strength of a rivet is  $350 \text{ N/mm}^2$  and its failure strength in compression is  $600 \text{ N/mm}^2$ , determine the minimum allowable values of rivet diameter and plate thickness.

Answer: The rivet diameter is 4.2 mm, plate thickness is 1.93 mm.

**P.12.7. MATLAB** Use MATLAB to repeat Problem P.12.7 for load values (P) from 13 kN to 17 kN in increments of 0.5 kN.

- Answer:
- (i) For  $P = 13 \text{ kN}$ , rivet diameter is 3.91 mm, plate thickness is 1.8 mm.
  - (ii) For  $P = 13.5 \text{ kN}$ , rivet diameter is 3.99 mm, plate thickness is 1.83 mm.
  - (iii) For  $P = 14 \text{ kN}$ , rivet diameter is 4.06 mm, plate thickness is 1.86 mm.
  - (iv) For  $P = 14.5 \text{ kN}$ , rivet diameter is 4.13 mm, plate thickness is 1.9 mm.
  - (v) For  $P = 15 \text{ kN}$ , rivet diameter is 4.2 mm, plate thickness is 1.93 mm.
  - (vi) For  $P = 15.5 \text{ kN}$ , rivet diameter is 4.27 mm, plate thickness is 1.96 mm.
  - (vii) For  $P = 16 \text{ kN}$ , rivet diameter is 4.34 mm, plate thickness is 1.99 mm.
  - (viii) For  $P = 16.5 \text{ kN}$ , rivet diameter is 4.41 mm, plate thickness is 2.02 mm.
  - (ix) For  $P = 17 \text{ kN}$ , rivet diameter is 4.48 mm, plate thickness is 2.05 mm.

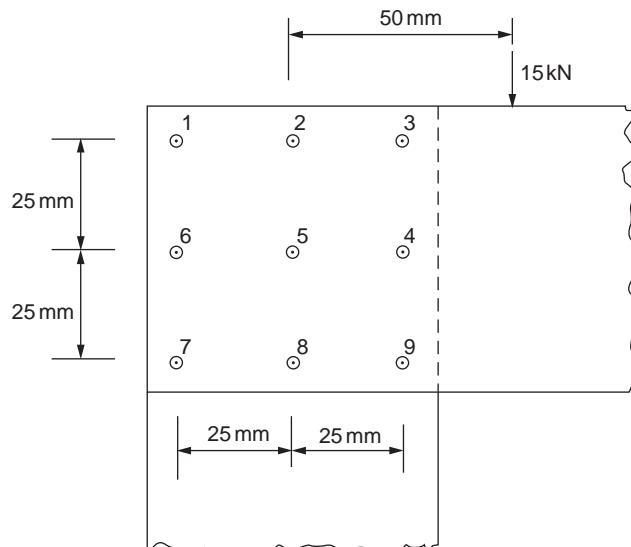


FIGURE P.12.7

# Airworthiness

# 13

The airworthiness of an aircraft is concerned with the standards of safety incorporated in all aspects of its construction. These range from structural strength to the provision of certain safeguards in the event of crash landings and include design requirements relating to aerodynamics, performance, and electrical and hydraulic systems. The selection of minimum standards of safety is largely the concern of “national and international” airworthiness authorities, who prepare handbooks of official requirements. The handbooks include operational requirements, minimum safety requirements, recommended practices, design data, and so forth.

In this chapter, we concentrate on the structural aspects of airworthiness that depend chiefly on the strength and stiffness of the aircraft. Stiffness problems may be conveniently grouped under the heading *aeroelasticity* and are discussed in Section B6. Strength problems arise, as we have seen, from ground and air loads, and their magnitudes depend on the selection of maneuvering and other conditions applicable to the operational requirements of a particular aircraft.

The airworthiness requirements for spacecraft are reviewed in Section 13.3.

## 13.1 FACTORS OF SAFETY: FLIGHT ENVELOPE

The control of weight in aircraft design is of extreme importance. Increases in weight require stronger structures to support them, which in turn lead to further increases in weight, and so on. Excesses of structural weight mean lesser amounts of payload, thereby affecting the economic viability of the aircraft. The aircraft designer is therefore constantly seeking to pare his aircraft’s weight to the minimum compatible with safety. However, to ensure general minimum standards of strength and safety, airworthiness regulations lay down several factors that the primary structure of the aircraft must satisfy. These are the *limit load*, which is the maximum load that the aircraft is expected to experience in normal operation; the *proof load*, which is the product of the limit load and the *proof factor* (1.0–1.25); and the *ultimate load*, which is the product of the limit load and the *ultimate factor* (usually 1.5). The aircraft’s structure must withstand the proof load without detrimental distortion and should not fail until the ultimate load has been achieved. The proof and ultimate factors may be regarded as factors of safety and provide for various contingencies and uncertainties, which are discussed in greater detail in Section 13.2.

The basic strength and flight performance limits for a particular aircraft are selected by the airworthiness authorities and are contained in the *flight envelope* or *V-n* diagram shown in Fig. 13.1. The curves OA and OF correspond to the stalled condition of the aircraft and are obtained from the well-known aerodynamic relationship,

$$\text{Lift} = nW = \frac{1}{2} \rho V^2 SC_{L,\max}$$

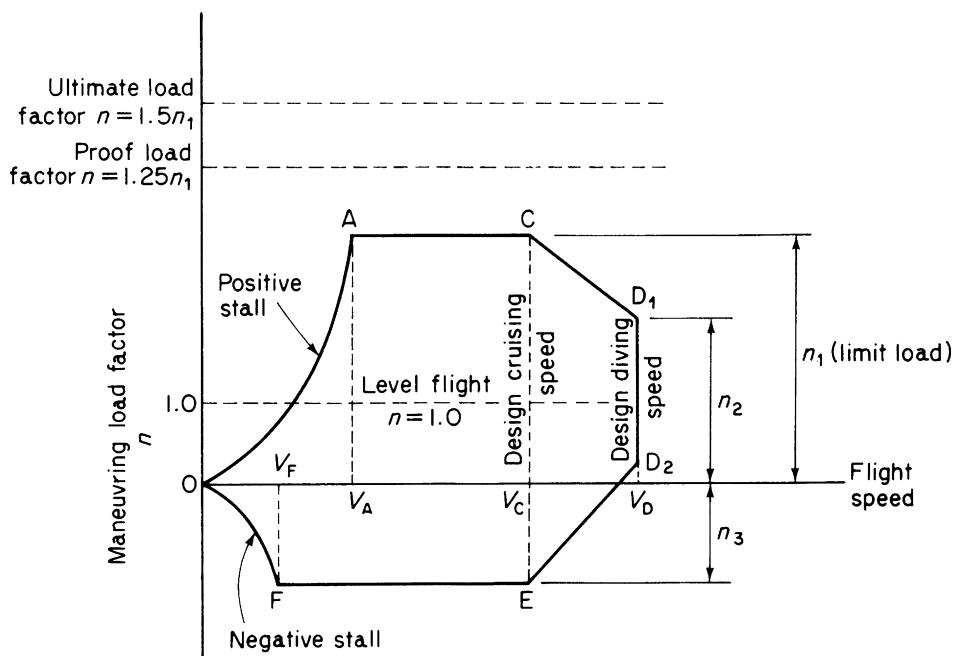


FIGURE 13.1 Flight Envelope

Therefore, for speeds below  $V_A$  (positive wing incidence) and  $V_F$  (negative incidence), the maximum loads that can be applied to the aircraft are governed by  $C_{L,\max}$ . As the speed increases, it is possible to apply the positive and negative limit loads, corresponding to  $n_1$  and  $n_3$ , without stalling the aircraft, so that AC and FE represent maximum operational load factors for the aircraft. Above the design cruising speed  $V_C$ , the cut-off lines  $CD_1$  and  $D_2E$  relieve the design cases to be covered, since it is not expected that the limit loads are applied at maximum speed. Values of  $n_1$ ,  $n_2$ , and  $n_3$  are specified by the airworthiness authorities for particular aircraft; typical load factors are shown in [Table 13.1](#).

A particular flight envelope is applicable to one altitude only, since  $C_{L,\max}$  is generally reduced with an increase in altitude, and the speed of sound decreases with altitude, thereby reducing the critical Mach number and hence the design diving speed  $V_D$ . Flight envelopes are therefore drawn for a range of altitudes from sea level to the operational ceiling of the aircraft.

**Table 13.1** Typical load factors

Load factor $n$	Category		
	Normal	Semi-aerobic	Aerobic
$n_1$	$2.1 + 24,000/(W + 10,000)$	4.5	6.0
$n_2$	$0.75n_1$ but $n_2 < 2.0$	3.1	4.5
$n_3$	1.0	1.8	3.0

## 13.2 LOAD FACTOR DETERMINATION

Several problems require solution before values for the various load factors in the flight envelope can be determined. The limit load, for example, may be produced by a specified maneuver or by an encounter with a particularly severe gust (gust cases and the associated gust envelope are discussed in [Section 14.4](#)). Clearly some knowledge of possible gust conditions is required to determine the limiting case. Furthermore, the fixing of the proof and ultimate factors also depends upon the degree of uncertainty in design, variations in structural strength, structural deterioration, and the like. We now investigate some of these problems to see their comparative influence on load factor values.

### 13.2.1 Limit load

An aircraft is subjected to a variety of loads during its operational life, the main classes of which are maneuver loads, gust loads, undercarriage loads, cabin pressure loads, buffeting, and induced vibrations. Of these, maneuver, undercarriage, and cabin pressure loads are determined with reasonable simplicity, since maneuver loads are controlled design cases, undercarriages are designed for given maximum descent rates, and cabin pressures are specified. The remaining loads depend to a large extent on the atmospheric conditions encountered during flight. Estimates of the magnitudes of such loads are possible, therefore, only if in-flight data on these loads are available. It obviously requires a great number of hours of flying if the experimental data are to include possible extremes of atmospheric conditions. In practice, the amount of data required to establish the probable period of flight time before an aircraft encounters, say, a gust load of a given severity is a great deal more than that available. It therefore becomes a problem in statistics to extrapolate the available data and calculate the probability of an aircraft being subjected to its proof or ultimate load during its operational life. The aim would be for a zero or negligible rate of occurrence of its ultimate load and an extremely low rate of occurrence of its proof load. Having decided on an ultimate load, the limit load may be fixed as defined in [Section 13.1](#), although the value of the ultimate factor includes, as already noted, allowances for uncertainties in design, variation in structural strength, and structural deterioration.

### 13.2.2 Uncertainties in design and structural deterioration

Neither of these presents serious problems in modern aircraft construction and therefore does not require large factors of safety to minimize their effects. Modern methods of aircraft structural analysis are refined and, in any case, tests to determine actual failure loads are carried out on representative full-scale components to verify design estimates. The problem of structural deterioration due to corrosion and wear may be largely eliminated by close inspection during service and the application of suitable protective treatments.

### 13.2.3 Variation in structural strength

To minimize the effect of the variation in structural strength between two apparently identical components, strict controls are employed in the manufacture of materials and in the fabrication of the structure. Material control involves the observance of strict limits in chemical composition and close supervision of manufacturing methods such as machining, heat treatment, and rolling. In addition, the

inspection of samples by visual, radiographic, and other means and the carrying out of strength tests on specimens enable below-limit batches to be isolated and rejected. Thus, if a sample of a batch of material falls below a specified minimum strength, then the batch is rejected. This means, of course, that an actual structure always comprises materials with properties equal to or better than those assumed for design purposes, an added but unallowed for “bonus” in considering factors of safety.

Similar precautions are applied to assembled structures with regard to dimension tolerances, quality of assembly, welding, and so forth. Again, visual and other inspection methods are employed and, in certain cases, strength tests are carried out on sample structures.

### 13.2.4 Fatigue

Although adequate precautions are taken to ensure that an aircraft’s structure possesses sufficient strength to withstand the most severe expected gust or maneuver load, the problem of fatigue remains. Practically all components of the aircraft’s structure are subjected to fluctuating loads, which occur a great many times during the life of the aircraft. It has been known for many years that materials fail under fluctuating loads at much lower values of stress than their normal static failure stress. A graph of failure stress against number of repetitions of this stress has the typical form shown in Fig. 13.2. For some materials, such as mild steel, the curve (usually known as an S–N curve or diagram) is asymptotic to a certain minimum value, which means that the material has an actual infinite-life stress. Curves for other materials – for example, aluminum and its alloys – do not always appear to have asymptotic values so that these materials may not possess an infinite-life stress. We discuss the implications of this a little later.

Prior to the mid-1940s, little attention had been paid to fatigue considerations in the design of aircraft structures. It was felt that sufficient static strength would eliminate the possibility of fatigue failure. However, evidence began to accumulate that several aircraft crashes had been caused by fatigue failure. The seriousness of the situation was highlighted in the early 1950s by catastrophic fatigue failures of two Comet airliners. These were caused by the once-per-flight cabin pressurization cycle that produced circumferential and longitudinal stresses in the fuselage skin. Although these stresses were well below the allowable stresses for single cycle loading, stress concentrations occurred at the corners of the windows and around rivets, which raised local stresses considerably above the general stress

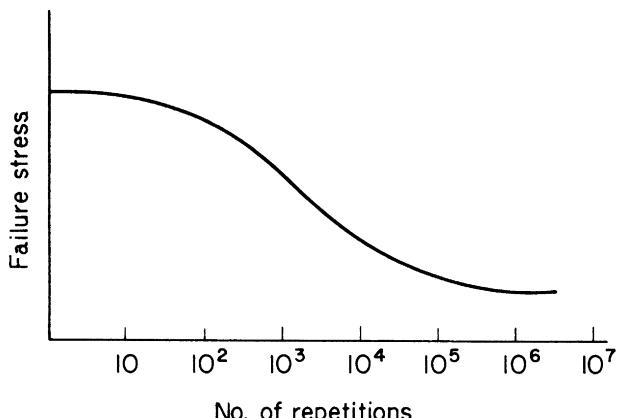


FIGURE 13.2 Typical Form of S–N Diagram

level. Repeated cycles of pressurization produced fatigue cracks that propagated disastrously, causing an explosion of the fuselage at high altitude.

Several factors contributed to the emergence of fatigue as a major factor in design. For example, aircraft speeds and sizes increased, calling for higher wing and other loadings. Consequently, the effect of turbulence was magnified and the magnitudes of the fluctuating loads became larger. In civil aviation, airliners had a greater utilization and a longer operational life. The new zinc-rich alloys, used for their high static strength properties, did not show a proportional improvement in fatigue strength, exhibited high crack propagation rates, and were extremely notch sensitive.

Even though the causes of fatigue were reasonably clear at that time, its elimination as a threat to aircraft safety was a different matter. The fatigue problem has two major facets: the prediction of the fatigue strength of a structure and a knowledge of the loads causing fatigue. Information was lacking on both counts. The Royal Aircraft Establishment (RAE) and the aircraft industry therefore embarked on an extensive test program to determine the behavior of complete components, joints, and other detail parts under fluctuating loads. These included fatigue testing by the RAE of some 50 Meteor 4 tailplanes at a range of temperatures, plus research, also by the RAE, into the fatigue behavior of joints and connections. Further work was undertaken by some universities and by the industry itself into the effects of stress concentrations.

In conjunction with their fatigue strength testing, the RAE initiated research to develop a suitable instrument for counting and recording gust loads over long periods of time. Such an instrument was developed by J. Taylor in 1950 and was designed so that the response fell off rapidly above 10 Hz. Crossings of  $g$  thresholds from 0.2 to 1.8  $g$  at 0.1  $g$  intervals were recorded (note that steady level flight is 1  $g$  flight) during experimental flying at the RAE on three different aircraft over 28,000 km, and the best techniques for extracting information from the data established. Civil airlines cooperated by carrying the instruments on their regular air services for a number of years. Eight types of aircraft were equipped, so that by 1961, records had been obtained for regions including Europe, the Atlantic, Africa, India, and the Far East, representing 19,000 hours and 8 million km of flying.

Atmospheric turbulence and the cabin pressurization cycle are only two of the many fluctuating loads that cause fatigue damage in aircraft. On the ground, the wing is supported on the undercarriage and experiences tensile stresses in its upper surfaces and compressive stresses in its lower surfaces. In flight, these stresses are reversed, as aerodynamic lift supports the wing. Also, the impact of landing and ground maneuvering on imperfect surfaces cause stress fluctuations, while, during landing and takeoff, flaps are lowered and raised, producing additional load cycles in the flap support structure. Engine pylons are subjected to fatigue loading from thrust variations in takeoff and landing and also from inertia loads produced by lateral gusts on the complete aircraft.

A more detailed investigation of fatigue and its associated problems is presented in [Chapter 15](#) while a fuller discussion of airworthiness as applied to civil jet aircraft is presented in Jenkinson, Simpkin, and Rhodes.<sup>1</sup>

### 13.3 AIRWORTHINESS: SPACECRAFT

Similar definitions of load to those for aircraft apply to spacecraft. For example, the *flight limit load* for a given design condition is the maximum occurring load expected during a mission. The *design limit load* is the flight limit load multiplied by a *design factor*; this is also known as the *qualification load*. The *ultimate load* is the design limit load multiplied by a *factor of safety* and is usually the most critical load in the design procedure. The *proof load* is the limit load multiplied by the *proof load factor* and is

used to test parts of the structure before the entire spacecraft or launch vehicle is tested. Such a test, for example, would be the testing of fuel tanks under internal pressure.

### 13.3.1 Factors of safety

The factors of safety vary in magnitude depending on which condition is being considered. For example, the factor of safety used to determine the design limit load from the flight limit load is of the order 1.4–1.5 and the factor of safety used to obtain the ultimate load from the design limit load is in the range 1.25–1.5.

## Reference

- [1] Jenkinson LR, Simpkin P, Rhodes D. Civil jet aircraft design. London: Arnold; 1999.

## PROBLEMS

**P.13.1** A radar dome weighing 300 kg is positioned on the top of the fuselage of a surveillance aircraft. If the dome is attached to the fuselage frames by four bolts, each having an ultimate shear strength of 5000 N and the aircraft is subjected to a maximum acceleration of 3 g in a vertical climb, determine (a) the limit load per bolt, (b) the ultimate load per bolt, and (c) the ultimate margin of safety. The special factor of safety is 1.5.

Answer: (a) 2943 N, (b) 4415 N, (c) 0.133

**P.13.2** The relative positions of the center of gravity and the centers of pressure of the wing and tailplane of an aircraft are shown in Fig. P.13.2; the total weight of the aircraft is 667.5 kN. If the fuselage and its contents weigh 26.3 kN/m and the weight of the tailplane is 8.9 kN, calculate the ultimate shear force in the fuselage at the section AA for a maneuver load factor of 3 g including gravity. Assume a factor of safety of 1.5.

Answer: 212.2 kN

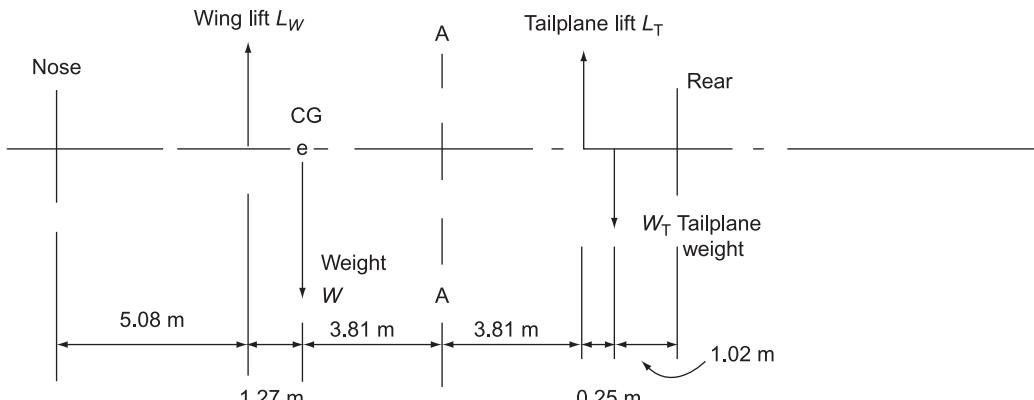


FIGURE P.13.2

# Airframe loads

# 14

In [Chapter 12](#), we discussed in general terms the types of load to which aircraft and spacecraft are subjected during their operational life. We shall now examine in more detail the loads produced by various maneuvers.

## 14.1 AIRCRAFT INERTIA LOADS

The maximum loads on the components of an aircraft's structure generally occur when the aircraft is undergoing some form of acceleration or deceleration, such as in landings, takeoffs, and maneuvers within the flight and gust envelopes. Therefore, before a structural component can be designed, the inertia loads corresponding to these accelerations and decelerations must be calculated. For these purposes, we shall suppose that an aircraft is a rigid body and represent it by a rigid mass,  $m$ , as shown in [Fig. 14.1](#). We also, at this stage, consider motion in the plane of the mass that would correspond to pitching of the aircraft without roll or yaw. We further suppose that the center of gravity (CG) of the mass has coordinates  $\bar{x}, \bar{y}$ , referred to  $x$  and  $y$  axes having an arbitrary origin  $O$ ; the mass is rotating about an axis through  $O$  perpendicular to the  $xy$  plane with a constant angular velocity  $\omega$ .

The acceleration of any point, a distance  $r$  from  $O$ , is  $\omega^2 r$  and is directed toward  $O$ . Thus, the inertia force acting on the element,  $\delta m$ , is  $\omega^2 r \delta m$  in a direction opposite to the acceleration, as shown in [Fig. 14.1](#). The components of this inertia force, parallel to the  $x$  and  $y$  axes, are  $\omega^2 r \delta m \cos\theta$  and  $\omega^2 r \delta m \sin\theta$ , respectively, or, in terms of  $x$  and  $y$ ,  $\omega^2 x \delta m$  and  $\omega^2 y \delta m$ . The resultant inertia forces,  $F_x$  and  $F_y$ , are given by

$$\begin{aligned} F_x &= \int \omega^2 x \, dm = \omega^2 \int x \, dm \\ F_y &= \int \omega^2 y \, dm = \omega^2 \int y \, dm \end{aligned}$$

in which we note that the angular velocity  $\omega$  is constant and may therefore be taken outside the integral sign. In these expressions,  $\int x \, dm$  and  $\int y \, dm$  are the moments of the mass,  $m$ , about the  $y$  and  $x$  axes, respectively, so that

$$F_x = \omega^2 \bar{x} m \tag{14.1}$$

and

$$F_y = \omega^2 \bar{y} m \tag{14.2}$$

If the CG lies on the  $x$  axis,  $\bar{y} = 0$  and  $F_y = 0$ . Similarly, if the CG lies on the  $y$  axis,  $F_x = 0$ . Clearly, if  $O$  coincides with the CG,  $\bar{x} = \bar{y} = 0$  and  $F_x = F_y = 0$ .

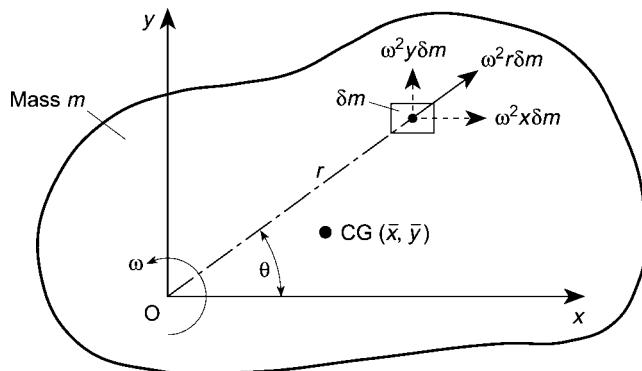


FIGURE 14.1 Inertia Forces on a Rigid Mass Having a Constant Angular Velocity

Suppose now that the rigid body is subjected to an angular acceleration (*or deceleration*)  $\alpha$  in addition to the constant angular velocity,  $\omega$ , as shown in Fig. 14.2. An additional inertia force,  $\alpha r \delta m$ , acts on the element  $\delta m$  in a direction perpendicular to  $r$  and in the opposite sense to the angular acceleration. This inertia force has components  $\alpha r \delta m \cos\theta$  and  $\alpha r \delta m \sin\theta$ , that is,  $\alpha x \delta m$  and  $\alpha y \delta m$ , in the  $y$  and  $x$  directions, respectively. The resultant inertia forces,  $F_x$  and  $F_y$ , are then given by

$$F_x = \int \alpha y \, dm = \alpha \int y \, dm$$

and

$$F_y = - \int \alpha x \, dm = -\alpha \int x \, dm$$

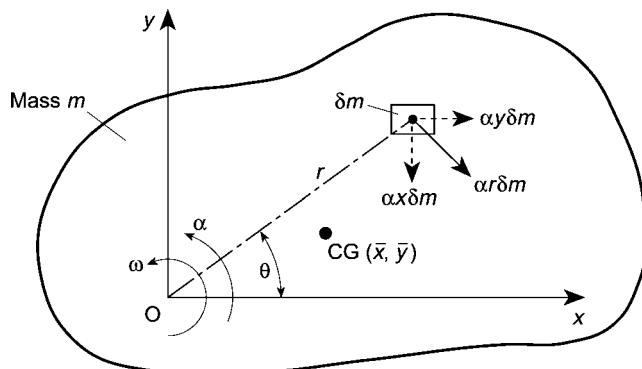


FIGURE 14.2 Inertia Forces on a Rigid Mass Subjected to an Angular Acceleration

for  $\alpha$  in the direction shown. Then, as before,

$$F_x = \alpha \bar{y} m \quad (14.3)$$

and

$$F_y = \alpha \bar{x} m \quad (14.4)$$

Also, if the CG lies on the  $x$  axis,  $\bar{y} = 0$  and  $F_x = 0$ . Similarly, if the CG lies on the  $y$  axis,  $\bar{x} = 0$  and  $F_y = 0$ .

The torque about the axis of rotation produced by the inertia force corresponding to the angular acceleration on the element  $\delta m$  is given by

$$\delta T_O = \alpha r^2 \delta m$$

Thus, for the complete mass,

$$T_O = \int \alpha r^2 dm = \alpha \int r^2 dm$$

The integral term in this expression is the moment of inertia,  $I_O$ , of the mass about the axis of rotation. Thus,

$$T_O = \alpha I_O \quad (14.5)$$

Equation (14.5) may be rewritten in terms of  $I_{CG}$ , the moment of inertia of the mass about an axis perpendicular to the plane of the mass through the CG. Hence, using the parallel axes theorem (see Section 16.4),

$$I_O = m(\bar{r})^2 + I_{CG}$$

where  $\bar{r}$  is the distance between O and the CG. Then,

$$I_O = m[(\bar{x})^2 + (\bar{y})^2] + I_{CG}$$

and

$$T_O = m[(\bar{x})^2 + (\bar{y})^2]\alpha + I_{CG}\alpha \quad (14.6)$$

### Example 14.1

An aircraft having a total weight of 45 kN lands on the deck of an aircraft carrier and is brought to rest by means of a cable engaged by an arrester hook, as shown in Fig. 14.3. If the deceleration induced by the cable is  $3g$ , determine the tension,  $T$ , in the cable, the load on an undercarriage strut, and the shear and axial loads in the fuselage at the section AA; the weight of the aircraft aft of AA is 4.5 kN. Calculate also the length of deck covered by the aircraft before it is brought to rest if the touch-down speed is 25 m/s.

The aircraft is subjected to a horizontal inertia force  $ma$  where  $m$  is the mass of the aircraft and  $a$  its deceleration. Thus, resolving forces horizontally,

$$T \cos 10^\circ - ma = 0$$

that is,

$$T \cos 10^\circ - \frac{45}{g} 3g = 0$$

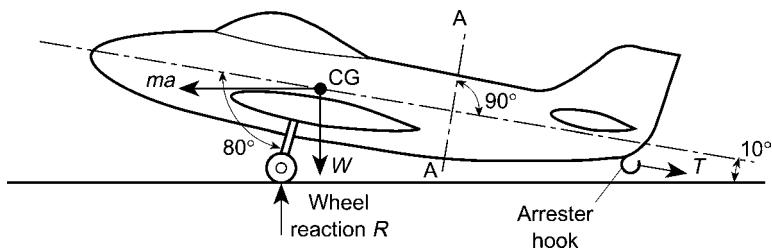


FIGURE 14.3 Forces on the Aircraft of Example 14.1

which gives

$$T = 137.1 \text{ kN}$$

Now, resolving forces vertically,

$$R - W - T \sin 10^\circ = 0$$

that is,

$$R = 45 + 137.1 \sin 10^\circ = 68.8 \text{ kN}$$

Assuming two undercarriage struts, the load in each strut is  $(R/2)/\cos 20^\circ = 36.6 \text{ kN}$ .

Let  $N$  and  $S$  be the axial and shear loads at the section AA, as shown in Fig. 14.4. The inertia load acting at the CG of the fuselage aft of AA is  $m_1 a$ , where  $m_1$  is the mass of the fuselage aft of AA. Then,

$$m_1 a = \frac{4.5}{g} 3 g = 13.5 \text{ kN}$$

Resolving forces parallel to the axis of the fuselage,

$$N - T + m_1 a \cos 10^\circ - 4.5 \sin 10^\circ = 0$$

that is,

$$N - 137.1 + 13.5 \cos 10^\circ - 4.5 \sin 10^\circ = 0$$

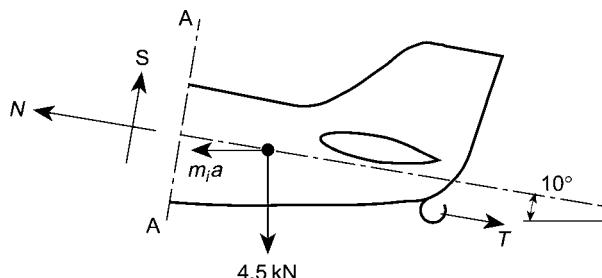


FIGURE 14.4 Shear and Axial Loads at the Section AA of the Aircraft of Example 14.1

from which

$$N = 124.6 \text{ kN}$$

Now, resolving forces perpendicular to the axis of the fuselage,

$$S - m_1 a \sin 10^\circ - 4.5 \cos 10^\circ = 0$$

that is,

$$S - 13.5 \sin 10^\circ - 4.5 \cos 10^\circ = 0$$

so that

$$S = 6.8 \text{ kN}$$

Note that, in addition to the axial load and shear load at the section AA, there will also be a bending moment.

Finally, from elementary dynamics,

$$v^2 = v_0^2 + 2as$$

where  $v_0$  is the touch-down speed,  $v$  the final speed ( $= 0$ ), and  $s$  the length of deck covered. Then,

$$v_0^2 = -2as$$

that is,

$$25^2 = -2(-3 \times 9.81)s$$

which gives

$$s = 10.6 \text{ m}$$

### Example 14.2

An aircraft having a weight of 250 kN and a tricycle undercarriage lands at a vertical velocity of 3.7 m/s, such that the vertical and horizontal reactions on the main wheels are 1,200 kN and 400 kN, respectively; at this instant, the nosewheel is 1.0 m from the ground, as shown in Fig. 14.5. If the moment of inertia of the aircraft about its CG is  $5.65 \times 10^8 \text{ Ns}^2 \text{ mm}$ , determine the inertia forces on the aircraft, the time taken for its vertical velocity to become zero, and its angular velocity at this instant.

The horizontal and vertical inertia forces  $ma_x$  and  $ma_y$  act at the CG, as shown in Fig. 14.5;  $m$  is the mass of the aircraft; and  $a_x$  and  $a_y$  its accelerations in the horizontal and vertical directions, respectively. Then, resolving forces horizontally,

$$ma_x - 400 = 0$$

from which

$$ma_x = 400 \text{ kN}$$

Now, resolving forces vertically,

$$ma_y + 250 - 1,200 = 0$$

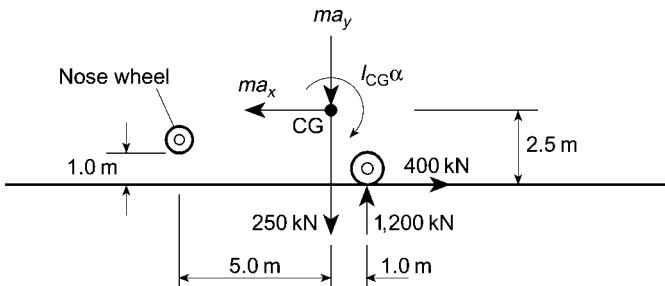


FIGURE 14.5 Geometry of the Aircraft of Example 14.2

which gives

$$ma_y = 950 \text{ kN}$$

Then,

$$a_y = \frac{950}{m} = \frac{950}{250/g} = 3.8 g \quad (\text{i})$$

Now, taking moments about the CG,

$$I_{CG}\alpha - 1,200 \times 1.0 - 400 \times 2.5 = 0 \quad (\text{ii})$$

from which

$$I_{CG}\alpha = 2,200 \text{ mkN}$$

Note that the units of  $I_{CG}\alpha$  are  $\text{mkNrad}$  but radians are nondimensional, so are not included but implicit.

Hence,

$$\alpha = \frac{I_{CG}\alpha}{I_{CG}} = \frac{2,200 \times 10^6}{5.65 \times 10^8} = 3.9 \text{ rad/s}^2 \quad (\text{iii})$$

From Eq. (i), the aircraft has a vertical deceleration of  $3.8 g$  from an initial vertical velocity of  $3.7 \text{ m/s}$ . Therefore, from elementary dynamics, the time,  $t$ , taken for the vertical velocity to become zero is given by

$$v = v_0 + a_y t \quad (\text{iv})$$

in which  $v = 0$  and  $v_0 = 3.7 \text{ m/s}$ . Hence,

$$0 = 3.7 - 3.8 \times 9.81t$$

from which

$$t = 0.099 \text{ s}$$

In a similar manner to Eq. (iv), the angular velocity of the aircraft after  $0.099 \text{ s}$  is given by

$$\omega = \omega_0 + \alpha t$$

in which  $\omega_0 = 0$  and  $\alpha = 3.9 \text{ rad/s}^2$ . Hence,

$$\omega = 3.9 \times 0.099$$

that is,

$$\omega = 0.39 \text{ rad/s}$$

### Example 14.3

The aircraft shown in Fig. 14.6 has a moment of inertia of  $5,460 \text{ kNs}^2\text{m}$  and an all-up weight of  $430 \text{ kN}$ . During landing the aerodynamic lift is equal to 90 percent of its all-up weight and it is subjected to the ground loads shown. Determine (a) the limit load factor at its center of gravity, (b) the limit pitching acceleration.

- (a) Wing lift  $= 0.9 \times 430 = 387 \text{ kN}$ . Resolving vertically, the total vertical load on the aircraft  $= 387 + 1,112.5 - 430 = 1,069.5 \text{ kN}$ . Therefore the limit load factor  $= (1,069.5/430) + 1 = 3.5$
- (b) Taking moments about the CG,

$$3.81L_w + 222.5 \times 2.54 + 1,112.5 \times 2.54 = I_O\alpha = 5460\alpha$$

from which

$$\alpha = 0.89 \text{ rad/s}^2$$

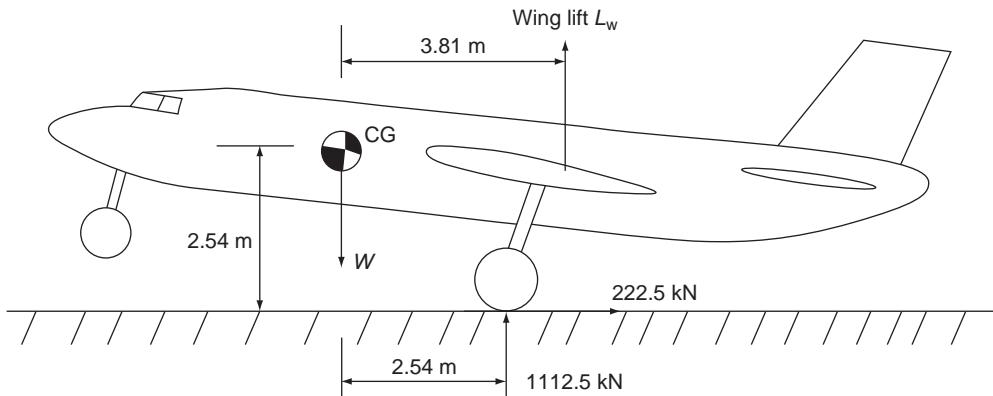


FIGURE 14.6 Aircraft of Example 14.3

## 14.2 SYMMETRIC MANEUVER LOADS

We shall now consider the calculation of aircraft loads corresponding to the flight conditions specified by flight envelopes. In fact, an infinite number of flight conditions lie within the boundary of the flight envelope, although, structurally, those represented by the boundary are the most severe. Furthermore, it

is usually found that the corners A, C, D<sub>1</sub>, D<sub>2</sub>, E, and F (see Fig. 13.1) are more critical than points on the boundary between the corners, so that, in practice, only the six conditions corresponding to these corner points need be investigated for each flight envelope.

In symmetric maneuvers, we consider the motion of the aircraft initiated by movement of the control surfaces in the plane of symmetry. Examples of such maneuvers are loops, straight pullouts, and bunts; and the calculations involve the determination of lift, drag, and tailplane loads at given flight speeds and altitudes. The effects of atmospheric turbulence and gusts are discussed in Section 14.4.

### 14.2.1 Level flight

Although steady level flight is not a maneuver in the strict sense of the word, it is a useful condition to investigate initially, since it establishes points of load application and gives some idea of the equilibrium of an aircraft in the longitudinal plane. The loads acting on an aircraft in steady flight are shown in Fig. 14.7, with the following notation:

- $L$  is the lift acting at the aerodynamic center of the wing.
- $D$  is the aircraft drag.
- $M_0$  is the aerodynamic pitching moment of the aircraft *less* its horizontal tail.
- $P$  is the vertical load on the horizontal tail acting at the aerodynamic center of the tail, usually taken to be at approximately one-third of the tailplane chord.
- $W$  is the aircraft weight acting at its CG.
- $T$  is the engine thrust, assumed here to act parallel to the direction of flight in order to simplify calculation.

The loads are in static equilibrium, since the aircraft is in a steady, unaccelerated, level flight condition. Thus, for vertical equilibrium,

$$L + P - W = 0 \quad (14.7)$$

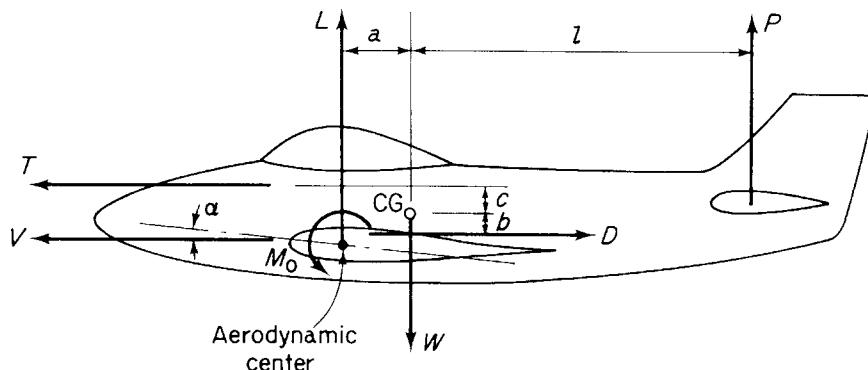


FIGURE 14.7 Aircraft Loads in Level Flight

for horizontal equilibrium,

$$T - D = 0 \quad (14.8)$$

and taking moments about the aircraft's CG in the plane of symmetry,

$$La - Db - Tc - M_0 - Pl = 0 \quad (14.9)$$

For a given aircraft weight, speed, and altitude, Eqs. (14.7)–(14.9) may be solved for the unknown lift, drag, and tail loads. However, other parameters in these equations, such as  $M_0$ , depend upon the wing incidence  $\alpha$ , which in turn is a function of the required wing lift, so that, in practice, a method of successive approximation is found to be the most convenient means of solution.

As a first approximation, we assume that the tail load  $P$  is small compared with the wing lift  $L$ , so that, from Eq. (14.7),  $L \approx W$ . From aerodynamic theory with the usual notation,

$$L = \frac{1}{2} \rho V^2 S C_L$$

Hence,

$$\frac{1}{2} \rho V^2 S C_L \approx W \quad (14.10)$$

Equation (14.10) gives the approximate lift coefficient  $C_L$  and thus (from  $C_L-\alpha$  curves established by wind tunnel tests), the wing incidence  $\alpha$ . The drag load  $D$  follows (knowing  $V$  and  $\alpha$ ), and hence we obtain the required engine thrust  $T$  from Eq. (14.8). Also  $M_0$ ,  $a$ ,  $b$ ,  $c$ , and  $l$  may be calculated (again, since  $V$  and  $\alpha$  are known) and Eq. (14.9) solved for  $P$ . As a second approximation, this value of  $P$  is substituted in Eq. (14.7) to obtain a more accurate value for  $L$  and the procedure is repeated. Usually, three approximations are sufficient to produce reasonably accurate values.

In most cases  $P$ ,  $D$ , and  $T$  are small compared with the lift and aircraft weight. Therefore, from Eq. (14.7),  $L \approx W$  and substitution in Eq. (14.9) gives, neglecting  $D$  and  $T$ ,

$$P \approx W \frac{a}{l} - \frac{M_0}{l} \quad (14.11)$$

We see, from Eq. (14.11), that if  $a$  is large, then  $P$  most likely will be positive. In other words, the tail load acts upward when the CG of the aircraft is far aft. When  $a$  is small or negative – that is, a forward CG – then  $P$  probably will be negative and act downward.

### 14.2.2 General case of a symmetric maneuver

In a rapid pullout from a dive, a downward load is applied to the tailplane, causing the aircraft to pitch nose upward. The downward load is achieved by a backward movement of the control column, thereby applying negative incidence to the elevators, or horizontal tail if the latter is all-moving. If the maneuver is carried out rapidly, the forward speed of the aircraft remains practically constant, so that increases in lift and drag result from the increase in wing incidence only. Since the lift is now greater than that required to balance the aircraft weight, the aircraft experiences an upward acceleration normal to its flight path. This normal acceleration combined with the aircraft's speed in the dive results in the curved flight path shown in Fig. 14.8. As the drag load builds up with an increase of incidence, the

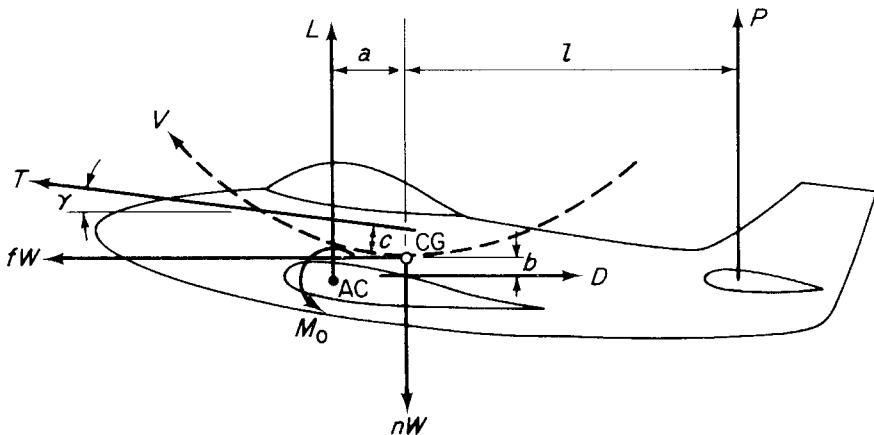


FIGURE 14.8 Aircraft Loads in a Pullout From a Dive

forward speed of the aircraft falls, since the thrust is assumed to remain constant during the maneuver. It is usual, as we observed in the discussion of the flight envelope, to describe the maneuvers of an aircraft in terms of a maneuvering load factor  $n$ . For steady level flight  $n = 1$ , giving  $1 g$  flight, although in fact the acceleration is zero. What is implied in this method of description is that the inertia force on the aircraft in the level flight condition is 1.0 times its weight. It follows that the vertical inertia force on an aircraft carrying out an  $ng$  maneuver is  $nW$ . We therefore replace the dynamic conditions of the accelerated motion by an equivalent set of static conditions, in which the applied loads are in equilibrium with the inertia forces. Thus, in Fig. 14.8,  $n$  is the maneuver load factor, while  $f$  is a similar factor giving the horizontal inertia force. Note that the actual normal acceleration in this particular case is  $(n - 1) g$ .

For vertical equilibrium of the aircraft, we have, referring to Fig. 14.8, where the aircraft is shown at the lowest point of the pullout,

$$L + P + T \sin\gamma - nW = 0 \quad (14.12)$$

For horizontal equilibrium,

$$T \cos\gamma + fW - D = 0 \quad (14.13)$$

and, for pitching moment equilibrium about the aircraft's CG,

$$La - Db - Tc - M_0 - Pl = 0 \quad (14.14)$$

Equation (14.14) contains no terms representing the effect of pitching acceleration of the aircraft; this is assumed to be negligible at this stage.

Again, the method of successive approximation is found to be most convenient for the solution of Eqs. (14.12)–(14.14). There is, however, a difference to the procedure described for the steady level flight case. The engine thrust  $T$  is no longer directly related to the drag  $D$ , as the latter changes during the maneuver. Generally, the thrust is regarded as remaining constant and equal to the value appropriate to conditions before the maneuver began.

**Example 14.4**

The curves  $C_D$ ,  $\alpha$ , and  $C_{M,CG}$  for a light aircraft are shown in Fig. 14.9(a). The aircraft weight is 8,000 N, its wing area  $14.5 \text{ m}^2$ , and its mean chord 1.35 m. Determine the lift, drag, tail load, and forward inertia force for a symmetric maneuver corresponding to  $n = 4.5$  and a speed of 60 m/s. Assume that engine-off conditions apply and that the air density is  $1.223 \text{ kg/m}^3$ . Figure 14.9(b) shows the relevant aircraft dimensions.

As a first approximation, we neglect the tail load  $P$ . Therefore, from Eq. (14.12), since  $T = 0$ , we have

$$L \approx nW \quad (\text{i})$$

Hence,

$$C_L = \frac{L}{\frac{1}{2} \rho V^2 S} \approx \frac{4.5 \times 8000}{\frac{1}{2} \times 1.223 \times 60^2 \times 14.5} = 1.113$$

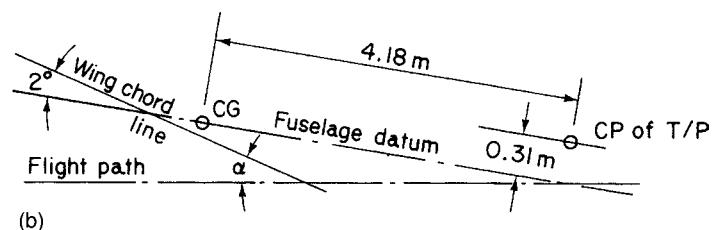
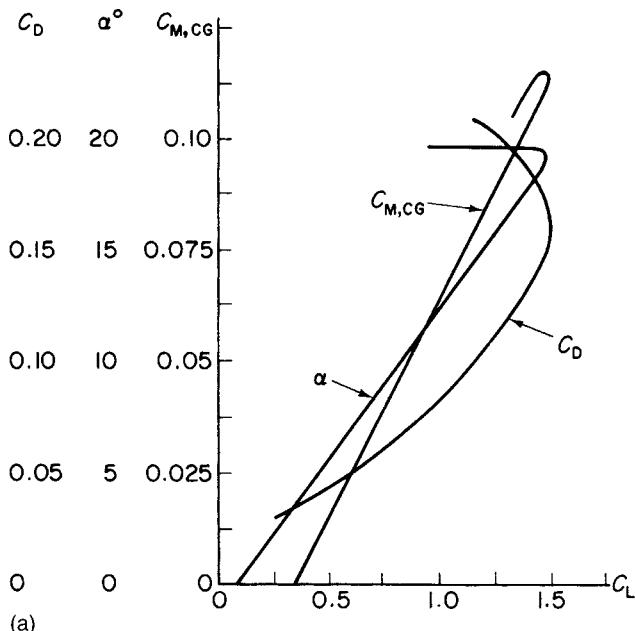


FIGURE 14.9 (a)  $C_D$ ,  $\alpha$ ,  $C_{M,CG}$ — $C_L$  Curves for Example 14.4; (b) Geometry of Example 14.4

From Fig. 14.9(a),  $\alpha = 13.75^\circ$  and  $C_{M,CG} = 0.075$ . The tail arm  $l$ , from Fig. 14.9(b), is

$$l = 4.18 \cos(\alpha - 2) + 0.31 \sin(\alpha - 2) \quad (\text{ii})$$

Substituting this value of  $\alpha$  gives  $l = 4.123$  m. In Eq. (14.14), the terms  $La - Db - M_0$  are equivalent to the aircraft pitching moment  $M_{CG}$  about its CG. Equation (14.14) may therefore be written

$$M_{CG} - Pl = 0$$

or

$$Pl = \frac{1}{2} \rho V^2 Sc C_{M,CG} \quad (\text{ii})$$

where  $c$  = wing mean chord. Substituting  $P$  from Eq. (iii) into Eq. (14.12), we have

$$L + \frac{\frac{1}{2} \rho V^2 Sc C_{M,CG}}{l} = nW$$

or dividing through by  $\frac{1}{2} \rho V^2 S$ ,

$$C_L + \frac{c}{l} C_{M,CG} = \frac{nW}{\frac{1}{2} \rho V^2 S} \quad (\text{iv})$$

We now obtain a more accurate value for  $C_L$  from Eq. (iv),

$$C_L = 1.113 - \frac{1.35}{4.123} \times 0.075 = 1.088$$

giving  $\alpha = 13.3^\circ$  and  $C_{M,CG} = 0.073$ .

Substituting this value of  $\alpha$  into Eq. (ii) gives a second approximation for  $l$ , namely  $l = 4.161$  m.

Equation (iv) now gives a third approximation for  $C_L$ , that is,  $C_L = 1.099$ . Since all three calculated values of  $C_L$  are extremely close, further approximations will not give values of  $C_L$  very much different to them. Therefore, we shall take  $C_L = 1.099$ . From Fig. 14.9(a),  $C_D = 0.0875$ .

The values of lift, tail load, drag, and forward inertia force then follow:

$$\text{Lift } L = \frac{1}{2} \rho V^2 S C_L = \frac{1}{2} \times 1.223 \times 60^2 \times 14.5 \times 1.099 = 35,000 \text{ N}$$

$$\text{Tail load } P = nW - L = 4.5 \times 8,000 - 35,000 = 1,000 \text{ N}$$

$$\text{Drag } D = \frac{1}{2} \rho V^2 S C_D = \frac{1}{2} \times 1.223 \times 60^2 \times 14.5 \times 0.0875 = 2,790 \text{ N}$$

$$\text{Forward inertia force } fW = D \text{ (from Eq.(14.13))} = 2,790 \text{ N}$$



### 14.3 NORMAL ACCELERATIONS ASSOCIATED WITH VARIOUS TYPES OF MANEUVER

In Section 14.2, we determined aircraft loads corresponding to a given maneuver load factor  $n$ . Clearly, it is necessary to relate this load factor to given types of maneuver. Two cases arise: the first involves a steady pullout from a dive; the second, a correctly banked turn. Although the latter is not a symmetric maneuver in the strict sense of the word, it gives rise to normal accelerations in the plane of symmetry and is therefore included.

### 14.3.1 Steady pullout

Let us suppose that the aircraft has just begun its pullout from a dive, so that it is describing a curved flight path but is not yet at its lowest point. The loads acting on the aircraft at this stage of the maneuver are shown in Fig. 14.10, where  $R$  is the radius of curvature of the flight path. In this case, the lift vector must equilibrate the normal (to the flight path) component of the aircraft weight and provide the force producing the centripetal acceleration  $V^2/R$  of the aircraft toward the center of curvature of the flight path. Thus,

$$L = \frac{WV^2}{gR} + W \cos\theta$$

or, since  $L = nW$  (see Section 14.2),

$$n = \frac{V^2}{gR} + \cos\theta \quad (14.15)$$

At the lowest point of the pullout,  $\theta = 0$ , and

$$n = \frac{V^2}{gR} + 1 \quad (14.16)$$

We see, from either Eq. (14.15) or (14.16), that the smaller the radius of the flight path – that is, the more severe the pullout – the greater is the value of  $n$ . It is quite possible therefore for a severe pullout to overstress the aircraft by subjecting it to loads that lie outside the flight envelope and which may even exceed the proof or ultimate loads. In practice, the control surface movement may be limited by stops incorporated in the control circuit. These stops usually operate only above a

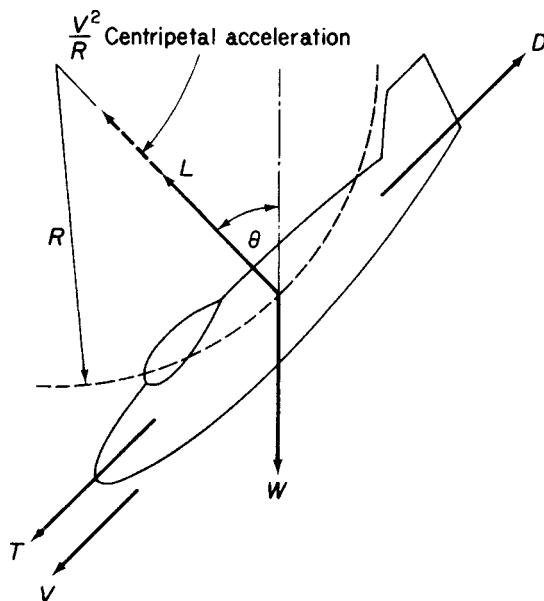


FIGURE 14.10 Aircraft Loads and Acceleration During a Steady Pullout

certain speed, giving the aircraft adequate maneuverability at lower speeds. For hydraulically operated controls, “artificial feel” is built in to the system, whereby the stick force increases progressively as the speed increases, a necessary precaution in this type of system, since the pilot is merely opening and closing valves in the control circuit and therefore receives no direct physical indication of control surface forces.

Alternatively, at low speeds, a severe pullout or pull-up may stall the aircraft. Again, safety precautions are usually incorporated in the form of stall warning devices, since, for modern high-speed aircraft, a stall can be disastrous, particularly at low altitude.

### Example 14.5

A semi-aerobatic aircraft has reached its design diving speed of 185 m/s in a dive inclined at  $45^\circ$  to the horizontal ground. If the maximum maneuver load factor for the aircraft is 5.5, determine the height at which the pullout from the dive must begin for straight and level flight to be achieved at a height of 500 m.

Assuming the design diving speed is maintained throughout the pullout, from Eq.(14.16),

$$5.5 = (185^2/gR) + 1$$

so that

$$R = 775.3 \text{ m}$$

Referring to Fig. 14.11,

$$a = R - R \cos 45^\circ$$

Substituting for  $R$  from the preceding,

$$a = 227 \text{ m}$$

Therefore,

$$h = 727 \text{ m}$$

#### 14.3.2 Correctly banked turn

In this maneuver, the aircraft flies in a horizontal turn with no sideslip at constant speed. If the radius of the turn is  $R$  and the angle of bank  $\phi$ , then the forces acting on the aircraft are those shown in Fig. 14.12. The horizontal component of the lift vector in this case provides the force necessary to produce the centripetal acceleration of the aircraft toward the center of the turn. Then,

$$L \sin\phi = \frac{WV^2}{gR} \quad (14.17)$$

and, for vertical equilibrium,

$$L \cos\phi = W \quad (14.18)$$

or

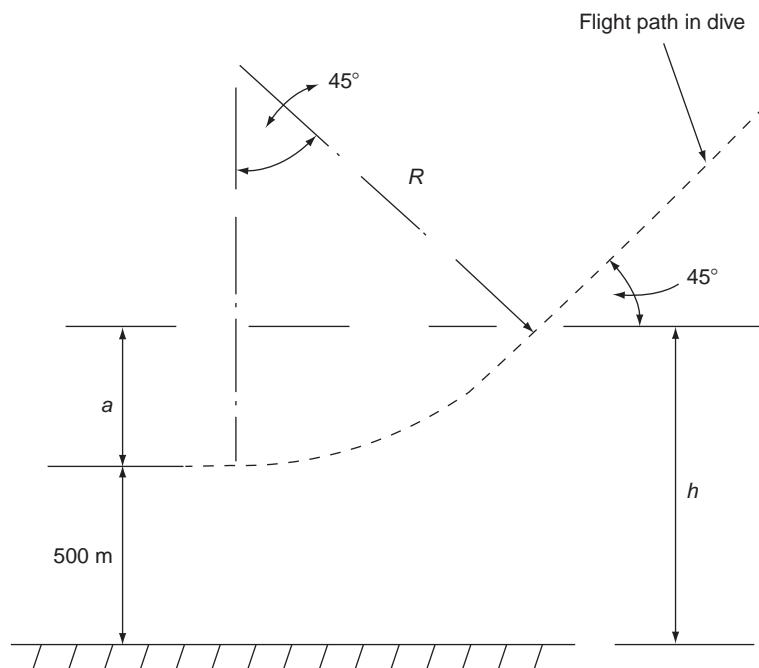


FIGURE 14.11 Pullout From a dive, Example 14.5

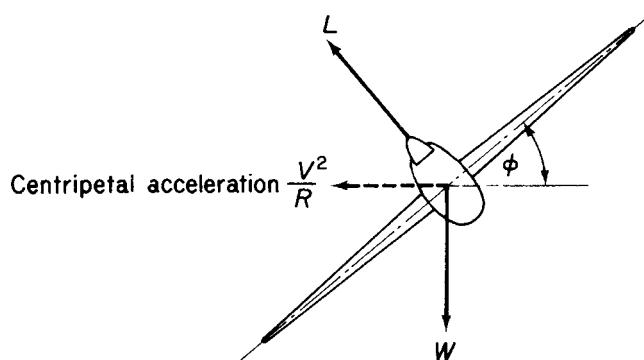


FIGURE 14.12 Correctly Banked Turn

$$L = W \sec \phi \quad (14.19)$$

From Eq. (14.19), we see that the load factor  $n$  in the turn is given by

$$n = \sec \phi \quad (14.20)$$

Also, dividing Eq. (14.17) by Eq. (14.18),

$$\tan\phi = \frac{V^2}{gR} \quad (14.21)$$

Examination of Eq. (14.21) reveals that the tighter the turn, the greater is the angle of bank required to maintain horizontal flight. Furthermore, we see, from Eq. (14.20), that an increase in bank angle results in an increased load factor. Aerodynamic theory shows that for a limiting value of  $n$ , the minimum time taken to turn through a given angle at a given value of engine thrust occurs when the lift coefficient  $C_L$  is a maximum; that is, with the aircraft on the point of stalling.

### Example 14.6

The wing of a military aircraft has a maximum lift coefficient of 1.25 and an area of  $16 \text{ m}^2$ ; the maximum maneuver load factor is 6.0. If the weight of the aircraft is 50 kN, determine the angle of bank required at a speed of 180 m/s. Calculate also the radius of turn. Take  $\rho = 1.223 \text{ kg/m}^3$ .

The maximum available lift is given by

$$L_{\max} = (1/2)\rho V^2 S C_{L,\max}$$

that is,

$$L_{\max} = (1/2) \times 1.223 \times 180^2 \times 16 \times 1.25 = 396.3 \text{ kN}$$

Then,

$$n = (396.3/50) = 7.9$$

which means that the aircraft exceeds the maximum maneuver load at the point of maximum lift. The maneuver load factor is therefore the critical criterion.

From Eq. (14.20),

$$\sec\phi = 6.0$$

which gives

$$\phi = 80.4^\circ$$

Then, from Eq.(14.21),

$$\tan 80.4^\circ = (180^2/9.81R)$$

So that

$$R = 558.6 \text{ m}$$

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## 14.4 GUST LOADS

In Section 14.2, we considered aircraft loads resulting from prescribed maneuvers in the longitudinal plane of symmetry. Other types of in-flight load are caused by air turbulence. The movements of the air in turbulence are generally known as *gusts* and produce changes in wing incidence, thereby subjecting the aircraft to sudden or gradual increases or decreases in lift from which normal accelerations result.

These may be critical for large, high-speed aircraft and may possibly cause higher loads than control initiated maneuvers.

At the present time two approaches are employed in gust analysis. One method, which has been in use for a considerable number of years, determines the aircraft response and loads due to a single or “discrete” gust of a given profile. This profile is defined as a distribution of vertical gust velocity over a given finite length or given period of time. Examples of these profiles are shown in Fig. 14.13.

Early airworthiness requirements specified an instantaneous application of gust velocity  $u$ , resulting in the “sharp-edged” gust of Fig. 14.13(a). Calculations of normal acceleration and aircraft response were based on the assumptions that the aircraft’s flight is undisturbed while the aircraft passes from still air into the moving air of the gust and during the time taken for the gust loads to build up; that the aerodynamic forces on the aircraft are determined by the instantaneous incidence of the particular lifting surface; and finally that the aircraft’s structure is rigid. The second assumption here relating the aerodynamic force on a lifting surface to its instantaneous incidence neglects the fact that, in a disturbance such as a gust, there is a gradual growth of circulation and hence of lift to a steady-state value (Wagner effect). This in general leads to an overestimation of the upward acceleration of an aircraft and therefore of gust loads.

The sharp-edged gust was replaced when it was realized that the gust velocity built up to a maximum over a period of time. Airworthiness requirements were modified, on the assumption that the gust velocity increased linearly to a maximum value over a specified gust gradient distance  $H$ ; hence, the “graded” gust of Fig. 14.13(b). In the United Kingdom,  $H$  is taken as 30.5 m. Since, as far as the aircraft is concerned, the gust velocity builds up to a maximum over a period of time, it is no longer allowable to ignore the change of flight path as the aircraft enters the gust. By the time the gust has attained its maximum value, the aircraft has developed a vertical component of velocity and, in addition, may be pitching, depending on its longitudinal stability characteristics. The effect of the former is to reduce the

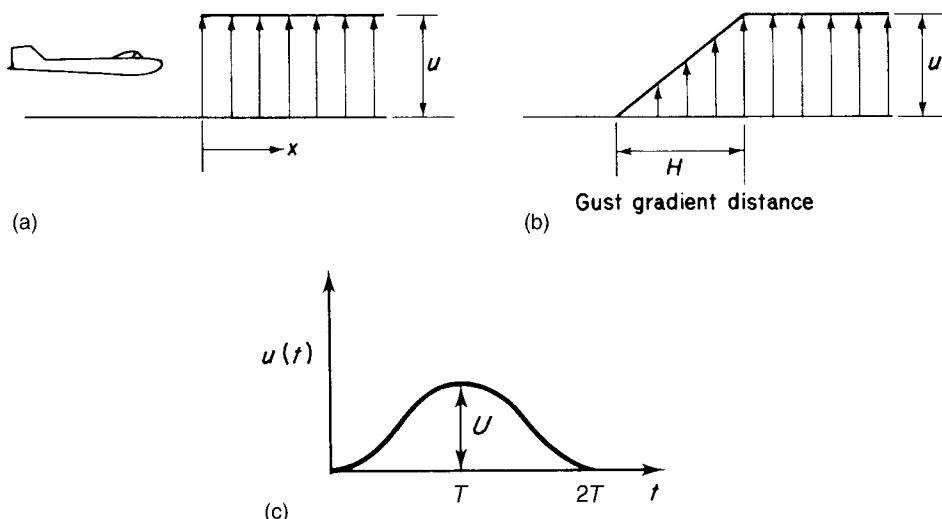


FIGURE 14.13 (a) Sharp-Edged Gust; (b) Graded Gust; (c) 1 – cosine Gust

severity of the gust, while the latter may either increase or decrease the loads involved. To evaluate the corresponding gust loads, the designer may either calculate the complete motion of the aircraft during the disturbance, and hence obtain the gust loads, or replace the graded gust by an equivalent sharp-edged gust producing approximately the same effect. We shall discuss the latter procedure in greater detail later.

The calculation of the complete response of the aircraft to a graded gust may be obtained from its response to a sharp-edged or “step” gust, by treating the former as comprising a large number of small steps and superimposing the responses to each of these. Such a process is known as *convolution* or *Duhamel integration*. This treatment is desirable for large or unorthodox aircraft, where aeroelastic (structural flexibility) effects on gust loads may be appreciable or unknown. In such cases, the assumption of a rigid aircraft may lead to an underestimation of gust loads. The equations of motion are therefore modified to allow for aeroelastic in addition to aerodynamic effects. For small and medium-sized aircraft having orthodox aerodynamic features, the equivalent sharp-edged gust procedure is satisfactory.

While the graded or “ramp” gust is used as a basis for gust load calculations, other shapes of gust profile are in current use. Typical of these is the “1 – cosine” gust of Fig. 14.13(c), where the gust velocity  $u$  is given by  $u(t) = (U/2)[1 - \cos(\pi t/T)]$ . Again the aircraft response is determined by superimposing the responses to each of a large number of small steps.

Although the discrete gust approach still finds widespread use in the calculation of gust loads, alternative methods based on *power spectral* analysis are being investigated. The advantage of the power spectral technique lies in its freedom from arbitrary assumptions of gust shapes and sizes. It is assumed that gust velocity is a random variable that may be regarded for analysis as consisting of a large number of sinusoidal components whose amplitudes vary with frequency. The *power spectrum* of such a function is then defined as the distribution of energy over the frequency range. This may then be related to gust velocity. Establishing appropriate amplitude and frequency distributions for a particular random gust profile requires a large amount of experimental data. The collection of such data was referred to in Section 13.2.

Calculations of the complete response of an aircraft and detailed assessments of the discrete gust and power spectral methods of analysis are outside the scope of this book. More information may be found in references [1–4] at the end of the chapter. Our present analysis is confined to the discrete gust approach, in which we consider the sharp-edged gust and the equivalent sharp-edged gust derived from the graded gust.

#### 14.4.1 Sharp-edged gust

The simplifying assumptions introduced in the determination of gust loads resulting from the sharp-edged gust have been discussed in the earlier part of this section. In Fig. 14.14, the aircraft is flying at a speed  $V$  with wing incidence  $\alpha_0$  in still air. After entering the gust of upward velocity  $u$ , the incidence increases by an amount  $\tan^{-1} u/V$ , or since  $u$  is usually small compared with  $V$ ,  $u/V$ . This is accompanied by an increase in aircraft speed from  $V$  to  $(V^2 + u^2)^{1/2}$ , but again this increase is neglected since  $u$  is small. The increase in wing lift  $\Delta L$  is then given by

$$\Delta L = \frac{1}{2} \rho V^2 S \frac{\partial C_L}{\partial \alpha} \frac{u}{V} = \frac{1}{2} \rho VS \frac{\partial C_L}{\partial \alpha} u \quad (14.22)$$

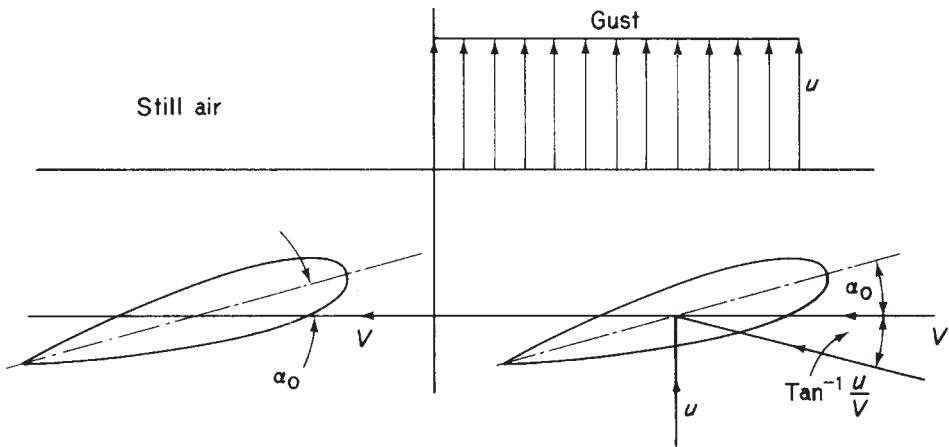


FIGURE 14.14 Increase in Wing Incidence Due to a Sharp-Edged Gust

where  $\partial C_L / \partial \alpha$  is the wing lift-curve slope. Neglecting the change of lift on the tailplane as a first approximation, the gust load factor  $\Delta n$  produced by this change of lift is

$$\Delta n = \frac{\frac{1}{2} \rho V S (\partial C_L / \partial \alpha) u}{W} \quad (14.23)$$

where  $W$  is the aircraft weight. Expressing Eq. (14.23) in terms of the wing loading,  $w = W/S$ , we have

$$\Delta n = \frac{\frac{1}{2} \rho V (\partial C_L / \partial \alpha) u}{w} \quad (14.24)$$

This increment in gust load factor is additional to the steady level flight value  $n = 1$ . Therefore, as a result of the gust, the total gust load factor is

$$n = 1 + \frac{\frac{1}{2} \rho V (\partial C_L / \partial \alpha) u}{w} \quad (14.25)$$

Similarly, for a downgust,

$$n = 1 - \frac{\frac{1}{2} \rho V (\partial C_L / \partial \alpha) u}{w} \quad (14.26)$$

If flight conditions are expressed in terms of equivalent sea-level conditions, then  $V$  becomes the equivalent airspeed (EAS),  $V_E$ ,  $u$  becomes  $u_E$ , and the air density  $\rho$  is replaced by the sea-level value  $\rho_0$ . Equations (14.25) and (14.26) are written

$$n = 1 + \frac{\frac{1}{2} \rho_0 V_E (\partial C_L / \partial \alpha) u_E}{w} \quad (14.27)$$

and

$$n = 1 - \frac{\frac{1}{2} \rho_0 V_E (\partial C_L / \partial \alpha) u_E}{w} \quad (14.28)$$

We observe from Eqs. (14.25)–(14.28) that the gust load factor is directly proportional to aircraft speed but inversely proportional to wing loading. It follows that high-speed aircraft with low or moderate wing loadings are most likely to be affected by gust loads.

The contribution to normal acceleration of the change in tail load produced by the gust may be calculated using the same assumptions as before. However, the change in tailplane incidence is not equal to the change in wing incidence, due to downwash effects at the tail. Thus, if  $\Delta P$  is the increase (or decrease) in tailplane load, then

$$\Delta P = \frac{1}{2} \rho_0 V_E^2 S_T \Delta C_{L,T} \quad (14.29)$$

where  $S_T$  is the tailplane area and  $\Delta C_{L,T}$  the increment of tailplane lift coefficient given by

$$\Delta C_{L,T} = \frac{\partial C_{L,T}}{\partial \alpha} \frac{u_E}{V_E} \quad (14.30)$$

in which  $\partial C_{L,T}/\partial \alpha$  is the rate of change of tailplane lift coefficient with wing incidence. From aerodynamic theory,

$$\frac{\partial C_{L,T}}{\partial \alpha} = \frac{\partial C_{L,T}}{\partial \alpha_T} \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right)$$

where  $\partial C_{L,T}/\partial \alpha_T$  is the rate of change of  $C_{L,T}$  with tailplane incidence and  $\partial \epsilon/\partial \alpha$  the rate of change of downwash angle with wing incidence. Substituting for  $\Delta C_{L,T}$  from Eq. (14.30) into Eq. (14.29), we have

$$\Delta P = \frac{1}{2} \rho_0 V_E S_T \frac{\partial C_{L,T}}{\partial \alpha} u_E \quad (14.31)$$

For positive increments of wing lift and tailplane load,

$$\Delta nW = \Delta L + \Delta P$$

or, from Eqs. (14.27) and (14.31),

$$\Delta n = \frac{\frac{1}{2} \rho_0 V_E (\partial C_L / \partial \alpha) u_E}{w} \left( 1 + \frac{S_T}{S} \frac{\partial C_{L,T} / \partial \alpha}{\partial C_L / \partial \alpha} \right) \quad (14.32)$$

#### 14.4.2 The graded gust

The graded gust of Fig. 14.13(b) may be converted to an equivalent sharp-edged gust by multiplying the maximum velocity in the gust by a *gust alleviation factor*,  $F$ . Equation (14.27) then becomes

$$n = 1 + \frac{\frac{1}{2} \rho_0 V_E (\partial C_L / \partial \alpha) F u_E}{w} \quad (14.33)$$

Similar modifications are carried out on Eqs. (14.25), (14.26), (14.28), and (14.32). The gust alleviation factor allows for some of the dynamic properties of the aircraft, including unsteady lift, and has been calculated taking into account the heaving motion (i.e., the up and down motion with zero rate of pitch) of the aircraft only.<sup>5</sup>

Horizontal gusts cause lateral loads on the vertical tail or fin. Their magnitudes may be calculated in an identical manner to the preceding, except that areas and values of lift–curve slope are referred to the vertical tail. Also, the gust alleviation factor in the graded gust case becomes  $F_1$  and includes allowances for the aerodynamic yawing moment produced by the gust and the yawing inertia of the aircraft.

### 14.4.3 Gust envelope

Airworthiness requirements usually specify that gust loads shall be calculated at certain combinations of gust and flight speed. The equations for gust load factor in the above analysis show that  $n$  is proportional to aircraft speed for a given gust velocity. Therefore, we plot a gust envelope similar to the flight envelope of Fig. 13.1, as shown in Fig. 14.15. The gust speeds  $\pm U_1$ ,  $\pm U_2$ , and  $\pm U_3$  are high-, medium- and low-velocity gusts, respectively. Cutoffs occur at points where the lines corresponding to each gust velocity meet specific aircraft speeds. For example, A and F denote speeds at which a gust of velocity  $\pm U_1$  would stall the wing.

The lift coefficient–incidence curve is, as we noted in connection with the flight envelope, affected by compressibility and therefore altitude, so that a series of gust envelopes should be drawn for different altitudes. An additional variable in the equations for gust load factor is the wing loading  $w$ . Further gust envelopes should therefore be drawn to represent different conditions of aircraft loading.

Typical values of  $U_1$ ,  $U_2$ , and  $U_3$  are 20 m/s, 15.25 m/s, and 7.5 m/s, respectively. It can be seen from the gust envelope that the maximum gust load factor occurs at the cruising speed  $V_C$ . If this value of  $n$  exceeds that for the corresponding flight envelope case, that is  $n_1$ , then the gust case is the most critical in the cruise.

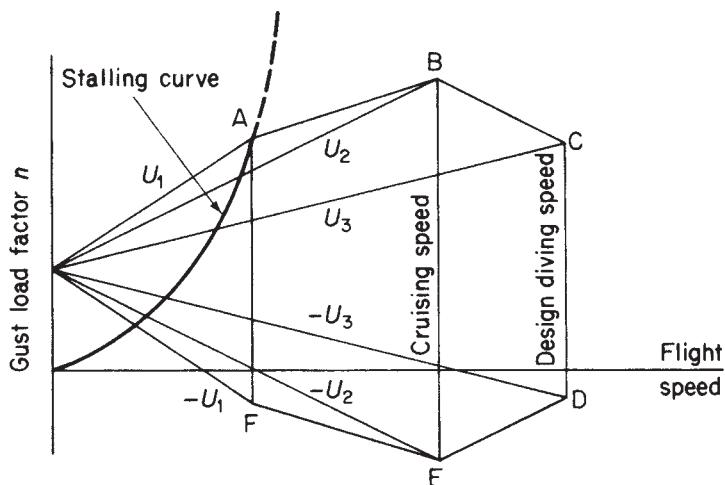


FIGURE 14.15 Typical Gust Envelope

**Example 14.7**

A civil, non-aerobatic aircraft has a wing loading of  $2,400 \text{ N/m}^2$  and  $\partial C_L/\partial\alpha = 5.0/\text{rad}$ . If  $n_1 = 2.5$  and  $F = 0.715$ , calculate the cruising speed for the gust case to be critical.

From Eq. (14.33),

$$n = 1 + \frac{\frac{1}{2} \times 1.223 V_C \times 5.0 \times 0.715 \times 15.25}{2,400}$$

giving  $n = 1 + 0.0139V_C$ , where the cruising speed  $V_C$  is expressed as an EAS. For the gust case to be critical,

$$1 + 0.0139V_C > 2.5$$

or

$$V_C > 108 \text{ m/s}$$

Thus, for civil aircraft of this type having cruising speeds in excess of 108 m/s, the gust case is the most critical. This, in fact, applies to most modern civil airliners.

Although the same combination of  $V$  and  $n$  in the flight and gust envelopes produce the same total lift on an aircraft, the individual wing and tailplane loads are different, as shown previously (see the derivation of Eq. (14.33)). This situation can be important for aircraft such as the Airbus, which has a large tailplane and a CG forward of the aerodynamic center. In the flight envelope case, the tail load is downward, whereas in the gust case, it is upward; clearly, there will be a significant difference in wing load.

The transference of maneuver and gust loads into bending, shear, and torsional loads on wings, fuselage, and tailplanes has been discussed in Section 12.1. Further loads arise from aileron application, in undercarriages during landing, on engine mountings, and during crash landings. Analysis and discussion of these may be found in the *Handbook of Aeronautics*.<sup>6</sup>

**Example 14.8**

An aircraft of all-up weight 150,000 N has wings of area  $60 \text{ m}^2$  and mean chord 2.6 m. The drag coefficient for the complete aircraft is given by  $C_D = 0.02 + 0.04C_L^2$  and the lift–curve slope of the wings is 4.5. The tailplane has an area of  $10 \text{ m}^2$  and a lift–curve slope of 2.2, allowing for the effects of downwash. The pitching moment coefficient about the aerodynamic center (of the complete aircraft less tailplane) based on wing area is  $-0.03$ . Geometric data are given in Fig. 14.16.

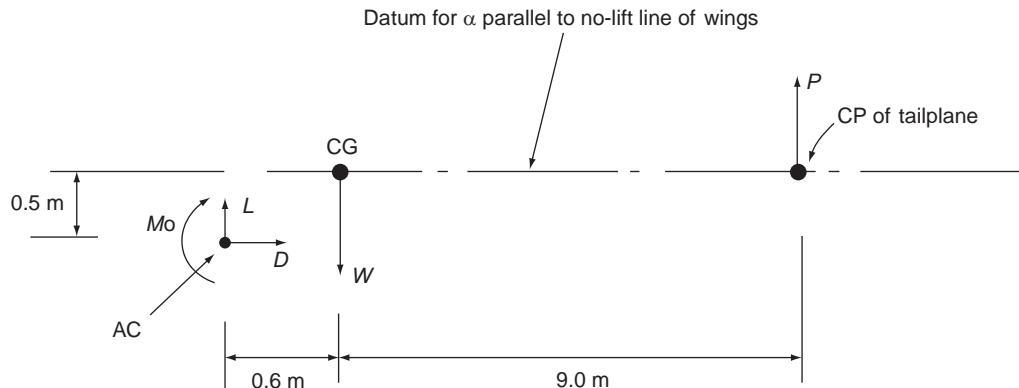
During a steady glide with zero thrust at 245 m/s EAS in which  $C_L = 0.09$ , the aircraft meets an upgust of equivalent sharp-edged speed 5 m/s. Calculate the tail load, the gust load factor, and the forward inertia force. Take  $\rho_0 = 1.223 \text{ kg/m}^3$ .

As a first approximation, suppose that the wing lift is equal to the aircraft weight. Then,

$$L = (1/2)\rho V^2 S (\partial C_L / \partial \alpha) \alpha_w = 150,000 \text{ N}$$

so that

$$\alpha_w = 150,000 / [(1/2) \times 1.223 \times 245^2 \times 60 \times 4.6] = 0.015 \text{ rad} = 0.87^\circ$$



**FIGURE 14.16 Geometric Data for Aircraft of Example 14.8**

Also,

$$C_D = 0.02 + 0.04 \times 0.09^2 = 0.02$$

Taking moments about the CG and noting that  $\cos 0.87^\circ = 1$  approximately,

$$0.6L - 0.5D + M_o = 9.0P$$

that is,

$$0.6(150,000 - P) - 0.5 \times (1/2) \rho V^2 S C_D + (1/2) \rho V^2 c C_{M,o} = 9.0P$$

Substituting the appropriate values gives

$$P = -10,813 \text{ N}$$

Then,

$$L = W - P = 150,000 + 10,813 = 160,813 \text{ N}$$

The change,  $\Delta P$ , in the tailplane load due to the gust is, from Eqs. (14.29) and (14.30),

$$\Delta P = (1/2) \rho V^2 S_T (\partial L_{T,T} / \partial \alpha) (u_E / V_E)$$

so that

$$\Delta P = (1/2) \times 1.223 \times 245^2 \times 10 \times 2.2 \times (5/245)$$

that is,

$$\Delta P = 16,480 \text{ N}$$

The total load on the tail is then  $16,480 - 10,813 = 5,667 \text{ N}$  (upward).

The increase in wing lift,  $\Delta L$ , due to the gust is given by

$$\Delta L = (1/2) \rho V^2 S (\partial C_L / \partial \alpha) (u_E / V_E)$$

Substituting the appropriate values gives

$$\Delta L = 202,254 \text{ N}$$

Then,

$$n = 1 + [(202,254 + 16,480)/150,000] = 2.46$$

The forward inertia force,  $FW$ , is given by

$$FW = D = (1/2)\rho V^2 SC_D = (1/2) \times 1.223 \times 245^2 \times 60 \times 0.02$$

that is,

$$FW = 44,046 \text{ N}$$



## 14.5 DESIGN LOADS: SPACECRAFT

The loads that govern the design of spacecraft depend upon a variety of conditions of which those produced during the launch are generally the most severe as far as the basic structure is concerned. These loads, caused by the acceleration of the launch vehicle, reach a maximum value at the end of the propulsion phase of the rocket stage and can be in the range 4–8 G depending on the launch vehicle. The acceleration increases during the propulsion phase because the mass of the launch vehicle decreases as fuel is burned while the overall thrust remains the same.

In addition to the steady-state loads, significant vibrations are generated in the structure during the launch so that the natural, or resonant, frequency of the structure must be determined. These vibrations, which are of low frequency (5–100 Hz) and are sinusoidal in form, are excited by the shock load caused by the rapid buildup of thrust and also by the combustion of the engines. Also higher-frequency (20–2000 Hz) random vibrations can be generated by unsteady combustion in the propulsion system, by the turbulent airflow along the rocket, and by noise from the exhaust.

Other loads that must be accounted for include handling and transportation loads (unless the spacecraft is specially protected), re-entry loads, shock loads caused by separation of the stages and by separation of the spacecraft from the launch vehicle, acoustic loads, temperature gradients, maneuver loads, pressure variations, etc.

To validate the spacecraft design, an exhaustive test program is carried out. This includes static load tests, shock load tests, and vibration tests. A full review of such a test program may be found in Wijker.<sup>7</sup>

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- [4] Babister AW. Aircraft stability and control. London: Pergamon Press; 1961.
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- [6] Handbook of Aeronautics, No. 1, Structural principles and data. 4th ed. The Royal Aeronautical Society; 1952.
- [7] Wijker JJ. Spacecraft structures. Berlin: Springer Verlag; 2010.

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## PROBLEMS

**P.14.1** The aircraft shown in [Fig. P.14.1\(a\)](#) weighs 135 kN and has landed such that, at the instant of impact, the ground reaction on each main undercarriage wheel is 200 kN and its vertical velocity is 3.5 m/s. If each undercarriage wheel weighs 2.25 kN and is attached to an oleo strut, as shown in [Fig. P.14.1\(b\)](#), calculate the axial load and bending moment in the strut; the strut may be assumed to be vertical. Determine also the shortening of the strut when the vertical velocity of the aircraft is zero. Finally, calculate the shear force and bending moment in the wing at the section AA if the wing, outboard of this section, weighs 6.6 kN and has its CG 3.05 m from AA.

Answer: 193.3 kN, 29.0 kNm (clockwise); 0.32 m; 19.5 kN, 59.6 kNm (counterclockwise)

**P.14.1 MATLAB** Use MATLAB to repeat Problem P.14.1 for the following aircraft weight values:

	(i) W 125 kN	(ii) 130 kN	(iii) 135 kN	(iv) 140 kN	(v) 145 kN
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- Answer: (i) 192.8 kN, 28.9 kNm (clockwise); 0.28 m; 21.1 kN, 64.4 kNm (counterclockwise)  
(ii) 193.1 kN, 29 kNm (clockwise); 0.3 m; 20.3 kN, 62 kNm (counterclockwise)  
(iii) 193.3 kN, 29 kNm (clockwise); 0.32 m; 19.5 kN, 59.6 kNm (counterclockwise)  
(iv) 193.6 kN, 29 kNm (clockwise); 0.34 m; 18.9 kN, 57.6 kNm (counterclockwise)  
(v) 193.8 kN, 29.1 kNm (clockwise); 0.35 m; 18.2 kN, 55.6 kNm (counterclockwise)

**P.14.2** Determine, for the aircraft of Example 14.2, the vertical velocity of the nosewheel when it hits the ground.

Answer: 3.1 m/s

**P.14.3** [Figure P.14.3](#) shows the plan view of an aircraft with wing-tip drop tanks, each of which, when full, weighs 15,575 N and has a radius of gyration in the yaw plane of 0.6 m about its own center of gravity. The aircraft is in steady level flight at 260 m/s and has a total weight of 169,000 N with tip tanks full. The radius of gyration in yaw of the aircraft without its tip tanks is 2.5 m, and the drag coefficient of the complete configuration is 0.02 (based on a wing area of  $60 \text{ m}^2$ ), of which 8 percent

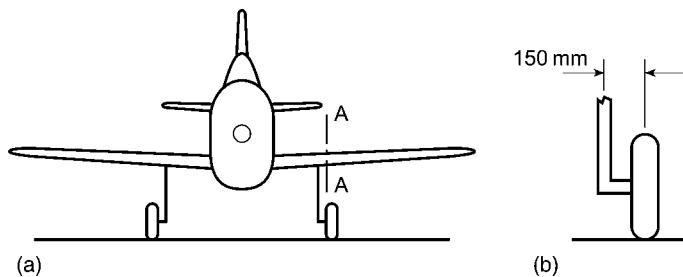


FIGURE P.14.1

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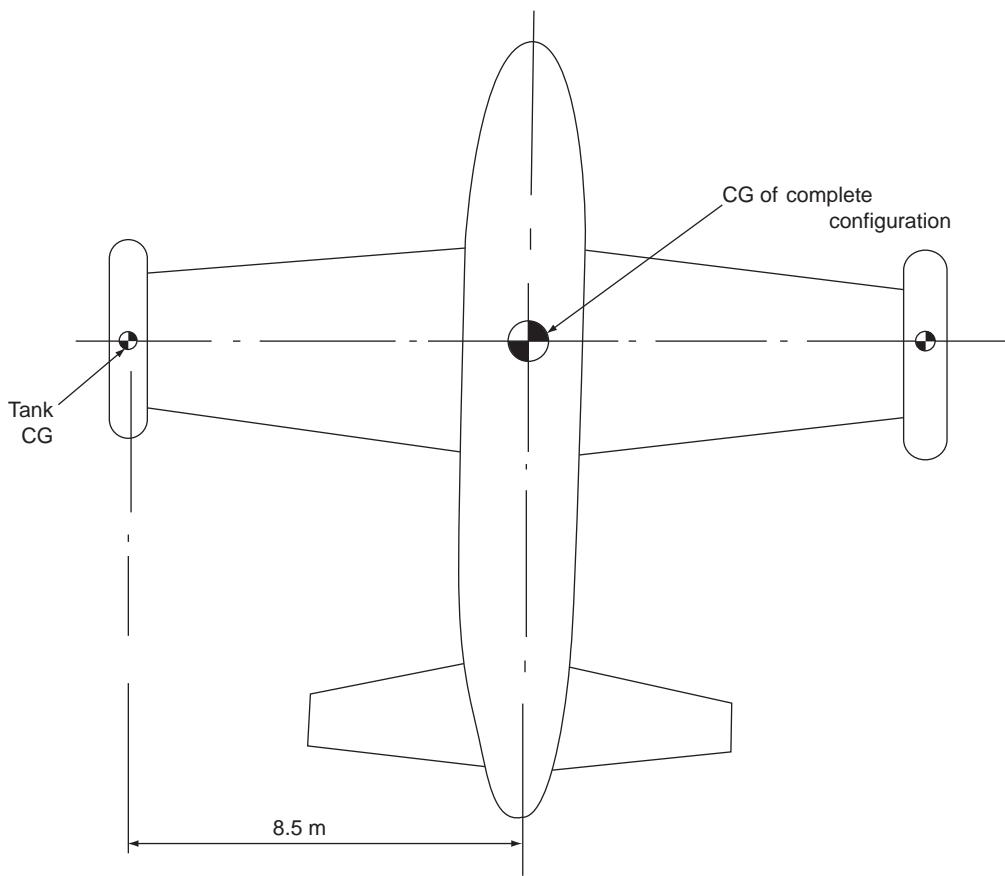


FIGURE P.14.3

is due to each tip tank. If, at the instant of release, only one tank is dropped, calculate the total fore-and-aft loading on the mounting of the remaining tip tank. Take  $\rho = 1.223 \text{ kg/m}^3$ .

Answer: 2,312 N

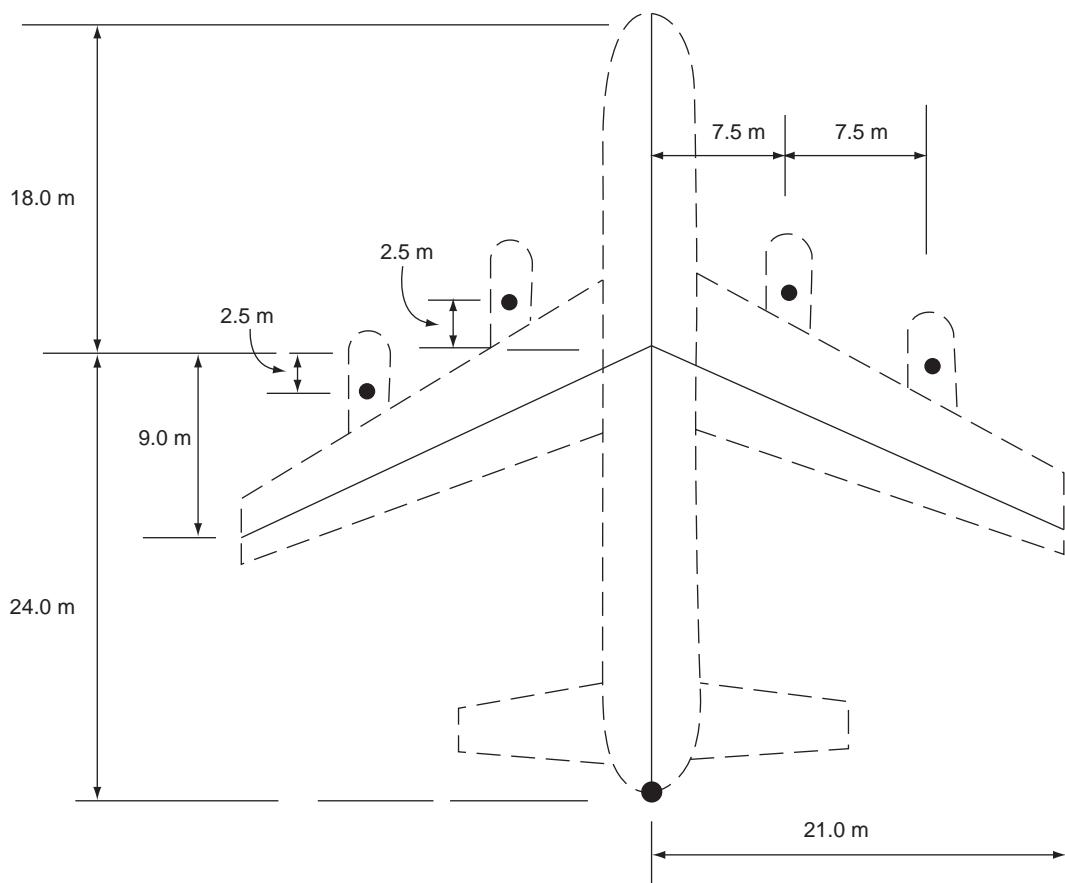
**P.14.4** Figure P.14.4 shows the idealized plan view of a four-engine jet aircraft. The weight distribution of the wing plus fuel is assumed to be uniform and to act along the solid lines shown. Similarly, the fuselage plus payload weight distribution is also assumed to be uniform and to act along the fuselage center line. The weights of the engines and tail unit are assumed to be concentrated at the positions shown. If each engine delivers 66,750 N of thrust in a certain flight condition, determine the instantaneous yawing acceleration if two engines on one side instantaneously lose thrust.

The fraction of the all-up weight,  $W$ , contributed by the major components of the aircraft are

Wings + fuel: 0.564, Fuselage + payload: 0.310,

Each engine: 0.027, Tail unit: 0.018

Answer:  $17,182/W \text{ rad/s}^2$



**FIGURE P.14.4**

**P.14.5** Figure P.14.5 shows the flight envelope at sea level for an aircraft of wing span 27.5 m, average wing chord 3.05 m, and total weight 196,000 N. The aerodynamic center is 0.915 m forward of the CG, and the center of lift for the tail unit is 16.7 m aft of the CG. The pitching moment coefficient is

$$C_{M,0} = -0.0638 \text{ (nose-up positive)}$$

both  $C_{M,0}$  and the position of the aerodynamic center are specified for the complete aircraft less tail unit. For steady cruising flight at sea level, the fuselage bending moment at the CG is 600,000 Nm. Calculate the maximum value of this bending moment for the given flight envelope. For this purpose, it may be assumed that the aerodynamic loadings on the fuselage itself can be neglected, that is, the only loads on the fuselage structure aft of the CG are those due to the tail lift and the inertia of the fuselage.

Answer: 1,549,500 Nm at  $n = 3.5, V = 152.5\text{m/s}$

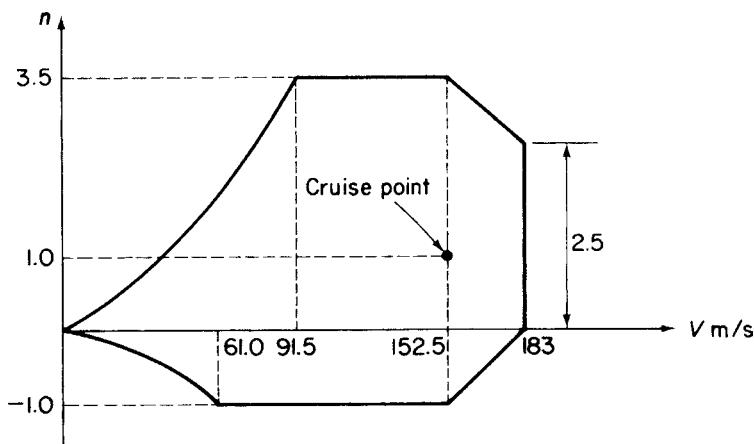


FIGURE P.14.5

**P.14.6** An aircraft weighing 238,000 N has wings  $88.5 \text{ m}^2$  in area, for which  $C_D = 0.0075 + 0.045C_L^2$ . The extra-to-wing drag coefficient based on wing area is 0.0128, and the pitching moment coefficient for all parts excluding the tailplane about an axis through the CG is given by  $C_M \cdot c = (0.427C_L - 0.061) \text{ m}$ . The radius from the CG to the line of action of the tail lift may be taken as constant at 12.2 m. The moment of inertia of the aircraft for pitching is  $204,000 \text{ kg m}^2$ . During a pullout from a dive with zero thrust at 215 m/s EAS when the flight path is at  $40^\circ$  to the horizontal with a radius of curvature of 1525 m, the angular velocity of pitch is checked by applying a retardation of  $0.25 \text{ rad/s}^2$ . Calculate the maneuver load factor both at the CG and at the tailplane CP, the forward inertia coefficient, and the tail lift.

Answer:  $n = 3.78(\text{CG})$ ,  $n = 5.19$  at TP,  $f = -0.370$ ,  $P = 18,925 \text{ N}$

**P.14.7** An aircraft flies at sea level in a correctly banked turn of radius 610 m at a speed of 168 m/s. Figure P.14.7 shows the relative positions of the CG, aerodynamic center of the complete aircraft less tailplane, and the tailplane center of pressure for the aircraft at zero lift incidence. Calculate the tail load necessary for equilibrium in the turn. The necessary data are given in the usual notation as follows:

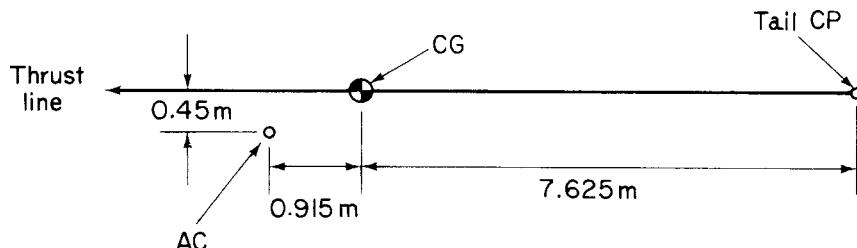


FIGURE P.14.7

$$\begin{aligned} \text{Weight } W &= 133,500 \text{ N}, & dC_L/d\alpha &= 4.5/\text{rad}, \\ \text{Wing area } S &= 46.5 \text{ m}^2, & C_D &= 0.01 + 0.05C_L^2, \\ \text{Wing mean chord } \bar{c} &= 3.0 \text{ m}, & C_{M,0} &= -0.03 \end{aligned}$$

Answer: 73,160 N

**P.14.8** The aircraft for which the stalling speed  $V_s$  in level flight is 46.5 m/s has a maximum allowable maneuver load factor  $n_1$  of 4.0. In assessing gyroscopic effects on the engine mounting, the following two cases are to be considered:

- (a) Pullout at maximum permissible rate from a dive in symmetric flight, the angle of the flight path to the horizontal being limited to  $60^\circ$  for this aircraft.
  - (b) Steady, correctly banked turn at the maximum permissible rate in horizontal flight.
- Find the corresponding maximum angular velocities in yaw and pitch.

Answer: (a) Pitch, 0.37 rad/s, (b) Pitch, 0.41 rad/s, Yaw, 0.103 rad/s

**P.14.9** A tail-first supersonic airliner, whose essential geometry is shown in Fig. P.14.9, flies at 610 m/s true airspeed at an altitude of 18,300 m. Assuming that thrust and drag forces act in the same straight line, calculate the tail lift in steady straight and level flight.

If, at the same altitude, the aircraft encounters a sharp-edged vertical upgust of 18 m/s true airspeed, calculate the changes in the lift and tail load and also the resultant load factor  $n$ .

The relevant data in the usual notation are as follows:

$$\begin{aligned} \text{Wing: } S &= 280 \text{ m}^2, & \partial C_L / \partial \alpha &= 1.5, \\ \text{Tail: } S_T &= 28 \text{ m}^2, & \partial C_{L,T} / \partial \alpha &= 2.0, \\ \text{Weight } W &= 1,600,000 \text{ N}, \\ C_{M,0} &= -0.01, \\ \text{Mean chord } \bar{c} &= 22.8 \text{ m} \end{aligned}$$

At 18,300 m,  $\rho = 0.116 \text{ kg/m}^3$

Answer:  $P = 267,852 \text{ N}$ ,  $\Delta P = 36,257 \text{ N}$ ,  $\Delta L = 271,931 \text{ N}$ ,  $n = 1.19$

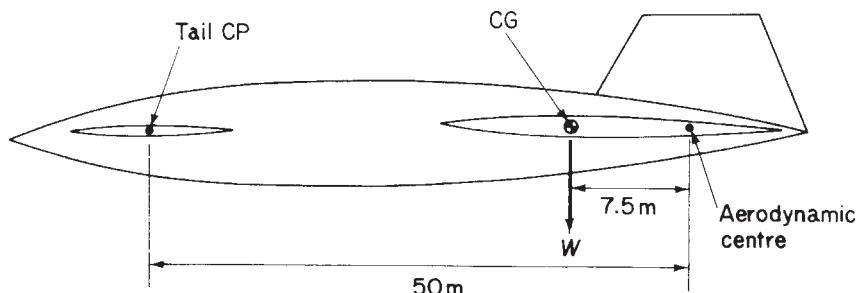


FIGURE P.14.9

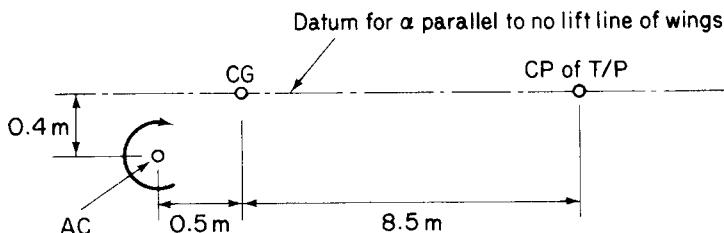


FIGURE P.14.10

**P.14.10** An aircraft of all-up weight 145,000 N has wings of area  $50 \text{ m}^2$  and mean chord 2.5 m. For the whole aircraft,  $C_D = 0.021 + 0.041 C_L^2$ , for the wings  $dC_L/d\alpha = 4.8$ , for the tailplane of area  $9.0 \text{ m}^2$ ,  $dC_{L,T}/d\alpha = 2.2$  allowing for the effects of downwash, and the pitching moment coefficient about the aerodynamic center (of complete aircraft less tailplane) based on wing area is  $C_{M,0} = -0.032$ . Geometric data are given in Fig. P.14.10. During a steady glide with zero thrust at 250 m/s EAS in which  $C_L = 0.08$ , the aircraft meets a downgust of equivalent sharp-edged speed 6 m/s. Calculate the tail load, the gust load factor, and the forward inertia force,  $\rho_0 = 1.223 \text{ kg/m}^3$ .

Answer:  $P = -28,902 \text{ N}$  (down),  $n = -0.64$ , forward inertia force = 40,703 N

**P.14.11** A launch vehicle is subjected to an average acceleration of  $4.5 \text{ g}$  for a period of 5 minutes after launch. Determine its velocity and height after this period.

Answer: 13,243.5 m/s,  $1.99 \times 106 \text{ m}$ .

# Fatigue

# 15

Fatigue was discussed briefly in [Section 11.7](#), when we examined the properties of materials, and also in [Section 13.2](#), as part of the chapter on airworthiness. We now look at fatigue in greater detail and consider factors affecting the life of an aircraft and spacecraft, including safe-life and fail-safe structures, designing against fatigue, the fatigue strength of components, the prediction of aircraft fatigue life, and crack propagation.

*Fatigue* is defined as the progressive deterioration of the strength of a material or structural component during service, such that failure can occur at much lower stress levels than the ultimate stress level. As we have seen, fatigue is a dynamic phenomenon that initiates small (micro) cracks in the material or component and causes them to grow into large (macro) cracks; these, if not detected, can result in catastrophic failure.

Fatigue damage can be produced in a variety of ways. *Cyclic fatigue* is caused by repeated fluctuating loads. *Corrosion fatigue* is fatigue accelerated by surface corrosion of the material penetrating inward so that the material strength deteriorates. Small-scale rubbing movements and abrasion of adjacent parts cause *fretting fatigue*, while *thermal fatigue* is produced by stress fluctuations induced by thermal expansions and contractions; the last does not include the effect on material strength of heat. Finally, high-frequency stress fluctuations, due to vibrations excited by jet or propeller noise or the propulsion system of a spacecraft, cause *sonic* or *acoustic fatigue*.

In this chapter, we shall concentrate on the aeronautical aspects of the fatigue problem although some of the topics such as crack propagation, for example, apply equally to spacecraft structures. Clearly an aircraft's structure must be designed so that fatigue does not become a problem. For aircraft in general, the requirement that the strength of an aircraft throughout its operational life shall be such as to ensure that the possibility of a disastrous fatigue failure shall be extremely remote (i.e., the probability of failure is less than  $10^{-7}$ ) under the action of the repeated loads of variable magnitude expected in service. Also, it is required that the principal parts of the primary structure of the aircraft be subjected to a detailed analysis and to load tests that demonstrate a *safe life* or that the parts of the primary structure have *fail-safe* characteristics. These requirements do not apply to light aircraft, provided that zinc-rich aluminum alloys are not used in their construction and that wing stress levels are kept low, that is, provided that a 3.05 m/s upgust causes no greater stress than  $14 \text{ N/mm}^2$ .

## 15.1 SAFE LIFE AND FAIL-SAFE STRUCTURES

The danger of a catastrophic fatigue failure in the structure of an aircraft may be eliminated completely or may become extremely remote if the structure is designed to have a safe life or to be fail safe. In the former approach, the structure is designed to have a minimum life during which it is known that no catastrophic damage will occur. At the end of this life, the structure must be replaced even though there

may be no detectable signs of fatigue. If a structural component is not economically replaceable when its safe life has been reached, the complete structure must be written off. Alternatively, it is possible for easily replaceable components, such as undercarriage legs and mechanisms, to have a safe life less than that of the complete aircraft, since it would probably be more economical to use, say, two lightweight undercarriage systems during the life of the aircraft rather than carry a heavier undercarriage which has the same safe life as the aircraft.

The fail-safe approach relies on the fact that the failure of a member in a redundant structure does not necessarily lead to the collapse of the complete structure, provided the remaining members are able to carry the load shed by the failed member and can withstand further repeated loads until the presence of the failed member is discovered. Such a structure is called a *fail-safe structure* or a *damage tolerant structure*.

Generally, it is more economical to design some parts of the structure to be fail-safe rather than to have a long safe life, since such components can be lighter. When failure is detected, either through a routine inspection or by some malfunction, such as fuel leakage from a wing crack, the particular aircraft may be taken out of service and repaired. However, the structure must be designed and the inspection intervals arranged such that a failure – for example, a crack – too small to be noticed at one inspection must not increase to a catastrophic size before the next. The determination of crack propagation rates is discussed later.

Some components must be designed to have a safe life; these include landing gear, major wing joints, wing–fuselage joints, and hinges on all-moving tailplanes or on variable geometry wings. Components that may be designed to be fail safe include wing skins stiffened by stringers and fuselage skins stiffened by frames and stringers; the stringers and frames prevent skin cracks spreading disastrously for a sufficient period of time for them to be discovered at a routine inspection.

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## 15.2 DESIGNING AGAINST FATIGUE

Various precautions may be taken to ensure that an aircraft has an adequate fatigue life. We saw, in [Chapter 11](#), that the early aluminum–zinc alloys possessed high ultimate and proof stresses but were susceptible to early failure under fatigue loading; choice of materials is therefore important. The naturally aged aluminum–copper alloys possess good fatigue resistance but with lower static strengths. Modern research is concentrating on alloys that combine high strength with high fatigue resistance.

Attention to detail design is equally important. Stress concentrations can arise at sharp corners and abrupt changes in section. Fillets should therefore be provided at re-entrant corners, and cutouts, such as windows and access panels, should be reinforced. In machined panels, the material thickness should be increased around bolt holes, while holes in primary bolted joints should be reamed to improve surface finish; surface scratches and machine marks are sources of fatigue crack initiation. Joggles in highly stressed members should be avoided, while asymmetry can cause additional stresses due to bending.

In addition to sound structural and detail design, an estimation of the number, frequency, and magnitude of the fluctuating loads an aircraft encounters is necessary. The *fatigue load spectrum* begins when the aircraft taxis to its takeoff position. During taxiing, the aircraft may be maneuvering over uneven ground with a full payload, so that wing stresses, for example, are greater than in the static case. Also, during takeoff and climb and descent and landing, the aircraft is subjected to the greatest load fluctuations. The undercarriage is retracted and lowered; flaps are raised and lowered; there is the impact on landing; the aircraft has to carry out maneuvers; and, finally, the aircraft, as we shall see, experiences a greater number of gusts than during the cruise.

The loads corresponding to these various phases must be calculated before the associated stresses can be obtained. For example, during takeoff, wing bending stresses and shear stresses due to shear and torsion are based on the total weight of the aircraft, including full fuel tanks, and maximum payload all factored by 1.2 to allow for a bump during each takeoff on a hard runway or by 1.5 for a takeoff from grass. The loads produced during level flight and symmetric maneuvers are calculated using the methods described in [Section 14.2](#). From these values, distributions of shear force, bending moment, and torque may be found in, say, the wing by integrating the lift distribution. Loads due to gusts are calculated using the methods described in [Section 14.4](#). Thus, due to a single equivalent sharp-edged gust, the load factor is given by either [Eq. \(14.25\)](#) or [\(14.26\)](#).

Although it is a relatively simple matter to determine the number of load fluctuations during a ground-air-ground cycle caused by standard operations, such as raising and lowering flaps or retracting and lowering the undercarriage, it is more difficult to estimate the number and magnitude of gusts an aircraft will encounter. For example, there is a greater number of gusts at low altitude (during take-off, climb, and descent) than at high altitude (during cruise). Terrain (sea, flat land, mountains) also affects the number and magnitude of gusts, as does weather. The use of radar enables aircraft to avoid cumulus, where gusts are prevalent, but has little effect at low altitude in the climb and descent, where clouds cannot easily be avoided. The ESDU (Engineering Sciences Data Unit) produced gust data based on information collected by gust recorders carried by aircraft. These show, in graphical form ( $l_{10}$  versus  $h$  curves,  $h$  is altitude), the average distance flown at various altitudes for a gust having a velocity greater than  $\pm 3.05$  m/s to be encountered. In addition, *gust frequency curves* give the number of gusts of a given velocity per 1000 gusts of velocity 3.05 m/s. Combining both sets of data enables the *gust exceedance* to be calculated, that is, the number of gust cycles having a velocity greater than or equal to a given velocity encountered per kilometre of flight.

Since an aircraft is subjected to the greatest number of load fluctuations during taxi-take-off-climb and descent-standoff-landing while little damage is caused during cruise, the fatigue life of an aircraft does not depend on the number of flying hours but on the number of flights. However, the operational requirements of aircraft differ from class to class. The Airbus is required to have a life free from fatigue cracks of 24,000 flights or 30,000 hours, while its economic repair life is 48,000 flights or 60,000 hours; its landing gear, however, is designed for a safe life of 32,000 flights, after which it must be replaced. On the other hand, the BAe 146, with a greater number of shorter flights per day than the Airbus, has a specified crack-free life of 40,000 flights and an economic repair life of 80,000 flights. Although these figures are operational requirements, the nature of fatigue is such that it is unlikely that all of a given type of aircraft satisfies them. Of the total number of Airbus aircraft, at least 90 percent achieve these values and 50 percent are better; clearly, frequent inspections are necessary during an aircraft's life.

## 15.3 FATIGUE STRENGTH OF COMPONENTS

In [Section 13.2.4](#), we discussed the effect of stress level on the number of cycles to failure of a material such as mild steel. As the stress level is decreased, the number of cycles to failure increases, resulting in a fatigue endurance curve (the  $S-N$  curve) of the type shown in [Fig. 13.2](#). Such a curve corresponds to the average value of  $N$  at each stress amplitude, since the given stress has a wide range of values of  $N$ ; even under carefully controlled conditions, the ratio of maximum  $N$  to minimum  $N$  may be as high as 10:1. Two other curves may therefore be drawn, as shown in [Fig. 15.1](#), enveloping all or nearly all the

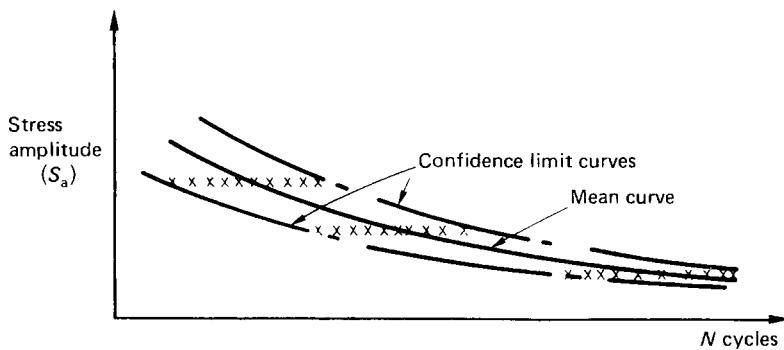


FIGURE 15.1 S-N Diagram

experimental results; these curves are known as the *confidence limits*. If 99.9 percent of all the results lie between the curves – that is, only 1 in 1000 falls outside – they represent the 99.9 percent confidence limits. If 99.9999 percent of results lie between the curves, only 1 in  $10^7$  results falls outside them and they represent the 99.9999 percent confidence limits.

The results from tests on a number of specimens may be represented as a histogram, in which the number of specimens failing within certain ranges  $R$  of  $N$  is plotted against  $N$ . Then, if  $N_{av}$  is the average value of  $N$  at a given stress amplitude, the probability of failure occurring at  $N$  cycles is given by

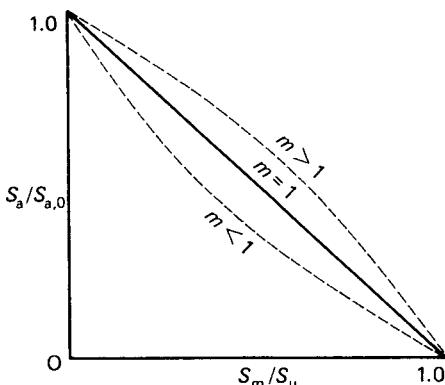
$$p(N) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{N - N_{av}}{\sigma}\right)^2\right] \quad (15.1)$$

in which  $\sigma$  is the standard deviation of the whole population of  $N$  values. The derivation of Eq. (15.1) depends on the histogram approaching the profile of a continuous function close to the *normal distribution*, which it does as the interval  $N_{av}/R$  becomes smaller and the number of tests increases. The *cumulative probability*, which gives the probability that a particular specimen fails at or below  $N$  cycles, is defined as

$$P(N) = \int_{-\infty}^N p(N) dN \quad (15.2)$$

The probability that a specimen endures more than  $N$  cycles is then  $1 - P(N)$ . The normal distribution allows negative values of  $N$ , which is clearly impossible in a fatigue testing situation. Other distributions, *extreme value distributions*, are more realistic and allow the existence of minimum fatigue endurance and fatigue limits.

The damaging portion of a fluctuating load cycle occurs when the stress is tensile; this causes cracks to open and grow. Therefore, if a steady tensile stress is superimposed on a cyclic stress, the maximum tensile stress during the cycle increases and the number of cycles to failure decreases. Conversely, if the steady stress is compressive, the maximum tensile stress decreases and the number of cycles to failure increases. An approximate method of assessing the effect of a steady mean value of stress is provided by a Goodman diagram, shown in Fig. 15.2. This shows the cyclic stress amplitudes that can be superimposed upon different mean stress levels to give a constant fatigue life. In Fig. 15.2,  $S_a$  is the



**FIGURE 15.2 Goodman Diagram**

allowable stress amplitude,  $S_{a,0}$  is the stress amplitude required to produce fatigue failure at  $N$  cycles with zero mean stress,  $S_m$  is the mean stress, and  $S_u$  the ultimate tensile stress. If  $S_m = S_u$ , any cyclic stress causes failure, while if  $S_m = 0$ , the allowable stress amplitude is  $S_{a,0}$ . The equation of the straight line portion of the diagram is

$$\frac{S_a}{S_{a,0}} = \left(1 - \frac{S_m}{S_u}\right) \quad (15.3)$$

Experimental evidence suggests a nonlinear relationship for particular materials. [Equation \(15.3\)](#) then becomes

$$\frac{S_a}{S_{a,0}} = \left[1 - \left(\frac{S_m}{S_u}\right)^m\right] \quad (15.4)$$

in which  $m$  lies between 0.6 and 2. The relationship for the case of  $m = 2$  is known as the *Gerber parabola*.

### Example 15.1

The allowable stress amplitude for a low-carbon steel is  $\pm 225 \text{ N/mm}^2$  and its ultimate tensile stress is  $750 \text{ N/mm}^2$ . The steel is subjected to a repeated cycle of stress in which the minimum stress is zero. Calculate the safe range of stress based on the Goodman and Gerber predictions.

For the Goodman prediction, [Eq. \(15.3\)](#) applies in which  $S_{a,0} = 450 \text{ N/mm}^2$  and  $S_m = S_a/2$ . Then,

$$S_a = 450[1 - (S_a/2 \times 750)]$$

from which

$$S_a = 346 \text{ N/mm}^2$$

For the Gerber prediction,  $m = 2$  in [Eq. \(15.4\)](#). Then,

$$S_a = 450[1 - (S_a/2 \times 750)^2]$$

which simplifies to

$$S_a^2 + 5000S_a - 2.25 \times 10^6 = 0$$

Solving gives

$$S_a = 415 \text{ N/mm}^2$$

### Example 15.2

If the steel in [Example 15.1](#) is subjected to an alternating cycle of tensile stress about a mean stress of 180 N/mm<sup>2</sup>, calculate the safe range of stress based on the Goodman prediction and also the maximum and minimum stress values.

From [Eq. \(15.3\)](#),

$$S_a = 450[1 - (180/750)]$$

that is,

$$S_a = 342 \text{ N/mm}^2$$

Now

$$S_a = \sigma_{\max} - \sigma_{\min} \quad (\text{i})$$

and

$$S_m = (\sigma_{\max} + \sigma_{\min})/2 \quad (\text{ii})$$

Adding [Eqs. \(i\) and \(ii\)](#),

$$S_a + 2S_m = 2\sigma_{\max}$$

that is,

$$342 + 2 \times 180 = 2\sigma_{\max}$$

so that

$$\sigma_{\max} = 351 \text{ N/mm}^2$$

Then, from [Eq. \(i\)](#),

$$\sigma_{\min} = 351 - 342 = 9 \text{ N/mm}^2$$

In practical situations, fatigue is not caused by a large number of identical stress cycles but by many stress amplitude cycles. The prediction of the number of cycles to failure therefore becomes complex. Miner and Palmgren proposed a *linear cumulative damage law*, as follows. If  $N$  cycles of stress

amplitude  $S_a$  cause fatigue failure, then one cycle produces  $1/N$  of the total damage to cause failure. Therefore, if  $r$  different cycles are applied, in which a stress amplitude  $S_j$  ( $j = 1, 2, \dots, r$ ) would cause failure in  $N_j$  cycles, the number of cycles  $n_j$  required to cause total fatigue failure is given by

$$\sum_{j=1}^r \frac{n_j}{N_j} = 1 \quad (15.5)$$

### Example 15.3

In a fatigue test, a steel specimen is subjected to a reversed cyclic loading in a continuous sequence of four stages as follows:

- 200 cycles at  $\pm 150 \text{ N/mm}^2$
- 250 cycles at  $\pm 125 \text{ N/mm}^2$
- 400 cycles at  $\pm 120 \text{ N/mm}^2$
- 550 cycles at  $\pm 100 \text{ N/mm}^2$

If the loading is applied at the rate of 80 cycles/hour and the fatigue lives at these stress levels are  $10^4$ ,  $10^5$ ,  $1.5 \times 10^5$ , and  $2 \times 10^5$ , respectively, calculate the life of the specimen.

Suppose that the specimen fails after  $P$  sequences of the four stages. Then, from Eq. (15.5),

$$P[(200/10^4) + (250/10^5) + (400/1.5 \times 10^5) + (550/2 \times 10^5)] = 1$$

which gives

$$P = 35.8$$

The total number of cycles in the four stages is 1400, so that, for a loading rate of 80 cycles/hour, the total number of hours to fracture is given by

$$\text{Total hours to fracture} = 35.8 \times 1400/80 = 626.5$$

Although  $S-N$  curves may be readily obtained for different materials by testing a large number of small specimens (*coupon tests*), it is not practical to adopt the same approach for aircraft components, since these are expensive to manufacture and the test program too expensive to run for long periods of time. However, such a program was initiated in the early 1950s to test the wings and tailplanes of Meteor and Mustang fighters. These were subjected to constant amplitude loading until failure, with different specimens being tested at different load levels. Stresses were measured at points where fatigue was expected (and actually occurred) and  $S-N$  curves plotted for the complete structure. The curves had the usual appearance and at low stress levels had such large endurances that fatigue did not occur; thus, a fatigue limit existed. It was found that the average  $S-N$  curve could be approximated to by the equation

$$S_a = 10.3(1 + 1000/\sqrt{N}) \quad (15.6)$$

in which the mean stress was  $90 \text{ N/mm}^2$ . In general terms, Eq. (15.6) may be written as

$$S_a = S_\infty(1 + C/\sqrt{N}) \quad (15.7)$$

in which  $S_\infty$  is the fatigue limit and  $C$  is a constant. Thus,  $S_a \rightarrow S_\infty$  as  $N \rightarrow \infty$ . [Equation \(15.7\)](#) may be rearranged to give the endurance directly:

$$N = C^2 \left( \frac{S_\infty}{S_a - S_\infty} \right)^2 \quad (15.8)$$

which shows clearly that, as  $S_a \rightarrow S_\infty$ ,  $N \rightarrow \infty$ .

### Example 15.4

Determine the number of cycles to fracture for a specimen of low-carbon steel whose fatigue limit is  $60 \text{ N/mm}^2$  and which is subjected to a stress amplitude of  $\pm 100 \text{ N/mm}^2$ .

Assume that  $C$  in [Eq. \(15.8\)](#) is 1000, then

$$N = 1000^2 [60 / (200 - 60)]^2$$

which gives

$$N = 1.8 \times 10^5 \text{ cycles}$$

It has been found experimentally that  $N$  is inversely proportional to the mean stress, as the latter varies in the region of  $90 \text{ N/mm}^2$  while  $C$  is virtually constant. This suggests a method of determining a “standard” endurance curve (corresponding to a mean stress level of  $90 \text{ N/mm}^2$ ) from tests carried out on a few specimens at other mean stress levels. Suppose  $S_m$  is the mean stress level, not  $90 \text{ N/mm}^2$ , in tests carried out on a few specimens at an alternating stress level  $S_{a,m}$ , where failure occurs at a mean number of cycles  $N_m$ . Then, assuming that the  $S$ - $N$  curve has the same form as [Eq. \(15.7\)](#),

$$S_{a,m} = S_{\infty,m} (1 + C / \sqrt{N_m}) \quad (15.9)$$

in which  $C = 1000$  and  $S_{\infty,m}$  is the fatigue limit stress corresponding to the mean stress  $S_m$ . Rearranging [Eq. \(15.9\)](#), we have

$$S_{\infty,m} = S_{a,m} / (1 + C / \sqrt{N_m}) \quad (15.10)$$

The number of cycles to failure at a mean stress of  $90 \text{ N/mm}^2$  would have been, from the above,

$$N' = \frac{S_m}{90} N_m \quad (15.11)$$

The corresponding fatigue limit stress would then have been, from a comparison with [Eq. \(15.10\)](#),

$$S'_{\infty,m} = S_{a,m} / (1 + C / \sqrt{N'}) \quad (15.12)$$

The standard endurance curve for the component at a mean stress of  $90 \text{ N/mm}^2$  is, from [Eq. \(15.7\)](#),

$$S_a = S'_{\infty,m} / (1 + C / \sqrt{N}) \quad (15.13)$$

Substituting in Eq. (15.13) for  $S'_{\infty,m}$  from Eq. (15.12), we have

$$S_a = \frac{S_{a,m}}{(1 + C/\sqrt{N'})} (1 + C/\sqrt{N}) \quad (15.14)$$

in which  $N'$  is given by Eq. (15.11).

Equation (15.14) is based on a few test results so that a “safe” fatigue strength is usually taken to be three standard deviations below the mean fatigue strength. Hence, we introduce a *scatter factor*  $K_n (>1)$  to allow for this; Eq. (15.14) then becomes

$$S_a = \frac{S_{a,m}}{K_n(1 + C/\sqrt{N'})} (1 + C/\sqrt{N}) \quad (15.15)$$

$K_n$  varies with the number of test results available and, for a coefficient of variation of 0.1,  $K_n = 1.45$  for 6 specimens,  $K_n = 1.445$  for 10 specimens,  $K_n = 1.44$  for 20 specimens, and for 100 specimens or more,  $K_n = 1.43$ . For typical  $S$ - $N$  curves, a scatter factor of 1.43 is equivalent to a life factor of 3 to 4.

### Example 15.5

A fatigue test is carried out on 20 specimens at a mean stress level of 120 N/mm<sup>2</sup>, where the alternating stress is  $\pm 150$  N/mm<sup>2</sup>. If the average number of cycles at fracture is  $10^5$ , determine the equation of the  $S$ - $N$  curve. Assume that the constant  $C$  is equal to 1000 and determine also the number of cycles to fracture for a stress amplitude of 200 N/mm<sup>2</sup>.

From Eq. (15.11),

$$N' = (120/90) \times 10^5 = 1.33 \times 10^5$$

Therefore, from Eq. (15.15) and since  $K_n = 1.44$  for 20 specimens,

$$S_a = \{300/[1.44(1 + 1000/(1.33 \times 10^5)^{1/2})]\}[1 + (1000/(N)^{1/2})]$$

so that

$$S_a = 55.7[1 + 1000/(N)^{1/2}] \quad (i)$$

From Eq. (i),

$$200 = 55.7[1 + 1000/(N)^{1/2}]$$

which gives

$$N = 1.5 \times 10^5 \text{ cycles}$$

## 15.4 PREDICTION OF AIRCRAFT FATIGUE LIFE

We see that an aircraft suffers fatigue damage during all phases of the ground–air–ground cycle. The various contributions to this damage may be calculated separately and thus the safe life of the aircraft in terms of the number of flights calculated.

In the ground-air-ground cycle, the maximum vertical acceleration during takeoff is  $1.2\ g$  for a takeoff from a runway or  $1.5\ g$  for a takeoff from grass. It is assumed that these accelerations occur at zero lift and therefore produce compressive (negative) stresses,  $-S_{TO}$ , in critical components such as the undersurface of wings. The maximum positive stress for the same component occurs in level flight (at  $1g$ ) and is  $+S_{1g}$ . The ground-air-ground cycle produces, on the undersurface of the wing, a fluctuating stress  $S_{GAG} = (S_{1g} + S_{TO})/2$  about a mean stress  $S_{GAG(\text{mean})} = (S_{1g} - S_{TO})/2$  (see Fig. 11.20 for clarification). Suppose that tests show that for this stress cycle and mean stress, failure occurs after  $N_G$  cycles. For a life factor of 3, the safe life is  $N_G/3$ , so that the damage done during one cycle is  $3/N_G$ . This damage is multiplied by a factor of 1.5 to allow for the variability of loading between different aircraft of the same type, so that the damage per flight  $D_{GAG}$  from the ground-air-ground cycle is given by

$$D_{GAG} = 4.5/N_G \quad (15.16)$$

Fatigue damage is also caused by gusts encountered in flight, particularly during the climb and descent. Suppose that a gust of velocity  $u_e$  causes a stress  $S_u$  about a mean stress corresponding to level flight and suppose also that the number of stress cycles of this magnitude required to cause failure is  $N(S_u)$ ; the damage caused by one cycle is then  $1/N(S_u)$ . Therefore, from the Palmgren-Miner hypothesis, when sufficient gusts of this and all other magnitudes together with the effects of all other load cycles produce a cumulative damage of 1.0, fatigue failure occurs. It is therefore necessary to know the number and magnitude of gusts likely to be encountered in flight.

Gust data have been accumulated over a number of years from accelerometer records from aircraft flying over different routes and terrains, at different heights, and at different seasons. The ESDU data sheets<sup>1</sup> present the data in two forms, as we previously noted. First,  $l_{10}$  against altitude curves show the distance that must be flown at a given altitude in order that a gust (positive or negative) having a velocity  $\geq 3.05\ \text{m/s}$  be encountered. It follows that  $1/l_{10}$  is the number of these gusts encountered in unit distance (1 km) at a particular height. Second, gust frequency distribution curves,  $r(u_e)$  against  $u_e$ , give the number of gusts of velocity  $u_e$  for every 1000 gusts of velocity  $3.05\ \text{m/s}$ .

From these two curves, the gust exceedance  $E(u_e)$  is obtained;  $E(u_e)$  is the number of times a gust of a given magnitude ( $u_e$ ) will be equaled or exceeded in 1 km of flight. Thus, from the preceding,

$$\text{Number of gusts } \geq 3.05\ \text{m/s per km} = 1/l_{10}$$

$$\text{Number of gusts equal to } u_e \text{ per 1000 gusts equal to } 3.05\ \text{m/s} = r(u_e)$$

Hence,

$$\text{Number of gusts equal to } u_e \text{ per single gust equal to } 3.05\ \text{m/s} = r(u_e)/1000$$

It follows that the gust exceedance  $E(u_e)$  is given by

$$E(u_e) = \frac{r(u_e)}{1000l_{10}} \quad (15.17)$$

in which  $l_{10}$  is dependent on height. A good approximation for the curve of  $r(u_e)$  against  $u_e$  in the region  $u_e = 3.05\ \text{m/s}$  is

$$r(u_e) = 3.23 \times 10^5 u_e^{-5.26} \quad (15.18)$$

Consider now the typical gust exceedance curve shown in Fig. 15.3. In 1 km of flight, there are likely to be  $E(u_e)$  gusts exceeding  $u_e$  m/s and  $E(u_e) - \delta E(u_e)$  gusts exceeding  $u_e + \delta u_e$  m/s. Thus, there will be

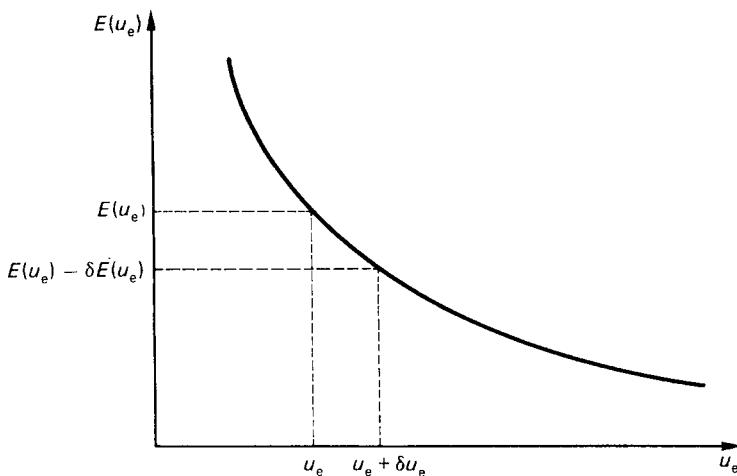


FIGURE 15.3 Gust Exceedance Curve

$\delta E(u_e)$  fewer gusts exceeding  $u_e + \delta u_e$  m/s than  $u_e$  m/s and the increment in gust speed  $\delta u_e$  corresponds to a number  $-\delta E(u_e)$  of gusts at a gust speed close to  $u_e$ . Half of these gusts will be positive (upgusts) and half negative (downgusts), so that if it is assumed that each upgust is followed by a downgust of equal magnitude, the number of complete gust cycles will be  $-\delta E(u_e)/2$ . Suppose that each cycle produces a stress  $S(u_e)$  and that the number of these cycles required to produce failure is  $N(S_{u,e})$ . The damage caused by one cycle is then  $1/N(S_{u,e})$ , and over the gust velocity interval  $\delta u_e$  the total damage  $\delta D$  is given by

$$\delta D = -\frac{\delta E(u_e)}{2N(S_{u,e})} = -\frac{dE(u_e)}{du_e} \frac{\delta u_e}{2N(S_{u,e})} \quad (15.19)$$

Integrating Eq. (15.19) over the whole range of gusts likely to be encountered, we obtain the total damage  $D_g$  per kilometer of flight:

$$D_g = - \int_0^{\infty} \frac{1}{2N(S_{u,e})} \frac{dE(u_e)}{du_e} du_e \quad (15.20)$$

Further, if the average block length journey of an aircraft is  $R_{av}$ , the average gust damage per flight is  $D_g R_{av}$ . Also, some aircraft in a fleet experience more gusts than others, since the distribution of gusts is random. Therefore, if, for example, it is found that one particular aircraft encounters 50 percent more gusts than the average, its gust fatigue damage is  $1.5 D_g$ /km.

The gust damage predicted by Eq. (15.20) is obtained by integrating over a complete gust velocity range from zero to infinity. Clearly, there will be a gust velocity below which no fatigue damage occurs, since the cyclic stress produced is below the fatigue limit stress of the particular component. Equation (15.20) is therefore rewritten

$$D_g = - \int_{u_f}^{\infty} \frac{1}{2N(S_{u,e})} \frac{dE(u_e)}{du_e} du_e \quad (15.21)$$

in which  $u_f$  is the gust velocity required to produce the fatigue limit stress.

We noted previously that more gusts are encountered during climb and descent than during cruise. Altitude therefore affects the amount of fatigue damage caused by gusts and its effects may be determined as follows. Substituting for the gust exceedance  $E(u_e)$  in Eq. (15.21) from Eq. (15.17), we obtain

$$D_g = -\frac{1}{1000l_{10}} \int_{u_t}^{\infty} \frac{1}{2N(S_{u,e})} \frac{dr(u_e)}{du_e} du_e$$

or

$$D_g = \frac{1}{l_{10}} d_g \text{ per kilometer} \quad (15.22)$$

in which  $l_{10}$  is a function of height  $h$  and

$$d_g = -\frac{1}{1000} \int_{u_t}^{\infty} \frac{1}{2N(S_{u,e})} \frac{dr(u_e)}{du_e} du_e$$

Suppose that the aircraft is climbing at a speed  $V$  with a rate of climb (ROC). The time taken for the aircraft to climb from a height  $h$  to a height  $h + \delta h$  is  $\delta h/\text{ROC}$ , during which time it travels a distance  $V\delta h/\text{ROC}$ . Hence, from Eq. (15.22), the fatigue damage experienced by the aircraft in climbing through a height  $\delta h$  is

$$\frac{1}{l_{10}} d_g \frac{V}{\text{ROC}} \delta h$$

The total damage produced during a climb from sea level to an altitude  $H$  at a constant speed  $V$  and ROC is

$$D_{g,\text{climb}} = d_g \frac{V}{\text{ROC}} \int_0^H \frac{dh}{l_{10}} \quad (15.23)$$

Plotting  $1/l_{10}$  against  $h$  from ESDU data sheets for aircraft having cloud warning radar and integrating gives

$$\int_0^{3000} \frac{dh}{l_{10}} = 303, \quad \int_{3000}^{6000} \frac{dh}{l_{10}} = 14, \quad \int_{6000}^{9000} \frac{dh}{l_{10}} = 3.4$$

From this,  $\int_0^{9000} dh/l_{10} = 320.4$ , from which it can be seen that approximately 95 percent of the total damage in the climb occurs in the first 3000 m.

An additional factor influencing the amount of gust damage is forward speed. For example, the change in wing stress produced by a gust may be represented by

$$S_{u,e} = k_1 u_e V_e \quad (\text{see Eq. (14.24)}) \quad (15.24)$$

in which the forward speed of the aircraft is in equivalent airspeed (EAS). From Eq. (15.24), we see that the gust velocity  $u_f$  required to produce the fatigue limit stress  $S_\infty$  is

$$u_f = S_\infty / k_1 V_e \quad (15.25)$$

The gust damage per kilometer at different forward speeds  $V_e$  is then found using Eq. (15.21) with the appropriate value of  $u_f$  as the lower limit of integration. The integral may be evaluated by using the known approximate forms of  $N(S_{u,e})$  and  $E(u_e)$  from Eqs. (15.15) and (15.17). From Eq. (15.15),

$$S_a = S_{u,e} = \frac{S'_{\infty,m}}{K_n} \left[ 1 + C / \sqrt{N(S_{u,e})} \right]$$

from which

$$N(S_{u,e}) = \left( \frac{C}{K_n} \right)^2 \left( \frac{S'_{\infty,m}}{S_{u,e} - S'_{\infty,m}} \right)^2$$

where  $S_{u,e} = k_1 V_e u_e$  and  $S'_{\infty,m} = k_1 V_e u_f$ . Also, Eq. (15.17) is

$$E(u_e) = \frac{r(u_e)}{1000 l_{10}}$$

or, substituting for  $r(u_e)$  from Eq. (15.18),

$$E(u_e) = \frac{3.23 \times 10^5 u_e^{-5.26}}{1000 l_{10}}$$

Equation (15.21) then becomes

$$D_g = - \int_{u_f}^{\infty} \frac{1}{2} \left( \frac{K_n}{C} \right)^2 \left( \frac{S_{u,e} - S'_{\infty,m}}{S'_{\infty,m}} \right)^2 \left( \frac{-3.23 \times 5.26 \times 10^5 u_e^{-5.26}}{1000 l_{10}} \right) du_e$$

Substituting for  $S_{u,e}$  and  $S'_{\infty,m}$ , we have

$$D_g = \frac{16.99 \times 10^2}{2 l_{10}} \left( \frac{K_n}{C} \right)^2 \int_{u_f}^{\infty} \left( \frac{u_e - u_f}{u_f} \right)^2 u_e^{-6.26} du_e$$

or

$$D_g = \frac{16.99 \times 10^2}{2 l_{10}} \left( \frac{K_n}{C} \right)^2 \int_{u_f}^{\infty} \left( \frac{u_e^{-4.26}}{u_f^2} - \frac{2u_e^{-5.26}}{u_f} + u_e^{-6.26} \right) du_e$$

from which

$$D_g = \frac{46.55}{2 l_{10}} \left( \frac{K_n}{C} \right)^2 u_f^{-5.26}$$

or, in terms of the aircraft speed  $V_e$ ,

$$D_g = \frac{46.55}{2 l_{10}} \left( \frac{K_n}{C} \right)^2 \left( \frac{k_1 V_e}{S'_{\infty,m}} \right)^{5.26} \text{ per kilometer} \quad (15.26)$$

It can be seen from Eq. (15.26) that gust damage increases in proportion to  $V_e^{5.26}$ , so that increasing forward speed has a dramatic effect on gust damage.

The total fatigue damage suffered by an aircraft per flight is the sum of the damage caused by the ground-air-ground cycle, the damage produced by gusts, and the damage due to other causes, such as pilot-induced maneuvers, ground turning and braking, and landing and takeoff load fluctuations. The damage produced by these other causes can be determined from load exceedance data. Thus, if this extra damage per flight is  $D_{\text{extra}}$ , the total fractional fatigue damage per flight is

$$D_{\text{total}} = D_{\text{GAG}} + D_g R_{\text{av}} + D_{\text{extra}}$$

or

$$D_{\text{total}} = 4.5/N_G + D_g R_{\text{av}} + D_{\text{extra}} \quad (15.27)$$

and the life of the aircraft in terms of flights is

$$N_{\text{flight}} = 1/D_{\text{total}} \quad (15.28)$$

### Example 15.6

A passenger aircraft cruises at 250 m/s EAS and its average block length journey is 1,500 km. The fatigue limit stress for critical components of the aircraft is  $\pm 230$  N/mm $^2$  based on a mean stress of 90 N/mm $^2$ . Assuming that the damage caused during the ground-air-ground cycle is 10 percent of the total damage per flight, calculate the life of the aircraft in terms of the number of flights. Take  $k_1 = 2.97$ .

The damage per kilometer due to gusts is given by Eq. (15.26), where we assume that  $K_n = 1.43$  and  $C = 1000$  (see Section 15.3). From gust data  $l_{10} = 5,000$  m. Substituting the appropriate values in Eq.(15.26),

$$D_g = (46.55/2 \times 5000)(1.43/1000)^2(2.97 \times 250/460)^{5.26}$$

so that

$$D_g = 1.18 \times 10^{-7}/\text{km}$$

Now,

$$D_{\text{total}} = D_g R_{\text{av}} + 0.1D_{\text{total}}$$

Then,

$$0.9D_{\text{total}} = 1.18 \times 10^{-7} \times 1500 = 1.77 \times 10^{-4}$$

and

$$D_{\text{total}} = 1.97 \times 10^{-4}$$

Therefore, the life of the aircraft in terms of the number of flights is

$$\text{No. of flights} = 1/1.97 \times 10^{-4} = 5,076, \text{ say, } 5,000$$

## 15.5 CRACK PROPAGATION

We have seen that the concept of fail-safe structures in aircraft construction relies on a damaged structure being able to retain enough of its load-carrying capacity to prevent catastrophic failure, at least until the damage is detected. It is therefore essential that the designer be able to predict how and at what rate a fatigue crack grows. The ESDU data sheets provide a useful introduction to the study of crack propagation; some of the results are presented here.

The analysis of stresses close to a crack tip using elastic stress concentration factors breaks down, since the assumption that the crack tip radius approaches zero results in the stress concentration factor tending to infinity. Instead, linear elastic fracture mechanics analyses the stress field around the crack tip and identifies features of the field common to all cracked elastic bodies.

### 15.5.1 Stress concentration factor

The three basic modes of crack growth are shown in Fig. 15.4. Generally, the stress field in the region of the crack tip is described by a two-dimensional model, which may be used as an approximation for many practical three-dimensional loading cases. Thus, the stress system at a distance  $r$  ( $r \leq a$ ) from the tip of a crack of length  $2a$ , shown in Fig. 15.5, can be expressed in the form<sup>2</sup>

$$S_r, S_\theta, S_{r,\theta} = \frac{K}{(2\pi r)^{\frac{1}{2}}} f(\theta) \quad (15.29)$$

in which  $f(\theta)$  is a different function for each of the three stresses and  $K$  is the *stress intensity factor*;  $K$  is a function of the nature and magnitude of the applied stress levels and also of the crack size. The terms  $(2\pi r)^{\frac{1}{2}}$  and  $f(\theta)$  map the stress field in the vicinity of the crack and are the same for all cracks under external loads that cause crack openings of the same type.

Equation (15.29) applies to all modes of crack opening, with  $K$  having different values, depending on the geometry of the structure, the nature of the applied loads, and the type of crack.

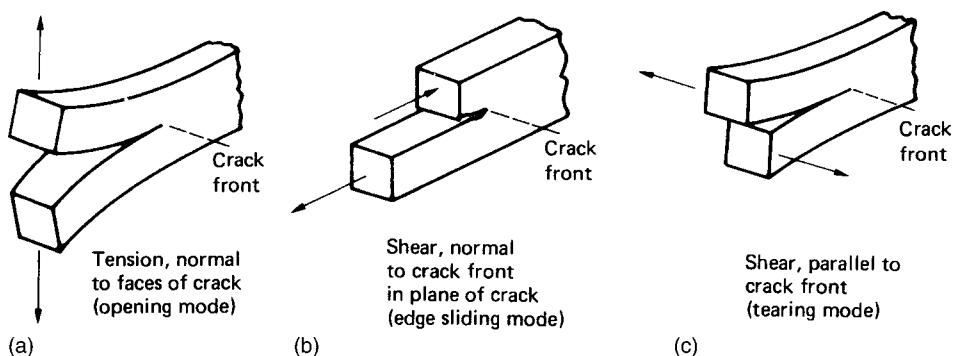


FIGURE 15.4 Basic Modes of Crack Growth

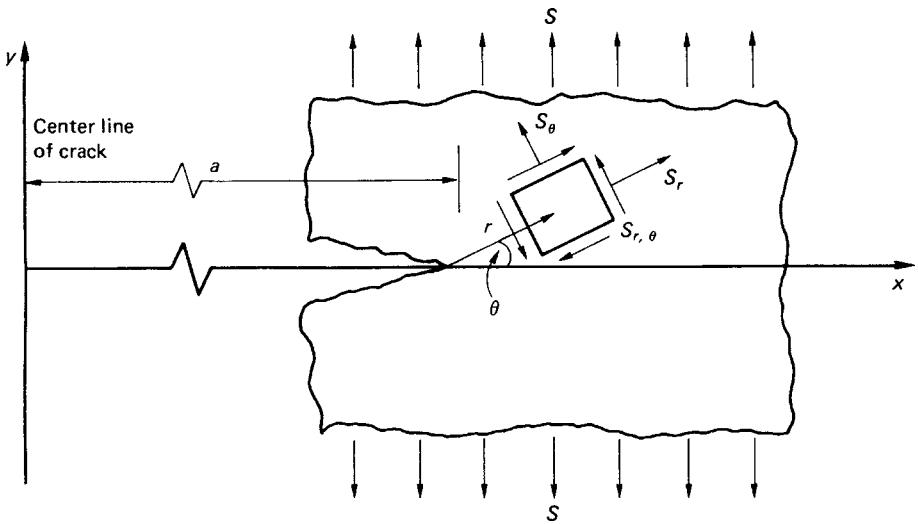


FIGURE 15.5 Stress Field in the Vicinity of a Crack

Experimental data show that crack growth and residual strength data are better correlated using  $K$  than any other parameter.  $K$  may be expressed as a function of the nominal applied stress  $S$  and the crack length in the form

$$K = S(\pi a)^{\frac{1}{2}}\alpha \quad (15.30)$$

in which  $\alpha$  is a non-dimensional coefficient usually expressed as the ratio of crack length to any convenient local dimension in the plane of the component; for a crack in an infinite plate under an applied uniform stress level  $S$  remote from the crack,  $\alpha = 1.0$ . Alternatively, in cases where opposing loads  $P$  are applied at points close to the plane of the crack,

$$K = \frac{P\alpha}{(\pi a)^{\frac{1}{2}}} \quad (15.31)$$

in which  $P$  is the load/unit thickness. Equations (15.30) and (15.31) may be rewritten as

$$K = K_0\alpha \quad (15.32)$$

where  $K_0$  is a reference value of the stress intensity factor, which depends upon the loading. For the simple case of a remotely loaded plate in tension,

$$K_0 = S(\pi a)^{\frac{1}{2}} \quad (15.33)$$

and Eqs. (15.32) and (15.30) are identical, so that for a given ratio of crack length to plate width,  $\alpha$  is the same in both formulations. In more complex cases, for example, the in-plane bending of a plate of width  $2b$  and having a central crack of length  $2a$ ,

$$K_0 = \frac{3Ma}{4b^3} (\pi a)^{\frac{1}{2}} \quad (15.34)$$

in which  $M$  is the bending moment per unit thickness. Comparing Eqs. (15.34) and (15.30), we see that  $S = 3 Ma/4b^3$ , which is the value of direct stress given by basic bending theory at a point a distance  $\pm a/2$  from the central axis. However, if  $S$  is specified as the bending stress in the outer fibers of the plate, that is, at  $\pm b$ , then  $S = 3 M/2b^2$ ; clearly, the different specifications of  $S$  require different values of  $\alpha$ . On the other hand, the final value of  $K$  must be independent of the form of presentation used. Use of Eqs. (15.30)–(15.32) depends on the form of the solution for  $K_0$ , and care must be taken to ensure that the formula used and the way in which the nominal stress is defined are compatible with those used in the derivation of  $\alpha$ .

A number of methods are available for determining the values of  $K$  and  $\alpha$ . In one method, the solution for a component subjected to more than one type of loading is obtained from available standard solutions using superposition or, if the geometry is not covered, two or more standard solutions may be compounded.<sup>1</sup> Alternatively, a finite element analysis may be used.

The coefficient  $\alpha$  in Eq. (15.30) has, as we have noted, different values depending on the plate and crack geometries. The following are values of  $\alpha$  for some of the more common cases.

- (i) A semi-infinite plate having an edge crack of length  $a$ ;  $\alpha = 1.12$ .
- (ii) An infinite plate having an embedded circular crack or a semicircular surface crack, each of radius  $a$ , lying in a plane normal to the applied stress;  $\alpha = 0.64$ .
- (iii) An infinite plate having an embedded elliptical crack of axes  $2a$  and  $2b$  or a semi-elliptical crack of width  $2b$  in which the depth  $a$  is less than half the plate thickness, each lying in a plane normal to the applied stress;  $\alpha = 1.12\Phi$  in which  $\Phi$  varies with the ratio  $b/a$  as follows:

$b/a$	0	0.2	0.4	0.6	0.8
$\Phi$	1.0	1.05	1.15	1.28	1.42

For  $b/a = 1$ , the situation is identical to case (ii).

- (iv) A plate of finite width  $w$  having a central crack of length  $2a$ , where  $a \leq 0.3w$ ;  $\alpha = [\sec(a\pi/w)]^{1/2}$ .
- (v) For a plate of finite width  $w$  having two symmetrical edge cracks each of depth  $2a$ , Eq. (15.30) becomes

$$K = S [w \tan(\pi a/w) + (0.1w) \sin(2\pi a/w)]^{1/2}$$

From Eq. (15.29), it can be seen that the stress intensity at a point ahead of a crack can be expressed in terms of the parameter  $K$ . Failure occurs when  $K$  reaches a critical value  $K_c$ . This is known as the *fracture toughness* of the material and has units  $\text{MN/m}^{3/2}$  or  $\text{N/mm}^{3/2}$ .

### Example 15.7

An infinite plate has a fracture toughness of  $3,300 \text{ N/mm}^{3/2}$ . If the plate contains an embedded circular crack of  $3 \text{ mm}$  radius, calculate the maximum allowable stress that could be applied around the boundary of the plate.

In this case, Eq. (15.30) applies with  $\alpha = 0.64$ . Then,

$$S = 3,300 / [(\pi \times 3)^{1/2} \times 0.64]$$

So that

$$S = 1,680 \text{ N/mm}^2$$

**Example 15.8**

If the steel plate of [Example 15.7](#) develops an elliptical crack of length 6 mm and width 2.4 mm, calculate the allowable stress that could be applied around the boundary of the plate.

In this case,  $b/a = 1.2/3 = 0.4$ . Then,  $\alpha = 1.12 \times 1.15 = 1.164$  and, from [Eq. \(15.30\)](#),

$$S = 3,300/[(\pi \times 3)^{1/2} \times 1.164]$$

that is,

$$S = 931.5 \text{ N/mm}^2$$

**Example 15.9**

Suppose that the plate of [Example 15.7](#) has a finite width of 50 mm and develops a central crack of length 6 mm. What is the allowable stress that could be applied around the boundary of the plate?

For this case,

$$\alpha = [\sec(\pi a/w)]^{1/2}$$

that is,

$$\alpha = [\sec(\pi \times 3/50)]^{1/2} = 1.018$$

so that

$$S = 3,300/[(\pi \times 3)^{1/2} \times 1.018]$$

which gives

$$S = 1,056 \text{ N/mm}^2$$

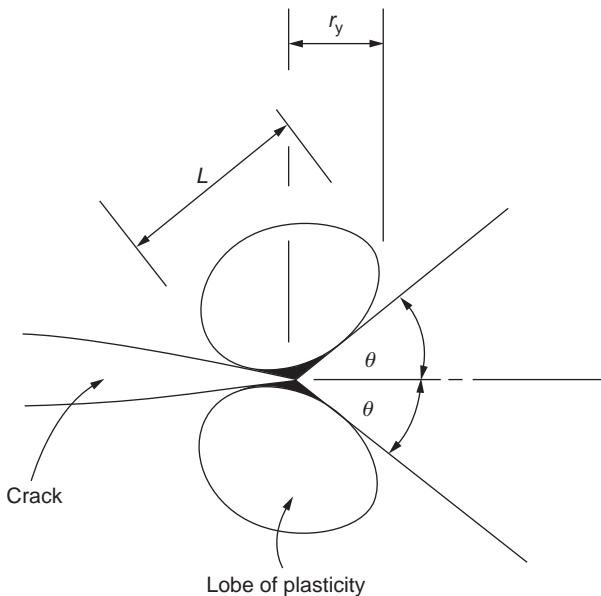
**15.5.2 Crack tip plasticity**

In certain circumstances, it may be necessary to account for the effect of plastic flow in the vicinity of the crack tip. This may be allowed for by estimating the size of the plastic zone and adding this to the actual crack length to form an effective crack length  $2a_1$ . Thus, if  $r_p$  is the radius of the plastic zone,  $a_1 = a + r_p$  and [Eq. \(15.30\)](#) becomes

$$K_p = S(\pi a_1)^{1/2} \alpha_1 \quad (15.35)$$

in which  $K_p$  is the stress intensity factor corrected for plasticity and  $\alpha_1$  corresponds to  $a_1$ . Thus, for  $r_p/t > 0.5$ , that is, a condition of plane stress,

$$r_p = \frac{1}{2\pi} \left( \frac{K}{f_y} \right)^2 \quad \text{or} \quad r_p = \frac{a}{2} \left( \frac{S}{f_y} \right)^2 \alpha^2 \quad (\text{Ref. 3}) \quad (15.36)$$



**FIGURE 15.6 Plane Strain Plasticity**

in which  $f_y$  is the yield proof stress of the material. For  $r_p/t < 0.02$ , a condition of plane strain,

$$r_p = \frac{1}{6\pi} \left( \frac{K}{f_y} \right)^2 \quad (15.37)$$

For intermediate conditions the correction should be such as to produce a conservative solution.

Dugdale<sup>4</sup> showed that the fracture toughness parameter  $K_c$  is highly dependent on plate thickness. In general, since the toughness of a material decreases with decreasing plasticity, it follows that the true fracture toughness is that corresponding to a plane strain condition. This lower limiting value is particularly important to consider in high-strength alloys, since these are prone to brittle failure. In addition, the assumption that the plastic zone is circular is not representative in plane strain conditions. Rice and Johnson<sup>5</sup> showed that, for a small amount of plane strain yielding, the plastic zone extends as two lobes (Fig. 15.6), each inclined at an angle  $\theta$  to the axis of the crack, where  $\theta = 70^\circ$  and the greatest extent  $L$  and forward penetration ( $r_y$  for  $\theta = 0$ ) of plasticity are given by

$$L = 0.155(K/f_y)^2$$

$$r_y = 0.04(K/f_y)^2$$

### 15.5.3 Crack propagation rates

Having obtained values of the stress intensity factor and the coefficient  $\alpha$ , fatigue crack propagation rates may be estimated. From these, the life of a structure containing cracks or crack-like defects may be determined; alternatively, the loading condition may be modified or inspection periods arranged so that the crack is detected before failure.

Under constant amplitude loading, the rate of crack propagation may be represented graphically by curves described in general terms by the law<sup>6</sup>

$$\frac{da}{dN} = f(R, \Delta K) \quad (15.38)$$

in which  $\Delta K$  is the stress intensity factor range and  $R = S_{\min}/S_{\max}$ . If Eq. (15.30) is used,

$$\Delta K = (S_{\max} - S_{\min})(\pi a)^{\frac{1}{2}}\alpha \quad (15.39)$$

Equation (15.39) may be corrected for plasticity under cyclic loading and becomes

$$\Delta K_p = (S_{\max} - S_{\min})(\pi a_1)^{\frac{1}{2}}\alpha_1 \quad (15.40)$$

in which  $a_1 = a + r_p$ , where, for plane stress,<sup>7</sup>

$$r_p = \frac{1}{8\pi} \left( \frac{\Delta K}{f_y} \right)^2$$

The curves represented by Eq. (15.38) may be divided into three regions. The first corresponds to a very slow crack growth rate ( $< 10^{-8}$  m/cycle), where the curves approach a threshold value of stress intensity factor  $\Delta K^{th}$  corresponding to  $4 \times 10^{-11}$  m/cycle, that is, no crack growth. In the second region ( $10^{-8}$ – $10^{-6}$  m/cycle), much of the crack life takes place and, for small ranges of  $\Delta K$ , Eq. (15.38) may be represented by<sup>8</sup>

$$\frac{da}{dN} = C(\Delta K)^n \quad (15.41)$$

in which  $C$  and  $n$  depend on the material properties; over small ranges of  $da/dN$  and  $\Delta K$ ,  $C$  and  $n$  remain approximately constant. The third region corresponds to crack growth rates  $> 10^{-6}$  m/cycle, where instability and final failure occur.

An attempt has been made to describe the complete set of curves by the relationship<sup>9</sup>

$$\frac{da}{dN} = \frac{C(\Delta K)^n}{(1 - R)K_c - \Delta K} \quad (15.42)$$

in which  $K_c$  is the fracture toughness of the material obtained from toughness tests. Integration of Eq. (15.41) or (15.42) analytically or graphically gives an estimate of the crack growth life of the structure, that is, the number of cycles required for a crack to grow from an initial size to an unacceptable length, or the crack growth rate for failure, whichever is the design criterion.

For example, for a plate of infinite width for which  $\alpha = 1.0$ , inversion of Eq. (15.41) gives

$$\frac{dN}{da} = \frac{1}{C} (\Delta K)^{-n}$$

Substituting for  $\Delta K$  from Eq. (15.39),

$$\frac{dN}{da} = \frac{1}{C} \left[ (S_{\max} - S_{\min})(\pi a)^{1/2} \right]^{-n}$$

Then

$$[N]_{N_i}^{N_f} = \frac{1}{C[(S_{\max} - S_{\min})\pi^{1/2}]^n} \int_{a_i}^{a_f} a^{-n/2} da$$

which gives

$$[N]_{N_i}^{N_f} = \frac{1}{C[(S_{\max} - S_{\min})\pi^{1/2}]^n} \left[ \frac{a^{(-n/2)+1}}{(-n/2) + 1} \right]_{a_i}^{a_f}$$

from which

$$[N]_{N_i}^{N_f} = \frac{1}{C[(S_{\max} - S_{\min})\pi^{1/2}]^n} \left[ \frac{a^{(1-n/2)}}{1 - n/2} \right]_{a_i}^{a_f} \quad (15.43)$$

for  $n > 2$ . An analytical integration may be carried out only if  $n$  is an integer and  $\alpha$  is in the form of a polynomial; otherwise, graphical or numerical techniques must be employed.

Substituting the limits in Eq. (15.43) and taking  $N_i = 0$ , the number of cycles to failure is given by

$$N_f = \frac{2}{C(n-2)[(S_{\max} - S_{\min})\pi^{1/2}]^n} \left[ \frac{1}{a_i^{(n-2)/2}} - \frac{1}{a_f^{(n-2)/2}} \right] \quad (15.44)$$

### Example 15.10

An infinite plate contains a crack having an initial length of 0.2 mm and is subjected to a cyclic repeated stress range of 175 N/mm<sup>2</sup>. If the fracture toughness of the plate is 1708 N/mm<sup>3/2</sup> and the rate of crack growth is  $40 \times 10^{-15} (\Delta K)^4$  mm/cycle, determine the number of cycles to failure.

The crack length at failure is given by Eq.(15.30), in which  $\alpha = 1$ ,  $K = 1,708$  N/mm<sup>3/2</sup>, and  $S = 175$  N/mm<sup>2</sup>; that is,

$$a_f = \frac{1,708^2}{\pi \times 175^2} = 30.3 \text{ mm}$$

Also, it can be seen from Eq. (15.41) that  $C = 40 \times 10^{-15}$  and  $n = 4$ . Then, substituting the relevant parameters in Eq. (15.44) gives

$$N_f = 1/(40 \times 10^{-15} [175 \times \pi^{1/2}]^4)(1/0.1 - 1/30.3)$$

From which

$$N_f = 26,919 \text{ cycles}$$

### Example 15.10 MATLAB

Use MATLAB to repeat Example 15.10 for the following initial crack lengths:

	(i)	(ii)	(iii)	(iv)
$a_i$	0.1 m	0.2 m	0.3 m	0.4 m

The number of cycles to failure for each case is obtained through the following MATLAB file:

```
% Declare any needed variables
a_i = [0.1 0.2 0.3 0.4];
alpha = 1;
```

```

K = 1708;
S = 175;
C = 40e-15;
n = 4;

% Calculate a_f using Eq. (15.30)
a_f = K^2/(pi*S^2*alpha^2);

% Calculate the number of cycles using Eq. (15.44)
N_f = 2/(C*(n-2)*(S*sqrt(pi))^n)*(1./a_i.^((n-2)/2) - 1/a_f.^((n-2)/2));

% Output the results for each case to the Command Window
for i=1:1:length(a_i)
    disp(['For a_i = ' num2str(a_i(i)) ' m: N_f = ' num2str(round(N_f(i))) ' cycles'])
    disp(' ')
end

```

The Command Window outputs resulting from this MATLAB file are as follows:

```

For a_i = 0.1 m: N_f = 26919 cycles
For a_i = 0.2 m: N_f = 13415 cycles
For a_i = 0.3 m: N_f = 8914 cycles
For a_i = 0.4 m: N_f = 6663 cycles

```

## References

- [1] ESDU data sheets. Fatigue. No. 80036.
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- [3] McClintock FA, Irwin GR. Plasticity aspects of fracture mechanics. In: Fracture toughness testing and its applications. Philadelphia: American Society for Testing Materials; ASTM STP 381, April 1965.
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- [6] Paris PC, Erdogan F. A critical analysis of crack propagation laws. Trans Am Soc Mech Engrs, Series D, December 1963;85:(4).
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- [8] Paris PC. The fracture mechanics approach to fatigue. In: Fatigue—an interdisciplinary approach. Syracuse, NY: Syracuse University Press; 1964.
- [9] Forman RG. Numerical analysis of crack propagation in cyclic-loaded structures. Trans Am Soc Mech Engrs, Series D, September 1967;89(3).

## Further reading

Freudenthal AM. Fatigue in aircraft structures. New York: Academic Press; 1956.

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## PROBLEMS

**P.15.1.** A material has a fatigue limit of  $\pm 230 \text{ N/mm}^2$  and an ultimate tensile strength of  $870 \text{ N/mm}^2$ . If the safe range of stress is determined by the Goodman prediction, calculate its value.

Answer:  $363 \text{ N/mm}^2$

**P.15.2.** A more accurate estimate for the safe range of stress for the material of P.15.1 is given by the Gerber prediction. Calculate its value.

Answer:  $432 \text{ N/mm}^2$

**P.15.3.** A steel component is subjected to a reversed cyclic loading of 100 cycles/day over a period of time in which  $\pm 160 \text{ N/mm}^2$  is applied for 200 cycles,  $\pm 140 \text{ N/mm}^2$  is applied for 200 cycles, and  $\pm 100 \text{ N/mm}^2$  is applied for 600 cycles. If the fatigue life of the material at each of these stress levels is  $10^4$ ,  $10^5$ , and  $2 \times 10^5$  cycles, respectively, estimate the life of the component using Miner's law.

Answer: 400 days

**P.15.3. MATLAB** Use MATLAB to repeat Problem P.15.3 for the following application cycles of the  $160 \text{ N/mm}^2$  load:

(i)	(ii)	(iii)	(iv)	(v)	(vi)
100 cycles	200 cycles	300 cycles	400 cycles	500 cycles	600 cycles

All other loads and cycles are as provided in P.15.3.

- Answer:
- (i) 600 days
  - (ii) 400 days
  - (iii) 314 days
  - (iv) 276 days
  - (v) 236 days
  - (vi) 215 days

**P.15.4.** An aircraft's cruise speed is increased from 200 m/s to 220 m/s. Determine the percentage increase in gust damage this would cause.

Answer: 65 percent

**P.15.5.** The average block length journey of an executive jet airliner is 1,000 km and its cruise speed is 240 m/s. If the damage during the ground-air-ground cycle may be assumed to be 10 percent of the total damage during a complete flight, determine the percentage increase in the life of the aircraft when the cruising speed is reduced to 235 m/s.

Answer: 12 percent

**P.15.6.** A small executive jet cruises at 220 m/s EAS and has an average block journey distance of 1,200 km. The most critical portion of the aircraft's structure is the undersurface of the wing, which has a fatigue limit stress of  $\pm 200 \text{ N/mm}^2$  based on a mean stress of  $90 \text{ N/mm}^2$ . If the ground-air-ground cycle and other minor loads produce 8 percent of the total damage, estimate the number of hours of the life of the aircraft. Take  $k_1 = 3.0$  and  $l_{10} = 6000 \text{ m}$ .

Answer: 11,400 hours

**P.15.7.** An infinite steel plate has a fracture toughness of  $3,320 \text{ N/mm}^{3/2}$  and contains a 4-mm long crack. Calculate the maximum allowable design stress that could be applied around the boundary of the plate.

Answer:  $1,324 \text{ N/mm}^2$

**P.15.8.** A semi-infinite plate has an edge crack of length 0.4 mm. If the plate is subjected to a cyclic repeated stress loading of  $180 \text{ N/mm}^2$ , its fracture toughness is  $1,800 \text{ N/mm}^{3/2}$ , and the rate of crack growth is  $30 \times 10^{-15} (\Delta K)^4 \text{ mm/cycle}$ , determine the crack length at failure and the number of cycles to failure.

Answer: 25.4 mm, 7,916 cycles

**P.15.8 MATLAB** Use MATLAB to repeat Problem P.15.8 for the following cyclic repeated stress loadings:

X	(i) $S = 150 \text{ N/mm}^2$	(ii) $S = 160 \text{ N/mm}^2$	(iii) $S = 170 \text{ N/mm}^2$	(iv) $S = 180 \text{ N/mm}^2$	(v) $S = 190 \text{ N/mm}^2$	(vi) $S = 200 \text{ N/mm}^2$
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- Answer:
- (i) 36.5 mm, 16,496 cycles
  - (ii) 32.1 mm, 12,723 cycles
  - (iii) 28.4 mm, 9,967 cycles
  - (iv) 25.4 mm, 7,916 cycles
  - (v) 22.8 mm, 6,365 cycles
  - (vi) 20.6 mm, 5,174 cycles

**P.15.9.** A steel plate 50 mm wide is 5 mm thick and carries an in-plane bending moment. If the plate develops an elliptical crack of length 6 mm and width 2.4 mm, calculate the maximum bending moment the plate can withstand if the fracture toughness of the steel is  $3,500 \text{ N/mm}^{3/2}$ .

Answer: 4,300 Nmm

# Bending of open and closed thin-walled beams

# 16

In [Chapter 12](#), we discussed the various types of structural component found in aircraft and spacecraft construction and the various loads they support. We saw that an aircraft is basically an assembly of stiffened shell structures, ranging from the single-cell closed section fuselage to multicellular wings and tail surfaces, each subjected to bending, shear, torsional, and axial loads. Other, smaller portions of the structure consist of thin-walled channel, T, Z, “top-hat,” or I sections, which are used to stiffen the thin skins of the cellular components and provide support for internal loads from floors, engine mountings, etc. Structural members such as these are known as *open section* beams, while the cellular components are termed *closed section* beams; clearly, both types of beam are subjected to axial, bending, shear, and torsional loads.

In this chapter, we investigate the stresses and displacements in thin-walled open and single-cell closed section beams produced by bending loads.

In [Chapter 1](#), we saw that an axial load applied to a member produces a uniform direct stress across the cross-section of the member. A different situation arises when the applied loads cause a beam to bend, which, if the loads are vertical, takes up a sagging ( $\smile$ ) or hogging ( $\frown$ ) shape. This means that, for loads that cause a beam to sag, the upper surface of the beam must be shorter than the lower surface, as the upper surface becomes concave and the lower one convex; the reverse is true for loads that cause hogging. The strains in the upper regions of the beam, therefore, are different from those in the lower regions, and since we established that stress is directly proportional to strain ([Eq. \(1.40\)](#)), it follows that the stress varies through the depth of the beam.

The truth of this can be demonstrated by a simple experiment. Take a reasonably long rectangular rubber eraser and draw three or four lines on its longer faces as shown in [Fig. 16.1\(a\)](#); the reason for this becomes clear a little later. Now hold the eraser between the thumb and forefinger at each end and apply pressure as shown by the direction of the arrows in [Fig. 16.1\(b\)](#). The eraser bends into the shape shown and the lines on the side of the eraser *remain straight* but are now further apart at the top than at the bottom.

Since, in [Fig. 16.1\(b\)](#), the upper fibers are stretched and the lower fibers compressed, there are fibers somewhere in between which are neither stretched nor compressed; the plane containing these fibers is called the *neutral plane*.

Now, rotate the eraser so that its shorter sides are vertical and apply the same pressure with your fingers. The eraser again bends but now requires much less effort. It follows that the geometry and orientation of a beam section must affect its *bending stiffness*. This is more readily demonstrated with a plastic ruler. When flat, it requires hardly any effort to bend it but when held with its width vertical it becomes almost impossible to bend.

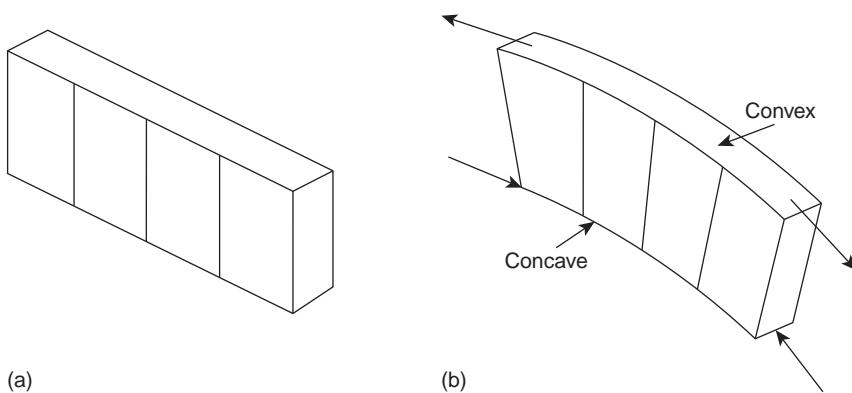


FIGURE 16.1 Bending a Rubber Eraser

## 16.1 SYMMETRICAL BENDING

Although symmetrical bending is a special case of the bending of beams of arbitrary cross-section, we shall investigate the former first, so that the more complex, general case may be more easily understood.

Symmetrical bending arises in beams which have either singly or doubly symmetrical cross-sections; examples of both types are shown in Fig. 16.2.

Suppose that a length of beam, of rectangular cross-section, say, is subjected to a pure, sagging bending moment,  $M$ , applied in a vertical plane. We define this later as a negative bending moment. The length of beam bends into the shape shown in Fig. 16.3(a), in which the upper surface is concave and the lower convex. It can be seen that the upper longitudinal fibers of the beam are compressed while the lower fibers are stretched. It follows that, as in the case of the eraser, between these two extremes are fibers that remain unchanged in length.

The direct stress therefore varies through the depth of the beam, from compression in the upper fibers to tension in the lower. Clearly, the direct stress is zero for the fibers that do not change in length; we called the plane containing these fibers the *neutral plane*. The line of intersection of the neutral plane and any cross-section of the beam is termed the *neutral axis* (Fig. 16.3(b)).

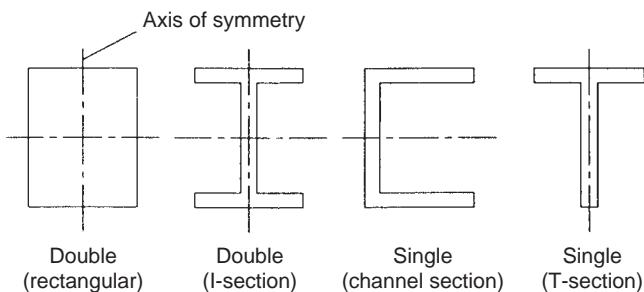
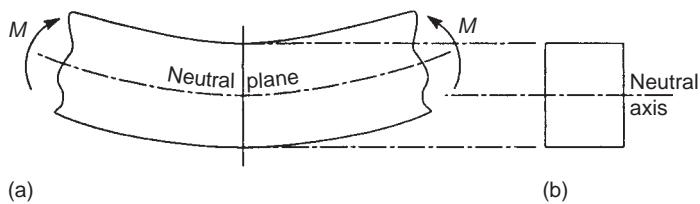


FIGURE 16.2 Symmetrical Section Beams



**FIGURE 16.3 Beam Subjected to a Pure Sagging Bending Moment**

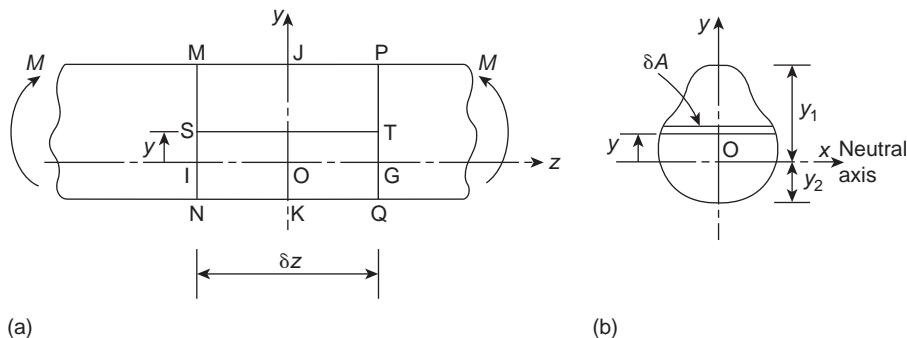
The problem, therefore, is to determine the variation of direct stress through the depth of the beam, determine the values of the stresses, and subsequently find the corresponding beam deflection.

### 16.1.1 Assumptions

The primary assumption made in determining the direct stress distribution produced by pure bending is that plane cross-sections of the beam remain plane and normal to the longitudinal fibers of the beam after bending. Again, we see this from the lines on the side of the eraser. We also assume that the material of the beam is linearly elastic, that is, it obeys Hooke's law, and that the material of the beam is homogeneous.

### 16.1.2 Direct stress distribution

Consider a length of beam (Fig. 16.4(a)) that is subjected to a pure, sagging bending moment,  $M$ , applied in a vertical plane; the beam cross-section has a vertical axis of symmetry, as shown in Fig. 16.4(b). The bending moment causes the length of beam to bend in a manner similar to that shown in Fig. 16.3(a), so that a neutral plane exists, which is, as yet, unknown distances  $y_1$  and  $y_2$  from the top and bottom of the beam, respectively. Coordinates of all points in the beam are referred to axes Oxyz in which the origin O lies in the neutral plane of the beam. We now investigate the behavior of an elemental length,  $\delta z$ , of the beam formed by parallel sections MIN and PGQ (Fig. 16.4(a)) and also



**FIGURE 16.4 Bending of a Symmetrical Section Beam**

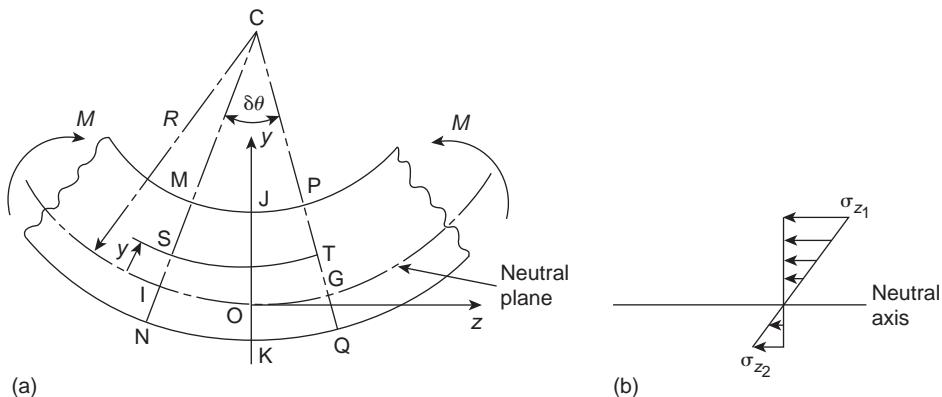


FIGURE 16.5 Length of Beam Subjected to a Pure Bending Moment

the fiber ST of cross-sectional area  $\delta A$  a distance  $y$  above the neutral plane. Clearly, before bending takes place,  $MP = IG = ST = NQ = \delta z$ .

The bending moment  $M$  causes the length of beam to bend about a *center of curvature*  $C$  as shown in Fig. 16.5(a). Since the element is small in length and a pure moment is applied, we can take the curved shape of the beam to be circular with a *radius of curvature*  $R$  measured to the neutral plane. This is a useful reference point, since, as we saw, strains and stresses are zero in the neutral plane.

The previously parallel plane sections MIN and PGQ remain plane, as we demonstrated, but are now inclined at an angle  $\delta\theta$  to each other. The length MP is now shorter than  $\delta z$ , as is ST, while NQ is longer; IG, being in the neutral plane, is still of length  $\delta z$ . Since the fiber ST has suffered a strain  $\varepsilon_z$  which is given by

$$\varepsilon_z = \frac{\text{change in length}}{\text{original length}}$$

Then,

$$\varepsilon_z = \frac{(R - y)\delta\theta - \delta z}{\delta z}$$

that is,

$$\varepsilon_z = \frac{(R - y)\delta\theta - R\delta\theta}{R\delta\theta}$$

so that

$$\varepsilon_z = -\frac{y}{R} \quad (16.1)$$

The negative sign in Eq. (16.1) indicates that fibers in the region where  $y$  is positive shorten when the bending moment is negative. Then, from Eq. (1.40), the direct stress  $\sigma_z$  in the fiber ST is given by

$$\sigma_z = -E \frac{y}{R} \quad (16.2)$$

The direct or normal force on the cross-section of the fiber ST is  $\sigma_z \delta A$ . However, since the direct stress in the beam section is due to a pure bending moment – in other words, there is no axial load – the resultant normal force on the complete cross-section of the beam must be zero. Then,

$$\int_A \sigma_z dA = 0 \quad (16.3)$$

where  $A$  is the area of the beam cross-section.

Substituting for  $\sigma_z$  in Eq. (16.3) from (16.2) gives

$$-\frac{E}{R} \int_A y dA = 0 \quad (16.4)$$

in which both  $E$  and  $R$  are constants for a beam of a given material subjected to a given bending moment. Therefore,

$$\int_A y dA = 0 \quad (16.5)$$

Equation (16.5) states that the first moment of the area of the cross-section of the beam with respect to the neutral axis, that is, the  $x$  axis, is equal to zero. Therefore, we see that *the neutral axis passes through the centroid of area of the cross-section*. Since the  $y$  axis in this case is also an axis of symmetry, it must also pass through the centroid of the cross-section. Hence, the origin, O, of the coordinate axes coincides with the centroid of area of the cross-section.

Equation (16.2) shows that, for a sagging (i.e., negative) bending moment, the direct stress in the beam section is negative (i.e., compressive) when  $y$  is positive and positive (i.e., tensile) when  $y$  is negative.

Consider now the elemental strip  $\delta A$  in Fig. 16.4(b); this is, in fact, the cross-section of the fiber ST. The strip is above the neutral axis, so that there will be a *compressive* force acting on its cross-section of  $\sigma_z \delta A$ , which is *numerically* equal to  $(Ey/R)\delta A$  from Eq. (16.2). Note that this force acts on all sections along the length of ST. At S, this force exerts a clockwise moment  $(Ey/R)y\delta A$  about the neutral axis, while at T the force exerts an identical counterclockwise moment about the neutral axis. Considering either end of ST, we see that the moment resultant about the neutral axis of the stresses on all such fibers must be *equivalent* to the applied negative moment  $M$ ; that is,

$$M = - \int_A E \frac{y^2}{R} dA$$

or

$$M = - \frac{E}{R} \int_A y^2 dA \quad (16.6)$$

The term  $\int_A y^2 dA$  is known as the *second moment of area* of the cross-section of the beam about the neutral axis and is given the symbol  $I$ . Rewriting Eq. (16.6), we have

$$M = - \frac{EI}{R} \quad (16.7)$$

or, combining this expression with Eq. (16.2),

$$\frac{M}{I} = -\frac{E}{R} = \frac{\sigma_z}{y} \quad (16.8)$$

From Eq. (16.8), we see that

$$\sigma_z = \frac{My}{I} \quad (16.9)$$

The direct stress,  $\sigma_z$ , at any point in the cross-section of a beam is therefore directly proportional to the distance of the point from the neutral axis and so varies linearly through the depth of the beam as shown, for the section JK, in Fig. 16.5(b). Clearly, for a positive bending moment,  $\sigma_z$  is positive, that is, tensile, when  $y$  is positive, and compressive (i.e., negative) when  $y$  is negative. Thus, in Fig. 16.5(b),

$$\sigma_{z,1} = \frac{My_1}{I} \text{ (compression)}, \quad \sigma_{z,2} = \frac{My_2}{I} \text{ (tension)} \quad (16.10)$$

Furthermore, we see from Eq. (16.7) that the curvature,  $1/R$ , of the beam is given by

$$\frac{1}{R} = -\frac{M}{EI} \quad (16.11)$$

and is therefore directly proportional to the applied bending moment and inversely proportional to the product  $EI$ , which is known as the *flexural rigidity* of the beam. Note that for a negative, i.e., sagging, bending moment the curvature is positive with the center of curvature above the beam.

### Example 16.1

The cross-section of a beam has the dimensions shown in Fig. 16.6(a). If the beam is subjected to a negative bending moment of 100 kNm applied in a vertical plane, determine the distribution of direct stress through the depth of the section.

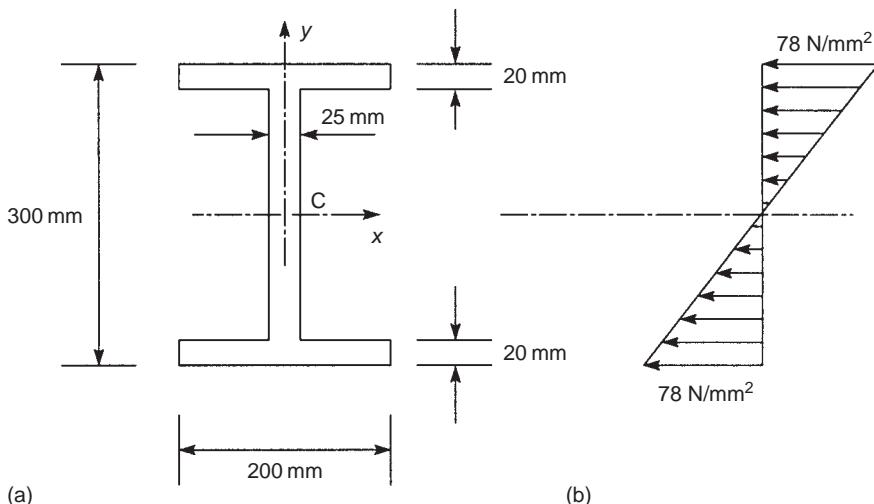


FIGURE 16.6 Direct Stress Distribution in the Beam of Example 16.1

The cross-section of the beam is doubly symmetrical, so that the centroid, C, of the section, and therefore the origin of axes, coincides with the mid-point of the web. Furthermore, the bending moment is applied to the beam section in a vertical plane, so that the  $x$  axis becomes the neutral axis of the beam section; we therefore need to calculate the second moment of area,  $I_{xx}$ , about this axis.

$$I_{xx} = \frac{200 \times 300^3}{12} - \frac{175 \times 260^3}{12} = 193.7 \times 10^6 \text{ mm}^4 \quad (\text{see Section 16.4})$$

From Eq. (16.9), the distribution of direct stress,  $\sigma_z$ , is given by

$$\sigma_z = -\frac{100 \times 10^6}{193.7 \times 10^6} y = -0.52y \quad (\text{i})$$

The direct stress, therefore, varies linearly through the depth of the section from a value

$$-0.52 \times (+150) = -78 \text{ N/mm}^2 \text{ (compression)}$$

at the top of the beam to

$$-0.52 \times (-150) = +78 \text{ N/mm}^2 \text{ (tension)}$$

at the bottom, as shown in Fig. 16.6(b).

### Example 16.2

Now determine the distribution of direct stress in the beam of Example 16.1 if the bending moment is applied on a horizontal plane and in a clockwise sense about Cy when viewed in the direction yC. Note that this will cause the length of beam to bow to the right when viewed from above.

In this case, the beam bends about the vertical  $y$  axis, which therefore becomes the neutral axis of the section. Thus, Eq. (16.9) becomes

$$\sigma_z = \frac{M}{I_{yy}} x \quad (\text{i})$$

where  $I_{yy}$  is the second moment of area of the beam section about the  $y$  axis. Again, from Section 16.4,

$$I_{yy} = 2 \times \frac{20 \times 200^3}{12} + \frac{260 \times 25^3}{12} = 27.0 \times 10^6 \text{ mm}^4$$

Hence, substituting for  $M$  and  $I_{yy}$  in Eq. (i),

$$\sigma_z = \frac{100 \times 10^6}{27.0 \times 10^6} x = 3.7x$$

We have not specified a sign convention for bending moments applied in a horizontal plane. However, a physical appreciation of the problem shows that the left-hand edges of the beam are in compression while the right-hand edges are in tension. Again, the distribution is linear and varies from  $3.7 \times (-100) = -370 \text{ N/mm}^2$  (compression) at the left-hand edges of each flange to  $3.7 \times (+100) = +370 \text{ N/mm}^2$  (tension) at the right-hand edges.

We note that the maximum stresses in this example are very much greater than those in Example 16.1. This is because the bulk of the material in the beam section is concentrated in the region of the neutral axis, where the stresses are low. The use of an I section in this manner would therefore be structurally inefficient.

**Example 16.3**

The beam section of Example 16.1 is subjected to a bending moment of 100 kNm applied in a plane parallel to the longitudinal axis of the beam but inclined at  $30^\circ$  to the left of vertical. The sense of the bending moment is clockwise when viewed from the left-hand edge of the beam section. Determine the distribution of direct stress.

The bending moment is first resolved into two components,  $M_x$  in a vertical plane and  $M_y$  in a horizontal plane. Equation (16.9) may then be written in two forms:

$$\sigma_z = \frac{M_x}{I_{xx}}y, \quad \sigma_z = \frac{M_y}{I_{yy}}x \quad (\text{i})$$

The separate distributions can then be determined and superimposed. A more direct method is to combine the two equations (i) to give the total direct stress at any point  $(x, y)$  in the section. Thus

$$\sigma_z = \frac{M_x}{I_{xx}}y + \frac{M_y}{I_{yy}}x \quad (\text{ii})$$

Now,

$$\left. \begin{aligned} M_x &= 100 \cos 30^\circ = 86.6 \text{ kNm} \\ M_y &= 100 \sin 30^\circ = 50.0 \text{ kNm} \end{aligned} \right\} \quad (\text{iii})$$

$M_x$  is, in this case, a positive bending moment producing tension in the upper half of the beam, where  $y$  is positive. Also,  $M_y$  produces tension in the left-hand half of the beam, where  $x$  is negative; we therefore call  $M_y$  a *negative bending moment*. Substituting the values of  $M_x$  and  $M_y$  from Eq. (iii) but with the appropriate sign in Eq. (ii), together with the values of  $I_{xx}$  and  $I_{yy}$  from Examples 16.1 and 16.2, we obtain

$$\sigma_z = \frac{86.6 \times 10^6}{193.7 \times 10^6}y - \frac{50.0 \times 10^6}{27.0 \times 10^6}x \quad (\text{iv})$$

or

$$\sigma_z = 0.45y - 1.85x \quad (\text{v})$$

Equation (v) gives the value of direct stress at any point in the cross-section of the beam and may also be used to determine the distribution over any desired portion. Thus, on the upper edge of the top flange,  $y = +150 \text{ mm}$ ,  $100 \text{ mm} \geq x \geq -100 \text{ mm}$ , so that the direct stress varies linearly with  $x$ . At the top left-hand corner of the top flange,

$$\sigma_z = 0.45 \times (+150) - 1.85 \times (-100) = +252.5 \text{ N/mm}^2 \text{ (tension)}$$

At the top right-hand corner,

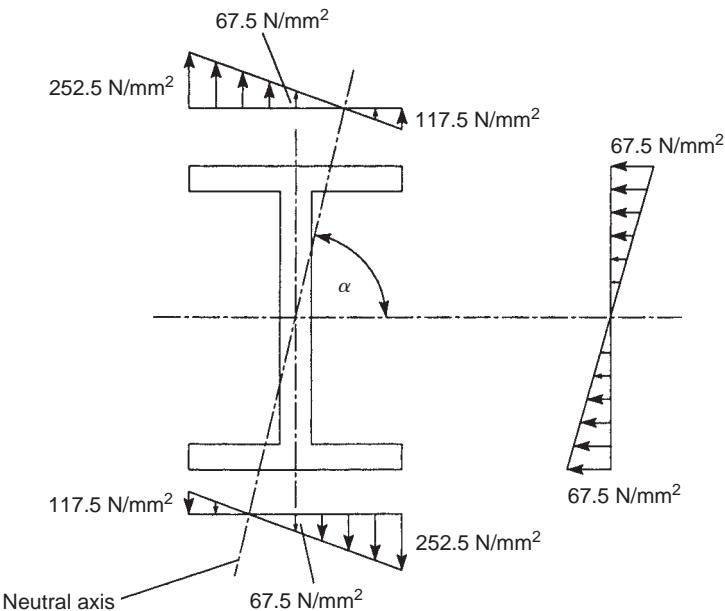
$$\sigma_z = 0.45 \times (+150) - 1.85 \times (+100) = -117.5 \text{ N/mm}^2 \text{ (compression)}$$

The distributions of direct stress over the outer edge of each flange and along the vertical axis of symmetry are shown in Fig. 16.7. Note that the neutral axis of the beam section does not, in this case, coincide with either the  $x$  or  $y$  axis, although it still passes through the centroid of the section. Its inclination,  $\alpha$ , to the  $x$  axis can be found by setting  $\sigma_z = 0$  in Eq. (v). Then,

$$0 = 0.45y - 1.85x$$

or

$$\frac{y}{x} = \frac{1.85}{0.45} = 4.11 = \tan \alpha$$



**FIGURE 16.7 Direct Stress Distribution in the Beam of Example 16.3**

which gives

$$\alpha = 76.3^\circ$$

Note that  $\alpha$  may be found in general terms from Eq. (ii) by again setting  $\sigma_z = 0$ . Hence,

$$\frac{y}{x} = -\frac{M_y I_{xx}}{M_x I_{yy}} = \tan \alpha \quad (\text{vi})$$

or

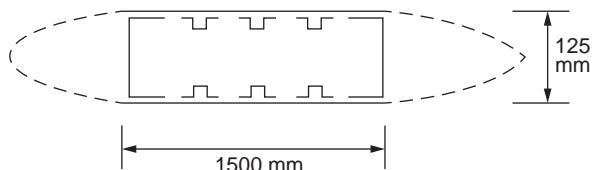
$$\tan \alpha = \frac{M_y I_{xx}}{M_x I_{yy}}$$

since  $y$  is positive and  $x$  is positive for a positive value of  $\alpha$ . We define  $\alpha$  in a slightly different way in Section 16.2.4 for beams of unsymmetrical section.

#### Example 16.4

The wing of the aircraft in P.14.1 at the Section AA has the configuration shown in Fig. 16.8. For the bending moment of 59.6 kNm calculate the maximum direct stress in the cross-section.

The channel section spars have a wall thickness of 2.5 mm, a web depth of 125 mm, and a flange width of 50 mm. The “top hat” stiffeners each have a cross-sectional area of  $50\text{mm}^2$  while the top and bottom skins are each 2 mm thick.

**FIGURE 16.8 Wing Section of Example 16.4**

Assume that the cross-sectional area of each “top hat” section stiffener is concentrated along the centerline of the skin (see Chapter 20). Then the second moment of area of the cross-section of the wing is given by (see Section 16.4.5)

$$I_{xx} = 2 \left( 2 \times 50 \times 2.5 \times 62.5^2 + \frac{2.5 \times 125^3}{12} + 2 \times 1500 \times 62.5^2 + 3 \times 50 \times 62.5^2 \right)$$

that is,

$$I_{xx} = 27.4 \times 10^6 \text{ mm}^4$$

Then, from Eq. (16.9),

$$\sigma_z = \pm \frac{59.6 \times 10^6 \times 62.5}{27.4 \times 10^6}$$

which gives

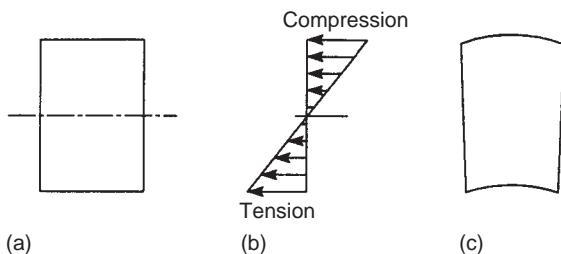
$$\sigma_z = \pm 135.9 \text{ N/mm}^2$$

The bending moment from P.14.1 is positive and therefore induces tension ( $+135.9 \text{ N/mm}^2$ ) on the upper surface of the wing and compression ( $-135.9 \text{ N/mm}^2$ ) on the lower surface.



### 16.1.3 Anticlastic bending

In the rectangular beam section shown in Fig. 16.9(a), the direct stress distribution due to a negative bending moment applied in a vertical plane varies from compression in the upper half of the beam to tension in the lower half (Fig. 16.9(b)). However, due to the Poisson effect, the compressive stress produces a lateral elongation of the upper fibers of the beam section, while the tensile stress produces a

**FIGURE 16.9 Anticlastic Bending of a Beam Section**

lateral contraction of the lower. The section does not therefore remain rectangular but distorts as shown in Fig. 16.9(c); the effect is known as *anticlastic bending*.

Anticlastic bending is of interest in the analysis of thin-walled box beams in which the cross-sections are maintained by stiffening ribs. The prevention of anticlastic distortion induces local variations in stress distributions in the webs and covers of the box beam and also in the stiffening ribs.

## 16.2 UNSYMMETRICAL BENDING

We have shown that the value of direct stress at a point in the cross-section of a beam subjected to bending depends on the position of the point, the applied loading, and the geometric properties of the cross-section. It follows that it is of no consequence whether or not the cross-section is open or closed. We therefore derive the theory for a beam of arbitrary cross-section and discuss its application to thin-walled open and closed section beams subjected to bending moments.

The assumptions are identical to those made for symmetrical bending and are listed in Section 16.1.1. However, before we derive an expression for the direct stress distribution in a beam subjected to bending, we establish sign conventions for moments, forces, and displacements; investigate the effect of choice of section on the positive directions of these parameters; and discuss the determination of the components of a bending moment applied in any longitudinal plane.

### 16.2.1 Sign conventions and notation

Forces, moments, and displacements are referred to an arbitrary system of axes Oxyz, of which Oz is parallel to the longitudinal axis of the beam and Oxy are axes in the plane of the cross-section. We assign the symbols  $M$ ,  $S$ ,  $P$ ,  $T$ , and  $w$  to bending moment, shear force, axial or direct load, torque, and distributed load intensity, respectively, with suffixes where appropriate to indicate sense or direction. Thus,  $M_x$  is a bending moment about the  $x$  axis,  $S_x$  is a shear force in the  $x$  direction, and so on. Figure 16.10 shows positive directions and senses for these loads and moments applied externally to a beam and also the positive directions of the components of displacement  $u$ ,  $v$ , and  $w$  of any point in the beam cross-section parallel to the  $x$ ,  $y$ , and  $z$  axes, respectively. A further condition defining the signs of the bending moments  $M_x$  and  $M_y$  is that they are positive when they induce tension in the positive  $xy$  quadrant of the beam cross-section. Note that this agrees with the hogging (+) and sagging (-) sign convention adopted in Section 16.1.

If we refer internal forces and moments to that face of a section which is seen when viewed in the direction  $zO$ , then, as shown in Fig. 16.11, positive internal forces and moments are in the same direction and sense as the externally applied loads, whereas on the opposite face they form an opposing system. The former system, which we use, has the advantage that direct and shear loads are always positive in the positive directions of the appropriate axes, whether they are internal loads or not. It must be realized, though, that internal stress resultants then become equivalent to externally applied forces and moments and are not in equilibrium with them.

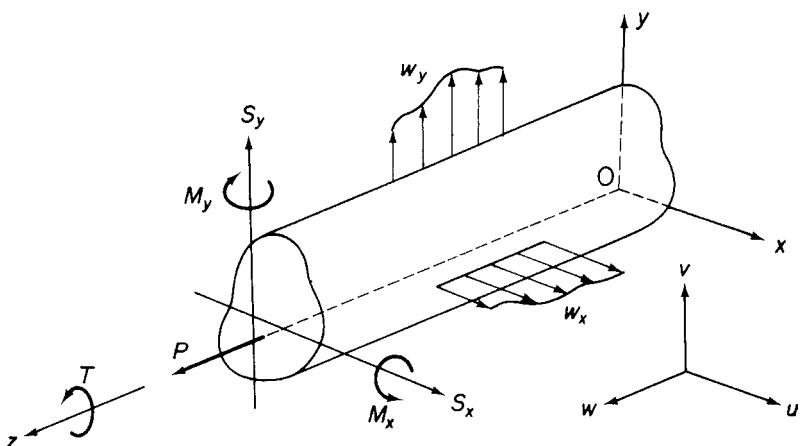


FIGURE 16.10 Notation and Sign Convention for Forces, Moments, and Displacements

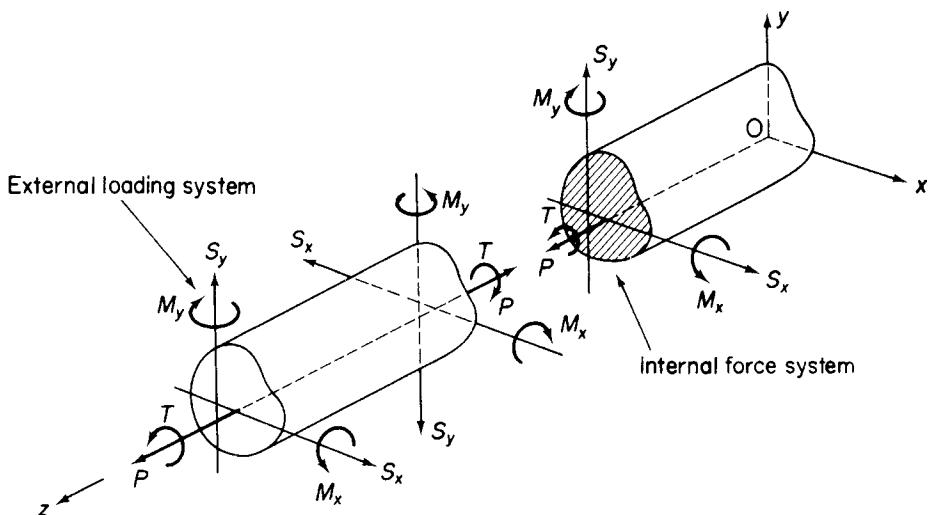
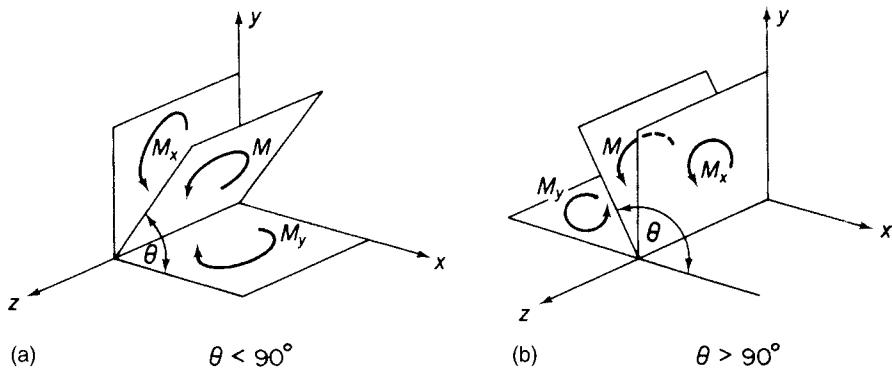


FIGURE 16.11 Internal Force System

### 16.2.2 Resolution of bending moments

A bending moment  $M$  applied in any longitudinal plane parallel to the  $z$  axis may be resolved into components  $M_x$  and  $M_y$  by the normal rules of vectors. However, a visual appreciation of the situation is often helpful. Referring to Fig. 16.12, we see that a bending moment  $M$  in a plane at an angle  $\theta$  to Ox may have components of differing sign depending on the size of  $\theta$ . In both cases, for the sense of  $M$  shown,

$$M_x = M \sin \theta$$



**FIGURE 16.12 Resolution of Bending Moments**

$$M_y = M \cos \theta$$

which give, for  $\theta < \pi/2$ ,  $M_x$  and  $M_y$  positive (Fig. 16.12(a)) and for  $\theta > \pi/2$ ,  $M_x$  positive and  $M_y$  negative (Fig. 16.12(b)).

### 16.2.3 Direct stress distribution due to bending

Consider a beam having the arbitrary cross-section shown in Fig. 16.13(a). The beam supports bending moments  $M_x$  and  $M_y$  and bends about some axis in its cross-section, which is therefore an axis of zero stress or a neutral axis (NA). Let us suppose that the origin of axes coincides with the centroid C of the cross-section and that the neutral axis is a distance  $p$  from C. The direct stress  $\sigma_z$  on an element of area  $dA$  at a point  $(x, y)$  and a distance  $\xi$  from the neutral axis is, from the third of Eq. (1.42),

$$\sigma_z = E\varepsilon_z \quad (16.12)$$

If the beam is bent to a radius of curvature  $\rho$  about the neutral axis at this particular section, then, since plane sections are assumed to remain plane after bending and by a comparison with symmetrical bending theory,

$$\varepsilon_z = \frac{\xi}{\rho} \quad (\text{see Eq. (16.1)})$$

Substituting for  $\varepsilon_z$  in Eq. (16.12), we have

$$\sigma_z = \frac{E\xi}{\rho} \quad (16.13)$$

The beam supports pure bending moments so that the resultant normal load on any section must be zero. Hence,

$$\int_A \sigma_z dA = 0$$

Therefore, replacing  $\sigma_z$  in this equation from Eq. (16.13) and cancelling the constant  $E/\rho$  gives

$$\int_A \xi dA = 0$$

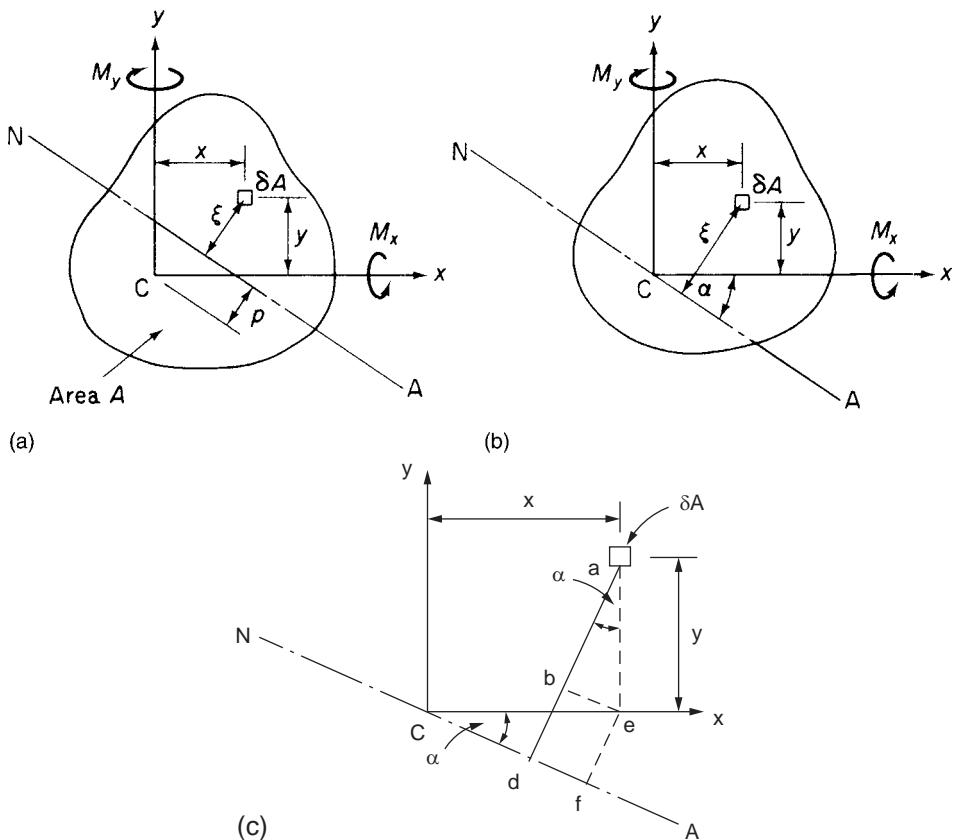


FIGURE 16.13 Determination of Neutral Axis Position and Direct Stress Due to Bending

that is, the first moment of area of the cross-section of the beam about the neutral axis is zero. It follows that the neutral axis passes through the centroid of the cross-section as shown in Fig. 16.13(b) which is the result we obtained for the case of symmetrical bending.

Suppose that the inclination of the neutral axis to  $C_x$  is  $\alpha$  (measured clockwise from  $C_x$ ); then referring to Fig. 16.13(c),

$$\xi = bd + ab = ef + ab$$

that is,

$$\xi = x \sin \alpha + y \cos \alpha \quad (16.14)$$

and from Eq. (16.13),

$$\sigma_z = \frac{E}{\rho} (x \sin \alpha + y \cos \alpha) \quad (16.15)$$

The moment resultants of the internal direct stress distribution have the same sense as the applied moments  $M_x$  and  $M_y$ . Therefore,

$$M_x = \int_A \sigma_z y \, dA, \quad M_y = \int_A \sigma_z x \, dA \quad (16.16)$$

Substituting for  $\sigma_z$  from Eq. (16.15) in (16.16) and defining the second moments of area of the section about the axes Cx, Cy as

$$I_{xx} = \int_A y^2 \, dA, \quad I_{yy} = \int_A x^2 \, dA, \quad I_{xy} = \int_A xy \, dA$$

gives

$$M_x = \frac{E \sin \alpha}{\rho} I_{xy} + \frac{E \cos \alpha}{\rho} I_{xx}, \quad M_y = \frac{E \sin \alpha}{\rho} I_{yy} + \frac{E \cos \alpha}{\rho} I_{xy}$$

or, in matrix form,

$$\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = \frac{E}{\rho} \begin{bmatrix} I_{xy} & I_{xx} \\ I_{yy} & I_{xy} \end{bmatrix} \begin{Bmatrix} \sin \alpha \\ \cos \alpha \end{Bmatrix}$$

from which

$$\frac{E}{\rho} \begin{Bmatrix} \sin \alpha \\ \cos \alpha \end{Bmatrix} = \begin{bmatrix} I_{xy} & I_{xx} \\ I_{yy} & I_{xy} \end{bmatrix}^{-1} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix}$$

that is,

$$\frac{E}{\rho} \begin{Bmatrix} \sin \alpha \\ \cos \alpha \end{Bmatrix} = \frac{1}{I_{xx}I_{yy} - I_{xy}^2} \begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix}$$

so that, from Eq. (16.15),

$$\sigma_z = \left( \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) x + \left( \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) y \quad (16.17)$$

Alternatively, Eq. (16.17) may be rearranged in the form

$$\sigma_z = \frac{M_x(I_{yy}y - I_{xy}x)}{I_{xx}I_{yy} - I_{xy}^2} + \frac{M_y(I_{xx}x - I_{xy}y)}{I_{xx}I_{yy} - I_{xy}^2} \quad (16.18)$$

From Eq. (16.18), it can be seen that, if, say,  $M_y = 0$ , the moment  $M_x$  produces a stress that varies with both  $x$  and  $y$ ; similarly for  $M_y$  if  $M_x = 0$ . In the case where the beam cross-section has either (or both) Cx or Cy as an axis of symmetry, the product second moment of area  $I_{xy}$  is zero and Cxy are *principal axes*. Equation (16.18) then reduces to

$$\sigma_z = \frac{M_x}{I_{xx}} y + \frac{M_y}{I_{yy}} x \quad (16.19)$$

Further, if either  $M_y$  or  $M_x$  is zero, then

$$\sigma_z = \frac{M_x}{I_{xx}} y \quad \text{or} \quad \sigma_z = \frac{M_y}{I_{yy}} x \quad (16.20)$$

Equations (16.19) and (16.20) are those derived for the bending of beams having at least a singly symmetrical cross-section (see Section 16.1). It may also be noted that, in Eq. (16.20),  $\sigma_z = 0$  when, for the first equation,  $y = 0$  and, for the second equation, when  $x = 0$ . Therefore, in symmetrical bending theory, the  $x$  axis becomes the neutral axis when  $M_y = 0$  and the  $y$  axis becomes the neutral axis when  $M_x = 0$ . We see that the position of the neutral axis depends on the form of the applied loading as well as the geometrical properties of the cross-section.

There exists, in any unsymmetrical cross-section, a centroidal set of axes for which the product second moment of area is zero.<sup>1</sup> These axes are then principal axes and the direct stress distribution referred to these axes takes the simplified form of Eq. (16.19) or (16.20). It therefore appears that the amount of computation can be reduced if these axes are used. This is not the case, however, unless the principal axes are obvious from inspection, since the calculation of the position of the principal axes, the principal sectional properties, and the coordinates of points at which the stresses are to be determined consumes a greater amount of time than direct use of Eqs. (16.17) or (16.18) for an arbitrary but convenient set of centroidal axes.

#### 16.2.4 Position of the neutral axis

The neutral axis always passes through the centroid of area of a beam's cross-section, but its inclination  $\alpha$  (see Fig. 16.13(b)) to the  $x$  axis depends on the form of the applied loading and the geometrical properties of the beam's cross-section.

At all points on the neutral axis, the direct stress is zero. Therefore, from Eq. (16.17),

$$0 = \left( \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) x_{NA} + \left( \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) y_{NA}$$

where  $x_{NA}$  and  $y_{NA}$  are the coordinates of any point on the neutral axis. Hence,

$$\frac{y_{NA}}{x_{NA}} = - \frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}}$$

or, referring to Fig. 16.13(b) and noting that, when  $\alpha$  is positive,  $x_{NA}$  and  $y_{NA}$  are of opposite sign,

$$\tan \alpha = \frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}} \quad (16.21)$$

---

#### Example 16.5

A beam having the cross-section shown in Fig. 16.14 is subjected to a bending moment of 1500 Nm in a vertical plane. Calculate the maximum direct stress due to bending stating the point at which it acts.

The position of the centroid of the section may be found by taking moments of areas about some convenient point. Thus,

$$(120 \times 8 + 80 \times 8)\bar{y} = 120 \times 8 \times 4 + 80 \times 8 \times 48$$

giving

$$\bar{y} = 21.6 \text{ mm}$$

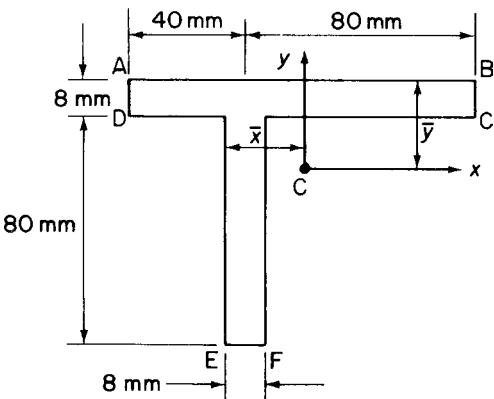


FIGURE 16.14 Cross-Section of Beam in Example 16.5

and

$$(120 \times 8 + 80 \times 8)\bar{x} = 80 \times 8 \times 4 + 120 \times 8 \times 24$$

giving

$$\bar{x} = 16 \text{ mm}$$

The next step is to calculate the section properties referred to axes Cxy (see Section 16.4):

$$I_{xx} = \frac{120 \times (8)^3}{12} + 120 \times 8 \times (17.6)^2 + \frac{8 \times (80)^3}{12} + 80 \times 8 \times (26.4)^2$$

$$= 1.09 \times 10^6 \text{ mm}^4$$

$$I_{yy} = \frac{8 \times (120)^3}{12} + 120 \times 8 \times (8)^2 + \frac{80 \times (8)^3}{12} + 80 \times 8 \times (12)^2$$

$$= 1.31 \times 10^6 \text{ mm}^4$$

$$I_{xy} = 120 \times 8 \times 8 \times 17.6 + 80 \times 8 \times (-12) \times (-26.4)$$

$$= 0.34 \times 10^6 \text{ mm}^4$$

Since  $M_x = 1500 \text{ Nm}$  and  $M_y = 0$ , we have, from Eq. (16.18),

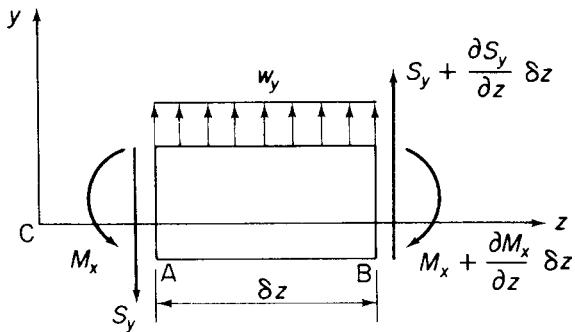
$$\sigma_z = 1.5y - 0.39x \quad (i)$$

in which the units are N and mm.

By inspection of Eq. (i), we see that  $\sigma_z$  is a maximum at F, where  $x = -8 \text{ mm}$ ,  $y = -66.4 \text{ mm}$ . Thus,

$$\sigma_{z,\max} = -96 \text{ N/mm}^2 \text{ (compressive)}$$

In some cases the maximum value cannot be obtained by inspection, so that values of  $\sigma_z$  at several points must be calculated.

FIGURE 16.15 Equilibrium of a Beam Element Supporting a General Force System in the  $yz$  Plane

### 16.2.5 Load intensity, shear force, and bending moment relationships, the general case

Consider an element of length  $\delta z$  of a beam of unsymmetrical cross-section subjected to shear forces, bending moments, and a distributed load of varying intensity, all in the  $yz$  plane, as shown in Fig. 16.15. The forces and moments are positive in accordance with the sign convention previously adopted. Over the length of the element, we assume that the intensity of the distributed load is constant. Therefore, for equilibrium of the element in the  $y$  direction,

$$\left( S_y + \frac{\partial S_y}{\partial z} \delta z \right) + w_y \delta z - S_y = 0$$

from which

$$w_y = - \frac{\partial S_y}{\partial z}$$

Taking moments about A, we have

$$\left( M_x + \frac{\partial M_x}{\partial z} \delta z \right) - \left( S_y + \frac{\partial S_y}{\partial z} \delta z \right) \delta z - w_y \frac{(\delta z)^2}{2} - M_x = 0$$

or, when second-order terms are neglected,

$$S_y = \frac{\partial M_x}{\partial z}$$

We may combine these results into a single expression:

$$-w_y = \frac{\partial S_y}{\partial z} = \frac{\partial^2 M_x}{\partial z^2} \quad (16.22)$$

Similarly, for loads in the  $xz$  plane,

$$-w_x = \frac{\partial S_x}{\partial z} = \frac{\partial^2 M_y}{\partial z^2} \quad (16.23)$$

### 16.3 DEFLECTIONS DUE TO BENDING

We noted that a beam bends about its neutral axis whose inclination relative to arbitrary centroidal axes is determined from Eq. (16.21). Suppose that, at some section of an unsymmetrical beam, the deflection normal to the neutral axis (and therefore an absolute deflection) is  $\zeta$ , as shown in Fig. 16.16. In other words, the centroid C is displaced from its initial position  $C_I$  through an amount  $\zeta$  to its final position  $C_F$ . Suppose also that the center of curvature R of the beam at this particular section is on the opposite side of the neutral axis to the direction of the displacement  $\zeta$  and that the radius of curvature is  $\rho$ . For this position of the center of curvature and from the usual approximate expression for curvature, we have

$$\frac{1}{\rho} = \frac{d^2\zeta}{dz^2} \quad (16.24)$$

The components  $u$  and  $v$  of  $\zeta$  are in the negative directions of the  $x$  and  $y$  axes, respectively, so that

$$u = -\zeta \sin \alpha, \quad v = -\zeta \cos \alpha \quad (16.25)$$

Differentiating Eqs. (16.25) twice with respect to  $z$  and substituting for  $\zeta$  from Eq. (16.24), we obtain

$$\frac{\sin \alpha}{\rho} = -\frac{d^2u}{dz^2}, \quad \frac{\cos \alpha}{\rho} = -\frac{d^2v}{dz^2} \quad (16.26)$$

In the derivation of Eq. (16.17), we see that

$$\frac{1}{\rho} \begin{Bmatrix} \sin \alpha \\ \cos \alpha \end{Bmatrix} = \frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)} \begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix} \quad (16.27)$$

Substituting in Eqs. (16.27) for  $\sin \alpha / \rho$  and  $\cos \alpha / \rho$  from Eqs (16.26) and writing  $u'' = d^2u/dz^2$ ,  $v'' = d^2v/dz^2$ , we have

$$\begin{Bmatrix} u'' \\ v'' \end{Bmatrix} = \frac{-1}{E(I_{xx}I_{yy} - I_{xy}^2)} \begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix} \quad (16.28)$$

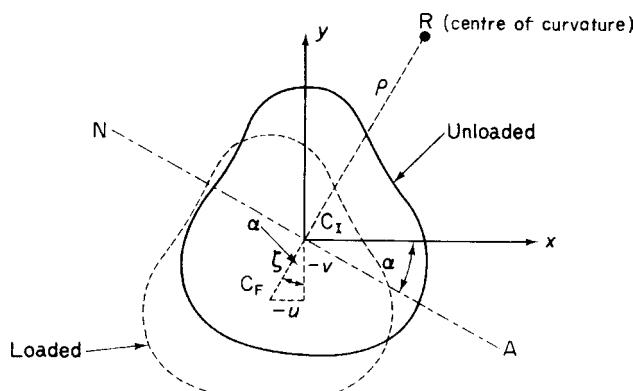


FIGURE 16.16 Determination of Beam Deflection Due to Bending

It is instructive to rearrange Eq. (16.28) as follows:

$$\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = -E \begin{bmatrix} I_{xy} & I_{xx} \\ I_{yy} & I_{xy} \end{bmatrix} \begin{Bmatrix} u'' \\ v'' \end{Bmatrix} \quad (\text{see derivation of Eq. (16.17)}) \quad (16.29)$$

that is,

$$\begin{aligned} M_x &= -EI_{xy}u'' - EI_{xx}v'' \\ M_y &= -EI_{yy}u'' - EI_{xy}v'' \end{aligned} \quad (16.30)$$

The first of Eqs. (16.30) shows that  $M_x$  produces curvatures, that is, deflections, in both the  $xz$  and  $yz$  planes even though  $M_y=0$ ; similarly for  $M_y$  when  $M_x=0$ . Thus, for example, an unsymmetrical beam deflects both vertically and horizontally, even though the loading is entirely in a vertical plane. Similarly, vertical and horizontal components of deflection in an unsymmetrical beam are produced by horizontal loads.

For a beam having either  $C_x$  or  $C_y$  (or both) as an axis of symmetry,  $I_{xy}=0$  and Eqs. (16.28) reduce to

$$u'' = -\frac{M_y}{EI_{yy}}, \quad v'' = -\frac{M_x}{EI_{xx}} \quad (16.31)$$

### Example 16.6

Determine the deflection curve and the deflection of the free end of the cantilever shown in Fig. 16.17(a); the flexural rigidity of the cantilever is  $EI$  and its section is doubly symmetrical.

The load  $W$  causes the cantilever to deflect such that its neutral plane takes up the curved shape shown in Fig. 16.17(b); the deflection at any section  $Z$  is then  $v$ , while that at its free end is  $v_{\text{tip}}$ . The axis system is chosen so that the origin coincides with the built-in end, where the deflection is clearly zero.

The bending moment,  $M$ , at the section  $Z$  is, from Fig. 16.17(a),

$$M = W(L-z) \quad (\text{i})$$

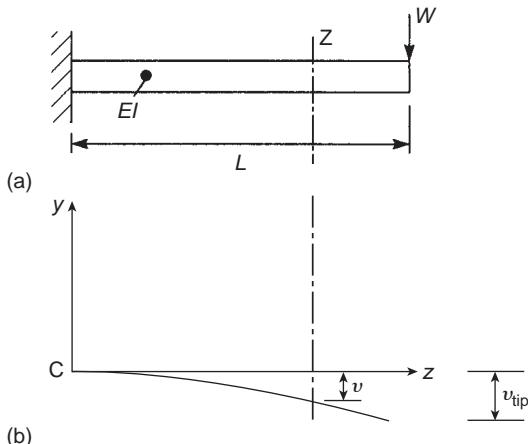


FIGURE 16.17 Deflection of a Cantilever Beam Carrying a Concentrated Load at Its Free End (Example 16.6)

Substituting for  $M$  in the second of Eq. (16.31),

$$v'' = -\frac{W}{EI}(L-z)$$

or, in more convenient form,

$$EIv'' = -W(L-z) \quad (\text{ii})$$

Integrating Eq. (ii) with respect to  $z$  gives

$$EIv' = -W\left(Lz - \frac{z^2}{2}\right) + C_1$$

where  $C_1$  is a constant of integration, which is obtained from the boundary condition that  $v'=0$  at the built-in end, where  $z=0$ . Hence,  $C_1=0$  and

$$EIv' = -W\left(Lz - \frac{z^2}{2}\right) \quad (\text{iii})$$

Integrating Eq. (iii), we obtain

$$EIv = -W\left(\frac{Lz^2}{2} - \frac{z^3}{6}\right) + C_2$$

in which  $C_2$  is again a constant of integration. At the built-in end,  $v=0$  when  $z=0$ , so that  $C_2=0$ . Hence, the equation of the deflection curve of the cantilever is

$$v = -\frac{W}{6EI}(3Lz^2 - z^3) \quad (\text{iv})$$

The deflection,  $v_{\text{tip}}$ , at the free end is obtained by setting  $z=L$  in Eq. (iv). Then,

$$v_{\text{tip}} = -\frac{WL^3}{3EI} \quad (\text{v})$$

and is clearly negative and downward.

### Example 16.7

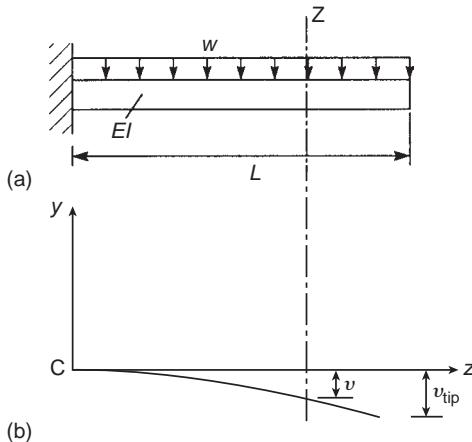
Determine the deflection curve and the deflection of the free end of the cantilever shown in Fig. 16.18(a). The cantilever has a doubly symmetrical cross-section.

The bending moment,  $M$ , at any section  $Z$  is given by

$$M = \frac{w}{2}(L-z)^2 \quad (\text{i})$$

Substituting for  $M$  in the second of Eqs. (16.31) and rearranging, we have

$$EIv'' = -\frac{w}{2}(L-z)^2 = -\frac{w}{2}(L^2 - 2Lz + z^2) \quad (\text{ii})$$

**FIGURE 16.18 Deflection of a Cantilever Beam Carrying a Uniformly Distributed Load**

Integration of Eq. (ii) yields

$$EIv' = -\frac{w}{2} \left( L^2 z - Lz^2 + \frac{z^3}{3} \right) + C_1$$

When  $z=0$  at the built-in end,  $v'=0$ , so that  $C_1=0$  and

$$EIv' = -\frac{w}{2} \left( L^2 z - Lz^2 + \frac{z^3}{3} \right) \quad (\text{iii})$$

Integrating Eq. (iii), we have

$$EIv = -\frac{w}{2} \left( L^2 \frac{z^2}{2} - \frac{Lz^3}{3} + \frac{z^4}{12} \right) + C_2$$

and since  $v=0$  when  $z=0$ ,  $C_2=0$ . The deflection curve of the beam therefore has the equation

$$v = -\frac{w}{24EI} (6L^2 z^2 - 4Lz^3 + z^4) \quad (\text{iv})$$

and the deflection at the free end when  $z=L$  is

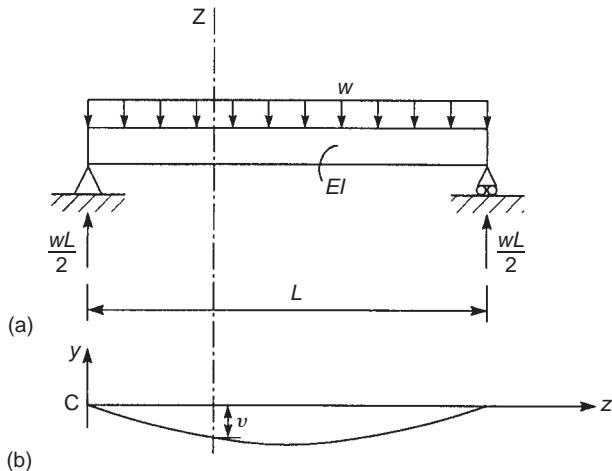
$$v_{\text{tip}} = -\frac{wL^4}{8EI} \quad (\text{v})$$

which is again negative and downward.

### Example 16.8

Determine the deflection curve and the mid-span deflection of the simply supported beam shown in Fig. 16.19(a); the beam has a doubly symmetrical cross-section.

The support reactions are each  $wL/2$  and the bending moment,  $M$ , at any section  $Z$ , a distance  $z$  from the left-hand support is



**FIGURE 16.19 Deflection of a Simply Supported Beam Carrying a Uniformly Distributed Load (Example 16.8)**

$$M = -\frac{wL}{2}z + \frac{wz^2}{2} \quad (\text{i})$$

Substituting for  $M$  in the second of Eqs. (16.31), we obtain

$$EIv'' = \frac{w}{2}(Lz - z^2) \quad (\text{ii})$$

Integrating, we have

$$EIv' = \frac{w}{2} \left( \frac{Lz^2}{2} - \frac{z^3}{3} \right) + C_1$$

From symmetry, it is clear that, at the mid-span section, the gradient  $v'=0$ . Hence,

$$0 = \frac{w}{2} \left( \frac{L^3}{8} - \frac{L^3}{24} \right) + C_1$$

which gives

$$C_1 = -\frac{wL^3}{24}$$

Therefore,

$$EIv' = \frac{w}{24}(6Lz^2 - 4z^3 - L^3) \quad (\text{iii})$$

Integrating again gives

$$EIv = \frac{w}{24}(2Lz^3 - z^4 - L^3z) + C_2$$

Since  $v=0$  when  $z=0$  (or since  $v=0$  when  $z=L$ ), it follows that  $C_2=0$  and the deflected shape of the beam has the equation

$$v = \frac{w}{24EI}(2Lz^3 - z^4 - L^3z) \quad (\text{iv})$$

The maximum deflection occurs at mid-span, where  $z=L/2$  and is

$$v_{\text{mid-span}} = -\frac{5wL^4}{384EI} \quad (\text{v})$$

So far, the constants of integration were determined immediately they arose. However, in some cases, a relevant boundary condition – say, a value of gradient – is not obtainable. The method is then to carry the unknown constant through the succeeding integration and use known values of deflection at two sections of the beam. Thus, in the previous example, Eq. (ii) is integrated twice to obtain

$$EIv = \frac{w}{2} \left( \frac{Lz^3}{6} - \frac{z^4}{12} \right) + C_1z + C_2$$

The relevant boundary conditions are  $v=0$  at  $z=0$  and  $z=L$ . The first of these gives  $C_2=0$ , while from the second, we have  $C_1=-wL^3/24$ . Thus, the equation of the deflected shape of the beam is

$$v = \frac{w}{24EI} (2Lz^3 - z^4 - L^3z)$$

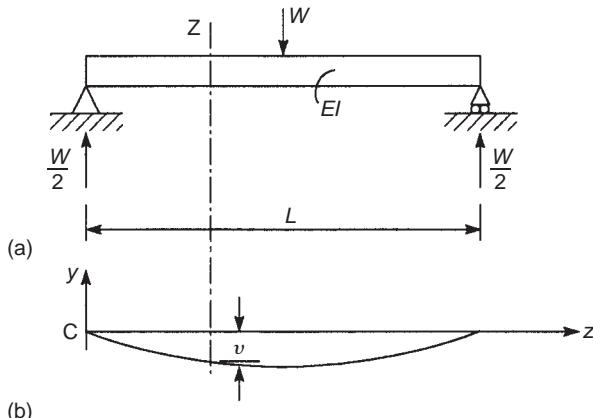
as before.

### Example 16.9

Figure 16.20(a) shows a simply supported beam carrying a concentrated load  $W$  at mid-span. Determine the deflection curve of the beam and the maximum deflection if the beam section is doubly symmetrical.

The support reactions are each  $W/2$  and the bending moment  $M$  at a section  $Z$  a distance  $z$  ( $\leq L/2$ ) from the left-hand support is

$$M = -\frac{W}{2}z \quad (\text{i})$$



**FIGURE 16.20 Deflection of a Simply Supported Beam Carrying a Concentrated Load at Mid-span (Example 16.9)**

From the second of Eqs. (16.31), we have

$$EIv'' = \frac{W}{2}z \quad (\text{ii})$$

Integrating, we obtain

$$EIv' = \frac{W}{2} \frac{z^2}{2} + C_1$$

From symmetry, the slope of the beam is zero at mid-span, where  $z=L/2$ . Thus,  $C_1=-WL^2/16$  and

$$EIv' = \frac{W}{16}(4z^2 - L^2) \quad (\text{iii})$$

Integrating Eq. (iii), we have

$$EIv = \frac{W}{16} \left( \frac{4z^3}{3} - L^2 z \right) + C_2$$

and when  $z=0$ ,  $v=0$ , so that  $C_2=0$ . The equation of the deflection curve is, therefore,

$$v = \frac{W}{48EI}(4z^3 - 3L^2 z) \quad (\text{iv})$$

The maximum deflection occurs at mid-span and is

$$v_{\text{mid-span}} = -\frac{WL^3}{48EI} \quad (\text{v})$$

Note that, in this problem, we could not use the boundary condition that  $v=0$  at  $z=L$  to determine  $C_2$ , since Eq. (i) applies only for  $0 \leq z \leq L/2$ ; it follows that Eqs. (iii) and (iv) for slope and deflection apply only for  $0 \leq z \leq L/2$ , although the deflection curve is clearly symmetrical about mid-span.

Examples 16.6–16.9 are frequently regarded as ‘standard’ cases of beam deflection.

### 16.3.1 Singularity functions

The double integration method used in Examples 16.6–16.9 becomes extremely lengthy when even relatively small complications, such as the lack of symmetry due to an offset load, are introduced. For example, the addition of a second concentrated load on a simply supported beam results in a total of six equations for slope and deflection, producing six arbitrary constants. Clearly, the computation involved in determining these constants is tedious, even though a simply supported beam carrying two concentrated loads is a comparatively simple practical case. An alternative approach is to introduce so-called *singularity* or *half-range* functions. Such functions were first applied to beam deflection problems by Macauley in 1919, and hence the method is frequently known as *Macauley’s method*.

We now introduce a quantity  $[z-a]$  and define it to be zero if  $(z-a)<0$ , that is,  $z < a$ , and to be simply  $(z-a)$  if  $z > a$ . The quantity  $[z-a]$  is known as a *singularity* or *half-range function* and is defined to have a value only when the argument is positive, in which case the square brackets behave in an identical manner to ordinary parentheses.

**Example 16.10**

Determine the position and magnitude of the maximum upward and downward deflections of the beam shown in Fig. 16.21.

A consideration of the overall equilibrium of the beam gives the support reactions; thus,

$$R_A = \frac{3}{4}W \text{ (upward)}, \quad R_F = \frac{3}{4}W \text{ (downward)}$$

Using the method of singularity functions and taking the origin of axes at the left-hand support, we write an expression for the bending moment,  $M$ , at any section  $Z$  between D and F, *the region of the beam furthest from the origin*:

$$M = -R_A z + W[z - a] + W[z - 2a] - 2W[z - 3a] \quad (\text{i})$$

Substituting for  $M$  in the second of Eqs. (16.31), we have

$$EIv'' = \frac{3}{4}Wz - W[z - a] - W[z - 2a] + 2W[z - 3a] \quad (\text{ii})$$

Integrating Eq. (ii) and retaining the square brackets, we obtain

$$EIv' = \frac{3}{8}Wz^2 - \frac{W}{2}[z - a]^2 - \frac{W}{2}[z - 2a]^2 + W[z - 3a]^2 + C_1 \quad (\text{iii})$$

and

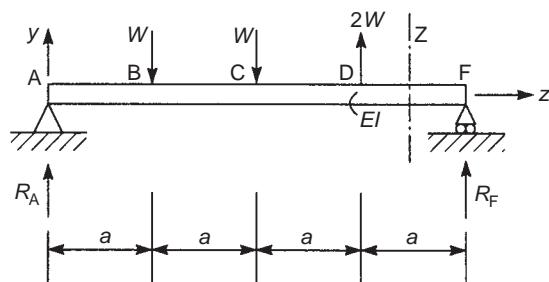
$$Elv = \frac{1}{8}Wz^3 - \frac{W}{6}[z - a]^3 - \frac{W}{6}[z - 2a]^3 + \frac{W}{3}[z - 3a]^3 + C_1z + C_2 \quad (\text{iv})$$

in which  $C_1$  and  $C_2$  are arbitrary constants. When  $z = 0$  (at A),  $v = 0$ , and hence  $C_2 = 0$ . Note that the second, third, and fourth terms on the right-hand side of Eq. (iv) disappear for  $z < a$ . Also  $v = 0$  at  $z = 4a$  (F), so that, from Eq. (iv), we have

$$0 = \frac{W}{8}64a^3 - \frac{W}{6}27a^3 - \frac{W}{6}8a^3 + \frac{W}{3}a^3 + 4aC_1$$

which gives

$$C_1 = -\frac{5}{8}Wa^2$$



**FIGURE 16.21 Macauley's Method for the Deflection of a Simply Supported Beam (Example 16.10)**

Equations (iii) and (iv) now become

$$EIv' = \frac{3}{8}Wz^2 - \frac{W}{2}[z-a]^2 - \frac{W}{2}[z-2a]^2 + W[z-3a]^2 - \frac{5}{8}Wa^2 \quad (\text{v})$$

and

$$EIv = \frac{1}{8}Wz^3 - \frac{W}{6}[z-a]^3 - \frac{W}{6}[z-2a]^3 + \frac{W}{3}[z-3a]^3 - \frac{5}{8}Wa^2z \quad (\text{vi})$$

respectively.

To determine the maximum upward and downward deflections, we need to know in which bays  $v' = 0$  and thereby which terms in Eq. (v) disappear when the exact positions are being located. One method is to select a bay and determine the sign of the slope of the beam at the extremities of the bay. A change of sign indicates that the slope is zero within the bay.

By inspection of Fig. 16.21, it seems likely that the maximum downward deflection occurs in BC. At B, using Eq. (v),

$$Elv' = \frac{3}{8}Wa^2 - \frac{5}{8}Wa^2$$

which is clearly negative. At C,

$$EIv' = \frac{3}{8}W4a^2 - \frac{W}{2}a^2 - \frac{5}{8}Wa^2$$

which is positive. Therefore, the maximum downward deflection does occur in BC and its exact position is located by equating  $v'$  to zero for any section in BC. Thus, from Eq. (v),

$$0 = \frac{3}{8}Wz^2 - \frac{W}{2}[z-a]^2 - \frac{5}{8}Wa^2$$

or, simplifying,

$$0 = z^2 - 8az + 9a^2 \quad (\text{vii})$$

Solution of Eq. (vii) gives

$$z = 1.35a$$

so that the maximum downward deflection is, from Eq. (vi),

$$EIv = \frac{1}{8}W(1.35a)^3 - \frac{W}{6}(0.35a)^3 - \frac{5}{8}Wa^2(1.35a)$$

that is,

$$v_{\max}(\text{downward}) = -\frac{0.54Wa^3}{EI}$$

In a similar manner it can be shown that the maximum upward deflection lies between D and F at  $z=3.42a$  and that its magnitude is

$$v_{\max}(\text{upward}) = \frac{0.04Wa^3}{EI}$$

An alternative method of determining the position of maximum deflection is to select a possible bay, set  $v' = 0$  for that bay and solve the resulting equation in  $z$ . If the solution gives a value of  $z$  that lies within the bay, then the selection is correct; otherwise, the procedure must be repeated for a second and possibly a third and a fourth bay. This method is quicker than the former if the correct bay is selected initially; if not, the equation corresponding to each selected bay must be completely solved, a procedure clearly longer than determining the sign of the slope at the extremities of the bay.

### Example 16.11

Determine the position and magnitude of the maximum deflection in the beam of Fig. 16.22.

Following the method of Example 16.10, we determine the support reactions and find the bending moment,  $M$ , at any section  $Z$  in the bay furthest from the origin of the axes.

Then,

$$M = -R_A z + w \frac{L}{4} \left[ z - \frac{5L}{8} \right] \quad (i)$$

Examining Eq. (i), we see that the singularity function  $[z - 5L/8]$  does not become zero until  $z \leq 5L/8$ , although Eq. (i) is valid only for  $z \geq 3L/4$ . To obviate this difficulty, we extend the distributed load to the support D while simultaneously restoring the status quo by applying an upward-distributed load of the same intensity and length as the additional load (Fig. 16.23).

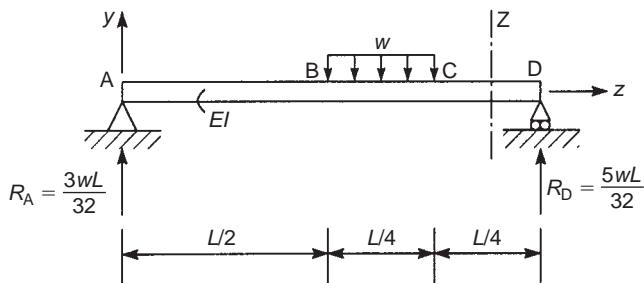


FIGURE 16.22 Deflection of a Beam Carrying a Part Span Uniformly Distributed Load (Example 16.11)

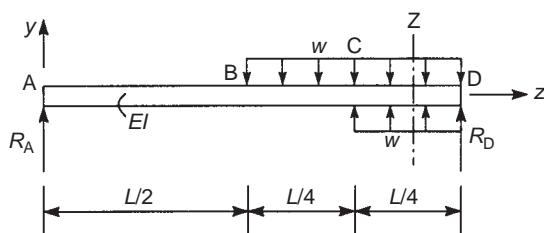


FIGURE 16.23 Method of Solution for a Part Span Uniformly Distributed Load

At the section Z, a distance  $z$  from A, the bending moment is now given by

$$M = -R_A z + \frac{w}{2} \left[ z - \frac{L}{2} \right]^2 - \frac{w}{2} \left[ z - \frac{3L}{4} \right]^2 \quad (\text{ii})$$

Equation (ii) is valid for all sections of the beam if the singularity functions are discarded as they become zero. Substituting Eq. (ii) into the second of Eqs. (16.31), we obtain

$$EI\nu'' = \frac{3}{32} w L z - \frac{w}{2} \left[ z - \frac{L}{2} \right]^2 + \frac{w}{2} \left[ z - \frac{3L}{4} \right]^2 \quad (\text{iii})$$

Integrating Eq. (iii) gives

$$EI\nu' = \frac{3}{64} w L z^2 - \frac{w}{6} \left[ z - \frac{L}{2} \right]^3 + \frac{w}{6} \left[ z - \frac{3L}{4} \right]^3 + C_1 \quad (\text{iv})$$

$$EI\nu = \frac{w L z^3}{64} - \frac{w}{24} \left[ z - \frac{L}{2} \right]^4 + \frac{w}{24} \left[ z - \frac{3L}{4} \right]^4 + C_1 z + C_2 \quad (\text{v})$$

where  $C_1$  and  $C_2$  are arbitrary constants. The required boundary conditions are  $\nu=0$  when  $z=0$  and  $z=L$ . From the first of these, we obtain  $C_2=0$ , while the second gives

$$0 = \frac{w L^4}{64} - \frac{w}{24} \left( \frac{L}{2} \right)^4 + \frac{w}{24} \left( \frac{L}{4} \right)^4 + C_1 L$$

from which

$$C_1 = -\frac{27wL^3}{2,048}$$

Equations (iv) and (v) then become

$$EI\nu' = \frac{3}{64} w L z^2 - \frac{w}{6} \left[ z - \frac{L}{2} \right]^3 + \frac{w}{6} \left[ z - \frac{3L}{4} \right]^3 - \frac{27wL^3}{2,048} \quad (\text{vi})$$

and

$$EI\nu = \frac{w L z^3}{64} - \frac{w}{24} \left[ z - \frac{L}{2} \right]^4 + \frac{w}{24} \left[ z - \frac{3L}{4} \right]^4 - \frac{27wL^3}{2,048} z \quad (\text{vii})$$

In this problem, the maximum deflection clearly occurs in the region BC of the beam. Therefore, equating the slope to zero for BC, we have

$$0 = \frac{3}{64} w L z^2 - \frac{w}{6} \left[ z - \frac{L}{2} \right]^3 - \frac{27wL^3}{2,048}$$

which simplifies to

$$z^3 - 1.78 L z^2 + 0.75 z L^2 - 0.046 L^3 = 0 \quad (\text{viii})$$

Solving Eq. (viii) by trial and error, we see that the slope is zero at  $z \approx 0.6 L$ . Hence, from Eq. (vii), the maximum deflection is

$$\nu_{\max} = -\frac{4.53 \times 10^{-3} w L^4}{EI}$$

**Example 16.12**

Determine the deflected shape of the beam shown in Fig. 16.24.

In this problem, an external moment  $M_0$  is applied to the beam at B. The support reactions are found in the normal way and are

$$R_A = -\frac{M_0}{L} \text{ (downward)}, \quad R_C = \frac{M_0}{L} \text{ (upward)}$$

The bending moment at any section Z between B and C is then given by

$$M = -R_A z - M_0 \quad (\text{Note: } [z - b]^0 = 1) \quad (\text{i})$$

Equation (i) is valid only for the region BC and clearly does not contain a singularity function, which would cause  $M_0$  to vanish for  $z \leq b$ . We overcome this difficulty by writing

$$M = -R_A z - M_0 [z - b]^0 \quad (\text{Note: } [z - b]^0 = 1) \quad (\text{ii})$$

Equation (ii) has the same value as Eq. (i) but is now applicable to all sections of the beam, since  $[z - b]^0$  disappears when  $z \leq b$ . Substituting for  $M$  from Eq. (ii) in the second of Eqs. (16.31), we obtain

$$EIv'' = R_A z + M_0 [z - b]^0 \quad (\text{iii})$$

Integration of Eq. (iii) yields

$$Elv' = R_A \frac{z^2}{2} + M_0 [z - b] + C_1 \quad (\text{vi})$$

and

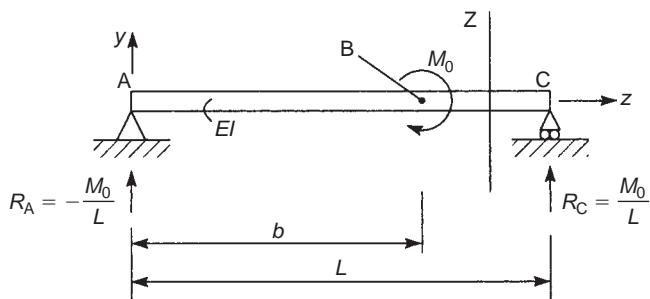
$$EIv = R_A \frac{z^3}{6} + \frac{M_0}{2} [z - b]^2 + C_1 z + C_2 \quad (\text{v})$$

where  $C_1$  and  $C_2$  are arbitrary constants. The boundary conditions are  $v = 0$  when  $z = 0$  and  $z = L$ . From the first of these, we have  $C_2 = 0$ , while the second gives

$$0 = -\frac{M_0 L^3}{L} + \frac{M_0}{2} [L - b]^2 + C_1 L$$

from which

$$C_1 = -\frac{M_0}{6L} (2L^2 - 6Lb + 3b^2) \quad (\text{vi})$$



**FIGURE 16.24 Deflection of a Simply Supported Beam Carrying a Point Moment (Example 16.12)**

The equation of the deflection curve of the beam is then

$$v = \frac{M_0}{6EI} \{z^3 + 3L[z - b]^2 - (2L^2 - 6Lb + 3b^2)z\} \quad (\text{vii})$$

### Example 16.12 MATLAB

Use MATLAB to repeat Example 16.12.

The beam deflection curve equation is obtained through the following MATLAB file:

```
% Declare any needed variables
syms M_0 L b EI z a o C_1 C_2

% Define the support reactions shown in Fig. 16.24
R_A=-M_0/L;
R_C=M_0/L;

% Define the equation for the bending moment at any section Z between B and C
M=[-R_A*z -M_0*(a)^o]; % For z<=b: a=0; For z>b: a=z-b

% Substitute M into the second of Eq. (16.32)
v_zz=-M/EI;

% Integrate v_zz to get v_z and v
v_z=[int(v_zz(1),z)+C_1/EI int(v_zz(2),a)];
v=[int(v_z(1),z)+C_2/EI int(v_z(2),a)];
v=sum(subs(v,o,0));

% Use boundary conditions to determine C_1 and C_2
% BC #1: v=0 when z=0
c_2=solve(subs(subs(v*EI,a,0),z,0),C_2);
v=subs(v,C_2,c_2);
% BC #1: v=0 when z=0
c_1=solve(subs(subs(v*EI,a,z-b),z,L),C_1,'IgnoreAnalyticConstraints',true);
v=simplify(subs(v,C_1,c_1));

% Output the resulting deflection equation to the Command Window
disp('The equation for the deflection curve of the beam is:')
disp(['v=' char(v)])
disp('Where: a=0 for z<=b, and a=z-b for z>b')
```

The Command Window outputs resulting from this MATLAB file are as follows. The equation for the deflection curve of the beam is

```
v=-(M_0*(2*L^2*z - 3*L*a^2 - 6*L*b*z+3*b^2*z+z^3))/(6*EI*L)
Where: a=0 for z<=b, and a=z-b for z>b
```

**Example 16.13**

Determine the horizontal and vertical components of the tip deflection of the cantilever shown in Fig. 16.25. The second moments of area of its unsymmetrical section are  $I_{xx}$ ,  $I_{yy}$ , and  $I_{xy}$ .

From Eqs. (16.28),

$$u'' = \frac{M_x I_{xy} - M_y I_{xx}}{E(I_{xx} I_{yy} - I_{xy}^2)} \quad (\text{i})$$

In this case,  $M_x = W(L - z)$ ,  $M_y = 0$ , so that Eq. (i) simplifies to

$$u'' = \frac{WI_{xy}}{E(I_{xx} I_{yy} - I_{xy}^2)} (L - z) \quad (\text{ii})$$

Integrating Eq. (ii) with respect to  $z$ ,

$$u' = \frac{WI_{xy}}{E(I_{xx} I_{yy} - I_{xy}^2)} \left( Lz - \frac{z^2}{2} + A \right) \quad (\text{iii})$$

and

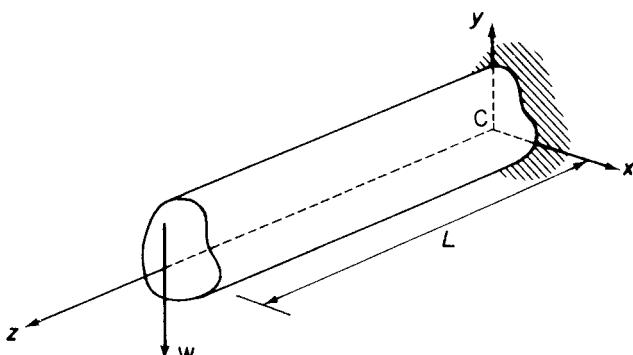
$$u = \frac{WI_{xy}}{E(I_{xx} I_{yy} - I_{xy}^2)} \left( L \frac{z^2}{2} - \frac{z^3}{6} + Az + B \right) \quad (\text{iv})$$

in which  $u'$  denotes  $du/dz$  and the constants of integration  $A$  and  $B$  are found from the boundary conditions, namely  $u' = 0$  and  $u = 0$  when  $z = 0$ . From the first of these and Eq. (iii),  $A = 0$ , while from the second and Eq. (iv),  $B = 0$ . Hence, the deflected shape of the beam in the  $xz$  plane is given by

$$u = \frac{WI_{xy}}{E(I_{xx} I_{yy} - I_{xy}^2)} \left( L \frac{z^2}{2} - \frac{z^3}{6} \right) \quad (\text{v})$$

At the free end of the cantilever ( $z = L$ ), the horizontal component of deflection is

$$u_{\text{f.e.}} = \frac{WI_{xy} L^3}{3E(I_{xy} I_{yy} - I_{xy}^2)} \quad (\text{vi})$$



**FIGURE 16.25 Determination of the Deflection of a Cantilever**

Similarly, the vertical component of the deflection at the free end of the cantilever is

$$v_{f.e.} = \frac{-WL^3}{3E(I_{xx}I_{yy} - I_{xy}^2)} \quad (vii)$$

The actual deflection  $\delta_{f.e.}$  at the free end is then given by

$$\delta_{f.e.} = (u_{f.e.}^2 + v_{f.e.}^2)^{\frac{1}{2}}$$

at an angle of  $\tan^{-1} u_{f.e.}/v_{f.e.}$  to the vertical.

Note that if either Cx or Cy were an axis of symmetry,  $I_{xy}=0$  and Eqs. (vi) and (vii) reduce to

$$u_{f.e.} = 0, \quad v_{f.e.} = \frac{-WL^3}{3EI_{xx}}$$

the well-known results for the bending of a cantilever having a symmetrical cross-section and carrying a concentrated vertical load at its free end (see Example 16.6). ■

### 16.3.2 Moment-area method for symmetrical bending

The double integration method and the method of singularity functions are used when the complete deflection curve of a beam is required. However, if only the deflection of a particular point is required, the moment-area method is generally more suitable.

Consider the second of the curvature-moment equations (Eq. (16.31)), that is,

$$\frac{d^2v}{dz^2} = -\frac{M}{EI}$$

which applies to deflections in a vertical plane the more usual deflection mode.

Integrating this expression between two sections A and B of a beam

$$\int_A^B \frac{d^2v}{dz^2} dz = - \int_A^B \frac{M}{EI} dz$$

so that

$$\left( \frac{dv}{dz} \right)_B - \left( \frac{dv}{dz} \right)_A = - \int_A^B \frac{M}{EI} dz \quad (16.32)$$

Qualitatively Eq. (16.32) states that the change of slope between two sections A and B of a beam is numerically equal to the area of the  $M/EI$  diagram between the two sections.

Now multiply both sides of the curvature-moment equation by  $z$  thereby retaining the relationship. Then

$$\frac{d^2v}{dz^2} z = -\frac{M}{EI} z$$

Integrating this equation between the two sections A and B of the beam we have

$$\int_A^B \frac{d^2v}{dz^2} z dz = - \int_A^B \frac{M}{EI} z dz$$

The left-hand side of this equation may be integrated by parts and gives

$$\left[ \frac{z \, dv}{dz} \right]_A^B - [v]_A^B = - \int_A^B \frac{M}{EI} z \, dz$$

from which

$$z_B \left( \frac{dv}{dz} \right)_B - z_A \left( \frac{dv}{dz} \right)_A - (v_B - v_A) = - \int_A^B \frac{M}{EI} z \, dz \quad (16.33)$$

in which  $z_A$  and  $z_B$  are the  $z$  coordinates of the sections A and B of the beam while  $(dv/dz)_A$  and  $(dv/dz)_B$  are the corresponding slopes and  $v_A$  and  $v_B$  the corresponding deflections. The right-hand side of Eq. (16.33) represents the moment of the area of the  $M/EI$  diagram about A. Equations (16.32) and (16.33) enable the slope and deflection to be calculated at any section of a beam. We note that in both equations we are concerned with the geometry of the  $M/EI$  diagram, which will have the same shape as the bending moment diagram unless there is a change in the beam cross-section. Further, the form of the right-hand side of both equations allows two possible approaches. In the case where the geometry of the  $M/EI$  diagram is relatively simple we can employ a *semi-graphical* approach utilizing the actual geometry of the  $M/EI$  diagram or, if the geometry is complex, we can use a completely *analytical* approach. Both methods are illustrated in the following examples.

### Example 16.14

Determine the slope and deflection of the free end of the cantilever shown in Fig. 16.26(a).

In this case, the cross-section of the cantilever is uniform so that the  $M/EI$  diagram is identical in shape to the bending moment diagram as shown in Fig. 16.26(b).

We choose the free end B of the cantilever as the origin for  $z$ , since this will give a slightly simpler expression for the bending moment at any section of the beam and will, as we shall see, enable the deflection at the free end of the cantilever to be determined directly.

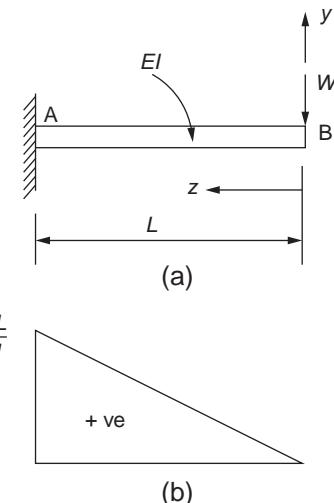


FIGURE 16.26 Cantilever Beam of Example 16.14

Also  $(dv/dz)_A = v_A = 0$ . Eq. (16.32) therefore reduces to

$$-\left(\frac{dv}{dz}\right)_B = -\int_B^A \frac{M}{EI} dz \quad (i)$$

At this stage we decide which approach is the most suitable; in this example, both methods are used for illustrative purposes, the semigraphical first.

From Fig. 16.26(b), the area of the  $M/EI$  diagram is  $WL^2/2EI$ . Then, from Eq. (i),

$$\left(\frac{dv}{dz}\right)_B = \frac{WL^2}{2EI}$$

Compare this with the value given by Eq. (iii) of Example 16.6. Alternatively using the analytical approach, the bending moment at any section a distance  $z$  from B is  $Wz$ . Substituting in Eq. (i) and integrating between the limits  $z = 0$  and  $z = L$  gives

$$\left(\frac{dv}{dz}\right)_B = \frac{WL^2}{2EI}$$

as before.

The bending moment at any section of the beam a distance  $z$  from B is  $Wz$ .

Substituting in Eq. (i), we have

$$-\left(\frac{dv}{dz}\right)_B = -\int_0^L \frac{Wz}{EI} dz$$

which gives

$$\left(\frac{dv}{dz}\right)_B = \frac{WL^2}{2EI}$$

With the origin of  $z$  at B,  $v_B = v_A = (dv/dz)_A = 0$  so that Eq. (16.33) reduces to

$$v_B = -\int_0^L \frac{Wz}{EI} dz$$

From the geometry of the  $M/EI$  diagram, the moment of the area about B is

$$\frac{1}{2}L\left(\frac{WL}{EI}\right)\frac{2L}{3}$$

so that

$$v_B = -\frac{WL^3}{3EI}$$

which is negative and therefore downward as expected. Compare also with Eq. (v) in Example 16.6.

Alternatively,

$$v_B = -\int_0^L \frac{W_z}{EI} z dz = -\left[\frac{Wz^3}{3EI}\right]_0^L$$

from which

$$v_B = -\frac{WL^3}{3EI}$$

Note that if the built-in end had been chosen as the origin for  $z$ , we could not have determined  $v_B$  directly, since the term  $z_B (dv/dz)_B$  would not have vanished from Eq. (16.33). Then,  $(dv/dz)_B$  would have had to have been determined first using Eq. (16.32). Note also, in this example, B has been chosen as the origin for  $z$ , not A.

### Example 16.15

Determine the maximum deflection in the simply supported beam shown in Fig. 16.27(a).

From symmetry, the support reactions are each  $wL/2$  and the maximum deflection will occur at mid-span. The  $M/EI$  diagram is shown in Fig. 16.27(b) and, as in the previous example, has the same shape as the bending moment diagram. If we choose the origin of axes to be at A, then  $z_A = v_A = (dv/dz)_C = 0$  and Eq. (16.33) reduces to

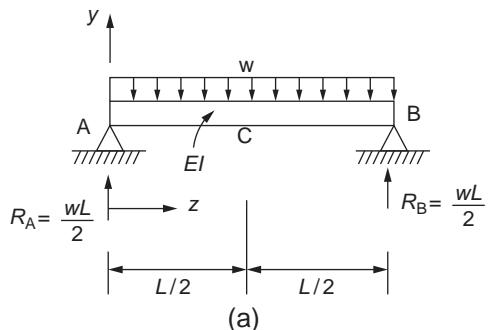
$$-v_C = - \int_0^{L/2} \frac{M}{EI} z \, dz \quad (i)$$

The geometry of the  $M/EI$  diagram is reasonably well known so that the semigraphical approach may be readily used. Referring to Fig. 16.27(b) and taking the moment of the area between A and C about A

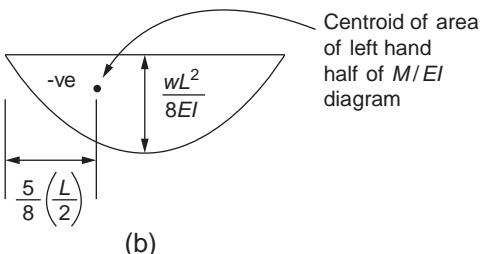
$$v_C = - \frac{2wL^2}{384EI} \frac{L}{2} \left( \frac{5}{8} \right)$$

that is,

$$v_C = - \frac{5wL^4}{384EI} \quad (\text{see Eq.(v) in Example 16.8})$$



(a)



(b)

FIGURE 16.27 Beam of Example 16.15

Note that the area of the  $M/EI$  diagram is negative, since the bending moment is negative at all sections of the beam.

Now using the analytical approach the bending moment at a section, a distance  $z$  from A is given by

$$M = \frac{-wLz}{2} + \frac{wz^2}{2}$$

Substituting for  $M$  in Eq. (i),

$$v_C = \int_0^{L/2} \frac{1}{EI} \left( \frac{-wLz}{2} + \frac{wz^2}{2} \right) z \, dz$$

which gives

$$v_C = -\frac{w}{2EI} \left[ \frac{Lz^3}{3} - \frac{wz^4}{4} \right]_0^{L/2}$$

from which

$$v_C = -\frac{5wL^4}{384EI}$$

### Example 16.16

The cantilever beam shown in Fig. 16.28(a) has a cross-section that changes midway along its length so that the second moment of area of its cross-section is reduced by half. Determine the deflection of its free end.

In this problem, the  $M/EI$  diagram and the bending moment diagram have different shapes as shown in Fig. 16.28(c) and (b), respectively. Choosing C as the origin of axes, Eq. (16.33) reduces to

$$v_C = - \int_0^L \frac{M}{EI} z \, dz$$

From the geometry of the  $M/EI$  diagram and taking moments of areas about C, we have

$$v_C = - \left[ \left( \frac{W}{2EI} \right) \frac{L}{2} \frac{3L}{4} + \frac{1}{2} \left( \frac{WL}{2EI} \right) \frac{L}{2} \frac{5L}{6} + \frac{1}{2} \left( \frac{WL}{EI} \right) \frac{L}{2} \frac{2L}{2} \right]$$

which gives

$$v_C = -\frac{3WL^3}{8EI}$$

Now using the analytical approach,

$$v_C = - \left[ \int_0^{L/2} \frac{Wz^2}{EI/2} \, dz + \int_{L/2}^L \frac{Wz^2}{EI} \, dz \right]$$

from which

$$v_C = -\frac{3WL^3}{8EI}$$

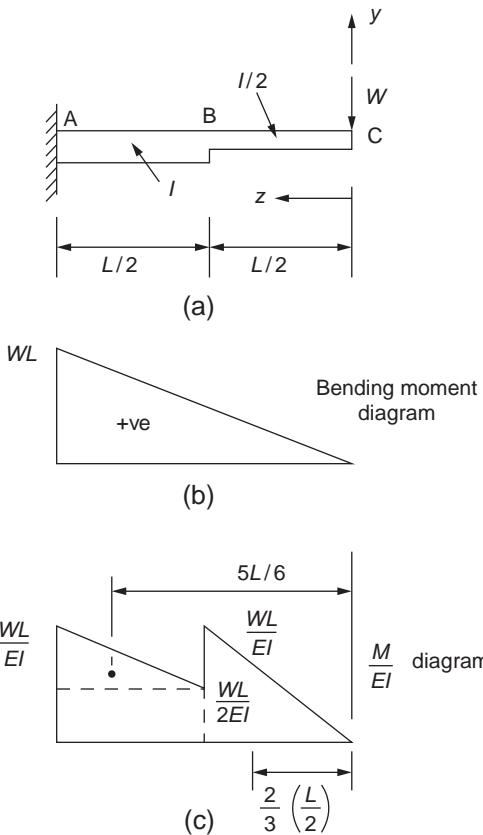


FIGURE 16.28 Cantilever Beam of Example 16.16

**Example 16.17**

The cantilever beam shown in Fig. 16.29 tapers along its length so that the second moment of area of its cross-section varies linearly from  $I_0$  at the free end to  $2I_0$  at the built-in end. Determine the deflection at the free end.

The second moment of area at any section of the beam is given by

$$I_z = I_0 \left( 1 + \frac{z}{L} \right)$$

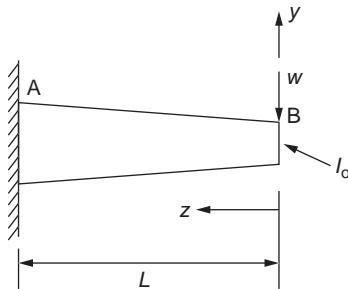


FIGURE 16.29 Cantilever Beam of Example 16.17

Choosing the free end of the cantilever as the origin of axes for the reasons stated in Example 16.14, Eq. (16.33) reduces to

$$v_B = - \int_0^L \frac{Mz}{EI_0(1+z/L)} dz \quad (i)$$

The geometry of the  $M/EI$  diagram will be complicated in this case, so the analytical approach will be the most suitable. The bending moment  $M$  at any section of the beam is  $Wz$  so that Eq. (i) becomes

$$v_B = - \frac{WL}{EI_0} \int_0^L \frac{z^2}{L+z} dz \quad (ii)$$

Rearranging Eq. (ii),

$$v_B = - \frac{WL}{EI_0} \left[ \int_0^L (z-L) dz + \int_0^L \frac{L^2}{L+z} dz \right]$$

which may be written in the form

$$v_B = - \frac{WL}{EI_0} \left[ \int_0^L (z-L) dz + L^2 \int_0^L \frac{dz}{L(1+z/L)} \right]$$

which gives

$$v_B = - \frac{WL}{EI_0} \left[ \frac{z^2}{2} - Lz + L^2 \log_e(1+z/L) \right]_0^L$$

Then

$$v_B = - \frac{0.19WL^3}{EI_0}$$

Other methods of calculating beam deflections include the elasticity approach, which was demonstrated in Chapter 2, and the virtual work and energy methods described in Chapters 4 and 5.

## 16.4 CALCULATION OF SECTION PROPERTIES

It will be helpful at this stage to discuss the calculation of the various section properties required in the analysis of beams subjected to bending. Initially, however, two useful theorems are quoted.

### 16.4.1 Parallel axes theorem

Consider the beam section shown in Fig. 16.30 and suppose that the second moment of area,  $I_C$ , about an axis through its centroid C is known. The second moment of area,  $I_N$ , about a parallel axis, NN, a distance  $b$  from the centroidal axis is then given by

$$I_N = I_C + Ab^2 \quad (16.34)$$

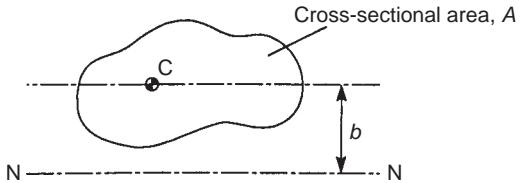


FIGURE 16.30 Parallel Axes Theorem

### 16.4.2 Theorem of perpendicular axes

In Fig. 16.31, the second moments of area,  $I_{xx}$  and  $I_{yy}$ , of the section about Ox and Oy are known. The second moment of area about an axis through O perpendicular to the plane of the section (i.e., a *polar second moment of area*) is

$$I_o = I_{xx} + I_{yy} \quad (16.35)$$

### 16.4.3 Second moments of area of standard sections

Many sections may be regarded as comprising a number of rectangular shapes. The problem of determining the properties of such sections is simplified if the second moments of area of the rectangular components are known and use is made of the parallel axes theorem. Thus, for the rectangular section of Fig. 16.32,

$$I_{xx} = \int_A y^2 dA = \int_{-d/2}^{d/2} b y^2 dy = b \left[ \frac{y^3}{3} \right]_{-d/2}^{d/2}$$

which gives

$$I_{xx} = \frac{bd^3}{12} \quad (16.36)$$

Similarly,

$$I_{yy} = \frac{db^3}{12} \quad (16.37)$$

Frequently, it is useful to know the second moment of area of a rectangular section about an axis that coincides with one of its edges. Therefore, in Fig. 16.32 and using the parallel axes theorem,

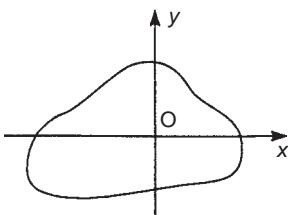
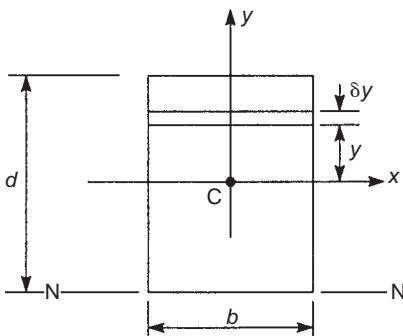


FIGURE 16.31 Theorem of Perpendicular Axes



**FIGURE 16.32 Second Moments of Area of a Rectangular Section**

$$I_N = \frac{bd^3}{12} + bd \left( -\frac{d}{2} \right)^2 = \frac{bd^3}{3} \quad (16.38)$$

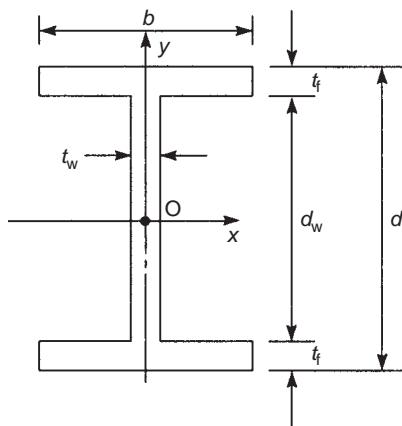
### Example 16.18

Determine the second moments of area  $I_{xx}$  and  $I_{yy}$  of the I section shown in Fig. 16.33.

Using Eq. (16.36),

$$I_{xx} = \frac{bd^3}{12} - \frac{(b - t_w)d_w^3}{12}$$

Alternatively, using the parallel axes theorem in conjunction with Eq. (16.36),



**FIGURE 16.33 Second Moments of Area of an I-Section**

$$I_{xx} = 2 \left[ \frac{bt_f^3}{12} + bt_f \left( \frac{d_w + t_f}{2} \right)^2 \right] + \frac{t_w d_w^3}{12}$$

The equivalence of these two expressions for  $I_{xx}$  is most easily demonstrated by a numerical example.

Also, from Eq. (16.37),

$$I_{yy} = 2 \frac{t_f b^3}{12} + \frac{d_w t_w^3}{12}$$



It is also useful to determine the second moment of area, about a diameter, of a circular section. In Fig. 16.34, where the  $x$  and  $y$  axes pass through the centroid of the section,

$$I_{xx} = \int_A y^2 dA = \int_{-d/2}^{d/2} 2 \left( \frac{d}{2} \cos \theta \right) y^2 dy \quad (16.39)$$

Integration of Eq. (16.39) is simplified if an angular variable,  $\theta$ , is used. Thus,

$$I_{xx} = \int_{-\pi/2}^{\pi/2} d \cos \theta \left( \frac{d}{2} \sin \theta \right)^2 \frac{d}{2} \cos \theta d\theta$$

that is,

$$I_{xx} = \frac{d^4}{8} \int_{-\pi/2}^{\pi/2} \cos^2 \theta \sin^2 \theta d\theta$$

which may be written

$$I_{xx} = \frac{d^4}{8} \int_{-\pi/2}^{\pi/2} \frac{1}{4} \sin^2 2\theta d\theta$$

or

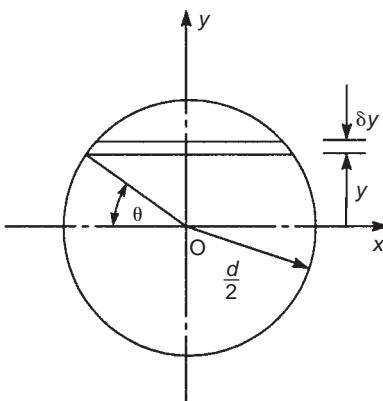


FIGURE 16.34 Second Moments of Area of a Circular Section

$$I_{xx} = \frac{d^4}{32} \int_{-\pi/2}^{\pi/2} \frac{1}{2} (1 - \cos 4\theta) d\theta$$

that is,

$$I_{xx} = \frac{d^4}{64} \left[ \theta - \frac{\sin 4\theta}{4} \right]_{-\pi/2}^{\pi/2}$$

which gives

$$I_{xx} = \frac{\pi d^4}{64} \quad (16.40)$$

Clearly, from symmetry,

$$I_{yy} = \frac{\pi d^4}{64} \quad (16.41)$$

Using the theorem of perpendicular axes, the polar second moment of area,  $I_o$ , is given by

$$I_o = I_{xx} + I_{yy} = \frac{\pi d^4}{32} \quad (16.42)$$

#### 16.4.4 Product second moment of area

The product second moment of area,  $I_{xy}$ , of a beam section with respect to  $x$  and  $y$  axes is defined by

$$I_{xy} = \int_A xy \, dA \quad (16.43)$$

Thus, each element of area in the cross-section is multiplied by the product of its coordinates and the integration is taken over the complete area. Although second moments of area are always positive, since elements of area are multiplied by the square of one of their coordinates, it is possible for  $I_{xy}$  to be negative if the section lies predominantly in the second and fourth quadrants of the axes system. Such a situation would arise in the case of the Z section of Fig. 16.35(a), where the product second moment of area of each flange is clearly negative.

A special case arises when one (or both) of the coordinate axes is an axis of symmetry, so that for any element of area,  $\delta A$ , having the product of its coordinates positive, there is an identical element for which the product of its coordinates is negative (Fig. 16.35 (b)). Summation (i.e., integration) over the entire section of the product second moment of area of all such pairs of elements results in a zero value for  $I_{xy}$ .

We have shown previously that the parallel axes theorem may be used to calculate second moments of area of beam sections comprising geometrically simple components. The theorem can be extended to the calculation of product second moments of area. Let us suppose that we wish to calculate the product

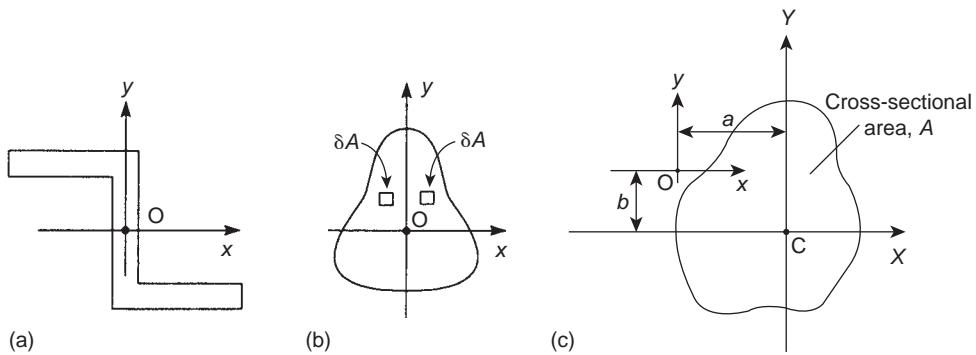


FIGURE 16.35 Product Second Moment of Area

second moment of area,  $I_{xy}$ , of the section shown in Fig. 16.35(c) about axes  $xy$  when  $I_{XY}$  about its own, say, centroidal, axes system  $CXY$  is known. From Eq. (16.43),

$$I_{xy} = \int_A xy \, dA$$

or

$$I_{xy} = \int_A (X - a)(Y - b) \, dA$$

which, on expanding, gives

$$I_{xy} = \int_A XY \, dA - b \int_A X \, dA - a \int_A Y \, dA + ab \int_A \, dA$$

If  $X$  and  $Y$  are centroidal axes, then  $\int_A X \, dA = \int_A Y \, dA = 0$ . Hence,

$$I_{xy} = I_{XY} + abA \quad (16.44)$$

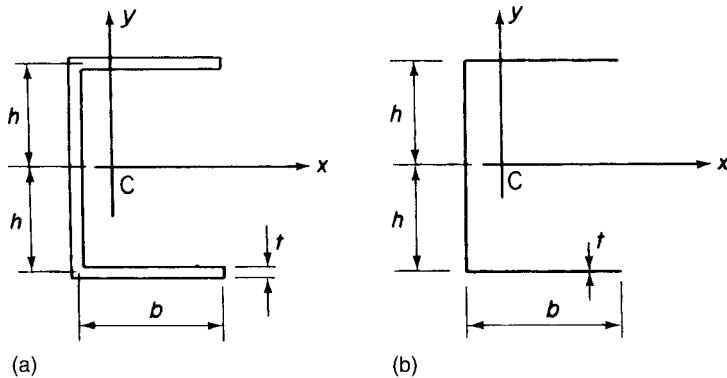
It can be seen from Eq. (16.44) that if either  $CX$  or  $CY$  is an axis of symmetry, that is,  $I_{XY} = 0$ , then

$$I_{xy} = abA \quad (16.45)$$

Therefore, for a section component having an axis of symmetry that is parallel to either of the section reference axes, the product second moment of area is the product of the coordinates of its centroid multiplied by its area.

#### 16.4.5 Approximations for thin-walled sections

We may exploit the thin-walled nature of aircraft structures to make simplifying assumptions in the determination of stresses and deflections produced by bending. Thus, the thickness  $t$  of thin-walled sections is assumed to be small compared with their cross-sectional dimensions, so that stresses may be regarded as constant across the thickness. Furthermore, we neglect squares and higher



**FIGURE 16.36 (a) Actual Thin-Walled Channel Section; (b) Approximate Representation of Section**

powers of  $t$  in the computation of sectional properties and take the section to be represented by the mid-line of its wall. As an illustration of the procedure we shall consider the channel section of Fig. 16.36(a). The section is singly symmetric about the  $x$  axis so that  $I_{xy} = 0$ . The second moment of area  $I_{xx}$  is then given by

$$I_{xx} = 2 \left[ \frac{(b + t/2)t^3}{12} + \left( b + \frac{t}{2} \right) th^2 \right] + t \frac{[2(h - t/2)]^3}{12}$$

Expanding the cubed term, we have

$$I_{xx} = 2 \left[ \frac{(b + t/2)t^3}{12} + \left( b + \frac{t}{2} \right) th^2 \right] + \frac{t}{12} \left[ (2)^3 \left( h^3 - 3h^2 \frac{t}{2} + 3h \frac{t^2}{4} - \frac{t^3}{8} \right) \right]$$

which reduces, after powers of  $t^2$  and upward are ignored, to

$$I_{xx} = 2bth^2 + t \frac{(2h)^3}{12}$$

The second moment of area of the section about  $Cy$  is obtained in a similar manner.

We see therefore that, for the purpose of calculating section properties, we may regard the section as represented by a single line, as shown in Fig. 16.36(b).

Thin-walled sections frequently have inclined or curved walls, which complicate the calculation of section properties. Consider the inclined thin section of Fig. 16.37. Its second moment of area about a horizontal axis through its centroid is given by

$$I_{xx} = 2 \int_0^{a/2} ty^2 \, ds = 2 \int_0^{a/2} t(s \sin \beta)^2 \, ds$$

from which

$$I_{xx} = \frac{a^3 t \sin^2 \beta}{12}$$

Similarly,

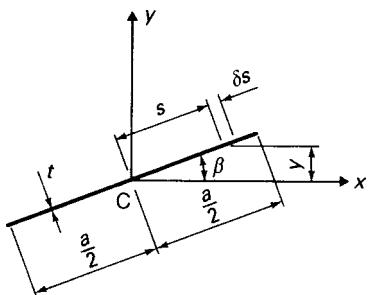


FIGURE 16.37 Second Moments of Area of an Inclined Thin Section

$$I_{yy} = \frac{a^3 t \cos^2 \beta}{12}$$

The product second moment of area is

$$\begin{aligned} I_{xy} &= 2 \int_0^{a/2} txy \, ds \\ &= 2 \int_0^{a/2} t(s \cos \beta)(s \sin \beta) \, ds \end{aligned}$$

which gives

$$I_{xy} = \frac{a^3 t \sin 2\beta}{24}$$

We note here that these expressions are approximate, in that their derivation neglects powers of  $t^2$  and upward by ignoring the second moments of area of the element  $\delta s$  about axes through its own centroid.

Properties of thin-walled curved sections are found in a similar manner. Thus,  $I_{xx}$  for the semicircular section of Fig. 16.38 is

$$I_{xx} = \int_0^{\pi r} ty^2 \, ds$$

Expressing  $y$  and  $s$  in terms of a single variable  $\theta$  simplifies the integration; hence,

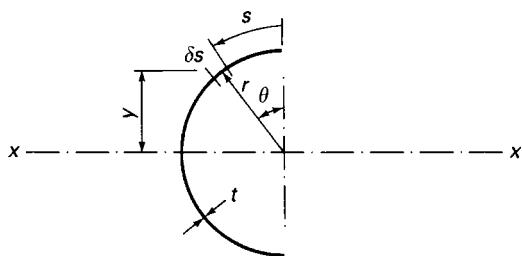


FIGURE 16.38 Second Moment of Area of a Semicircular Section

$$I_{xx} = \int_0^\pi t(r \cos \theta)^2 r d\theta$$

from which

$$I_{xx} = \frac{\pi r^3 t}{2}$$

### Example 16.19

Determine the direct stress distribution in the thin-walled Z section shown in Fig. 16.39, produced by a positive bending moment  $M_x$ .

The section is antisymmetrical, with its centroid at the mid-point of the vertical web. Therefore, the direct stress distribution is given by either of Eq. (16.17) or (16.18), in which  $M_y = 0$ . From Eq. (16.18),

$$\sigma_z = \frac{M_x(I_{yy}y - I_{xy}x)}{I_{xx}I_{yy} - I_{xy}^2} \quad (i)$$

The section properties are calculated as follows:

$$\begin{aligned} I_{xx} &= 2 \frac{ht}{2} \left( \frac{h}{2} \right)^2 + \frac{th^3}{12} = \frac{h^3 t}{3} \\ I_{yy} &= 2 \frac{t}{3} \left( \frac{h}{2} \right)^3 = \frac{h^3 t}{12} \\ I_{xy} &= \frac{ht}{2} \left( \frac{h}{4} \right) \left( \frac{h}{2} \right) + \frac{ht}{2} \left( -\frac{h}{4} \right) \left( -\frac{h}{2} \right) = \frac{h^3 t}{8} \end{aligned}$$

Substituting these values in Eq. (i),

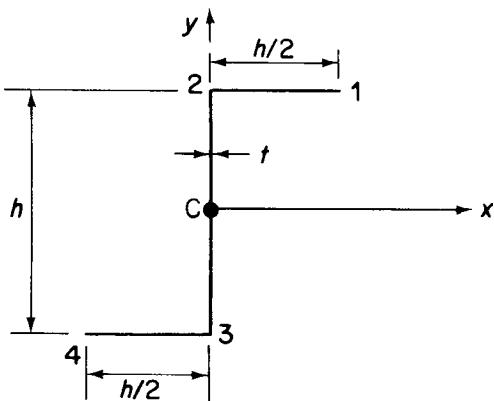


FIGURE 16.39 Z Section Beam of Example 16.19

$$\sigma_z = \frac{M_x}{h^3 t} (6.86y - 10.30x) \quad (\text{ii})$$

On the top flange,  $y=h/2$ ,  $0 \leq x \leq h/2$  and the distribution of direct stress is given by

$$\sigma_z = \frac{M_x}{h^3 t} (3.43h - 10.30x)$$

which is linear. Hence,

$$\sigma_{z,1} = -\frac{1.72M_x}{h^2 t} \quad (\text{compressive})$$

$$\sigma_{z,2} = +\frac{3.43M_x}{h^2 t} \quad (\text{tensile})$$

In the web,  $-h/2 \leq y \leq h/2$  and  $x=0$ . Again, the distribution is of linear form and is given by the equation

$$\sigma_z = \frac{M_x}{h^3 t} 6.86y$$

from which

$$\sigma_{z,2} = +\frac{3.43M_x}{h^2 t} \quad (\text{tensile})$$

and

$$\sigma_{z,3} = -\frac{3.43M_x}{h^2 t} \quad (\text{compressive})$$

The distribution in the lower flange may be deduced from antisymmetry; the complete distribution is as shown in Fig. 16.40.

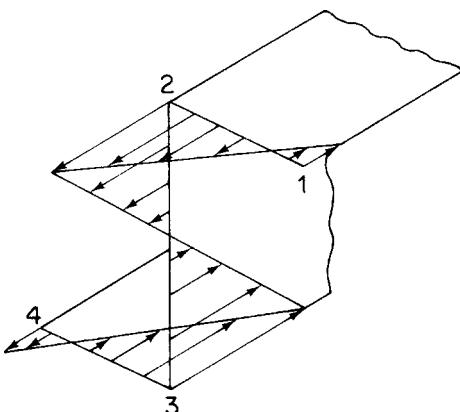


FIGURE 16.40 Distribution of Direct Stress in Z Section Beam of Example 16.19

## 16.5 APPLICABILITY OF BENDING THEORY

The expressions for direct stress and displacement derived in the preceding theory are based on the assumptions that the beam is of uniform, homogeneous cross-section and that plane sections remain plane after bending. The latter assumption is strictly true only if the bending moments  $M_x$  and  $M_y$  are constant along the beam. Variation of bending moment implies the presence of shear loads, and the effect of these is to deform the beam section into a shallow reversed S (see [Section 2.6](#)). However, shear stresses in beams whose cross-sectional dimensions are small in relation to their lengths are comparatively low so that the basic theory of bending may be used with reasonable accuracy.

In thin-walled sections, shear stresses produced by shear loads are not small and must be calculated, although the direct stresses may still be obtained from the basic theory of bending so long as axial constraint stresses are absent; this effect is discussed in [Chapters 27](#) and [28](#). Deflections in thin-walled structures are assumed to result primarily from bending strains; the contribution of shear strains may be calculated separately if required.

## 16.6 TEMPERATURE EFFECTS

In [Section 1.15.1](#), we considered the effect of temperature change on stress-strain relationships, while in [Section 5.11](#), we examined the effect of a simple temperature gradient on a cantilever beam of rectangular cross-section using an energy approach. However, as we saw, beam sections in aircraft structures are generally thin walled and do not necessarily have axes of symmetry. We now investigate how the effects of temperature on such sections may be determined.

We saw that the strain produced by a temperature change  $\Delta T$  is given by

$$\varepsilon = \alpha\Delta T \quad (\text{see Eq.(1.55)})$$

It follows from [Eq. \(1.40\)](#) that the direct stress on an element of cross-sectional area  $\delta A$  is

$$\sigma = E\alpha\Delta T \quad (16.46)$$

Consider now the beam section shown in [Fig. 16.41](#) and suppose that a temperature variation  $\Delta T$  is applied to the complete cross-section, that is,  $\Delta T$  is a function of both  $x$  and  $y$ .

The total normal force due to the temperature change on the beam cross-section is then given by

$$N_T = \iint_A E\alpha\Delta T \, dA \quad (16.47)$$

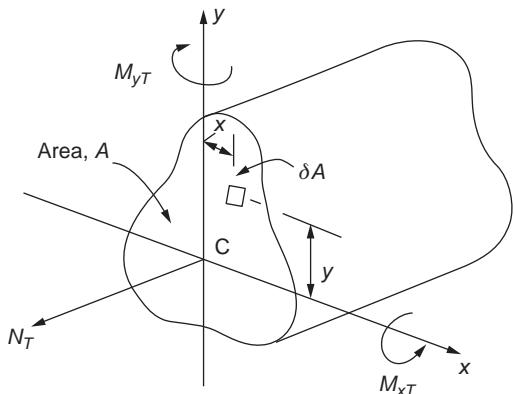
Further, the moments about the  $x$  and  $y$  axes are

$$M_{xT} = \iint_A E\alpha\Delta Ty \, dA \quad (16.48)$$

and

$$M_{yT} = \iint_A E\alpha\Delta Tx \, dA \quad (16.49)$$

respectively.



**FIGURE 16.41 Beam Section Subjected to a Temperature Rise**

We have noted that beam sections in aircraft structures are generally thin-walled, so that Eqs. (16.43)–(16.45) may be more easily integrated for such sections by dividing them into thin rectangular components, as we did when calculating section properties. We then use the Riemann integration technique, in which we calculate the contribution of each component to the normal force and moments and sum them to determine each result. Equations (16.47)–(16.49) then become

$$N_T = \sum E\alpha\Delta T A_i \quad (16.50)$$

$$M_{xT} = \sum E\alpha\Delta T \bar{y}_i A_i \quad (16.51)$$

$$M_{yT} = \sum E\alpha\Delta T \bar{x}_i A_i \quad (16.52)$$

in which  $A_i$  is the cross-sectional area of a component and  $\bar{x}_i$  and  $\bar{y}_i$  are the coordinates of its centroid.

### Example 16.20

The beam section shown in Fig. 16.42 is subjected to a temperature rise of  $2T_0$  in its upper flange, a temperature rise of  $T_0$  in its web, and no temperature change in its lower flange. Determine the normal force on the beam section and the moments about the centroidal  $x$  and  $y$  axes. The beam section has a Young's modulus  $E$  and the coefficient of linear expansion of the material of the beam is  $\alpha$ .

From Eq. (16.50),

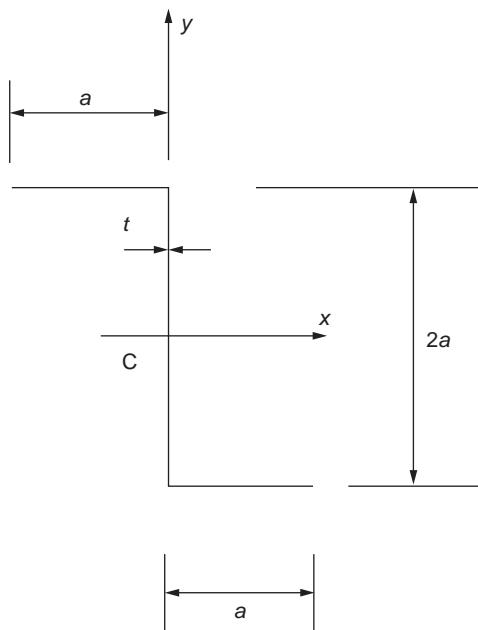
$$N_T = E\alpha(2T_0at + T_0 2at) = 4E\alpha at T_0$$

From Eq. (16.51),

$$M_{xT} = E\alpha[2T_0at(a) + T_0 2at(0)] = 2E\alpha a^2 t T_0$$

and from Eq. (16.52),

$$M_{yT} = E\alpha[2T_0at(-a/2) + T_0 2at(0)] = -E\alpha a^2 t T_0$$



**FIGURE 16.42 Beam Section of Example 16.20**

Note that  $M_{yT}$  is negative, which means that the upper flange tends to rotate out of the paper about the web, which agrees with a temperature rise for this part of the section. The stresses corresponding to these stress resultants are calculated in the normal way and are added to those produced by any applied loads.

In some cases, the temperature change is not conveniently constant in the components of a beam section and must then be expressed as a function of  $x$  and  $y$ . Consider the thin-walled beam section shown in Fig. 16.43 and suppose that a temperature change  $\Delta T(x, y)$  is applied.

The direct stress on an element  $ds$  in the wall of the section is then, from Eq. (16.46),

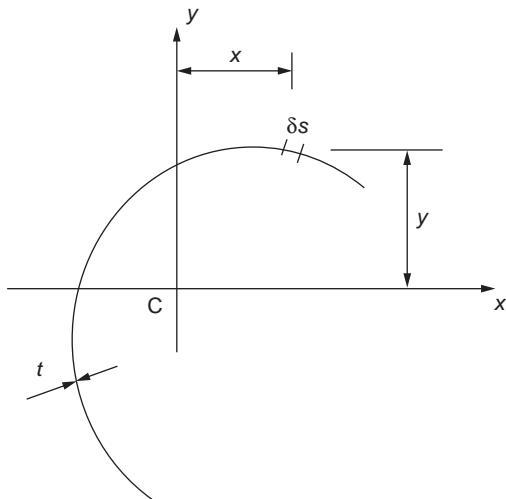
$$\sigma = E\alpha\Delta T(x, y)$$

Equations (16.47)–(16.49) then become

$$N_T = \int_A E\alpha\Delta T(x, y)t \, ds \quad (16.53)$$

$$M_{xT} = \int_A E\alpha\Delta T(x, y)ty \, ds \quad (16.54)$$

$$M_{yT} = \int_A E\alpha\Delta T(x, y)tx \, ds \quad (16.55)$$



**FIGURE 16.43 Thin-walled Beam Section Subjected to a Varying Temperature Change**

### Example 16.21

If, in the beam section of Example 16.20, the temperature change in the upper flange is \$2T\_0\$ but in the web varies linearly from \$2T\_0\$ at its junction with the upper flange to zero at its junction with the lower flange, determine the values of the stress resultants; the temperature change in the lower flange remains zero.

The temperature change at any point in the web is given by

$$T_w = 2T_0(a + y)/2a = \frac{T_0}{a}(a + y)$$

Then, from Eqs. (16.50) and (16.53),

$$N_T = E\alpha 2T_0at + \int_{-a}^a E\alpha \frac{T_0}{a}(a + y)t \, ds$$

that is,

$$N_T = E\alpha T_0 \left\{ 2at + \frac{1}{a} \left[ ay + \frac{y^2}{2} \right]_{-a}^a \right\}$$

which gives

$$N_T = 4E\alpha T_0at$$

Note that, in this case, the answer is identical to that in Example 16.20, which is to be expected, since the average temperature change in the web is \$(2T\_0 + 0)/2 = T\_0\$, which is equal to the constant temperature change in the web in Example 16.20.

From Eqs. (16.51) and (16.54),

$$M_{xT} = E\alpha 2T_0 at(a) + \int_{-a}^a E\alpha \frac{T_0}{a}(a+y)yt \, ds$$

that is,

$$M_{xT} = E\alpha T_0 \left\{ 2a^2 t + \frac{1}{a} \left[ \frac{ay^2}{2} + \frac{y^3}{3} \right]_{-a}^a \right\}$$

from which

$$M_{xT} = \frac{8E\alpha a^2 t T_0}{3}$$

Alternatively, the average temperature change  $T_0$  in the web may be considered to act at the centroid of the temperature change distribution. Then,

$$M_{xT} = E\alpha 2T_0 at(a) + E\alpha T_0 2at \left( \frac{a}{3} \right)$$

that is,

$$M_{xT} = \frac{8E\alpha a^2 t T_0}{3}$$

as before. The contribution of the temperature change in the web to  $M_{yT}$  remains zero, since the section centroid is in the web; the value of  $M_{yT}$  is therefore  $-E\alpha a^2 t T_0$  as in Example 16.19.

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## Reference

- [1] Megson THG. Structures and stress analysis. 4th ed. Oxford: Elsevier; 2019.

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## PROBLEMS

**P.16.1** Figure P.16.1 shows the section of an angle purlin. A bending moment of 3,000 Nm is applied to the purlin in a plane at an angle of  $30^\circ$  to the vertical  $y$  axis. If the sense of the bending moment is such that both its components  $M_x$  and  $M_y$  produce tension in the positive  $xy$  quadrant, calculate the maximum direct stress in the purlin, stating clearly the point at which it acts.

Answer:  $\sigma_{z,\max} = -63.3 \text{ N/mm}^2$  at C

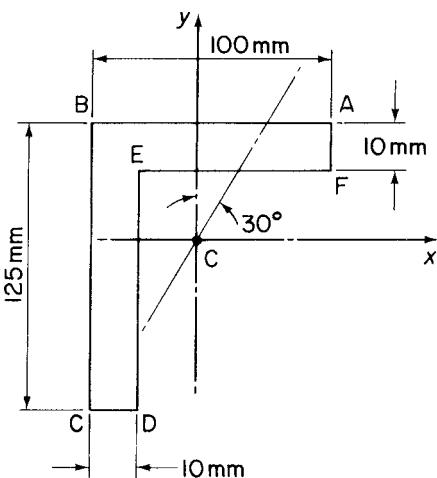


FIGURE P.16.1

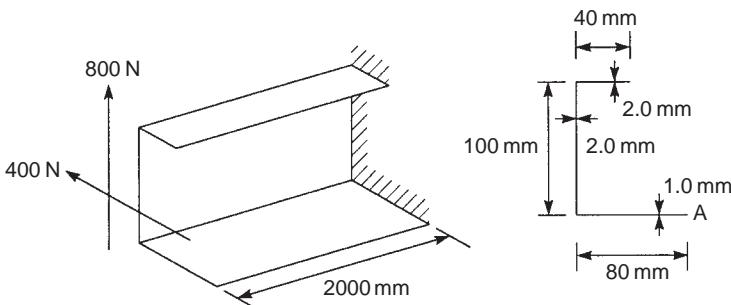


FIGURE P.16.2

**P.16.2** A thin-walled, cantilever beam of unsymmetrical cross-section supports shear loads at its free end as shown in Fig. P.16.2. Calculate the value of direct stress at the extremity of the lower flange (point A) at a section halfway along the beam if the position of the shear loads is such that no twisting of the beam occurs.

Answer: 194.5 N/mm<sup>2</sup> (tension)

**P.16.2 MATLAB** If the 400 and 800 N shear loads in Fig. P.16.2 are labeled  $S_x$  and  $S_y$ , respectively, use MATLAB to repeat Problem P.16.2 for the following  $(S_x, S_y)$  combinations, assuming positive values are in the direction of the load arrows shown:

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)
$S_y$ (N)	400	500	600	700	800	900	1000	1100	1200
$S_x$ (N)	600	600	600	400	400	-400	-400	-600	-600

- Answer: (i) 269.9 N/mm<sup>2</sup> (tension)  
(ii) 272.6 N/mm<sup>2</sup> (tension)  
(iii) 275.3 N/mm<sup>2</sup> (tension)  
(iv) 191.8 N/mm<sup>2</sup> (tension)  
(v) 194.5 N/mm<sup>2</sup> (tension)  
(vi) 148.1 N/mm<sup>2</sup> (compression)  
(vii) 145.3 N/mm<sup>2</sup> (compression)  
(viii) 228.9 N/mm<sup>2</sup> (compression)  
(ix) 226.2 N/mm<sup>2</sup> (compression)

**P.16.3** A beam, simply supported at each end, has a thin-walled cross-section, as shown in Fig. P.16.3. If a uniformly distributed loading of intensity  $w/\text{unit length}$  acts on the beam in the plane of the lower, horizontal flange, calculate the maximum direct stress due to bending of the beam and show diagrammatically the distribution of the stress at the section where the maximum occurs. The thickness  $t$  is to be taken as small in comparison with the other cross-sectional dimensions in calculating the section properties  $I_{xx}$ ,  $I_{yy}$ , and  $I_{xy}$ .

Answer:  $\sigma_{z,\max} = \sigma_{z,3} = 13wl^2/384a^2t$ ,  $\sigma_{z,1} = wl^2/96a^2t$ ,  $\sigma_{z,2} = -wl^2/48a^2t$

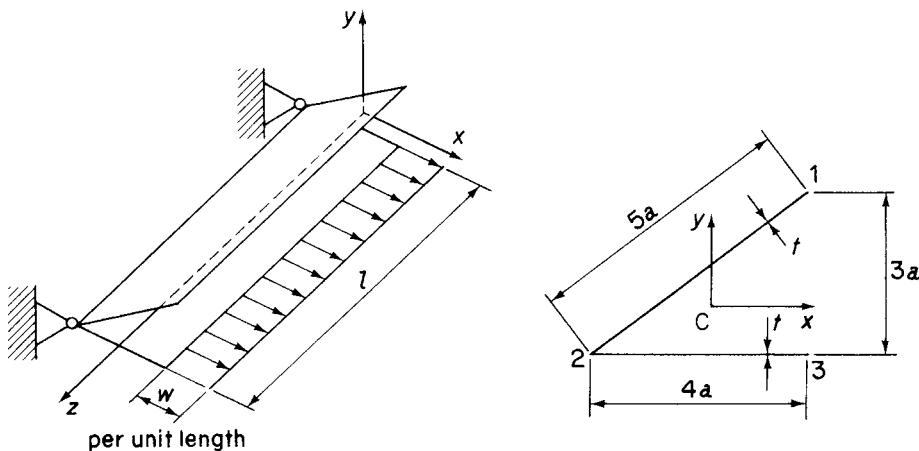


FIGURE P.16.3

**P.16.4** A thin-walled cantilever with walls of constant thickness  $t$  has the cross-section shown in Fig. P.16.4. It is loaded by a vertical force  $W$  at the tip and a horizontal force  $2W$  at the mid-section, both forces acting through the shear center. Determine and sketch the distribution of direct stress, according to the basic theory of bending, along the length of the beam for the points 1 and 2 of the cross-section. The wall thickness  $t$  can be taken as very small in comparison with  $d$  in calculating the sectional properties  $I_{xx}$ ,  $I_{xy}$ , and so forth.

Answer:  $\sigma_{z,1} \text{ (mid-point)} = -0.05 Wl/td^2$ ,  $\sigma_{z,1} \text{ (built-in end)} = -1.85 Wl/td^2$ ,  
 $\sigma_{z,2} \text{ (mid-point)} = -0.63 Wl/td^2$ ,  $\sigma_{z,2} \text{ (built-in end)} = 0.1 Wl/td^2$

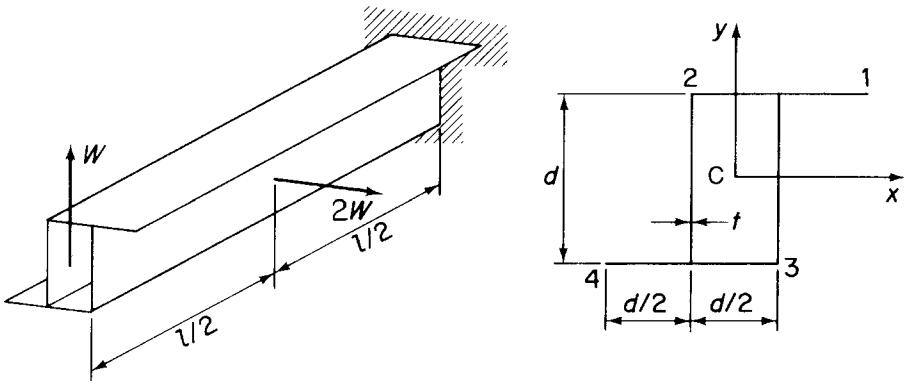


FIGURE P.16.4

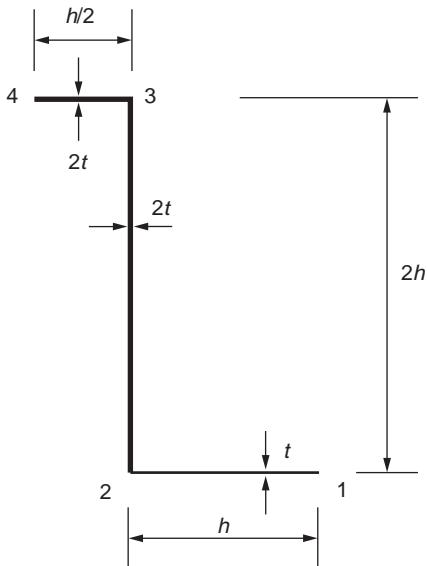


FIGURE P.16.5

**P. 16.5** A thin-walled beam has the cross-section shown in Fig. P.16.5. If the beam is subjected to a bending moment  $M_x$  in the plane of the web 23, calculate and sketch the distribution of direct stress in the beam cross-section.

Answer: At 1,  $0.92M_x/th^2$ ; at 2,  $-0.65M_x/th^2$ ; at 3,  $0.65M_x/th^2$ ; at 4,  $-0.135M_x/th^2$

**P.16.6 MATLAB** If the angle from horizontal of the web in the section shown in Fig. P.16.6 is labeled  $\theta$ , use MATLAB to repeat Problem P.16.6 for the following values of  $\theta$ :

(i)	(ii)	(iii)	(iv)	(v)	(vi)	
$\theta$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$	$70^\circ$	$80^\circ$

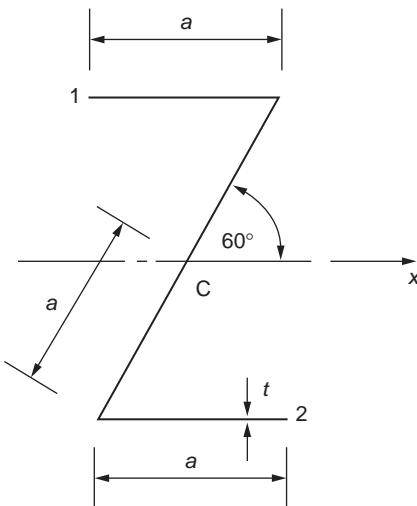


FIGURE P.16.6

- Answer:
- (i) NA inclined at  $23.4^\circ$  to Cx.  $\pm 1.03M_x/a^2t$  at 1 and 2, respectively
  - (ii) NA inclined at  $30.5^\circ$  to Cx.  $\pm 0.86M_x/a^2t$  at 1 and 2, respectively
  - (iii) NA inclined at  $36.6^\circ$  to Cx.  $\pm 0.78M_x/a^2t$  at 1 and 2, respectively
  - (iv) NA inclined at  $40.9^\circ$  to Cx.  $\pm 0.74M_x/a^2t$  at 1 and 2, respectively
  - (v) NA inclined at  $41.2^\circ$  to Cx.  $\pm 0.70M_x/a^2t$  at 1 and 2, respectively
  - (vi) NA inclined at  $31.4^\circ$  to Cx.  $\pm 0.59M_x/a^2t$  at 1 and 2, respectively

**P.16.7** A thin-walled cantilever has a constant cross-section of uniform thickness with the dimensions shown in Fig. P.16.7. It is subjected to a system of point loads acting in the planes of the walls of the section in the directions shown. Calculate the direct stresses according to the basic theory of bending at the points 1, 2, and 3 of the cross-section at the built-in end and halfway along the beam. Illustrate your answer by means of a suitable sketch. The thickness is to be taken as small in comparison with the other cross-sectional dimensions in calculating the section properties  $I_{xx}$ ,  $I_{xy}$ , and so on.

Answer: At built-in end,  $\sigma_{z,1} = -11.4 \text{ N/mm}^2$ ,  $\sigma_{z,2} = -18.9 \text{ N/mm}^2$ ,  $\sigma_{z,3} = 39.1 \text{ N/mm}^2$   
Halfway,  $\sigma_{z,1} = -20.3 \text{ N/mm}^2$ ,  $\sigma_{z,2} = -1.1 \text{ N/mm}^2$ ,  $\sigma_{z,3} = 15.4 \text{ N/mm}^2$ .

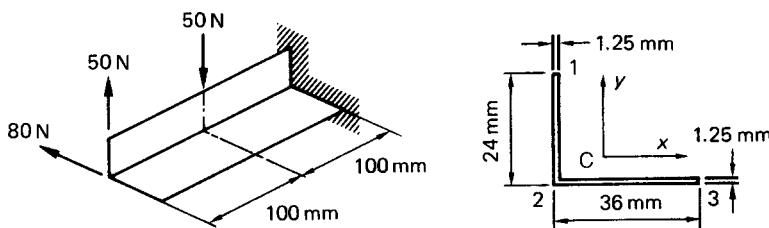


FIGURE P.16.7

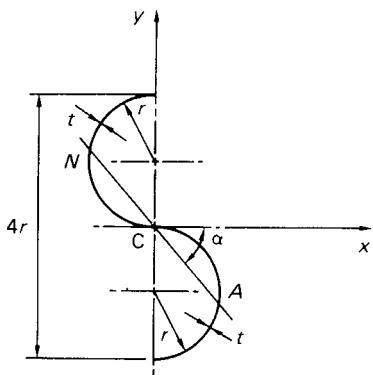


FIGURE P.16.8

**P.16.8** A uniform thin-walled beam has the open cross-section shown in Fig. P.16.8. The wall thickness  $t$  is constant. Calculate the position of the neutral axis and the maximum direct stress for a bending moment  $M_x = 3.5 \text{ Nm}$  applied about the horizontal axis Cx. Take  $r = 5 \text{ mm}$ ,  $t = 0.64 \text{ mm}$ .

Answer:  $\alpha = 51.9^\circ$ ,  $\sigma_{z,\max} = 101 \text{ N/mm}^2$ .

**P.16.9** A uniform beam is simply supported over a span of 6 m. It carries a trapezoidally distributed load with intensity varying from 30 kN/m at the left-hand support to 90 kN/m at the right-hand support. Find the equation of the deflection curve and hence the deflection at the mid-span point. The second moment of area of the cross-section of the beam is  $120 \times 10^6 \text{ mm}^4$  and Young's modulus  $E = 206,000 \text{ N/mm}^2$ .

Answer: 41 mm (downward)

**P.16.10** A cantilever of length  $L$  and having a flexural rigidity  $EI$  carries a distributed load that varies in intensity from  $w/\text{unit length}$  at the built-in end to zero at the free end. Find the deflection of the free end.

Answer:  $wL^4/30EI$  (downward)

**P.16.11** Determine the position and magnitude of the maximum deflection of the simply supported beam shown in Fig. P.16.11 in terms of its flexural rigidity  $EI$ .

Answer:  $38.8/EI$  m downward at 2.9 m from left-hand support.

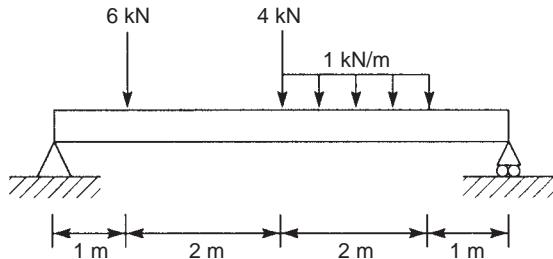


FIGURE P.16.11

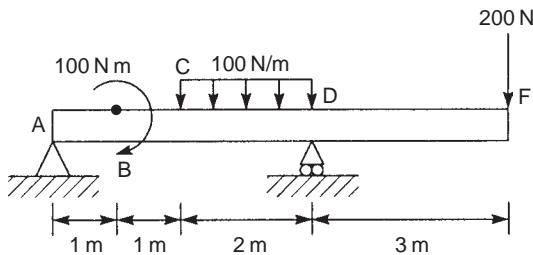


FIGURE P.16.12

**P.16.12** Determine the equation of the deflection curve of the beam shown in Fig. P.16.12. The flexural rigidity of the beam is  $EI$ .

$$\text{Answer: } v = -\frac{1}{EI} \left( \frac{125}{6}z^3 - 50[z-1]^2 + \frac{50}{12}[z-2]^4 - \frac{50}{12}[z-4]^4 - \frac{525}{6}[z-4]^3 + 237.5z \right)$$

**P.16.13** A uniform thin-walled beam ABD of open cross-section (Fig. P.16.13) is simply supported at points B and D with its web vertical. It carries a downward vertical force  $W$  at the end A in the plane of the web. Derive expressions for the vertical and horizontal components of the deflection of the beam midway between the supports B and D. The wall thickness  $t$  and Young's modulus  $E$  are constant throughout.

$$\text{Answer: } u = 0.186WL^3/Ea^3t, \quad v = 0.177WL^3/Ea^3t.$$

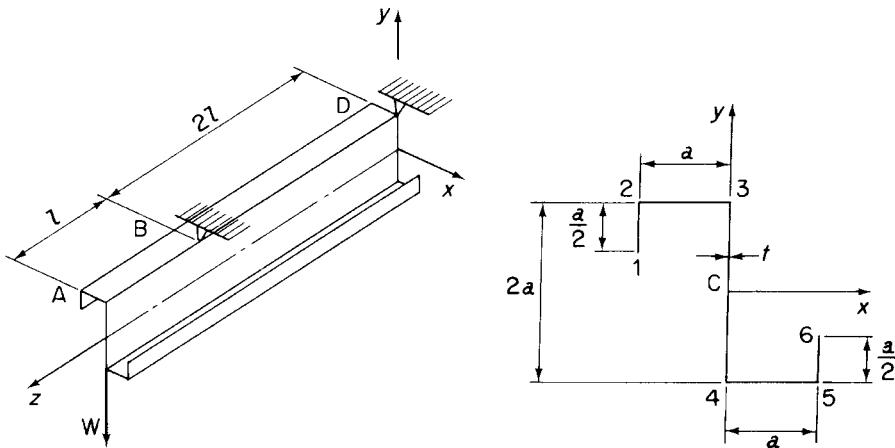


FIGURE P.16.13

**P.16.14** A uniform cantilever of arbitrary cross-section and length  $l$  has section properties  $I_{xx}$ ,  $I_{yy}$ , and  $I_{xy}$  with respect to the centroidal axes shown in Fig. P.16.14. It is loaded in the vertical ( $yz$ ) plane with a uniformly distributed load of intensity  $w/\text{unit length}$ . The tip of the beam is hinged to a horizontal link

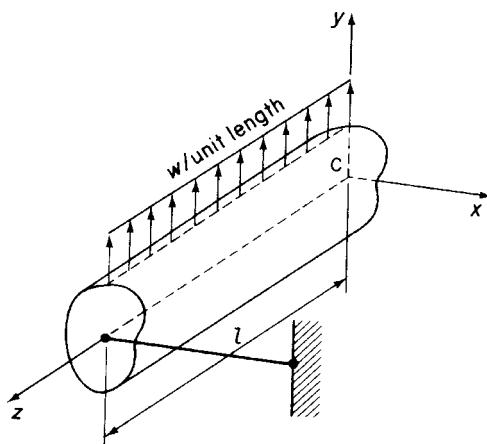


FIGURE P.16.14

which constrains it to move in the vertical direction only (provided that the actual deflections are small). Assuming that the link is rigid and that there are no twisting effects, calculate

- (a) The force in the link;
- (b) The deflection of the tip of the beam.

Answer: (a)  $3wlI_{xy}/8I_{xx}$ ; (b)  $wl^4/8EI_{xx}$ .

**P.16.15** A uniform beam of arbitrary, unsymmetrical cross-section and length  $2l$  is built in at one end and simply supported in the vertical direction at a point halfway along its length. This support, however, allows the beam to deflect freely in the horizontal  $x$  direction (Fig. P.16.15). For a vertical load  $W$  applied at the free end of the beam, calculate and draw the bending moment diagram, putting in the principal values.

Answer:  $M_C = 0$ ,  $M_B = Wl$ ,  $M_A = -Wl/2$ ; linear distribution

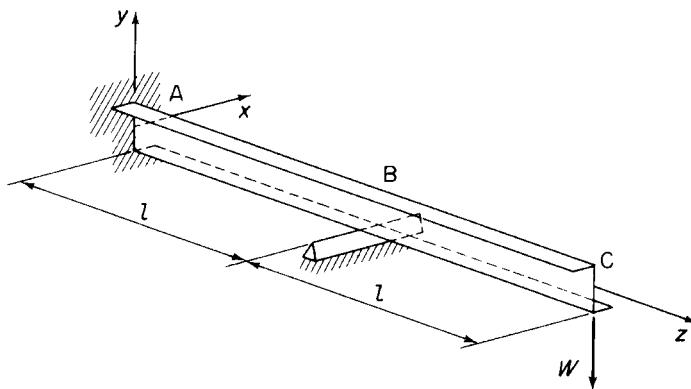


FIGURE P.16.15

**P. 16.16** For the aircraft at touchdown in P.14.1 calculate the deflection of the wing tip relative to the section AA if the span of the wing outboard of AA is 5.5 m and the weight of this part of the wing is distributed linearly from zero at the tip to 2.4 kN/m at AA. Further, the second moment of area of the wing section reduces linearly from a value of  $27.4 \times 10^6 \text{ mm}^4$  at AA (see Example 16.4) to  $2.4 \times 10^6 \text{ mm}^4$  at the tip. Take Young's modulus  $E = 70,000 \text{ N/mm}^2$ .

Answer: 0.14 m.

**P.16.17** The beam section of P.16.4 is subjected to a temperature rise of  $4 T_0$  in its upper flange 12, a temperature rise of  $2T_0$  in both vertical webs, and a temperature rise of  $T_0$  in its lower flange 34. Determine the changes in axial force and in the bending moments about the  $x$  and  $y$  axes. Young's modulus for the material of the beam is  $E$  and its coefficient of linear expansion is  $\alpha$ .

Answer:  $N_T = 9E\alpha dt T_0, M_{xT} = 3E\alpha d^2 t T_0/2, M_{yT} = 3E\alpha d^2 t T_0/4$

**P.16.18** The beam section shown in Fig. P.16.18 is subjected to a temperature change which varies with  $y$  such that  $T = T_0 y/2a$ . Determine the corresponding changes in the stress resultants. Young's modulus for the material of the beam is  $E$ , while its coefficient of linear expansion is  $\alpha$ .

Answer:  $N_T = 0, M_{xT} = 5E\alpha a^2 t T_0/3, M_{yT} = E\alpha a^2 t T_0/6$

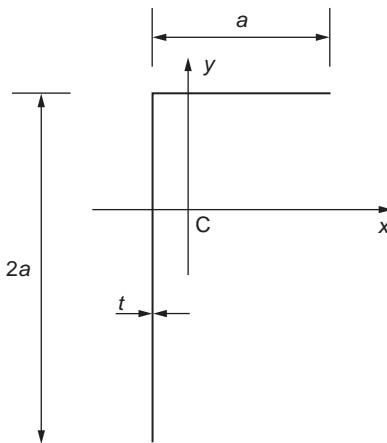


FIGURE P.16.18

# Shear of beams

# 17

In [Chapter 16](#), we developed the theory for the bending of beams by considering solid or thick beam sections and extended the theory to the thin-walled beam sections typical of aircraft structural components. In fact, it is only in the calculation of section properties that thin-walled sections subjected to bending are distinguished from solid and thick sections. However, for thin-walled beams subjected to shear, the theory is based on assumptions applicable only to thin-walled sections, so that we do not consider solid and thick sections; the relevant theory for such sections may be found in any text on structural and stress analysis.<sup>1</sup> The relationships between bending moments, shear forces, and load intensities derived in [Section 16.2.5](#) still apply.

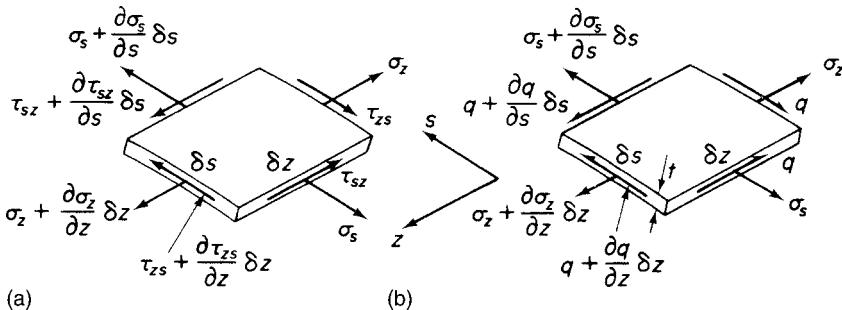
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## 17.1 GENERAL STRESS, STRAIN, AND DISPLACEMENT RELATIONSHIPS FOR OPEN AND SINGLE-CELL CLOSED SECTION THIN-WALLED BEAMS

We establish in this section the equations of equilibrium and expressions for strain necessary for the analysis of open section beams supporting shear loads and closed section beams carrying shear and torsional loads. The analysis of open section beams subjected to torsion requires a different approach and is discussed separately in [Chapter 18](#). The relationships are established from first principles for the particular case of thin-walled sections in preference to the adaption of [Eqs. \(1.6\), \(1.27\), and \(1.28\)](#), which refer to different coordinate axes; the form, however, will be seen to be the same. Generally, in the analysis, we assume that axial constraint effects are negligible, that the shear stresses normal to the beam surface may be neglected since they are zero at each surface and the wall is thin, that direct and shear stresses on planes normal to the beam surface are constant across the thickness, and finally that the beam is of uniform section so that the thickness may vary with distance around each section but is constant along the beam. In addition, we ignore squares and higher powers of the thickness  $t$  in the calculation of section properties (see [Section 16.4.5](#)).

The parameter  $s$  in the analysis is distance measured around the cross-section from some convenient origin.

An element  $\delta s \times \delta z \times t$  of the beam wall is maintained in equilibrium by a system of direct and shear stresses, as shown in [Fig. 17.1\(a\)](#). The direct stress  $\sigma_z$  is produced by bending moments or by the bending action of shear loads, while the shear stresses are due to shear and/or torsion of a closed section beam or shear of an open section beam. The hoop stress  $\sigma_s$  is usually zero but may be caused, in closed section beams, by internal pressure. Although we specified that  $t$  may vary with  $s$ , this variation is small for most thin-walled structures, so that we may reasonably make the approximation that  $t$  is constant



**FIGURE 17.1 (a) General Stress System on an Element of a Closed or Open Section Beam; (b) Direct Stress and Shear Flow System on the Element**

over the length  $\delta s$ . Also, from Eq. (1.4), we deduce that  $\tau_{zs} = \tau_{sz} = \tau$ , say. However, we find it convenient to work in terms of *shear flow*  $q$ , that is, shear force per unit length rather than in terms of shear stress. Hence, in Fig. 17.1(b),

$$q = \tau t \quad (17.1)$$

and is regarded as being positive in the direction of increasing  $s$ .

For equilibrium of the element in the  $z$  direction and neglecting body forces (see Section 1.2),

$$\left( \sigma_z + \frac{\partial \sigma_z}{\partial z} \delta z \right) t \delta s - \sigma_z t \delta s + \left( q + \frac{\partial q}{\partial s} \delta s \right) \delta z - q \delta z = 0$$

which reduces to

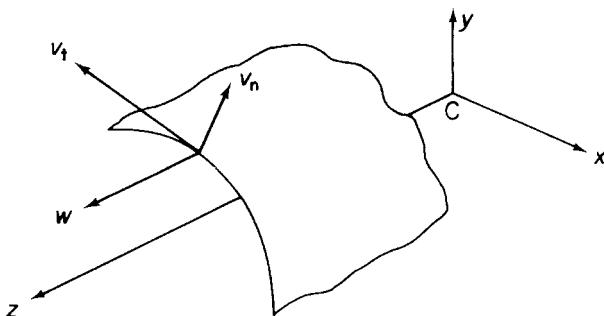
$$\frac{\partial q}{\partial s} + t \frac{\partial \sigma_z}{\partial z} = 0 \quad (17.2)$$

Similarly, for equilibrium in the  $s$  direction,

$$\frac{\partial q}{\partial z} + t \frac{\partial \sigma_s}{\partial s} = 0 \quad (17.3)$$

The direct stresses  $\sigma_z$  and  $\sigma_s$  produce direct strains  $\epsilon_z$  and  $\epsilon_s$ , while the shear stress  $\tau$  induces a shear strain  $\gamma (= \gamma_{zs} = \gamma_{sz})$ . We now proceed to express these strains in terms of the three components of the displacement of a point in the section wall (see Fig. 17.2). Of these components  $v_t$  is a tangential displacement in the  $xy$  plane and is taken to be positive in the direction of increasing  $s$ ;  $v_n$  is a normal displacement in the  $xy$  plane and is positive outward; and  $w$  is an axial displacement defined in Section 16.2.1. Immediately, from the third of Eqs. (1.18), we have

$$\epsilon_z = \frac{\partial w}{\partial z} \quad (17.4)$$



**FIGURE 17.2 Axial, Tangential, and Normal Components of Displacement of a Point in the Beam Wall**

It is possible to derive a simple expression for the direct strain \$\varepsilon\_s\$ in terms of \$v\_t\$, \$v\_n\$, \$s\$, and the curvature \$1/r\$ in the \$xy\$ plane of the beam wall. However, as we do not require \$\varepsilon\_s\$ in the subsequent analysis, for brevity, we merely quote the expression

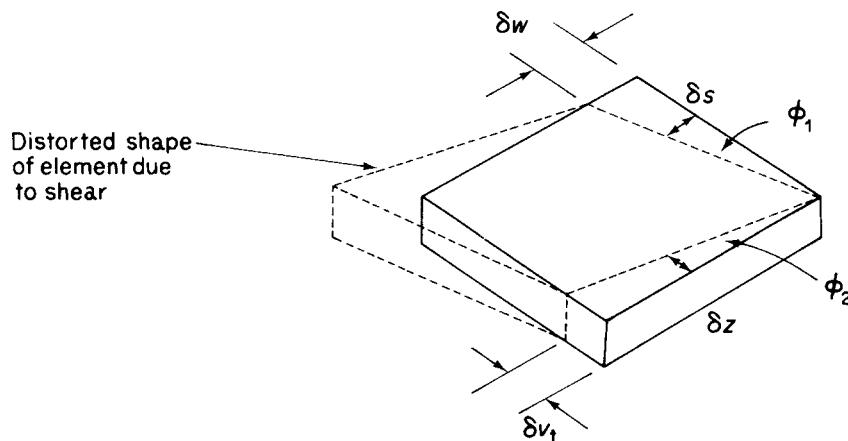
$$\varepsilon_s = \frac{\partial v_t}{\partial s} + \frac{v_n}{r} \quad (17.5)$$

The shear strain \$\gamma\$ is found in terms of the displacements \$w\$ and \$v\_t\$ by considering the shear distortion of an element \$\delta s \times \delta z\$ of the beam wall. From Fig. 17.3, we see that the shear strain is given by

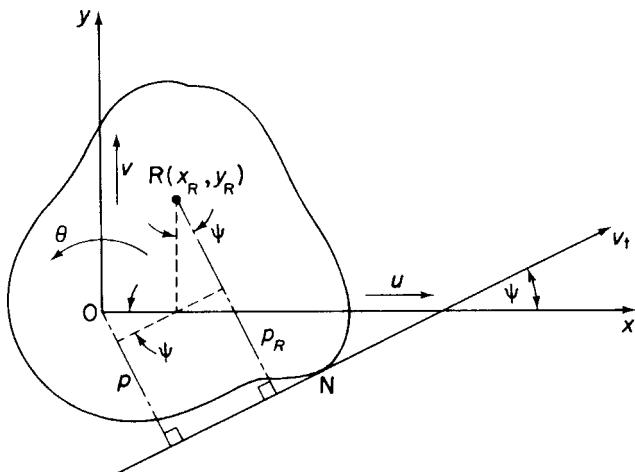
$$\gamma = \phi_1 + \phi_2$$

or, in the limit as both \$\delta s\$ and \$\delta z\$ tend to zero,

$$\gamma = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \quad (17.6)$$



**FIGURE 17.3 Determination of Shear Strain \$\gamma\$ in Terms of Tangential and Axial Components of Displacement**



**FIGURE 17.4 Establishment of Displacement Relationships and Position of Center of Twist of Beam (Open or Closed)**

In addition to the assumptions specified in the earlier part of this section, we further assume that, during any displacement, the shape of the beam cross-section is maintained by a system of closely spaced diaphragms, which are rigid on their own plane but are perfectly flexible normal to their own plane (CSRD assumption). There is, therefore, no resistance to axial displacement  $w$  and the cross-section moves as a rigid body in its own plane, the displacement of any point being completely specified by translations  $u$  and  $v$  and a rotation  $\theta$  (see Fig. 17.4).

At first sight this appears to be a rather sweeping assumption, but for aircraft structures of the thin-shell type described in Chapter 12 whose cross-sections are stiffened by ribs or frames positioned at frequent intervals along their lengths, it is a reasonable approximation of the actual behavior of such sections. The tangential displacement  $v_t$  of any point N in the wall of either an open or closed section beam is seen from Fig. 17.4 to be

$$v_t = p\theta + u \cos \psi + v \sin \psi \quad (17.7)$$

where clearly  $u$ ,  $v$ , and  $\theta$  are functions of  $z$  only ( $w$  may be a function of  $z$  and  $s$ ).

The origin O of the axes in Fig. 17.4 has been chosen arbitrarily, and the axes suffer displacements  $u$ ,  $v$ , and  $\theta$ . These displacements, in a loading case such as pure torsion, are equivalent to a pure rotation about some point  $R(x_R, y_R)$  in the cross-section, where  $R$  is the *center of twist*. Therefore, in Fig. 17.4,

$$v_t = p_R \theta \quad (17.8)$$

and

$$p_R = p - x_R \sin \psi + y_R \cos \psi$$

which gives

$$v_t = p\theta - x_R \theta \sin \psi + y_R \theta \cos \psi$$

and

$$\frac{\partial v_t}{\partial z} = p \frac{d\theta}{dz} - x_R \sin \psi \frac{d\theta}{dz} + y_R \cos \psi \frac{d\theta}{dz} \quad (17.9)$$

Also, from Eq. (17.7),

$$\frac{\partial v_t}{\partial z} = p \frac{d\theta}{dz} + \frac{du}{dz} \cos \psi + \frac{dv}{dz} \sin \psi \quad (17.10)$$

Comparing the coefficients of Eqs. (17.9) and (17.10), we see that

$$x_R = -\frac{dv/dz}{d\theta/dz}, \quad y_R = \frac{du/dz}{d\theta/dz} \quad (17.11)$$

## 17.2 SHEAR OF OPEN SECTION BEAMS

The open section beam of arbitrary section shown in Fig. 17.5 supports shear loads  $S_x$  and  $S_y$  such that there is no twisting of the beam cross-section. For this condition to be valid, both shear loads must pass through a particular point in the cross-section known as the *shear center*.

Since there are no hoop stresses in the beam, the shear flows and direct stresses acting on an element of the beam wall are related by Eq. (17.2), that is,

$$\frac{\partial q}{\partial s} + t \frac{\partial \sigma_z}{\partial z} = 0$$

We assume that the direct stresses are obtained with sufficient accuracy from basic bending theory so that, from Eq. (16.17),

$$\frac{\partial \sigma_z}{\partial z} = \frac{[(\partial M_y / \partial z) I_{xx} - (\partial M_x / \partial z) I_{xy}]}{I_{xx} I_{yy} - I_{xy}^2} x + \frac{[(\partial M_x / \partial z) I_{yy} - (\partial M_y / \partial z) I_{xy}]}{I_{xx} I_{yy} - I_{xy}^2} y$$

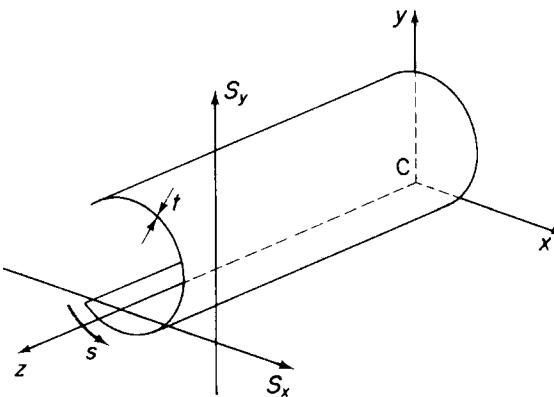


FIGURE 17.5 Shear Loading of an Open Section Beam

Using the relationships of Eqs. (16.22) and (16.23), that is,  $\partial M_y/\partial z = S_x$ , and so forth, this expression becomes

$$\frac{\partial \sigma_z}{\partial z} = \frac{(S_x I_{xx} - S_y I_{xy})}{I_{xx} I_{yy} - I_{xy}^2} x + \frac{(S_y I_{yy} - S_x I_{xy})}{I_{xx} I_{yy} - I_{xy}^2} y$$

Substituting for  $\partial \sigma_z/\partial z$  in Eq. (17.2) gives

$$\frac{\partial q}{\partial s} = -\frac{(S_x I_{xx} - S_y I_{xy})}{I_{xx} I_{yy} - I_{xy}^2} tx - \frac{(S_y I_{yy} - S_x I_{xy})}{I_{xx} I_{yy} - I_{xy}^2} ty \quad (17.12)$$

Integrating Eq. (17.12) with respect to  $s$  from some origin for  $s$  to any point around the cross-section, we obtain

$$\int_0^s \frac{\partial q}{\partial s} ds = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty ds \quad (17.13)$$

If the origin for  $s$  is taken at the open edge of the cross-section, then  $q = 0$  when  $s = 0$ , and Eq. (17.13) becomes

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty ds \quad (17.14)$$

For a section having either Cx or Cy as an axis of symmetry,  $I_{xy} = 0$  and Eq. (17.14) reduces to

$$q_s = -\frac{S_x}{I_{yy}} \int_0^s tx ds - \frac{S_y}{I_{xx}} \int_0^s ty ds$$

### Example 17.1

Determine the shear flow distribution in the thin-walled Z section shown in Fig. 17.6 due to a shear load  $S_y$  applied through the shear center of the section.

The origin for our system of reference axes coincides with the centroid of the section at the midpoint of the web. From antisymmetry, we also deduce by inspection that the shear center occupies the same position. Since  $S_y$  is applied through the shear center, no torsion exists and the shear flow distribution is given by Eq. (17.14), in which  $S_x = 0$ , that is,

$$q_s = \frac{S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s tx ds - \frac{S_y I_{yy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s ty ds$$

or

$$q_s = \frac{S_y}{I_{xx} I_{yy} - I_{xy}^2} \left( I_{xy} \int_0^s tx ds - I_{yy} \int_0^s ty ds \right) \quad (i)$$

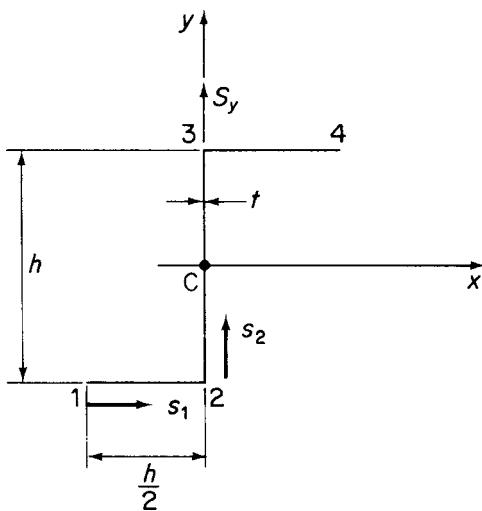


FIGURE 17.6 Shear Loaded Z Section of Example 17.1

The second moments of area of the section were determined in [Example 16.14](#) and are

$$I_{xx} = \frac{h^3 t}{3}, \quad I_{yy} = \frac{h^3 t}{12}, \quad I_{xy} = \frac{h^3 t}{8}$$

Substituting these values in [Eq. \(i\)](#), we obtain

$$q_s = \frac{S_y}{h^3} \int_0^s (10.32x - 6.84y) ds \quad (\text{ii})$$

On the bottom flange 12,  $y = -h/2$  and  $x = -h/2 + s_1$ , where  $0 \leq s_1 \leq h/2$ . Therefore,

$$q_{12} = \frac{S_y}{h^3} \int_0^{s_1} (10.32s_1 - 1.74h) ds_1$$

giving

$$q_{12} = \frac{S_y}{h^3} (5.16s_1^2 - 1.74hs_1) \quad (\text{iii})$$

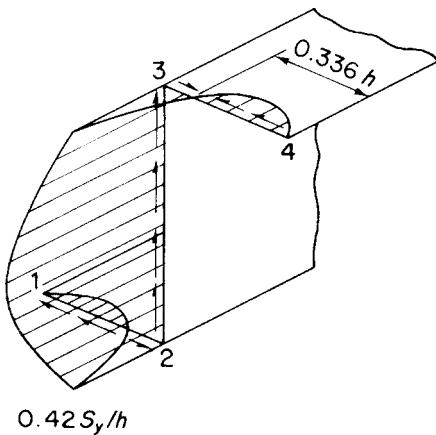
Hence, at 1, ( $s_1 = 0$ ),  $q_1 = 0$  and at 2, ( $s_1 = h/2$ ),  $q_2 = 0.42S_y/h$ . Further examination of [Eq. \(iii\)](#) shows that the shear flow distribution on the bottom flange is parabolic with a change of sign (i.e., direction) at  $s_1 = 0.336h$ . For values of  $s_1 < 0.336h$ ,  $q_{12}$  is negative and therefore in the opposite direction to  $s_1$ .

In the web 23,  $y = -h/2 + s_2$ , where  $0 \leq s_2 \leq h$  and  $x = 0$ . Then,

$$q_{23} = \frac{S_y}{h^3} \int_0^{s_2} (3.42h - 6.84s_2) ds_2 + q_2 \quad (\text{iv})$$

We note, in [Eq. \(iv\)](#), that the shear flow is not zero when  $s_2 = 0$  but equal to the value obtained by inserting  $s_1 = h/2$  in [Eq. \(iii\)](#), that is,  $q_2 = 0.42S_y/h$ . Integration of [Eq. \(iv\)](#) yields

$$q_{23} = \frac{S_y}{h^3} (0.42h^2 + 3.42hs_2 - 3.42s_2^2) \quad (\text{v})$$

**FIGURE 17.7 Shear Flow Distribution in Z Section of Example 17.1**

This distribution is symmetrical about  $C_x$  with a maximum value at  $s_2 = h/2(y = 0)$  and the shear flow is positive at all points in the web.

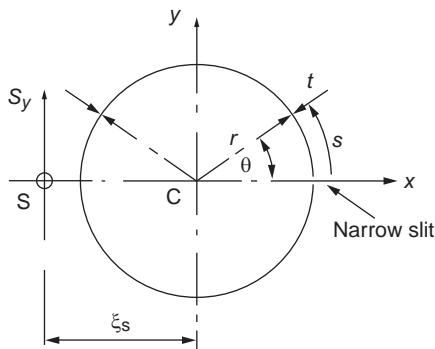
The shear flow distribution in the upper flange may be deduced from antisymmetry, so that the complete distribution is of the form shown in Fig. 17.7.

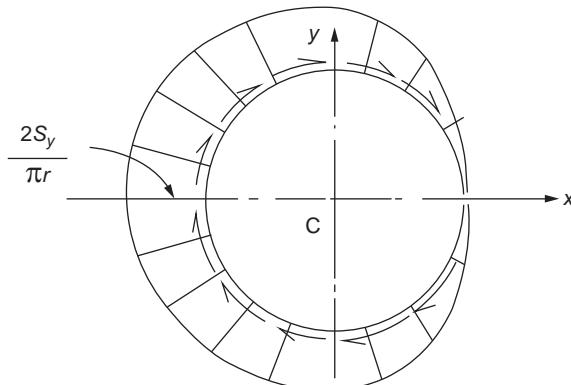
### Example 17.2

Calculate the shear flow distribution in the thin-walled open section shown in Fig. 17.8 produced by a vertical shear load,  $S_y$ , acting through its shear center.

The centroid of the section coincides with the center of the circle. Also the  $C_x$  axis is an axis of symmetry so that  $I_{xy} = 0$ , and since  $S_x = 0$ , Eq. (17.14) reduces to

$$q_s = -\left(S_y/I_{xx}\right) \int_0^s ty \, ds \quad (i)$$

**FIGURE 17.8 Beam Section of Example 17.2**



**FIGURE 17.9 Shear Flow Distribution in the Section of Example 17.2**

The second moment of area,  $I_{xx}$ , of the section about Cx may be deduced from the second moment of area of the semi-circular section shown in Fig. 16.38 and is  $\pi r^3 t$ . Then, at any point a distance  $s$  from one edge of the narrow slit,

$$q_s = -\left(S_y/\pi r^3 t\right) \int_0^s ty \, ds \quad (\text{ii})$$

Working with angular coordinates for convenience, Eq. (ii) becomes

$$q_\theta = -\left(S_y/\pi r^3 t\right) \int_0^\theta tr \sin\theta \, r \, d\theta$$

i.e.,

$$q_\theta = -\left(S_y/\pi r\right) \int_0^\theta \sin\theta \, d\theta$$

which gives

$$q_\theta = -\left(S_y/\pi r\right) [\cos\theta]_0^\theta$$

Then

$$q_\theta = \left(S_y/\pi r\right) (\cos\theta - 1) \quad (\text{iii})$$

From Eq. (iii), when  $\theta = 0$ ,  $q_\theta = 0$  (as expected at an open edge), and when  $\theta = \pi$ ,  $q_\theta = -2S_y/\pi r$ . Further analysis of Eq. (iii) shows that  $q_\theta$  is also zero at  $\theta = 2\pi$  and that  $q_\theta$  is a maximum when  $\theta = \pi$ . Also  $q_\theta$  is negative, i.e., in the opposite sense to increasing values of  $\theta$ , for all values of  $\theta$ . The complete shear flow distribution is shown in Fig. 17.9.

### 17.2.1 Shear center

We defined the position of the shear center as that point in the cross-section through which shear loads produce no twisting. It may be shown by use of the reciprocal theorem that this point is also the center of twist of sections subjected to torsion. There are, however, some important exceptions to this general

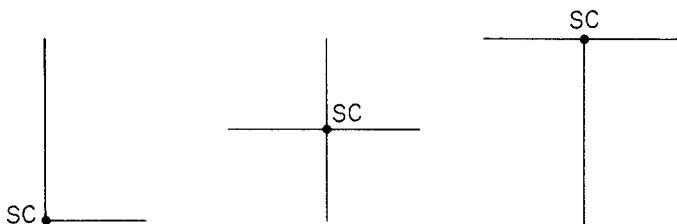


FIGURE 17.10 Shear Center Position for the Type of Open Section Beam Shown

rule, as we observe in [Section 27.1](#). Clearly, in the majority of practical cases, it is impossible to guarantee that a shear load will act through the shear center of a section. Equally apparent is the fact that any shear load may be represented by the combination of the shear load applied through the shear center and a torque. The stresses produced by the separate actions of torsion and shear may then be added by superposition. It is therefore necessary to know the location of the shear center in all types of section or to calculate its position. Where a cross-section has an axis of symmetry, the shear center must, of course, lie on this axis. For cruciform or angle sections of the type shown in [Fig. 17.10](#), the shear center is located at the intersection of the sides, since the resultant internal shear loads all pass through these points.

### Example 17.3

Calculate the position of the shear center of the thin-walled channel section shown in [Fig. 17.11](#). The thickness  $t$  of the walls is constant.

The shear center  $S$  lies on the horizontal axis of symmetry at some distance  $\xi_s$ , say, from the web. If we apply an arbitrary shear load  $S_y$  through the shear center, then the shear flow distribution is given by [Eq. \(17.14\)](#) and the moment about any point in the cross-section produced by these shear flows is *equivalent* to the moment of the applied shear load.  $S_y$  appears on both sides of the resulting equation and may therefore be eliminated to leave  $\xi_s$ .

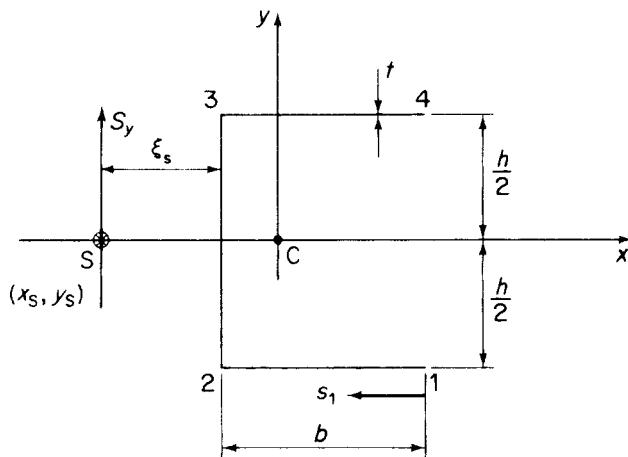


FIGURE 17.11 Determination of Shear Center Position of Channel Section of Example 17.3

For the channel section, Cx is an axis of symmetry so that  $I_{xy} = 0$ . Also  $S_x = 0$ , and therefore Eq. (17.14) simplifies to

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s ty \, ds \quad (\text{i})$$

where

$$I_{xx} = 2bt \left(\frac{h}{2}\right)^2 + \frac{th^3}{12} = \frac{h^3 t}{12} \left(1 + \frac{6b}{h}\right)$$

Substituting for  $I_{xx}$  in Eq. (i), we have

$$q_s = \frac{-12S_y}{h^3(1+6b/h)} \int_0^s y \, ds \quad (\text{ii})$$

The amount of computation involved may be reduced by giving some thought to the requirements of the problem. In this case, we are asked to find the position of the shear center only, not a complete shear flow distribution. From symmetry, it is clear that the moments of the resultant shears on the top and bottom flanges about the midpoint of the web are numerically equal and act in the same rotational sense. Furthermore, the moment of the web shear about the same point is zero. We deduce that it is necessary to obtain the shear flow distribution only on either the top or bottom flange for a solution. Alternatively, choosing a web–flange junction as a moment center leads to the same conclusion.

On the bottom flange,  $y = -h/2$ , so that, from Eq. (ii), we have

$$q_{12} = \frac{6S_y}{h^2(1+6b/h)} s_1 \quad (\text{iii})$$

Equating the clockwise moments of the internal shears about the midpoint of the web to the clockwise moment of the applied shear load about the same point gives

$$S_y \xi_s = 2 \int_0^b q_{12} \frac{h}{2} \, ds_1$$

or, by substitution from Eq. (iii),

$$S_y \xi_s = 2 \int_0^b \frac{6S_y}{h^2(1+6b/h)} \frac{h}{2} s_1 \, ds_1$$

from which

$$\xi_s = \frac{3b^2}{h(1+6b/h)} \quad (\text{iv})$$

In the case of an unsymmetrical section, the coordinates  $(\xi_s, \eta_s)$  of the shear center referred to some convenient point in the cross-section are obtained by first determining  $\xi_s$  in a similar manner to that of Example 17.3 then finding  $\eta_s$  by applying a shear load  $S_x$  through the shear center. In both cases, the choice of a web–flange junction as a moment center reduces the amount of computation.

**Example 17.4**

Determine the position of the shear center of the open section beam of Example 17.2.

The shear center S lies on the horizontal axis of symmetry a distance,  $\xi_S$ , say, from the centroid C. The shear flow distribution is given by Eq. (iii) of Ex. 17.2, i.e.,  $q_\theta = S_y(\cos\theta - 1)/\pi r$ . Therefore, taking moments about C,

$$S_y \xi_S = - \int_0^{2\pi} q_\theta r \, ds = - \int_0^{2\pi} q_\theta r^2 \, d\theta$$

Then

$$S_y \xi_S = -(S_y r / \pi) \int_0^{2\pi} (\cos\theta - 1) \, d\theta \quad (i)$$

Note that the assumed direction of  $q_\theta$  in Ex. 17.2 is in the direction of increasing  $\theta$  so that we introduce a negative sign to allow for this when equating the moment of the external shear force to that of the internal shear flow. Integrating Eq. (i), we obtain

$$\xi_S = -(r/\pi)[\sin\theta - \theta]_0^{2\pi}$$

which gives

$$\xi_S = 2r$$

**Example 17.5**

Calculate the position of the shear center of the thin-walled section shown in Fig. 17.12; the thickness of the section is 2 mm and is constant throughout.

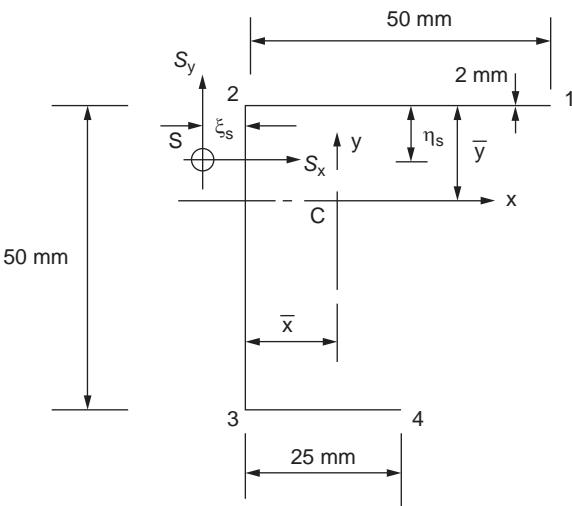


FIGURE 17.12 Beam Section of Example 17.5

Since we are only asked to find the position of the shear center of the section and not a complete shear flow distribution, the argument of Ex. 17.3 applies in that if we refer the position of the shear center to the web/flange junction 2 (or 3), it is only necessary to obtain the shear flow distribution in the flange 34 (or 12). First, we must calculate the section's properties.

Taking moments of area about the top flange,

$$2(50 + 50 + 25)\bar{y} = 2 \times 50 \times 25 + 2 \times 25 \times 50$$

which gives

$$\bar{y} = 20 \text{ mm}$$

Now taking moments of area about the vertical web,

$$2(50 + 50 + 25)\bar{x} = 2 \times 50 \times 25 + 2 \times 25 \times 12.5$$

from which

$$\bar{x} = 12.5 \text{ mm}$$

The second moments of area are then calculated using the methods of Section 16.4.5,

$$I_{xx} = 2 \times 50 \times 20^2 + (2 \times 50^3/12) + 2 \times 50 \times 5^2 + 2 \times 25 \times 30^2$$

i.e.,

$$I_{xx} = 108,333 \text{ mm}^4$$

$$I_{yy} = (2 \times 50^3/12) + 2 \times 50 \times 12.5^2 + 2 \times 50 \times 12.5^2 + (2 \times 25^3/12)$$

i.e.,

$$I_{yy} = 54,689 \text{ mm}^4$$

$$I_{xy} = 2 \times 50 (+12.5)(+20) + 2 \times 50 (-12.5)(-5)$$

i.e.,

$$I_{xy} = 31,250 \text{ mm}^4$$

Note that the flange 34 makes no contribution to  $I_{xy}$ , since its centroid coincides with the y axis of the section.

Considering the horizontal position of the shear center, we apply a vertical shear load,  $S_y$ , through the shear center and determine the shear flow distribution in the flange 34. Since  $S_x = 0$ , Eq. (17.14) reduces to

$$q_s = [(S_y I_{xy}) / (I_{xx} I_{yy} - I_{xy}^2)] \int_0^s t x \, ds - [(S_y I_{yy}) / (I_{xx} I_{yy} - I_{xy}^2)] \int_0^s t y \, ds \quad (\text{i})$$

Substituting the values of  $I_{xx}$  etc. gives

$$q_s = 1.26 \times 10^{-5} S_y \int_0^s x \, ds - 2.21 \times 10^{-5} S_y \int_0^s y \, ds \quad (\text{ii})$$

On the flange 34,  $x = 12.5 - s$  and  $y = -30 \text{ mm}$ . Eq. (ii) then becomes

$$q_{43} = S_y \times 10^{-5} \int_0^s (82.05 - 1.26s) \, ds$$

Integrating gives

$$q_{43} = S_y \times 10^{-5} (82.05s - 0.63s^2) \quad (\text{iii})$$

Now taking moments about the flange/web junction 2,

$$S_y \xi_S = \int_0^{25} q_{43} \times 50 \, ds$$

Substituting for  $q_{43}$  from Eq. (iii) and integrating gives

$$\xi_S = 11.2 \text{ mm}$$

We now apply  $S_x$  only through the shear center and carry out the same procedure as above. Then

$$q_{43} = S_x \times 10^{-5} (2.19s^2 - 92.55s) \quad (\text{iv})$$

Taking moments about the web/flange junction 2,

$$S_x \eta_S = - \int_0^{25} q_{43} \times 50 \, ds$$

Note that in this case the moment of the resultant of the internal shear flows is in the opposite sense to that of the applied force. Substituting for  $q_{43}$  from Eq. (iv) and carrying out the integration gives

$$\eta_S = 7.2 \text{ mm}$$

■

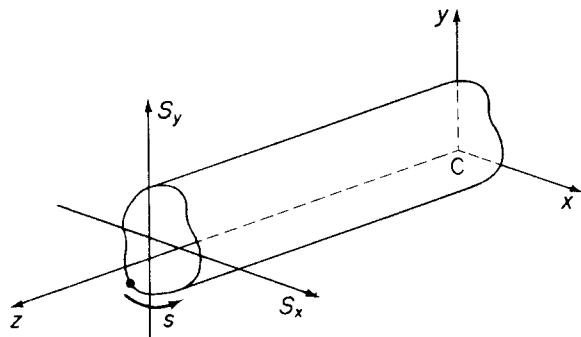
### 17.3 SHEAR OF CLOSED SECTION BEAMS

The solution for a shear-loaded closed section beam follows a similar pattern to that described in Section 17.2 for an open section beam but with two important differences. First, the shear loads may be applied through points in the cross-section other than the shear center, so that torsional as well as shear effects are included. This is possible, since, as we shall see, shear stresses produced by torsion in closed section beams have exactly the same form as shear stresses produced by shear, unlike shear stresses due to shear and torsion in open section beams. Second, it is generally not possible to choose an origin for  $s$  at which the value of shear flow is known. Consider the closed section beam of arbitrary section shown in Fig. 17.13. The shear loads  $S_x$  and  $S_y$  are applied through any point in the cross-section and, in general, cause direct bending stresses and shear flows related by the equilibrium equation (17.2). We assume that hoop stresses and body forces are absent. Therefore,

$$\frac{\partial q}{\partial s} + t \frac{\partial \sigma_z}{\partial z} = 0$$

From this point, the analysis is identical to that for a shear-loaded open section beam, until we reach the stage of integrating Eq. (17.13), namely

$$\int_0^s \frac{\partial q}{\partial s} \, ds = - \left( \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \int_0^s tx \, ds - \left( \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \int_0^s ty \, ds$$



**FIGURE 17.13 Shear of Closed Section Beams**

Let us suppose that we choose an origin for  $s$  where the shear flow has the unknown value  $q_{s,0}$ . Integration of Eq. (17.13) then gives

$$q_s - q_{s,0} = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s t x \, ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s t y \, ds$$

or

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s t x \, ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s t y \, ds + q_{s,0} \quad (17.15)$$

We observe by comparison of Eqs. (17.15) and (17.14) that the first two terms on the right-hand side of Eq. (17.15) represent the shear flow distribution in an open section beam loaded through its shear center. This fact indicates a method of solution for a shear-loaded closed section beam. Representing this “open” section or “basic” shear flow by  $q_b$ , we write Eq. (17.15) in the form

$$q_s = q_b + q_{s,0} \quad (17.16)$$

We obtain  $q_b$  by supposing that the closed beam section is “cut” at some convenient point, thereby producing an “open” section (see Fig. 17.14(b)). The shear flow distribution ( $q_b$ ) around this open section is given by

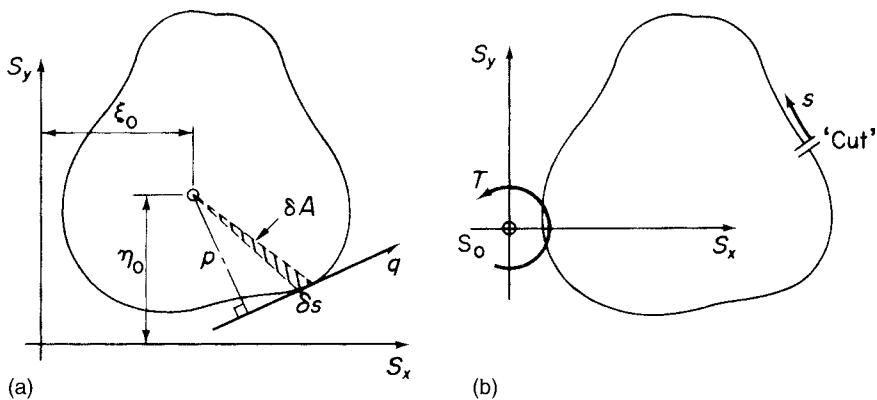
$$q_b = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s t x \, ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s t y \, ds$$

as in Section 17.2. The value of shear flow at the cut ( $s = 0$ ) is found by equating applied and internal moments taken about some convenient moment center. Then, from Fig. 17.14(a),

$$S_x \eta_0 - S_y \xi_0 = \oint p q \, ds = \oint p q_b \, ds + q_{s,0} \oint p \, ds$$

where  $\oint$  denotes integration completely around the cross-section. In Fig. 17.14(a),

$$\delta A = \frac{1}{2} p \delta s$$

FIGURE 17.14 (a) Determination of  $q_{s,0}$ ; (b) Equivalent Loading on an "Open" Section Beam

so that

$$\oint dA = \frac{1}{2} \oint p \, ds$$

Hence,

$$\oint p \, ds = 2A$$

where  $A$  is the area enclosed by the midline of the beam section wall. Hence,

$$S_x \eta_0 - S_y \xi_0 = \oint p q_b \, ds + 2A q_{s,0} \quad (17.17)$$

If the moment center is chosen to coincide with the lines of action of  $S_x$  and  $S_y$ , then Eq. (17.17) reduces to

$$0 = \oint p q_b \, ds + 2A q_{s,0} \quad (17.18)$$

The unknown shear flow  $q_{s,0}$  follows from either Eq. (17.17) or (17.18).

It is worthwhile to consider some of the implications of this process. Equation (17.14) represents the shear flow distribution in an open section beam for the condition of zero twist. Therefore, by "cutting" the closed section beam of Fig. 17.14(a) to determine  $q_b$ , we are, in effect, replacing the shear loads of Fig. 17.14(a) by shear loads  $S_x$  and  $S_y$  acting through the shear center of the resulting "open" section beam together with a torque  $T$ , as shown in Fig. 17.14(b). We show in Section 18.1 that the application of a torque to a closed section beam results in a constant shear flow. In this case, the constant shear flow  $q_{s,0}$  corresponds to the torque but has different values for different positions of the cut, since the corresponding various open-section beams have different locations for their shear centers. An additional effect of cutting the beam is to produce a statically determinate structure, since the  $q_b$  shear flows are obtained from statical equilibrium considerations. It follows that a single-cell closed section beam supporting shear loads is singly redundant.

**Example 17.6**

Determine the shear flow distribution in the walls of the thin-walled closed section beam shown in Fig. 17.15; the wall thickness,  $t$ , is constant throughout.

Since the  $x$  axis is an axis of symmetry,  $I_{xy} = 0$ , and since  $S_x = 0$ , Eq. (17.15) reduces to

$$q_s = -\left(S_y/I_{xx}\right) \int_0^s ty \, ds + q_{s,0} \quad (i)$$

where

$$I_{xx} = (\pi tr^3/2) + 2 \times 2rt \times r^2 + \left[t(2r)^3/12\right] = 6.24tr^3$$

(see Section 16.4.5)

We now “cut” the beam section at 1. Any point may be chosen for the “cut,” but the amount of computation will be reduced if a point is chosen that coincides with the axis of symmetry. Then

$$q_{b,12} = -\left(S_y/I_{xx}\right) \int_0^\theta tr \sin\theta r d\theta$$

which gives

$$q_{b,12} = 0.16(S_y/r) [\cos\theta]_0^\theta$$

so that

$$q_{b,12} = 0.16(S_y/r)(\cos\theta - 1) \quad (ii)$$

When  $\theta = \pi/2$ ,  $q_{b,12} = -0.16(S_y/r)$ . The  $q_b$  shear flow in the wall 23 is then

$$q_{b,23} = -\left(S_y/I_{xx}\right) \int_0^{s_1} tr \, ds_1 - 0.16(S_y/r)$$

i.e.,

$$q_{b,23} = -0.16(S_y/r^2)(s_1 + r) \quad (iii)$$

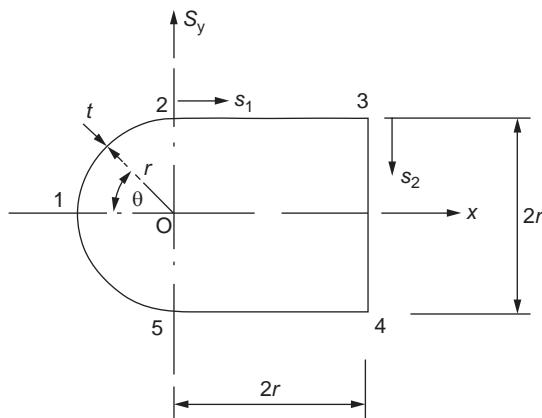


FIGURE 17.15 Beam Section of Example 17.6

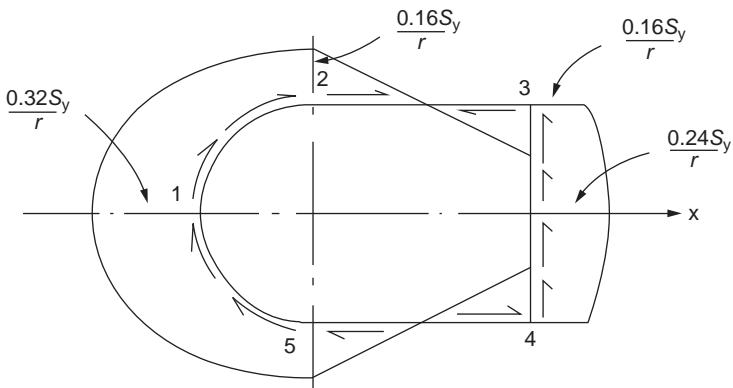


FIGURE 17.16 Shear Flow Distribution in Beam Section of Example 17.6

and when  $s_1 = 2r$ ,  $q_{b,3} = -0.48(S_y/r)$ . Then, in the wall 34,

$$q_{b,34} = -0.16(S_y/tr^3) \int_0^{s_2} t(r - s_2) ds_2 - 0.48(S_y/r)$$

i.e.,

$$q_{b,34} = -0.16(S_y/r^3)(rs_2 - 0.5s_2^2 + 3r^2) \quad (\text{iv})$$

The remaining distribution follows from symmetry. Now taking moments about O and using Eq. (17.18),

$$0 = 2 \left[ \int_0^{\pi/2} q_{b,12} r^2 d\theta + \int_0^{2r} q_{b,23} r ds_1 + \int_0^r q_{b,34} 2r ds_2 \right] + 2[4r^2 + (\pi r^2/2)] q_{s,0} \quad (\text{v})$$

Substituting in Eq. (v) for  $q_{b,12}$  etc from Eqs. (ii), (iii), and (iv) gives

$$q_{s,0} = 0.32S_y/r$$

Clearly, the substitution of  $q_{b,12}$ , etc. in Eq. (v) and the subsequent integration is a lengthy but straightforward procedure, and therefore, for brevity, is omitted here.

Adding  $q_{s,0}$  to the  $q_b$  distributions of Eqs. (ii), (iii), and (iv) gives

$$\begin{aligned} q_{12} &= 0.16(S_y/r^3)(r^2 \cos \theta + r^2) \\ q_{23} &= 0.16(S_y/r^3)(r^2 - rs_1) \\ q_{34} &= 0.16(S_y/r^3)(0.5s_2^2 - rs_2 - r^2) \end{aligned}$$

Note that  $q_{23}$  changes sign at  $s_1 = r$ . The shear flow distribution in the lower half of the section follows from symmetry and the complete distribution is shown in Fig. 17.16.

### 17.3.1 Twist and warping of shear-loaded closed section beams

Shear loads that are not applied through the shear center of a closed section beam cause cross-sections to twist and warp; that is, in addition to rotation, they suffer out-of-plane axial displacements. Expressions for these quantities may be derived in terms of the shear flow distribution  $q_s$  as follows. Since

$q = \tau t$  and  $\tau = G\gamma$  (see Chapter 1), we can express  $q_s$  in terms of the warping and tangential displacements  $w$  and  $v_t$  of a point in the beam wall by using Eq. (17.6). Thus,

$$q_s = Gt \left( \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \right) \quad (17.19)$$

Substituting for  $\partial v_t/\partial z$  from Eq. (17.10), we have

$$\frac{q_s}{Gt} = \frac{\partial w}{\partial s} + p \frac{d\theta}{dz} + \frac{du}{dz} \cos \psi + \frac{dv}{dz} \sin \psi \quad (17.20)$$

Integrating Eq. (17.20) with respect to  $s$  from the chosen origin for  $s$  and noting that  $G$  may also be a function of  $s$ , we obtain

$$\int_0^s \frac{q_s}{Gt} ds = \int_0^s \frac{\partial w}{\partial s} ds + \frac{d\theta}{dz} \int_0^s p ds + \frac{du}{dz} \int_0^s \cos \psi ds + \frac{dv}{dz} \int_0^s \sin \psi ds$$

or

$$\int_0^s \frac{q_s}{Gt} ds = \int_0^s \frac{\partial w}{\partial s} ds + \frac{d\theta}{dz} \int_0^s p ds + \frac{du}{dz} \int_0^s dx + \frac{dv}{dz} \int_0^s dy$$

which gives

$$\int_0^s \frac{q_s}{Gt} ds = (w_s - w_0) + 2A_{Os} \frac{d\theta}{dz} + \frac{du}{dz} (x_s - x_0) + \frac{dv}{dz} (y_s - y_0) \quad (17.21)$$

where  $A_{Os}$  is the area swept out by a generator, center at the origin of axes, O, from the origin for  $s$  to any point  $s$  around the cross-section. Continuing the integration completely around the cross-section yields, from Eq. (17.21),

$$\oint \frac{q_s}{Gt} ds = 2A \frac{d\theta}{dz}$$

from which

$$\frac{d\theta}{dz} = \frac{1}{2A} \oint \frac{q_s}{Gt} ds \quad (17.22)$$

Substituting for the rate of twist in Eq. (17.21) from Eq. (17.22) and rearranging, we obtain the warping distribution around the cross-section:

$$w_s - w_0 = \int_0^s \frac{q_s}{Gt} ds - \frac{A_{Os}}{A} \oint \frac{q_s}{Gt} ds - \frac{du}{dz} (x_s - x_0) - \frac{dv}{dz} (y_s - y_0) \quad (17.23)$$

Using Eqs. (17.11) to replace  $du/dz$  and  $dv/dz$  in Eq. (17.23), we have

$$w_s - w_0 = \int_0^s \frac{q_s}{Gt} ds - \frac{A_{Os}}{A} \oint \frac{q_s}{Gt} ds - y_R \frac{d\theta}{dz} (x_s - x_0) + x_R \frac{d\theta}{dz} (y_s - y_0) \quad (17.24)$$

The last two terms in Eq. (17.24) represent the effect of relating the warping displacement to an arbitrary origin that itself suffers axial displacement due to warping. In the case where the origin coincides with the center of twist R of the section (i.e.,  $x_R = y_R = 0$ ), Eq. (17.24) simplifies to

$$w_s - w_0 = \int_0^s \frac{q_s}{Gt} ds - \frac{A_{Os}}{A} \oint \frac{q_s}{Gt} ds \quad (17.25)$$

In problems involving singly or doubly symmetrical sections, the origin for  $s$  may be taken to coincide with a point of zero warping, which occurs where an axis of symmetry and the wall of the section intersect. For unsymmetrical sections, the origin for  $s$  may be chosen arbitrarily. The resulting warping distribution has exactly the same form as the actual distribution but is displaced axially by the unknown warping displacement at the origin for  $s$ . This value may be found by referring to the torsion of closed section beams subject to axial constraint (see Section 27.3). In the analysis of such beams, it is assumed that the direct stress distribution set up by the constraint is directly proportional to the free warping of the section; that is,

$$\sigma = \text{constant} \times w$$

Also, since a pure torque is applied, the result of any internal direct stress system must be zero; in other words, it is self-equilibrating. Thus,

$$\text{Resultant axial load} = \oint \sigma t ds$$

where  $\sigma$  is the direct stress at any point in the cross-section. Then, from the preceding assumption,

$$0 = \oint w t ds$$

or

$$0 = \oint (w_s - w_0)t ds$$

so that

$$w_0 = \frac{\oint w_s t ds}{\oint t ds} \quad (17.26)$$

### 17.3.2 Shear center

The shear center of a closed section beam is located in a similar manner to that described in Section 17.2.1 for open section beams. Therefore, to determine the coordinate  $\xi_S$  (referred to any convenient point in the cross-section) of the shear center S of the closed section beam shown in Fig. 17.17, we apply an arbitrary shear load  $S_y$  through S, calculate the distribution of shear flow  $q_s$  due to  $S_y$  and equate internal and external moments. However, a difficulty arises in obtaining  $q_{s,0}$  since, at this stage, it is impossible to equate internal and external moments to produce an equation similar to Eq. (17.17) as the position of  $S_y$  is unknown. We therefore use the condition that a

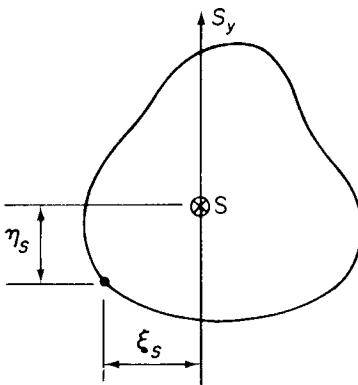


FIGURE 17.17 Shear Center of a Closed Section Beam

shear load acting through the shear center of a section produces zero twist. It follows that  $d\theta/dz$  in Eq. (17.22) is zero, so that

$$0 = \oint \frac{q_s}{Gt} ds$$

or

$$0 = \oint \frac{1}{Gt} (q_b + q_{s,0}) ds$$

which gives

$$q_{s,0} = - \frac{\oint (q_b/Gt) ds}{\oint ds/Gt} \quad (17.27)$$

If  $Gt = \text{constant}$ , then Eq. (17.27) simplifies to

$$q_{s,0} = - \frac{\oint q_b ds}{\oint ds} \quad (17.28)$$

The coordinate  $\eta_s$  is found in a similar manner by applying  $S_x$  through S.

### Example 17.7

A thin-walled closed section beam has the singly symmetrical cross-section shown in Fig. 17.18. Each wall of the section is flat and has the same thickness  $t$  and shear modulus  $G$ . Calculate the distance of the shear center from point 4.

The shear center clearly lies on the horizontal axis of symmetry, so that it is necessary only to apply a shear load  $S_y$  through S and to determine  $\xi_s$ . If we take the  $x$  reference axis to coincide with the axis of symmetry, then  $I_{xy} = 0$ , and since  $S_x = 0$ , Eq. (17.15) simplifies to

$$q_s = - \frac{S_y}{I_{xx}} \int_0^s ty ds + q_{s,0} \quad (i)$$

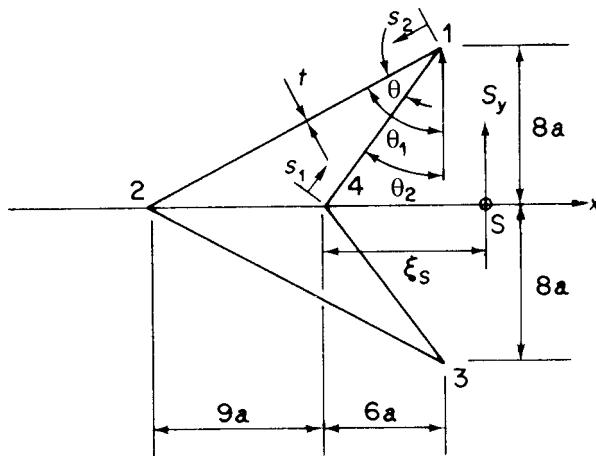


FIGURE 17.18 Closed Section Beam of Example 17.7

in which

$$I_{xx} = 2 \left[ \int_0^{10a} t \left( \frac{8}{10} s_1 \right)^2 ds_1 + \int_0^{17a} t \left( \frac{8}{17} s_2 \right)^2 ds_2 \right]$$

Evaluating this expression gives  $I_{xx} = 1,152a^3t$ .

The basic shear flow distribution  $q_b$  is obtained from the first term in Eq. (i). Then, for the wall 41,

$$q_{b,41} = \frac{-S_y}{1,152a^3t} \int_0^{s_1} t \left( \frac{8}{10} s_1 \right) ds_1 = \frac{-S_y}{1,152a^3} \left( \frac{2}{5} s_1^2 \right) \quad (\text{ii})$$

In the wall 12,

$$q_{b,12} = \frac{-S_y}{1,152a^3} \left[ \int_0^{s_2} (17a - s_2) \frac{8}{17} ds_2 + 40a^2 \right] \quad (\text{iii})$$

which gives

$$q_{b,12} = \frac{-S_y}{1,152a^3} \left( -\frac{4}{17} s_2^2 + 8as_2 + 40a^2 \right) \quad (\text{iv})$$

The  $q_b$  distributions on the walls 23 and 34 follow from symmetry. Hence, from Eq. (17.28) and noting that  $\oint ds = 2 \left[ 10a + \sqrt{(8a)^2 + (15a)^2} \right] = 54a$ ,

$$q_{s,0} = \frac{2S_y}{54a \times 1,152a^3} \left[ \int_0^{10a} \frac{2}{5} s_1^2 ds_1 + \int_0^{17a} \left( -\frac{4}{17} s_2^2 + 8as_2 + 40a^2 \right) ds_2 \right]$$

giving

$$q_{s,0} = \frac{S_y}{1,152a^3} (58.7a^2) \quad (\text{v})$$

Taking moments about the point 2, we have

$$S_y(\xi_s + 9a) = 2 \int_0^{10a} q_{41} 17a \sin \theta \, ds_1$$

or

$$S_y(\xi_s + 9a) = \frac{S_y 34a \sin \theta}{1,152a^3} \int_0^{10a} \left( -\frac{2}{5}s_1^2 + 58.7a^2 \right) \, ds_1 \quad (\text{vi})$$

We may replace  $\sin \theta$  by  $\sin(\theta_1 - \theta_2) = \sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2$ , where  $\sin \theta_1 = 15/17$ ,  $\cos \theta_2 = 8/10$ ,  $\cos \theta_1 = 8/17$ , and  $\sin \theta_2 = 6/10$ . Substituting these values and integrating Eq. (vi) gives

$$\xi_s = -3.35a$$

which means that the shear center is inside the beam section. ■

## Reference

- [1] Megson THG. Structural and stress analysis. 4th ed. Oxford: Elsevier; 2019.

## PROBLEMS

**P.17.1.** A beam has the singly symmetrical, thin-walled cross-section shown in Fig. P.17.1. The thickness  $t$  of the walls is constant throughout. Show that the distance of the shear center from the web is given by

$$\xi_s = -d \frac{\rho^2 \sin \alpha \cos \alpha}{1 + 6\rho + 2\rho^3 \sin^2 \alpha}$$

where

$$\rho = d/h$$

**P.17.2.** A beam has the singly symmetrical, thin-walled cross-section shown in Fig. P.17.2. Each wall of the section is flat and has the same length  $a$  and thickness  $t$ . Calculate the distance of the shear center from the point 3.

Answer:  $5a \cos \alpha/8$

**P.17.3.** Determine the position of the shear center S relative to the point 1 for the thin-walled, open cross-section shown in Fig. P.17.3. The thickness  $t$  is constant.

Answer:  $\pi r/3$

**P.17.4.** Figure P.17.4 shows the cross-section of a thin, singly symmetrical I section. Show that the distance  $\xi_s$  of the shear center from the vertical web is given by

$$\frac{\xi_s}{d} = \frac{3\rho(1 - \beta)}{(1 + 12\rho)}$$

where  $\rho = d/h$ . The thickness  $t$  is taken to be negligibly small in comparison with the other dimensions.

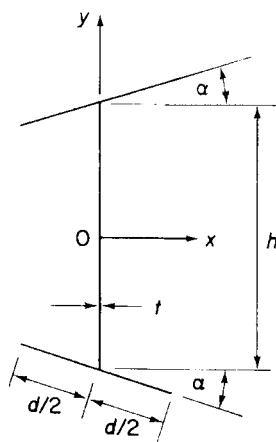


FIGURE P.17.1

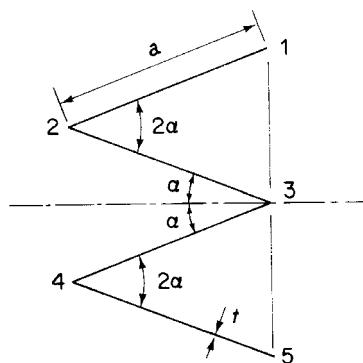


FIGURE P.17.2

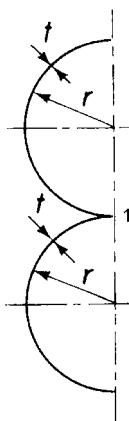


FIGURE P.17.3

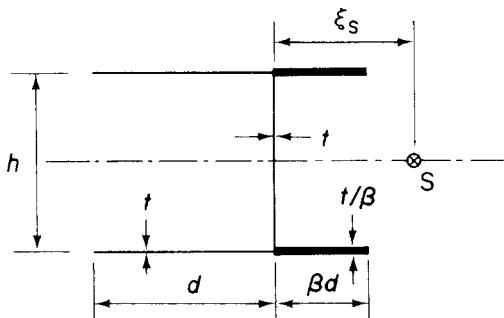


FIGURE P.17.4

**P.17.5.** A thin-walled beam has the cross-section shown in Fig. P.17.5. The thickness of each flange varies linearly from  $t_1$  at the tip to  $t_2$  at the junction with the web. The web itself has a constant thickness  $t_3$ . Calculate the distance  $ξ_s$  from the web to the shear center S.

Answer:  $d^2(2t_1 + t_2)/[3d(t_1 + t_2) + ht_3]$

**P.17.6.** Figure P.17.6 shows the singly symmetrical cross-section of a thin-walled open section beam of constant wall thickness  $t$ , which has a narrow longitudinal slit at the corner 15. Calculate and sketch the distribution of shear flow due to a vertical shear force  $S_y$  acting through the shear center S and note the principal values. Show also that the distance  $ξ_s$  of the shear center from the nose of the section is  $ξ_s = l/2(1 + a/b)$ .

Answer:  $q_2 = q_4 = 3bS_y/2h(b+a)$ ,  $q_3 = 3S_y/2h$ , parabolic distributions

**P.17.7.** Show that the position of the shear center S with respect to the intersection of the web and lower flange of the thin-walled section shown in Fig. P.17.7, is given by

$$\xi_s = -45a/97, \quad \eta_s = 46a/97$$

**P.17.8.** Determine the position of the shear center of the beam section shown in Fig. P.17.8.

Answer: 13.5 mm to the left of the web 24 and 32.7 mm below the flange 123.

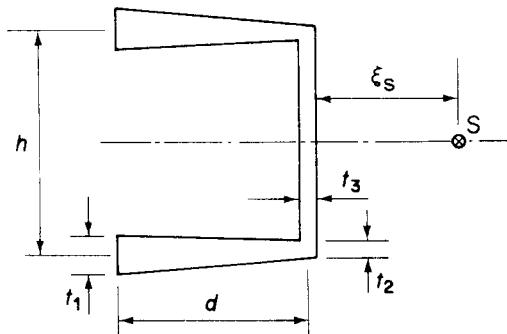


FIGURE P.17.5

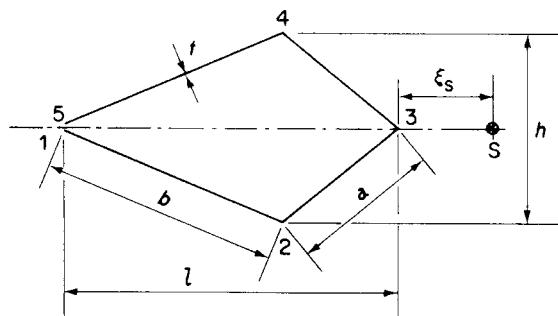


FIGURE P.17.6

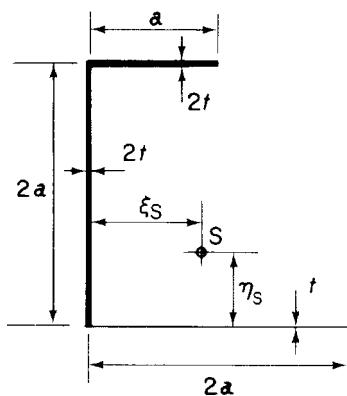


FIGURE P.17.7

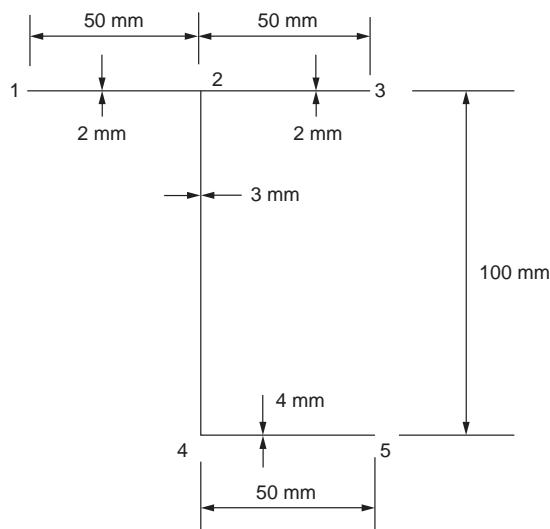


FIGURE P.17.8

**P.17.9.** Define the term *shear center* of a thin-walled open section and determine the position of the shear center of the thin-walled open section shown in Fig. P.17.9.

Answer:  $2.66r$  from center of semi-circular wall

**P.17.10.** Determine the position of the shear center of the cold-formed, thin-walled section shown in Fig. P.17.10. The thickness of the section is constant throughout.

Answer: 87.5 mm above center of semi-circular wall

**P.17.11.** Find the position of the shear center of the thin-walled beam section shown in Fig. P.17.11.

Answer:  $1.2r$  on axis of symmetry to the left of the section

**P.17.12.** Calculate the position of the shear center of the thin-walled section shown in Fig. P.17.12.

Answer: 20.2 mm to the left of the vertical web on axis of symmetry

**P.17.12. MATLAB** If the total length of the vertical web of the thin-walled section shown in Fig. P.17.12 is labeled  $L_{\text{web}}$ , use MATLAB to repeat Problem P.17.12 for values of  $L_{\text{web}}$  ranging from 40 mm to 80 mm in increments of 5 mm (i.e., 40, 45, 50, . . . , 80).

Answer: (i)  $L_{\text{web}} = 40 \text{ mm}$ : 16.2 mm to the left of the vertical web on axis of symmetry  
(ii)  $L_{\text{web}} = 45 \text{ mm}$ : 17.3 mm to the left of the vertical web on axis of symmetry

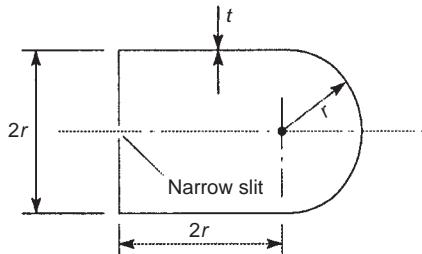


FIGURE P.17.9

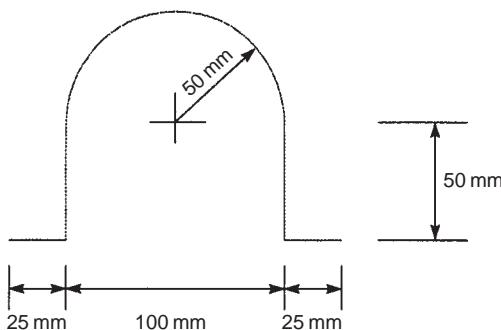


FIGURE P.17.10

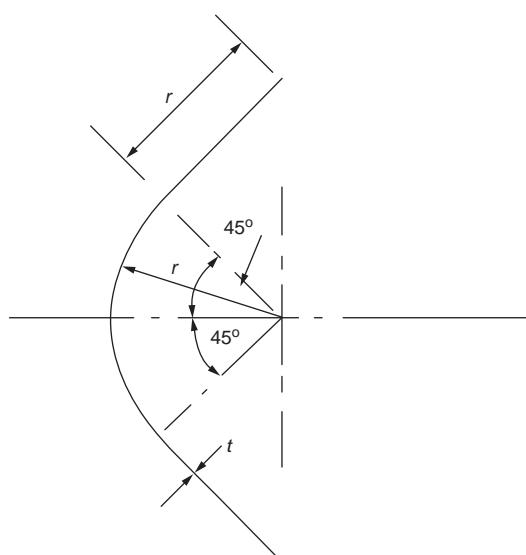


FIGURE P.17.11

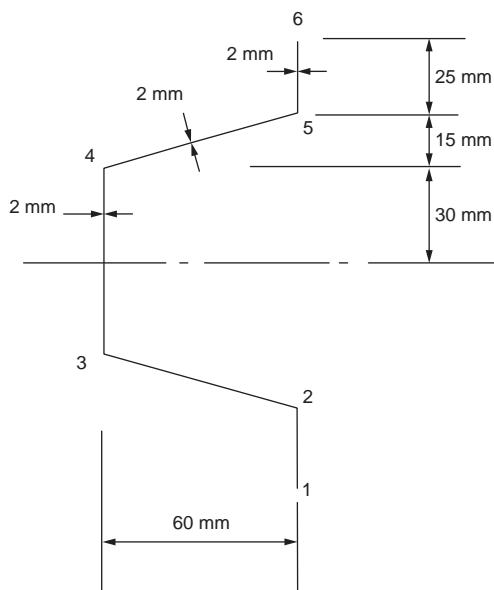


FIGURE P.17.12

- (iii)  $L_{\text{web}} = 50 \text{ mm}$ : 18.3 mm to the left of the vertical web on axis of symmetry
- (iv)  $L_{\text{web}} = 55 \text{ mm}$ : 19.3 mm to the left of the vertical web on axis of symmetry
- (v)  $L_{\text{web}} = 60 \text{ mm}$ : 20.2 mm to the left of the vertical web on axis of symmetry
- (vi)  $L_{\text{web}} = 65 \text{ mm}$ : 21.0 mm to the left of the vertical web on axis of symmetry
- (vii)  $L_{\text{web}} = 70 \text{ mm}$ : 21.8 mm to the left of the vertical web on axis of symmetry
- (viii)  $L_{\text{web}} = 75 \text{ mm}$ : 22.5 mm to the left of the vertical web on axis of symmetry
- (ix)  $L_{\text{web}} = 80 \text{ mm}$ : 23.1 mm to the left of the vertical web on axis of symmetry

**P.17.13.** Determine the horizontal distance from O of the shear center of the beam section shown in Fig. P.17.13. Note that the position of the centroid of area, C, is given.

Answer:  $2.0r$  to the left of O.

**P.17.14.** Calculate the horizontal distance from the vertical flange 12 of the shear center of the beam section shown in Fig. P.17.14. What would be the horizontal movement of the shear center if the circular arc 34 was removed? Note that the section properties are given.

Answer:  $0.36r$ .  $0.36r$  to the left.

**P.17.15.** A thin-walled closed section beam of constant wall thickness  $t$  has the cross-section shown in Fig. P.17.15. Assuming that the direct stresses are distributed according to the basic theory of bending, calculate and sketch the shear flow distribution for a vertical shear force  $S_y$  applied tangentially to the curved part of the beam.

$$\text{Answer: } q_{01} = S_y(1.61 \cos \theta - 0.80)/r$$

$$q_{12} = \frac{S_y}{r^3} (0.57s^2 - 1.14rs + 0.33r^2)$$

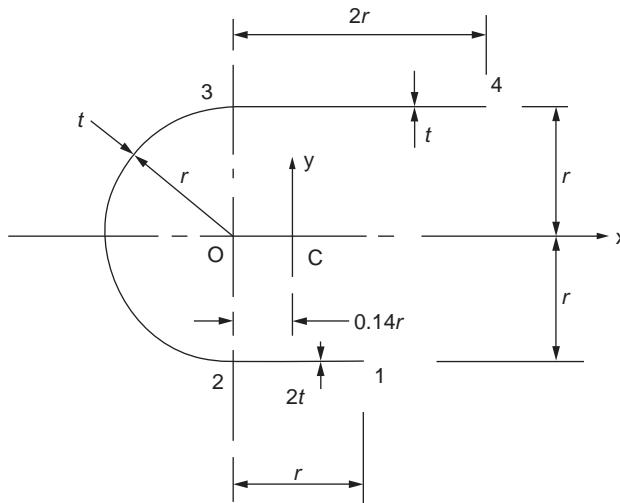


FIGURE P.17.13

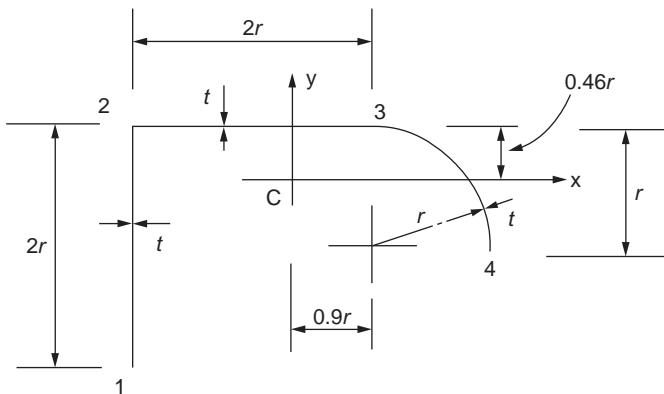


FIGURE P.17.14

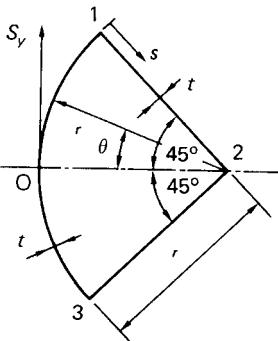


FIGURE P.17.15

**P.17.16.** A uniform thin-walled beam of constant wall thickness  $t$  has a cross-section in the shape of an isosceles triangle and is loaded with a vertical shear force  $S_y$  applied at the apex. Assuming that the distribution of shear stress is according to the basic theory of bending, calculate the distribution of shear flow over the cross-section. Illustrate your answer with a suitable sketch, marking in carefully with arrows the direction of the shear flows and noting the principal values.

$$\text{Answer: } q_{12} = S_y(3s_1^2/d - h - 3d)/h(h + 2d)$$

$$q_{23} = S_y(-6s_2^2 + 6hs_2 - h^2)/h^2(h + 2d)$$

**P.17.17.** Figure P.17.17 shows the regular hexagonal cross-section of a thin-walled beam of sides  $a$  and constant wall thickness  $t$ . The beam is subjected to a transverse shear force  $S$ , its line of action being along a side of the hexagon, as shown. Plot the shear flow distribution around the section, with values in terms of  $S$  and  $a$ .

$$\text{Answer: } q_1 = -0.52S/a, q_2 = q_8 = -0.47S/a, q_3 = q_7 = -0.17S/a,$$

$$q_4 = q_6 = 0.13S/a, q_5 = 0.18S/a$$

Parabolic distributions,  $q$  positive clockwise

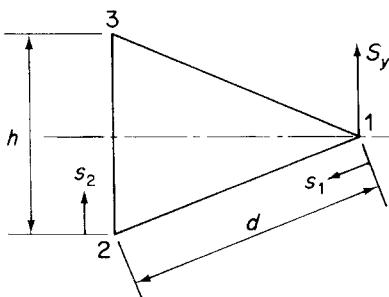


FIGURE P.17.16

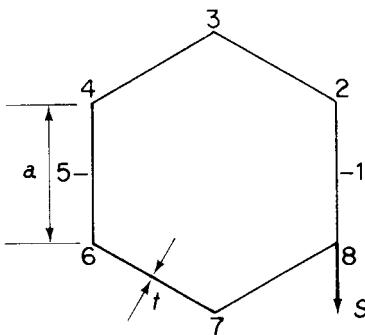


FIGURE P.17.17

**P.17.18.** A box girder has the singly symmetrical trapezoidal cross-section shown in Fig. P.17.18. It supports a vertical shear load of 500 kN applied through its shear center and in a direction perpendicular to its parallel sides. Calculate the shear flow distribution and the maximum shear stress in the section.

$$\begin{aligned} \text{Answer: } q_{OA} &= 0.25s_A \\ q_{AB} &= 0.21s_B - 2.14 \times 10^{-4}s_B^2 + 250 \\ q_{BC} &= -0.17s_C + 246 \\ \tau_{\max} &= 30.2 \text{ N/mm}^2 \end{aligned}$$

**P.17.19.** Calculate the position of the shear center of the beam section shown in Fig. P.17.15.

Answer:  $0.61r$  from the corner 2.

**P.17.20.** Determine the radius of the inscribed circle of the beam section shown in Fig. P.17.16 and show that the shear center of the section coincides with its center.

$$\text{Answer: } h[(2d - h)(2d + h)]^{1/2}/2$$

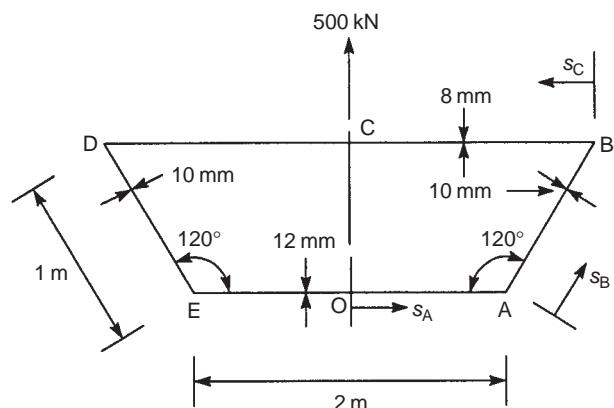


FIGURE P.17.18

# Torsion of beams

# 18

In [Chapter 3](#), we developed the theory for the torsion of solid sections using both the Prandtl stress function approach and the St. Venant warping function solution. From that point, we looked, via the membrane analogy, at the torsion of a narrow rectangular strip. We use the results of this analysis to investigate the torsion of thin-walled open section beams, but first we shall examine the torsion of thin-walled closed section beams, since the theory for this relies on the general stress, strain, and displacement relationships, which we established in [Chapter 17](#).

## 18.1 TORSION OF CLOSED SECTION BEAMS

A closed section beam subjected to a pure torque  $T$ , as shown in [Fig. 18.1](#), does not, in the absence of an axial constraint and internal pressure, develop a direct stress system. It follows that the equilibrium conditions of [Eqs. \(17.2\)](#) and [\(17.3\)](#) reduce to  $\partial q/\partial s = 0$  and  $\partial q/\partial z = 0$ , respectively. These relationships may be satisfied simultaneously only by a constant value of  $q$ . We deduce, therefore, that the application of a pure torque to a closed section beam results in the development of a constant shear flow in the beam wall. However, the shear stress  $\tau$  may vary around the cross-section, since we allow the wall thickness  $t$  to be a function of  $s$ . The relationship between the applied torque and this constant shear flow is simply derived by considering the torsional equilibrium of the section shown in [Fig. 18.2](#). The torque produced by the shear flow acting on an element  $ds$  of the beam wall is  $pqds$ . Hence,

$$T = \oint pq \, ds$$

or, since  $q$  is constant and  $\oint p \, ds = 2A$  (see [Section 17.3](#)),

$$T = 2Aq \quad (18.1)$$

Note that the origin O of the axes in [Fig. 18.2](#) may be positioned in or outside the cross-section of the beam, since the moment of the internal shear flows (whose resultant is a pure torque) is the same about any point in their plane. For an origin outside the cross-section, the term  $\oint p \, ds$  involves the summation of positive and negative areas. The sign of an area is determined by the sign of  $p$ , which itself is associated with the sign convention for torque as follows. If the movement of the foot of  $p$  along the tangent at any point in the positive direction of  $s$  leads to a counterclockwise rotation of  $p$  about the origin of axes,  $p$  is positive. The positive direction of  $s$  is in the positive direction of  $q$ , which is counterclockwise (corresponding to a positive torque). Thus, in [Fig. 18.3](#), a generator OA, rotating about O, initially sweeps out a negative area, since  $p_A$  is negative. At B, however,  $p_B$  is positive, so that the area swept out by the generator has changed sign (at the point where the tangent passes through O

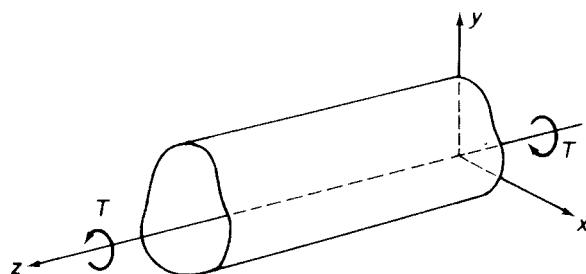


FIGURE 18.1 Torsion of a Closed Section Beam

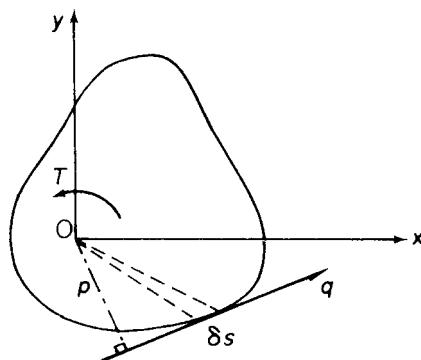


FIGURE 18.2 Determination of the Shear Flow Distribution in a Closed Section Beam Subjected to Torsion

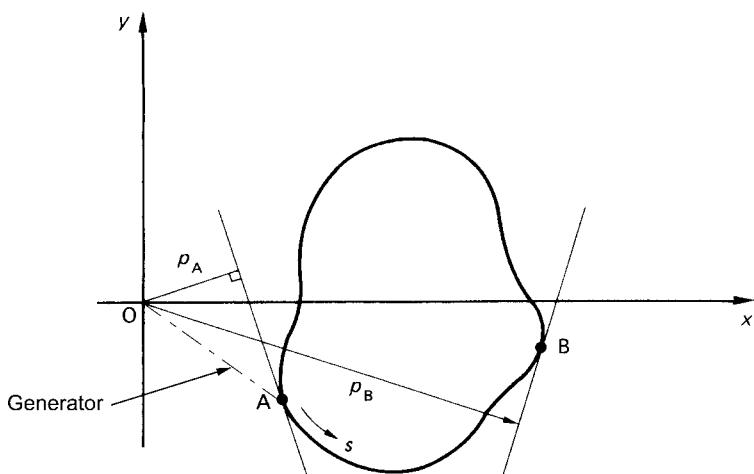


FIGURE 18.3 Sign Convention for Swept Areas

and  $p = 0$ ). Positive and negative areas cancel each other out, as they overlap, so that, as the generator moves completely around the section, starting and returning to A, say, the resultant area is that enclosed by the profile of the beam.

The theory of the torsion of closed section beams is known as the *Bredt–Batho theory*, and Eq. (18.1) is often referred to as the *Bredt–Batho formula*.

### 18.1.1 Displacements associated with the Bredt–Batho shear flow

The relationship between  $q$  and shear strain  $\gamma$ , established in Eq. (17.19), namely

$$q = Gt \left( \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \right)$$

is valid for the pure torsion case where  $q$  is constant. Differentiating this expression with respect to  $z$ , we have

$$\frac{\partial q}{\partial z} = Gt \left( \frac{\partial^2 w}{\partial z \partial s} + \frac{\partial^2 v_t}{\partial z^2} \right) = 0$$

or

$$\frac{\partial}{\partial s} \left( \frac{\partial w}{\partial z} \right) + \frac{\partial^2 v_t}{\partial z^2} = 0 \quad (18.2)$$

In the absence of direct stresses, the longitudinal strain  $\partial w / \partial z$  ( $= \varepsilon_z$ ) is zero, so that

$$\frac{\partial^2 v_t}{\partial z^2} = 0$$

Hence, from Eq. (17.7),

$$p \frac{d^2 \theta}{dz^2} + \frac{d^2 u}{dz^2} \cos \psi + \frac{d^2 v}{dz^2} \sin \Psi = 0 \quad (18.3)$$

For Eq. (18.3) to hold for all points around the section wall – in other words for all values of  $\Psi$  –

$$\frac{d^2 \theta}{dz^2} = 0, \quad \frac{d^2 u}{dz^2} = 0, \quad \frac{d^2 v}{dz^2} = 0$$

It follows that  $\theta = Az + B$ ,  $u = Cz + D$ ,  $v = Ez + F$ , where  $A, B, C, D, E$ , and  $F$  are unknown constants. Thus,  $\theta$ ,  $u$ , and  $v$  are all linear functions of  $z$ .

Equation (17.22), relating the rate of twist to the variable shear flow  $q_s$  developed in a shear-loaded closed section beam, is also valid for the case  $q_s = q = \text{constant}$ . Hence,

$$\frac{d\theta}{dz} = \frac{q}{2A} \oint \frac{ds}{Gt}$$

which becomes, on substituting for  $q$  from Eq. (18.1),

$$\frac{d\theta}{dz} = \frac{T}{4A^2} \oint \frac{ds}{Gt} \quad (18.4)$$

The warping distribution produced by a varying shear flow, as defined by Eq. (17.25) for axes having their origin at the center of twist, is also applicable to the case of a constant shear flow. Thus,

$$w_s - w_0 = q \int_0^s \frac{ds}{Gt} - \frac{A_{Os}}{A} q \oint \frac{ds}{Gt}$$

Replacing  $q$  from Eq. (18.1), we have

$$w_s - w_0 = \frac{T\delta}{2A} \left( \frac{\delta_{Os}}{\delta} - \frac{A_{Os}}{A} \right) \quad (18.5)$$

where

$$\delta = \oint \frac{ds}{Gt} \quad \text{and} \quad \delta_{Os} = \int_0^s \frac{ds}{Gt}$$

The sign of the warping displacement in Eq. (18.5) is governed by the sign of the applied torque  $T$  and the signs of the parameters  $\delta_{Os}$  and  $A_{Os}$ . Having specified initially that a positive torque is counterclockwise, the signs of  $\delta_{Os}$  and  $A_{Os}$  are fixed, in that  $\delta_{Os}$  is positive when  $s$  is positive, that is,  $s$  is taken as positive in a counterclockwise sense, and  $A_{Os}$  is positive when, as before,  $p$  (see Fig. 18.3) is positive.

We noted that the longitudinal strain  $\varepsilon_z$  is zero in a closed section beam subjected to a pure torque. This means that all sections of the beam must possess identical warping distributions. In other words, longitudinal generators of the beam surface remain unchanged in length although subjected to axial displacement.

### Example 18.1

A thin-walled circular section beam has a diameter of 200 mm and is 2 m long; it is firmly restrained against rotation at each end. A concentrated torque of 30 kNm is applied to the beam at its mid-span point. If the maximum shear stress in the beam is limited to 200 N/mm<sup>2</sup> and the maximum angle of twist to 2°, calculate the minimum thickness of the beam walls. Take  $G = 25,000$  N/mm<sup>2</sup>.

The minimum thickness of the beam corresponding to the maximum allowable shear stress of 200 N/mm<sup>2</sup> is obtained directly using Eq. (18.1), in which  $T_{max} = 15$  kNm.

Then,

$$t_{min} = \frac{15 \times 10^6 \times 4}{2 \times \pi \times 200^2 \times 200} = 1.2 \text{ mm}$$

The rate of twist along the beam is given by Eq. (18.4), in which

$$\oint \frac{ds}{t} = \frac{\pi \times 200}{t_{min}}$$

Hence,

$$\frac{d\theta}{dz} = \frac{T}{4A^2G} \times \frac{\pi \times 200}{t_{min}} \quad (i)$$

Taking the origin for  $z$  at one of the fixed ends and integrating Eq. (i) for half the length of the beam, we obtain

$$\theta = \frac{T}{4A^2G} \times \frac{200\pi}{t_{min}} z + C_1$$

where  $C_1$  is a constant of integration. At the fixed end, where  $z = 0$ ,  $\theta = 0$ , so that  $C_1 = 0$ .

Hence,

$$\theta = \frac{T}{4A^2G} \times \frac{200\pi}{t_{\min}} z$$

The maximum angle of twist occurs at the mid-span of the beam, where  $z = 1$  m. Hence,

$$t_{\min} = \frac{15 \times 10^6 \times 200 \times \pi \times 1 \times 10^3 \times 180}{4 \times (\pi \times 200^2/4)^2 \times 25,000 \times 2 \times \pi} = 2.7 \text{ mm}$$

The minimum allowable thickness that satisfies both conditions is therefore 2.7 mm.

### Example 18.2

Determine the warping distribution in the doubly symmetrical rectangular, closed section beam, shown in Fig. 18.4, when subjected to a counterclockwise torque  $T$ .

From symmetry, the center of twist R coincides with the mid-point of the cross-section and points of zero warping lie on the axes of symmetry at the mid-points of the sides. We therefore take the origin for  $s$  at the mid-point of side 14 and measure  $s$  in the positive, counterclockwise sense around the section. Assuming the shear modulus  $G$  to be constant, we rewrite Eq. (18.5) in the form

$$w_s - w_0 = \frac{T\delta}{2AG} \left( \frac{\delta_{Os}}{\delta} - \frac{A_{Os}}{A} \right) \quad (i)$$

where

$$\delta = \oint \frac{ds}{t} \quad \text{and} \quad \delta_{Os} = \int_0^s \frac{ds}{t}$$

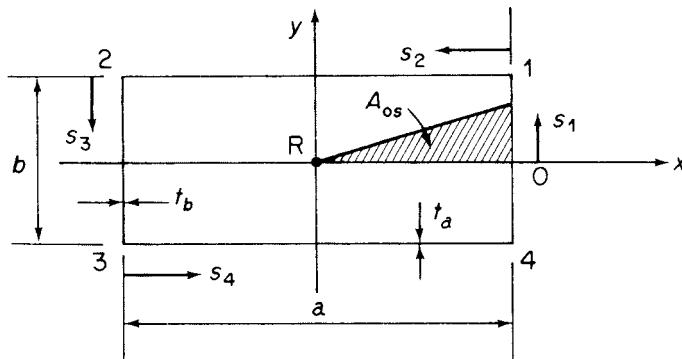


FIGURE 18.4 Torsion of a Rectangular Section Beam

In Eq. (i),

$$w_0 = 0, \quad \delta = 2 \left( \frac{b}{t_b} + \frac{a}{t_a} \right), \quad \text{and} \quad A = ab$$

From 0 to 1,  $0 \leq s_1 \leq b/2$  and

$$\delta_{Os} = \int_0^s \frac{ds_1}{t_b} = \frac{s_1}{t_b}, \quad A_{Os} = \frac{as_1}{4} \quad (\text{ii})$$

Note that both  $\delta_{Os}$  and  $A_{Os}$  are positive.

Substitution for  $\delta_{Os}$  and  $A_{Os}$  from Eq. (ii) in (i) shows that the warping distribution in the wall 01,  $w_{01}$ , is linear. Also,

$$w_1 = \frac{T}{2abG} 2 \left( \frac{b}{t_b} + \frac{a}{t_a} \right) \left[ \frac{b/2t_b}{2(b/t_b + a/t_a)} - \frac{ab/8}{ab} \right]$$

that is,

$$w_1 = \frac{T}{8abG} \left( \frac{b}{t_b} + \frac{a}{t_a} \right) \left[ \frac{2b/t_b}{(b/t_b + a/t_a)} - 1 \right]$$

which simplifies to

$$w_1 = \frac{T}{8abG} \left( \frac{b}{t_b} + \frac{a}{t_a} \right) \left[ \frac{2b/t_b - b/t_b - a/t_a}{(b/t_b + a/t_a)} \right]$$

Then

$$w_1 = \frac{T}{8abG} \left( \frac{b}{t_b} - \frac{a}{t_a} \right) \quad (\text{iii})$$

The remainder of the warping distribution may be deduced from symmetry and the fact that the warping must be zero at points where the axes of symmetry and the walls of the cross-section intersect. It follows that

$$w_2 = -w_1 = -w_3 = w_4$$

giving the distribution shown in Fig. 18.5. Note that the warping distribution takes the form shown in Fig. 18.5 as long as  $T$  is positive and  $b/t_b > a/t_a$ . If either of these conditions is reversed,  $w_1$  and  $w_3$  become negative and  $w_2$  and  $w_4$  positive. In the case when  $b/t_b = a/t_a$ , the warping is zero at all points in the cross-section.

Suppose now that the origin for  $s$  is chosen arbitrarily at, say, point 1. Then, from Fig. 18.6,  $\delta_{Os}$  in the wall 12 =  $s_1/t_a$  and  $A_{Os} = \frac{1}{2}s_1b/2 = s_1b/4$  and both are positive.

Substituting in Eq. (i) and setting  $w_0 = 0$ ,

$$w'_{12} = \frac{T\delta}{2abG} \left( \frac{s_1}{\delta t_a} - \frac{s_1}{4a} \right) \quad (\text{iv})$$

so that  $w'_{12}$  varies linearly from zero at 1 to

$$w'_2 = \frac{T}{2abG} 2 \left( \frac{b}{t_b} + \frac{a}{t_a} \right) \left[ \frac{a}{2(b/t_b + a/t_a)t_a} - \frac{1}{4} \right]$$

at 2. Thus,

$$w'_2 = \frac{T}{4abG} \left( \frac{a}{t_a} - \frac{b}{t_b} \right)$$

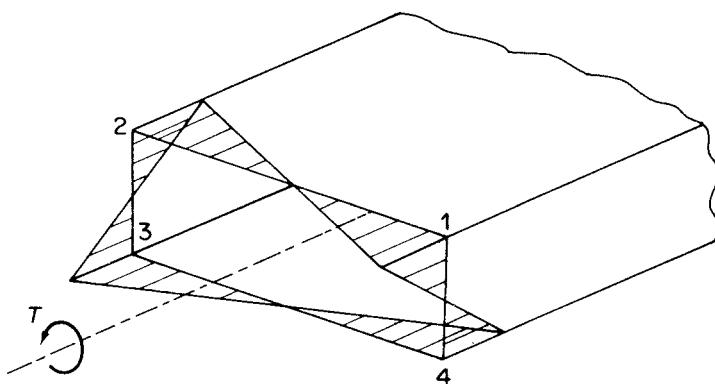
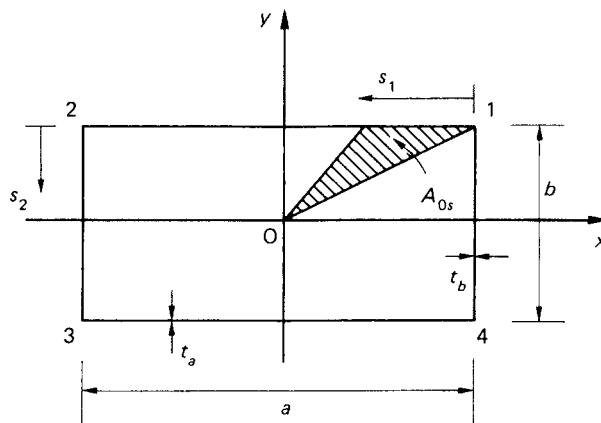


FIGURE 18.5 Warping Distribution in the Rectangular Section Beam of Example 18.2

FIGURE 18.6 Arbitrary Origin for  $s$ 

or

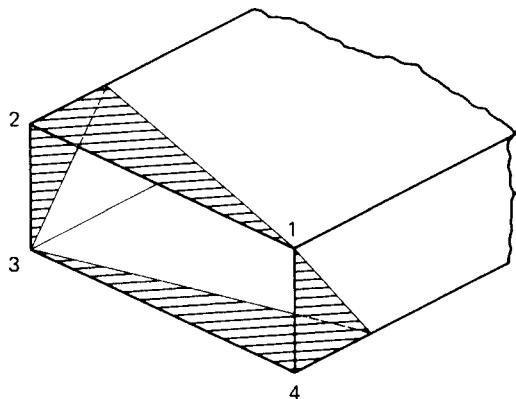
$$w'_2 = -\frac{T}{4abG} \left( \frac{b}{t_b} - \frac{a}{t_a} \right) \quad (\text{v})$$

Similarly,

$$w'_{23} = \frac{T\delta}{2abG} \left[ \frac{1}{\delta} \left( \frac{a}{t_a} + \frac{s_2}{t_b} \right) - \frac{1}{4b} (b + s_2) \right] \quad (\text{vi})$$

The warping distribution therefore varies linearly from a value  $-T(b/t_b - a/t_a)/4abG$  at 2 to zero at 3. The remaining distribution follows from symmetry, so that the complete distribution takes the form shown in Fig. 18.7.

Comparing Figs. 18.5 and 18.7, it can be seen that the form of the warping distribution is the same but that, in the latter case, the complete distribution has been displaced axially. The actual value of the warping at the origin for  $s$  is found using Eq. (17.26).



**FIGURE 18.7 Warping Distribution Produced by Selecting an Arbitrary Origin for  $s$**

Thus,

$$w_0 = \frac{2}{2(at_a + bt_b)} \left( \int_0^a w'_{12} t_a \, ds_1 + \int_0^b w'_{23} t_b \, ds_2 \right) \quad (\text{vii})$$

Substituting in Eq. (vii) for  $w'_{12}$  and  $w'_{23}$  from Eqs. (iv) and (vi), respectively, and evaluating gives

$$w_0 = -\frac{T}{8abG} \left( \frac{b}{t_b} - \frac{a}{t_a} \right) \quad (\text{viii})$$

Subtracting this value from the values of  $w'_1 (= 0)$  and  $w'_2 (= -T(b/t_b - a/t_a)/4abG)$ , we have

$$w_1 = \frac{T}{8abG} \left( \frac{b}{t_b} - \frac{a}{t_a} \right), \quad w_2 = -\frac{T}{8abG} \left( \frac{b}{t_b} - \frac{a}{t_a} \right)$$

as before. Note that setting  $w_0 = 0$  in Eq. (i) implies that  $w_0$ , the actual value of warping at the origin for  $s$ , has been added to all warping displacements. This value must therefore be *subtracted* from the calculated warping displacements (i.e., those based on an arbitrary choice of origin) to obtain true values.

It is instructive at this stage to examine the mechanics of warping to see how it arises. Suppose that each end of the rectangular section beam of Example 18.2 rotates through opposite angles  $\theta$  giving a total angle of twist  $2\theta$  along its length  $L$ . The corner 1 at one end of the beam is displaced by amounts  $a\theta/2$  vertically and  $b\theta/2$  horizontally, as shown in Fig. 18.8. Consider now the displacements of the web and cover of the beam due to rotation. From Figs. 18.8 and 18.9(a) and (b), it can be seen that the angles of rotation of the web and the cover are, respectively,

$$\phi_b = (a\theta/2)/(L/2) = a\theta/L$$

and

$$\phi_a = (b\theta/2)/(L/2) = b\theta/L$$

The axial displacements of the corner 1 in the web and cover are then

$$\frac{b}{2} \frac{a\theta}{L}, \quad \frac{a}{2} \frac{b\theta}{L}$$

respectively, as shown in Figs. 18.9(a) and (b). In addition to displacements produced by twisting, the webs and covers are subjected to shear strains  $\gamma_b$  and  $\gamma_a$  corresponding to the shear stress system given by Eq. (18.1). Due to  $\gamma_b$ , the axial displacement of corner 1 in the web is  $\gamma_b b/2$  in the positive  $z$  direction, while in the cover the

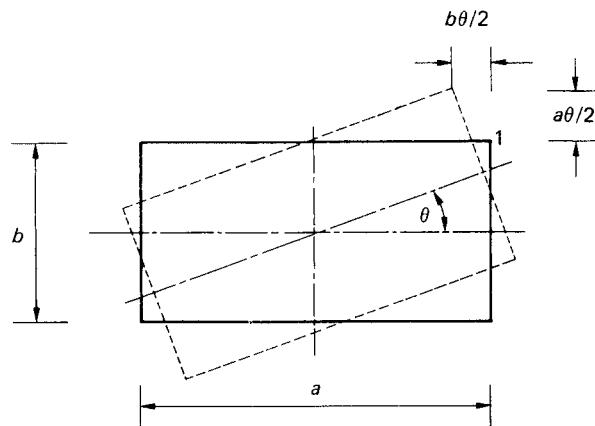


FIGURE 18.8 Twisting of a Rectangular Section Beam

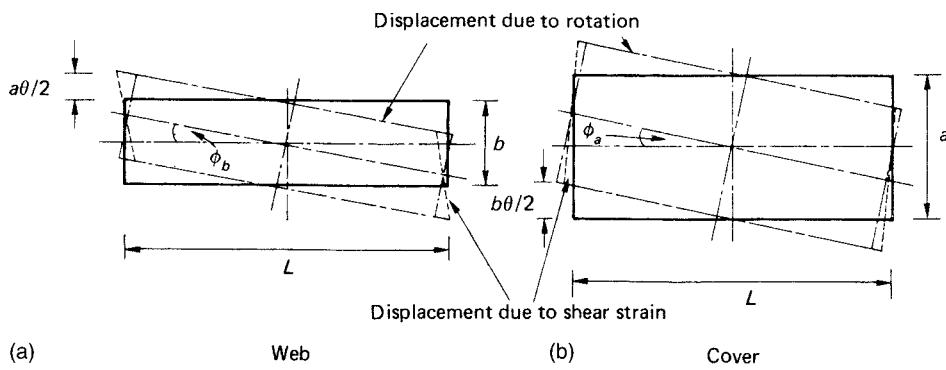


FIGURE 18.9 Displacements Due to Twist and Shear Strain

displacement is  $\gamma_a a/2$  in the negative  $z$  direction. Note that the shear strains  $\gamma_b$  and  $\gamma_a$  correspond to the shear stress system produced by a positive counterclockwise torque. Clearly, the total axial displacement of the point 1 in the web and cover must be the same, so that

$$-\frac{b a \theta}{2 L} + \gamma_b \frac{b}{2} = \frac{a b \theta}{2 L} - \gamma_a \frac{a}{2}$$

from which

$$\theta = \frac{L}{2ab} (\gamma_a a + \gamma_b b)$$

The shear strains are obtained from Eq. (18.1) and are

$$\gamma_a = \frac{T}{2abGt_a}, \quad \gamma_b = \frac{T}{2abGt_b}$$

from which

$$\theta = \frac{TL}{4a^2b^2G} \left( \frac{a}{t_a} + \frac{b}{t_b} \right)$$

The total angle of twist from end to end of the beam is  $2\theta$ ; therefore,

$$\frac{2\theta}{L} = \frac{TL}{4a^2b^2G} \left( \frac{2a}{t_a} + \frac{2b}{t_b} \right)$$

or

$$\frac{d\theta}{dz} = \frac{T}{4A^2G} \oint \frac{ds}{t}$$

as in Eq. (18.4).

Substituting for  $\theta$  in either of the expressions for the axial displacement of the corner 1 gives the warping  $w_1$  at 1. Thus,

$$w_1 = \frac{ab}{2L} \frac{TL}{4a^2b^2G} \left( \frac{a}{t_a} + \frac{b}{t_b} \right) - \frac{T}{2abGt_a} \frac{a}{2}$$

that is,

$$w_1 = \frac{T}{8abG} \left( \frac{b}{t_b} - \frac{a}{t_a} \right)$$

as before. It can be seen that the warping of the cross-section is produced by a combination of the displacements caused by twisting and the displacements due to the shear strains; these shear strains correspond to the shear stresses whose values are fixed by statics. The angle of twist must therefore be such as to ensure compatibility of displacement between the webs and covers.



### 18.1.2 Condition for zero warping at a section

The geometry of the cross-section of a closed section beam subjected to torsion may be such that no warping of the cross-section occurs. From Eq. (18.5), we see that this condition arises when

$$\frac{\delta_{Os}}{\delta} = \frac{A_{Os}}{A}$$

or

$$\frac{1}{\delta} \int_0^s \frac{ds}{Gt} = \frac{1}{2A} \int_0^s p_R ds \quad (18.6)$$

Differentiating Eq. (18.6) with respect to  $s$  gives

$$\frac{1}{\delta Gt} = \frac{p_R}{2A}$$

or

$$p_R Gt = \frac{2A}{\delta} = \text{constant} \quad (18.7)$$

A closed section beam for which  $p_R Gt = \text{constant}$  does not warp and is known as a *Neuber beam*. For closed section beams having a constant shear modulus, the condition becomes

$$p_R t = \text{constant} \quad (18.8)$$

Examples of such beams are a circular section beam of constant thickness, a rectangular section beam for which  $at_b = bt_a$  (see [Example 18.2](#)), and a triangular section beam of constant thickness. In the last case, the shear center and hence the center of twist may be shown to coincide with the center of the inscribed circle so that  $p_R$  for each side is the radius of the inscribed circle.

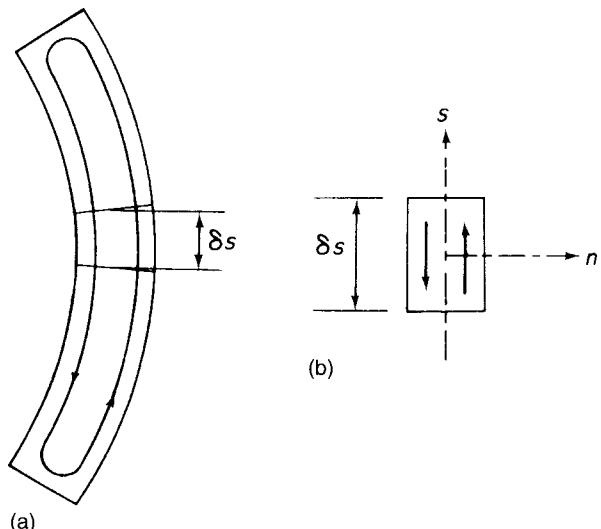
## 18.2 TORSION OF OPEN SECTION BEAMS

An approximate solution for the torsion of a thin-walled open section beam may be found by applying the results obtained in [Section 3.4](#) for the torsion of a thin rectangular strip. If such a strip is bent to form an open section beam, as shown in [Fig. 18.10\(a\)](#), and if the distance  $s$  measured around the cross-section is large compared with its thickness  $t$ , then the contours of the membrane – that is, lines of shear stress – are still approximately parallel to the inner and outer boundaries. It follows that the shear lines in an element  $\delta s$  of the open section must be nearly the same as those in an element  $\delta y$  of a rectangular strip, as demonstrated in [Fig. 18.10\(b\)](#). Equations (3.27)–(3.29) may therefore be applied to the open beam but with reduced accuracy. Referring to [Fig. 18.10\(b\)](#), we observe that [Eq. \(3.27\)](#) becomes

$$\tau_{zs} = 2Gn \frac{d\theta}{dz}, \quad \tau_{zn} = 0 \quad (18.9)$$

[Equation \(3.28\)](#) becomes

$$\tau_{zs, \max} = \pm Gt \frac{d\theta}{dz} \quad (18.10)$$



**FIGURE 18.10 (a) Shear Lines in a Thin-Walled Open Section Beam Subjected to Torsion; (b) Approximation of Elemental Shear Lines to Those in a Thin Rectangular Strip**

and Eq. (3.29) is

$$J = \sum \frac{st^3}{3} \quad \text{or} \quad J = \frac{1}{3} \int_{\text{sect}} t^3 ds \quad (18.11)$$

In Eq. (18.11), the second expression for the torsion constant is used if the cross-section has a variable wall thickness. Finally, the rate of twist is expressed in terms of the applied torque by Eq. (3.12), namely

$$T = GJ \frac{d\theta}{dz} \quad (18.12)$$

The shear stress distribution and the maximum shear stress are sometimes more conveniently expressed in terms of the applied torque. Therefore, substituting for  $d\theta/dz$  in Eqs. (18.9) and (18.10) gives

$$\tau_{zs} = \frac{2n}{J} T, \quad \tau_{zs,\max} = \pm \frac{tT}{J} \quad (18.13)$$

We assume, in open beam torsion analysis, that the cross-section is maintained by the system of closely spaced diaphragms described in Section 17.1 and that the beam is of uniform section. Clearly, in this problem, the shear stresses vary across the thickness of the beam wall, whereas other stresses, such as axial constraint stresses, which we shall discuss in Chapter 28, are assumed constant across the thickness.

### 18.2.1 Warping of the cross-section

We saw, in Section 3.4, that a thin rectangular strip suffers warping across its thickness when subjected to torsion. In the same way, a thin-walled open section beam warps across its thickness. This warping,  $w_t$ , may be deduced by comparing Fig. 18.10(b) with Fig. 3.10 and using Eq. (3.32), thus,

$$w_t = ns \frac{d\theta}{dz} \quad (18.14)$$

In addition to warping across the thickness, the cross-section of the beam warps in a manner similar to that of a closed section beam. From Fig. 17.3,

$$\gamma_{zs} = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \quad (18.15)$$

Referring the tangential displacement  $v_t$  to the center of twist R of the cross-section, we have, from Eq. (17.8),

$$\frac{\partial v_t}{\partial z} = p_R \frac{d\theta}{dz} \quad (18.16)$$

Substituting for  $\partial v_t/\partial z$  in Eq. (18.15) gives

$$\gamma_{zs} = \frac{\partial w}{\partial s} + p_R \frac{d\theta}{dz}$$

from which

$$\tau_{zs} = G \left( \frac{\partial w}{\partial s} + p_R \frac{d\theta}{dz} \right) \quad (18.17)$$

On the midline of the section wall  $\tau_{zs} = 0$  (see Eq. (18.9)), so that, from Eq. (18.17),

$$\frac{\partial w}{\partial s} = -p_R \frac{d\theta}{dz}$$

Integrating this expression with respect to  $s$  and taking the lower limit of integration to coincide with the point of zero warping, we obtain

$$w_s = - \frac{d\theta}{dz} \int_0^s p_R ds \quad (18.18)$$

From Eqs. (18.14) and (18.18), it can be seen that two types of warping exist in an open section beam. Equation (18.18) gives the warping of the midline of the beam; this is known as *primary warping* and is assumed to be constant across the wall thickness. Equation (18.14) gives the warping of the beam across its wall thickness. This is called *secondary warping*, is very much less than primary warping, and is usually ignored in the thin-walled sections common to aircraft structures.

Equation (18.18) may be rewritten in the form

$$w_s = -2A_R \frac{d\theta}{dz} \quad (18.19)$$

or, in terms of the applied torque,

$$w_s = -2A_R \frac{T}{GJ} \quad (\text{see Eq. (18.12)}) \quad (18.20)$$

in which  $A_R = \frac{1}{2} \int_0^s p_R ds$  is the area swept out by a generator, rotating about the center of twist, from the point of zero warping, as shown in Fig. 18.11. The sign of  $w_s$ , for a given direction of torque, depends

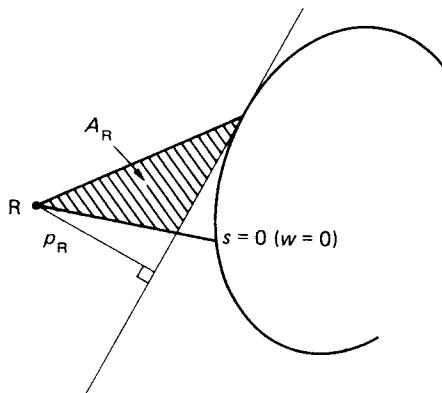


FIGURE 18.11 Warping of an Open Section Beam

upon the sign of  $A_R$ , which in turn depends upon the sign of  $p_R$ , the perpendicular distance from the center of twist to the tangent at any point. Again, as for closed section beams, the sign of  $p_R$  depends upon the assumed direction of a positive torque, in this case counterclockwise. Therefore,  $p_R$  (and therefore  $A_R$ ) is positive if movement of the foot of  $p_R$  along the tangent in the assumed direction of  $s$  leads to a counterclockwise rotation of  $p_R$  about the center of twist. Note that, for open section beams, the positive direction of  $s$  may be chosen arbitrarily, since, for a given torque, the sign of the warping displacement depends only on the sign of the swept area  $A_R$ .

### Example 18.3

Determine the maximum shear stress and the warping distribution in the channel section shown in Fig. 18.12 when it is subjected to a counterclockwise torque of 10 Nm.  $G = 25,000 \text{ N/mm}^2$ .

From the second of Eqs. (18.13), it can be seen that the maximum shear stress occurs in the web of the section where the thickness is greatest. Also, from the first of Eqs. (18.11),

$$J = \frac{1}{3}(2 \times 25 \times 1.5^3 + 50 \times 2.5^3) = 316.7 \text{ mm}^4$$

so that

$$\tau_{\max} = \pm \frac{2.5 \times 10 \times 10^3}{316.7} = \pm 78.9 \text{ N/mm}^2$$

The warping distribution is obtained using Eq. (18.20), in which the origin for  $s$  (and hence  $A_R$ ) is taken at the intersection of the web and the axis of symmetry where the warping is zero. Further, the center of twist R of the section coincides with its shear center S, whose position is found using the method described in Section 17.2.1, this gives  $\xi_S = 8.04 \text{ mm}$ . In the wall O2,

$$A_R = \frac{1}{2} \times 8.04s_1 \quad (p_R \text{ is positive})$$

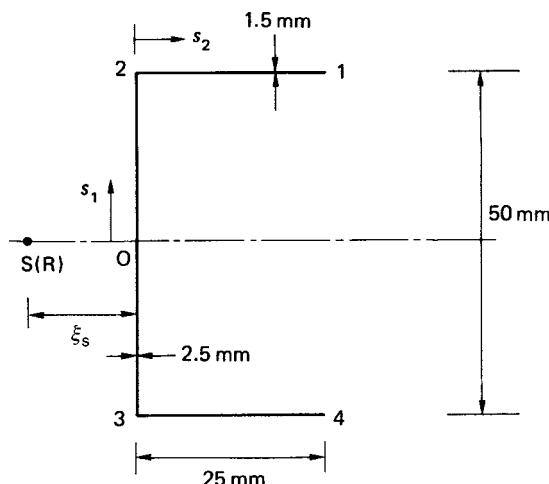


FIGURE 18.12 Channel Section of Example 18.3

so that

$$w_{O2} = -2 \times \frac{1}{2} \times 8.04s_1 \times \frac{10 \times 10^3}{25,000 \times 316.7} = -0.01s_1 \quad (\text{i})$$

that is, the warping distribution is linear in O2 and

$$w_2 = -0.01 \times 25 = -0.25 \text{ mm}$$

In the wall 21,

$$A_R = \frac{1}{2} \times 8.04 \times 25 - \frac{1}{2} \times 25s_2$$

in which the area swept out by the generator in the wall 21 provides a negative contribution to the total swept area  $A_R$ . Thus,

$$w_{21} = -25(8.04 - s_2) \frac{10 \times 10^3}{25,000 \times 316.7}$$

or

$$w_{21} = -0.03(8.04 - s_2) \quad (\text{ii})$$

Again, the warping distribution is linear and varies from  $-0.25 \text{ mm}$  at 2 to  $+0.54 \text{ mm}$  at 1. Examination of Eq. (ii) shows that  $w_{21}$  changes sign at  $s_2 = 8.04 \text{ mm}$ . The remaining warping distribution follows from symmetry and the complete distribution is shown in Fig. 18.13. In unsymmetrical section beams, the position of the point of zero warping is not known but may be found using the method described in Section 28.2 for the restrained warping of an open section beam. From the derivation of Eq. (28.3), we see that

$$2A'_R = \frac{\int_{\text{sect}} 2A_{R,0}t \, ds}{\int_{\text{sect}} t \, ds} \quad (\text{iii})$$

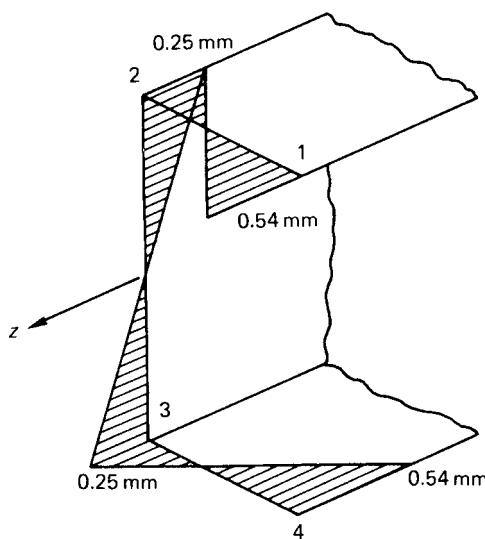


FIGURE 18.13 Warping Distribution in Channel Section of Example 18.3

in which  $A_{R,O}$  is the area swept out by a generator rotating about the center of twist from some convenient origin and  $A'_R$  is the value of  $A_{R,O}$  at the point of zero warping. As an illustration, we apply the method to the beam section of [Example 18.3](#).

Suppose that the position of the center of twist (i.e., the shear center) has already been calculated, and suppose also that we choose the origin for  $s$  to be at the point 1. Then, in [Fig. 18.14](#),

$$\int_{\text{sect}} t \, ds = 2 \times 1.5 \times 25 + 2.5 \times 50 = 200 \text{ mm}^2$$

In the wall 12,

$$A_{12} = \frac{1}{2} \times 25s_1 \quad (\text{for the wall } 12) \quad (\text{iv})$$

from which

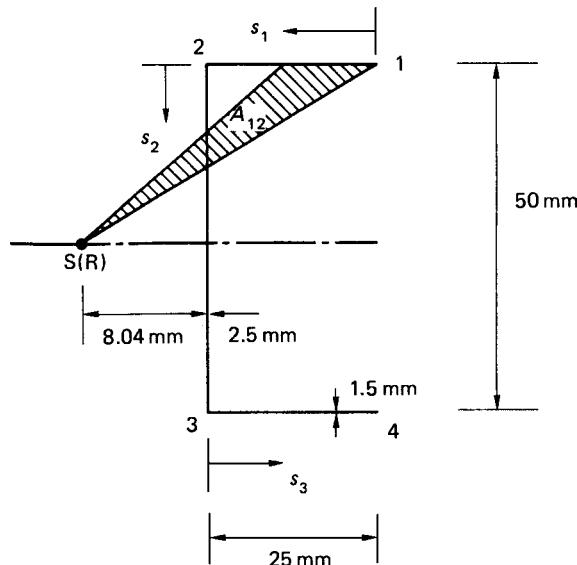
$$A_2 = \frac{1}{2} \times 25 \times 25 = 312.5 \text{ mm}^2$$

Also,

$$A_{23} = 312.5 - \frac{1}{2} \times 8.04s_2 \quad (\text{v})$$

and

$$A_3 = 312.5 - \frac{1}{2} \times 8.04 \times 50 = 111.5 \text{ mm}^2$$



**FIGURE 18.14 Determination of Points of Zero Warping**

Finally,

$$A_{34} = 111.5 + \frac{1}{2} \times 25s_3 \quad (\text{vi})$$

Substituting for  $A_{12}$ ,  $A_{23}$ , and  $A_{34}$  from Eqs. (iv)–(vi) in Eq. (iii), we have

$$2A'_R = \frac{1}{200} \left[ \int_0^{25} 25 \times 1.15s_1 \, ds_1 + \int_0^{50} 2(312.5 - 4.02s_2)2.5 \, ds_2 + \int_0^{25} 2(111.5 + 12.5s_3)1.5 \, ds_3 \right] \quad (\text{vii})$$

Evaluation of Eq. (vii) gives

$$2A'_R = 424 \text{ mm}^2$$

We now examine each wall of the section in turn to determine points of zero warping. Suppose that, in the wall 12, a point of zero warping occurs at a value of  $s_1$  equal to  $s_{1,0}$ . Then,

$$2 \times \frac{1}{2} \times 25s_{1,0} = 424$$

from which

$$s_{1,0} = 16.96 \text{ mm}$$

so that a point of zero warping occurs in the wall 12 at a distance of 8.04 mm from the point 2 as before. In the web 23, let the point of zero warping occur at  $s_2 = s_{2,0}$ . Then,

$$2 \times \frac{1}{2} \times 25 \times 25 - 2 \times \frac{1}{2} \times 8.04s_{2,0} = 424$$

which gives  $s_{2,0} = 25$  mm (i.e., on the axis of symmetry). Clearly, from symmetry, a further point of zero warping occurs in the flange 34 at a distance of 8.04 mm from the point 3. The warping distribution is then obtained directly using Eq. (18.20), in which

$$A_R = A_{R,O} - A'_R$$

## PROBLEMS

**P.18.1** A uniform, thin-walled, cantilever beam of closed rectangular cross-section has the dimensions shown in Fig. P.18.1. The shear modulus  $G$  of the top and bottom covers of the beam is 18,000 N/mm<sup>2</sup> while that of the vertical webs is 26,000 N/mm<sup>2</sup>. The beam is subjected to a uniformly distributed torque of 20 Nm/mm along its length. Calculate the maximum shear stress according to the Bredt–Batho theory of torsion. Calculate also, and sketch, the distribution of twist along the length of the cantilever, assuming that axial constraint effects are negligible.

Answer:  $\tau_{\max} = 83.3 \text{ N/mm}^2$ ,  $\theta = 8.14 \times 10^{-9} \left( 2,500z - \frac{z^2}{2} \right) \text{ rad}$

**P.18.2** A single-cell, thin-walled beam with the double trapezoidal cross-section shown in Fig. P.18.2, is subjected to a constant torque  $T = 90,500 \text{ Nm}$  and is constrained to twist about an axis through the point R. Assuming that the shear stresses are distributed according to the Bredt–Batho theory of torsion,

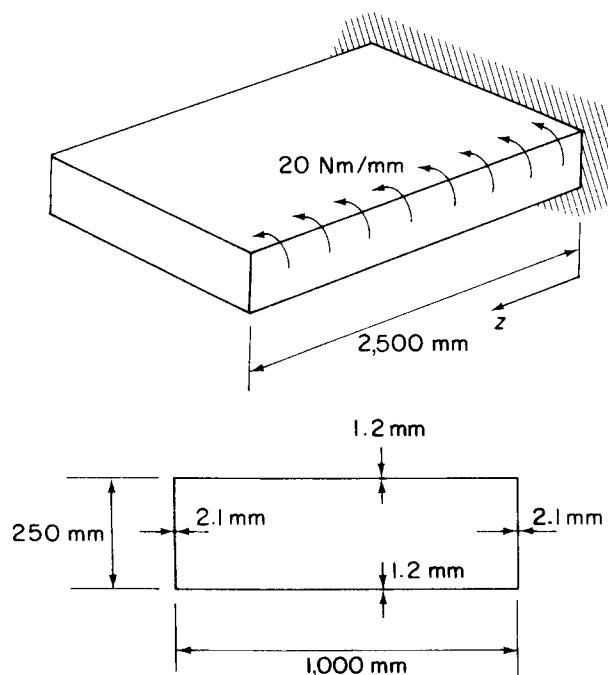


FIGURE P.18.1

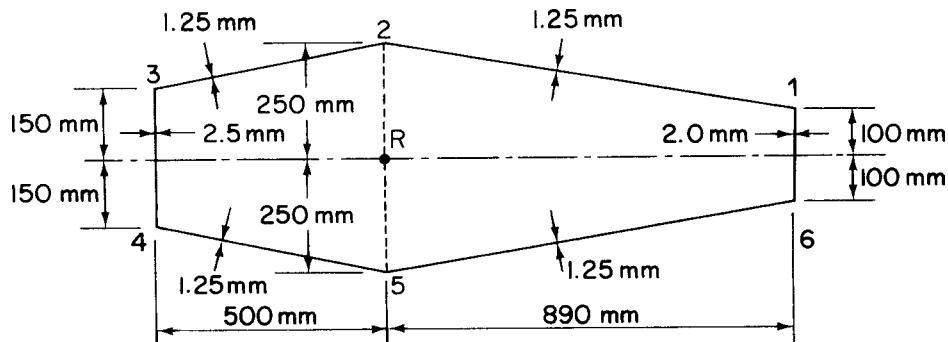


FIGURE P.18.2

calculate the distribution of warping around the cross-section. Illustrate your answer clearly by means of a sketch and insert the principal values of the warping displacements. The shear modulus  $G = 27,500 \text{ N/mm}^2$  and is constant throughout.

Answer:  $w_1 = -w_6 = -0.53 \text{ mm}$ ,  $w_2 = -w_5 = 0.05 \text{ mm}$ ,  $w_3 = -w_4 = 0.38 \text{ mm}$

Linear distribution.

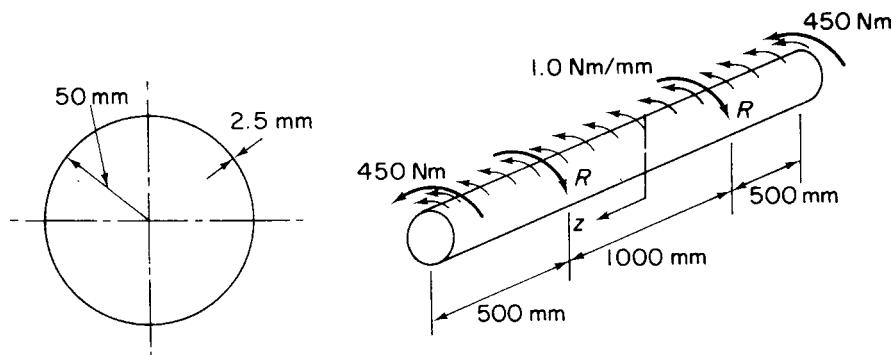


FIGURE P.18.3

**P.18.3** A uniform thin-walled beam is circular in cross-section and has a constant thickness of 2.5 mm. The beam is 2,000 mm long, carrying end torques of 450 Nm and, in the same sense, a distributed torque loading of 1.0 Nm/mm. The loads are reacted by equal couples  $R$  at sections 500 mm distant from each end (Fig. P.18.3). Calculate the maximum shear stress in the beam and sketch the distribution of twist along its length. Take  $G = 30,000 \text{ N/mm}^2$  and neglect axial constraint effects.

Answer:  $\tau_{\max} = 24.2 \text{ N/mm}^2$ ,  $\theta = -0.85 \times 10^{-8} z^2 \text{ rad}$ ,  $0 \leq z \leq 500 \text{ mm}$ ,  
 $\theta = 1.7 \times 10^{-8}(1,450z - z^2/2) - 12.33 \times 10^{-3} \text{ rad}$ ,  $500 \leq z \leq 1,000 \text{ mm}$ .

**P.18.4** The thin-walled box section beam ABCD shown in Fig. P.18.4 is attached at each end to supports that allow rotation of the ends of the beam in the longitudinal vertical plane of symmetry but prevent rotation of the ends in vertical planes perpendicular to the longitudinal axis of the beam. The beam is subjected to a uniform torque loading of 20 Nm/mm over the portion BC of its span. Calculate the maximum shear stress in the cross-section of the beam and the distribution of angle of twist along its length,  $G = 70,000 \text{ N/mm}^2$ .

Answer:  $71.4 \text{ N/mm}^2$ ,  $\theta_B = \theta_C = 0.36^\circ$ ,  $\theta$  at mid-span =  $0.72^\circ$

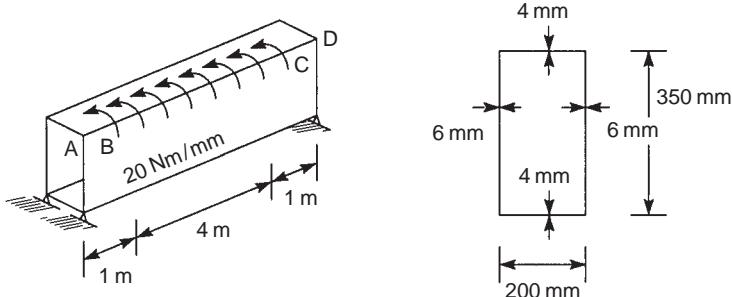


FIGURE P.18.4

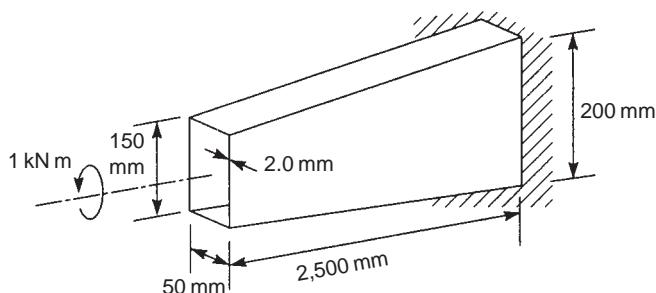


FIGURE P.18.5

**P.18.5** Figure P.18.5 shows a thin-walled cantilever box beam having a constant width of 50 mm and a depth which decreases linearly from 200 mm at the built-in end to 150 mm at the free end. If the beam is subjected to a torque of 1 kNm at its free end, plot the angle of twist of the beam at 500 mm intervals along its length and determine the maximum shear stress in the beam section. Take  $G = 25,000 \text{ N/mm}^2$ .

$$\text{Answer: } \tau_{\max} = 33.3 \text{ N/mm}^2$$

**P.18.6** A uniform closed section beam, of the thin-walled section shown in Fig. P.18.6, is subjected to a twisting couple of 4,500 Nm. The beam is constrained to twist about a longitudinal axis through the center C of the semi-circular arc 12. For the curved wall 12, the thickness is 2 mm and the shear modulus is  $22,000 \text{ N/mm}^2$ . For the plane walls 23, 34, and 41, the corresponding figures are 1.6 mm and  $27,500 \text{ N/mm}^2$ . (Note:  $Gt = \text{constant}$ .) Calculate the rate of twist in radians per millimeter. Give a sketch illustrating the distribution of warping displacement in the cross-section and quote values at points 1 and 4.

$$\text{Answer: } d\theta/dz = 29.3 \times 10^{-6} \text{ rad/mm,}$$

$$w_3 = -w_4 = -0.19 \text{ mm}, \quad w_2 = -w_1 = -0.056 \text{ mm}$$

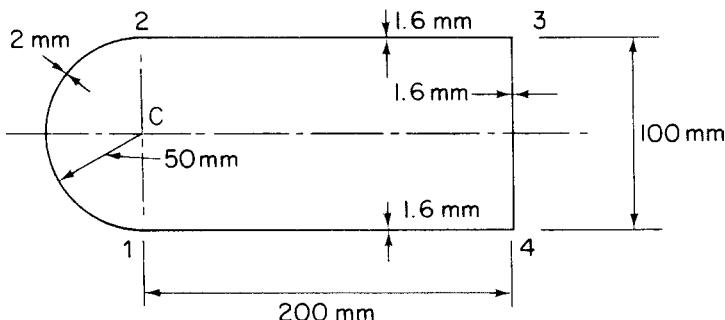


FIGURE P.18.6

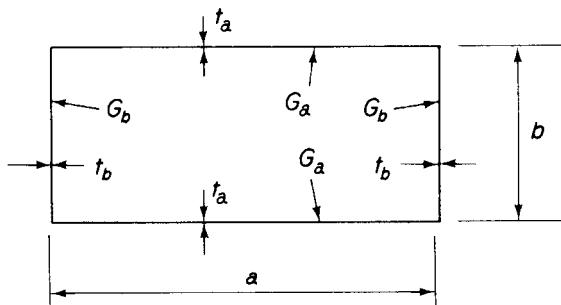


FIGURE P.18.7

**P.18.7** A uniform beam with the doubly symmetrical cross-section shown in Fig. P.18.7 has horizontal and vertical walls made of different materials, which have shear moduli  $G_a$  and  $G_b$ , respectively. If, for any material, the ratio mass density/shear modulus is constant, find the ratio of the wall thicknesses  $t_a$  and  $t_b$ , so that, for a given torsional stiffness and given dimensions  $a, b$ , the beam has minimum weight per unit span. Assume the Bredt–Batho theory of torsion is valid. If this thickness requirement is satisfied, find the  $a/b$  ratio (previously regarded as fixed), which gives minimum weight for given torsional stiffness.

Answer:  $t_b/t_a = G_a/G_b$ ,  $b/a = 1$

**P.18.8** The cold-formed section shown in Fig. P.18.8 is subjected to a torque of 50 Nm. Calculate the maximum shear stress in the section and its rate of twist.  $G = 25,000 \text{ N/mm}^2$ .

Answer:  $\tau_{\max} = 220.6 \text{ N/mm}^2$ ,  $d\theta/dz = 0.0044 \text{ rad/mm}$

**P.18.9** Determine the rate of twist per unit torque of the beam section shown in Fig. P.17.12 if the shear modulus  $G$  is  $25,000 \text{ N/mm}^2$ .

Answer:  $6.42 \times 10^{-8} \text{ rad/mm}$

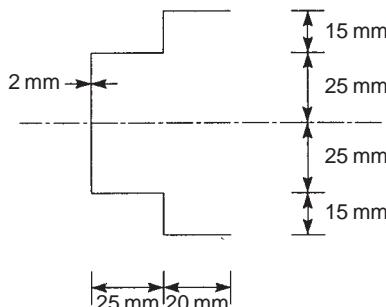


FIGURE P.18.8

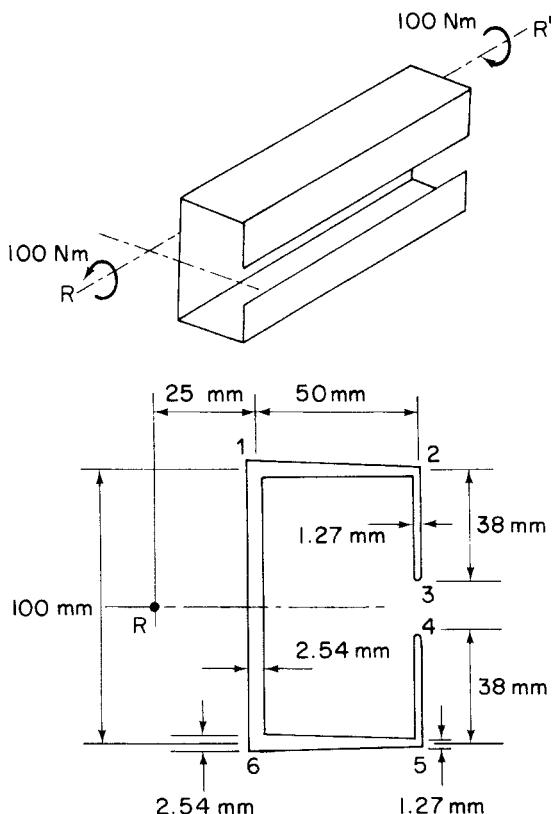


FIGURE P.18.10

**P.18.10** Figure P.18.10 shows the cross-section of a thin-walled beam in the form of a channel with lipped flanges. The lips are of constant thickness 1.27 mm, while the flanges increase linearly in thickness from 1.27 mm where they meet the lips to 2.54 mm at their junctions with the web. The web has a constant thickness of 2.54 mm. The shear modulus  $G$  is 26,700 N/mm<sup>2</sup> throughout. The beam has an enforced axis of twist  $RR'$  and is supported in such a way that warping occurs freely but is zero at the mid-point of the web. If the beam carries a torque of 100 Nm, calculate the maximum shear stress according to the St. Venant theory of torsion for thin-walled sections. Ignore any effects of stress concentration at the corners. Find also the distribution of warping along the middle line of the section, illustrating your results by means of a sketch.

$$\text{Answer: } \tau_{\max} = \pm 297.4 \text{ N/mm}^2, \quad w_1 = -5.48 \text{ mm} = -w_6 \\ w_2 = 5.48 \text{ mm} = -w_5, \quad w_3 = 17.98 \text{ mm} = -w_4$$

**P.18.11** The thin-walled section shown in Fig. P.18.11 is symmetrical about the  $x$  axis. The thickness  $t_0$  of the center web 34 is constant, while the thickness of the other walls varies linearly from  $t_0$  at points 3 and 4 to zero at the open ends 1, 6, 7, and 8. Determine the St. Venant torsion constant  $J$  for the section and also the maximum value of the shear stress due to a torque  $T$ . If the section is

constrained to twist about an axis through the origin O, plot the relative warping displacements of the section per unit rate of twist.

Answer:  $J = 4at_0^3/3$ ,  $\tau_{\max} = \pm 3T/4at_0^2$ ,  
 $w_1 = +a^2(1 + 2\sqrt{2})$ ,  $w_2 = +\sqrt{2}a^2$ ,  $w_7 = -a^2$ ,  $w_3 = 0$

**P.18.11 MATLAB** If the length of the vertical web of the section shown in Fig. P.18.11 is labeled  $L_{\text{web}}$ , use MATLAB to repeat the calculation of  $J$  and  $\tau_{\max}$  in Problem P.18.11 for values of  $L_{\text{web}}$  ranging from  $a$  to  $4a$  in increments of  $a/2$  (i.e.,  $a, 3a/2, 2a, \dots, 4a$ ).

- Answer:
- (i)  $L_{\text{web}} = a$ :  $J = at_0^3/3$ ,  $\tau_{\max} = \pm T/at_0^2$
  - (ii)  $L_{\text{web}} = 3a/2$ :  $J = 7at_0^3/6$ ,  $\tau_{\max} = \pm 6T/7at_0^2$
  - (iii)  $L_{\text{web}} = 2a$ :  $J = 4at_0^3/3$ ,  $\tau_{\max} = \pm 3T/4at_0^2$
  - (iv)  $L_{\text{web}} = 5a/2$ :  $J = 3at_0^3/2$ ,  $\tau_{\max} = \pm 2T/3at_0^2$
  - (v)  $L_{\text{web}} = 3a$ :  $J = 5at_0^3/3$ ,  $\tau_{\max} = \pm 3T/5at_0^2$
  - (vi)  $L_{\text{web}} = 7a/2$ :  $J = 11at_0^3/6$ ,  $\tau_{\max} = \pm 6T/11at_0^2$
  - (vii)  $L_{\text{web}} = 4a$ :  $J = 2at_0^3$ ,  $\tau_{\max} = \pm T/2at_0^2$

**P.18.12** The thin-walled section shown in Fig. P.18.12 is constrained to twist about an axis through R, the center of the semi-circular wall 34. Calculate the maximum shear stress in the section per unit torque and the warping distribution per unit rate of twist. Also compare the value of warping displacement at the point 1 with that corresponding to the section being constrained to twist about an axis through the

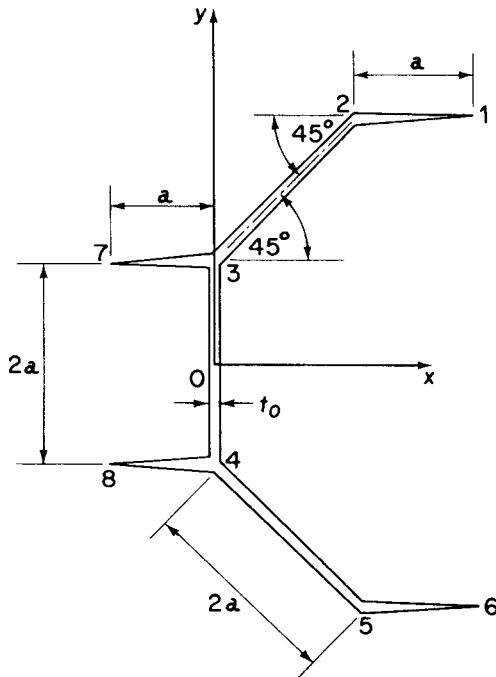


FIGURE P.18.11

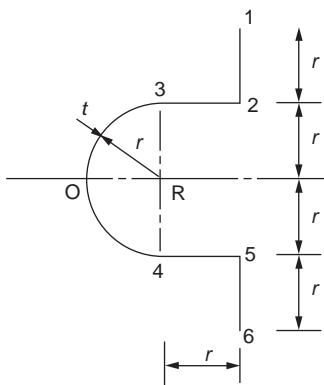


FIGURE P.18.12

point O and state what effect this movement has on the maximum shear stress and the torsional stiffness of the section.

Answer: Maximum shear stress is  $\pm 0.42/rt^2$  per unit torque

$$w_{03} = +r^2\theta, \quad w_{32} = +\frac{r}{2}(\pi r + 2s_1), \quad w_{21} = -\frac{r}{2}(2s_2 - 5.142r)$$

With center of twist at O,  $w_1 = -0.43r^2$ . Maximum shear stress is unchanged, torsional stiffness increased since warping reduced.

**P.18.13** Determine the maximum shear stress in the beam section shown in Fig. P.18.13, stating clearly the point at which it occurs. Determine also the rate of twist of the beam section if the shear modulus  $G$  is 25,000 N/mm<sup>2</sup>.

Answer: 70.2 N/mm<sup>2</sup> on the underside of 24 at 2 or on the upper surface of 32 at  $2.90 \times 10^{-4}$  rad/mm

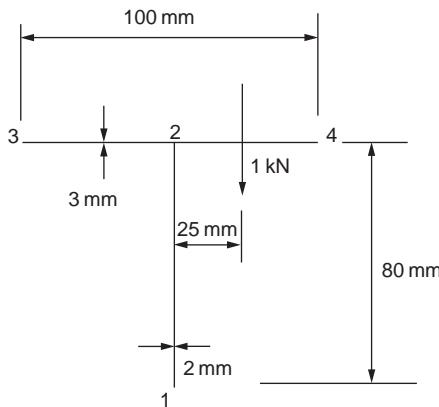


FIGURE P.18.13

# 19

## Combined open and closed section beams

So far, in [Chapters 16–18](#), we analyzed thin-walled beams that consist of either completely closed cross-sections or completely open cross-sections. Frequently, aircraft components comprise combinations of open and closed section beams. For example, the section of a wing in the region of an undercarriage bay could take the form shown in [Fig. 19.1](#). Clearly, part of the section is an open channel section, while the nose portion is a single-cell closed section. We now examine the methods of analysis of such sections when subjected to bending, shear, and torsional loads.

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### 19.1 BENDING

It is immaterial what form the cross-section of a beam takes; the direct stresses due to bending are given by either [Eq. \(16.17\)](#) or [\(16.18\)](#).

---

#### Example 19.1

Determine the direct stress distribution in the wing section shown in [Fig. 19.2](#) when it is subjected to a bending moment of 5000 Nm applied in a vertical plane.

The beam section has the  $x$  axis as an axis of symmetry so that  $I_{xy} = 0$  and also, since  $M_y = 0$ , [Eq. \(16.18\)](#) reduces to

$$\sigma_z = (M_x/I_{xx})y \quad (i)$$

From [Section 16.4.5](#), the second moment of area of the semicircular nose portion is

$$I_{xx}(\text{nose portion}) = \pi \times 50^3 \times 2/2 = 0.4 \times 10^6 \text{ mm}^4$$

For the rectangular box,

$$I_{xx}(\text{rect.box}) = (2 \times 4 \times 100^3)/12 + 2 \times 200 \times 2 \times 50^2 = 2.7 \times 10^6 \text{ mm}^4$$

For the arcs 23 and 56,

$$I_{xx}(\text{arcs } 23, 56) = 2 \int_{\pi/4}^{\pi/2} 2(50 \sin \theta)^2 50 \, d\theta$$

which gives

$$I_{xx}(\text{arcs}) = 0.3 \times 10^6 \text{ mm}^4$$

Therefore, for the complete beam section,

$$I_{xx} = 3.4 \times 10^6 \text{ mm}^4$$

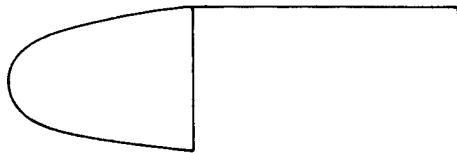


FIGURE 19.1 Wing Section Comprising Open and Closed Components

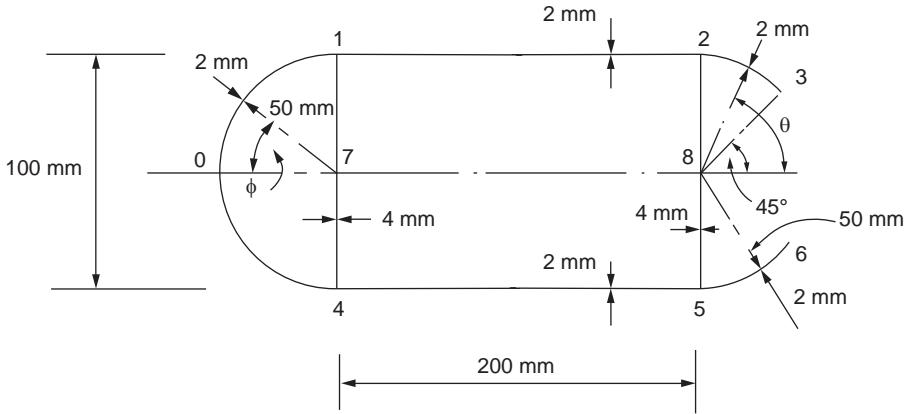


FIGURE 19.2 Beam Section of Example 19.1

and Eq. (i) becomes

$$\sigma_z = (5000 \times 10^3 / 3.4 \times 10^6)y = 1.47y \quad (\text{ii})$$

In 01,

$$\sigma_{z,01} = 1.47 \times 50 \sin \phi = 73.5 \sin \phi \quad (\text{iii})$$

so that when  $\phi = 0$ ,  $\sigma_{z,01} = 0$ , and when  $\phi = 90^\circ$ ,  $\sigma_{z,1} = +73.5 \text{ N/mm}^2$ .

In 71,

$$\sigma_{z,71} = 1.47y \quad (\text{iv})$$

When  $y = 0$ ,  $\sigma_{z,7} = 0$ , and when  $y = 50 \text{ mm}$ ,  $\sigma_{z,1} = +73.5 \text{ N/mm}^2$ .

By inspection,  $\sigma_{z,12} = +73.5 \text{ N/mm}^2$  and the direct stress distribution in 82 will be identical to that in 71.

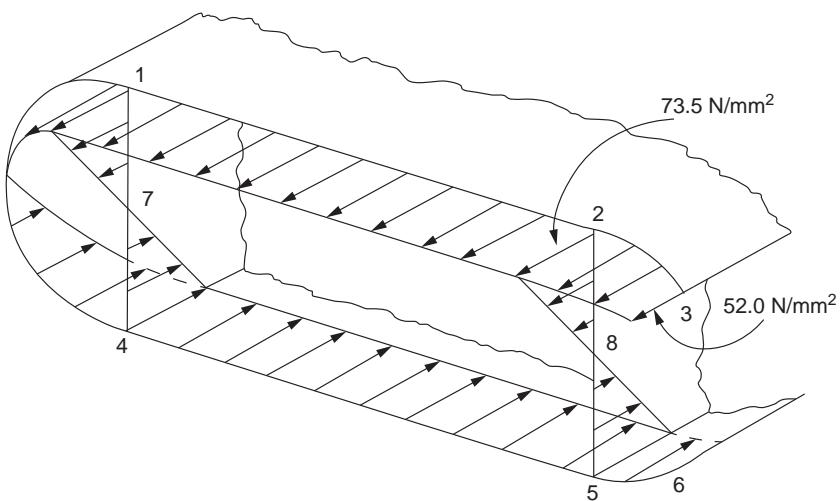
In 32,

$$\sigma_{z,32} = 1.47y = 1.47 \times 50 \sin \theta = 73.5 \sin \theta \quad (\text{v})$$

When  $\theta = \pi/4$ ,

$$\sigma_{z,3} = +52.0 \text{ N/mm}^2.$$

The direct stress distribution in the lower half of the wing section follows from symmetry; the complete distribution is shown in Fig. 19.3.



**FIGURE 19.3** Direct Stress Distribution in Beam Section of Example 19.1

## 19.2 SHEAR

The methods described in Sections 17.2 and 17.3 are used to determine the shear stress distribution, although, unlike the completely closed section case, shear loads must be applied through the shear center of the combined section; otherwise, shear stresses of the type described in Section 18.2 due to torsion arise. Where shear loads do not act through the shear center, its position must be found and the loading system replaced by shear loads acting through the shear center together with a torque; the two loading cases are then analyzed separately. Again, we assume that the cross-section of the beam remains undistorted by the loading.

### Example 19.2

Determine the shear flow distribution in the beam section shown in Fig. 19.4 when it is subjected to a shear load in its vertical plane of symmetry. The thickness of the walls of the section is 2 mm throughout.

The centroid of area C lies on the axis of symmetry at some distance  $\bar{y}$  from the upper surface of the beam section. Taking moments of area about this upper surface,

$$(4 \times 100 \times 2 + 4 \times 200 \times 2)\bar{y} = 2 \times 100 \times 2 \times 50 + 2 \times 200 \times 2 \times 100 + 200 \times 2 \times 200$$

which gives  $\bar{y} = 75$  mm.

The second moment of area of the section about Cx is given by

$$\begin{aligned} I_{xx} &= 2 \left( \frac{2 \times 100^3}{12} + 2 \times 100 \times 25^2 \right) + 400 \times 2 \times 75^2 + 200 \times 2 \times 125^2 \\ &\quad + 2 \left( \frac{2 \times 200^3}{12} + 2 \times 200 \times 25^2 \right) \end{aligned}$$

that is,

$$I_{xx} = 14.5 \times 10^6 \text{ mm}^4$$

The section is symmetrical about Cy so that  $I_{xy} = 0$  and, since  $S_x = 0$ , the shear flow distribution in the closed section 3456 is, from Eq. (17.15),

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s ty \, ds + q_{s,0} \quad (\text{i})$$

Also, the shear load is applied through the shear center of the complete section, that is, along the axis of symmetry, so that, in the open portions 123 and 678, the shear flow distribution is, from Eq. (17.14),

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s ty \, ds \quad (\text{ii})$$

We note that the shear flow is zero at the points 1 and 8 and therefore the analysis may conveniently, though not necessarily, begin at either of these points. Thus, referring to Fig. 19.4,

$$q_{12} = -\frac{100 \times 10^3}{14.5 \times 10^6} \int_0^{s_1} 2(-25 + s_1) \, ds_1$$

that is,

$$q_{12} = -69.0 \times 10^{-4} (-50s_1 + s_1^2) \quad (\text{iii})$$

from which  $q_2 = -34.5 \text{ N/mm}$ .

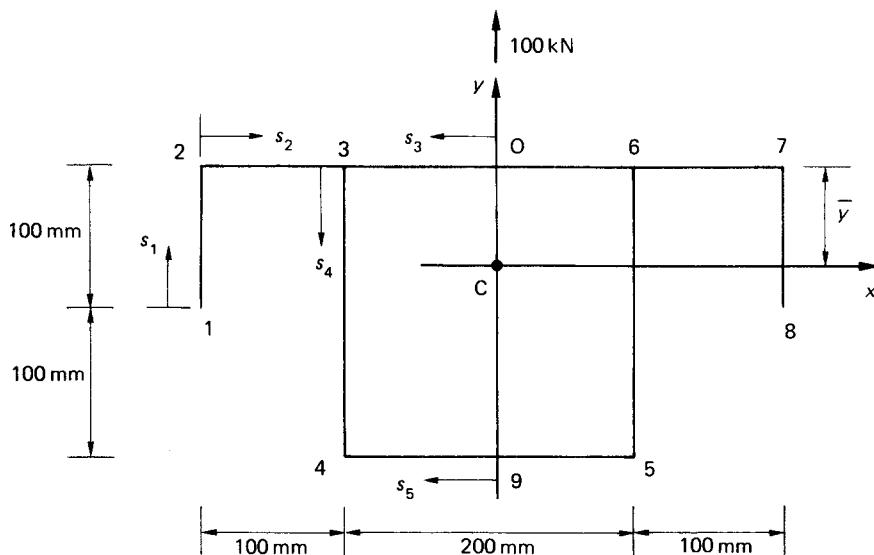


FIGURE 19.4 Beam Section of Example 19.2

Examination of Eq. (iii) shows that  $q_{12}$  is initially positive and changes sign when  $s_1 = 50$  mm. Further,  $q_{12}$  has a turning value ( $dq_{12}/ds_1 = 0$ ) at  $s_1 = 25$  mm of 4.3 N/mm. In the wall 23,

$$q_{23} = -69.0 \times 10^{-4} \int_0^{s_2} 2 \times 75 \, ds_2 - 34.5$$

that is,

$$q_{23} = -1.04s_2 - 34.5 \quad (\text{iv})$$

Hence,  $q_{23}$  varies linearly from a value of -34.5 N/mm at 2 to -138.5 N/mm at 3 in the wall 23.

The analysis of the open part of the beam section is now complete, since the shear flow distribution in the walls 67 and 78 follows from symmetry. To determine the shear flow distribution in the closed part of the section, we must use the method described in Section 17.3, in which the line of action of the shear load is known. Thus, we "cut" the closed part of the section at some convenient point, obtain the  $q_b$  or "open section" shear flows for the complete section, then take moments as in Eq. (17.16) or (17.17). However, in this case, we may use the symmetry of the section and loading to deduce that the final value of shear flow must be zero at the midpoints of the walls 36 and 45, that is,  $q_s = q_{s,0} = 0$  at these points. Hence,

$$q_{03} = -69.0 \times 10^{-4} \int_0^{s_3} 2 \times 75 \, ds_3$$

so that

$$q_{03} = -1.04s_3 \quad (\text{v})$$

and  $q_3 = -104$  N/mm in the wall 03. It follows that, for equilibrium of shear flows at 3,  $q_3$ , in the wall 34, must be equal to  $-138.5 - 104 = -242.5$  N/mm. Hence,

$$q_{34} = -69.0 \times 10^{-4} \int_0^{s_4} 2(75 - s_4) \, ds_4 - 242.5$$

which gives

$$q_{34} = -1.04s_4 + 69.0 \times 10^{-4}s_4^2 - 242.5 \quad (\text{vi})$$

Examination of Eq. (vi) shows that  $q_{34}$  has a maximum value of -281.7 N/mm at  $s_4 = 75$  mm; also,  $q_4 = -172.5$  N/mm. Finally, the distribution of shear flow in the wall 94 is given by

$$q_{94} = -69.0 \times 10^{-4} \int_0^{s_5} 2(-125) \, ds_5$$

that is,

$$q_{94} = 1.73s_5 \quad (\text{vii})$$

The complete distribution is shown in Fig. 19.5.

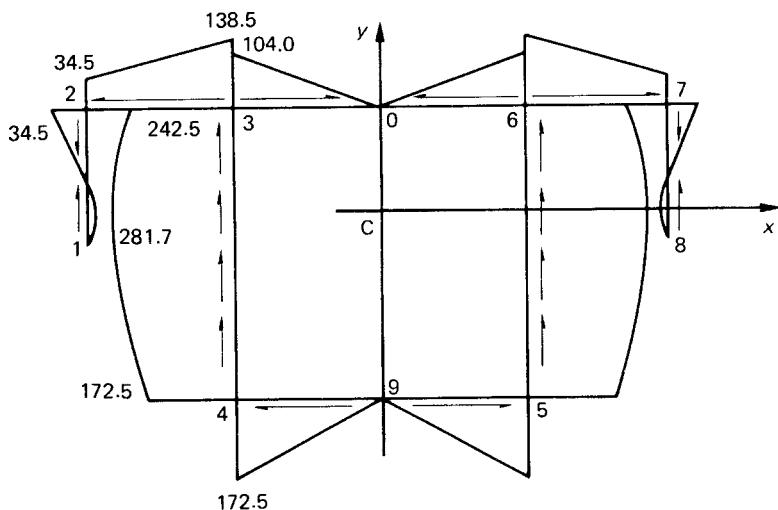


FIGURE 19.5 Shear Flow Distribution in Beam of Example 19.2 (All Shear Flows in N/mm)

### 19.3 TORSION

Generally, in the torsion of composite sections, the closed portion is dominant, since its torsional stiffness is far greater than that of the attached open section portion, which frequently may be ignored in the calculation of torsional stiffness; shear stresses should, however, be checked in this part of the section.

#### Example 19.3

Find the angle of twist per unit length in the wing whose cross-section is shown in Fig. 19.6 when it is subjected to a torque of 10 kNm. Find also the maximum shear stress in the section.  $G = 25,000 \text{ N/mm}^2$ .

$$\text{Wall 12 (outer)} = 900 \text{ mm}; \text{ nose cell area} = 20,000 \text{ mm}^2$$

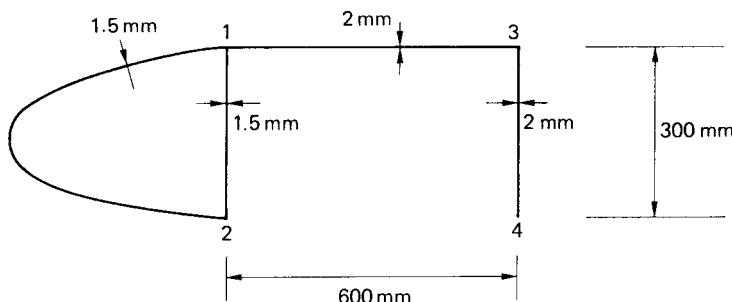


FIGURE 19.6 Wing Section of Example 19.3

It may be assumed, in a simplified approach, that the torsional rigidity  $GJ$  of the complete section is the sum of the torsional rigidities of the open and closed portions. For the closed portion, the torsional rigidity is, from Eq. (18.4),

$$(GJ)_{\text{cl}} = \frac{4A^2G}{\oint ds/t} = \frac{4 \times 20,000^2 \times 25,000}{(900 + 300)/1.5}$$

which gives

$$(GJ)_{\text{cl}} = 5000 \times 10^7 \text{ Nmm}^2$$

The torsional rigidity of the open portion is found using Eq. (18.11), thus,

$$(GJ)_{\text{op}} = G \sum \frac{st^3}{3} = \frac{25,000 \times 900 \times 2^3}{3}$$

that is,

$$(GJ)_{\text{op}} = 6 \times 10^7 \text{ Nmm}^2$$

The torsional rigidity of the complete section is then

$$GJ = 5000 \times 10^7 + 6 \times 10^7 = 5006 \times 10^7 \text{ Nmm}^2$$

In all unrestrained torsion problems, the torque is related to the rate of twist by the expression

$$T = GJ \frac{d\theta}{dz}$$

The angle of twist per unit length is therefore given by

$$\frac{d\theta}{dz} = \frac{T}{GJ} = \frac{10 \times 10^6}{5006 \times 10^7} = 0.0002 \text{ rad/mm}$$

Substituting for  $T$  in Eq. (18.1) from Eq. (18.4), we obtain the shear flow in the closed section. Thus,

$$q_{\text{cl}} = \frac{(GJ)_{\text{cl}}}{2A} \frac{d\theta}{dz} = \frac{5000 \times 10^7}{2 \times 20,000} \times 0.0002$$

from which

$$q_{\text{cl}} = 250 \text{ N/mm}$$

The maximum shear stress in the closed section is then  $250/1.5 = 166.7 \text{ N/mm}^2$ .

In the open portion of the section, the maximum shear stress is obtained directly from Eq. (18.10) and is

$$\tau_{\text{max,op}} = 25,000 \times 2 \times 0.0002 = 10 \text{ N/mm}^2$$

It can be seen from this that, in terms of strength and stiffness, the closed portion of the wing section dominates. This dominance may be used to determine the warping distribution. Having first found the position of the center of twist (the shear center), the warping of the closed portion is calculated using the method described in Section 18.1. The warping in the walls 13 and 34 is then determined using Eq. (18.19), in which the origin of the swept area  $A_R$  is taken at the point 1 and the value of warping is that previously calculated for the closed portion at 1.

**Example 19.4**

The wing section shown in Fig. 19.7 is subjected to a vertical shear load of 50 kN in the plane of the web 43. Determine the rate of twist of the section and the maximum shear stress. Take  $G=25,000 \text{ N/mm}^2$ .

The 50 kN load acting in the plane of the web 43 must be replaced by a 50 kN load acting through the shear center of the section together with a torque. Initially, therefore, we must find the position of the shear center that will, however, lie on the horizontal axis of symmetry.

From Section 16.4.5,

$$I_{xx} = (\pi \times 50^3 \times 2)/2 + (4 \times 100^3/12) + 2[2 \times 100 \times 50^2 + (2 \times 25^3/12) + 2 \times 25 \times 37.5^2]$$

i.e.,

$$I_{xx} = 1.87 \times 10^6 \text{ mm}^4$$

To find the position of the shear center we apply an arbitrary shear load,  $S_y$ , through the shear center S as shown in Fig. 19.7. Then, since  $S_x=0$  and  $I_{xy}=0$ , Eq. (17.15) reduces to

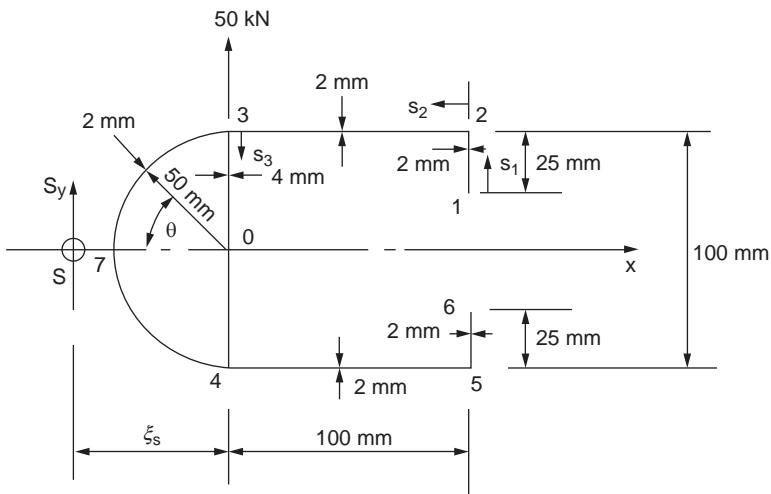
$$q_s = -(S_y/I_{xx}) \int_0^s ty \, ds + q_{s,0} \quad (i)$$

For the open part of the wing section,  $q_{s,0}=0$  at the open edges 1 and 6, while for the closed part of the section,  $q_{s,0}$  may be found using Eq. (17.27), since the rate of twist will be zero for a shear load applied through the shear center. Then

$$q_{12} = -(S_y/I_{xx}) \int_0^{s_1} ty \, ds_1$$

i.e.,

$$q_{12} = -(S_y/I_{xx}) \int_0^{s_1} 2(25 + s_1) \, ds_1$$



**FIGURE 19.7 Wing Section of Example 19.4**

which gives

$$q_{12} = -(S_y/I_{xx}) (50s_1 + s_1^2) \quad (\text{ii})$$

At the point 2, when  $s_1 = 25$  mm,  $q_2 = -1,875(S_y/I_{xx})$ .

In the wall 23,

$$q_{23} = -(S_y/I_{xx}) \left( \int_0^{s_2} 2 \times 50 \, ds_2 + 1875 \right)$$

so that

$$q_{23} = -(S_y/I_{xx}) (100s_2 + 1875) \quad (\text{iii})$$

When  $s_2 = 100$  mm,  $q_3 = -11,875(S_y/I_{xx})$ .

Now we “cut” the closed part of the section at 7. Then

$$q_{b,73} = -(S_y/I_{xx}) \int_0^\theta 2 \times 50 \sin \theta \times 50 \, d\theta$$

which gives

$$q_{b,73} = 5000(S_y/I_{xx})(\cos \theta - 1) \quad (\text{iv})$$

When  $\theta = \pi/2$  at 3,  $q_{b,3} = -5,000(S_y/I_{xx})$ .

In the wall 34

$$q_{b,34} = -(S_y/I_{xx}) \left[ \int_0^{s_3} 4(50 - s_3) \, ds_3 + 5000 + 11,875 \right]$$

which gives,

$$q_{b,34} = (S_y/I_{xx}) (2s_3^2 - 200s_3 - 16,875) \quad (\text{v})$$

From Eq. (17.27),

$$q_{s,0} = -[\oint (q_b/t) \, ds] / (\oint (ds/t)) \quad (\text{vi})$$

where  $\oint (ds/t) = (\pi \times 50/2) + (100/4) = 103.5$  and

$$\oint (q_b/t) \, ds = 2(S_y/I_{xx}) \left[ \int_0^{\pi/2} (5000/2)(\cos \theta - 1) 50 \, d\theta + \int_0^{50} (1/4)(2s_3^2 - 200s_3 - 16,875) \, ds_3 \right]$$

Integrating and substituting the limits gives

$$\oint (q_b/t) \, ds = -647,908(S_y/I_{xx})$$

so that from Eq. (vi),

$$q_{s,0} = 6,260(S_y/I_{xx})$$

Then

$$q_{73} = (S_y/I_{xx})(5,000 \cos \theta + 1,260) \quad (\text{vii})$$

and

$$q_{34} = (S_y/I_{xx})(2s_3^2 - 200s_3 - 10,615) \quad (\text{viii})$$

Now taking moments about the center of the web 34,

$$S_y \xi_S = 2 \left[ \int_0^{\pi/2} q_{73} \times 50^2 d\theta - \int_0^{25} q_{12} \times 100 ds_1 - \int_0^{100} q_{23} \times 50 ds_2 \right]$$

Substituting for  $q_{73}$  etc. from Eqs. (vii), (ii), and (iii) and carrying out the integration gives

$$\xi_S = 57.7 \text{ mm}$$

The actual loading on the wing section of 50 kN in the plane of the web 43 is therefore equivalent to a 50 kN load acting through the shear center together with an counterclockwise torque of  $50 \times 10^3 \times 57.7 = 2885 \text{ Nm}$ .

The shear flow distribution due to the 50 kN load acting through the shear center is given by Eqs. (ii), (iii), (vii), and (viii) in which  $S_y = 50 \text{ kN}$ . Then

$$q_{12} = -(50 \times 10^3 / 1.87 \times 10^6)(50s_1 + s_1^2) = -0.027(50s_1 + s_1^2) \quad (\text{ix})$$

$$q_{23} = -0.027(100s_2 + 1,875) \quad (\text{x})$$

$$q_{73} = 0.027(5000 \cos \theta + 1260) \quad (\text{xi})$$

$$q_{34} = 0.027(2s_3^2 - 200s_3 - 10,615) \quad (\text{xii})$$

The maximum shear flow in the wall 12 occurs, from Eq. (ix), at  $s_1 = 25 \text{ mm}$  and is equal to  $-50.1 \text{ N/mm}$ . From Eq. (x), the maximum shear flow in the wall 23 occurs when  $s_2 = 100 \text{ mm}$  and is equal to  $-320.6 \text{ N/mm}$ . From Eq. (xi), the maximum shear flow in the wall 73 occurs when  $\theta = 0$  and is  $+169.0 \text{ N/mm}$ . At 3 in the wall 73, the shear flow is  $34.0 \text{ N/mm}$ . Also, in the wall 34 at 3, the shear flow, from Eq. (xii), is  $286.6 \text{ N/mm}$ , and the maximum value of the shear flow in the wall 34 occurs when  $s_3 = 50 \text{ mm}$  and is equal to  $-421.6 \text{ N/mm}$ .

We shall assume, as in Ex. 19.3, that the torsional rigidity,  $GJ$ , of the complete section is the sum of the torsional rigidities of the closed and open sections. For the closed portion, from Eq. (18.4),

$$(GJ)_{\text{cl}} = (4A^2/G)/(\phi ds/t) = [4(\pi \times 50^2/2)^2 \times 25,000]/103.5 = 1487 \times 10^7 \text{ Nmm}^2$$

From Eq. (18.11),

$$(GJ)_{\text{op}} = 25,000 \times 2(100 \times 2^3 + 25 \times 2^3)/3 = 1.7 \times 10^7 \text{ Nmm}^2$$

The torsional rigidity of the complete section is then

$$GJ = 1487 \times 10^7 + 1.7 \times 10^7 = 1489 \times 10^7 \text{ Nmm}^2$$

The rate of twist in the combined section is given by

$$(d\theta/dz) = (T/GJ) = 2,885 \times 10^3 / 1,489 \times 10^7 = 1.94 \times 10^{-4} \text{ rad/mm}$$

Substituting for the torque,  $T$ , in Eq. (18.1) from Eq.(18.4) we obtain the shear flow in the closed part of the section, i.e.,

$$q_{\text{cl}} = [(GJ)_{\text{cl}}/2A](d\theta/dz) = [1,487 \times 10^7 / 2(\pi \times 50^2/2)] \times 1.94 \times 10^{-4} = 367.3 \text{ N/mm}$$

which is due to torsion.

In the open part of the section the maximum shear stress is obtained directly from Eq. (18.10) and is

$$\tau_{\text{max,op}} = \pm 25,000 \times 2 \times 1.94 \times 10^{-4} = \pm 9.7 \text{ N/mm}^2$$

We shall now examine the complete shear stress distribution to determine its maximum value. The maximum shear flow due to shear in the open part of the section occurs at 3 in the wall 23 and is  $320.6 \text{ N/mm}$  so that the corresponding shear stress is  $320.6/2 = 160.3 \text{ N/mm}^2$  giving a maximum shear stress of  $160.3 + 9.7 = 170.0 \text{ N/mm}^2$ . Note that  $q_{23}$  is negative at all points along the wall 23 and is therefore in the direction 32. The shear stress due to torsion varies linearly across the thickness and for the counterclockwise torque applied will be in the direction 23 on the upper surface of the wall 23 and in the direction 32 on the lower surface. The maximum shear stress in the wall 23 due to shear and torsion therefore occurs on the lower surface of the wall. The maximum shear flow due to shear in the closed part of the section is  $421.6 \text{ N/mm}$  in the direction 43 while the shear flow due to torsion is  $363.5 \text{ N/mm}$  in the direction 34 giving a resultant shear flow of  $421.6 - 363.5 = 58.1 \text{ N/mm}$  in the direction 43; the corresponding shear stress is  $58.1/4 = 14.5 \text{ N/mm}^2$ . At 3 in the web 34, the combined shear flow due to shear and torsion is  $363.5 - 286.6 = 76.9 \text{ N/mm}$  in the direction 34; the corresponding shear stress is  $76.9/4 = 19.2 \text{ N/mm}^2$ . In the wall 73, the maximum shear flow due to shear and torsion is  $169.0 + 363.5 = 532.5 \text{ N/mm}$  giving a shear stress of  $532.5/2 = 266.3 \text{ N/mm}^2$ .

We conclude, therefore, that the maximum shear stress in the combined section is  $266.3 \text{ N/mm}^2$  and occurs at the point 7 in the wall 73.

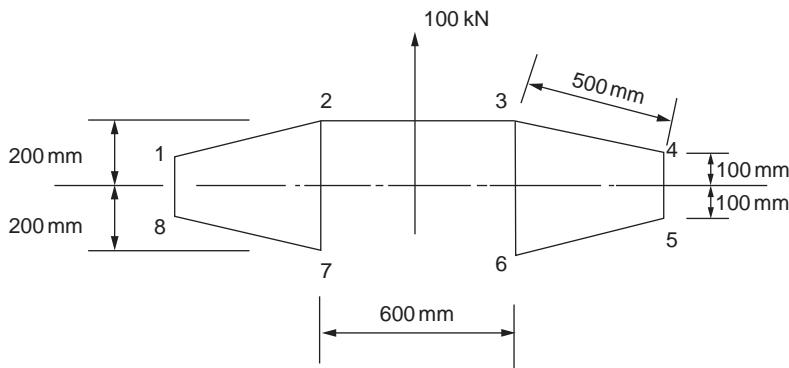
## PROBLEMS

**P.19.1.** The beam section of [Example 19.2](#) (see [Fig. 19.4](#)) is subjected to a bending moment in a vertical plane of  $20 \text{ kNm}$ . Calculate the maximum direct stress in the cross-section of the beam.

Answer:  $172.5 \text{ N/mm}^2$

**P.19.2.** A wing box has the cross-section shown diagrammatically in [Fig. P.19.2](#) and supports a shear load of  $100 \text{ kN}$  in its vertical plane of symmetry. Calculate the shear stress at the midpoint of the web 36 if the thickness of all walls is  $2 \text{ mm}$ .

Answer:  $89.7 \text{ N/mm}^2$



**FIGURE P.19.2**

**P.19.3.** If the wing box of P.19.2 is subjected to a torque of 100 kNm, calculate the rate of twist of the section and the maximum shear stress. The shear modulus  $G$  is 25,000 N/mm<sup>2</sup>.

Answer:  $18.5 \times 10^{-6}$  rad/mm, 170 N/mm<sup>2</sup>

**P.19.3. MATLAB** Use MATLAB to repeat Problem P.19.3 for values of  $t$  ranging from 1 to 3 mm in increments of 0.25 mm.

- Answer:
- (i)  $t = 1$  mm:  $37 \times 10^{-6}$  rad/mm, 340 N/mm<sup>2</sup>
  - (ii)  $t = 1.25$  mm:  $29.6 \times 10^{-6}$  rad/mm, 272 N/mm<sup>2</sup>
  - (iii)  $t = 1.5$  mm:  $24.7 \times 10^{-6}$  rad/mm, 227 N/mm<sup>2</sup>
  - (iv)  $t = 1.75$  mm:  $21.2 \times 10^{-6}$  rad/mm, 194 N/mm<sup>2</sup>
  - (v)  $t = 2$  mm:  $18.5 \times 10^{-6}$  rad/mm, 170 N/mm<sup>2</sup>
  - (vi)  $t = 2.25$  mm:  $16.5 \times 10^{-6}$  rad/mm, 151 N/mm<sup>2</sup>
  - (vii)  $t = 2.5$  mm:  $14.8 \times 10^{-6}$  rad/mm, 136 N/mm<sup>2</sup>
  - (viii)  $t = 2.75$  mm:  $13.5 \times 10^{-6}$  rad/mm, 124 N/mm<sup>2</sup>
  - (ix)  $t = 3$  mm:  $12.3 \times 10^{-6}$  rad/mm, 113 N/mm<sup>2</sup>

**P.19.4.** If the shear load of 100 kN acting in the vertical plane of symmetry of the wing box shown in Fig. P.19.2 now acts in the plane of the web 72, calculate the rate of twist in the section and the maximum shear stress.

Answer:  $5.6 \times 10^{-6}$  rad/mm, 140.7 N/mm<sup>2</sup>.

**P.19.5.** The singly symmetrical wing section shown in Fig. P.19.5 carries a vertical shear load of 50 kN in the plane of the web 63. Calculate the rate of twist of the section and the maximum shear stress in its walls. Take  $G = 25,000$  N/mm<sup>2</sup>.

Answer:  $3.23 \times 10^{-4}$  rad/mm, 441.3 N/mm<sup>2</sup>.

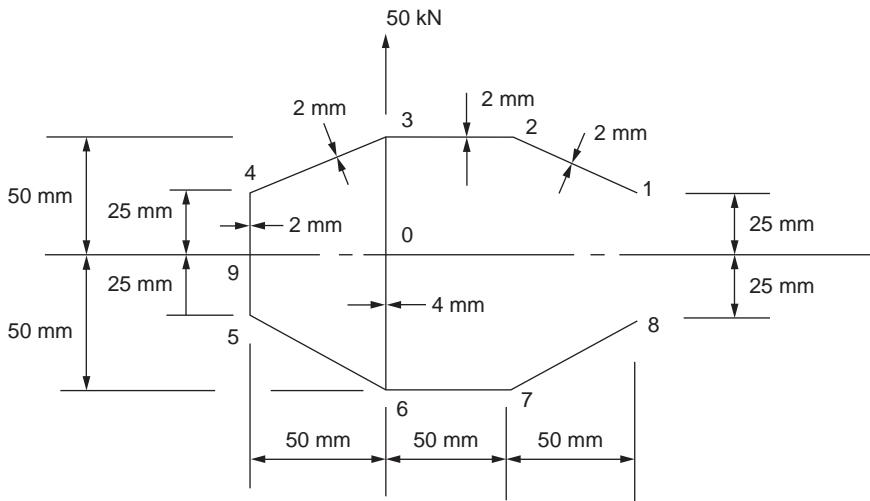


FIGURE P.19.5

# Structural idealization

# 20

So far, we have been concerned with relatively uncomplicated structural sections that, in practice, are formed from thin plate or by the extrusion process. While these sections exist as structural members in their own right, they are frequently used, as we saw in [Chapter 12](#), to stiffen more complex structural shapes such as fuselages, wings, and tail surfaces. Thus, a two spar wing section could take the form shown in [Fig. 20.1](#), in which Z-section stringers are used to stiffen the thin skin while angle sections form the spar flanges. Clearly, the analysis of a section of this type is complicated and tedious unless some simplifying assumptions are made. Generally, the number and nature of these simplifying assumptions determine the accuracy and the degree of complexity of the analysis; the more complex the analysis, the greater the accuracy obtained. The degree of simplification introduced is governed by the particular situation surrounding the problem. For a preliminary investigation, speed and simplicity are often of greater importance than extreme accuracy; on the other hand, a final solution must be as exact as circumstances allow.

Complex structural sections may be idealized into simpler “mechanical model” forms, which behave, under given loading conditions, in the same, or very nearly the same, way as the actual structure. We shall see, however, that different models of the same structure are required to simulate actual behavior under different systems of loading.

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## 20.1 PRINCIPLE

In the wing section of [Fig. 20.1](#), the stringers and spar flanges have small cross-sectional dimensions compared with the complete section. Therefore, the variation in stress over the cross-section of a stringer due to, say, bending of the wing is small. Furthermore, the difference between the distances of the stringer centroids and the adjacent skin from the wing section axis is small. It is reasonable to assume therefore that the direct stress is constant over the stringer cross-sections. We therefore replace the stringers and spar flanges by concentrations of area, known as *booms*, over which the direct stress is constant and which are located along the mid-line of the skin, as shown in [Fig. 20.2](#). In wing and fuselage sections of the type shown in [Fig. 20.1](#), the stringers and spar flanges carry most of the direct stresses while the skin is effective mainly in resisting shear stresses, although it also carries some of the direct stresses. The idealization shown in [Fig. 20.2](#) may therefore be taken a stage further by assuming that all direct stresses are carried by the booms while the skin is effective only in shear. The direct stress carrying capacity of the skin may be allowed for by increasing each boom area by an area equivalent to the direct stress carrying capacity of the adjacent skin panels. The calculation of these equivalent areas generally depends upon an initial assumption as to the form of the distribution of direct stress in a boom–skin panel.

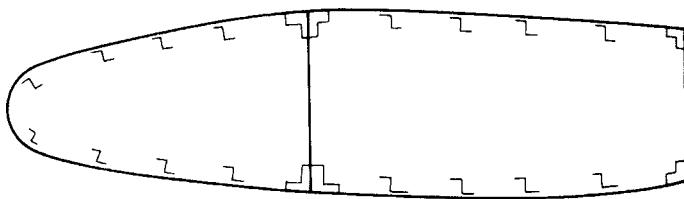


FIGURE 20.1 Typical Wing Section

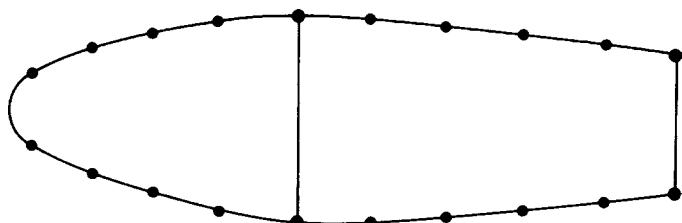


FIGURE 20.2 Idealization of a Wing Section

## 20.2 IDEALIZATION OF A PANEL

Suppose we wish to idealize the panel of Fig. 20.3(a) into a combination of direct stress carrying booms and shear stress only carrying skin, as shown in Fig. 20.3(b). In Fig. 20.3(a), the direct stress carrying thickness  $t_D$  of the skin is equal to its actual thickness  $t$ ; while, in Fig. 20.3(b),  $t_D = 0$ . Suppose also that the direct stress distribution in the actual panel varies linearly from an unknown value  $\sigma_1$  to an unknown value  $\sigma_2$ . Clearly, the analysis should predict the extremes of stress  $\sigma_1$  and  $\sigma_2$ , although the distribution of direct stress is obviously lost. Since the loading producing the direct stresses in the actual and idealized panels must be the same, we can equate moments to obtain expressions for the boom areas  $B_1$  and  $B_2$ . Thus, taking moments about the right-hand edge of each panel,

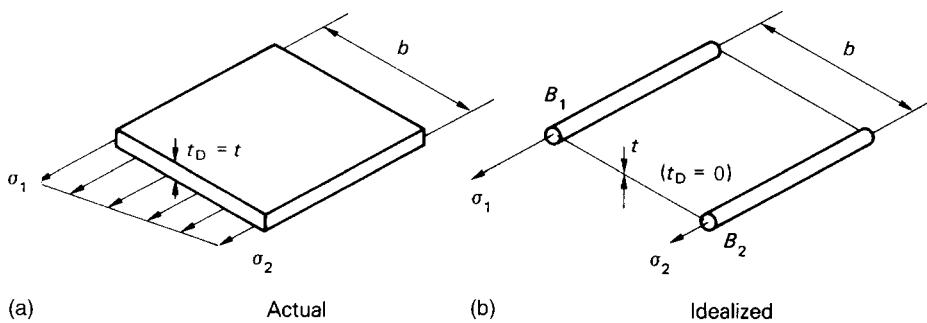


FIGURE 20.3 Idealization of a Panel

$$\sigma_2 t_D \frac{b^2}{2} + \frac{1}{2}(\sigma_1 - \sigma_2) t_D b \frac{2}{3} b = \sigma_1 B_1 b$$

from which

$$B_1 = \frac{t_D b}{6} \left( 2 + \frac{\sigma_2}{\sigma_1} \right) \quad (20.1)$$

Similarly,

$$B_2 = \frac{t_D b}{6} \left( 2 + \frac{\sigma_1}{\sigma_2} \right) \quad (20.2)$$

In Eqs. (20.1) and (20.2), the ratio of  $\sigma_1$  to  $\sigma_2$ , if not known, may frequently be assumed.

The direct stress distribution in Fig. 20.3(a) is caused by a combination of axial load and bending moment. For axial load only,  $\sigma_1/\sigma_2 = 1$  and  $B_1 = B_2 = t_D b/2$ ; for a pure bending moment,  $\sigma_1/\sigma_2 = -1$  and  $B_1 = B_2 = t_D b/6$ . Thus, different idealizations of the same structure are required for different loading conditions.

### Example 20.1

Part of a wing section is in the form of the two-cell box shown in Fig. 20.4(a), in which the vertical spars are connected to the wing skin through angle sections all having a cross-sectional area of  $300 \text{ mm}^2$ . Idealize the section into an arrangement of direct stress carrying booms and shear stress only carrying panels suitable for resisting bending moments in a vertical plane. Position the booms at the spar/skin junctions.

The idealized section is shown in Fig. 20.4(b), in which, from symmetry,  $B_1 = B_6$ ,  $B_2 = B_5$ ,  $B_3 = B_4$ . Since the section is required to resist bending moments in a vertical plane, the direct stress at any point in the actual wing section is directly proportional to its distance from the horizontal axis of symmetry. Further, the distribution of direct stress in all the panels is linear, so that either Eq. (20.1) or (20.2) may be used. We note that, in addition to contributions from adjacent panels, the boom areas include the existing spar flanges. Hence,

$$B_1 = 300 + \frac{3.0 \times 400}{6} \left( 2 + \frac{\sigma_6}{\sigma_1} \right) + \frac{2.0 \times 600}{6} \left( 2 + \frac{\sigma_2}{\sigma_1} \right)$$

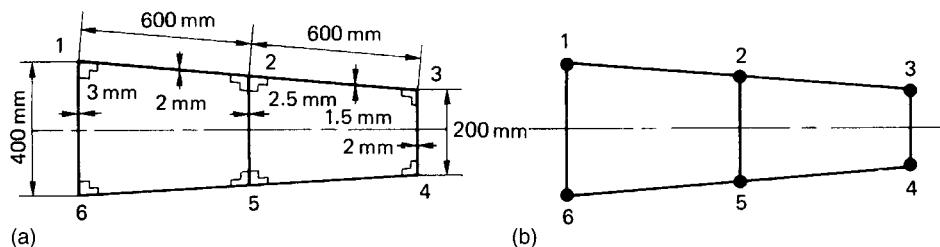


FIGURE 20.4 Idealization of a Wing Section

or

$$B_1 = 300 + \frac{3.0 \times 400}{6} (2 - 1) + \frac{2.0 \times 600}{6} \left( 2 + \frac{150}{200} \right)$$

which gives

$$B_1 (= B_6) = 1,050 \text{ mm}^2$$

Also,

$$B_2 = 2 \times 300 + \frac{2.0 \times 600}{6} \left( 2 + \frac{\sigma_1}{\sigma_2} \right) + \frac{2.5 \times 300}{6} \left( 2 + \frac{\sigma_5}{\sigma_2} \right) + \frac{1.5 \times 600}{6} \left( 2 + \frac{\sigma_3}{\sigma_2} \right)$$

that is,

$$B_2 = 2 \times 300 + \frac{2.0 \times 600}{6} \left( 2 + \frac{200}{150} \right) + \frac{2.5 \times 300}{6} (2-1) + \frac{1.5 \times 600}{6} \left( 2 + \frac{100}{150} \right)$$

from which

$$B_2 (= B_5) = 1,791.7 \text{ mm}^2$$

Finally,

$$B_3 = 300 + \frac{1.5 \times 600}{6} \left( 2 + \frac{\sigma_2}{\sigma_3} \right) + \frac{2.0 \times 200}{6} \left( 2 + \frac{\sigma_4}{\sigma_3} \right)$$

that is,

$$B_3 = 300 + \frac{1.5 \times 600}{6} \left( 2 + \frac{150}{100} \right) + \frac{2.0 \times 200}{6} (2-1)$$

so that

$$B_3 (= B_4) = 891.7 \text{ mm}^2$$

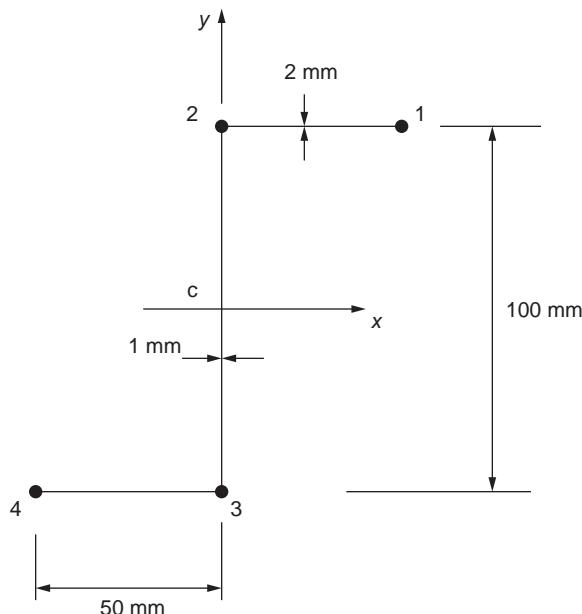


### 20.2.1 Unsymmetrical sections

The idealization of a beam having an unsymmetrical cross-section presents a problem in that the ratio of direct stresses at the ends of a particular panel is unlikely to be known with any degree of certainty. One approach to circumvent the problem would be to make an assumption, determine the stresses, and therefore obtain a more accurate value for their ratio. This iterative procedure could be repeated until an accurate estimate of the boom areas is obtained. Possibly, though, the amount of computation required could be greater than that involved in analyzing the unidealized structure, particularly with the availability of modern computational techniques. However, one particular unsymmetrical beam section may be idealized with facility as demonstrated below.

**Example 20.2**

Idealize the Z-section beam shown in Fig. 20.5 into one comprising direct stress carrying booms and shear stress only carrying panels and which is subjected to a bending moment  $M_x$ .



**FIGURE 20.5 Beam Section of Example 20.2**

The beam cross-section is antisymmetrical and therefore possesses a value for the product second moment of area  $I_{xy}$ . Initially we shall calculate the second moments of area of the unidealized beam section. Then (see Section 16.4),

$$I_{xx} = 2 \times 50 \times 2 \times 50^2 + \frac{1 \times 100^3}{12} = 0.58 \times 10^6 \text{ mm}^4$$

$$I_{yy} = 2 \times \frac{2 \times 50^3}{3} = 0.17 \times 10^6 \text{ mm}^4$$

$$I_{xy} = 2 \times 50(+25)(+50) + 2 \times 50(-25)(-50) = 0.25 \times 10^6 \text{ mm}^4$$

We note in the above that the calculation of  $I_{xx}$  is identical to that for a singly symmetrical channel section having the same flange and web dimensions and supporting a bending moment  $M_x$ . Similarly the calculation of  $I_{yy}$  is identical to that for a doubly symmetrical cruciform section carrying a bending moment  $M_y$ . Therefore, the same stress ratios can be applied to the Z-section as would be applied to the channel and cruciform sections. Then, for the boom area  $B_1$ ,  $(\sigma_2/\sigma_1) = 1$  and for boom area  $B_2$ ,  $(\sigma_1/\sigma_2) = 1$  and  $(\sigma_3/\sigma_2) = -1$ . Therefore, for the “channel” section and referring to either Eq. (20.1) or (20.2),

$$B_1 = \frac{50 \times 2}{6} (2 + 1) = 50 \text{ mm}^2 (= B_4)$$

$$B_2 = \frac{100 \times 1}{6} (2 - 1) + \frac{50 \times 2}{6} (2 + 1) = 66.67 \text{ mm}^4 (= B_3)$$

Then

$$I_{xx} = 2 \times 50 \times 50^2 + 2 \times 66.67 \times 50^2 = 0.58 \times 10^6 \text{ mm}^4$$

For the “cruciform” section,

$$B_1 = 100 \times 2 (2 - 1) = 33.33 \text{ mm}^4 (= B_4)$$

Then

$$I_{yy} = 2 \times 33.33 \times 50^2 = 0.17 \times 10^6 \text{ mm}^4$$

In the calculation of  $I_{xy}$ , the boom area  $B_1 (=B_4)$  corresponding to the “channel” section must be used, since the applied moment is  $M_x$ . Further, the boom areas  $B_2$  and  $B_3$  make no contribution to  $I_{xy}$ , since they both lie on the  $y$ -axis. Then

$$I_{xy} = 50 (+50)(+50) + 50 (-50)(-50) = 0.25 \times 10^5 \text{ mm}^4$$

and we see that all three values of second moment of area are identical to those calculated for the unidealized Z-section.

### 20.3 EFFECT OF IDEALIZATION ON THE ANALYSIS OF OPEN AND CLOSED SECTION BEAMS

The addition of direct stress carrying booms to open and closed section beams clearly modifies the analyses presented in Chapters 16–18. Before considering individual cases, we shall discuss the implications of structural idealization. Generally, in any idealization, different loading conditions require different idealizations of the same structure. In Example 20.1, the loading is applied in a vertical plane. If, however, the loading had been applied in a horizontal plane, the assumed stress distribution in the panels of the section would have been different, resulting in different values of boom area.

Suppose that an open or closed section beam is subjected to given bending or shear loads and that the required idealization has been completed. The analysis of such sections usually involves the determination of the neutral axis position and the calculation of sectional properties. The position of the neutral axis is derived from the condition that the resultant load on the beam cross-section is zero, that is,

$$\int_A \sigma_z \, dA = 0 \quad (\text{see Eq.(16.3)})$$

The area  $A$  in this expression is clearly the direct stress carrying area. It follows that the centroid of the cross-section is the centroid of the direct stress carrying area of the section, depending on the degree and method of idealization. The sectional properties,  $I_{xx}$ , and so forth, must also refer to the direct stress carrying area.

### 20.3.1 Bending of open and closed section beams

The analysis presented in Sections 16.1 and 16.2 applies and the direct stress distribution is given by any of Eqs. (16.9), (16.17), or (16.18), depending on the beam section being investigated. In these equations the coordinates ( $x, y$ ) of points in the cross-section are referred to axes having their origin at the centroid of the direct stress carrying area. Furthermore, the section properties  $I_{xx}$ ,  $I_{yy}$ , and  $I_{xy}$  are calculated for the direct stress carrying area only.

In the case where the beam cross-section has been completely idealized into direct stress carrying booms and shear stress only carrying panels, the direct stress distribution consists of a series of direct stresses concentrated at the centroids of the booms.

#### Example 20.3

The fuselage section shown in Fig. 20.6 is subjected to a bending moment of 100 kNm applied in the vertical plane of symmetry. If the section has been completely idealized into a combination of direct stress carrying booms and shear stress only carrying panels, determine the direct stress in each boom.

The section has  $Cy$  as an axis of symmetry and resists a bending moment  $M_x = 100$  kNm. Equation (16.17) therefore reduces to

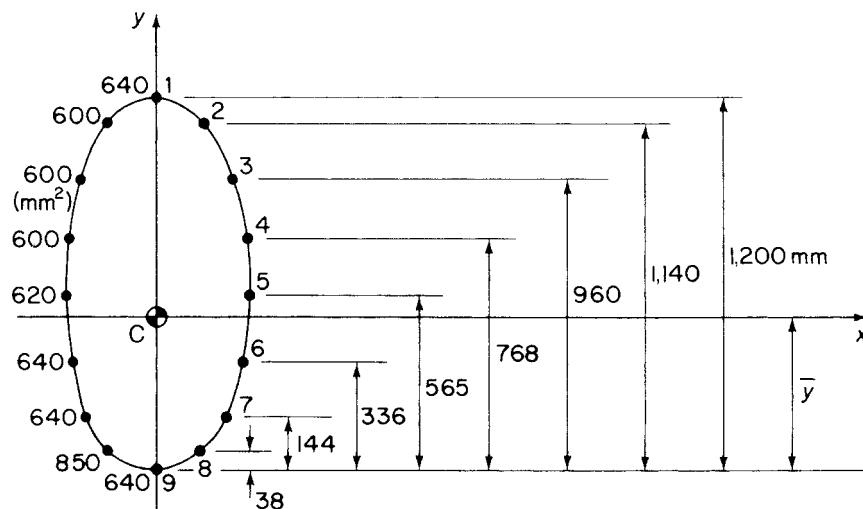


FIGURE 20.6 Idealized Fuselage Section of Example 20.3

$$\sigma_z = \frac{M_x}{I_{xx}} y \quad (i)$$

The origin of axes  $Cxy$  coincides with the position of the centroid of the direct stress carrying area, which, in this case, is the centroid of the boom areas. Thus, taking moments of area about boom 9,

$$\begin{aligned} & (6 \times 640 + 6 \times 600 + 2 \times 620 + 2 \times 850)\bar{y} \\ &= 640 \times 1,200 + 2 \times 600 \times 1140 + 2 \times 600 \times 960 + 2 \times 600 \times 768 \\ &+ 2 \times 620 \times 565 + 2 \times 640 \times 336 + 2 \times 640 \times 144 + 2 \times 850 \times 38 \end{aligned}$$

**Table 20.1** Example 20.3

① Boom	② $y$ (mm)	③ $B$ (mm $^2$ )	④ $\Delta I_{xx} = By^2$ (mm $^4$ )	⑤ $\sigma_z$ (N/mm $^2$ )
1	+660	640	$278 \times 10^6$	35.6
2	+600	600	$216 \times 10^6$	32.3
3	+420	600	$106 \times 10^6$	22.6
4	+228	600	$31 \times 10^6$	12.3
5	+25	620	$0.4 \times 10^6$	1.3
6	-204	640	$27 \times 10^6$	-11.0
7	-396	640	$100 \times 10^6$	-21.4
8	-502	850	$214 \times 10^6$	-27.0
9	-540	640	$187 \times 10^6$	-29.0

which gives

$$\bar{y} = 540 \text{ mm}$$

The solution is now completed in [Table 20.1](#).

From column ④,

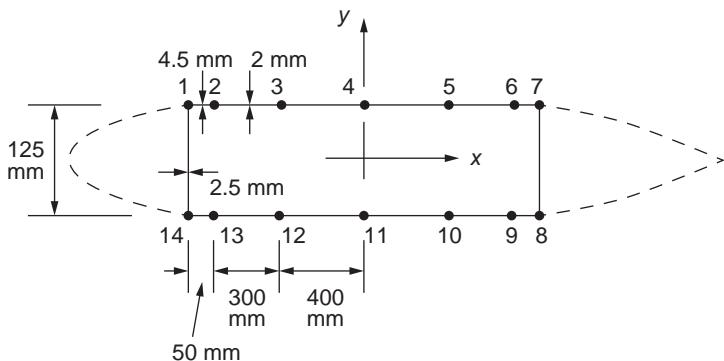
$$I_{xx} = 1,854 \times 10^6 \text{ mm}^4$$

and column ⑤ is completed using [Eq. \(i\)](#).

### Example 20.4

Idealize the wing section of Example 16.4 into an arrangement of direct stress carrying booms and shear stress only carrying skin where the outer “top hat” stiffeners are each 400 mm either side of the vertical axis of symmetry. Then calculate the maximum direct stress in the cross-section due to the bending moment of 59.6 kNm.

The booms corresponding to the “top hat” stiffeners are positioned as shown in [Fig. 20.7](#) but will now receive contributions from the adjacent skin panels. Also the flanges of the channel section spars cause, in effect, discontinuities in skin thickness so that booms must be positioned at their extremities.

**FIGURE 20.7** Idealized Wing Section of Example 20.4

From Eq. (20.1) or (20.2),

$$\begin{aligned} B_1 &= \frac{50 \times 4.5}{6} \left( 2 + \frac{\sigma_2}{\sigma_1} \right) + \frac{125 \times 2.5}{6} \left( 2 + \frac{\sigma_{14}}{\sigma_1} \right) \\ &= \frac{50 \times 4.5}{6} (2+1) + \frac{125 \times 2.5}{6} (2-1) \end{aligned}$$

that is,

$$B_1 = 164.6 \text{ mm}^2 \quad (= B_7 = B_8 = B_{14})$$

Similarly,

$$B_2 = \frac{50 \times 4.5}{6} (2+1) + \frac{300 \times 2}{6} (2+1)$$

that is,

$$\begin{aligned} B_2 &= 412.5 \text{ mm}^2 \quad (= B_6 = B_9 = B_{13}) \\ B_3 &= 50 + \frac{300 \times 2}{6} (2+1) + \frac{400 \times 2}{6} (2+1) \end{aligned}$$

that is,

$$\begin{aligned} B_3 &= 750 \text{ mm}^2 \quad (= B_5 = B_{10} = B_{12}) \\ B_4 &= 50 + 2 \times \frac{400 \times 2}{6} (2+1) \end{aligned}$$

that is,

$$B_4 = 850 \text{ mm}^2 \quad (= B_{11})$$

Then

$$I_{xx} = 4 \times 164.6 \times 62.5^2 + 4 \times 412.5 \times 62.5^2 + 4 \times 750 \times 62.5^2 + 2 \times 850 \times 62.5^2$$

so that

$$I_{xx} = 27.4 \times 10^6 \text{ mm}^4$$

From Eq. (16.9),

$$\sigma_z (\text{max.}) = \pm \frac{59.6 \times 10^6 \times 62.5}{27.4 \times 10^6}$$

which gives

$$\sigma_z (\text{max.}) = \pm 135.9 \text{ N/mm}^2$$

which is identical to the solution of Example 16.4.

### 20.3.2 Shear of open section beams

The derivation of Eq. (17.14) for the shear flow distribution in the cross-section of an open section beam is based on the equilibrium equation (17.2). The thickness  $t$  in this equation refers to the direct stress carrying thickness  $t_D$  of the skin. Equation (17.14) may therefore be rewritten

$$q_s = - \left( \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \int_0^s t_D x \, ds - \left( \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \int_0^s t_D y \, ds \quad (20.3)$$

in which  $t_D = t$  if the skin is fully effective in carrying direct stress or  $t_D = 0$  if the skin is assumed to carry only shear stresses. Again, the section properties in Eq. (20.3) refer to the direct stress carrying area of the section, since they are those which feature in Eqs. (16.17) and (16.18).

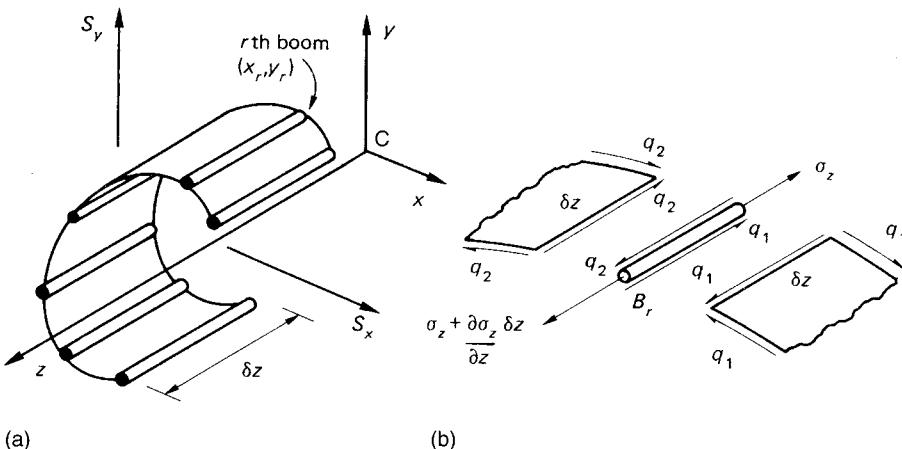
Equation (20.3) makes no provision for the effects of booms that cause discontinuities in the skin and therefore interrupt the shear flow. Consider the equilibrium of the  $r$ th boom in the elemental length of beam shown in Fig. 20.8(a), which carries shear loads  $S_x$  and  $S_y$  acting through its shear center  $S$ . These shear loads produce direct stresses due to bending in the booms and skin and shear stresses in the skin. Suppose that the shear flows in the skin adjacent to the  $r$ th boom of cross-sectional area  $B_r$  are  $q_1$  and  $q_2$ . Then, from Fig. 20.8(b),

$$\left( \sigma_z + \frac{\partial \sigma_z}{\partial z} \delta z \right) B_r - \sigma_z B_r + q_2 \delta z - q_1 \delta z = 0$$

which simplifies to

$$q_2 - q_1 = - \frac{\partial \sigma_z}{\partial z} B_r \quad (20.4)$$

Substituting for  $\sigma_z$  in Eq. (20.4) from (16.17), we have



**FIGURE 20.8 (a) Elemental Length of Shear Loaded Open Section Beam With Booms; (b) Equilibrium of Boom Element**

$$q_2 - q_1 = - \left[ \frac{(\partial M_y / \partial z) I_{xx} - (\partial M_x / \partial z) I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right] B_r x_r - \left[ \frac{(\partial M_x / \partial z) I_{yy} - (\partial M_y / \partial z) I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right] B_r y_r$$

or, using the relationships of Eqs. (16.22) and (16.23),

$$q_2 - q_1 = - \left( \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) B_r x_r - \left( \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) B_r y_r \quad (20.5)$$

Equation (20.5) gives the change in shear flow induced by a boom that itself is subjected to a direct load ( $\sigma_z B_r$ ). Each time a boom is encountered, the shear flow is incremented by this amount, so that if, at any distance  $s$  around the profile of the section,  $n$  booms have been passed, the shear flow at the point is given by

$$q_s = - \left( \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D x \, ds + \sum_{r=1}^n B_r x_r \right) - \left( \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D y \, ds + \sum_{r=1}^n B_r y_r \right) \quad (20.6)$$

### Example 20.5

Calculate the shear flow distribution in the channel section shown in Fig. 20.9 produced by a vertical shear load of 4.8 kN acting through its shear center. Assume that the walls of the section are effective in resisting only shear stresses, while the booms, each of area 300 mm<sup>2</sup>, carry all the direct stresses.

The effective direct stress carrying thickness  $t_D$  of the walls of the section is zero so that the centroid of area and the section properties refer to the boom areas only. Since  $C_x$  (and  $C_y$  as far as the boom areas are concerned) is an axis of symmetry,  $I_{xy} = 0$ ; also  $S_x = 0$ , and Eq. (20.6) thereby reduces to

$$q_s = - \frac{S_y}{I_{xx}} \sum_{r=1}^n B_r y_r \quad (i)$$

in which  $I_{xx} = 4 \times 300 \times 200^2 = 48 \times 10^6$  mm<sup>4</sup>. Substituting the values of  $S_y$  and  $I_{xx}$  in Eq. (i) gives

$$q_s = - \frac{4.8 \times 10^3}{48 \times 10^6} \sum_{r=1}^n B_r y_r = -10^{-4} \sum_{r=1}^n B_r y_r \quad (ii)$$

At the outside of boom 1,  $q_s = 0$ . As boom 1 is crossed, the shear flow changes by an amount given by

$$\Delta q_1 = -10^{-4} \times 300 \times 200 = -6 \text{ N/mm}$$

Hence,  $q_{12} = -6 \text{ N/mm}$ , since, from Eq. (i), it can be seen that no further changes in shear flow occur until the next boom (2) is crossed. Hence,

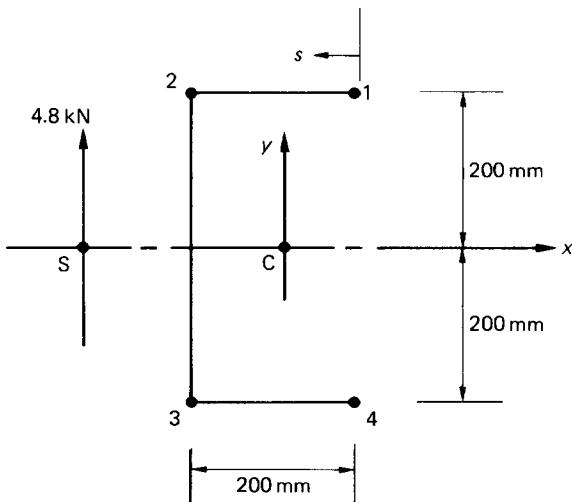


FIGURE 20.9 Idealized Channel Section of Example 20.5

$$q_{23} = -6 \times 10^{-4} \times 300 \times 200 = -12 \text{ N/mm}$$

Similarly,

$$q_{34} = -12 \times 10^{-4} \times 300 \times (-200) = -6 \text{ N/mm}$$

while, finally, at the outside of boom 4, the shear flow is

$$-6 \times 10^{-4} \times 300 \times (-200) = 0$$

as expected. The complete shear flow distribution is shown in Fig. 20.10.

It can be seen, from Eq. (i) in Example 20.5, that the analysis of a beam section which is idealized into a combination of direct stress carrying booms and shear stress only carrying skin gives constant values of the shear flow in the skin between the booms; the actual distribution of shear flows is therefore lost. What remains is in fact the average of the shear flow, as can be seen by referring to Example 20.5. Analysis of the unidealized channel section results in a parabolic distribution of shear flow in the web 23, whose resultant is statically equivalent to the externally applied shear load of 4.8 kN. In Fig. 20.10, the resultant of the constant shear flow in the web 23 is  $12 \times 400 = 4800 \text{ N} = 4.8 \text{ kN}$ . It follows that this constant value of shear flow is the average of the parabolically distributed shear flows in the unidealized section.

The result, from the idealization of a beam section, of a constant shear flow between booms may be used to advantage in parts of the analysis. Suppose that the curved web 12 in Fig. 20.11 has booms at its extremities and that the shear flow  $q_{12}$  in the web is constant. The shear force on an element  $\delta s$  of the web is  $q_{12}\delta s$ , whose components horizontally and vertically are  $q_{12}\delta s \cos \phi$  and  $q_{12}\delta s \sin \phi$ . The resultant, parallel to the  $x$  axis,  $S_x$ , of  $q_{12}$  is therefore given by

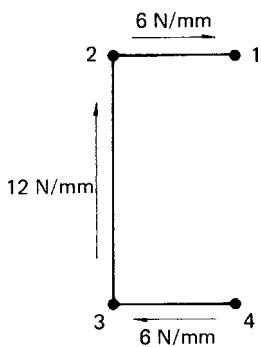


FIGURE 20.10 Shear Flow in Channel Section of Example 20.5

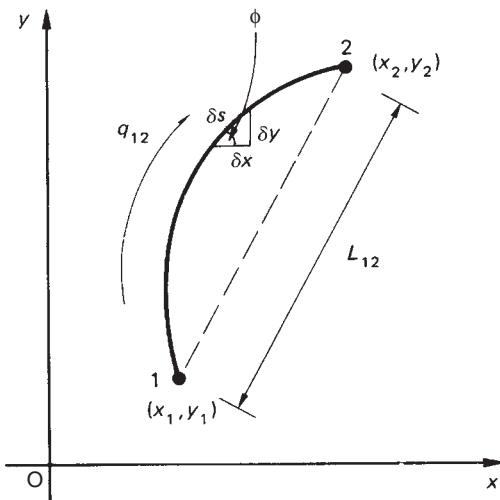


FIGURE 20.11 Curved Web With Constant Shear Flow

$$S_x = \int_1^2 q_{12} \cos \phi \, ds$$

or

$$S_x = q_{12} \int_1^2 \cos \phi \, ds$$

which, from Fig. 20.11, may be written as

$$S_x = q_{12} \int_1^2 dx = q_{12}(x_2 - x_1) \quad (20.7)$$

Similarly, the resultant of  $q_{12}$  parallel to the  $y$  axis is

$$S_y = q_{12}(y_2 - y_1) \quad (20.8)$$

Thus, the resultant, in a given direction, of a constant shear flow acting on a web is the value of the shear flow multiplied by the projection on that direction of the web.

The resultant shear force  $S$  on the web of Fig. 20.11 is

$$S = \sqrt{S_x^2 + S_y^2} = q_{12}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

that is,

$$S = q_{12}L_{12} \quad (20.9)$$

Therefore, the resultant shear force acting on the web is the product of the shear flow and the length of the straight line joining the ends of the web; clearly, the direction of the resultant is parallel to this line.

The moment  $M_q$  produced by the shear flow  $q_{12}$  about any point O in the plane of the web is, from Fig. 20.12,

$$M_q = \int_1^2 q_{12}p \, ds = q_{12} \int_1^2 2 \, dA$$

or

$$M_q = 2Aq_{12} \quad (20.10)$$

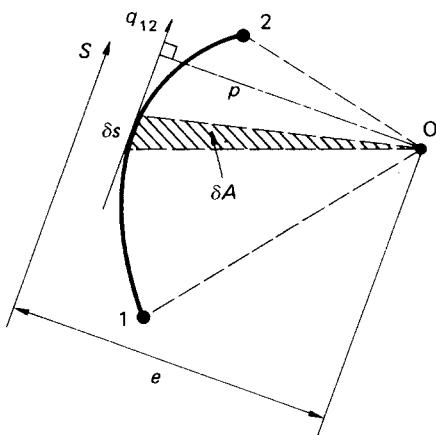


FIGURE 20.12 Moment Produced by a Constant Shear Flow

in which  $A$  is the area enclosed by the web and the lines joining the ends of the web to the point O. This result may be used to determine the distance of the line of action of the resultant shear force from any point. From Fig. 20.12,

$$Se = 2Aq_{12}$$

from which

$$e = \frac{2A}{S}q_{12}$$

Substituting for  $q_{12}$  from Eq. (20.9) gives

$$e = \frac{2A}{L_{12}}$$

### Example 20.5 MATLAB

Use MATLAB to repeat Example 20.5.

The shear flow distribution is obtained through the following MATLAB file:

```
% Declare any needed variables
S_y = 4.8e3;
B = 300*ones(1,4);

% Define the lengths of all open section walls
L = [200 400 200];

% Calculate the section properties
y_r = [L(2)/2 L(2)/2 -L(2)/2 -L(2)/2];
I_xx = sum(B.*y_r.^2); % I_yy and I_xy not needed due to symmetry and S_x=0

% Due to symmetry and since S_x,t_D=0, Eq. (20.6) simplifies
% Calculate the shear flow in wall 1-2 using the reduced Eq. (20.6)
A = B.*y_r;
q_12 = -S_y/I_xx*sum(A(1));

% Calculate the subsequent shear flow in walls 2-3 and 3-4
q_23 = -S_y/I_xx*sum(A(1:2));
q_34 = -S_y/I_xx*sum(A(1:3));

% Output the calculated shear flows to the Command Window
disp(['q_12 = ' num2str(q_12) ' N/mm'])
disp(['q_23 = ' num2str(q_23) ' N/mm'])
disp(['q_34 = ' num2str(q_34) ' N/mm'])
```

The Command Window outputs resulting from this MATLAB file are as follows:

```
q_12 = - 6 N/mm
q_23 = - 12 N/mm
q_34 = - 6 N/mm
```

### 20.3.3 Shear loading of closed section beams

Arguments identical to those in the shear of open section beams apply in this case. Thus, the shear flow at any point around the cross-section of a closed section beam comprising booms and skin of direct stress carrying thickness  $t_D$  is, by a comparison of Eqs. (20.6) and (17.15),

$$q_s = - \left( \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D dx \, ds + \sum_{r=1}^n B_r x_r \right) - \left( \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D dy \, ds + \sum_{r=1}^n B_r y_r \right) + q_{s,0} \quad (20.11)$$

Note that the zero value of the “basic” or “open section” shear flow at the “cut” in a skin for which  $t_D = 0$  extends from the cut to the adjacent booms.

---

#### Example 20.6

The thin-walled single cell beam shown in Fig. 20.13 has been idealized into a combination of direct stress carrying booms and shear stress only carrying walls. If the section supports a vertical shear load of 10 kN acting in a vertical plane through booms 3 and 6, calculate the distribution of shear flow around the section.

Boom areas:  $B_1 = B_8 = 200 \text{ mm}^2$ ,  $B_2 = B_7 = 250 \text{ mm}^2$ ,  $B_3 = B_6 = 400 \text{ mm}^2$ ,  $B_4 = B_5 = 100 \text{ mm}^2$

The centroid of the direct stress carrying area lies on the horizontal axis of symmetry, so that  $I_{xy} = 0$ . Also, since  $t_D = 0$  and only a vertical shear load is applied, Eq. (20.11) reduces to

$$q_s = - \frac{S_y}{I_{xx}} \sum_{r=1}^n B_r y_r + q_{s,0} \quad (i)$$

in which

$$I_{xx} = 2(200 \times 30^2 + 250 \times 100^2 + 400 \times 100^2 + 100 \times 50^2) = 13.86 \times 10^6 \text{ mm}^4$$

Equation (i) then becomes

$$q_s = - \frac{10 \times 10^3}{13.86 \times 10^6} \sum_{r=1}^n B_r y_r + q_{s,0}$$

that is,

$$q_s = -7.22 \times 10^{-4} \sum_{r=1}^n B_r y_r + q_{s,0} \quad (ii)$$

"Cutting" the beam section in the wall 23 (any wall may be chosen) and calculating the basic shear flow distribution  $q_b$  from the first term on the right-hand side of Eq. (ii), we have

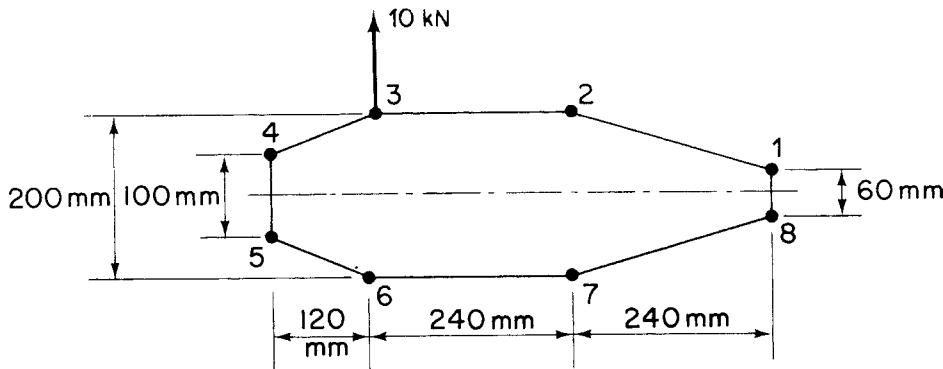


FIGURE 20.13 Closed Section of Beam of Example 20.6

$$q_{b,23} = 0$$

$$q_{b,34} = -7.22 \times 10^{-4} (400 \times 100) = -28.9 \text{ N/mm}$$

$$q_{b,45} = -28.9 - 7.22 \times 10^{-4} (100 \times 50) = -32.5 \text{ N/mm}$$

$$q_{b,56} = q_{b,34} = -28.9 \text{ N/mm} \text{ (by symmetry)}$$

$$q_{b,67} = q_{b,23} = 0 \text{ (by symmetry)}$$

$$q_{b,21} = -7.22 \times 10^{-4} (250 \times 100) = -18.1 \text{ N/mm}$$

$$q_{b,18} = -18.1 - 7.22 \times 10^{-4} (200 \times 30) = -22.4 \text{ N/mm}$$

$$q_{b,87} = q_{b,21} = -18.1 \text{ N/mm} \text{ (by symmetry)}$$

Taking moments about the intersection of the line of action of the shear load and the horizontal axis of symmetry and referring to the results of Eqs. (20.7) and (20.8), we have, from Eq. (17.18),

$$0 = [q_{b,81} \times 60 \times 480 + 2q_{b,12} (240 \times 100 + 70 \times 240) + 2q_{b,23} \times 240 \times 100 - 2q_{b,43} \times 120 \times 100 - q_{b,54} \times 100 \times 120] + 2 \times 97,200q_{s,0}$$

Substituting these values of  $q_b$  in this equation gives

$$q_{s,0} = -5.4 \text{ N/mm}$$

the negative sign indicating that  $q_{s,0}$  acts in a clockwise sense.

In any wall, the final shear flow is given by  $q_s = q_b + q_{s,0}$ , so that

$$q_{21} = -18.1 + 5.4 = -12.7 \text{ N/mm} = q_{87}$$

$$q_{23} = -5.4 \text{ N/mm} = q_{67}$$

$$q_{34} = -34.3 \text{ N/mm} = q_{56}$$

$$q_{45} = -37.9 \text{ N/mm}$$

$$q_{81} = 17.0 \text{ N/mm}$$

giving the shear flow distribution shown in Fig. 20.14. ■

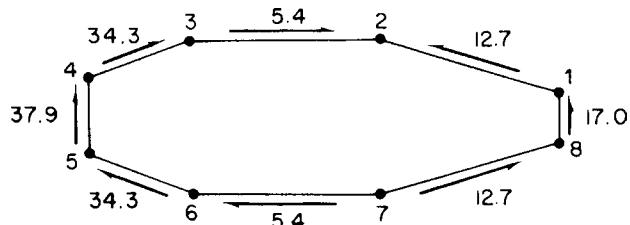


FIGURE 20.14 Shear Flow Distribution N/mm in Walls of the Beam Section of Example 20.6

### 20.3.4 Alternative method for the calculation of shear flow distribution

Equation (20.4) may be rewritten in the form

$$q_2 - q_1 = \frac{-\partial P_r}{\partial z} \quad (20.12)$$

in which  $P_r$  is the direct load in the  $r$ th boom. This form of the equation suggests an alternative approach to the determination of the effect of booms on the calculation of shear flow distributions in open and closed section beams.

Let us suppose that the boom load varies linearly with  $z$ . This is the case for a length of beam over which the shear force is constant. Equation (20.12) then becomes

$$q_2 - q_1 = -\Delta P_r \quad (20.13)$$

in which  $\Delta P_r$  is the *change* in boom load over unit length of the  $r$ th boom.  $\Delta P_r$  may be calculated by first determining the *change* in bending moment between two sections of a beam a unit distance apart and then calculating the corresponding change in boom stress using either Eq. (16.17) or (16.18); the change in boom load follows by multiplying the change in boom stress by the boom area  $B_r$ . Note that the section properties contained in Eqs. (16.17) and (16.18) refer to the direct stress carrying area of the beam section. In cases where the shear force is not constant over the unit length of beam, the method is approximate.

We illustrate the method by applying it to Example 20.5. In Fig. 20.9, the shear load of 4.8 kN is applied to the face of the section, which is seen when a view is taken along the  $z$  axis toward the origin. Thus, when considering unit length of the beam, we must ensure that this situation is unchanged. Figure 20.15 shows a unit (say, 1 mm) length of beam. The change in bending moment between the front and rear faces of the length of beam is  $4.8 \times 1$  kNmm, which produces a change in boom load given by (see Eq. (16.9))

$$\Delta P_r = \sigma_z B_r = \frac{4.8 \times 10^3 \times 200}{48 \times 10^6} \times 300 = 6 \text{ kN}$$

The change in boom load is compressive in booms 1 and 2 and tensile in booms 3 and 4.

Equation (20.12), and hence Eq. (20.13), is based on the tensile load in a boom increasing with increasing  $z$ . If the tensile load increases with decreasing  $z$ , the right-hand side of these equations is positive. It follows that in the case where a compressive load increases with decreasing  $z$ , as for booms

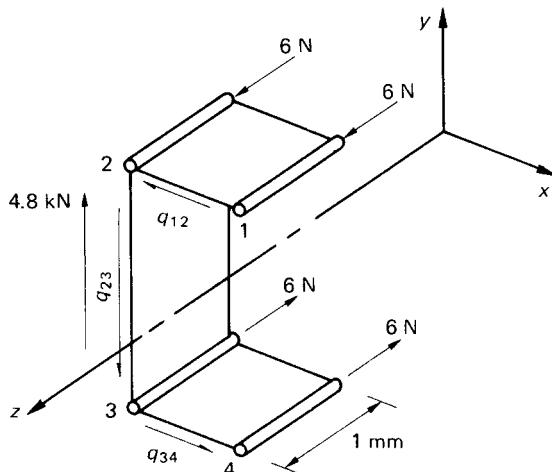


FIGURE 20.15 Alternative Solution to Example 20.5

1 and 2 in Fig. 20.15, the right-hand side is negative; similarly for booms 3 and 4, the right-hand side is positive. Thus,

$$\begin{aligned}q_{12} &= -6 \text{ N/mm} \\q_{23} &= -6 + q_{12} = -12 \text{ N/mm}\end{aligned}$$

and

$$q_{34} = +6 + q_{23} = -6 \text{ N/mm}$$

giving the same solution as before. Note that, if the unit length of beam had been taken to be 1 m, the solution would have been  $q_{12} = -6000 \text{ N/m}$ ,  $q_{23} = -12000 \text{ N/m}$ ,  $q_{34} = -6000 \text{ N/m}$ .

### 20.3.5 Torsion of open and closed section beams

No direct stresses are developed in either open or closed section beams subjected to a pure torque unless axial constraints are present. The shear stress distribution is therefore unaffected by the presence of booms and the analyses presented in Chapter 18 apply.

---

## 20.4 DEFLECTION OF OPEN AND CLOSED SECTION BEAMS

Bending, shear, and torsional deflections of thin-walled beams are readily obtained by application of the unit load method described in Section 5.5.

The displacement in a given direction due to torsion is given directly by the last of Eqs. (5.21), thus,

$$\Delta_T = \int_L \frac{T_0 T_1}{GJ} dz \quad (20.14)$$

where  $J$ , the torsion constant, depends on the type of beam under consideration. For an open section beam,  $J$  is given by either of Eqs. (18.11), whereas in the case of a closed section beam,  $J = 4A^2/(\oint ds/t)$  (Eq. (18.4)) for a constant shear modulus.

Expressions for the bending and shear displacements of unsymmetrical thin-walled beams may also be determined by the unit load method. They are complex for the general case and are most easily derived from first principles by considering the complementary energy of the elastic body in terms of stresses and strains rather than loads and displacements. In Chapter 5, we observed that the theorem of the principle of the stationary value of the total complementary energy of an elastic system is equivalent to the application of the principle of virtual work where virtual forces act through real displacements. We may therefore specify that, in our expression for total complementary energy, the displacements are the actual displacements produced by the applied loads, while the virtual force system is the unit load.

Considering deflections due to bending, we see, from Eq. (5.6), that the increment in total complementary energy due to the application of a virtual unit load is

$$-\int_L \left( \int_A \sigma_{z,1} \varepsilon_{z,0} dA \right) dz + 1\Delta_M$$

where  $\sigma_{z,1}$  is the direct bending stress at any point in the beam cross-section corresponding to the unit load and  $\varepsilon_{z,0}$  is the strain at the point produced by the actual loading system. Further,  $\Delta_M$  is the actual displacement due to bending at the point of application and in the direction of the unit load. Since the system is in equilibrium under the action of the unit load this expression must equal zero (see Eq. (5.6)). Hence,

$$\Delta_M = \int_L \left( \int_A \sigma_{z,1} \varepsilon_{z,0} dA \right) dz \quad (20.15)$$

From Eq. (16.17) and the third of Eqs. (1.42),

$$\begin{aligned} \sigma_{z,1} &= \left( \frac{M_{y,1}I_{xx} - M_{x,1}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) x + \left( \frac{M_{x,1}I_{yy} - M_{y,1}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) y \\ \varepsilon_{z,0} &= \frac{1}{E} \left[ \left( \frac{M_{y,0}I_{xx} - M_{x,0}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) x + \left( \frac{M_{x,0}I_{yy} - M_{y,0}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) y \right] \end{aligned}$$

where the suffixes 1 and 0 refer to the unit and actual loading systems and  $x, y$  are the coordinates of any point in the cross-section referred to a centroidal system of axes. Substituting for  $\sigma_{z,1}$  and  $\varepsilon_{z,0}$  in Eq. (20.15) and remembering that  $\int_A x^2 dA = I_{yy}$ ,  $\int_A y^2 dA = I_{xx}$ , and  $\int_A xydA = I_{xy}$ , we have

$$\begin{aligned} \Delta_M &= \frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)^2} \int_L \{ (M_{y,1}I_{xx} - M_{x,1}I_{xy})(M_{y,0}I_{xx} - M_{x,0}I_{xy})I_{yy} \\ &\quad + (M_{x,1}I_{yy} - M_{y,1}I_{xy})(M_{x,0}I_{yy} - M_{y,0}I_{xy})I_{xx} \\ &\quad + [(M_{y,1}I_{xx} - M_{x,1}I_{xy})(M_{x,0}I_{yy} - M_{y,0}I_{xy}) \\ &\quad + (M_{x,1}I_{yy} - M_{y,1}I_{xy})(M_{y,0}I_{xx} - M_{x,0}I_{xy})]I_{xy} \} dz \end{aligned} \quad (20.16)$$

For a section having either the  $x$  or  $y$  axis as an axis of symmetry,  $I_{xy} = 0$  and Eq. (20.16) reduces to

$$\Delta_M = \frac{1}{E} \int_L \left( \frac{M_{y,1}M_{y,0}}{I_{yy}} + \frac{M_{x,1}M_{x,0}}{I_{xx}} \right) dz \quad (20.17)$$

The derivation of an expression for the shear deflection of thin-walled sections by the unit load method is achieved in a similar manner. By comparison with Eq. (20.15), we deduce that the deflection  $\Delta_S$ , due to shear of a thin-walled open or closed section beam of thickness  $t$ , is given by

$$\Delta_S = \int_L \left( \int_{\text{sect}} \tau_1 \gamma_0 t \, ds \right) dz \quad (20.18)$$

where  $\tau_1$  is the shear stress at an arbitrary point  $s$  around the section produced by a unit load applied at the point and in the direction  $\Delta_S$ , and  $\gamma_0$  is the shear strain at the arbitrary point corresponding to the actual loading system. The integral in parentheses is taken over all the walls of the beam. In fact, both the applied and unit shear loads must act through the shear center of the cross-section, otherwise additional torsional displacements occur. Where shear loads act at other points, these must be replaced by shear loads at the shear center plus a torque. The thickness  $t$  is the actual skin thickness and may vary around the cross-section but is assumed to be constant along the length of the beam. Rewriting Eq. (20.18) in terms of shear flows  $q_1$  and  $q_0$ , we obtain

$$\Delta_S = \int_L \left( \int_{\text{sect}} \frac{q_0 q_1}{Gt} \, ds \right) dz \quad (20.19)$$

where again the suffixes refer to the actual and unit loading systems. In the cases of both open and closed section beams, the general expressions for shear flow are long and are best evaluated before substituting in Eq. (20.19). For an open section beam comprising booms and walls of direct stress carrying thickness  $t_D$ , we have, from Eq. (20.6),

$$q_0 = - \left( \frac{S_{x,0}I_{xx} - S_{y,0}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D x \, ds + \sum_{r=1}^n B_r x_r \right) - \left( \frac{S_{y,0}I_{yy} - S_{x,0}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D y \, ds + \sum_{r=1}^n B_r y_r \right) \quad (20.20)$$

and

$$q_1 = - \left( \frac{S_{x,1}I_{xx} - S_{y,1}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D x \, ds + \sum_{r=1}^n B_r x_r \right) - \left( \frac{S_{y,1}I_{yy} - S_{x,1}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D y \, ds + \sum_{r=1}^n B_r y_r \right) \quad (20.21)$$

Similar expressions are obtained for a closed section beam from Eq. (20.11).

**Example 20.7**

Calculate the deflection of the free end of a cantilever 2000 mm long having a channel section identical to that in [Example 20.5](#) and supporting a vertical, upward load of 4.8 kN acting through the shear center of the section. The effective direct stress carrying thickness of the skin is zero while its actual thickness is 1 mm. Young's modulus  $E$  and the shear modulus  $G$  are 70,000 and 30,000 N/mm<sup>2</sup>, respectively.

The section is doubly symmetrical (i.e., the direct stress carrying area) and supports a vertical load producing a vertical deflection. Thus, we apply a unit load through the shear center of the section at the tip of the cantilever and in the same direction as the applied load. Since the load is applied through the shear center, there is no twisting of the section and the total deflection is given, from [Eqs. \(20.17\), \(20.19\), \(20.20\)](#), and [\(20.21\)](#), by

$$\Delta = \int_0^L \frac{M_{x,0}M_{x,1}}{EI_{xx}} dz + \int_0^L \left( \int_{\text{sect}} \frac{q_0 q_1}{Gt} ds \right) dz \quad (\text{i})$$

where  $M_{x,0} = -4.8 \times 10^3 (2000 - z)$ ,  $M_{x,1} = -1(2000 - z)$ ,

$$q_0 = -\frac{4.8 \times 10^3}{I_{xx}} \sum_{r=1}^n B_r y_r, \quad q_1 = -\frac{1}{I_{xx}} \sum_{r=1}^n B_r y_r$$

and  $z$  is measured from the built-in end of the cantilever. The actual shear flow distribution was calculated in [Example 20.5](#). In this case, the  $q_1$  shear flows may be deduced from the actual distribution shown in [Fig. 20.10](#), that is,

$$q_1 = q_0 / 4.8 \times 10^3$$

Evaluating the bending deflection, we have

$$\Delta_M = \int_0^{2000} \frac{4.8 \times 10^3 (2,000 - z)^2}{70,000 \times 48 \times 10^6} dz = 3.81 \text{ mm}$$

The shear deflection  $\Delta_S$  is given by

$$\begin{aligned} \Delta_S &= \int_0^{2000} \frac{1}{30,000 \times 1} \left[ \frac{1}{4.8 \times 10^3} (6^2 \times 200 + 12^2 \times 400 + 6^2 \times 200) \right] dz \\ &= 1.0 \text{ mm} \end{aligned}$$

The total deflection  $\Delta$  is then  $\Delta_M + \Delta_S = 4.81$  mm in a vertical upward direction.

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## PROBLEMS

**P.20.1.** Idealize the box section shown in [Fig. P.20.1](#) into an arrangement of direct stress carrying booms positioned at the four corners and panels that are assumed to carry only shear stresses. Hence, determine the distance of the shear center from the left-hand web.

Answer: 225 mm

**P.20.1. MATLAB** Assuming the width of the section shown in Fig. P.20.1 is labeled  $L_1$ , use MATLAB to repeat Problem P.20.1 for values of  $L_1$  from 300 to 700 mm in increments of 50 mm.

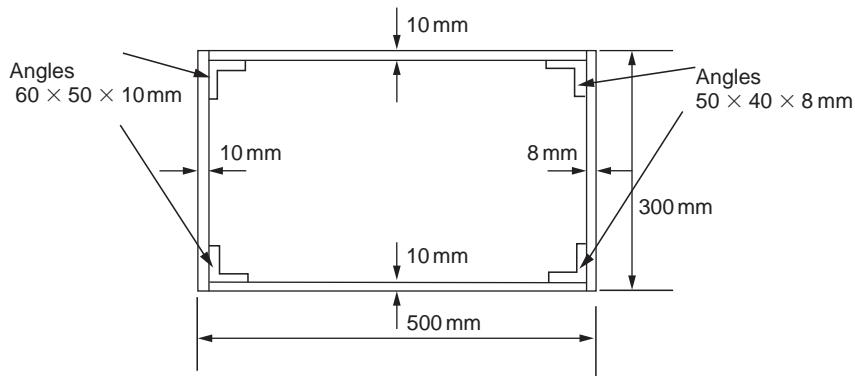


FIGURE P.20.1

- Answer:
- (i)  $L_1 = 300 \text{ mm}$ : 133 mm from left-hand web
  - (ii)  $L_1 = 350 \text{ mm}$ : 156 mm from left-hand web
  - (iii)  $L_1 = 400 \text{ mm}$ : 178 mm from left-hand web
  - (iv)  $L_1 = 450 \text{ mm}$ : 201 mm from left-hand web
  - (v)  $L_1 = 500 \text{ mm}$ : 225 mm from left-hand web
  - (vi)  $L_1 = 550 \text{ mm}$ : 248 mm from left-hand web
  - (vii)  $L_1 = 600 \text{ mm}$ : 271 mm from left-hand web
  - (viii)  $L_1 = 650 \text{ mm}$ : 295 mm from left-hand web
  - (ix)  $L_1 = 700 \text{ mm}$ : 319 mm from left-hand web

**P.20.2.** The beam section shown in Fig. P.20.2 has been idealized into an arrangement of direct stress carrying booms and shear stress only carrying panels. If the beam section is subjected to a vertical shear load of 1,495 N through its shear center, each of booms 1, 4, 5, and 8 has an area of  $200 \text{ mm}^2$ , and each of booms 2, 3, 6, and 7 has an area of  $250 \text{ mm}^2$ , determine the shear flow distribution and the position of the shear center.

Answer: Wall 12, 1.86 N/mm; 43, 1.49 N/mm; 32, 5.21 N/mm; 27, 10.79 N/mm; remaining distribution follows from symmetry. 122 mm to the left of the web 27

**P.20.3** Figure P.20.3 shows the cross-section of a single-cell, thin-walled beam with a horizontal axis of symmetry. The direct stresses are carried by the booms  $B_1$  to  $B_4$ , while the walls are effective only in carrying shear stresses. Assuming that the basic theory of bending is applicable, calculate the position of the shear center S. The shear modulus  $G$  is the same for all walls:

$$\text{Cell area} = 135,000 \text{ mm}^2; \text{Boom areas: } B_1 = B_4 = 450 \text{ mm}^2, B_2 = B_3 = 550 \text{ mm}^2$$

Wall	Length (mm)	Thickness (mm)
12, 34	500	0.8
23	580	1.0
41	200	1.2

Answer: 197.2 mm to the right of the vertical through booms 2 and 3

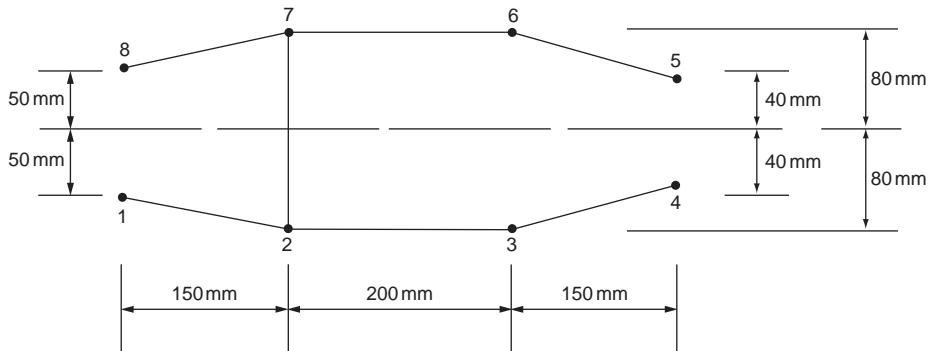


FIGURE P.20.2

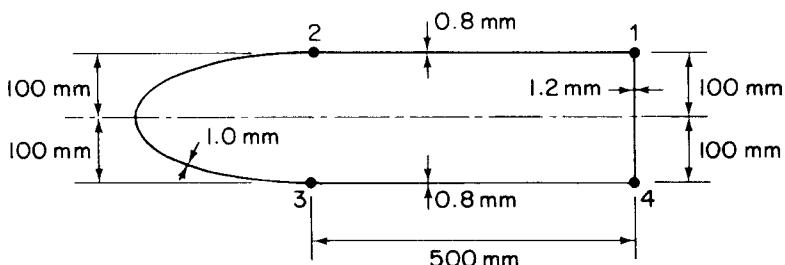


FIGURE P.20.3

### P.20.3 MATLAB

Use MATLAB to repeat Problem P.20.3 for the following boom area combinations:

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
$B_1$ ( $\text{mm}^2$ )	550	450	550	450	550	450
$B_2$ ( $\text{mm}^2$ )	550	450	450	550	450	550
$B_3$ ( $\text{mm}^2$ )	450	550	550	450	450	550
$B_4$ ( $\text{mm}^2$ )	450	550	450	550	550	450

- Answer:
- 197.2 mm from vertical through booms 2 and 3
  - 230.9 mm from vertical through booms 2 and 3
  - 215.4 mm from vertical through booms 2 and 3
  - 215.4 mm from vertical through booms 2 and 3
  - 209.9 mm from vertical through booms 2 and 3
  - 218.2 mm from vertical through booms 2 and 3

**P.20.4** Find the position of the shear center of the rectangular four boom beam section shown in Fig. P.20.4. The booms carry only direct stresses, but the skin is fully effective in carrying both shear and direct stress. The area of each boom is  $100 \text{ mm}^2$ .

Answer: 142.5 mm from side 23

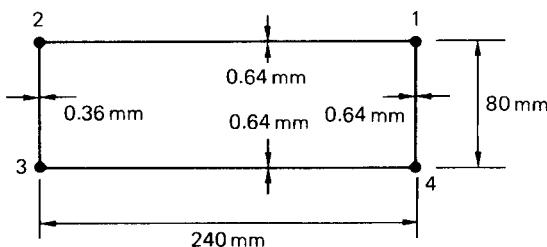
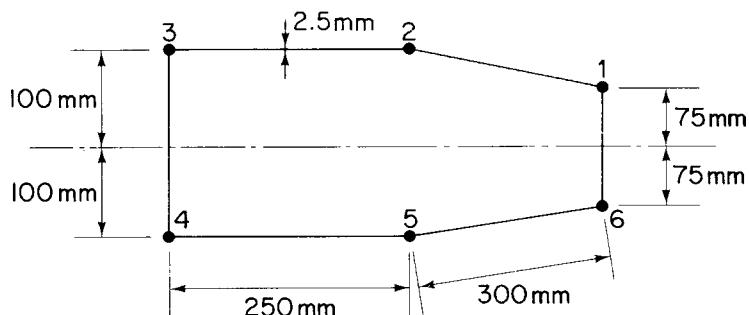
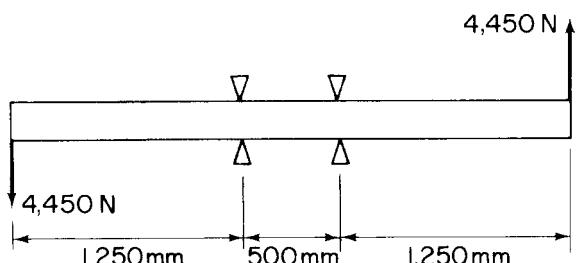


FIGURE P.20.4



(a)



(b)

FIGURE P.20.5

**P.20.5** A uniform beam with the cross-section shown in Fig. P.20.5(a) is supported and loaded as shown in Fig. P.20.5(b). If the direct and shear stresses are given by the basic theory of bending, the direct stresses being carried by the booms and the shear stresses by the walls, calculate the vertical deflection at the ends of the beam when the loads act through the shear centers of the end cross-sections, allowing for the effect of shear strains. Take  $E = 69,000 \text{ N/mm}^2$  and  $G = 26,700 \text{ N/mm}^2$ . Boom areas: 1, 3, 4, 6 =  $650 \text{ mm}^2$ ; 2, 5 =  $1,300 \text{ mm}^2$ .

Answer: 3.4 mm

**P.20.6** A cantilever, length  $L$ , has a hollow cross-section in the form of a doubly symmetric wedge, as shown in Fig. P.20.6. The chord line is of length  $c$ , wedge thickness is  $t$ , the length of a sloping side is  $a/2$ , and the wall thickness is constant and equal to  $t_0$ . Uniform pressure distributions of the magnitudes shown act on the faces of the wedge. Find the vertical deflection of point A due to this given loading. If  $G = 0.4E$ ,  $t/c = 0.05$ , and  $L = 2c$ , show that this deflection is approximately  $5,600p_0c^2/Et_0$ .

**P.20.7** A rectangular section thin-walled beam of length  $L$  and breadth  $3b$ , depth  $b$ , and wall thickness  $t$  is built-in at one end (Fig. P.20.7). The upper surface of the beam is subjected to a pressure which varies linearly across the breadth from a value  $p_0$  at edge AB to zero at edge CD. Note that the complete upper surface of the beam is subjected to this pressure. Find the vertical deflection of point A.

Answer:  $p_0L^2(9L^2/80Eb^2 + 1,609/2,000G)/t$

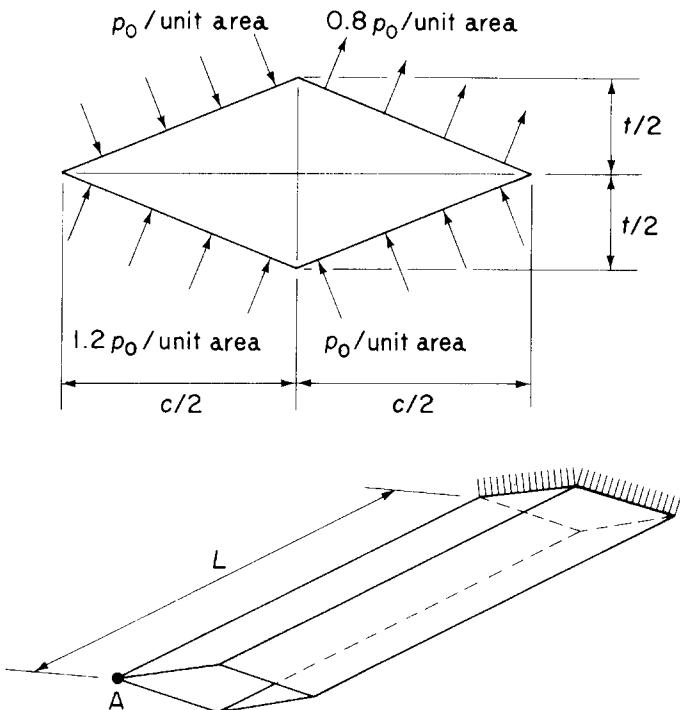


FIGURE P.20.6

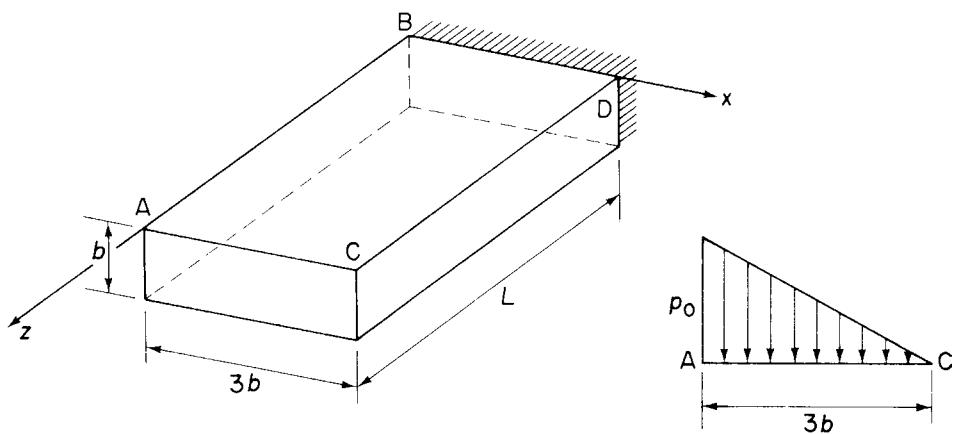


FIGURE P.20.7

# Wing spars and box beams

# 21

In [Chapters 16–18](#), we established the basic theory for the analysis of open and closed section thin-walled beams subjected to bending, shear, and torsional loads. In addition, in [Chapter 20](#), we saw how complex stringer stiffened sections could be idealized into sections more amenable to analysis. We now extend this analysis to actual aircraft components, including, in this chapter, wing spars and box beams. In subsequent chapters, we investigate the analysis of fuselages, wings, frames, and ribs and consider the effects of cut-outs in wings and fuselages. Also, in [Chapter 25](#) we shall briefly examine spacecraft structures and finally, in Chapter 26, introduce the analysis of components fabricated from composite materials.

Aircraft structural components are, as we saw in [Chapter 12](#), complex, consisting usually of thin sheets of metal stiffened by arrangements of stringers. These structures are highly redundant and require some degree of simplification or idealization before they can be analyzed. The analysis presented here is therefore approximate, and the degree of accuracy obtained depends on the number of simplifying assumptions made. A further complication arises in that factors such as warping restraint, structural and loading discontinuities, and shear lag significantly affect the analysis; we investigate these effects in some simple structural components in [Chapters 27 and 28](#). Generally, a high degree of accuracy can be obtained only by using computer-based techniques, such as the finite element method (see [Chapter 6](#)). However, the simpler, quicker, and cheaper approximate methods can be used to advantage in the preliminary stages of design, when several possible structural alternatives are being investigated; they also provide an insight into the physical behavior of structures which computer-based techniques do not.

Major aircraft structural components such as wings and fuselages are usually tapered along their lengths for greater structural efficiency. Thus, wing sections are reduced both chordwise and in depth along the wing span toward the tip and fuselage sections aft of the passenger cabin taper to provide a more efficient aerodynamic and structural shape.

The analysis of open and closed section beams, presented in [Chapters 16–18](#), assumes that the beam sections are uniform. The effect of taper on the prediction of direct stresses produced by bending is minimal if the taper is small and the section properties are calculated at the particular section being considered; [Eqs. \(16.17\)–\(16.21\)](#) may therefore be used with reasonable accuracy. On the other hand, the calculation of shear stresses in beam webs can be significantly affected by taper.

## 21.1 TAPERED WING SPAR

Consider first the simple case of a beam, for example, a wing spar, positioned in the  $yz$  plane and comprising two flanges and a web: an elemental length  $\delta z$  of the beam is shown in [Fig. 21.1](#). At the section  $z$ , the beam is subjected to a positive bending moment  $M_x$  and a positive shear force  $S_y$ . The bending

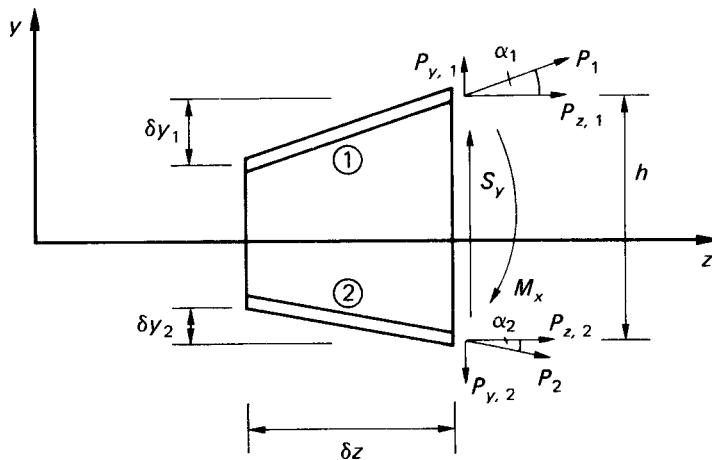


FIGURE 21.1 Effect of Taper on Beam Analysis

moment resultants  $P_{z,1}$  and  $P_{z,2}$  are parallel to the  $z$  axis of the beam. For a beam in which the flanges are assumed to resist all the direct stresses,  $P_{z,1} = M_x/h$  and  $P_{z,2} = -M_x/h$ . In the case where the web is assumed to be fully effective in resisting direct stress,  $P_{z,1}$  and  $P_{z,2}$  are determined by multiplying the direct stresses  $\sigma_{z,1}$  and  $\sigma_{z,2}$ , found using Eq. (16.17) or (16.18), by the flange areas  $B_1$  and  $B_2$ .  $P_{z,1}$  and  $P_{z,2}$  are the components in the  $z$  direction of the axial loads  $P_1$  and  $P_2$  in the flanges. These have components  $P_{y,1}$  and  $P_{y,2}$  parallel to the  $y$  axis given by

$$\left. \begin{aligned} P_{y,1} &= P_{z,1} \tan \alpha_1 = P_{z,1} \frac{\delta y_1}{\delta z} \\ P_{y,2} &= -P_{z,2} \tan \alpha_2 = -P_{z,2} \frac{\delta y_2}{\delta z} \end{aligned} \right\} \quad (21.1)$$

in which, for the direction of taper shown,  $\delta y_2$  is negative. The axial load in flange ① is given by

$$P_1 = (P_{z,1}^2 + P_{y,1}^2)^{1/2}$$

Substituting for  $P_{y,1}$  from Eq. (21.1), we have

$$P_1 = \left[ P_{z,1}^2 + P_{z,1}^2 \left( \frac{\delta y_1}{\delta z} \right)^2 \right]^{1/2}$$

so that

$$P_1 = P_{z,1} \frac{(\delta z^2 + \delta y_1^2)^{1/2}}{\delta z} = \frac{P_{z,1}}{\cos \alpha_1} \quad (21.2)$$

Similarly,

$$P_2 = \frac{P_{z,2}}{\cos \alpha_2} \quad (21.3)$$

It would appear that these expressions for  $P_1$  and  $P_2$  could have been found using the basic triangle of forces, that is,  $P_{z,1} = P_1 \cos \alpha_1$ . However, in this case,  $P_1$  and  $P_2$  are the unknowns so that the above approach is required.

The internal shear force  $S_y$  comprises the resultant  $S_{y,w}$  of the web shear flows together with the vertical components of  $P_1$  and  $P_2$ . Thus,

$$S_y = S_{y,w} + P_{y,1} - P_{y,2}$$

or

$$S_y = S_{y,w} + P_{z,1} \frac{\delta y_1}{\delta z} + P_{z,2} \frac{\delta y_2}{\delta z} \quad (21.4)$$

so that

$$S_{y,w} = S_y - P_{z,1} \frac{\delta y_1}{\delta z} - P_{z,2} \frac{\delta y_2}{\delta z} \quad (21.5)$$

Again we note that  $\delta y_2$  in Eqs. (21.4) and (21.5) is negative. Equation (21.5) may be used to determine the shear flow distribution in the web. For a completely idealized beam, the web shear flow is constant through the depth and is given by  $S_{y,w}/h$ . For a beam in which the web is fully effective in resisting direct stresses, the web shear flow distribution is found using Eq. (20.6) in which  $S_y$  is replaced by  $S_{y,w}$  and which, for the beam of Fig. 21.1, simplifies to

$$q_s = -\frac{S_{y,w}}{I_{xx}} \left( \int_0^s t_D y \, ds + B_1 y_1 \right) \quad (21.6)$$

or

$$q_s = -\frac{S_{y,w}}{I_{xx}} \left( \int_0^s t_D y \, ds + B_2 y_2 \right) \quad (21.7)$$

### Example 21.1

Determine the shear flow distribution in the web of the tapered beam shown in Fig. 21.2, at a section midway along its length. The web of the beam has a thickness of 2 mm and is fully effective in resisting direct stress. The beam tapers symmetrically about its horizontal centroidal axis and the cross-sectional area of each flange is  $400 \text{ mm}^2$ .

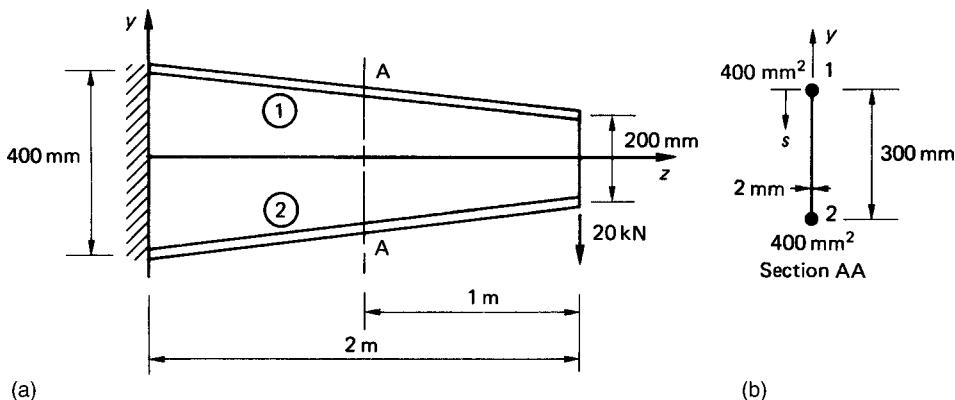


FIGURE 21.2 Tapered Beam of Example 21.1

The internal bending moment and shear load at the section AA produced by the externally applied load are, respectively,

$$M_x = 20 \times 1 = 20 \text{ kNm}, S_y = -20 \text{ kN}$$

The direct stresses parallel to the  $z$  axis in the flanges at this section are obtained from either Eq. (16.17) or (16.18) in which  $M_y = 0$  and  $I_{xy} = 0$ . Thus, from Eq. (16.17),

$$\sigma_z = \frac{M_x y}{I_{xx}} \quad (\text{i})$$

in which

$$I_{xx} = 2 \times 400 \times 150^2 + 2 \times 300^3 / 12$$

that is,

$$I_{xx} = 22.5 \times 10^6 \text{ mm}^4$$

Hence,

$$\sigma_{z,1} = -\sigma_{z,2} = \frac{20 \times 10^6 \times 150}{22.5 \times 10^6} = 133.3 \text{ N/mm}^2$$

The components parallel to the  $z$  axis of the axial loads in the flanges are therefore

$$P_{z,1} = -P_{z,2} = 133.3 \times 400 = 53,320 \text{ N}$$

The shear load resisted by the beam web is then, from Eq. (21.5),

$$S_{y,w} = -20 \times 10^3 - 53,320 \frac{\delta y_1}{\delta z} + 53,320 \frac{\delta y_2}{\delta z}$$

in which, from Figs. 21.1 and 21.2, we see that

$$\frac{\delta y_1}{\delta z} = \frac{-100}{2 \times 10^3} = -0.05, \quad \frac{\delta y_2}{\delta z} = \frac{100}{2 \times 10^3} = 0.05$$

Hence,

$$S_{y,w} = -20 \times 10^3 + 53,320 \times 0.05 + 53,320 \times 0.05 = -14,668 \text{ N}$$

The shear flow distribution in the web follows from either Eq. (21.6) or (21.7) and is (see Fig. 21.2(b))

$$q_{12} = \frac{14,668}{22.5 \times 10^6} \left( \int_0^s 2(150 - s) ds + 400 \times 150 \right)$$

that is,

$$q_{12} = 6.52 \times 10^{-4} (-s^2 + 300s + 60,000) \quad (\text{ii})$$

The maximum value of  $q_{12}$  occurs when  $s = 150$  mm and  $q_{12}(\text{max}) = 53.8 \text{ N/mm}$ . The values of shear flow at points 1 ( $s = 0$ ) and 2 ( $s = 300$  mm) are  $q_1 = 39.1 \text{ N/mm}$  and  $q_2 = 39.1 \text{ N/mm}$ ; the complete distribution is shown in Fig. 21.3.

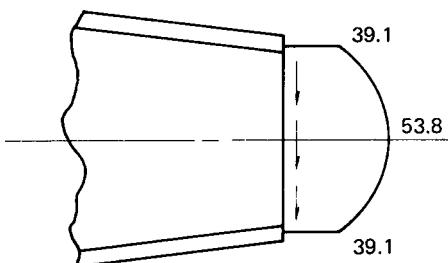


FIGURE 21.3 Shear Flow (N/mm) Distribution at Section AA in Example 21.1

## 21.2 OPEN AND CLOSED SECTION BEAMS

We now consider the more general case of a beam tapered in two directions along its length and comprising an arrangement of booms and skin. Practical examples of such a beam are complete wings and fuselages; note that in addition to the effects of taper, the method allows for sweep of a wing, which occurs in modern high speed jet aircraft (see Fig. 12.6). The beam may be of open or closed section; the effects of taper are determined in an identical manner in either case.

Figure 21.4(a) shows a short length  $\delta z$  of a beam carrying shear loads  $S_x$  and  $S_y$  at the section  $z$ ;  $S_x$ , and  $S_y$  are positive when acting in the directions shown. Note that if the beam is of open cross-section, the shear loads are applied through its shear center, so that no twisting of the beam occurs. In addition to shear loads, the beam is subjected to bending moments  $M_x$  and  $M_y$ , which produce direct stresses  $\sigma_z$  in the booms and skin. Suppose that, in the  $r$ th boom, the direct stress in a direction parallel to the  $z$  axis is  $\sigma_{z,r}$ , which may be found using either Eq. (16.17) or (16.18). The component  $P_{z,r}$  of the axial load  $P_r$  in the  $r$ th boom is then given by

$$P_{z,r} = \sigma_{z,r} B_r \quad (21.8)$$

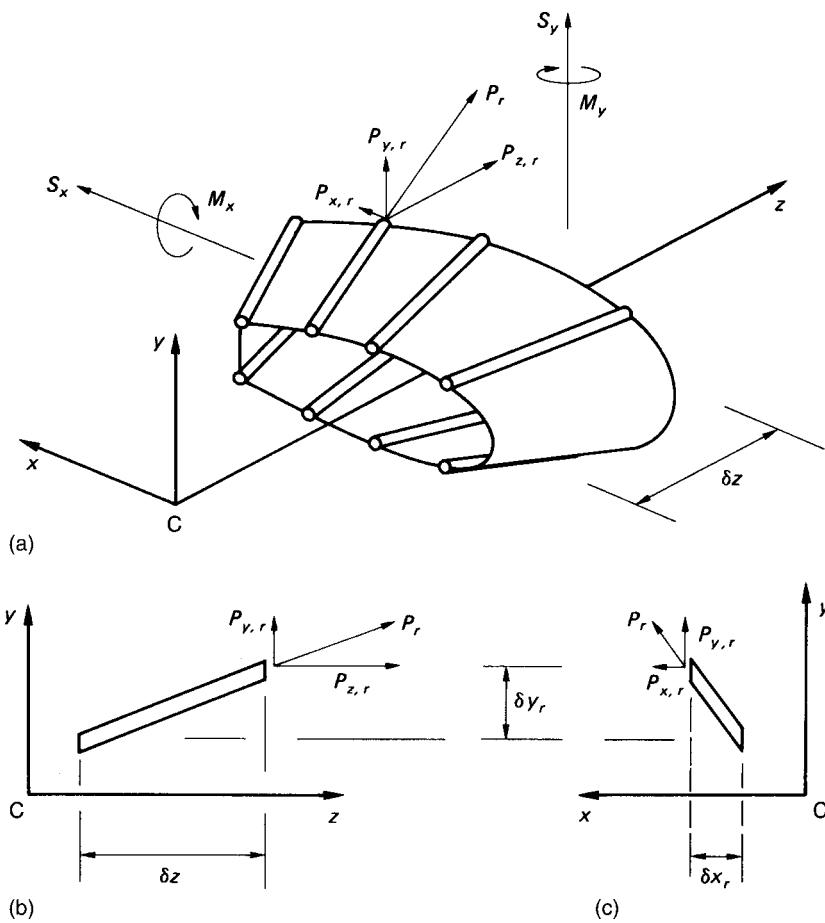


FIGURE 21.4 Effect of Taper on the Analysis of Open and Closed Section Beams

where  $B_r$  is the cross-sectional area of the  $r$ th boom.

From Fig. 21.4(b),

$$P_{y,r} = P_{z,r} \frac{\delta y_r}{\delta z} \quad (21.9)$$

Further, from Fig. 21.4(c),

$$P_{x,r} = P_{y,r} \frac{\delta x_r}{\delta y_r}$$

or, substituting for  $P_{y,r}$  from Eq. (21.9),

$$P_{x,r} = P_{z,r} \frac{\delta x_r}{\delta z} \quad (21.10)$$

The axial load  $P_r$  is then given by

$$P_r = \left( P_{x,r}^2 + P_{y,r}^2 + P_{z,r}^2 \right)^{1/2} \quad (21.11)$$

or, alternatively,

$$P_r = P_{z,r} \frac{(\delta x_r^2 + \delta y_r^2 + \delta z^2)^{1/2}}{\delta z} \quad (\text{see derivation of Eq. (21.2)}) \quad (21.12)$$

The applied shear loads  $S_x$  and  $S_y$  are reacted by the resultants of the shear flows in the skin panels and webs, together with the components  $P_{x,r}$  and  $P_{y,r}$  of the axial loads in the booms. Therefore, if  $S_{x,w}$  and  $S_{y,w}$  are the resultants of the skin and web shear flows and there is a total of  $m$  booms in the section,

$$S_x = S_{x,w} + \sum_{r=1}^m P_{x,r}, \quad S_y = S_{y,w} + \sum_{r=1}^m P_{y,r} \quad (21.13)$$

Substituting in Eq. (21.13) for  $P_{x,r}$  and  $P_{y,r}$  from Eqs. (21.10) and (21.9), we have

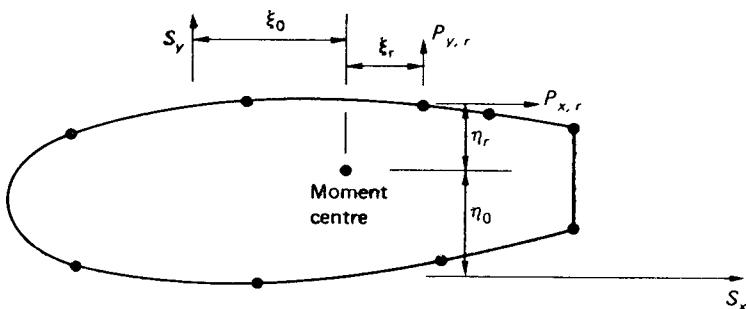
$$S_x = S_{x,w} + \sum_{r=1}^m P_{z,r} \frac{\delta x_r}{\delta z}, \quad S_y = S_{y,w} + \sum_{r=1}^m P_{z,r} \frac{\delta y_r}{\delta z} \quad (21.14)$$

Hence,

$$S_{x,w} = S_x - \sum_{r=1}^m P_{z,r} \frac{\delta x_r}{\delta z}, \quad S_{y,w} = S_y - \sum_{r=1}^m P_{z,r} \frac{\delta y_r}{\delta z} \quad (21.15)$$

The shear flow distribution in an open section beam is now obtained using Eq. (20.6), in which  $S_x$  is replaced by  $S_{x,w}$  and  $S_y$  by  $S_{y,w}$  from Eq. (21.15). Similarly, for a closed section beam,  $S_x$  and  $S_y$  in Eq. (20.11) are replaced by  $S_{x,w}$  and  $S_{y,w}$ . In the latter case, the moment equation (Eq. (17.17)) requires modification, due to the presence of the boom load components  $P_{x,r}$  and  $P_{y,r}$ . Thus, from Fig. 21.5, we see that Eq. (17.17) becomes

$$S_x \eta_0 - S_y \xi_0 = \oint q_b p \, ds + 2Aq_{s,0} - \sum_{r=1}^m P_{x,r} \eta_r + \sum_{r=1}^m P_{y,r} \xi_r \quad (21.16)$$



**FIGURE 21.5 Modification of Moment Equation in Shear of Closed Section Beams Due to Boom Load**

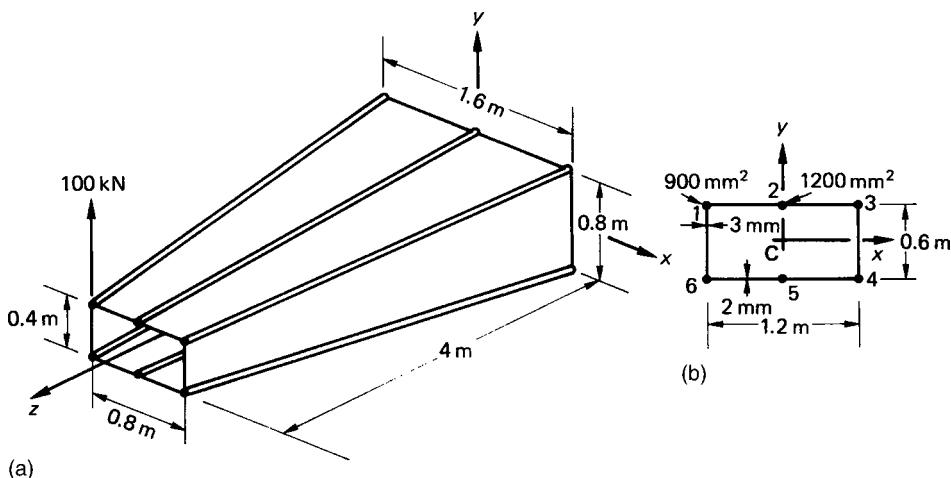
Equation (21.16) is directly applicable to a tapered beam subjected to forces positioned in relation to the moment centre, as shown. Care must be taken in a particular problem to ensure that the moments of the forces are given the correct sign.

### Example 21.2

The cantilever beam shown in Fig. 21.6 is uniformly tapered along its length in both  $x$  and  $y$  directions and carries a load of 100 kN at its free end. Calculate the forces in the booms and the shear flow distribution in the walls at a section 2 m from the built-in end if the booms resist all the direct stresses while the walls are effective only in shear. Each corner boom has a cross-sectional area of  $900 \text{ mm}^2$ , while both central booms have cross-sectional areas of  $1,200 \text{ mm}^2$ .

The internal force system at a section 2 m from the built-in end of the beam is

$$S_y = 100 \text{ kN}, S_x = 0, M_x = -100 \times 2 = -200 \text{ kNm}, M_y = 0$$



**FIGURE 21.6 (a) Beam of Example 21.2; (b) Section 2 m From Built-in End**

The beam has a doubly symmetrical cross-section, so that  $I_{xy} = 0$  and Eq. (16.17) reduces to

$$\sigma_z = \frac{M_x y}{I_{xx}} \quad (\text{i})$$

in which, for the beam section shown in Fig. 21.6(b),

$$I_{xx} = 4 \times 900 \times 300^2 + 2 \times 1,200 \times 300^2 = 5.4 \times 10^8 \text{ mm}^4$$

Then,

$$\sigma_{z,r} = \frac{-200 \times 10^6}{5.4 \times 10^8} y_r$$

or

$$\sigma_{z,r} = -0.37 y_r \quad (\text{ii})$$

Hence,

$$P_{z,r} = -0.37 y_r B_r \quad (\text{iii})$$

The value of  $P_{z,r}$  is calculated from Eq. (iii) in column ② of Table 21.1;  $P_{x,r}$  and  $P_{y,r}$  follow from Eqs. (21.10) and (21.9), respectively, in columns ⑤ and ⑥. The axial load  $P_r$ , column ⑦, is given by  $[②^2 + ⑤^2 + ⑥^2]^{1/2}$  and has the same sign as  $P_{z,r}$  (see Eq. (21.12)). The moments of  $P_{x,r}$  and  $P_{y,r}$  are calculated for a moment center at the center of symmetry with counterclockwise moments taken as positive. Note that, in Table 21.1,  $P_{x,r}$  and  $P_{y,r}$  are positive when they act in the positive directions of the section  $x$  and  $y$  axes, respectively; the distances  $\eta_r$  and  $\xi_r$  of the lines of action of  $P_{x,r}$  and  $P_{y,r}$  from the moment center are not given signs, since it is simpler to determine the sign of each moment,  $P_{x,r}\eta_r$  and  $P_{y,r}\xi_r$ , by referring to the directions of  $P_{x,r}$  and  $P_{y,r}$  individually.

From column ⑥,

$$\sum_{r=1}^6 P_{y,r} = 33.4 \text{ kN}$$

From column ⑩,

$$\sum_{r=1}^6 P_{x,r} \eta_r = 0$$

**Table 21.1** Example 21.2

① Boom	② $P_{z,r}$ (kN)	③ $\delta x_r / \delta z$	④ $\delta y_r / \delta z$	⑤ $P_{x,r}$ (kN)	⑥ $P_{y,r}$ (kN)	⑦ $P_r$ (kN)	⑧ $\xi_r$ (m)	⑨ $\eta_r$ (m)	⑪ $P_{x,r}\eta_r$ (kNm)	⑩ $P_{y,r}\xi_r$ (kNm)
1	-100	0.1	-0.05	-10	5	-101.3	0.6	0.3	3	-3
2	-133	0	-0.05	0	6.7	-133.2	0	0.3	0	0
3	-100	-0.1	-0.05	10	5	-101.3	0.6	0.3	-3	3
4	100	-0.1	0.05	-10	5	101.3	0.6	0.3	-3	3
5	133	0	0.05	0	6.7	133.2	0	0.3	0	0
6	100	0.1	0.05	10	5	101.3	0.6	0.3	3	-3

From column ①,

$$\sum_{r=1}^6 P_{y,r} \xi_r = 0$$

From Eq. (21.15),

$$S_{x,w} = 0, \quad S_{y,w} = 100 - 33.4 = 66.6 \text{ kN}$$

The shear flow distribution in the walls of the beam is now found using the method described in Section 20.3. Since, for this beam,  $I_{xy} = 0$  and  $S_x = S_{x,w} = 0$ , Eq. (20.11) reduces to

$$q_s = \frac{-S_{y,w}}{I_{xx}} \sum_{r=1}^n B_r y_r + q_{s,0} \quad (\text{iv})$$

We now “cut” one of the walls, say 16. The resulting “open section” shear flow is given by

$$q_b = -\frac{66.6 \times 10^3}{5.4 \times 10^8} \sum_{r=1}^n B_r y_r$$

or

$$q_b = -1.23 \times 10^{-4} \sum_{r=1}^n B_r y_r \quad (\text{v})$$

Thus,

$$\begin{aligned} q_{b,16} &= 0 \\ q_{b,12} &= 0 - 1.23 \times 10^{-4} \times 900 \times 300 = -33.2 \text{ N/mm} \\ q_{b,23} &= -33.2 - 1.23 \times 10^{-4} \times 1,200 \times 300 = -77.5 \text{ N/mm} \\ q_{b,34} &= -77.5 - 1.23 \times 10^{-4} \times 900 \times 300 = -110.7 \text{ N/mm} \\ q_{b,45} &= -77.5 \text{ N/mm (from symmetry)} \\ q_{b,56} &= -33.2 \text{ N/mm (from symmetry)} \end{aligned}$$

giving the distribution shown in Fig. 21.7. Taking moments about the center of symmetry, we have, from Eq. (21.16),

$$-100 \times 10^3 \times 600 = 2 \times 33.2 \times 600 \times 300 + 2 \times 77.5 \times 600 \times 300 + 110.7 \times 600 \times 600 + 2 \times 1,200 \times 600 q_{s,0}$$

from which  $q_{s,0} = -97.0 \text{ N/mm}$  (i.e., clockwise). The complete shear flow distribution is found by adding the value of  $q_{s,0}$  to the  $q_b$  shear flow distribution of Fig. 21.7 and is shown in Fig. 21.8.

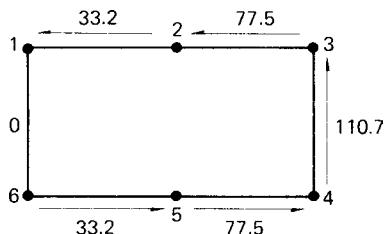


FIGURE 21.7 "Open Section" Shear Flow (N/mm) Distribution in Beam Section of Example 21.2

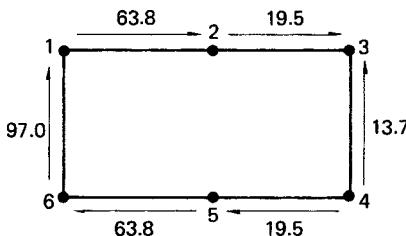


FIGURE 21.8 Shear Flow (N/mm) Distribution in Beam Section of Example 21.2

### Example 21.3

If the central vertical plane of the box beam of Example 21.2, i.e., the vertical plane through booms 2 and 5, has a sweep of  $20^\circ$  as shown in Fig. 21.9(a), calculate the forces in the booms and the shear flow distribution in the walls at a section BB, 2 m distant from the built-in end. Note that the span of the beam has been retained.

At the section 2 m from the built-in end the internal force system and second moment of area  $I_{xx}$  are identical to those in Example 21.2, i.e.,

$$S_y = 100 \text{ kN}, \quad S_x = 0, \quad M_x = -200 \text{ kNm}, \quad M_y = 0, \quad I_{xx} = 5.4 \times 10^6 \text{ mm}^4$$

Also  $P_{z,r} = -0.37y_rB_r$  as in Example 21.2.

The values of  $\delta x_r/\delta z$  are obtained by elementary trigonometry and are listed in Table 21.2 as are the values of  $\delta y_r/\delta z$ , which are identical to those in Example 21.2. The final boom loads  $P_r$  are listed in column 7.

From Table 21.2,  $\sum_{r=1}^n P_{x,r} = 0$ ,  $\sum_{r=1}^n P_{y,r} = 33.4 \text{ kN}$ ,  $\sum_{r=1}^n P_{y,r}\eta_r = -74.0 \text{ kNm}$  (i.e., counterclockwise),  $\sum_{r=1}^n P_{y,r}\xi_r = 0$ . Then  $S_{x,w} = 0$ ,  $S_{y,w} = 100 - 33.4 = 66.6 \text{ kN}$ .

The  $q_b$  shear flows are identical to those in Example 21.2. Therefore, taking moments about the center of symmetry and noting that the line of action of the externally applied load is 0.33 m to the right of the center of symmetry (again obtained from elementary trigonometry), we have

$$100 \times 10^3 \times 330 = 2 \times 33.2 \times 600 \times 300 + 2 \times 77.5 \times 600 \times 300 \\ + 110.7 \times 600 \times 600 + 74.0 \times 10^6 + 2 \times 1200 \times 600 \times q_{s,0}$$

from which  $q_{s,0} = -83.8 \text{ N/mm}$  (i.e., clockwise).

The final shear flow distribution is then as shown in Fig. 21.10.

## 21.3 BEAMS HAVING VARIABLE STRINGER AREAS

In many aircraft, structural beams, such as wings, have stringers whose cross-sectional areas vary in the spanwise direction. The effects of this variation on the determination of shear flow distribution cannot therefore be found by the methods described in Section 20.3, which assume constant boom areas. In fact, as we noted in Section 20.3, if the stringer stress is made constant by varying the area of cross-section, there is no change in shear flow, as the stringer-boom is crossed.

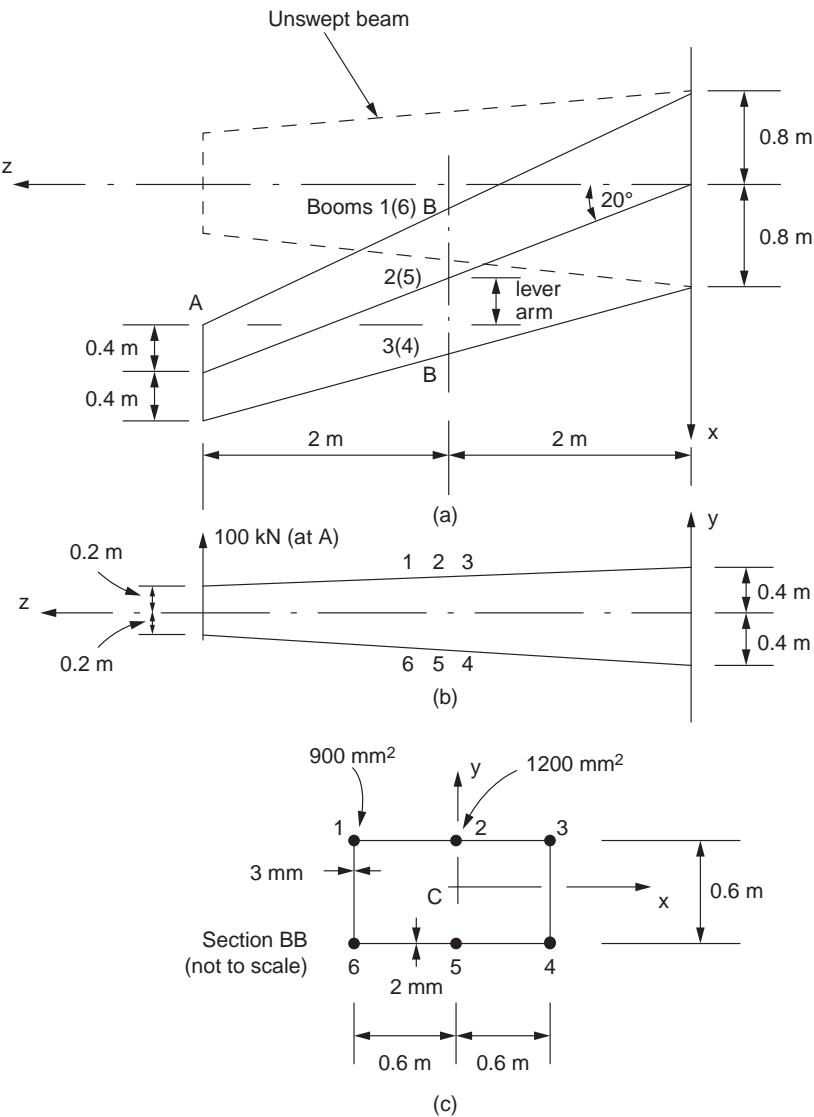
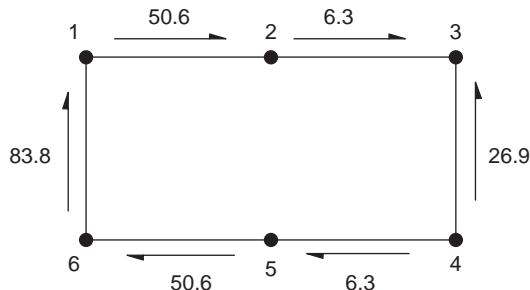


FIGURE 21.9 Box Beam of Example 21.3

Table 21.2 Example 21.3

① Boom	② $P_{z,r}$ (kN)	③ $\delta x_r / \delta z$	④ $\delta y_r / \delta z$	⑤ $P_{x,r}$ (kN)	⑥ $P_{y,r}$ (kN)	⑦ $P_r$ (kN)	⑧ $\xi_r$ (m)	⑨ $\eta_r$ (m)	⑩ $P_{x,r}\eta_r$ (kNm)	⑪ $P_{y,r}\xi_r$ (kNm)
1	-100	0.47	-0.05	-47.0	5.0	-110.6	-0.6	0.3	-14.1	-3
2	-133	0.37	-0.05	-49.2	6.7	-142.0	0	0.3	-14.8	0
3	-100	0.27	-0.05	-27.0	5.0	-103.7	0.6	0.3	-8.1	3
4	100	0.27	0.05	27.0	5.0	103.7	0.6	-0.3	-8.1	3
5	133	0.37	0.05	49.2	6.7	142.0	0	-0.3	-14.8	0
6	100	0.47	0.05	47.0	5.0	110.6	-0.6	-0.3	-14.1	-3

**FIGURE 21.10 Shear Flow (N/mm) Distribution at Section BB in Box Beam of Example 21.3**

The calculation of shear flow distributions in beams having variable stringer areas is based on the alternative method for the calculation of shear flow distributions described in [Section 20.3](#) and illustrated in the alternative solution to [Example 20.3](#). The stringer loads  $P_{z,1}$  and  $P_{z,2}$  are calculated at two sections  $z_1$  and  $z_2$  of the beam a convenient distance apart. We assume that the stringer load varies linearly along its length so that the change in stringer load per unit length of beam is given by

$$\Delta P = \frac{P_{z,1} - P_{z,2}}{z_1 - z_2}$$

The shear flow distribution follows as previously described.

### Example 21.4

Solve [Example 21.2](#) by considering the differences in boom load at sections of the beam either side of the specified section.

In this example, the stringer areas do not vary along the length of the beam, but the method of solution is identical.

We are required to find the shear flow distribution at a section 2 m from the built-in end of the beam. We therefore calculate the boom loads at sections, say, 0.1 m either side of this section. Thus, at a distance 2.1 m from the built-in end,

$$M_x = -100 \times 1.9 = -190 \text{ kNm}$$

The dimensions of this section are easily found by proportion and are width = 1.18 m, depth = 0.59 m. Therefore, the second moment of area is

$$I_{xx} = 4 \times 900 \times 295^2 + 2 \times 1200 \times 295^2 = 5.22 \times 10^8 \text{ mm}^4$$

and

$$\sigma_{z,r} = \frac{-190 \times 10^6}{5.22 \times 10^8} y_r = -0.364 y_r$$

Hence,

$$P_1 = P_3 = -P_4 = -P_6 = -0.364 \times 295 \times 900 = -96,642 \text{ N}$$

and

$$P_2 = -P_5 = -0.364 \times 295 \times 1200 = -128,856 \text{ N}$$

At a section 1.9 m from the built-in end,

$$M_x = -100 \times 2.1 = -210 \text{ kNm}$$

and the section dimensions are width = 1.22 m, depth = 0.61 m, so that

$$I_{xx} = 4 \times 900 \times 305^2 + 2 \times 1200 \times 305^2 = 5.58 \times 10^8 \text{ mm}^4$$

and

$$\sigma_{z,r} = \frac{-210 \times 10^6}{5.58 \times 10^8} y_r = -0.376 y_r$$

Hence,

$$P_1 = P_3 = -P_4 = -P_6 = -0.376 \times 305 \times 900 = -103,212 \text{ N}$$

and

$$P_2 = -P_5 = -0.376 \times 305 \times 1200 = -137,616 \text{ N}$$

Thus, there is an increase in compressive load of  $103,212 - 96,642 = 6570 \text{ N}$  in booms 1 and 3 and an increase in tensile load of  $6570 \text{ N}$  in booms 4 and 6 between the two sections. Also, the compressive load in boom 2 increases by  $137,616 - 128,856 = 8760 \text{ N}$ , while the tensile load in boom 5 increases by  $8760 \text{ N}$ . Therefore, the change in boom load per unit length is given by

$$\Delta P_1 = \Delta P_3 = -\Delta P_4 = -\Delta P_6 = \frac{6570}{200} = 32.85 \text{ N}$$

and

$$\Delta P_2 = -\Delta P_5 = \frac{8760}{200} = 43.8 \text{ N}$$

The situation is illustrated in Fig. 21.11. Suppose now that the shear flows in the panels 12, 23, 34, and so forth are  $q_{12}$ ,  $q_{23}$ ,  $q_{34}$ , and so on, and consider the equilibrium of boom 2, as shown in Fig. 21.12, with adjacent portions of the panels 12 and 23. Thus,

$$q_{23} + 43.8 - q_{12} = 0$$

or

$$q_{23} = q_{12} - 43.8$$

Similarly,

$$q_{34} = q_{23} - 32.85 = q_{12} - 76.65$$

$$q_{45} = q_{34} + 32.85 = q_{12} - 43.8$$

$$q_{56} = q_{45} + 43.8 = q_{12}$$

$$q_{61} = q_{45} + 32.85 = q_{12} + 32.85$$

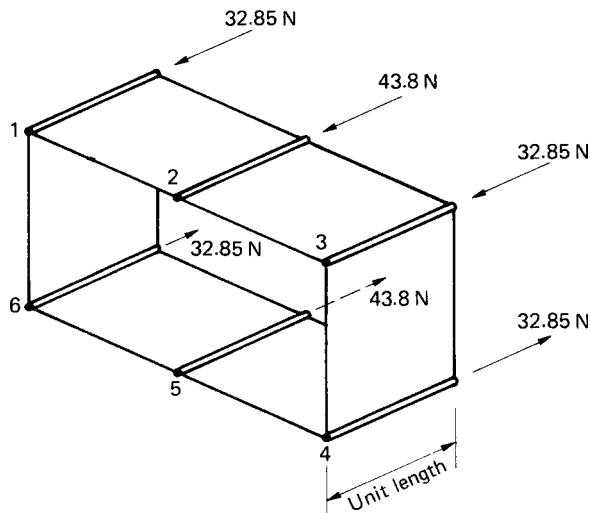


FIGURE 21.11 Change in Boom Loads/Unit Length of Beam

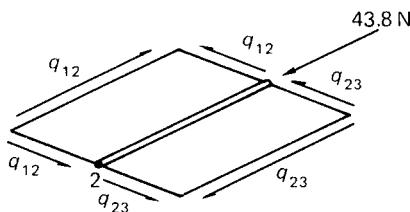


FIGURE 21.12 Equilibrium of Boom

The moment resultant of the internal shear flows, together with the moments of the components  $P_{y,r}$  of the boom loads about any point in the cross-section, is equivalent to the moment of the externally applied load about the same point. We note, from [Example 21.2](#), that for moments about the center of symmetry,

$$\sum_{r=1}^6 P_{x,r} \eta_r = 0, \quad \sum_{r=1}^6 P_{y,r} \xi_r = 0$$

Therefore, taking moments about the center of symmetry,

$$100 \times 10^3 \times 600 = 2q_{12} \times 600 \times 300 + 2(q_{12} - 43.8)600 \times 300 + (q_{12} - 76.65)600 \times 600 \\ + (q_{12} + 32.85)600 \times 600$$

from which

$$q_{12} = 62.5 \text{ N/mm}$$

whence

$$\begin{aligned} q_{23} &= 19.7 \text{ N/mm}, & q_{34} &= -13.2 \text{ N/mm}, & q_{45} &= 19.7 \text{ N/mm}, \\ q_{56} &= 63.5 \text{ N/mm}, & q_{61} &= 96.4 \text{ N/mm} \end{aligned}$$

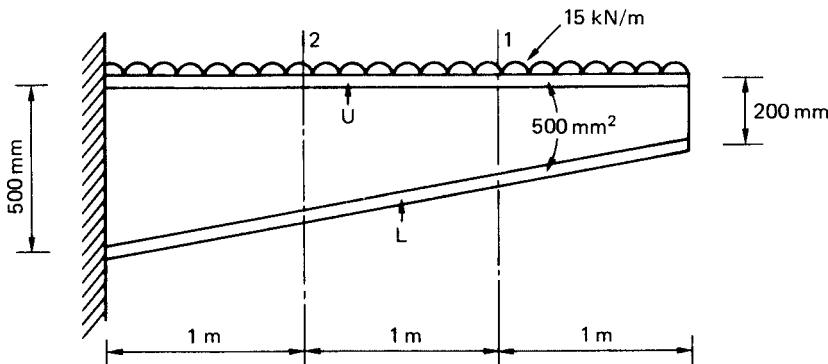
so that the solution is almost identical to the longer exact solution of [Example 21.2](#).

The shear flows  $q_{12}$ ,  $q_{23}$ , and so forth induce complementary shear flows  $q_{12}$ ,  $q_{23}$ , and so on in the panels in the longitudinal direction of the beam; these are, in fact, the average shear flows between the two sections considered. For a complete beam analysis, this procedure is applied to a series of sections along the span. The distance between adjacent sections may be taken to be any convenient value; for actual wings, distances of the order of 350–700 mm are usually chosen. However, for very small values, small percentage errors in  $P_{z,1}$  and  $P_{z,2}$  result in large percentage errors in  $\Delta P$ . On the other hand, if the distance is too large, the average shear flow between two adjacent sections may not be quite equal to the shear flow midway between the sections.

## PROBLEMS

**P.21.1.** A wing spar has the dimensions shown in [Fig. P.21.1](#) and carries a uniformly distributed load of 15 kN/m along its complete length. Each flange has a cross-sectional area of 500 mm<sup>2</sup> with the top flange being horizontal. If the flanges are assumed to resist all direct loads while the spar web is effective only in shear, determine the flange loads and the shear flows in the web at sections 1 and 2 m from the free end.

Answer: 1 m from free end:  $P_U = 25$  kN (tension),  $P_L = 25.1$  kN (compression),  
 $q = 41.7$  N/mm  
2 m from free end:  $P_U = 75$  kN (tension),  $P_L = 75.4$  kN (compression),  
 $q = 56.3$  N/mm



**FIGURE P.21.1**

**P.21.2.** If the web in the wing spar of Problem 21.1 has a thickness of 2 mm and is fully effective in resisting direct stresses, calculate the maximum value of shear flow in the web at a section 1 m from the free end of the beam.

Answer: 46.8 N/mm

**P.21.2 MATLAB** Use MATLAB to repeat Problem P.21.2 for web thickness values ( $t$ ) from 1 to 2.6 mm in increments of 0.2 mm.

- Answer:
- (i)  $t = 1 \text{ mm} : 44 \text{ N/mm}$
  - (ii)  $t = 1.2 \text{ mm} : 45.1 \text{ N/mm}$
  - (iii)  $t = 1.4 \text{ mm} : 45.4 \text{ N/mm}$
  - (iv)  $t = 1.6 \text{ mm} : 45.6 \text{ N/mm}$
  - (v)  $t = 1.8 \text{ mm} : 46.7 \text{ N/mm}$
  - (vi)  $t = 2 \text{ mm} : 46.8 \text{ N/mm}$
  - (vii)  $t = 2.2 \text{ mm} : 46.9 \text{ N/mm}$
  - (viii)  $t = 2.4 \text{ mm} : 47.9 \text{ N/mm}$
  - (ix)  $t = 2.6 \text{ mm} : 48 \text{ N/mm}$

**P.21.3.** Calculate the shear flow distribution and the stringer and flange loads in the beam shown in Fig. P.21.3 at a section 1.5 m from the built-in end. Assume that the skin and web panels are effective in resisting shear stress only; the beam tapers symmetrically in a vertical direction about its longitudinal axis.

- Answer:  $q_{13} = q_{42} = 19.2 \text{ N/mm}$ ,  $q_{35} = q_{64} = -10.4 \text{ N/mm}$ ,  $q_{21} = 78.5 \text{ N/mm}$ ,  
 $q_{65} = 40.0 \text{ N/mm}$   
 $P_2 = -P_1 = 133.3 \text{ kN}$ ,  $P_4 = P_6 = -P_3 = -P_5 = 66.7 \text{ kN}$

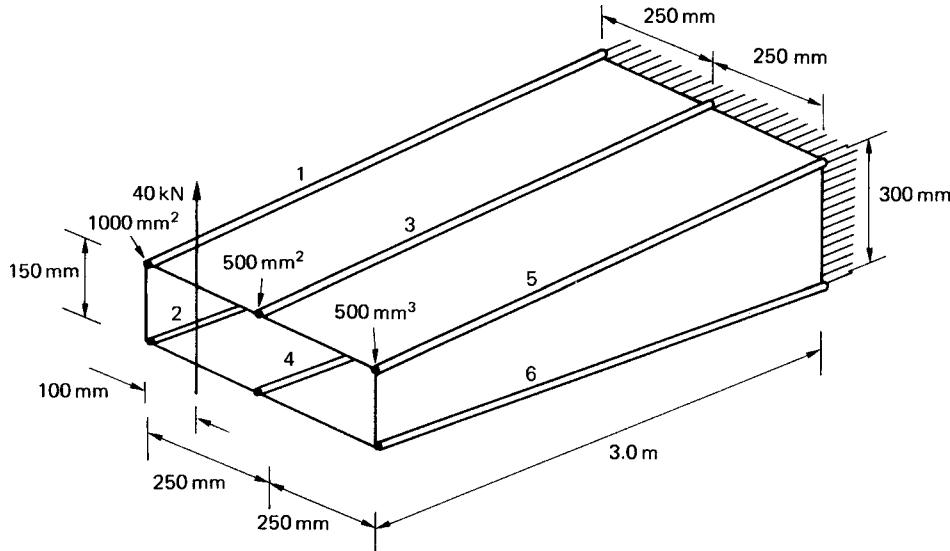


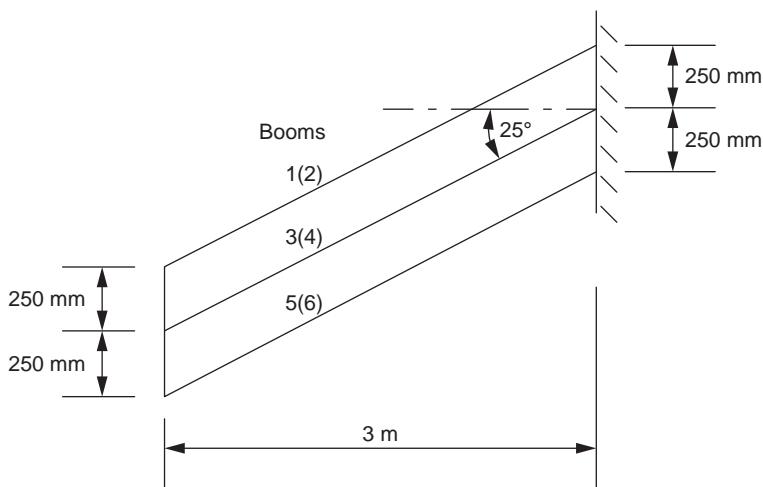
FIGURE P.21.3

**P.21.4** If the vertical plane of symmetry in the beam of P.21.3 is swept back by  $25^\circ$  as shown in plan in Fig. P.21.4, the span of 3 m is retained and the 40 kN load is applied as in P.21.3, calculate the flange and stringer loads and the shear flow distribution at a section 1.5 m from the built-in end.

Answer:  $P_1 = -P_2 = -147.3$  kN,  $P_3 = -P_4 = P_5 = -P_6 = -73.7$  kN

$$q_{12} = 141.6 \text{ N/mm}, \quad q_{56} = 23.1 \text{ N/mm}, \quad q_{13} = q_{42} = 82.3 \text{ N/mm}$$

$$q_{35} = q_{64} = 52.7 \text{ N/mm}$$



**FIGURE P.21.4**

# Fuselages

# 22

Aircraft fuselages consist, as we saw in [Chapter 12](#), of thin sheets of material stiffened by large numbers of longitudinal stringers together with transverse frames. Generally, they carry bending moments, shear forces, and torsional loads, which induce axial stresses in the stringers and skin, together with shear stresses in the skin; the resistance of the stringers to shear forces is generally ignored. Also, the distance between adjacent stringers is usually small, so that the variation in shear flow in the connecting panel is small. It is therefore reasonable to assume that the shear flow is constant between adjacent stringers, so that the analysis simplifies to the analysis of an idealized section in which the stringers/booms carry all the direct stresses while the skin is effective only in shear. The direct stress carrying capacity of the skin may be allowed for by increasing the stringer/boom areas as described in [Section 20.3](#). The analysis of fuselages therefore involves the calculation of direct stresses in the stringers and the shear stress distributions in the skin; the latter are also required in the analysis of transverse frames, as we shall see in [Chapter 24](#).

In addition to the above loading systems, modern aircraft, particularly large passenger aircraft, are pressurized so that the fuselage, with fore and aft bulkheads, comprises a closed cylinder subjected to internal pressure; this produces longitudinal and circumferential or “hoop” stresses in the cylinder wall. We shall examine such a situation in Section 22.4.

## 22.1 BENDING

The skin–stringer arrangement is idealized into one comprising booms and skin, as described in [Section 20.3](#). The direct stress in each boom is then calculated using either [Eq. \(16.17\)](#) or [\(16.18\)](#), in which the reference axes and the section properties refer to the direct stress carrying areas of the cross-section.

### Example 22.1

The fuselage of a light passenger-carrying aircraft has the circular cross-section shown in [Fig. 22.1\(a\)](#). The cross-sectional area of each stringer is 100 mm<sup>2</sup> and the vertical distances given in [Fig. 22.1\(a\)](#) are to the mid-line of the section wall at the corresponding stringer position. If the fuselage is subjected to a bending moment of 200 kNm applied in the vertical plane of symmetry, at this section, calculate the direct stress distribution.

The section is first idealized using the method described in [Section 20.3](#). As an approximation, we assume that the skin between adjacent stringers is flat, so that we may use either [Eq. \(20.1\)](#) or [\(20.2\)](#) to determine the boom areas. From symmetry,  $B_1=B_9$ ,  $B_2=B_8=B_{10}=B_{16}$ ,  $B_3=B_7=B_{11}=B_{15}$ ,  $B_4=B_6=B_{12}=B_{14}$ , and  $B_5=B_{13}$ . From [Eq. \(20.1\)](#),

$$B_1 = 100 + \frac{0.8 \times 149.6}{6} \left( 2 + \frac{\sigma_2}{\sigma_1} \right) + \frac{0.8 \times 149.6}{6} \left( 2 + \frac{\sigma_{16}}{\sigma_1} \right)$$

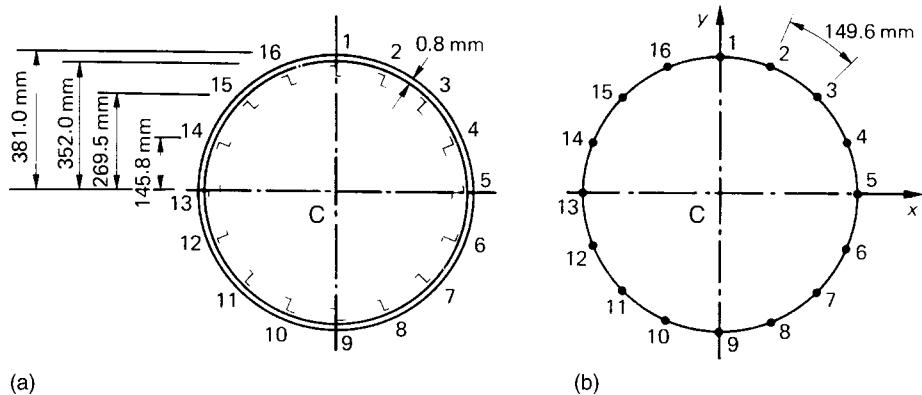


FIGURE 22.1 (a) Actual Fuselage Section; (b) Idealized Fuselage Section

that is,

$$B_1 = 100 + \frac{0.8 \times 149.6}{6} \left( 2 + \frac{352.0}{381.0} \right) \times 2 = 216.6 \text{ mm}^2$$

Similarly  $B_2 = 216.6 \text{ mm}^2$ ,  $B_3 = 216.6 \text{ mm}^2$ ,  $B_4 = 216.7 \text{ mm}^2$ . We note that stringers 5 and 13 lie on the neutral axis of the section and are therefore unstressed; the calculation of boom areas  $B_5$  and  $B_{13}$  does not then arise. For this particular section,  $I_{xy} = 0$ , since  $C_x$  (and  $C_y$ ) is an axis of symmetry. Further,  $M_y = 0$ , so that Eq. (16.17) reduces to

$$\sigma_z = \frac{M_x y}{I_{xx}}$$

in which

$$I_{xx} = 2 \times 216.6 \times 381.0^2 + 4 \times 216.6 \times 352.0^2 + 4 \times 216.6 \times 269.5^2 + 4 \times 216.7 \times 145.8^2 \\ = 2.52 \times 10^8 \text{ mm}^4$$

The solution is completed in Table 22.1.

Table 22.1 Example 22.1

Stringer/boom	y (mm)	$\sigma_z$ (N/mm <sup>2</sup> )
1	381.0	302.4
2,16	352.0	279.4
3,15	269.5	213.9
4,14	145.8	115.7
5,13	0	0
6,12	-145.8	-115.7
7,11	-269.5	-213.9
8,10	-352.0	-279.4
9	-381.0	-302.4

## 22.2 SHEAR

For a fuselage having a cross-section of the type shown in Fig. 22.1(a), the determination of the shear flow distribution in the skin produced by shear is basically the analysis of an idealized single-cell closed section beam. The shear flow distribution is therefore given by Eq. (20.11), in which the direct stress carrying capacity of the skin is assumed to be zero, that is,  $t_D = 0$ , thus,

$$q_s = - \left( \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \sum_{r=1}^n B_r y_r - \left( \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \sum_{r=1}^n B_r x_r + q_{s,0} \quad (22.1)$$

Equation (22.1) is applicable to loading cases in which the shear loads are not applied through the section shear center, so that the effects of shear and torsion are included simultaneously. Alternatively, if the position of the shear center is known, the loading system may be replaced by shear loads acting through the shear center together with a pure torque, and the corresponding shear flow distributions may be calculated separately and superimposed to obtain the final distribution.

### Example 22.2

The fuselage of Example 22.1 is subjected to a vertical shear load of 100 kN applied at a distance of 150 mm from the vertical axis of symmetry, as shown, for the idealized section, in Fig. 22.2. Calculate the distribution of shear flow in the section.

As in Example 22.1,  $I_{xy} = 0$  and, since  $S_x = 0$ , Eq. (22.1) reduces to

$$q_s = - \frac{S_y}{I_{xx}} \sum_{r=1}^n B_r y_r + q_{s,0} \quad (i)$$

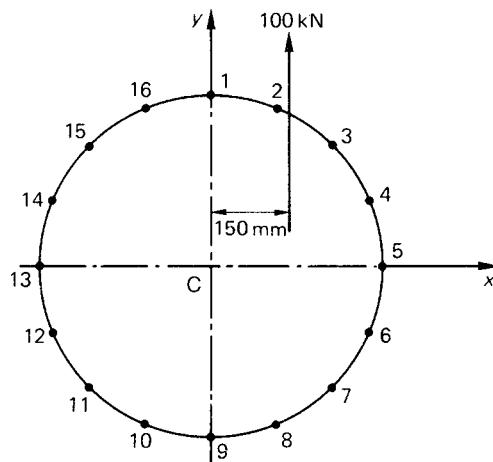


FIGURE 22.2 Idealized Fuselage Section of Example 22.2

in which  $I_{xx} = 2.52 \times 10^8 \text{ mm}^4$  as before. Then,

$$q_s = \frac{-100 \times 10^3}{2.52 \times 10^8} \sum_{r=1}^n B_r y_r + q_{s,0}$$

or

$$q_s = -3.97 \times 10^{-4} \sum_{r=1}^n B_r y_r + q_{s,0} \quad (\text{ii})$$

The first term on the right-hand side of Eq. (ii) is the “open section” shear flow  $q_b$ . We therefore “cut” one of the skin panels, say, 12, and calculate  $q_b$ . The results are presented in Table 22.2.

Note that in Table 22.2, the column headed Boom indicates the boom that is crossed when the analysis moves from one panel to the next. Note also that, as would be expected, the  $q_b$  shear flow distribution is symmetrical about the Cx axis. The shear flow  $q_{s,0}$  in panel 12 is now found by taking moments about a convenient moment center, say C. Therefore, from Eq. (17.17),

$$100 \times 10^3 \times 150 = \oint q_b p ds + 2Aq_{s,0} \quad (\text{iii})$$

in which  $A = \pi \times 381.0^2 = 4.56 \times 10^5 \text{ mm}^2$ . Since the  $q_b$  shear flows are constant between the booms, Eq. (iii) may be rewritten in the form (see Eq. (20.10))

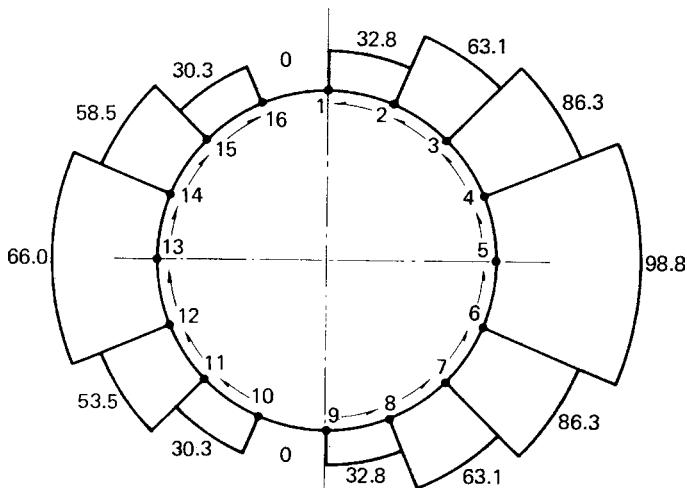
$$100 \times 10^3 \times 150 = -2A_{12}q_{b,12} - 2A_{23}q_{b,23} - \dots - 2A_{161}q_{b,161} + 2Aq_{s,0} \quad (\text{iv})$$

in which  $A_{12}, A_{23}, \dots, A_{161}$  are the areas subtended by the skin panels 12, 23, ..., 161 at the center C of the circular cross-section and counterclockwise moments are taken as positive. Clearly,  $A_{12} = A_{23} = \dots = A_{161} = 4.56 \times 10^5 / 16 = 28,500 \text{ mm}^2$ . Equation (iv) then becomes

$$100 \times 10^3 \times 150 = 2 \times 28,500(-q_{b,12} - q_{b,23} - \dots - q_{b,161}) + 2 \times 4.56 \times 10^5 q_{s,0} \quad (\text{v})$$

**Table 22.2 Example 22.2**

Skin panel	Boom	$B_r (\text{mm}^2)$	$y_r (\text{mm})$	$q_b (\text{N/mm})$
1 2	—	—	—	0
2 3	2	216.6	352.0	-30.3
3 4	3	216.6	269.5	-53.5
4 5	4	216.7	145.8	-66.0
5 6	5	—	0	-66.0
6 7	6	216.7	-145.8	-53.5
7 8	7	216.6	-269.5	-30.3
8 9	8	216.6	-352.0	0
1 16	1	216.6	381.0	-32.8
16 15	16	216.6	352.0	-63.1
15 14	15	216.6	269.5	-86.3
14 13	14	216.6	145.8	-98.8
13 12	13	—	0	-98.8
12 11	12	216.7	-145.8	-86.3
11 10	11	216.6	-269.5	-63.1
10 9	10	216.6	-352.0	-32.8



**FIGURE 22.3 Shear Flow (N/mm) Distribution in the Fuselage Section of Example 22.2**

Substituting the values of  $q_b$  from Table 22.2 in Eq. (v), we obtain

$$100 \times 10^3 \times 150 = 2 \times 28,500(-262.4) + 2 \times 4.56 \times 10^5 q_{s,0}$$

from which

$$q_{s,0} = 32.8 \text{ N/mm (acting in a counterclockwise sense)}$$

The complete shear flow distribution follows by adding the value of  $q_{s,0}$  to the  $q_b$  shear flow distribution, giving the final distribution shown in Fig. 22.3. The solution may be checked by calculating the resultant of the shear flow distribution parallel to the Cy axis. Thus,

$$2[(98.8 + 66.0)145.8 + (86.3 + 53.5)123.7 + (63.1 + 30.3)82.5 + (32.8 - 0)29.0] \times 10^{-3} = 99.96 \text{ kN}$$

which agrees with the applied shear load of 100 kN. The analysis of a fuselage tapered along its length is carried out using the method described in Section 21.2 and illustrated in Example 21.2.

## 22.3 TORSION

A fuselage section is basically a single-cell closed section beam. The shear flow distribution produced by a pure torque is therefore given by Eq. (18.1) and is

$$q = \frac{T}{2A} \quad (22.2)$$

It is immaterial whether or not the section has been idealized, since, in both cases, the booms are assumed not to carry shear stresses.

Equation (22.2) provides an alternative approach to that illustrated in Example 22.2 for the solution of shear loaded sections in which the position of the shear center is known. In Fig. 22.1, the shear center coincides with the center of symmetry, so that the loading system may be replaced

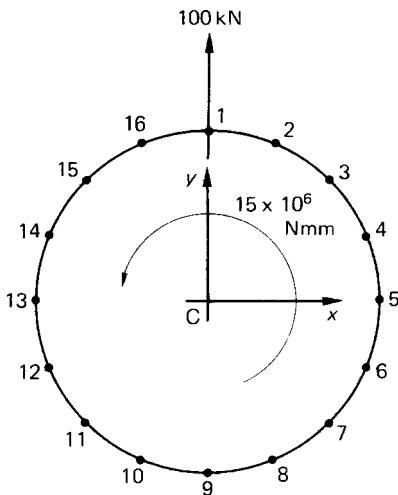


FIGURE 22.4 Alternative Solution of Example 22.2

by the shear load of 100kN acting through the shear center together with a pure torque equal to  $100 \times 10^3 \times 150 = 15 \times 10^6$  Nmm, as shown in Fig. 22.4. The shear flow distribution due to the shear load may be found using the method of Example 22.2 but with the left-hand side of the moment equation (iii) equal to zero for moments about the center of symmetry. Alternatively, use may be made of the symmetry of the section and the fact that the shear flow is constant between adjacent booms. Suppose that the shear flow in the panel 21 is  $q_{21}$ . Then, from symmetry and using the results of Table 22.2,

$$\begin{aligned} q_{98} &= q_{910} = q_{161} = q_{21} \\ q_{32} &= q_{87} = q_{1011} = q_{1516} = 30.3 + q_{21} \\ q_{43} &= q_{76} = q_{1112} = q_{1415} = 53.5 + q_{21} \\ q_{54} &= q_{65} = q_{1213} = q_{1314} = 66.0 + q_{21} \end{aligned}$$

The resultant of these shear flows is statically equivalent to the applied shear load, so that

$$4(29.0q_{21} + 82.5q_{32} + 123.7q_{43} + 145.8q_{54}) = 100 \times 10^3$$

Substituting for  $q_{32}$ ,  $q_{43}$ , and  $q_{54}$  from the preceding, we obtain

$$4(381q_{21} + 18740.5) = 100 \times 10^3$$

from which

$$q_{21} = 16.4 \text{ N/mm}$$

and

$$q_{32} = 46.7 \text{ N/mm}, \quad q_{43} = 69.9 \text{ N/mm}, \quad q_{54} = 83.4 \text{ N/mm}, \text{ and so forth}$$

The shear flow distribution due to the applied torque is, from Eq. (22.2),

$$q = \frac{15 \times 10^6}{2 \times 4.56 \times 10^5} = 16.4 \text{ N/mm}$$

acting in a counterclockwise sense completely around the section. This value of shear flow is now superimposed on the shear flows produced by the shear load; this gives the solution shown in Fig. 22.3; that is,

$$q_{21} = 16.4 + 16.4 = 32.8 \text{ N/mm}$$

$$q_{161} = 16.4 - 16.4 = 0, \text{ and so on}$$

The monocoque fuselage shell of Examples 2.3–2.5 is not a practical form of construction since, as we noted in Chapter 12 (see Fig. 12.7), fuselages are strengthened by arrangements of frames and longitudinal stiffeners. Also the cross-beams supporting the passenger cabin floor are additional reinforcement.

## 22.4 PRESSURIZED FUSELAGES

As we have previously noted, the fuselages of large passenger aircraft, which typically fly at altitudes of 10,000 m, are pressurized so that passengers and crew can breathe normally. The fuselage of such an aircraft then acts as a closed pressure vessel and is therefore subjected to both longitudinal and circumferential or hoop stresses in its walls.

Consider the closed cylinder shown in Fig. 22.5 which is subjected to an internal pressure  $p$ . This pressure acts against the sealed ends (i.e., the fuselage bulkheads) and produces a longitudinal stress  $\sigma_L$  in its wall as shown, for any section along its length, in Fig. 22.6. Suppose that the diameter of the cylinder is  $d$  and the thickness of its *thin* walls is  $t$ . Then, for equilibrium of the portion of the cylinder to the right (or left) of the section,

$$\sigma_L \pi dt = p \frac{\pi d^2}{4}$$

or

$$\sigma_L = \frac{pd}{4t} \quad (22.3)$$

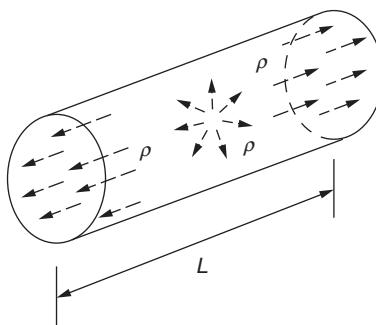


FIGURE 22.5 Thin Cylindrical Shell Under Internal Pressure

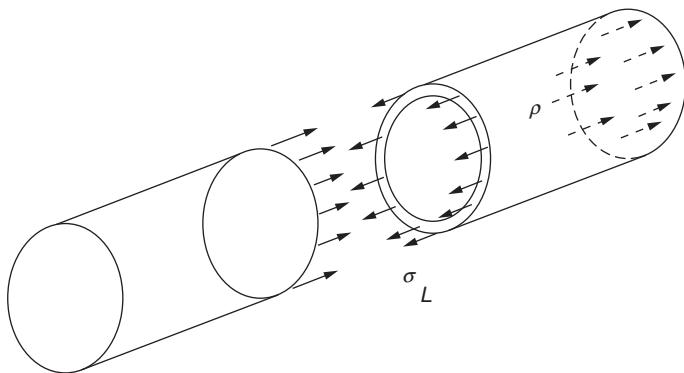


FIGURE 22.6 Longitudinal Stresses Due to Internal Pressure

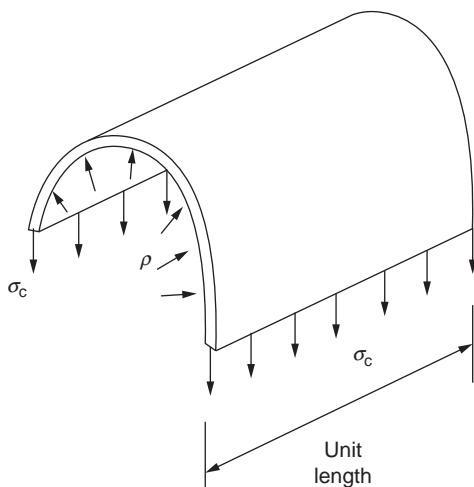


FIGURE 22.7 Circumferential Stress Due to Internal Pressure

In addition to these longitudinal stresses, circumferential stresses are induced. Now consider a unit length of part of the cylinder formed by a diametral plane (which need not necessarily be horizontal) as shown in Fig. 22.7.

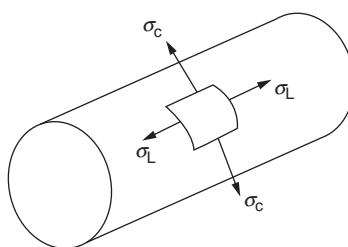
The force perpendicular to the diametral plane produced by the internal pressure is given by  $p \times$  projected area. This is equilibrated by the circumferential stress  $\sigma_C$  in the walls of the cylinder. Then

$$2\sigma_C \times t \times 1 = p \times 1 \times d$$

so that

$$\sigma_C = \frac{pd}{2t} \quad (22.4)$$

Note that  $\sigma_C = 2\sigma_L$ .



**FIGURE 22.8 Biaxial Stress System in a Thin-Walled Cylinder Under Internal Pressure**

The state of stress at any point in the cylinder wall is therefore that shown in Fig. 22.8 and is known as a *biaxial* stress system. Then, from Eqs. (1.40) and (1.41), the longitudinal strain  $\varepsilon_L$  is given by

$$\varepsilon_L = \frac{\sigma_L}{E} - v \frac{\sigma_C}{E}$$

or, substituting for  $\sigma_L$  and  $\sigma_C$  from Eqs. (22.3) and (22.4),

$$\varepsilon_L = \frac{pd}{2tE} \left( \frac{1}{2} - v \right) \quad (22.5)$$

Similarly the circumferential strain is given by

$$\varepsilon_C = \frac{pd}{2tE} \left( 1 - \frac{1}{2}v \right) \quad (22.6)$$

The increase in length of the cylinder is  $\varepsilon_L L$  and the increase in circumference is  $\varepsilon_C d$ , which corresponds to an increase in diameter  $\varepsilon_C d$  so that the circumferential strain is equal to the diametral (and radial) strain. The increase in volume is then given by

$$\Delta V = \frac{\pi}{4} (d + \varepsilon_C d)^2 (L + \varepsilon_L L) - \frac{\pi}{4} d^2 L$$

Expanding this expression

$$\Delta V = \frac{\pi}{4} (d^2 + 2\varepsilon_C d^2 + \varepsilon_C^2 d^2) (L + \varepsilon_L L) - \frac{\pi d^2}{4} L$$

so that, neglecting second-order terms,

$$\Delta V = \frac{\pi d^2}{4} L (2\varepsilon_C + \varepsilon_L)$$

Now substituting for  $\varepsilon_L$  and  $\varepsilon_C$  from Eqs. (22.5) and (22.6),

$$\Delta V = \frac{\pi d^2}{4} L \left[ \frac{pd}{tE} \left( 1 - \frac{1}{2}v \right) + \frac{pd}{tE} \left( \frac{1}{4} - \frac{1}{2}v \right) \right]$$

which simplifies to

$$\Delta V = \frac{\pi d^2}{4} L \left[ \frac{pd}{tE} \left( \frac{5}{4} - v \right) \right]$$

The volumetric strain is then

$$\frac{\Delta V}{\pi d^2 L / 4} = \frac{pd}{tE} \left( \frac{5}{4} - v \right) \quad (22.7)$$

### Example 22.3

A monocoque fuselage shell of diameter 2m is subjected to a pressure differential of 0.5 N/mm<sup>2</sup>. If the tensile strength of the material of the shell is 450 N/mm<sup>2</sup>, calculate the skin thickness required allowing for a factor of safety of 2.5.

If, for the purpose of testing, the ends are sealed, calculate the percentage increase in volume. Take  $E = 70,000 \text{ N/mm}^2$  and  $v = 0.3$ .

The maximum stress in the shell is circumferential. Therefore, from Eq. (22.4),

$$\frac{450}{2.5} = \frac{0.5 \times 2 \times 10^3}{2t}$$

which gives

$$t = 2.78 \text{ mm, say } 2.8 \text{ mm}$$

The volumetric strain is, from Eq. (22.7),

$$\text{Vol.strain} = \frac{0.5 \times 2 \times 10^3}{2.8 \times 70,000} \left( \frac{5 - 0.3}{4} \right)$$

that is,

$$\text{Vol.strain} = 0.0048$$

so that the percentage increase in volume is 0.48%.

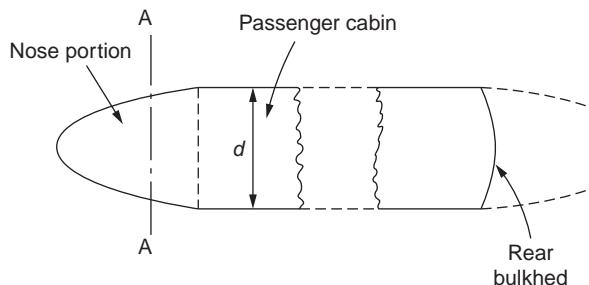
Generally the bulkheads at each end of the passenger cabin of an aircraft would not be flat but would have the shape shown in Fig. 22.9 where the rear bulkhead could be part spherical and the nose portion conical.

Initially we shall consider the rear bulkhead and assume that it is part spherical. To determine the stress at any point in the bulkhead, we shall make use of the membrane analogy presented in Section 3.3. From symmetry the stress system acting on an element of the bulkhead will be identical in form to that shown in Fig. 3.8 where  $N = \sigma t$  and  $q = p$ . Then, from Eq. (3.24),

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{-p}{\sigma t} \quad (22.8)$$

For a spherical shape, the curvature is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{r} \quad (\text{see Eq.(16.24)})$$



**FIGURE 22.9 Profile of an Aircraft Passenger Cabin**

where  $r$  is the radius of the spherical shape. We normally define curvature as positive when the center of curvature is positioned above the structural element. In this case and referring to Fig. 3.8, we see that the center of curvature is below the element and is therefore negative. Eq. (22.8) then becomes

$$\frac{2}{r} = \frac{p}{\sigma t}$$

or

$$\sigma = \frac{pr}{2t} \quad (22.9)$$

which is identical to the expression derived by considering a diametral plane in a complete spherical shell (see Ref. [1]).

We now consider the section AA of the nose portion as shown in Fig. 22.10. Unlike the part spherical rear bulkhead stresses, the nose portion will have longitudinal and circumferential stresses of different value. Further, the longitudinal stress will act in a direction tangential to the wall of the nose portion as shown in Fig. 22.10 and will have the same value at all positions round the section AA and be inclined at an angle  $\alpha$  to the longitudinal axis.

Suppose that the radius of the section AA is  $r_{AA}$ . Then, for equilibrium of forces in the longitudinal direction,

$$\sigma_L t 2\pi r_{AA} \cos \alpha = p \pi r_{AA}^2$$

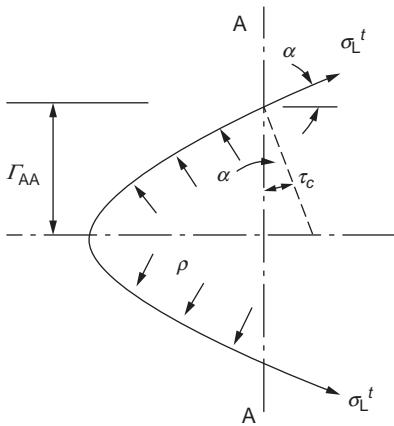
which gives

$$\sigma_L = \frac{pr_{AA}}{2t \cos \alpha} \quad (22.10)$$

Clearly the values of  $r_{AA}$  and  $\alpha$  depend on the exact shape of the nose cone and the position of the section AA.

Returning now to the membrane analogy and Fig. 3.8, let us suppose that the force on the side  $\delta x$  of the element corresponds to the circumferential stress  $\sigma_C$  and that the force on the side  $\delta y$  corresponds to the longitudinal stress  $\sigma_L$ . Then, for equilibrium of the element in the  $z$ -direction, we have

$$-\sigma_L t \delta y \frac{\partial w}{\partial x} - \sigma_L t \delta y \left( -\frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2} \delta x \right) - \sigma_C t \delta x \frac{\partial w}{\partial y} - \sigma_C t \delta x \left( -\frac{\partial w}{\partial y} - \frac{\partial^2 w}{\partial y^2} \delta y \right) + p \delta x \delta y = 0$$

**FIGURE 22.10 Nose Portion of a Pressurized Fuselage**

which simplifies to

$$\sigma_L t \frac{\partial^2 w}{\partial x^2} + \sigma_L t \frac{\partial^2 w}{\partial y^2} = -p$$

In this expression,

$$\frac{\partial^2 w}{\partial x^2} = -\frac{1}{r_L}, \quad \frac{\partial^2 w}{\partial y^2} = -\frac{1}{r_C}$$

which are the curvatures in the longitudinal and circumferential planes, respectively.

Then

$$\frac{\sigma_L t}{r_L} + \frac{\sigma_C t}{r_C} = p \quad (22.11)$$

In Eq. (22.11),  $r_C = r_{AA}/\cos \alpha$  (see Fig. 22.10). Now, from Eqs. (22.11) and (22.10),

$$\frac{\sigma_C t}{r_C} = p - \frac{\sigma_L t}{r_L} = p - \frac{pr_{AA}}{2r_L \cos \alpha}$$

or

$$\sigma_C = \frac{pr_{AA}}{t \cos \alpha} \left( 1 - \frac{r_{AA}}{2r_L \cos \alpha} \right) \quad (22.12)$$

Note that if  $\alpha = 0$ , that is, for a section where the nose portion joins the passenger cabin,  $r_{AA} = d/2$  and  $r_L$  is infinite so that Eq. (22.12) reduces to

$$\sigma_C = \frac{pd}{2t} \quad (\text{cf. Eq.(22.4)})$$

Suppose now that the section AA is close to the tip of the nose portion and that the nose portion has the form of a spherical cap. Also suppose that the radius of this spherical cap is 1/3 of that of the main passenger cabin, that is,  $r_{AA} = d/6$ . Then, by a comparison of Eqs. (22.4) and (22.9) we see that the circumferential stress in the nose portion is 1/6 of that in the main passenger cabin. This reduction in stress is an advantage for the designer, since the major structural discontinuity in the pressurized fuselage is the cockpit canopy, which is usually well forward in the nose portion.

---

### Example 22.4

The rear bulkhead of the fuselage shell of Example 22.3 is part spherical and has a maximum depth of 0.5 m measured from the vertical plane through its junction with the main passenger cabin and is fabricated from the same material. Calculate the required thickness of the bulkhead allowing for the same factor of safety.

If the maximum depth of the bulkhead is  $h$ , then, since its diameter at its junction with the main cabin is  $d$  its radius, from simple geometry, is given by

$$r = \frac{h}{2} + \frac{d^2}{8h} = \frac{0.5}{2} + \frac{2^2}{8 \times 0.5} = 1.25 \text{ m}$$

From Eq. (22.9),

$$\frac{450}{2.5} = \frac{0.5 \times 1.25 \times 10^3}{2t}$$

which gives  $t = 1.74$  mm.

---

### Example 22.5

The nose portion of the fuselage shell of Example 22.3 has the profile of half an ellipse; the distance from its junction with the main passenger cabin to its tip is 2 m. If the wall thickness of the nose portion is 2.8 mm and is fabricated from the same material as the main passenger cabin, calculate the longitudinal and circumferential stresses at a section 1 m from its tip.

The longitudinal stress is given by Eq. (22.10) and the circumferential stress by Eq. (22.12) from which we see that the radius of curvature in the longitudinal plane,  $r_L$ , the angle  $\alpha$  at the section and the radius  $r_{AA}$  of the section must be calculated. The equation of an ellipse referred to  $xz$ -axes is

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \quad (\text{i})$$

from which

$$z = b \left( 1 - \frac{x^2}{a^2} \right)^{1/2} \quad (\text{ii})$$

Then

$$\frac{dz}{dx} = -\frac{b}{a} \left( a^2 - x^2 \right)^{-1/2} (x) \quad (\text{iii})$$

and

$$\frac{d^2z}{dx^2} = -\frac{b}{a}(a^2 - x^2)^{-3/2}(-x^2) - \frac{b}{a}(a^2 - x^2)^{-1/2} \quad (\text{iv})$$

In this case,  $a = 2\text{ m}$  and  $b = 1\text{ m}$ . Substituting in Eqs. (iii) and (iv) gives

$$\frac{dz}{dx} = 0.289 = \tan \alpha$$

from which

$$\alpha = 16.1^\circ$$

and

$$\frac{d^2z}{dx^2} = -0.45 = -\frac{1}{r_L}$$

so that

$$r_L = 2.22 \text{ m}$$

Also, from Eq. (ii),

$$r_{AA} = 0.87 \text{ m}$$

Then, from Eq. (22.10),

$$\sigma_L = \frac{0.5 \times 0.87 \times 10^3}{2 \times 2.8 \cos 16.1^\circ} = 80.9 \text{ N/mm}^2$$

and from Eq. (22.12),

$$\sigma_C = \frac{0.5 \times 0.87 \times 10^3}{2.8 \cos 16.1} \left(1 - \frac{0.87 \times 10^3}{2 \times 2.22 \times 10^3 \times 0.96}\right) = 128.8 \text{ N/mm}^2$$



Further, a fuselage shell will contain cut-outs for windows and doors, which must be reinforced locally. Again, crack prevention features are employed to prevent the type of catastrophic failure of the Comet airliner in the 1950s.

For fuselage pressurization, a limit load cannot be found because pressurization occurs every flight. Therefore, a larger factor of safety, frequently as large as 2, is applied to the maximum pressurization load.

## 22.5 CUT-OUTS IN FUSELAGES

So far, we have considered fuselages to be closed sections stiffened by transverse frames and longitudinal stringers. In practice, it is necessary to provide openings in these closed stiffened shells for, for example, doors, cockpits, bomb bays, and windows in passenger cabins. These openings or

“cut-outs” produce discontinuities in the otherwise continuous shell structure, so that loads are redistributed in the vicinity of the cut-out, thereby affecting loads in the skin, stringers, and frames. Frequently, these regions must be heavily reinforced, resulting in unavoidable weight increases. In some cases, for example, door openings in passenger aircraft, it is not possible to provide rigid fuselage frames on each side of the opening, because the cabin space must not be restricted. In such situations, a rigid frame is placed around the opening to resist shear loads and to transmit loads from one side of the opening to the other.

The effects of smaller cut-outs such as those required for rows of windows in the cabin of a passenger aircraft may be found approximately as follows. Figure 22.11 shows a fuselage panel provided with cut-outs for windows. The dotted lines represent the fuselage frames and stringers required to stiffen the panel. In practice, of course, the panel would be curved, but for a large passenger aircraft the panel, over a depth of fuselage containing the windows, may be regarded as flat.

The panel carries a known shear flow,  $q_{av}$ , which would be constant throughout the panel were it not for the presence of the cut-outs. It will be assumed that the cut-outs only affect the shear flows in the line of shear panels above and below them (shear flows will be present in all the shear panels). There will, therefore, be three unknown shear flows,  $q_1$ ,  $q_2$ , and  $q_3$  as shown in the central area of the panel in Fig. 22.11. These may be found using the equations of statical equilibrium as follows.

Figure 22.12(a) shows the free-body diagram of an area of the panel between and above a cut-out. Considering horizontal forces,

$$q_1 l_1 = q_{av} (l_1 + l_w)$$

so that

$$q_1 = q_{av} \left( 1 + \frac{l_w}{l_1} \right) \quad (22.13)$$

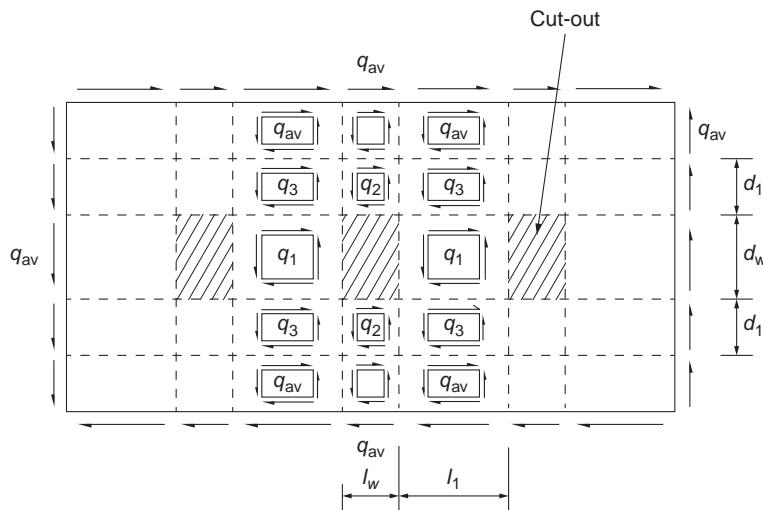


FIGURE 22.11 Fuselage Panel With Regularly Spaced Cut-outs

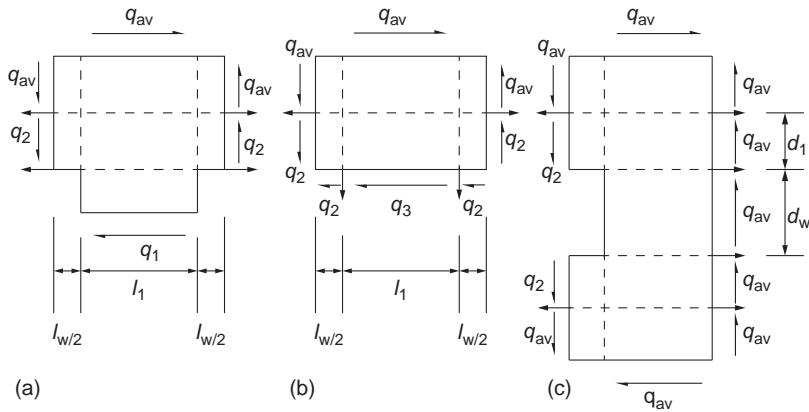


FIGURE 22.12 Free-Body Diagrams of Portions of the Fuselage Panel of Fig. 22.11

Note that there will be axial loads in the frames and stringers produced by the different values of shear flow in adjacent panels. However, from symmetry, the horizontal (stringer) loads will be equal while the vertical (frame) loads do not enter the equation of horizontal equilibrium.

Now consider the free-body diagram in Fig. 22.12(b). Resolving forces vertically,

$$2q_2d_1 = q_{av}(2d_1 + d_w)$$

from which

$$q_2 = q_{av} \left( 1 + \frac{d_w}{2d_1} \right) \quad (22.14)$$

Finally, from the free-body diagram in Fig. 22.12(c) and considering horizontal equilibrium,

$$q_3l_1 + q_2l_w = q_{av}(l_1 + l_w)$$

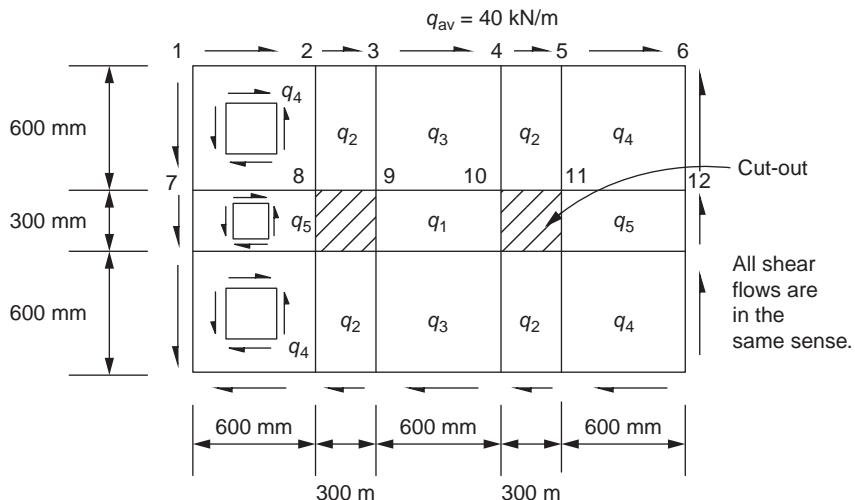
Substituting for  $q_2$  from Eq. (22.14) and rearranging,

$$q_3 = q_{av} \left( 1 - \frac{d_w l_w}{2d_1 l_1} \right) \quad (22.15)$$

The distribution of axial load in the stringers again may be found from statics as illustrated in Example 22.6.

### Example 22.6

The fuselage panel shown in Fig. 22.13 carries an average shear flow of 40 kN/mm along its outer edges. Cut-outs for windows are arranged regularly along its length, and stiffeners are provided as shown. Determine the shear flows in the panels and the distribution of axial load along the stiffeners bordering a cut-out.



**FIGURE 22.13 Fuselage Panel of Example 22.6**

In this example the shear flows in the outer shear panels cannot be assumed to be equal to the externally applied shear flow, since there are no intermediate frames and stiffeners between the cut-outs and the outer edges of the panel. Again we consider the equilibrium of a series of free-body diagrams.

From Fig. 22.14(a) and resolving forces horizontally,

$$600q_1 = 900q_{av}$$

so that

$$q_1 = 1.5 \times 40 = 60 \text{ kN/m} \quad (\text{or N/mm})$$

From Fig. 22.14(b) and resolving forces vertically,

$$2q_2 \times 600 = q_{av} \times 1500$$

which gives

$$q_2 = 1.25 \times 40 = 50 \text{ kN/m} \quad (\text{or N/mm})$$

From Fig. 22.14(c) and resolving forces vertically,

$$2 \times 600q_3 + 300q_1 = 1500q_{av}$$

from which

$$q_3 = 35 \text{ kN/m} \quad (\text{or N/mm})$$

From Fig. 22.14(d) and resolving forces horizontally,

$$2q_5 \times 600 + 600q_1 = 2400q_{av}$$

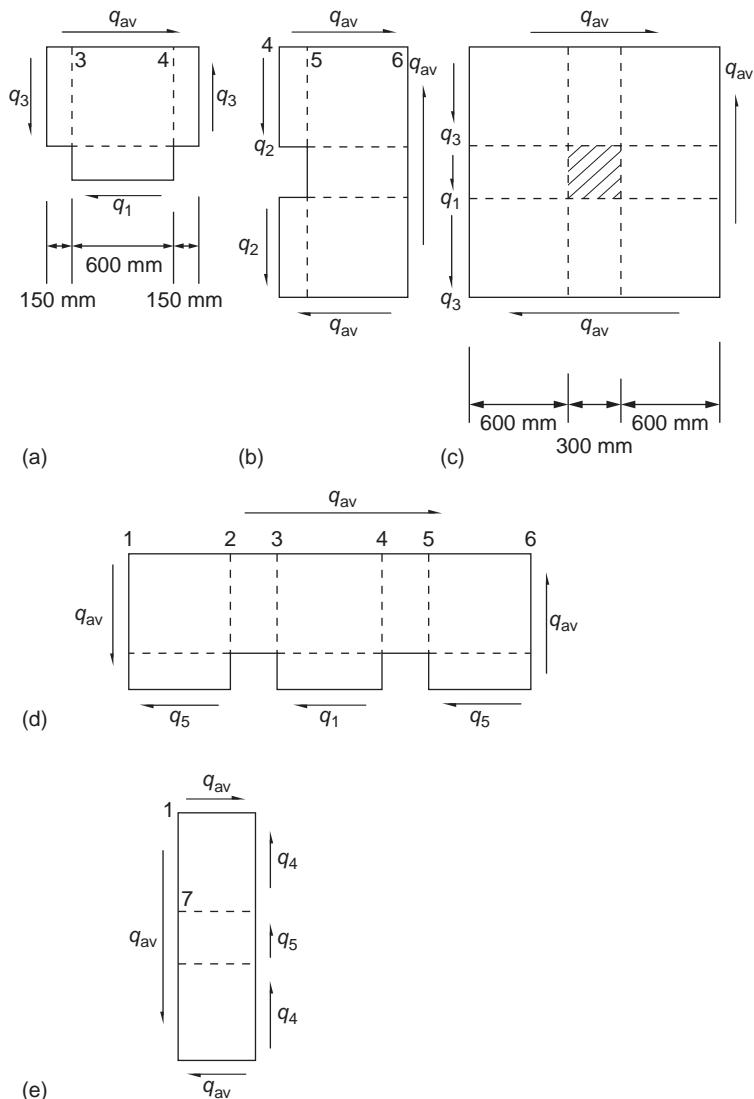


FIGURE 22.14 Free-Body Diagrams for the Fuselage Panel of Example 22.6

so that

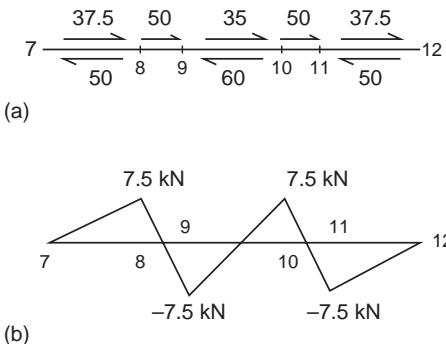
$$1200q_5 + 600 \times 60 = 2400 \times 40$$

which gives

$$q_5 = 50 \text{ kN/mm} \quad (\text{or N/mm})$$

Now from Fig. 22.14(c) and resolving forces vertically,

$$2q_4 \times 600 + 300q_5 = 1500q_{av}$$



**FIGURE 22.15 (a) Shear Flows (kN/m) Acting on Stringer 7 8 9 10 11 12; (b) Distribution of Axial Load in Stringer 7 8 9 10 11 12**

or

$$1200q_4 + 300 \times 50 = 1500 \times 40$$

from which

$$q_4 = 37.5 \text{ kN/m} \text{ (or N/mm)}$$

The shear flows acting on the stringer 7 8 9 10 11 12 are shown in Fig. 22.15(a), and the variation of axial load follows from simple statics. For example, at 8,

$$\text{Axial load} = 50 \times 0.6 - 37.5 \times 0.6 = 7.5 \text{ kN}$$

and so on. The complete distribution is shown in Fig. 22.15(b). ■

## Reference

- [1] Megson THG. Structural and stress analysis. 4th ed. Oxford: Elsevier; 2019.

## PROBLEMS

**P.22.1** The doubly symmetrical fuselage section shown in Fig. P.22.1 has been idealized into an arrangement of direct stress carrying booms and shear stress carrying skin panels; all the boom areas are 150 mm<sup>2</sup>. Calculate the direct stresses in the booms and the shear flows in the panels when the section is subjected to a shear load of 50 kN and a bending moment of 100 kNm.

Answer:  $\sigma_{z,1} = -\sigma_{z,6} = 180 \text{ N/mm}^2$ ,  $\sigma_{z,2} = \sigma_{z,10} = -\sigma_{z,5} = -\sigma_{z,7} = 144.9 \text{ N/mm}^2$ ,  $\sigma_{z,3} = \sigma_{z,9} = -\sigma_{z,4} = -\sigma_{z,8} = 60 \text{ N/mm}^2$

$q_{2,1} = q_{6,5} = 1.9 \text{ N/mm}$ ,  $q_{3,2} = q_{5,4} = 12.8 \text{ N/mm}$ ,  $q_{4,3} = 17.3 \text{ N/mm}$ ,  
 $q_{6,7} = q_{10,1} = 11.6 \text{ N/mm}$ ,  $q_{7,8} = q_{9,10} = 22.5 \text{ N/mm}$ ,  $q_{8,9} = 27.0 \text{ N/mm}$

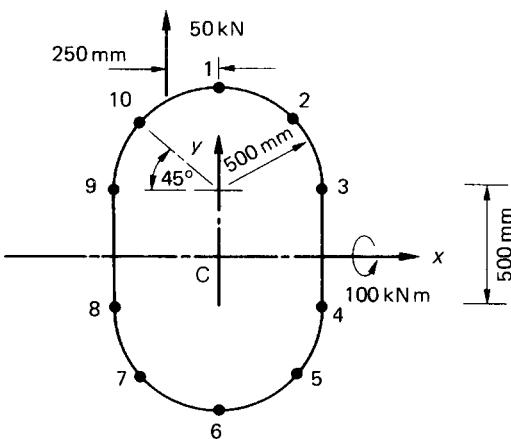


FIGURE P.22.1

**P.22.1 MATLAB** Use MATLAB to repeat [Problem P.22.1](#) for the following shear load ( $S_y$ ) and bending moment ( $M_x$ ) combinations.

	(i)	(ii)	(iii)	(iv)	(v)
$S_y$ (kN)	40	50	50	60	50
$M_x$ (kNm)	150	100	75	200	250

- Answer:
- (i)  $[\sigma_{z,1}, \sigma_{z,2}, \dots, \sigma_{z,10}] = [262.5, 211.2, 87.5, -87.5, -211.2, -262.5, -211.2, -87.5, 87.5, 211.2] \text{ N/mm}^2$   
 $[q_{1,2}, q_{2,3}, \dots, q_{10,1}] = [-1.4, -9.8, -13.3, -9.8, -1.4, 9.1, 17.5, 21, 17.5, 9.1] \text{ N/mm}$
  - (ii)  $[\sigma_{z,1}, \sigma_{z,2}, \dots, \sigma_{z,10}] = [180, 144.9, 60, -60, -144.9, -180, -144.9, -60, 60, 144.9] \text{ N/mm}^2;$   
 $[q_{1,2}, q_{2,3}, \dots, q_{10,1}] = [-1.9, -12.8, -17.3, -12.8, -1.9, 11.6, 22.5, 27, 22.5, 11.6] \text{ N/mm}$
  - (iii)  $[\sigma_{z,1}, \sigma_{z,2}, \dots, \sigma_{z,10}] = [135, 108.6, 45, -45, -108.6, -135, -108.6, -45, 45, 108.6] \text{ N/mm}^2;$   
 $[q_{1,2}, q_{2,3}, \dots, q_{10,1}] = [-2, -14.7, -20, -14.8, -2.1, 13.7, 26.4, 31.7, 26.4, 13.8] \text{ N/mm}$
  - (iv)  $[\sigma_{z,1}, \sigma_{z,2}, \dots, \sigma_{z,10}] = [352.5, 283.7, 117.5, -117.5, -283.7, -352.5, -283.7, -117.5, 117.5, 283.7] \text{ N/mm}^2;$   
 $[q_{1,2}, q_{2,3}, \dots, q_{10,1}] = [-1.9, -12.8, -17.3, -12.8, -1.9, 11.6, 22.5, 27, 22.5, 11.6] \text{ N/mm}$
  - (v)  $[\sigma_{z,1}, \sigma_{z,2}, \dots, \sigma_{z,10}] = [442.5, 356.1, 147.5, -147.5, -356.1, -442.5, -356.1, -147.5, 147.5, 356.1] \text{ N/mm}^2;$   
 $[q_{1,2}, q_{2,3}, \dots, q_{10,1}] = [-1.4, -9.8, -13.3, -9.8, -1.4, 9.1, 17.5, 21, 17.5, 9.1] \text{ N/mm}$

**P.22.2** Determine the shear flow distribution in the fuselage section of [P.22.1](#) by replacing the applied load by a shear load through the shear center together with a pure torque.

**P.22.3** Calculate the minimum wall thickness of a cast iron water pipe having an internal diameter of 1 m under a head of 120 m. The limiting tensile strength of cast iron is 20 N/mm<sup>2</sup> and the density of water is 1000 kg/m<sup>3</sup>.

Answer: 29.4 mm.

**P.22.4** A thin-walled spherical shell is fabricated from steel plates and has to withstand an internal pressure of 0.75 N/mm<sup>2</sup>. The internal diameter is 3 m and the joint efficiency is 80%. Calculate the minimum plate thickness required using a working stress of 80 N/mm<sup>2</sup>.

Answer: 8.8 mm.

**P.22.5** A passenger-carrying aircraft has a cabin of 3 m diameter and is subjected in flight to a differential pressure of 0.6 N/mm<sup>2</sup>. If the maximum allowable direct stress in the material of the cabin is 425 N/mm<sup>2</sup>, calculate the minimum allowable thickness of the cabin wall using a factor of safety of 1.75.

Answer: 3.7 mm.

**P.22.6** If the rear bulkhead of the passenger cabin of the aircraft in P.22.5 is part spherical, is fabricated from the same material, and has the same thickness as the skin of the passenger cabin, determine the maximum stress in the bulkhead if its depth is 0.5 m.

Answer: 202.7 N/mm<sup>2</sup>.

**P.22.7** The nose portion of the passenger cabin of the aircraft in P.22.5 is elliptic in profile with its tip 4 m forward of its junction with the passenger cabin. If it is fabricated from the same material as the passenger cabin and has the same skin thickness, calculate the longitudinal and circumferential stresses at a section 1.5 m from the tip of its nose.

Answer:  $\sigma_L = 116.0 \text{ N/mm}^2$ ,  $\sigma_C = 206.9 \text{ N/mm}^2$ .

**P.22.8** Determine the shear flows  $q_1$ ,  $q_2$ , and  $q_3$  in the panel shown in Fig. P.22.8 and also the distribution of axial load along the stiffeners 1234 and 5678.

Answer:  $q_1 = 75 \text{ kN/m}$ ,  $q_2 = 42.2 \text{ kN/m}$ ,  $q_3 = 65.6 \text{ kN/m}$

$P_2 = -P_3 = -3.1 \text{ kN}$ ,  $P_6 = -P_7 = 9.4 \text{ kN}$

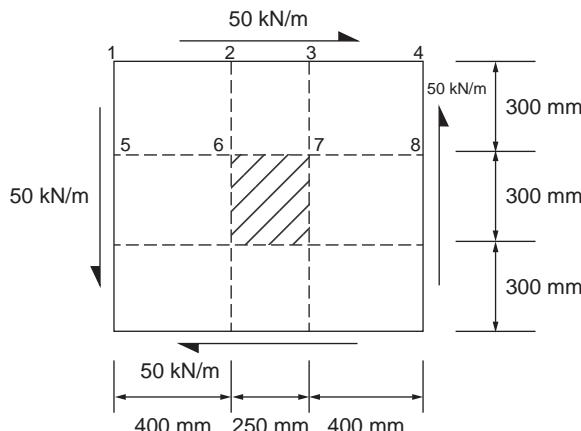


FIGURE P.22.8

## Wings

## 23

We have seen, in [Chapters 12 and 20](#), that wing sections consist of thin skins stiffened by combinations of stringers, spar webs, and caps and ribs. The resulting structure frequently comprises one, two, or more cells and is highly redundant. However, as in the case of fuselage sections, the large number of closely spaced stringers allows the assumption of a constant shear flow in the skin between adjacent stringers, so that a wing section may be analyzed as though it were completely idealized, as long as the direct stress carrying capacity of the skin is allowed for by additions to the existing stringer/boom areas. We shall investigate the analysis of multicellular wing sections subjected to bending, torsional, and shear loads, although, initially, it is instructive to examine the special case of an idealized three-boom shell.

### 23.1 THREE-BOOM SHELL

The wing section shown in [Fig. 23.1](#) has been idealized into an arrangement of direct stress carrying booms and shear stress only carrying skin panels. The part of the wing section aft of the vertical spar 31 performs only an aerodynamic role and is therefore unstressed. Lift and drag loads,  $S_y$  and  $S_x$ , induce shear flows in the skin panels, which are constant between adjacent booms, since the section has been completely idealized. Therefore, resolving horizontally and noting that the resultant of the internal shear flows is equivalent to the applied load, we have

$$S_x = -q_{12}l_{12} + q_{23}l_{23} \quad (23.1)$$

Now resolving vertically,

$$S_y = q_{31}(h_{12} + h_{23}) - q_{12}h_{12} - q_{23}h_{23} \quad (23.2)$$

Finally, taking moments about, say, boom 3,

$$S_x\eta_0 + S_y\xi_0 = -2A_{12}q_{12} - 2A_{23}q_{23} \quad (23.3)$$

(see [Eqs. \(20.9\) and \(20.10\)](#)). In the preceding, there are three unknown values of shear flow,  $q_{12}$ ,  $q_{23}$ ,  $q_{31}$ , and three equations of statical equilibrium. We conclude therefore that a three-boom idealized shell is statically determinate.

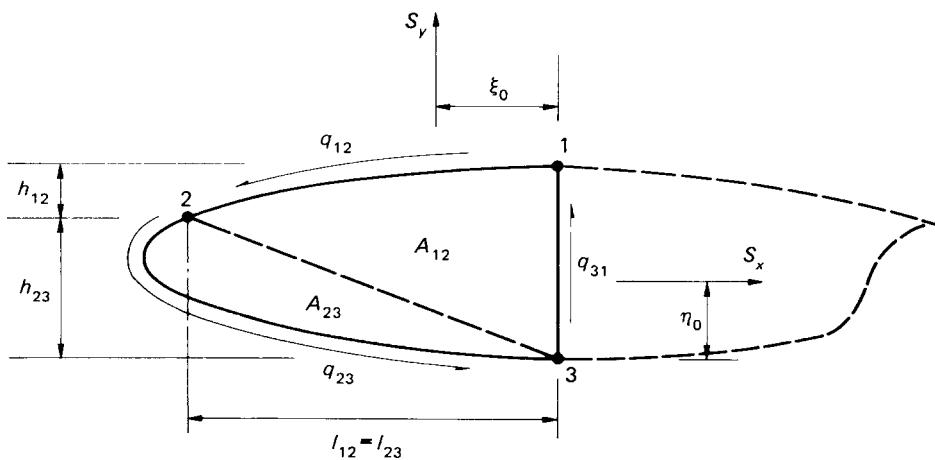


FIGURE 23.1 Three-Boom Wing Section

We shall return to the simple case of a three-boom wing section when we examine the distributions of direct load and shear flows in wing ribs. Meanwhile, we consider the bending, torsion, and shear of multicellular wing sections.

## 23.2 BENDING

Bending moments at any section of a wing are usually produced by shear loads at other sections of the wing. The direct stress system for such a wing section (Fig. 23.2) is given by either Eq. (16.17) or (16.18), in which the coordinates  $(x,y)$  of any point in the cross-section and the sectional properties are referred to axes Cxy, in which the origin C coincides with the centroid of the direct stress carrying area.

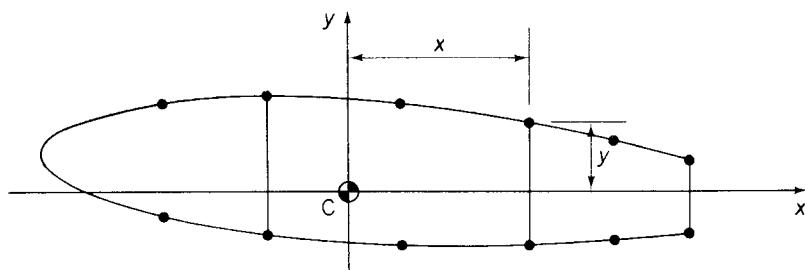


FIGURE 23.2 Idealized Section of a Multicell Wing

**Example 23.1**

The wing section shown in Fig. 23.3 has been idealized such that the booms carry all the direct stresses. If the wing section is subjected to a bending moment of 300 kNm applied in a vertical plane, calculate the direct stresses in the booms:

$$\text{Boom areas: } B_1 = B_6 = 2580 \text{ mm}^2, \quad B_2 = B_5 = 3880 \text{ mm}^2, \quad B_3 = B_4 = 3230 \text{ mm}^2$$

We note that the distribution of the boom areas is symmetrical about the horizontal  $x$  axis. Hence, in Eq. (16.17),  $I_{xy} = 0$ . Further,  $M_x = 300 \text{ kNm}$  and  $M_y = 0$ , so that Eq. (16.17) reduces to

$$\sigma_z = \frac{M_x y}{I_{xx}} \quad (\text{i})$$

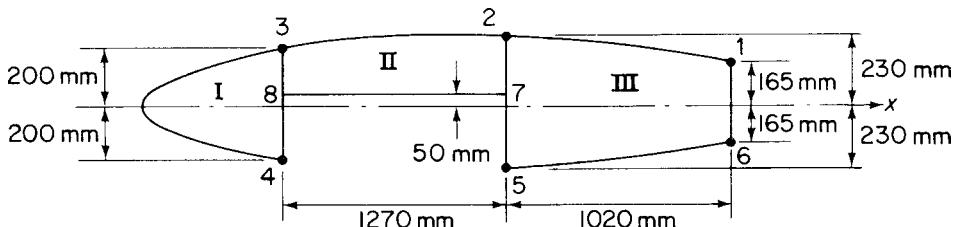
in which

$$I_{xx} = 2(2580 \times 165^2 + 3880 \times 230^2 + 3230 \times 200^2) = 809 \times 10^6 \text{ mm}^4$$

Hence,

$$\sigma_z = \frac{300 \times 10^6}{809 \times 10^6} y = 0.371y \quad (\text{ii})$$

The solution is now completed in Table 23.1, in which positive direct stresses are tensile and negative direct stresses compressive.



**FIGURE 23.3 Wing Section of Example 23.1**

**Table 23.1** Example 23.1

Boom	$y$ (mm)	$\sigma_z$ (N/mm $^2$ )
1	165	61.2
2	230	85.3
3	200	74.2
4	-200	-74.2
5	-230	-85.3
6	-165	-61.2

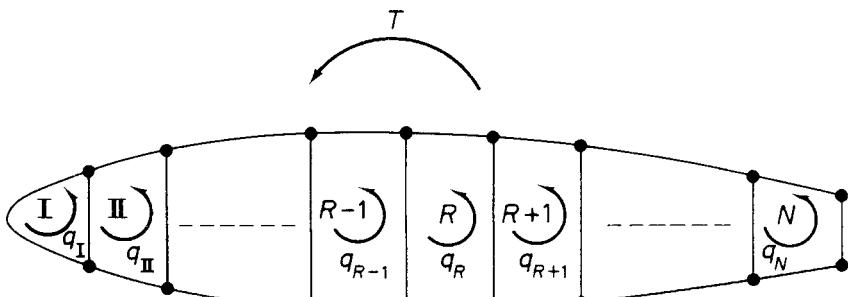


FIGURE 23.4 Multicell Wing Section Subjected to Torsion

### 23.3 TORSION

The chordwise pressure distribution on an aerodynamic surface may be represented by shear loads (lift and drag loads) acting through the aerodynamic center together with a pitching moment  $M_0$  (see Section 12.1). This system of shear loads may be transferred to the shear center of the section in the form of shear loads  $S_x$  and  $S_y$  together with a torque  $T$ . We shall consider the pure torsion case first, since this will be found to be helpful in the analysis of wings carrying shear loads. In the analysis, we assume that no axial constraint effects are present and that the shape of the wing section remains unchanged by the load application. In the absence of axial constraint, there is no development of direct stress in the wing section, so that only shear stresses are present. It follows that the presence of booms does not affect the analysis in the pure torsion case.

The wing section shown in Fig. 23.4 comprises  $N$  cells and carries a torque  $T$ , which generates individual but unknown torques in each of the  $N$  cells. Each cell therefore develops a constant shear flow  $q_1, q_{II}, \dots, q_R, \dots, q_N$  given by Eq. (18.1).

The total torque is therefore

$$T = \sum_{R=1}^N 2A_R q_R \quad (23.4)$$

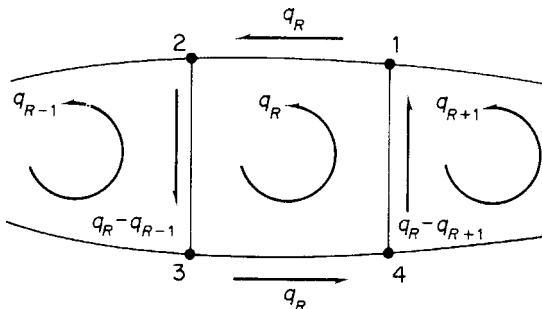
Although Eq. (23.4) is sufficient for the solution of the special case of a single-cell section, which is therefore statically determinate, additional equations are required for an  $N$ -cell section. These are obtained by considering the rate of twist in each cell and the compatibility of displacement condition that all  $N$  cells possess the same rate of twist  $d\theta/dz$ ; this arises directly from the assumption of an undistorted cross-section.

Consider the  $R$ th cell of the wing section, shown in Fig. 23.5. The rate of twist in the cell is, from Eq. (17.22),

$$\frac{d\theta}{dz} = \frac{1}{2A_R G} \oint_R q \frac{ds}{t} \quad (23.5)$$

The shear flow in Eq. (23.5) is constant along each wall of the cell and has the values shown in Fig. 23.5. Writing  $\int ds/t$  for each wall as  $\delta$ , Eq. (23.5) becomes

$$\frac{d\theta}{dz} = \frac{1}{2A_R G} [q_R \delta_{12} + (q_R - q_{R-1}) \delta_{23} + q_R \delta_{34} + (q_R - q_{R+1}) \delta_{41}]$$



**FIGURE 23.5 Shear Flow Distribution in the  $R$ th Cell of an  $N$ -Cell Wing Section**

or, rearranging the terms in square brackets,

$$\frac{d\theta}{dz} = \frac{1}{2A_R G} [-q_{R-1}\delta_{23} + q_R(\delta_{12} + \delta_{23} + \delta_{34} + \delta_{41}) - q_{R+1}\delta_{41}]$$

In general terms, this equation may be rewritten in the form

$$\frac{d\theta}{dz} = \frac{1}{2A_R G} (-q_{R-1}\delta_{R-1,R} + q_R\delta_R - q_{R+1}\delta_{R+1,R}) \quad (23.6)$$

in which  $\delta_{R-1,R}$  is  $\int ds/t$  for the wall common to the  $R$ th and  $(R-1)$ th cells,  $\delta_R$  is  $\int ds/t$  for all the walls enclosing the  $R$ th cell, and  $\delta_{R+1,R}$  is  $\int ds/t$  for the wall common to the  $R$ th and  $(R+1)$ th cells.

The general form of Eq. (23.6) is applicable to multicell sections in which the cells are connected consecutively: that is, cell I is connected to cell II, cell II to cells I and III, and so on. In some cases, cell I may be connected to cells II and III, and so forth (see Problem P.23.4), so that Eq. (23.6) cannot be used in its general form. For this type of section, the term  $\int q(ds/t)$  should be computed by considering  $\int q(ds/t)$  for each wall of a particular cell in turn.

There are  $N$  equations of the type (23.6), which, with Eq. (23.4), comprise the  $N + 1$  equations required to solve for the  $N$  unknown values of shear flow and the one unknown value of  $d\theta/dz$ .

Frequently, in practice, the skin panels and spar webs are fabricated from materials possessing different properties such that the shear modulus  $G$  is not constant. The analysis of such sections is simplified if the actual thickness  $t$  of a wall is converted to a modulus-weighted thickness  $t^*$  as follows. For the  $R$ th cell of an  $N$ -cell wing section in which  $G$  varies from wall to wall, Eq. (23.5) takes the form

$$\frac{d\theta}{dz} = \frac{1}{2A_R} \oint_R q \frac{ds}{Gt}$$

This equation may be rewritten as

$$\frac{d\theta}{dz} = \frac{1}{2A_R G_{\text{REF}}} \oint_R q \frac{ds}{(G/G_{\text{REF}})t} \quad (23.7)$$

in which  $G_{\text{REF}}$  is a convenient reference value of the shear modulus. Equation (23.7) is now rewritten as

$$\frac{d\theta}{dz} = \frac{1}{2A_R G_{\text{REF}}} \oint_R q \frac{ds}{t^*} \quad (23.8)$$

in which the modulus-weighted thickness  $t^*$  is given by

$$t^* = \frac{G}{G_{\text{REF}}} t \quad (23.9)$$

Then, in Eq. (23.6),  $\delta$  becomes  $\int ds/t^*$ .

### Example 23.2

Calculate the shear stress distribution in the walls of the three-cell wing section shown in Fig. 23.6 when it is subjected to a counterclockwise torque of 11.3 kNm. The data are in Table 23.2.

Since the wing section is loaded by a pure torque, the presence of the booms has no effect on the analysis.

Choosing  $G_{\text{REF}} = 27,600 \text{ N/mm}^2$ , then, from Eq. (23.9),

$$t_{12}^* = \frac{24,200}{27,600} \times 1.22 = 1.07 \text{ mm}$$

Similarly,

$$t_{13}^* = t_{24}^* = 1.07 \text{ mm}, \quad t_{35}^* = t_{46}^* = t_{56}^* = 0.69 \text{ mm}$$

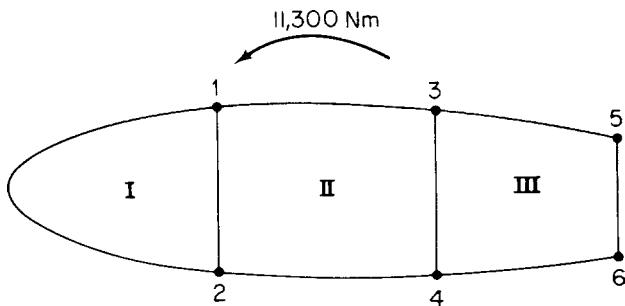


FIGURE 23.6 Wing Section of Example 23.2

**Table 23.2** Example 23.2

Wall	Length (mm)	Thickness (mm)	$G (\text{N/mm}^2)$	Cell area ( $\text{mm}^2$ )
12 <sup>o</sup>	1650	1.22	24,200	$A_I = 258,000$
12 <sup>i</sup>	508	2.03	27,600	$A_{II} = 355,000$
13, 24	775	1.22	24,200	$A_{III} = 161,000$
34	380	1.63	27,600	
35, 46	508	0.92	20,700	
56	254	0.92	20,700	

Note: The superscript symbols  $o$  and  $i$  are used to distinguish between outer and inner walls connecting the same two booms.

Hence,

$$\delta_{12^\circ} = \int_{12^\circ} \frac{ds}{t^*} = \frac{1,650}{1.07} = 1542$$

Similarly,

$$\delta_{12^\circ} = 250, \quad \delta_{13} = \delta_{24} = 725, \quad \delta_{34} = 233, \quad \delta_{35} = \delta_{46} = 736, \quad \delta_{56} = 368$$

Substituting the appropriate values of  $\delta$  in Eq. (23.6) for each cell in turn gives the following:

- For cell I,

$$\frac{d\theta}{dz} = \frac{1}{2 \times 258,000 G_{\text{REF}}} [q_I(1,542 + 250) - 250q_{II}] \quad (\text{i})$$

- For cell II,

$$\frac{d\theta}{dz} = \frac{1}{2 \times 355,000 G_{\text{REF}}} [-250q_I + q_{II}(250 + 725 + 233 + 725) - 233q_{III}] \quad (\text{ii})$$

- For cell III,

$$\frac{d\theta}{dz} = \frac{1}{2 \times 161,000 G_{\text{REF}}} [-233q_{II} + q_{III}(736 + 233 + 736 + 368)] \quad (\text{iii})$$

In addition, from Eq. (23.4),

$$11.3 \times 10^6 = 2(258,000q_I + 355,000q_{II} + 161,000q_{III}) \quad (\text{iv})$$

Solving Eqs. (i)–(iv) simultaneously gives

$$q_I = 7.1 \text{ N/mm}, \quad q_{II} = 8.9 \text{ N/mm}, \quad q_{III} = 4.2 \text{ N/mm}$$

The shear stress in any wall is obtained by dividing the shear flow by the *actual* wall thickness. Hence, the shear stress distribution is as shown in Fig. 23.7.

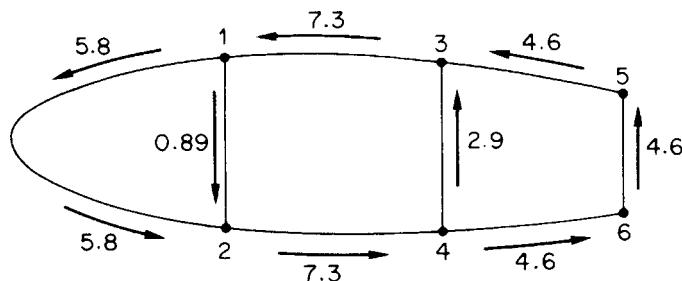


FIGURE 23.7 Shear Stress ( $\text{N/mm}^2$ ) Distribution in the Wing Section of Example 23.2

## 23.4 SHEAR

Initially, we consider the general case of an  $N$ -cell wing section comprising booms and skin panels, the latter being capable of resisting both direct and shear stresses. The wing section is subjected to shear loads  $S_x$  and  $S_y$ , whose lines of action do not necessarily pass through the shear center  $S$  (see Fig. 23.8); the resulting shear flow distribution is therefore due to the combined effects of shear and torsion.

The method for determining the shear flow distribution and the rate of twist is based on a simple extension of the analysis of a single-cell beam subjected to shear loads (Sections 17.3 and 20.3). Such a beam is statically indeterminate, the single redundancy being selected as the value of shear flow at an arbitrarily positioned “cut.” Thus, the  $N$ -cell wing section of Fig. 23.8 may be made statically determinate by cutting a skin panel in each cell, as shown. While the actual position of these cuts is theoretically immaterial, there are advantages to be gained from a numerical point of view if the cuts are made near the center of the top or bottom skin panel in each cell. Generally, at these points, the redundant shear flows ( $q_{s,0}$ ) are small, so that the final shear flows differ only slightly from those of the determinate structure. The system of simultaneous equations from which the final shear flows are found will then be “well conditioned” and produce reliable results. The solution of an “ill-conditioned” system of equations would probably involve the subtraction of large numbers of a similar size, which therefore need to be expressed to a large number of significant figures for reasonable accuracy. Although this reasoning does not apply to a completely idealized wing section, since the calculated values of shear flow are constant between the booms, it is again advantageous to cut either the top or bottom skin panels for, in the special case of a wing section having a horizontal axis of symmetry, a cut in, say, the top skin panels results in the “open section” shear flows ( $q_b$ ) being zero in the bottom skin panels. This decreases the arithmetical labor and simplifies the derivation of the moment equation, as will become obvious in Example 23.3.

The open section shear flow  $q_b$  in the wing section of Fig. 23.8 is given by Eq. (20.6), that is,

$$q_b = - \left( \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D x ds + \sum_{r=1}^n B_r x_r \right) - \left( \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \left( \int_0^s t_D y ds + \sum_{r=1}^n B_r y_r \right)$$

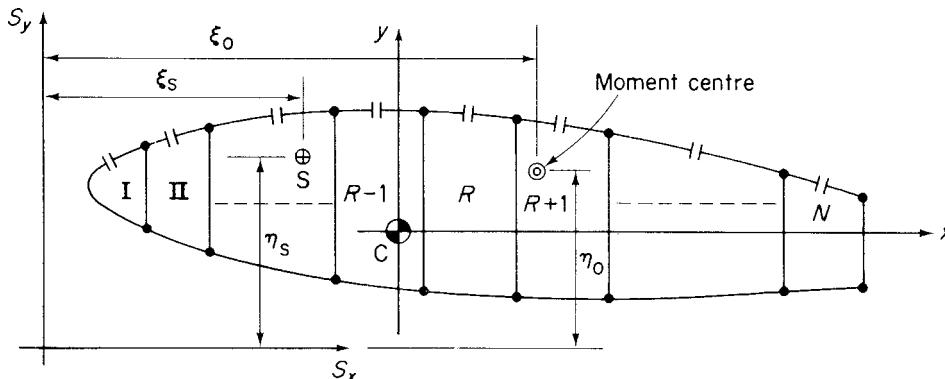


FIGURE 23.8  $N$ -Cell Wing Section Subjected to Shear Loads

We are left with an unknown value of shear flow at each of the cuts, that is,  $q_{s,0,I}, q_{s,0,II}, \dots, q_{s,0,N}$ , plus the unknown rate of twist  $d\theta/dz$ , which, from the assumption of an undistorted cross-section, is the same for each cell. Therefore, as in the torsion case, there are  $N + 1$  unknowns requiring  $N + 1$  equations for a solution.

Consider the  $R$ th cell shown in Fig. 23.9. The complete distribution of shear flow around the cell is given by the summation of the open section shear flow  $q_b$  and the value of shear flow at the cut,  $q_{s,0,R}$ . We may therefore regard  $q_{s,0,R}$  as a constant shear flow acting around the cell. The rate of twist is again given by Eq. (17.22); thus,

$$\frac{d\theta}{dz} = \frac{1}{2A_R G} \oint_R q \frac{ds}{t} = \frac{1}{2A_R G} \oint_R (q_b + q_{s,0,R}) \frac{ds}{t}$$

By comparison with the pure torsion case, we deduce that

$$\frac{d\theta}{dz} = \frac{1}{2A_R G} \left( -q_{s,0,R-1}\delta_{R-1,R} + q_{s,0,R}\delta_R - q_{s,0,R+1}\delta_{R+1,R} + \oint_R q_b \frac{ds}{t} \right) \quad (23.10)$$

in which  $q_b$  has previously been determined. There are  $N$  equations of the type (23.10), so that a further equation is required to solve for the  $N + 1$  unknowns. This is obtained by considering the moment equilibrium of the  $R$ th cell in Fig. 23.10.

The moment  $M_{q,R}$  produced by the total shear flow about any convenient moment center O is given by

$$M_{q,R} = \oint q_R p_0 ds \quad (\text{see Section 18.1})$$

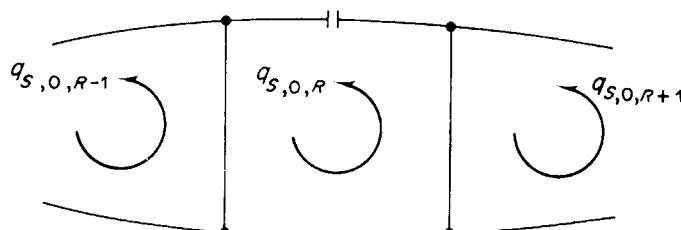


FIGURE 23.9 Redundant Shear Flow in the  $R$ th Cell of an  $N$ -Cell Wing Section Subjected to Shear

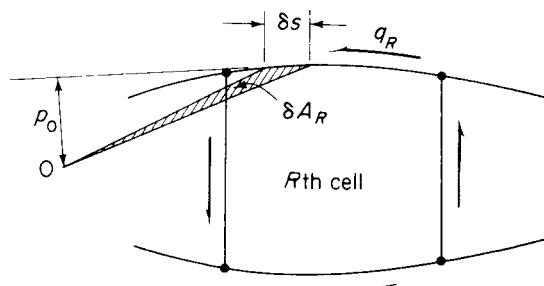


FIGURE 23.10 Moment Equilibrium of the  $R$ th Cell

Substituting for  $q_R$  in terms of the open section shear flow  $q_b$  and the redundant shear flow  $q_{s,0,R}$ , we have

$$M_{q,R} = \oint_R q_b p_0 \, ds + q_{s,0,R} \oint_R p_0 \, ds$$

or

$$M_{q,R} = \oint_R q_b p_0 \, ds + 2A_R q_{s,0,R}$$

The sum of the moments from the individual cells is equivalent to the moment of the externally applied loads about the same point. Thus, for the wing section of Fig. 23.8,

$$S_x \eta_0 - S_y \xi_0 = \sum_{R=1}^N M_{q,R} = \sum_{R=1}^N \oint_R q_b p_0 \, ds + \sum_{R=1}^N 2A_R q_{s,0,R} \quad (23.11)$$

If the moment center is chosen to coincide with the point of intersection of the lines of action of  $S_x$  and  $S_y$ , Eq. (23.11) becomes

$$0 = \sum_{R=1}^N \oint_R q_b p_0 \, ds + \sum_{R=1}^N 2A_R q_{s,0,R} \quad (23.12)$$

### Example 23.3

The wing section of Example 23.1 (Fig. 23.3) carries a vertically upward shear load of 86.8 kN in the plane of the web 572. The section has been idealized such that the booms resist all the direct stresses while the walls are effective only in shear. If the shear modulus of all walls is  $27,600 \text{ N/mm}^2$  except for wall 78, for which it is three times this value, calculate the shear flow distribution in the section and the rate of twist. Additional data are given in Table 23.3.

Choosing  $G_{\text{REF}}$  as  $27,600 \text{ N/mm}^2$ , then, from Eq. (23.9),

$$t_{78}^* = \frac{3 \times 27,600}{27,600} \times 1.22 = 3.66 \text{ mm}$$

**Table 23.3** Example 23.3

Wall	Length (mm)	Thickness (mm)	Cell area ( $\text{mm}^2$ )
12, 56	1023	1.22	$A_I = 265,000$
23	1274	1.63	$A_{II} = 213,000$
34	2200	2.03	$A_{III} = 413,000$
483	400	2.64	
572	460	2.64	
61	330	1.63	
78	1270	1.22	

Hence,

$$\delta_{78} = \frac{1,270}{3.66} = 347$$

Also,

$$\delta_{12} = \delta_{56} = 839, \delta_{23} = 782, \delta_{34} = 1,084, \delta_{38} = 57, \delta_{84} = 95, \delta_{87} = 347, \delta_{27} = 68, \delta_{75} = 106, \delta_{16} = 202$$

We now cut the top skin panels in each cell and calculate the open section shear flows using Eq. (20.6), which, since the wing section is idealized, singly symmetrical (as far as the direct stress carrying area is concerned), and subjected to a vertical shear load only, reduces to

$$q_b = \frac{-S_y}{I_{xx}} \sum_{r=1}^n B_r y_r \quad (i)$$

where, from Example 23.1,  $I_{xx} = 809 \times 10^6 \text{ mm}^4$ . Thus, from Eq. (i),

$$q_b = -\frac{86.8 \times 10^3}{809 \times 10^6} \sum_{r=1}^n B_r y_r = -1.07 \times 10^{-4} \sum_{r=1}^n B_r y_r \quad (ii)$$

Since  $q_b = 0$  at each "cut,"  $q_b = 0$  for the skin panels 12, 23, and 34. The remaining  $q_b$  shear flows are now calculated using Eq. (ii). Note that the order of the numerals in the subscript of  $q_b$  indicates the direction of movement from boom to boom.

$$\begin{aligned} q_{b,27} &= -1.07 \times 10^{-4} \times 3880 \times 230 = -95.5 \text{ N/mm} \\ q_{b,16} &= -1.07 \times 10^{-4} \times 2580 \times 165 = -45.5 \text{ N/mm} \\ q_{b,65} &= -45.5 - 1.07 \times 10^{-4} \times 2580 \times (-165) = 0 \\ q_{b,57} &= -1.07 \times 10^{-4} \times 3880 \times (-230) = 95.5 \text{ N/mm} \\ q_{b,38} &= -1.07 \times 10^{-4} \times 3230 \times 200 = -69.1 \text{ N/mm} \\ q_{b,48} &= -1.07 \times 10^{-4} \times 3230 \times (-200) = 69.1 \text{ N/mm} \end{aligned}$$

Therefore, as  $q_{b,83} = q_{b,48}$  (or  $q_{b,72} = q_{b,57}$ ),  $q_{b,78} = 0$ . The distribution of the  $q_b$  shear flows is shown in Fig. 23.11. The values of  $\delta$  and  $q_b$  are now substituted in Eq. (23.10) for each cell in turn.

- For cell I,

$$d\theta/dz = [q_{s,0,I}(1084 + 95 + 57) - 57q_{s,0,II} + 69.1 \times 95 + 69.1 \times 57]/(2 \times 265,000G_{REF}) \quad (iii)$$

- For cell II,

$$d\theta/dz = [-57q_{s,0,I} + q_{s,0,II}(782 + 57 + 347 + 68) - 68q_{s,0,III} + 95.5 \times 68 - 69.1 \times 57]/(2 \times 213,000G_{REF}) \quad (iv)$$

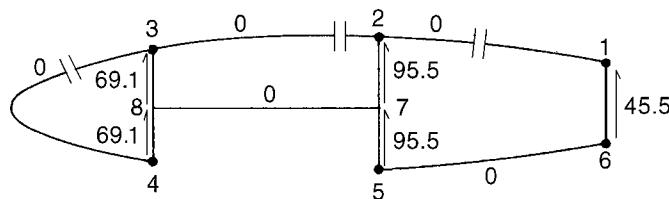


FIGURE 23.11 The  $q_b$  Distribution (N/mm)

- For cell III,

$$\frac{d\theta}{dz} = [-68q_{s,0,II} + q_{s,0,III}(839 + 68 + 106 + 839 + 202) + 45.5 \times 202 - 95.5 \times 68 - 95.5 \times 106]/(2 \times 413,000G_{REF}) \quad (v)$$

The solely numerical terms in Eqs. (iii)–(v) represent  $\int_R q_b (ds/t)$  for each cell. Care must be taken to ensure that the contribution of each  $q_b$  value to this term is interpreted correctly. The path of the integration follows the positive direction of  $q_{s,0}$  in each cell, that is, counterclockwise. Thus, the positive contribution of  $q_{b,83}$  to  $\int_I q_b (ds/t)$  becomes a negative contribution to  $\int_{II} q_b (ds/t)$  and so on.

The fourth equation required for a solution is obtained from Eq. (23.12) by taking moments about the intersection of the  $x$  axis and the web 572. Thus,

$$0 = -69.1 \times 250 \times 1270 - 69.1 \times 150 \times 1270 + 45.5 \times 330 \times 1020 + 2 \times 265,000q_{s,0,I} + 2 \times 213,000q_{s,0,II} + 2 \times 413,000q_{s,0,III} \quad (vi)$$

Simultaneous solution of Eqs. (iii)–(vi) gives

$$q_{s,0,I} = 4.9 \text{ N/mm}, \quad q_{s,0,II} = 9.3 \text{ N/mm}, \quad q_{s,0,III} = 16.0 \text{ N/mm}$$

Superimposing these shear flows on the  $q_b$  distribution of Fig. 23.11, we obtain the final shear flow distribution:

$$\begin{aligned} q_{34} &= 4.9 \text{ N/mm}, \quad q_{23} = q_{87} = 9.3 \text{ N/mm}, \quad q_{12} = q_{56} = 16.0 \text{ N/mm} \\ q_{61} &= 61.5 \text{ N/mm}, \quad q_{57} = 79.5 \text{ N/mm}, \quad q_{72} = 88.7 \text{ N/mm} \\ q_{48} &= 74.0 \text{ N/mm}, \quad q_{83} = 64.7 \text{ N/mm} \end{aligned}$$

Finally, from any of Eqs. (iii)–(v),

$$\frac{d\theta}{dz} = 1.09 \times 10^{-6} \text{ rad/mm}$$

### Example 23.3 MATLAB

Use MATLAB to repeat Example 23.3.

The shear flow distribution and rate of twist in the section are obtained through the following MATLAB file:

```
% Declare any needed variables
syms q_s0I q_s0II q_s0III
B = [2580 3880 3230 3230 3880 2580];
y = [165 230 200 -200 -230 -165];
e = 50;
w = 1020;
S_y = 86.8*10^3;
G = [27600 27600 27600 3*27600 27600 27600 27600 27600];
G_ref = 27600;

% Define the length, thickness, and cell areas provided
L = [1023 1274 2200 1270 1023 330 (y(2)-e) -(y(5)-e) (y(3)-e) -(y(4)-e)];
t = [1.22 1.63 2.03 1.22 1.22 1.63 2.64 2.64 2.64 2.64];
A = [265000 213000 413000];
```

```
% Calculate the modulus weighted thickness using Eq. (23.9)
t_s = t.*G/G_ref;

% Calculate the delta for each wall/cell
del = round(L./t_s);
delta_I = del(3)+sum(del(9:10));
delta_II = del(2)+del(9)+del(4)+del(7);
delta_III = del(1)+sum(del(5:8));

delta = [delta_I delta_II delta_III];
delta_I = [del(3) del(9) del(10)];
delta_II = [del(2) del(9) del(4) del(7)];
delta_III = [del(1) del(5) del(6) del(7) del(8)];

% Calculate the section properties
I_xx = sum(B.*y.^2);
% I_yy and I_xy not needed due to symmetry and M_y=0
% Due to symmetry and since S_x=0, Eq. (20.6) simplifies
% Calculate the base shear flows in the section using the reduced Eq. (20.6)
B_i = B.*y;
C = round(-S_y/I_xx*10^6)/(10^6);
q_12b = 0; % Cut cell in wall 1-2
q_23b = 0; % Cut cell in wall 2-3
q_34b = 0; % Cut cell in wall 3-4
q_38b = round(C*B_i(3)*10)/10;
q_48b = round(C*B_i(4)*10)/10;
q_78b = q_48b + q_38b;
q_27b = round(C*B_i(2)*10)/10;
q_57b = round(C*B_i(5)*10)/10;
q_16b = round(C*B_i(1)*10)/10;
q_65b = q_16b + round(C*B_i(6)*10)/10;
q_b = [q_12b q_23b q_34b -q_78b -q_65b -q_16b q_27b q_57b q_38b -q_48b];
q_bI = [q_34b -q_38b q_48b];
q_bII = [q_23b q_38b q_78b -q_27b];
q_bIII = [q_12b -q_65b -q_16b q_27b -q_57b];

% Substitute the values of delta and q_b into Eq. (23.10) for each cell
eqI = (q_s0I*delta(1)-q_s0II*del(9)+sum(q_bI.*delta_I))/(2*A(1)*G_ref);
eqII = (-q_s0I*del(9)+q_s0II*delta(2)-q_s0III*del(7)+sum(q_bII.*delta_II))/(
2*A(2)*G_ref);
eqIII = (-q_s0II*del(7)+q_s0III*delta(3)+sum(q_bIII.*delta_III))/(
2*A(3)*G_ref);

% Take moments about the intersection of the x-axis and web 572 using Eq. (23.12)
L_M = [y(1) y(2) y(3) e y(5) w 0 0 L(4) L(4)];
eqIV = sum(q_b.*L_M.*L)+sum(2*A.*[q_s0I q_s0II q_s0III]);
```

```
% Simultaneously solve eqI-eqIV
[q_s0i q_s0ii q_s0iii] = solve(eqI-eqII,eqII-eqIII,eqIV,q_s0I,q_s0II,q_s0III);
q_s0i = vpa(q_s0i,2);
q_s0ii = vpa(q_s0ii,2);
q_s0iii = vpa(q_s0iii,2);

% Calculate the shear flows using q_s0i, q_s0ii, and q_s0iii
q_12 = q_12b+q_s0iii;
q_23 = q_23b+q_s0ii;
q_34 = q_34b+q_s0i;
q_78 = q_78b-q_s0ii;
q_65 = q_65b-q_s0iii;
q_16 = q_16b-q_s0iii;
q_27 = q_27b+q_s0iii-q_s0ii;
q_57 = q_57b-q_s0iii;
q_38 = q_38b-q_s0i+q_s0ii;
q_48 = q_48b+q_s0i;
q = double([q_12 q_23 q_34 q_78 q_65 q_16 q_27 q_57 q_38 q_48]);
q = round(q*10)/10;

% Substitute q_s0i, q_s0ii, and q_s0iii back into any of eqI-eqIV
dtheta_dz = subs(subs(eqI,q_s0I,q_s0i),q_s0II,q_s0ii);
dtheta_dz = round(double(dtheta_dz)*10^8)/(10^8);

% Output the shear flows and angle of twist to the Command Window
disp('The shear flows (N/mm) are:')
text = '[q_12, q_23, q_34, q_78, q_65, q_16, q_27, q_57, q_38, q_48] = [';
c = length(text);
for j=1:1:10
    num = num2str(q(j));
    if j<10
        text(c+1:c+length(num)+2) = [num ','];
        c = length(text);
    else
        text(c+1:c+length(num)+1) = [num ']'];
        disp(text)
    end
end
disp(['The rate of twist in the section = ' num2str(dtheta_dz) 'rad/mm'])
```

The Command Window outputs resulting from this MATLAB file are as follows:

The shear flows (N/mm) are:

[q\_12, q\_23, q\_34, q\_78, q\_65, q\_16, q\_27, q\_57, q\_38, q\_48] = [16, 9.3, 4.9, -9.3, -16, -61.5, -88.7, 79.5, -64.7, 74]

The rate of twist in the section = 1.09e-006 rad/mm

## 23.5 SHEAR CENTER

The position of the shear center of a wing section is found in an identical manner to that described in [Section 17.3](#). Arbitrary shear loads  $S_x$  and  $S_y$  are applied in turn through the shear center S, the corresponding shear flow distributions determined, and moments taken about some convenient point. The shear flow distributions are obtained as described previously in the shear of multicell wing sections, except that the  $N$  equations of the type (23.10) are sufficient for a solution, since the rate of twist  $d\theta/dz$  is zero for shear loads applied through the shear center.

## 23.6 TAPERED WINGS

Wings are generally tapered in both spanwise and chordwise directions. The effects on the analysis of taper in a single-cell beam have been discussed in [Section 21.2](#). In a multicell wing section, the effects are dealt with in an identical manner, except that the moment [equation \(21.16\)](#) becomes, for an  $N$ -cell wing section (see [Figs 21.5](#) and [23.8](#)),

$$S_x \eta_0 - S_y \xi_0 = \sum_{R=1}^N \oint_R q_b p_0 \, ds + \sum_{R=1}^N 2A_R q_{s,0,R} - \sum_{r=1}^m P_{x,r} \eta_r + \sum_{r=1}^m P_{y,r} \xi_r \quad (23.13)$$

### Example 23.4

A two-cell beam has singly symmetrical cross-sections 1.2 m apart and tapers symmetrically in the  $y$  direction about a longitudinal axis ([Fig. 23.12](#)). The beam supports loads that produce a shear force  $S_y = 10 \text{ kN}$  and a bending moment  $M_x = 1.65 \text{ kNm}$  at the larger cross-section; the shear load is applied in the plane of the internal spar web. If booms 1 and 6 lie on a plane that is parallel to the  $yz$  plane, calculate the forces in the booms and the shear flow distribution in the walls at the larger cross-section. The booms are assumed to resist all the direct stresses while the walls are effective only in shear. The shear modulus is constant throughout, the vertical webs are all 1.0 mm thick, while the remaining walls are all 0.8 mm thick:

Boom areas:  $B_1 = B_3 = B_4 = B_6 = 600 \text{ mm}^2$ ,  $B_2 = B_5 = 900 \text{ mm}^2$

At the larger cross-section,

$$I_{xx} = 4 \times 600 \times 90^2 + 2 \times 900 \times 90^2 = 34.02 \times 10^6 \text{ mm}^4$$

The direct stress in a boom is given by [Eq. \(16.17\)](#) in which  $I_{xy} = 0$  and  $M_y = 0$ ; that is,

$$\sigma_{z,r} = \frac{M_x y_r}{I_{xx}}$$

from which

$$P_{z,r} = \frac{M_x y_r}{I_{xx}} B_r$$

or

$$P_{z,r} = \frac{1.65 \times 10^6 y_r B_r}{34.02 \times 10^6} = 0.08 y_r B_r \quad (\text{i})$$

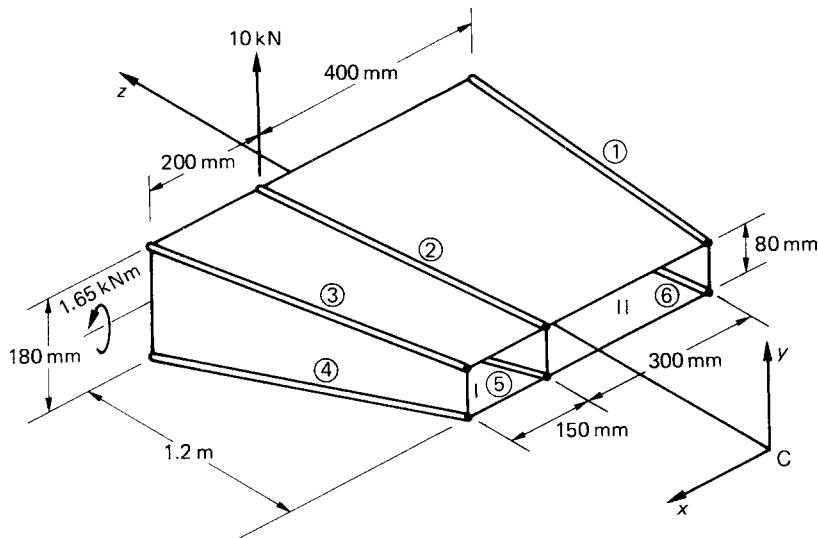


FIGURE 23.12 Tapered Beam of Example 23.4

The value of  $P_{z,r}$  is calculated from Eq. (i) in column ② of Table 23.4;  $P_{x,r}$  and  $P_{y,r}$  follow from Eqs. (21.10) and (21.9), respectively, in columns ⑤ and ⑥. The axial load  $P_r$  is given by  $\sqrt{②^2 + ⑤^2 + ⑥^2}$  in column ⑦ and has the same sign as  $P_{z,r}$  (see Eq. (21.12)). The moments of  $P_{x,r}$  and  $P_{y,r}$ , columns ⑩ and ⑪, are calculated for a moment center at the mid-point of the internal web taking counterclockwise moments as positive.

From column ⑤,

$$\sum_{r=1}^6 P_{x,r} = 0$$

(as would be expected from symmetry). From column ⑥,

$$\sum_{r=1}^6 P_{y,r} = 764.4 \text{ N}$$

Table 23.4 Example 23.4

① Boom	② $P_{z,r}$ (N)	③ $\frac{\delta x_r}{\delta z}$	④ $\frac{\delta y_r}{\delta z}$	⑤ $P_{x,r}$ (N)	⑥ $P_{y,r}$ (N)	⑦ $P_r$ (N)	⑧ $\xi_r$ (mm)	⑨ $\eta_r$ (mm)	⑩ $P_{x,r}\eta_r$ (N mm)	⑪ $P_{y,r}\xi_r$ (N mm)
1	2619.0	0	0.0417	0	109.2	2621.3	400	90	0	43,680
2	3928.6	0.0833	0.0417	327.3	163.8	3945.6	0	90	-29,457	0
3	2619.0	0.1250	0.0417	327.4	109.2	2641.6	200	90	-29,466	21,840
4	-2619.0	0.1250	-0.0417	-327.4	109.2	-2641.6	200	90	-29,466	21,840
5	-3928.6	0.0833	-0.0417	-327.3	163.8	-3945.6	0	90	-29,457	0
6	-2619.0	0	-0.0417	0	109.2	-2621.3	400	90	0	-43,680

From column ⑩,

$$\sum_{r=1}^6 P_{x,r} \eta_r = -117,846 \text{ Nmm}$$

From column ⑪,

$$\sum_{r=1}^6 P_{y,r} \xi_r = -43,680 \text{ Nmm}$$

From Eq. (21.15),

$$S_{x,w} = 0, \quad S_{y,w} = 10 \times 10^3 - 764.4 = 9235.6 \text{ N}$$

Also, since Cx is an axis of symmetry,  $I_{xy} = 0$  and Eq. (20.6) for the “open section” shear flow reduces to

$$q_b = -\frac{S_{y,w}}{I_{xx}} \sum_{r=1}^n B_r y_r$$

or

$$q_b = \frac{9,235.6}{34.02 \times 10^6} \sum_{r=1}^n B_r y_r = -2.715 \times 10^{-4} \sum_{r=1}^n B_r y_r \quad (\text{ii})$$

Cutting the top walls of each cell and using Eq. (ii), we obtain the  $q_b$  distribution shown in Fig. 23.13. Evaluating  $\delta$  for each wall and substituting in Eq. (23.10) gives, for cell I,

$$\frac{d\theta}{dz} = \frac{1}{2 \times 36,000G} (760q_{s,0,I} - 180q_{s,0,II} - 1314) \quad (\text{iii})$$

and, for cell II,

$$\frac{d\theta}{dz} = \frac{1}{2 \times 72,000G} (-180q_{s,0,I} + 1160q_{s,0,II} + 1314) \quad (\text{iv})$$

Taking moments about the mid-point of web 25, we have, using Eq. (23.13),

$$0 = -14.7 \times 180 \times 400 + 14.7 \times 180 \times 200 + 2 \times 36,000q_{s,0,I} + 2 \times 72,000q_{s,0,II} \\ -117,846 - 43,680$$

or

$$0 = -690,726 + 72,000q_{s,0,I} + 144,000q_{s,0,II} \quad (\text{v})$$

Solving Eqs (iii)–(v) gives

$$q_{s,0,I} = 4.6 \text{ N/mm}, \quad q_{s,0,II} = 2.5 \text{ N/mm}$$

and the resulting shear flow distribution is shown in Fig. 23.14.

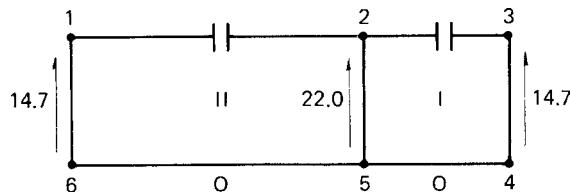


FIGURE 23.13 The  $q_b$  (N/mm) Distribution in the Beam Section of Example 23.4 (viewed along z axis toward C)

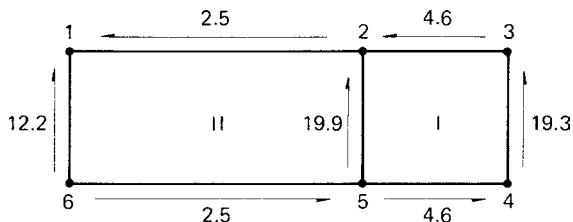


FIGURE 23.14 Shear Flow (N/mm) Distribution in the Tapered Beam of Example 23.4

## 23.7 DEFLECTIONS

Deflections of multicell wings may be calculated by the unit load method in an identical manner to that described in [Section 20.4](#) for open and single-cell beams.

### Example 23.5

Calculate the deflection at the free end of the two-cell beam shown in [Fig. 23.15](#), allowing for both bending and shear effects. The shear load of 44,500 N is applied through the shear center of the beam section. The booms carry all the direct stresses while the skin panels, of constant thickness throughout, are effective only in shear.

Take  $E = 69,000 \text{ N/mm}^2$  and  $G = 25,900 \text{ N/mm}^2$

Boom areas:  $B_1 = B_3 = B_4 = B_6 = 650 \text{ mm}^2$ ,  $B_2 = B_5 = 1,300 \text{ mm}^2$

The beam cross-section is symmetrical about a horizontal axis and carries a vertical load at its free end through the shear center. The deflection  $\Delta$  at the free end is then, from [Eqs. \(20.17\)](#) and [\(20.19\)](#),

$$\Delta = \int_0^{2000} \frac{M_{x,0}M_{x,1}}{EI_{xx}} dz + \int_0^{2000} \left( \int_{\text{section}} \frac{q_0 q_1}{Gt} ds \right) dz \quad (\text{i})$$

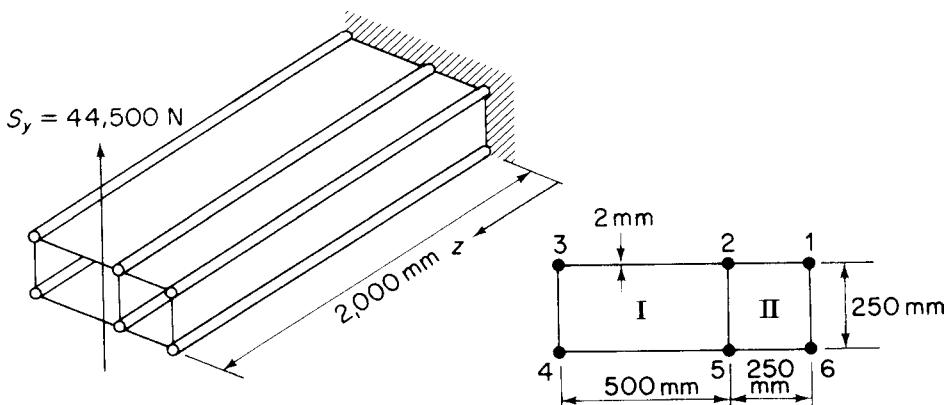


FIGURE 23.15 Deflection of a Two-Cell Wing Section

where

$$M_{x,0} = -44.5 \times 10^3(2000 - z), M_{x,1} = -(2000 - z)$$

and

$$I_{xx} = 4 \times 650 \times 125^2 + 2 \times 13,00 \times 125^2 = 81.3 \times 10^6 \text{ mm}^4$$

also

$$S_{y,0} = 44.5 \times 10^3 \text{ N}, S_{y,1} = 1$$

The  $q_0$  and  $q_1$  shear flow distributions are obtained as previously described (note  $d\theta/dz = 0$  for a shear load through the shear center) and are

$$\begin{aligned} q_{0,12} &= 9.6 \text{ N/mm}, \quad q_{0,23} = -5.8 \text{ N/mm}, \quad q_{0,43} = 50.3 \text{ N/mm} \\ q_{0,45} &= -5.8 \text{ N/mm}, \quad q_{0,56} = 9.6 \text{ N/mm}, \quad q_{0,61} = 54.1 \text{ N/mm} \\ q_{0,52} &= 73.6 \text{ N/mm at all sections of the beam} \end{aligned}$$

The  $q_1$  shear flows in this case are given by  $q_0/44.5 \times 10^3$ . Thus,

$$\begin{aligned} \int_{\text{section}} \frac{q_0 q_1}{Gt} ds &= \frac{1}{25,900 \times 2 \times 44.5 \times 10^3} (9.6^2 \times 250 \times 2 + 5.8^2 \times 500 \times 2 \\ &\quad + 50.3^2 \times 250 + 54.1^2 \times 250 + 73.6^2 \times 250) \\ &= 1.22 \times 10^{-3} \end{aligned}$$

Hence, from Eq. (i),

$$\Delta = \int_0^{2,000} \frac{44.5 \times 10^3 (2,000 - z)^2}{69,000 \times 81.3 \times 10^6} dz + \int_0^{2,000} 1.22 \times 10^{-3} dz$$

giving

$$\Delta = 23.5 \text{ mm}$$

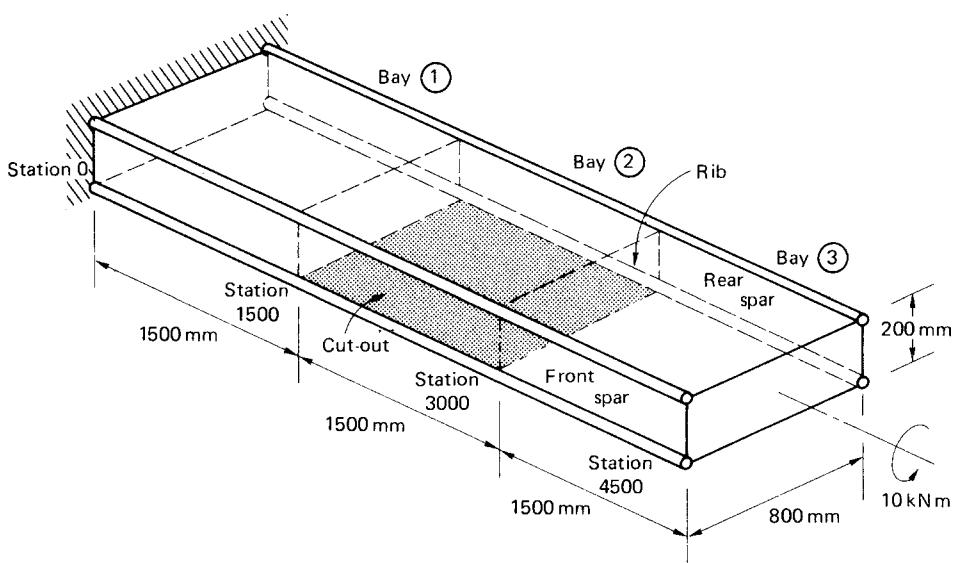
## 23.8 CUT-OUTS IN WINGS

Wings, as well as fuselages, have openings in their surfaces to accommodate undercarriages, engine nacelles, weapons installations, and the like. In addition, inspection panels are required at specific positions, so that, as for fuselages, the loads in adjacent portions of the wing structure are modified.

Initially, we consider the case of a wing subjected to a pure torque in which one bay of the wing has the skin on its undersurface removed. The method is best illustrated by a numerical example.

### Example 23.6

The structural portion of a wing consists of a three-bay rectangular section box that may be assumed to be firmly attached at all points around its periphery to the aircraft fuselage at its inboard end. The skin on the undersurface of the central bay has been removed and the wing is subjected to a torque of 10 kNm at its tip (Fig. 23.16). Calculate the shear flows in the skin panels and spar webs, the loads in the corner flanges, and the forces in the ribs on each



**FIGURE 23.16 Three-Bay Wing Structure With Cut-out of Example 23.6**

side of the cut-out, assuming that the spar flanges carry all the direct loads while the skin panels and spar webs are effective only in shear.

If the wing structure were continuous and the effects of restrained warping at the built-in end ignored, the shear flows in the skin panels would be given by Eq. (18.1), that is,

$$q = \frac{T}{2A} = \frac{10 \times 10^6}{2 \times 200 \times 800} = 31.3 \text{ N/mm}$$

and the flanges would be unloaded. However, the removal of the lower skin panel in bay ② results in a torsionally weak channel section for the length of bay ②, which must in any case still transmit the applied torque to bay ① and subsequently to the wing support points. Although open section beams are inherently weak in torsion (see Section 18.2), the channel section in this case is attached at its inboard and outboard ends to torsionally stiff closed boxes, so that, in effect, it is built in at both ends. We shall examine the effect of axial constraint on open section beams subjected to torsion in Chapter 28. An alternative approach is to assume that the torque is transmitted across bay ② by the differential bending of the front and rear spars. The bending moment in each spar is resisted by the flange loads  $P$  as shown, for the front spar, in Fig. 23.17(a). The shear loads in the front and rear spars form a couple at any station in bay ②, which is equivalent to the applied torque. Thus, from Fig. 23.17(b),

$$800S = 10 \times 10^6 \text{ Nmm}$$

that is,

$$S = 12,500 \text{ N}$$

The shear flow  $q_1$  in Fig. 23.17(a) is given by

$$q_1 = \frac{12,500}{200} = 62.5 \text{ N/mm}$$

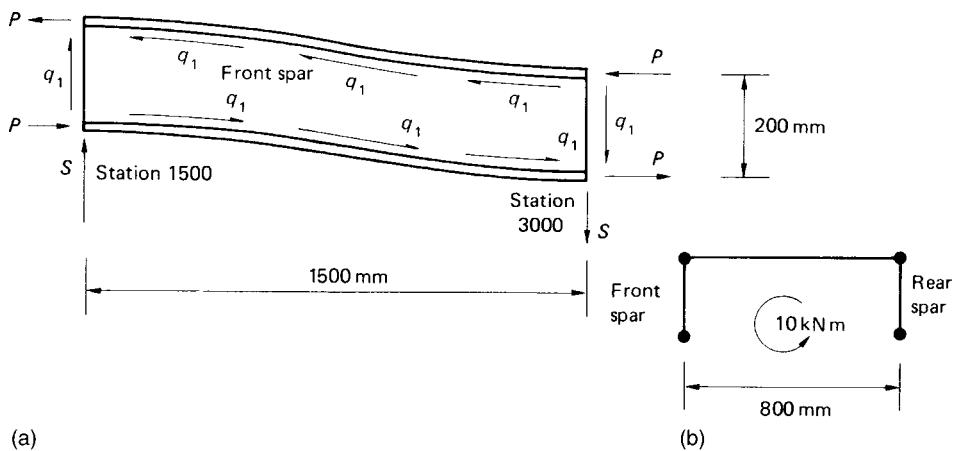


FIGURE 23.17 Differential Bending of Front Spar

Midway between stations 1500 and 3000 a point of contraflexure occurs in the front and rear spars, so that at this point the bending moment is zero. Hence,

$$200P = 12,500 \times 750 \text{ Nmm}$$

so that

$$P = 46,875 \text{ N}$$

Alternatively,  $P$  may be found by considering the equilibrium of either of the spar flanges. Thus,

$$2P = 1500q_1 = 1500 \times 62.5 \text{ N}$$

from which

$$P = 46,875 \text{ N}$$

The flange loads  $P$  are reacted by loads in the flanges of bays ① and ③. These flange loads are transmitted to the adjacent spar webs and skin panels, as shown in Fig. 23.18 for bay ③, and modify the shear flow distribution given by Eq. (18.1). For equilibrium of flange 1,

$$1500q_2 - 1500q_3 = P = 46,875 \text{ N}$$

or

$$q_2 - q_3 = 31.3 \quad (\text{i})$$

The resultant of the shear flows  $q_2$  and  $q_3$  must be equivalent to the applied torque. Hence, for moments about the center of symmetry at any section in bay ③ and using Eq. (20.10),

$$200 \times 800q_2 + 200 \times 800q_3 = 10 \times 10^6 \text{ Nmm}$$

or

$$q_2 + q_3 = 62.5 \quad (\text{ii})$$

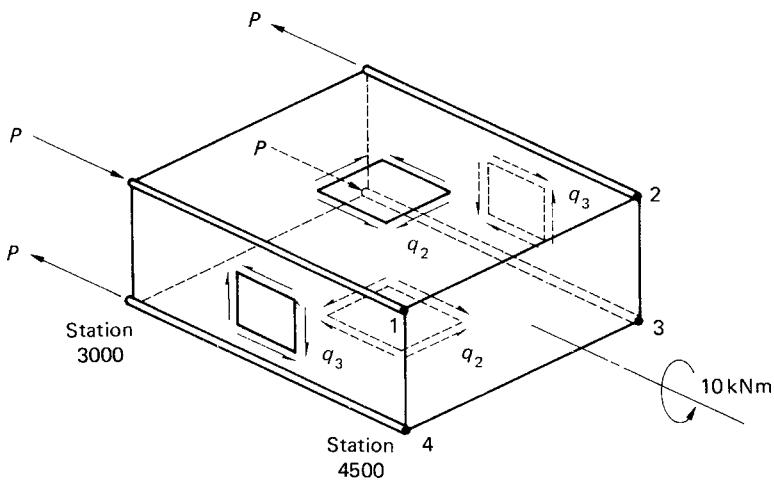


FIGURE 23.18 Loads on Bay ③ of the Wing of Example 23.6

Solving Eqs. (i) and (ii), we obtain

$$q_2 = 46.9 \text{ N/mm}, \quad q_3 = 15.6 \text{ N/mm}$$

Comparison with the results of Eq. (18.1) shows that the shear flows are increased by a factor of 1.5 in the upper and lower skin panels and decreased by a factor of 0.5 in the spar webs.

The flange loads are in equilibrium with the resultants of the shear flows in the adjacent skin panels and spar webs. Thus, for example, in the top flange of the front spar,

$$\begin{aligned} P(\text{st. 4500}) &= 0 \\ P(\text{st. 3000}) &= 1500q_2 - 1500q_3 = 46,875 \text{ N (compression)} \\ P(\text{st. 2250}) &= 1500q_2 - 1500q_3 - 750q_1 = 0 \end{aligned}$$

The loads along the remainder of the flange follow from antisymmetry, giving the distribution shown in Fig. 23.19. The load distribution in the bottom flange of the rear spar is identical to that shown in Fig. 23.19, while the distributions in the bottom flange of the front spar and the top flange of the rear spar are reversed. We note that the flange loads are zero at the built-in end of the wing (station 0). Generally, however, additional stresses are induced

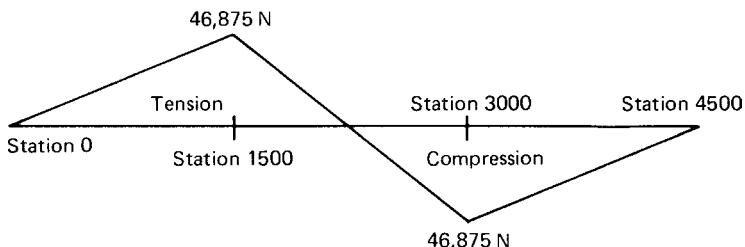
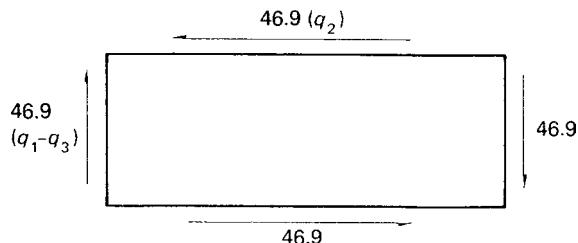


FIGURE 23.19 Distribution of Load in the Top Flange of the Front Spar of the Wing of Example 23.6



**FIGURE 23.20 Shear Flows (N/mm) on Wing Rib at Station 3000 in the Wing of Example 23.6**

by the warping restraint at the built-in end; these are investigated in Chapter 27. The loads on the wing ribs on either the inboard or outboard end of the cut-out are found by considering the shear flows in the skin panels and spar webs immediately inboard and outboard of the rib. Thus, for the rib at station 3000, we obtain the shear flow distribution shown in Fig. 23.20.

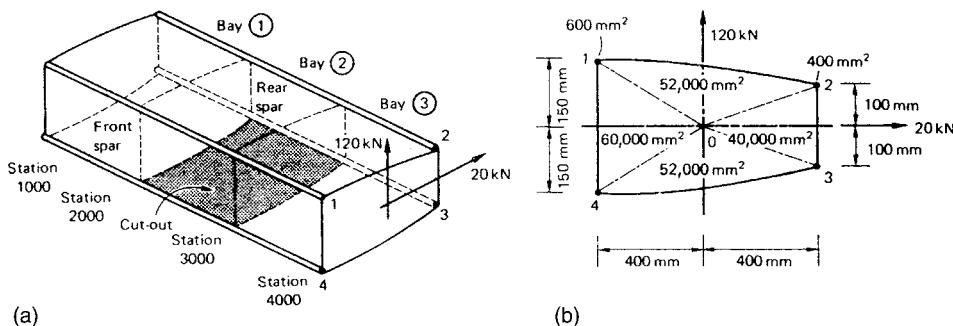
In Example 23.6, we implicitly assumed in the analysis that the local effects of the cut-out were completely dissipated within the length of the adjoining bays, which were equal in length to the cut-out bay. The validity of this assumption relies on St. Venant's principle (Section 2.4). It may generally be assumed therefore that the effects of a cut-out are restricted to spanwise lengths of the wing equal to the length of the cut-out on both the inboard and outboard ends of the cut-out bay.

We now consider the more complex case of a wing having a cut-out and subjected to shear loads which produce both bending and torsion. Again, the method is illustrated by a numerical example.

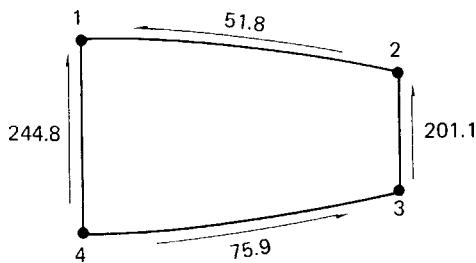
### Example 23.7

A wing box has the skin panel on its undersurface removed between stations 2000 and 3000 and carries lift and drag loads that are constant between stations 1000 and 4000, as shown in Fig. 23.21(a). Determine the shear flows in the skin panels and spar webs and also the loads on the wing ribs at the inboard and outboard ends of the cut-out bay.

Assume that all bending moments are resisted by the spar flanges while the skin panels and spar webs are effective only in shear.



**FIGURE 23.21 Wing Box of Example 23.7**



**FIGURE 23.22 Shear Flow (N/mm) Distribution at Any Station in the Wing Box of Example 23.7 Without the Cut-out**

The simplest approach is first to determine the shear flows in the skin panels and spar webs as though the wing box were continuous and to apply an equal and opposite shear flow to that calculated around the edges of the cut-out. The shear flows in the wing box without the cut-out are the same in each bay and are calculated using the method described in [Section 20.3](#) and illustrated in [Example 20.4](#). This gives the shear flow distribution shown in [Fig. 23.22](#).

We now consider bay ② and apply a shear flow of 75.9 N/mm in the wall 34 in the opposite sense to that shown in [Fig. 23.22](#). This reduces the shear flow in the wall 34 to zero and, in effect, restores the cut-out to bay ②. The shear flows in the remaining walls of the cut-out bay no longer are equivalent to the externally applied shear loads, so that corrections are required. Consider the cut-out bay ([Fig. 23.23](#)) with the shear flow of 75.9 N/mm applied in the opposite sense to that shown in [Fig. 23.22](#). The correction shear flows  $q'_{12}$ ,  $q'_{32}$ , and  $q'_{14}$  may be found using statics. Thus, resolving forces horizontally, we have

$$800q'_{12} = 800 \times 75.9 \text{ N}$$

from which

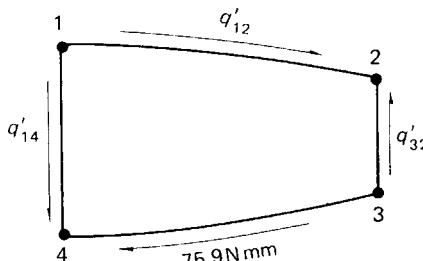
$$q'_{12} = 75.9 \text{ N/mm}$$

Resolving forces vertically,

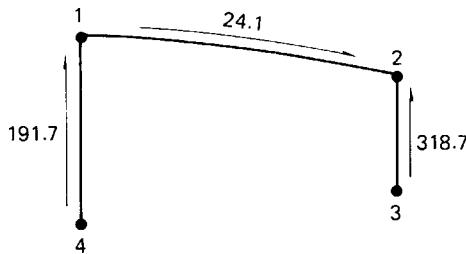
$$200q'_{32} = 50q'_{12} - 50 \times 75.9 - 300q'_{14} = 0 \quad (\text{i})$$

and, taking moments about O in [Fig. 23.21\(b\)](#), we obtain

$$2 \times 52,000q'_{12} - 2 \times 40,000q'_{32} + 2 \times 52,000 \times 75.9 - 2 \times 60,000q'_{14} = 0 \quad (\text{ii})$$



**FIGURE 23.23 Correction Shear Flows in the Cut-out Bay of the Wing Box of Example 23.7**



**FIGURE 23.24 Final Shear Flows (N/mm) in the Cut-out Bay of the Wing Box of Example 23.7**

Solving Eqs. (i) and (ii) gives

$$q'_{32} = 117.6 \text{ N/mm}, \quad q'_{14} = 53.1 \text{ N/mm}$$

The final shear flows in bay ② are found by superimposing  $q'_{12}$ ,  $q'_{32}$ , and  $q'_{14}$  on the shear flows in Fig. 23.22, giving the distribution shown in Fig. 23.24. Alternatively, these shear flows could be found directly by considering the equilibrium of the cut-out bay under the action of the applied shear loads.

The correction shear flows in bay ② (Fig. 23.23) also modify the shear flow distributions in bays ① and ③. The correction shear flows to be applied to those shown in Fig. 23.22 for bay ③ (those in bay ① are identical) may be found by determining the flange loads corresponding to the correction shear flows in bay ②.

It can be seen from the magnitudes and directions of these correction shear flows (Fig. 23.23) that, at any section in bay ②, the loads in the upper and lower flanges of the front spar are equal in magnitude but opposite in direction; similarly for the rear spar. Thus, the correction shear flows in bay ② produce an identical system of flange loads to that shown in Fig. 23.17 for the cut-out bays in the wing structure of Example 23.6.

It follows that these correction shear flows produce differential bending of the front and rear spars in bay ② and that the spar bending moments and hence the flange loads are zero at the mid-bay points. Therefore, at station 3000, the flange loads are

$$P_1 = (75.9 + 53.1) \times 500 = 64,500 \text{ N (compression)}$$

$$P_4 = 64,500 \text{ N (tension)}$$

$$P_2 = (75.9 + 117.6) \times 500 = 96,750 \text{ N (tension)}$$

$$P_3 = 96,750 \text{ N (compression)}$$

These flange loads produce correction shear flows  $q''_{21}$ ,  $q''_{43}$ ,  $q''_{23}$ , and  $q''_{41}$  in the skin panels and spar webs of bay ③, as shown in Fig. 23.25. Thus, for equilibrium of flange 1,

$$1000q''_{41} + 1000q''_{21} = 64,500 \text{ N} \quad (\text{iii})$$

and, for equilibrium of flange 2,

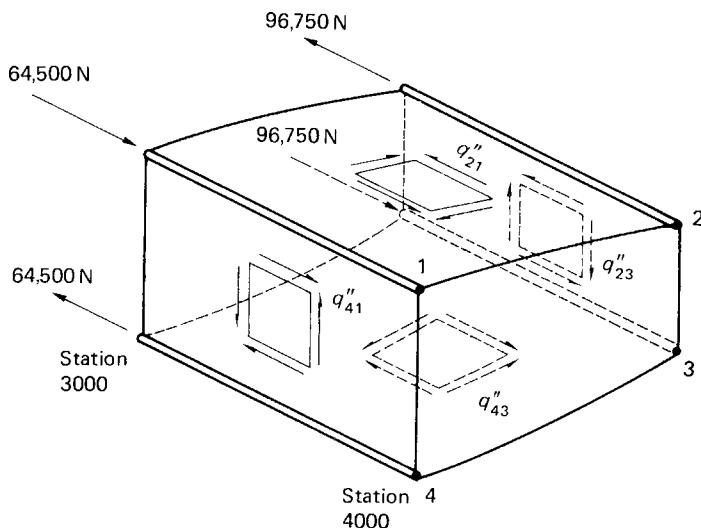
$$1000q''_{21} + 1000q''_{23} = 96,750 \text{ N} \quad (\text{iv})$$

For equilibrium in the chordwise direction at any section in bay ③,

$$800q''_{21} = 800q''_{43}$$

or

$$q''_{21} = q''_{43} \quad (\text{v})$$



**FIGURE 23.25 Correction Shear Flows in Bay ③ of the Wing Box of Example 23.7**

Finally, for vertical equilibrium at any section in bay ③,

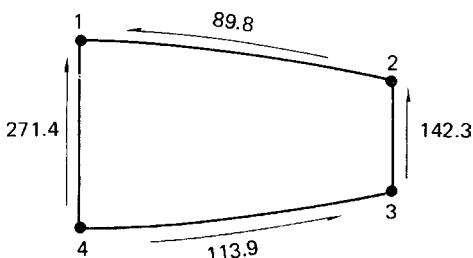
$$300q''_{41} + 50q''_{43} + 50q''_{23} - 200q''_{21} = 0 \quad (\text{vii})$$

Simultaneous solution of Eqs. (iii)–(vi) gives

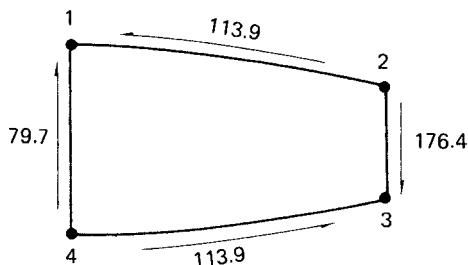
$$q''_{21} = q''_{43} = 38.0 \text{ N/mm}, \quad q''_{23} = 58.8 \text{ N/mm}, \quad q''_{41} = 26.6 \text{ N/mm}$$

Superimposing these correction shear flows on those shown in Fig. 23.22 gives the final shear flow distribution in bay ③, as shown in Fig. 23.26. The rib loads at stations 2000 and 3000 are found as before by adding algebraically the shear flows in the skin panels and spar webs on each side of the rib. Thus, at station 3000, we obtain the shear flows acting around the periphery of the rib, as shown in Fig. 23.27. The shear flows applied to the rib at the inboard end of the cut-out bay will be equal in magnitude but opposite in direction.

Note that, in this example, only the shear loads on the wing box between stations 1000 and 4000 are given. We cannot therefore determine the final values of the loads in the spar flanges, since we do not know the values of the bending moments at these positions caused by loads acting on other parts of the wing.



**FIGURE 23.26 Final Shear Flows (N/mm) in Bay ③ (and Bay ①) of the Wing Box of Example 23.7**



**FIGURE 23.27 Shear Flows (N/mm) Applied to the Wing Rib at Station 3000 in the Wing Box of Example 23.7**

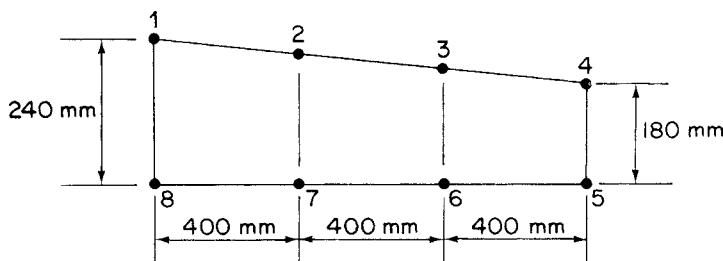
## PROBLEMS

**P.23.1** The central cell of a wing has the idealized section shown in Fig. P.23.1. If the lift and drag loads on the wing produce bending moments of  $-120,000$  Nm and  $-30,000$  Nm, respectively, at the section shown, calculate the direct stresses in the booms. Neglect axial constraint effects and assume that the lift and drag vectors are in the vertical and horizontal planes.

$$\begin{aligned} \text{Boom areas: } B_1 &= B_4 = B_5 = B_8 = 1000 \text{ mm}^2 \\ B_2 &= B_3 = B_6 = B_7 = 600 \text{ mm}^2 \end{aligned}$$

Answer:  $\sigma_1 = -190.7 \text{ N/mm}^2$ ,  $\sigma_2 = -181.7 \text{ N/mm}^2$ ,  $\sigma_3 = -172.8 \text{ N/mm}^2$   
 $\sigma_4 = -163.8 \text{ N/mm}^2$ ,  $\sigma_5 = 140 \text{ N/mm}^2$ ,  $\sigma_6 = 164.8 \text{ N/mm}^2$   
 $\sigma_7 = 189.6 \text{ N/mm}^2$ ,  $\sigma_8 = 214.4 \text{ N/mm}^2$ .

**P.23.1 MATLAB** Assuming the left-hand side height of the section shown in Fig. P.23.1 is labeled  $h_1$ , use MATLAB to repeat Problem P.23.1 for values of  $h_1$  ranging from 180 to 270 mm in increments of 10 mm. All other dimensions are as shown in Fig. P.23.1.



**FIGURE P.23.1**

Answer:

	$h_1$ (mm)	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$	$\sigma_5$	$\sigma_6$	$\sigma_7$	$\sigma_8$
(i)	180	-196.4	-204.4	-212.4	-220.4	196.4	204.4	212.4	220.4
(ii)	190	-196.4	-200.7	-205	-209.3	185.5	197.1	208.7	220.3
(iii)	200	-196	-196.9	-197.9	-198.8	175.2	190	204.8	219.6
(iv)	210	-195.3	-193.2	-191.1	-189	165.8	183.4	201	218.6
(v)	220	-193.8	-189.3	-184.7	-180.2	156.4	176.8	197.2	217.6
(vi)	230	-192.2	-185.4	-178.6	-171.8	147.8	170.6	193.4	216.2
(vii)	240	-190.7	-181.7	-172.8	-163.8	140	164.8	189.6	214.4
(viii)	250	-189.3	-178.2	-167.1	-156	133	159.4	185.8	212.2
(ix)	260	-187.4	-174.6	-161.8	-149	126.2	154.2	182.2	210.2
(x)	270	-185.1	-171	-156.9	-142.7	119.5	149.1	178.7	208.3

The units for these direct stresses are N/mm<sup>2</sup>.

**P.23.2** Figure P.23.2 shows the cross-section of a two-cell torque box. If the shear stress in any wall must not exceed 140 N/mm<sup>2</sup>, find the maximum torque that can be applied to the box. If this torque were applied at one end and resisted at the other end of such a box of span 2500 mm, find the twist in degrees of one end relative to the other and the torsional rigidity of the box. The shear modulus  $G = 26,600 \text{ N/mm}^2$  for all walls. Data:

Shaded areas:	$A_{34} = 6450 \text{ mm}^2, A_{16} = 7750 \text{ mm}^2$
Wall lengths:	$s_{34} = 250 \text{ mm}, s_{16} = 300 \text{ mm}$
Wall thickness:	$t_{12} = 1.63 \text{ mm}, t_{34} = 0.56 \text{ mm}$
	$t_{23} = t_{45} = t_{56} = 0.92 \text{ mm}$
	$t_{61} = 2.3 \text{ mm}$
	$t_{25} = 2.54 \text{ mm}$

Answer:  $T = 102,417 \text{ Nm}, \theta = 1.46^\circ, GJ = 10 \times 10^{12} \text{ Nmm}^2/\text{rad}$ .

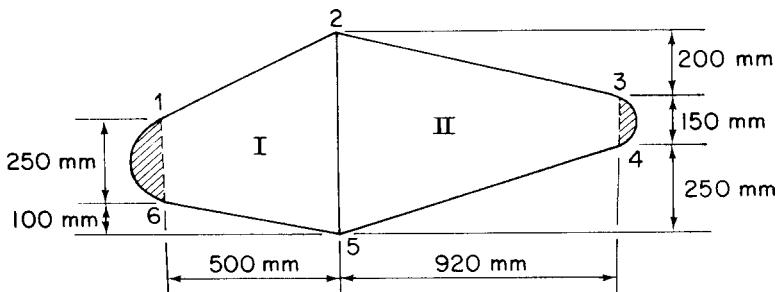


FIGURE P.23.2

**P.23.3** Determine the torsional stiffness of the four-cell wing section shown in Fig. P.23.3. Data are in Table 23.5.

Answer:  $522.5 \times 10^6 G \text{ Nmm}^2/\text{rad}$

**P.23.4** Determine the shear flow distribution for a torque of 56,500 Nm for the three-cell section shown in Fig. P.23.4 with the data shown in Table 23.6. The section has a constant shear modulus throughout.

Answer:  $q_{12}^U = 25.4 \text{ N/mm}$ ,  $q_{21}^L = 33.5 \text{ N/mm}$ ,  $q_{14} = q_{32} = 8.1 \text{ N/mm}$   
 $q_{43}^U = 13.4 \text{ N/mm}$ ,  $q_{34}^L = 5.3 \text{ N/mm}$

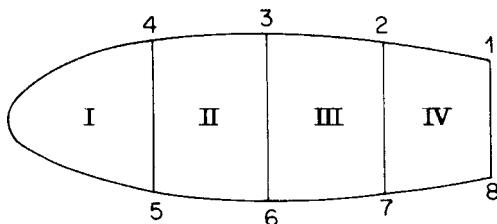


FIGURE P.23.3

**Table 23.5** Problem 23.3

Wall	12	23	34					
	78	67	56	$45^\circ$	$45^\circ$	36	27	18
Peripheral length (mm)	762	812	812	1,525	356	406	356	254
Thickness (mm)	0.915	0.915	0.915	0.711	1.220	1.625	1.220	0.915
Cell areas ( $\text{mm}^2$ )	$A_I = 161,500$		$A_{II} = 291,000$		$A_{III} = 291,000$		$A_{IV} = 226,000$	

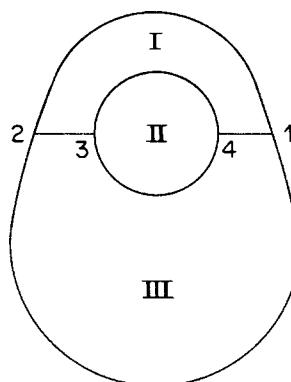


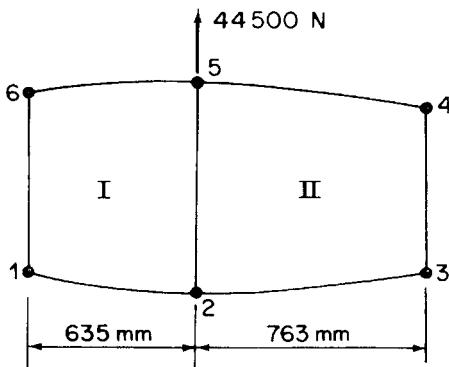
FIGURE P.23.4

**Table 23.6** Problem 23.4

Wall	Length (mm)	Thickness (mm)	Cell	Area (mm <sup>2</sup> )
12 <sup>U</sup>	1084	1.220	I	108,400
12 <sup>L</sup>	2160	1.625	II	202,500
14, 23	127	0.915	III	528,000
34 <sup>U</sup>	797	0.915		
34 <sup>L</sup>	797	0.915		

**P.23.5** The idealized cross-section of a two-cell thin-walled wing box is shown in Fig. P.23.5 with the data in Table 23.7. If the wing box supports a load of 44,500 N acting along web 25, calculate the shear flow distribution. The shear modulus  $G$  is the same for all walls of the wing box. The cell areas are  $A_I = 232,000 \text{ mm}^2$ ,  $A_{II} = 258,000 \text{ mm}^2$ .

Answer:  $q_{16} = 33.9 \text{ N/mm}$ ,  $q_{65} = q_{21} = 1.1 \text{ N/mm}$   
 $q_{45} = q_{23} = 7.2 \text{ N/mm}$ ,  $q_{34} = 20.8 \text{ N/mm}$   
 $q_{25} = 73.4 \text{ N/mm}$

**FIGURE P.23.5****Table 23.7** Problem 23.5

Wall	Length (mm)	Thickness (mm)	Boom	Area (mm <sup>2</sup> )
16	254	1.625	1,6	1290
25	406	2.032	2,5	1936
34	202	1.220	3,4	645
12, 56	647	0.915		
23, 45	775	0.559		

**P.23.6** Figure P.23.6 shows a singly symmetric, two-cell wing section in which all direct stresses are carried by the booms, shear stresses alone being carried by the walls. All walls are flat with the exception of the nose portion 45. Find the position of the shear center S and the shear flow distribution for a load of  $S_y = 66,750 \text{ N}$  through S. Tabulated in Table 23.8 are lengths, thicknesses, and shear moduli of the shear carrying walls. Note that dotted line 45 is not a wall. Nose area  $N_1 = 51,500 \text{ mm}^2$ .

Answer:  $x_S = 161.1 \text{ mm}$ ,  $q_{12} = q_{78} = 17.8 \text{ N/mm}$ ,  $q_{32} = q_{76} = 18.5 \text{ N/mm}$

$q_{63} = 88.4 \text{ N/mm}$ ,  $q_{43} = q_{65} = 2.7 \text{ N/mm}$ ,  $q_{54} = 39.0 \text{ N/mm}$

$q_{81} = 90.4 \text{ N/mm}$ .

**P.23.6 MATLAB** Assuming the length of cell wall 34 (and consequently wall 56) of the section shown in Fig. P.23.6 is labeled  $L_I$ , use MATLAB to repeat Problem P.23.6 for values of  $L_I$  ranging from 350 to 410 mm in increments of 10 mm. The units for the shear flows ( $q$ ) are N/mm; the units for  $L_I$  and  $x_S$  are mm.

Answer:

	$L_I$	$x_S$	$q_{12}$	$q_{23}$	$q_{34}$	$q_{45}$	$q_{56}$	$q_{67}$	$q_{78}$	$q_{81}$	$q_{36}$
(i)	350	166.4	17.8	-18.5	-2.8	-39.1	-2.8	-18.5	17.8	90.4	-88.3
(ii)	360	164.4	17.8	-18.5	-2.8	v39.1	-2.8	-18.5	17.8	90.4	-88.3
(iii)	370	163	17.8	-18.5	-2.7	-39	-2.7	-18.5	17.8	90.4	-88.3
(iv)	380	161.1	17.8	-18.5	-2.7	-39	-2.7	-18.5	17.8	90.4	-88.4
(v)	390	159.7	17.8	-18.5	-2.6	-38.9	-2.6	-18.5	17.8	90.4	-88.4
(vi)	400	157.8	17.8	-18.5	-2.6	-38.9	-2.6	-18.5	17.8	90.4	-88.5
(vii)	410	155.9	17.8	-18.5	-2.6	-38.9	-2.6	-18.5	17.8	90.4	-88.5

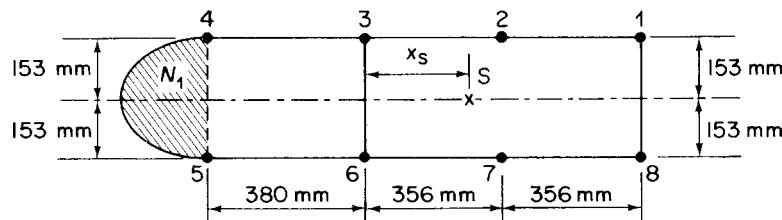


FIGURE P.23.6

**Table 23.8** Problem 23.6

Wall	Length (mm)	Thickness (mm)	$G(\text{N/mm}^2)$	Boom	Area ( $\text{mm}^2$ )
34, 56	380	0.915	20,700	1, 3, 6, 8	1290
12, 23, 67, 78	356	0.915	24,200	2, 4, 5, 7	645
36, 81	306	1.220	24,800		
45	610	1.220	24,800		

**P.23.7** A singly symmetric wing section consists of two closed cells and one open cell (see Fig. P.23.7 and Table 23.9). Webs 25, 34 and walls 12, 56 are straight, while all other walls are curved. All walls of the section are assumed to be effective in carrying shear stresses only, direct stresses being carried by booms 1–6. Calculate the distance  $x_S$  of the shear center S aft of web 34. The shear modulus  $G$  is the same for all walls.

Answer: 241.4 mm

**Table 23.9** Problem 23.7

Wall	Length (mm)	Thickness (mm)	Boom	Area ( $\text{mm}^2$ )	Cell	Area ( $\text{mm}^2$ )
12, 56	510	0.559	1, 6	645	I	93,000
23, 45	765	0.915	2, 5	1290	II	258,000
34°	1015	0.559	3, 4	1935		
34 <sup>i</sup>	304	2.030				
25	304	1.625				

**P.23.8** A portion of a tapered, three-cell wing has singly symmetrical idealized cross-sections 1000 mm apart, as shown in Fig. P.23.8. A bending moment  $M_x = 1800 \text{ Nm}$  and a shear load  $S_y = 12,000 \text{ N}$  in the plane of web 52 are applied at the larger cross-section. Calculate the forces in the booms and the shear

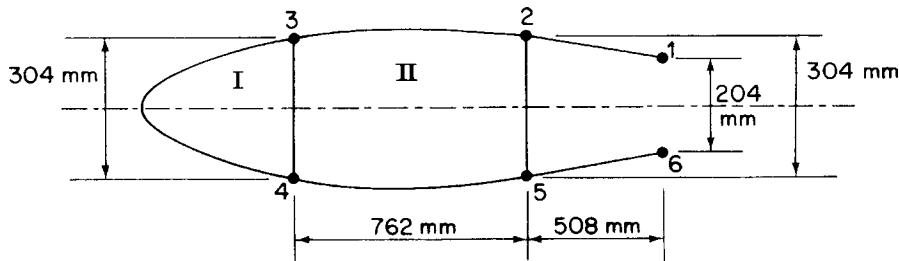


FIGURE P.23.7

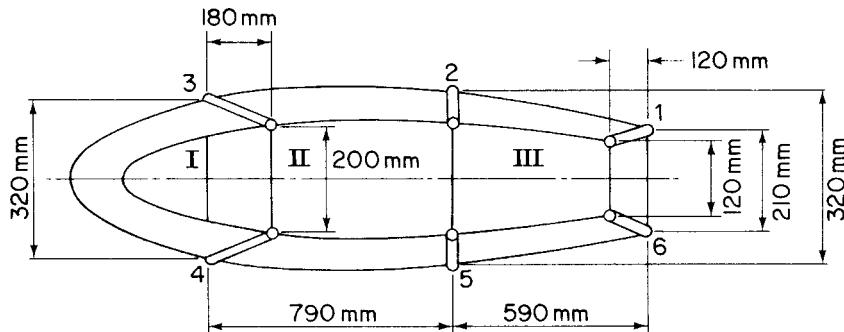


FIGURE P.23.8

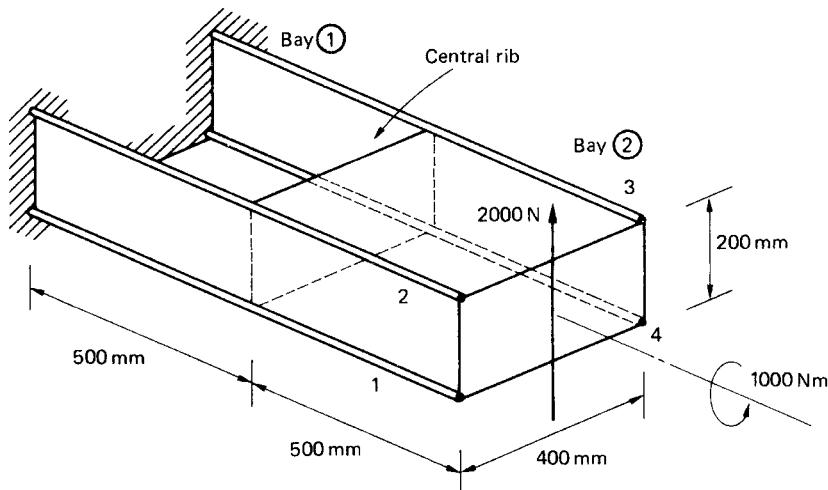
flow distribution at this cross-section. The modulus  $G$  is constant throughout. Section dimensions at the larger cross-section are given in [Table 23.10](#).

Answer:  $P_1 = -P_6 = 1200 \text{ N}$ ,  $P_2 = -P_5 = 2424 \text{ N}$ ,  $P_3 = -P_4 = 2462 \text{ N}$   
 $q_{12} = q_{56} = 3.74 \text{ N/mm}$ ,  $q_{23} = q_{45} = 3.11 \text{ N/mm}$ ,  $q_{34^\circ} = 0.06 \text{ N/mm}$   
 $q_{43^i} = 12.16 \text{ N/mm}$ ,  $q_{52} = 14.58 \text{ N/mm}$ ,  $q_{61} = 11.22 \text{ N/mm}$ .

**Table 23.10** Problem 23.8

Wall	Length (mm)	Thickness (mm)	Boom	Area ( $\text{mm}^2$ )	Cell	Area ( $\text{mm}^2$ )
12, 56	600	1.0	1, 6	600	I	100,000
23, 45	800	1.0	2, 5	800	II	260,000
$34^\circ$	1200	0.6	3, 4	800	III	180,000
$34^i$	320	2.0				
25	320	2.0				
16	210	1.5				

**P.23.9** A portion of a wing box is built in at one end and carries a shear load of 2000 N through its shear center and a torque of 1000 Nm, as shown in [Fig. P.23.9](#). If the skin panel in the upper surface of the inboard bay is removed, calculate the shear flows in the spar webs and remaining skin panels, the distribution of load in the spar flanges, and the loading on the central rib. Assume that the spar webs and skin panels are effective in resisting shear stresses only.



**FIGURE P.23.9**

Answer: Bay ①:  $q$  in spar webs = 7.5 N/mm

Bay ②:  $q$  in spar webs = 1.9 N/mm, in skin panels = 9.4 N/mm

Flange loads (2): at built-in end = 1875 N (compression)

at central rib = 5625 N (compression)

Rib loads:  $q$  (horizontal edges) = 9.4 N/mm

$q$  (vertical edges) = 9.4 N/mm.

# Fuselage frames and wing ribs

# 24

Aircraft are constructed primarily from thin metal skins that are capable of resisting in-plane tension and shear loads but buckle under comparatively low values of in-plane compressive loads. The skins are therefore stiffened by longitudinal stringers that resist the in-plane compressive loads and, at the same time, resist small distributed loads normal to the plane of the skin. The effective length in compression of the stringers is reduced, in the case of fuselages, by transverse frames or bulkheads or, in the case of wings, by ribs. In addition, the frames and ribs resist concentrated loads in transverse planes and transmit them to the stringers and the plane of the skin. Therefore, cantilever wings may be bolted to fuselage frames at the spar caps while undercarriage loads are transmitted to the wing through spar and rib attachment points.

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## 24.1 PRINCIPLES OF STIFFENER/WEB CONSTRUCTION

Generally, frames and ribs are themselves fabricated from thin sheets of metal and therefore require stiffening members to distribute the concentrated loads to the thin webs. If the load is applied in the plane of a web, the stiffeners must be aligned with the direction of the load. Alternatively, if this is not possible, the load should be applied at the intersection of two stiffeners, so that each stiffener resists the component of load in its direction. The basic principles of stiffener/web construction are illustrated in [Example 24.1](#).

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### Example 24.1

A cantilever beam ([Fig. 24.1](#)) carries concentrated loads as shown. Calculate the distribution of stiffener loads and the shear flow distribution in the web panels, assuming that the latter are effective only in shear.

We note that stiffeners HKD and JK are required at the point of application of the 4000 N load to resist its vertical and horizontal components. A further transverse stiffener GJC is positioned at the unloaded end J of the stiffener JK, since stress concentrations are produced if a stiffener ends in the center of a web panel. We note also that the web panels are effective only in shear, so that the shear flow is constant throughout a particular web panel; the assumed directions of the shear flows are shown in [Fig. 24.1](#).

It is instructive at this stage to examine the physical role of the different structural components in supporting the applied loads. Generally, stiffeners are assumed to withstand axial forces only, so that the horizontal component of the load at K is equilibrated locally by the axial load in the stiffener JK and not by the bending of stiffener HKD. By the same argument, the vertical component of the load at K is resisted by the axial load in the stiffener HKD. These axial stiffener loads are equilibrated in turn by the resultants of the shear flows  $q_1$  and  $q_2$  in the web panels CDKJ and JKHG. Thus, we see that the web panels resist the shear component of the externally applied load and at the

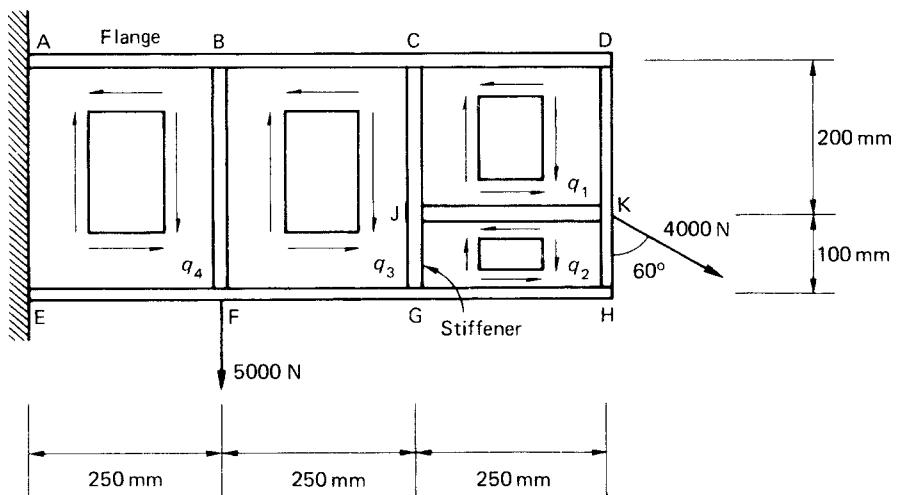


FIGURE 24.1 Cantilever Beam of Example 24.1

same time transmit the bending and axial load of the externally applied load to the beam flanges; subsequently, the flange loads are reacted at the support points A and E.

Consider the free body diagrams of the stiffeners JK and HKD shown in Fig. 24.2.

From the equilibrium of stiffener JK, we have

$$(q_1 - q_2) \times 250 = 4000 \sin 60^\circ = 3464.1 \text{ N} \quad (\text{i})$$

and, from the equilibrium of stiffener HKD,

$$200q_1 + 100q_2 = 4000 \cos 60^\circ = 2000 \text{ N} \quad (\text{ii})$$

Solving Eqs. (i) and (ii), we obtain

$$q_1 = 11.3 \text{ N/mm}, \quad q_2 = -2.6 \text{ N/mm}$$

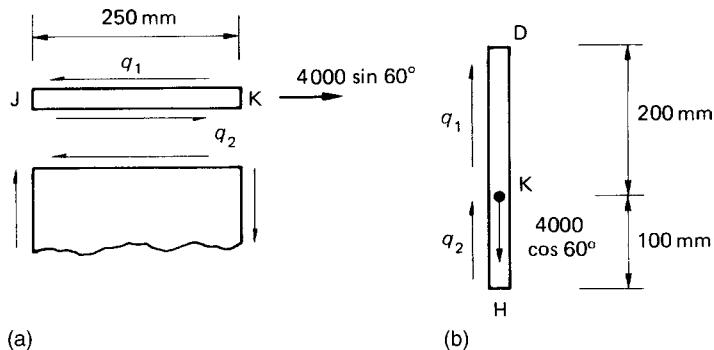


FIGURE 24.2 Free Body Diagrams of Stiffeners JK and HKD in the Beam of Example 24.1

The vertical shear force in the panel BCGF is equilibrated by the vertical resultant of the shear flow  $q_3$ . Thus,

$$300q_3 = 4000 \cos 60^\circ = 2000 \text{ N}$$

from which

$$q_3 = 6.7 \text{ N/mm}$$

Alternatively,  $q_3$  may be found by considering the equilibrium of the stiffener CJG. From Fig. 24.3,

$$300q_3 = 200q_1 + 100q_2$$

or

$$300q_3 = 200 \times 11.3 - 100 \times 2.6$$

from which

$$q_3 = 6.7 \text{ N/mm}$$

The shear flow  $q_4$  in the panel ABFE may be found using either of the above methods. Thus, considering the vertical shear force in the panel,

$$300q_4 = 4000 \cos 60^\circ + 5000 = 7000 \text{ N}$$

from which

$$q_4 = 23.3 \text{ N/mm}$$

Alternatively, from the equilibrium of stiffener BF,

$$300q_4 - 300q_3 = 5000 \text{ N}$$

from which

$$q_4 = 23.3 \text{ N/mm}$$

The flange and stiffener load distributions are calculated in the same way and are obtained from the algebraic summation of the shear flows along their lengths. For example, the axial load  $P_A$  at A in the flange ABCD is given by

$$P_A = 250q_1 + 250q_3 + 250q_4$$

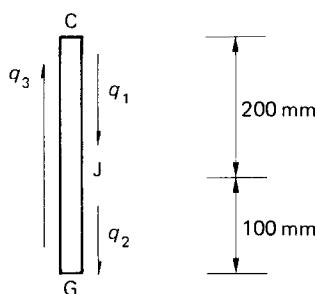


FIGURE 24.3 Equilibrium of Stiffener CJG in the Beam of Example 24.1

or

$$P_A = 250 \times 11.3 + 250 \times 6.7 + 250 \times 23.3 = 10,325 \text{ N (tension)}$$

Similarly,

$$P_E = -250q_2 - 250q_3 - 250q_4$$

that is,

$$P_E = 250 \times 2.6 - 250 \times 6.7 - 250 \times 23.3 = -6850 \text{ N (compression)}$$

The complete load distribution in each flange is shown in Fig. 24.4. The stiffener load distributions are calculated in the same way and are shown in Fig. 24.5.

The distribution of flange load in the bays ABFE and BCGF could be obtained by considering the bending and axial loads on the beam at any section. For example, at the section AE, we can replace the actual loading system by a bending moment

$$M_{AE} = 5000 \times 250 + 2000 \times 750 - 3464.1 \times 50 = 2,576,800 \text{ Nmm}$$

and an axial load acting midway between the flanges (irrespective of whether or not the flange areas are symmetrical about this point) of

$$P = 3464.1 \text{ N}$$

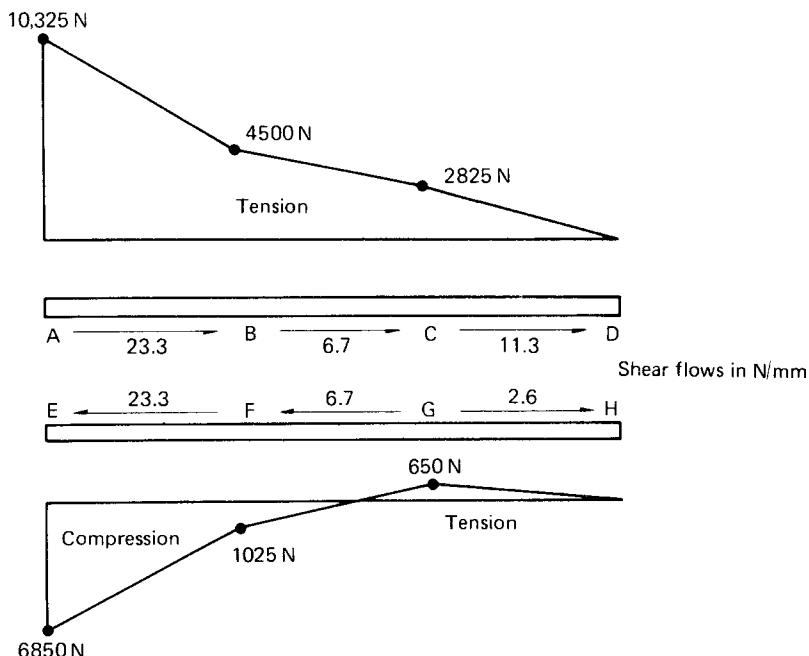


FIGURE 24.4 Load Distributions in Flanges of the Beam of Example 24.1

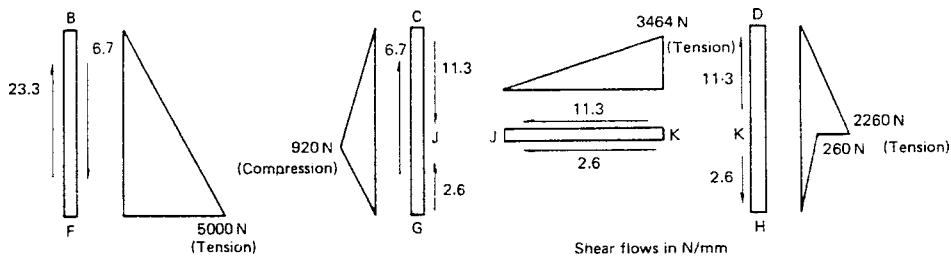


FIGURE 24.5 Load Distributions in Stiffeners of the Beam of Example 24.1

Thus,

$$P_A = \frac{2,576,800}{300} + \frac{3464.1}{2} = 10,321 \text{ N (tension)}$$

and

$$P_E = \frac{-2,576,800}{300} + \frac{3464.1}{2} = -6857 \text{ N (compression)}$$

This approach cannot be used in the bay CDHG except at the section CJG, since the axial load in the stiffener JK introduces an additional unknown.

This analysis assumes that the web panels in beams of the type shown in Fig. 24.1 resist pure shear along their boundaries. In Chapter 9, we saw that thin webs may buckle under the action of such shear loads, producing tension field stresses that, in turn, induce additional loads in the stiffeners and flanges of beams. The tension field stresses may be calculated separately by the methods described in Chapter 9 and then superimposed on the stresses determined as just described.

So far, we have been concerned with web stiffener arrangements in which loads have been applied in the plane of the web, so that two stiffeners are sufficient to resist the components of a concentrated load. Frequently, loads have an out-of-plane component; in which case, the structure should be arranged so that two webs meet at the point of load application with stiffeners aligned with the three

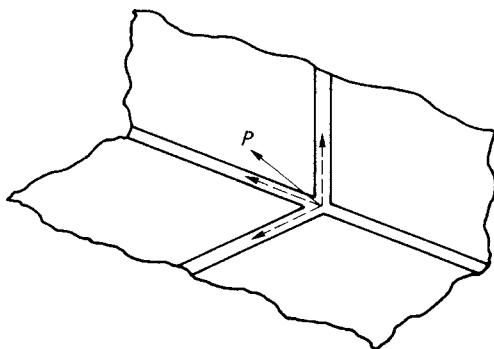
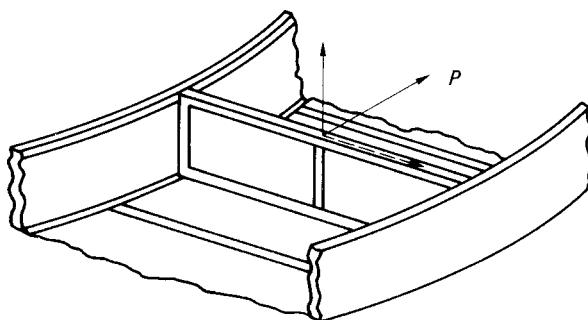


FIGURE 24.6 Structural Arrangement for an Out-of-Plane Load



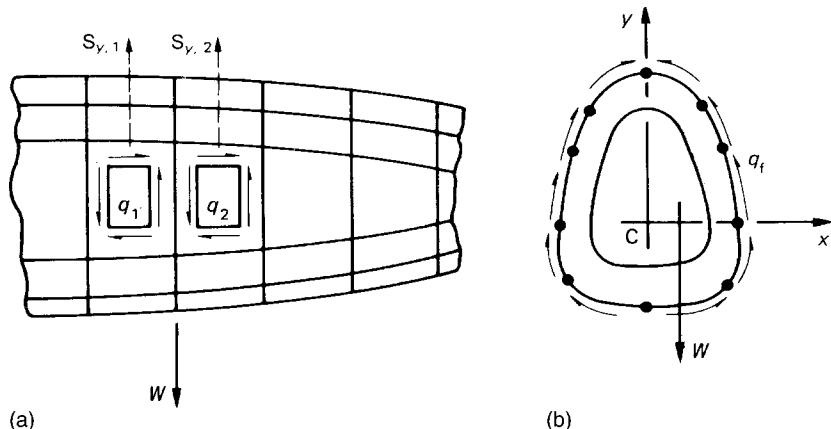
**FIGURE 24.7 Support of Load Having a Component Normal to a Web**

component directions (Fig. 24.6). In some situations, it is not practicable to have two webs meeting at the point of load application, so that a component normal to a web exists. If this component is small, it may be resisted in bending by an in-plane stiffener; otherwise, an additional member must be provided, spanning between adjacent frames or ribs, as shown in Fig. 24.7. In general, no normal loads should be applied to an unsupported web no matter how small their magnitude.

## 24.2 FUSELAGE FRAMES

We noted that fuselage frames transfer loads to the fuselage shell and provide column support for the longitudinal stringers. The frames generally take the form of open rings, so that the interior of the fuselage is not obstructed. They are connected continuously around their peripheries to the fuselage shell and are not necessarily circular in form but usually are symmetrical about a vertical axis.

A fuselage frame is in equilibrium under the action of any external loads and the reaction shear flows from the fuselage shell. Suppose that a fuselage frame has a vertical axis of symmetry and carries a vertical external load  $W$ , as shown in Fig. 24.8. The fuselage shell stringer section has been idealized



**FIGURE 24.8 Loads on a Fuselage Frame**

such that the fuselage skin is effective only in shear. Suppose also that the shear force in the fuselage immediately to the left of the frame is  $S_{y,1}$  and that the shear force in the fuselage immediately to the right of the frame is  $S_{y,2}$ ; clearly,  $S_{y,2} = S_{y,1} - W$ .  $S_{y,1}$  and  $S_{y,2}$  generate shear flow distributions  $q_1$  and  $q_2$ , respectively, in the fuselage skin, each given by Eq. (22.1), in which  $S_{x,1} = S_{x,2} = 0$  and  $I_{xy} = 0$  ( $C_y$  is an axis of symmetry). The shear flow  $q_f$  transmitted to the periphery of the frame is equal to the algebraic sum of  $q_1$  and  $q_2$ ; that is,

$$q_f = q_1 - q_2$$

Thus, substituting for  $q_1$  and  $q_2$  obtained from Eq. (22.1) and noting that  $S_{y,2} = S_{y,1} - W$ , we have

$$q_f = \frac{-W}{I_{xx}} \sum_{r=1}^n B_r y_r + q_{s,0}$$

in which  $q_{s,0}$  is calculated using Eq. (17.17), where the shear load is  $W$  and

$$q_b = \frac{-W}{I_{xx}} \sum_{r=1}^n B_r y_r$$

The method of determining the shear flow distribution applied to the periphery of a fuselage frame is identical to the method of solution (or the alternative method) of Example 22.2.

Having determined the shear flow distribution around the periphery of the frame, the frame itself may be analyzed for distributions of bending moment, shear force, and normal force, as described in Section 5.4.

## 24.3 WING RIBS

Wing ribs perform similar functions to those performed by fuselage frames. They maintain the shape of the wing section, assist in transmitting external loads to the wing skin, and reduce the column length of the stringers. Their geometry, however, is usually different, in that they are frequently of unsymmetrical shape and possess webs that are continuous except for lightness holes and openings for control runs.

Wing ribs are subjected to loading systems similar to those applied to fuselage frames. External loads applied in the plane of the rib produce a change in shear force in the wing across the rib; this induces reaction shear flows around its periphery. These are calculated using the methods described in Chapters 17 and 23.

To illustrate the method of rib analysis, we use the example of a three-flange wing section in which, as we noted in Section 23.1, the shear flow distribution is statically determinate.

### Example 24.2

Calculate the shear flows in the web panels and the axial loads in the flanges of the wing rib shown in Fig. 24.9. Assume that the web of the rib is effective only in shear while the resistance of the wing to bending moments is provided entirely by the three flanges 1, 2, and 3.

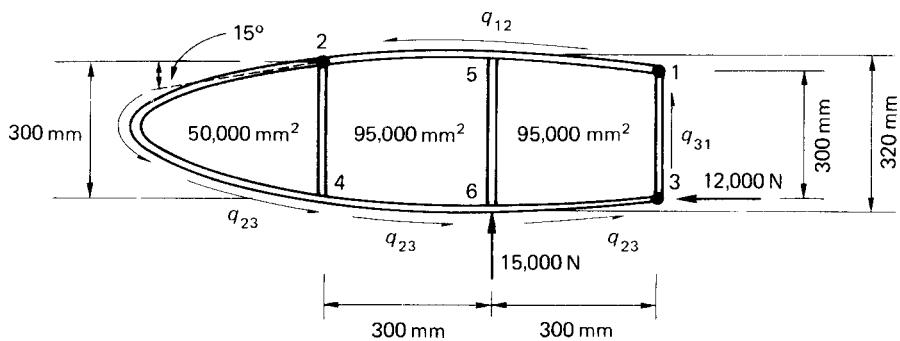


FIGURE 24.9 Wing Rib of Example 24.2

Since the wing bending moments are resisted entirely by the flanges 1, 2, and 3, the shear flows developed in the wing skin are constant between the flanges. Using the method described in Section 23.1 for a three-flange wing section, we have, resolving forces horizontally,

$$600q_{12} - 600q_{23} = 12,000 \text{ N} \quad (\text{i})$$

Resolving vertically,

$$300q_{31} - 300q_{23} = 15,000 \text{ N} \quad (\text{ii})$$

Taking moments about flange 3,

$$2(50,000 + 95,000)q_{23} + 2 \times 95,000q_{12} = -15,000 \times 300 \text{ Nmm} \quad (\text{iii})$$

Solution of Eqs. (i)–(iii) gives

$$q_{12} = 13.0 \text{ N/mm}, \quad q_{23} = -7.0 \text{ N/mm}, \quad q_{31} = 43.0 \text{ N/mm}$$

Consider now the nose portion of the rib shown in Fig. 24.10 and suppose that the shear flow in the web immediately to the left of the stiffener 24 is  $q_1$ . The total vertical shear force  $S_{y,1}$  at this section is given by

$$S_{y,1} = 7.0 \times 300 = 2100 \text{ N}$$

The horizontal components of the rib flange loads resist the bending moment at this section. Thus,

$$P_{x,4} = P_{x,2} = \frac{2 \times 50,000 \times 7.0}{300} = 2333.3 \text{ N}$$

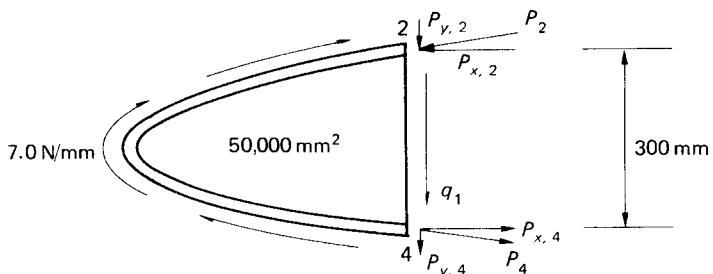


FIGURE 24.10 Equilibrium of Nose Portion of the Rib

The corresponding vertical components are then

$$P_{y,2} = P_{y,4} = 2333.3 \tan 15^\circ = 625.2 \text{ N}$$

Thus, the shear force carried by the web is  $2100 - 2 \times 625.2 = 849.6 \text{ N}$ . Hence,

$$q_1 = \frac{849.6}{300} = 2.8 \text{ N/mm}$$

The axial loads in the rib flanges at this section are given by

$$P_2 = P_4 = (2333.3^2 + 625.2^2)^{1/2} = 2415.6 \text{ N}$$

The rib flange loads and web panel shear flows, at a vertical section immediately to the left of the intermediate web stiffener 56, are found by considering the free body diagram shown in Fig. 24.11. At this section, the rib flanges have zero slope, so that the flange loads  $P_5$  and  $P_6$  are obtained directly from the value of bending moment at this section. Thus,

$$P_5 = P_6 = 2[(50,000 + 46,000) \times 7.0 - 49,000 \times 13.0]/320 = 218.8 \text{ N}$$

The shear force at this section is resisted solely by the web. Hence,

$$320q_2 = 7.0 \times 300 + 7.0 \times 10 - 13.0 \times 10 = 2040 \text{ N}$$

so that

$$q_2 = 6.4 \text{ N/mm}$$

The shear flow in the rib immediately to the right of stiffener 56 is found most simply by considering the vertical equilibrium of stiffener 56, as shown in Fig. 24.12. Thus,

$$320q_3 = 6.4 \times 320 + 15,000$$

which gives

$$q_3 = 53.3 \text{ N/mm}$$

Finally, we consider the rib flange loads and the web shear flow at a section immediately forward of stiffener 31. From Fig. 24.13, in which we take moments about the point 3,

$$M_3 = 2[(50,000 + 95,000) \times 7.0 - 95,000 \times 13.0] + 15,000 \times 300 = 4.06 \times 10^6 \text{ Nmm}$$

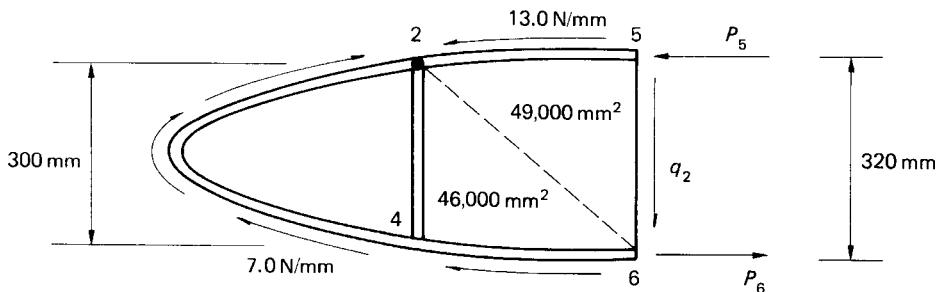


FIGURE 24.11 Equilibrium of Rib Forward of Intermediate Stiffener 56

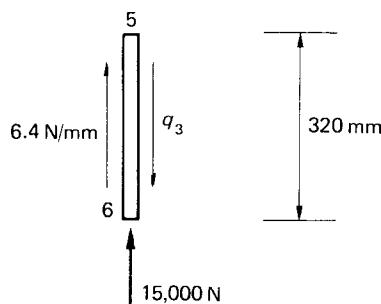


FIGURE 24.12 Equilibrium of Stiffener 56

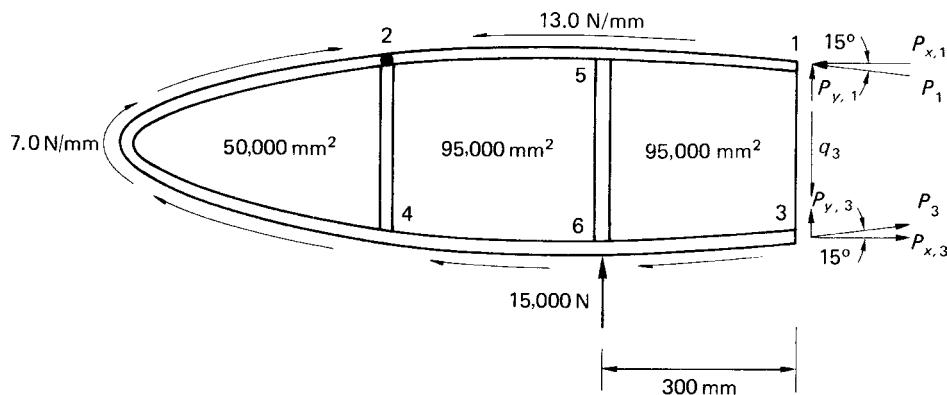


FIGURE 24.13 Equilibrium of the Rib Forward of Stiffener 31

The horizontal components of the flange loads at this section are then

$$P_{x,1} = P_{x,3} = \frac{4.06 \times 10^6}{300} = 13,533.3 \text{ N}$$

and the vertical components are

$$P_{y,1} = P_{y,3} = 3,626.2 \text{ N}$$

Hence,

$$P_1 = P_3 = \sqrt{13,533.3^2 + 3,626.2^2} = 14,010.7 \text{ N}$$

The total shear force at this section is  $15,000 + 300 \times 7.0 = 17,100 \text{ N}$ . Therefore, the shear force resisted by the web is  $17,100 - 2 \times 3626.2 = 9847.6 \text{ N}$ , so that the shear flow  $q_3$  in the web at this section is

$$q_3 = \frac{9847.6}{300} = 32.8 \text{ N/mm}$$

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**PROBLEMS**

**P.24.1** The beam shown in Fig. P.24.1 is simply supported at each end and carries a load of 6000 N. If all direct stresses are resisted by the flanges and stiffeners and the web panels are effective only in shear, calculate the distribution of axial load in the flange ABC, the stiffener BE, and the shear flows in the panels.

Answer:  $q(ABEF) = 4 \text{ N/mm}$ ,  $q(BCDE) = 2 \text{ N/mm}$

$P_{BE}$  increases linearly from zero at B to 6000 N (tension) at E.

$P_{AB}$  and  $P_{CB}$  increase linearly from zero at A and C to 4000 N (compression) at B.

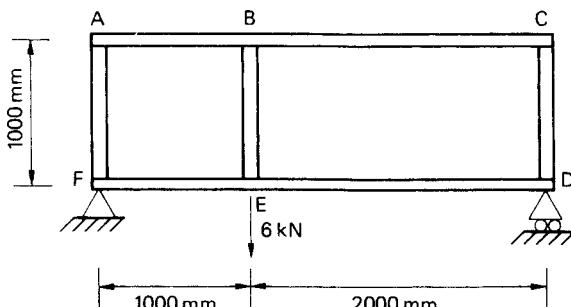


FIGURE P.24.1

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**P.24.2** Calculate the shear flows in the web panels and direct load in the flanges and stiffeners of the beam shown in Fig. P.24.2 if the web panels resist shear stresses only.

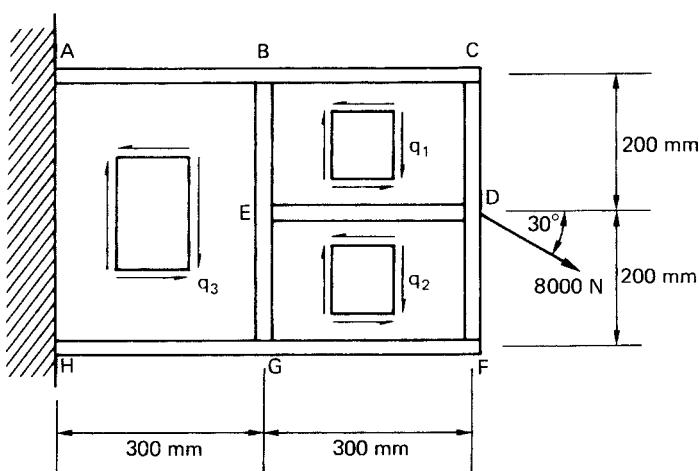


FIGURE P.24.2

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Answer:  $q_1 = 21.6 \text{ N/mm}$ ,  $q_2 = -1.6 \text{ N/mm}$ ,  $q_3 = 10 \text{ N/mm}$   
 $P_C = 0$ ,  $P_B = 6480 \text{ N}$  (tension),  $P_A = 9480 \text{ N}$  (tension)  
 $P_F = 0$ ,  $P_G = 480 \text{ N}$  (tension),  $P_H = 2520 \text{ N}$  (compression)  
 $P_E$  in BEG = 2320 N (compression),  $P_D$  in ED = 6928 N (tension)  
 $P_D$  in CD = 4320 N (tension),  $P_D$  in DF = 320 N (tension)

**P.24.2 MATLAB** Assuming the angle (+ = clockwise) from horizontal of the 8000 N load shown in Fig. P.24.2 is labeled  $\alpha$ , use MATLAB to repeat the calculations of the web panel shear flows in Problem P.24.2 for values of  $\alpha$  ranging from  $-50^\circ$  to  $50^\circ$  in increments of  $10^\circ$ . All other dimensions and loads are as shown in Fig. P.24.2.

Answer:	$\alpha$	$q_1$	$q_2$	$q_3$
(i)	$-50^\circ$	-6.8 N/mm	-23.9 N/mm	-15.3 N/mm
(ii)	$-40^\circ$	-2.7 N/mm	-23.1 N/mm	-12.9 N/mm
(iii)	$-30^\circ$	1.6 N/mm	-21.6 N/mm	-10 N/mm
(iv)	$-20^\circ$	5.7 N/mm	-19.4 N/mm	-6.8 N/mm
(v)	$-10^\circ$	9.7 N/mm	-16.6 N/mm	-3.5 N/mm
(vi)	$0^\circ$	13.4 N/mm	-13.4 N/mm	0 N/mm
(vii)	$10^\circ$	16.6 N/mm	-9.7 N/mm	3.5 N/mm
(viii)	$20^\circ$	19.4 N/mm	-5.7 N/mm	6.8 N/mm
(ix)	$30^\circ$	21.6 N/mm	-1.6 N/mm	10 N/mm
(x)	$40^\circ$	23.1 N/mm	2.7 N/mm	12.9 N/mm
(xi)	$50^\circ$	23.9 N/mm	6.8 N/mm	15.3 N/mm

**P.24.3** A three-flange wing section is stiffened by the wing rib shown in Fig. P.24.3. If the rib flanges and stiffeners carry all the direct loads while the rib panels are effective only in shear, calculate the shear flows in the panels and the direct loads in the rib flanges and stiffeners.

Answer:  $q_1 = 4.0 \text{ N/mm}$ ,  $q_2 = 26.0 \text{ N/mm}$ ,  $q_3 = 6.0 \text{ N/mm}$   
 $P_2$  in 12 =  $-P_3$  in 43 = 1200 N (tension),  $P_5$  in 154 = 2000 N (tension)  
 $P_3$  in 263 = 8000 N (compression),  $P_5$  in 56 = 12,000 N (tension)  
 $P_6$  in 263 = 6000 N (compression)

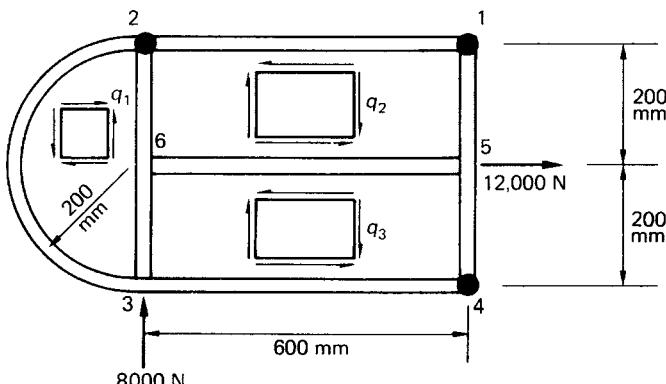


FIGURE P.24.3

## Spacecraft

## 25

In Section 12.2, we discussed the types of load to which spacecraft are subjected. The discussion was mainly qualitative apart from typical strength and stability calculations for the stage section of a launch vehicle in which the thrust was specified (see P.12.4 and P.12.5). However, in addition to strength, a spacecraft must possess sufficient stiffness to ensure that the frequency of the oscillations generated during launch does not approach its resonant or natural frequency. Therefore, the spacecraft must be designed such that its lowest natural frequency is well above the minimum frequency specified in the launcher's user manual.

In the calculation of natural frequencies, the spacecraft may be idealized into a relatively simple model consisting, say, of a beam and concentrated mass, the former representing the launch vehicle structure and the latter the payload.

We shall now investigate spacecraft frequencies using the methods established in Chapter 10.

## 25.1 NATURAL FREQUENCIES

The vertically positioned beam shown in Fig. 25.1 carries a concentrated mass  $m$  at its upper end (1). Assuming that the beam is weightless but has a flexural rigidity  $EI$ , we require to determine its normal mode and frequency of vibration.

From Eq. (10.3),

$$m\ddot{v}_1 \delta_{11} + v_1 = 0 \quad (25.1)$$

in which  $\delta_{11} = h^3/3EI$  (see Example 10.1). Rewriting Eq. (25.1),

$$\ddot{v}_1 + \omega^2 v_1 = 0 \quad (25.2)$$

where  $\omega^2 = 3EI/mh^3$ . The solution of Eq. (25.2) is of standard form and is

$$v_1 = A \cos \omega t + B \sin \omega t$$

When  $t = 0$ ,  $v_1 = 0$  so that  $A = 0$ . When  $v_1 = v_{1(\max)}$ ,  $dv_1/dt (\dot{v}) = 0$  so that  $\omega B \cos \omega t = 0$ . For a nontrivial solution (i.e.,  $B \neq 0$ ),  $\cos \omega t = 0$ , which gives  $\omega t = \pi/2$ . Then  $v_{1(\max)} = B$  and the equation for the normal mode of oscillation of the mass  $m$  is

$$v_1 = v_{1(\max)} \sin \omega t \quad (25.3)$$

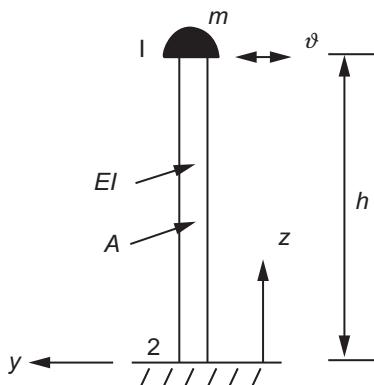


FIGURE 25.1 Oscillation of a Beam Carrying a Concentrated Mass

The time taken for the mass to oscillate from  $v_1 = 0$  to  $v_1 = v_{1(\max)}$  is, from Eq. (25.3), given by  $\sin \omega t = 1$  from which  $\omega t = \pi/2$ , i.e.,  $t = \pi/2\omega$ . Therefore, the time taken for one complete oscillation is  $4 \times (\pi/2\omega) = 2\pi/\omega$ . It follows that the frequency  $f_z$  of the oscillations is given by

$$f_z = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{3EI}{mh^3}} \quad (25.4)$$

It is possible for the mass  $m$  to oscillate vertically as well as laterally. In this case, the stiffness of the beam is the axial stiffness so that, if the beam has a cross-sectional area  $A$ , the increase in length due to a unit axial load is  $\epsilon_z h$  where  $\epsilon_z = \sigma_z/E$  and  $\sigma_z = 1/A$ . Then, in Eq. (25.1),  $\delta_{11} = h/AE$  and the natural frequency of oscillation is

$$f_y = \frac{1}{2\pi} \sqrt{\frac{AE}{mh}} \quad (25.5)$$

## 25.2 AXIAL STRESS

Although the calculation of natural frequencies is important in spacecraft design, the structure must also be designed to resist so-called quasi-static loads that arise from the acceleration of the spacecraft during launch. These loads produce axial stresses where, referring to Fig. 25.1, the direct stress is given by

$$\sigma_z = \frac{P}{A} \text{ (see Chapter 1)} \quad (25.6)$$

in which the axial load  $P = mg_z$  where  $m$  is the spacecraft mass and  $g_z$  its acceleration in the axial direction;  $A$  is the cross-sectional area of the load bearing structure. The direct stress due to the bending action of the launch loads is given by Eq. (16.9) for the case of a spacecraft having a symmetrical (at least about one of the cross-sectional axes) cross-section subjected to a bending moment

$M$  where  $M = mg_v(h - z)$  (see Fig. 25.1);  $g_v$  is the acceleration in the lateral direction. The maximum value of bending stress will occur at the bottom of the beam and is

$$\sigma_z = \pm \frac{mg_y h y_{\max}}{I} \quad (25.7)$$

The direct stresses given by Eqs. (25.6) and (25.7) must be superimposed to give a total direct stress  $\sigma_{\text{tot}}$ . Then

$$\sigma_{\text{tot}} \leq \sigma_{\text{allowable}}$$

where  $\sigma_{\text{allowable}}$  is the ultimate stress divided by a factor of safety.

### 25.3 STABILITY

In addition to the direct stresses caused by the axial and bending forces produced at launch, the space-craft structure must be checked for stability. The critical load for a column of height  $h$  that is built in at one end and free at the other is, from Table 8.1,

$$P_{\text{CR}} = \frac{\pi^2 EI}{4h^2} \quad (25.8)$$

Therefore, for stability, the axial load generated by the launch must be less than the critical load given by Eq. (25.8), i.e.,  $mg_z < P_{\text{CR}}$ .

#### Example 25.1

A spacecraft, together with its launch vehicle, has the idealized arrangement shown in Fig. 25.2. The mass of the payload is 2750 kg, and the cylindrical shell of the launch vehicle is fabricated from aluminum alloy having a Young's modulus of 70,000 N/mm<sup>2</sup>. If the natural frequencies that the launch vehicle must meet are 30 Hz axially

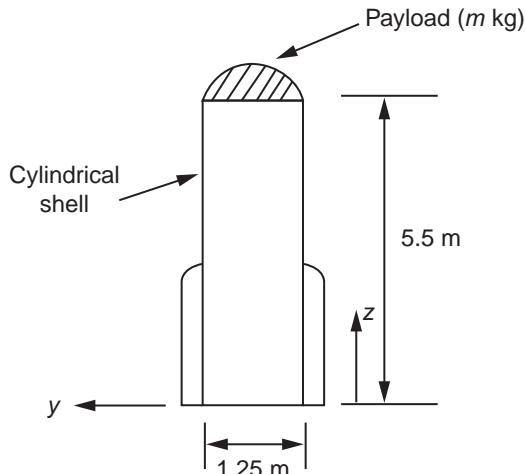


FIGURE 25.2 Launch Vehicle of Example 25.1

and 10 Hz laterally, determine the minimum allowable thickness of the launch vehicle shell; the maximum axial acceleration at launch is  $6g$  and that, laterally, is  $1.5g$ . Also, if the ultimate stress of the aluminum alloy is  $480 \text{ N/mm}^2$ , check the strength of the launch vehicle using a factor of safety of 2. Finally, evaluate the stability of the structure.

In the launch ( $z$ ) direction, the natural frequency is given by Eq. (25.5). Then

$$f_z = 30 = \frac{1}{2\pi} \sqrt{\frac{A \times 70,000}{2750 \times 5.5}}$$

which gives  $A = 7677.1 \text{ mm}^2$ .

For a thin-walled cylindrical shell of diameter  $d$  and wall thickness  $t$ ,  $A = \pi dt$ . Then

$$7677.1 = \pi \times 1.25 \times 10^3 t$$

from which  $t = 1.95 \text{ mm}$ .

The frequency of oscillation in the  $y$  (lateral) direction is given by Eq. (25.4), i.e.,

$$f_y = 10 = \frac{1}{2\pi} \sqrt{\frac{3 \times 70,000 \times I \times 10^{-6}}{2750 \times 5.5^3}}$$

which gives  $I = 8.6 \times 10^9 \text{ mm}^4$ .

Note that in the basic equation of motion (25.1), the units of acceleration are  $\text{m/s}^2$ . (Acceleration due to gravity is  $9.81 \text{ m/s}^2$ ). Therefore, in Eqs. (25.4) and (25.5), the linear units will be meters. The linear unit of  $AE$  disappears in Eq. (25.5) so that if we specify  $E$  in  $\text{N/mm}^2$ ,  $A$  will be in  $\text{mm}^2$ . However, in Eq. (25.4), the linear units of  $EI$  are  $\text{mm}^2$  so that, in terms of meters, we convert the linear units of  $EI$  from  $\text{mm}^2$  to  $\text{m}^2$  by multiplying by  $10^{-6}$ .

From Section 16.4.5, the second moment of area of a thin-walled circular shell of radius  $r$  is  $\pi r^3 t$ . Therefore

$$\pi \times (0.625 \times 10^3)^3 t = 8.6 \times 10^9$$

from which  $t = 11.2 \text{ mm}$ .

The governing thickness is therefore 11.2 mm.

The strength of the launch vehicle must now be checked. The direct stress produced by the axial acceleration is given by Eq. (25.6), i.e.,

$$\sigma_{z,\text{axial}} = \frac{2750 \times 6 \times 9.81}{\pi \times 1.25 \times 10^3 \times 11.2} = 3.7 \text{ N/mm}^2$$

The direct stress due to the bending action of the lateral acceleration is given by Eq. (25.7), i.e.,

$$\sigma_{z,\text{bending}} = \frac{2750 \times 1.5 \times 9.81 \times 5.5 \times 10^3 \times 0.625 \times 10^3}{8.6 \times 10^9} = 16.2 \text{ N/mm}^2$$

The total direct stress is then  $\sigma_{z,\text{tot}} = 3.7 + 16.2 = 19.9 \text{ N/mm}^2$ . Note that both values of direct stress are compressive by inspection. The allowable value of direct stress is  $480/2 = 240 \text{ N/mm}^2$  so that the strength of the launch is adequate.

The critical load for stability is given by Eq. (25.8). Then

$$P_{\text{CR}} = \frac{\pi^2 \times 70,000 \times 8.6 \times 10^9}{4 \times (5.5 \times 10^3)^2 \times 10^3} = 49,103 \text{ kN}$$

The axial load during launch is  $2750 \times 6 \times 9.81/10^3 = 161.9 \text{ kN}$ . The launch vehicle is therefore stable.

The discussion presented in Section 12.2 and Chapter 25 forms a brief introduction to spacecraft design. A more detailed and comprehensive study may be found in Ref. [1].

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**REFERENCE**

- [1] Wijker JJ. Spacecraft structures. Berlin: Springer-Verlag; 2010.

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**PROBLEMS**

**P.25.1** The launch vehicle of a spacecraft comprises an aluminum alloy cylindrical shell of diameter 1.0 m and wall thickness 10 mm. If the launch vehicle carries a payload of 2500 kg, which may be regarded as concentrated at a height of 5 m, calculate the axial and lateral natural frequencies.  $E = 70,000 \text{ N/mm}^2$ .

Answer:  $f_{\text{axial}} = 66.8 \text{ Hz}$ ,  $f_{\text{lateral}} = 8.1 \text{ Hz}$ .

**P.25.2** If the natural frequencies of the launch vehicle of P.25.1 are limited to 45 Hz (axial) and 10 Hz (lateral), calculate the required minimum thickness.

Answer: 15.0 mm.

**P.25.3** If the axial and lateral accelerations at launch experienced by the launch vehicle of P.25.1 are 6.5g and 2.0g, respectively, check the strength of the vehicle and its stability when its shell thickness is increased to 15 mm. The ultimate stress of the aluminum alloy is  $460 \text{ N/mm}^2$  and the factor of safety is 2.

Answer:  $\sigma_{\text{tot}} = 24.2 \text{ N/mm}^2$ , Axial load = 159.4 kN.

Strength and stability satisfactory.

# Laminated composite structures

# 26

An increasingly large proportion of the structures of many modern aircraft are fabricated from composite materials. These, as we saw in [Chapter 12](#), consist of laminas in which a stiff, high-strength filament – for example, carbon fiber – is embedded in a matrix such as epoxy or polyester. The use of composites can lead to considerable savings in weight over conventional metallic structures. They also have the advantage that the direction of the filaments in a multilamina structure may be aligned with the direction of the major loads at a particular point, resulting in a more efficient design.

There are two approaches to the analysis of composite materials. In the first, micromechanics, the constituent materials – that is, the fibers and resin (the matrix) – are considered separately. The properties of the composite change from point to point in a particular direction, depending on whether the fiber or the resin is being examined. In the second approach, macromechanics, the composite material is regarded as a whole, so that the properties do not change from point to point in a particular direction. Generally, the design and analysis of composite materials are based on the macro rather than the micro approach.

Initially, but briefly, we consider the micro approach, in which the elastic constants of a lamina are determined in terms of the known properties of the constituent materials; we then determine the corresponding stresses.

## 26.1 ELASTIC CONSTANTS OF A SIMPLE LAMINA

A simple lamina of a composite structure can be considered as orthotropic with two principal material directions in its own plane: one parallel, the other perpendicular to the direction of the filaments; we designate the former the longitudinal direction ( $l$ ), the latter the transverse direction ( $t$ ).

In [Fig. 26.1](#), a portion of a lamina containing a single filament is subjected to a stress,  $\sigma_l$ , in the longitudinal direction, which produces an extension  $\Delta l$ . If it is assumed that plane sections remain plane during deformation, then the strain  $\varepsilon_l$  corresponding to  $\sigma_l$  is given by

$$\varepsilon_l = \frac{\Delta l}{l} \quad (26.1)$$

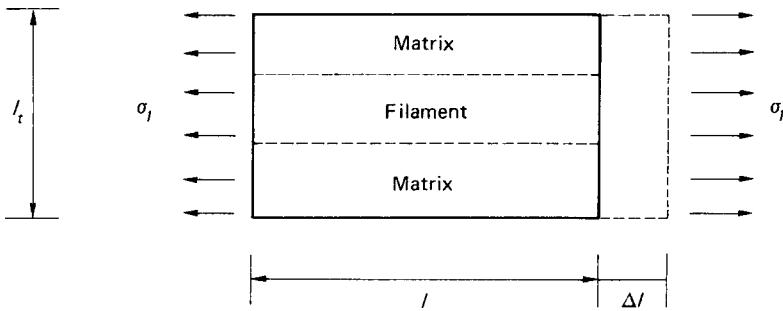
and

$$\sigma_l = E_l \varepsilon_l \quad (26.2)$$

where  $E_l$  is the modulus of elasticity of the lamina in the direction of the filament. Also, using the suffixes  $f$  and  $m$  to designate filament and matrix parameters, we have

$$\sigma_f = E_f \varepsilon_l \quad \sigma_m = E_m \varepsilon_l \quad (26.3)$$

Note that the strain in the filament and the matrix must be the same.

FIGURE 26.1 Determination of  $E_l$ 

Further, if  $A$  is the total area of cross-section of the lamina in Fig. 26.1,  $A_f$  is the cross-sectional area of the filament and  $A_m$  the cross-sectional area of the matrix, then, for equilibrium in the direction of the filament,

$$\sigma_l A = \sigma_f A_f + \sigma_m A_m$$

or, substituting for  $\sigma_l$ ,  $\sigma_f$ , and  $\sigma_m$  from Eqs. (26.2) and (26.3),

$$E_l \varepsilon_l A = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m$$

so that

$$E_l = E_f \frac{A_f}{A} + E_m \frac{A_m}{A} \quad (26.4)$$

Writing  $A_f/A = v_f$  and  $A_m/A = v_m$ , Eq. (26.4) becomes

$$E_l = v_f E_f + v_m E_m \quad (26.5)$$

Equation (26.5) is generally referred to as the *law of mixtures*.

A similar approach may be used to determine the modulus of elasticity in the transverse direction ( $E_t$ ). In Fig. 26.2, the total extension in the transverse direction is produced by  $\sigma_t$  and is given by

$$\varepsilon_t l_t = \varepsilon_m l_m + \varepsilon_f l_f$$

or

$$\frac{\sigma_t}{E_t} l_t = \frac{\sigma_t}{E_m} l_m + \frac{\sigma_t}{E_f} l_f$$

so that

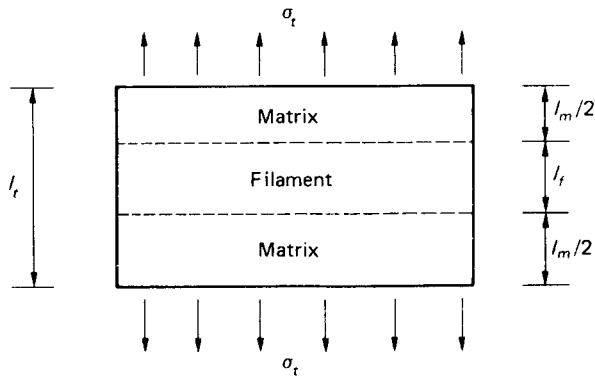
$$\frac{1}{E_t} = \frac{1}{E_m} \frac{l_m}{l_t} + \frac{1}{E_f} \frac{l_f}{l_t}$$

But for a lamina of, say, unit thickness

$$1 \times l_m = A_m, \quad 1 \times l_t = A$$

Then

$$\frac{1}{E_t} = \frac{v_m}{E_m} + \frac{v_f}{E_f}$$



**FIGURE 26.2 Determination of  $E_t$**

Rearranging this, we obtain

$$E_t = \frac{E_m E_f}{v_m E_f + v_f E_m} \quad (26.6)$$

The major Poisson's ratio  $v_{lt}$  may be found by referring to the stress system of Fig. 26.1 and the dimensions given in Fig. 26.2. The total displacement in the transverse direction produced by  $\sigma_t$  is given by

$$\Delta_t = v_{lt} \epsilon_l l_t$$

that is,

$$\Delta_t = v_{lt} \epsilon_l l_t = v_m \epsilon_l l_m + v_f \epsilon_l l_f$$

from which

$$v_{lt} = v_m v_m + v_f v_f \quad (26.7)$$

The minor Poisson's ratio  $v_{tl}$  is found by referring to Fig. 26.2. The strain in the longitudinal direction produced by the transverse stress  $\sigma_t$  is given by

$$v_{tl} \frac{\sigma_t}{E_t} = v_m \frac{\sigma_t}{E_m} = v_f \frac{\sigma_t}{E_f} \quad (26.8)$$

From the last two of Eqs. (26.8),

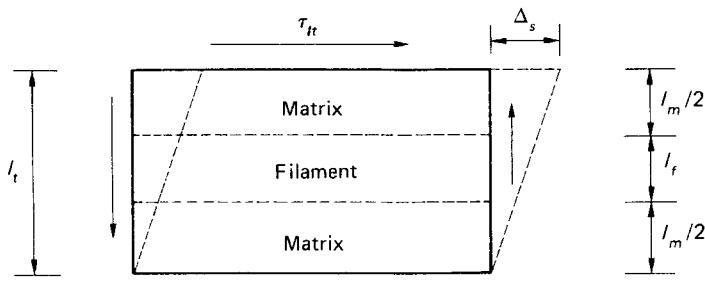
$$v_f = \frac{E_f}{E_m} v_m$$

Substituting in Eq. (26.7),

$$v_{lt} = v_m \left( v_m + \frac{E_f}{E_m} v_f \right) = \frac{v_m}{E_m} (v_m E_m + v_f E_f)$$

or, from Eq. (26.5),

$$v_{lt} = v_m \frac{E_l}{E_m}$$

FIGURE 26.3 Determination of  $G_{lt}$ 

Now, substituting for  $v_m$  in the first two of Eqs. (26.8),

$$\frac{v_{tl}}{E_t} = \frac{v_{lt}}{E_l}$$

or, from Eq. (26.7),

$$v_{tl} = \frac{E_t}{E_l} v_{lt} = \frac{E_t}{E_l} (v_m v_m + v_f v_f) \quad (26.9)$$

Finally, the shear modulus  $G_{lt}(=G_{tl})$  is determined by assuming that the constituent materials are subjected to the same shear stress  $\tau_{lt}$  as shown in Fig. 26.3. The displacement  $\Delta_s$  produced by shear is

$$\Delta_s = \frac{\tau_{lt}}{G_{lt}} l_t = \frac{\tau_{lt}}{G_m} l_m + \frac{\tau_{lt}}{G_f} l_f$$

in which  $G_m$  and  $G_f$  are the shear moduli of the matrix and filament, respectively. Then,

$$\frac{l_t}{G_{lt}} = \frac{l_m}{G_m} + \frac{l_f}{G_f}$$

from which

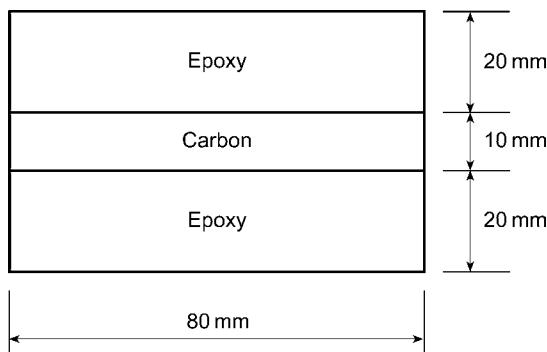
$$G_{lt} = \frac{G_m G_f}{v_m G_f + v_f G_m} \quad (26.10)$$

### Example 26.1

A laminated bar, whose cross-section is shown in Fig. 26.4, is 500 mm long and is composed of an epoxy resin matrix reinforced by a carbon filament having moduli equal to  $5,000 \text{ N/mm}^2$  and  $200,000 \text{ N/mm}^2$ , respectively; the corresponding values of Poisson's ratio are 0.2 and 0.3. If the bar is subjected to an axial tensile load of 100 kN, determine the lengthening of the bar and the reduction in its thickness. Calculate also the stresses in the epoxy resin and the carbon filament.

From Eq. (26.5), the modulus of the bar is given by

$$E_l = 200,000 \times \frac{80 \times 10}{80 \times 50} + 5000 \times \frac{80 \times 40}{80 \times 50}$$



**FIGURE 26.4** Cross-section of the Bar of Example 26.1

that is,

$$E_l = 44,000 \text{ N/mm}^2$$

The direct stress,  $\sigma_l$ , in the longitudinal direction, i.e., perpendicular to the plane of the page, is given by

$$\sigma_l = \frac{100 \times 10^3}{80 \times 50} = 25.0 \text{ N/mm}^2$$

Therefore, from Eq. (26.2), the longitudinal strain in the bar is

$$\varepsilon_l = \frac{25.0}{44,000} = 5.68 \times 10^{-4}$$

The lengthening,  $\Delta_l$ , of the bar is then

$$\Delta_l = 5.68 \times 10^{-4} \times 500$$

that is,

$$\Delta_l = 0.284 \text{ mm}$$

The major Poisson's ratio for the bar is found from Eq. (26.7); that is,

$$\nu_{ll} = \frac{80 \times 40}{80 \times 50} \times 0.2 + \frac{80 \times 10}{80 \times 50} \times 0.3 = 0.22$$

The strain in the bar across its thickness is then

$$\varepsilon_t = -0.22 \times 5.68 \times 10^{-4} = -1.25 \times 10^{-4}$$

The reduction in thickness,  $\Delta_t$ , of the bar is then

$$\Delta_t = 1.25 \times 10^{-4} \times 50$$

that is,

$$\Delta_t = 0.006 \text{ mm}$$

The stresses in the epoxy and the carbon are found using Eq. (26.3). Thus,

$$\sigma_m(\text{epoxy}) = 5000 \times 5.68 \times 10^{-4} = 2.84 \text{ N/mm}^2$$

$$\sigma_f(\text{carbon}) = 200,000 \times 5.68 \times 10^{-4} = 113.6 \text{ N/mm}^2$$

**Example 26.2**

The wings of a replica biplane are connected by a system of composite cables, each of which is 1.5 m long and can only carry tensile loads. The cross-section of a cable is shown in Fig. 26.5 where the polyester matrix has a modulus of  $2800 \text{ N/mm}^2$  while that of the Kevlar filament is  $145,000 \text{ N/mm}^2$  with corresponding Poisson's ratios of 0.15 and 0.30 respectively.

If the maximum tensile load in a cable is 10 kN, determine the corresponding extension and the stresses in the polyester and Kevlar.

From Eq. (26.5), the modulus of the cable is

$$E_l = 145,000 \times (\pi \times 5^2 / 4) / (\pi \times 15^2 / 4) + 2800 \times [\pi(15^2 - 5^2) / 4] / (\pi \times 15^2 / 4)$$

which gives

$$E_l = 18,600 \text{ N/mm}^2.$$

The direct stress is given by

$$\sigma_l = 10 \times 10^3 / (\pi \times 15^2 / 4) = 56.6 \text{ N/mm}^2$$

Therefore, from Eq. (26.2), the longitudinal strain in the cable is

$$\epsilon_l = 56.6 / 18,600 = 3.0 \times 10^{-3}$$

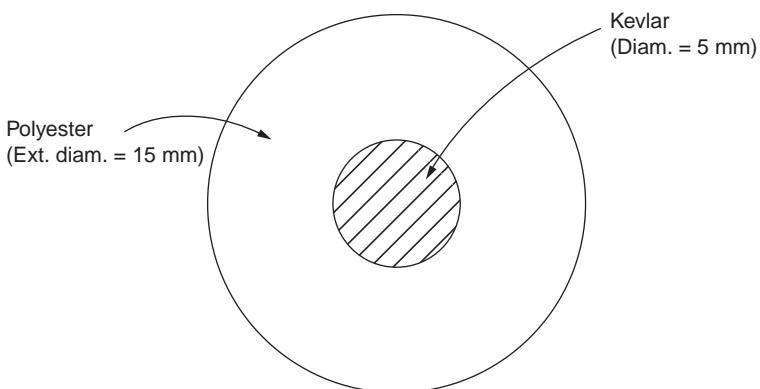
so that the lengthening,  $\Delta_l$ , of the cable is given by

$$\Delta_l = 3.0 \times 10^{-3} \times 1.5 \times 10^3 = 4.5 \text{ mm}$$

The stresses in the polyester and Kevlar are found using Eq. (26.3), i.e.,

$$\sigma_m \text{ (polyester)} = 2800 \times 3.0 \times 10^{-3} = 8.4 \text{ N/mm}^2$$

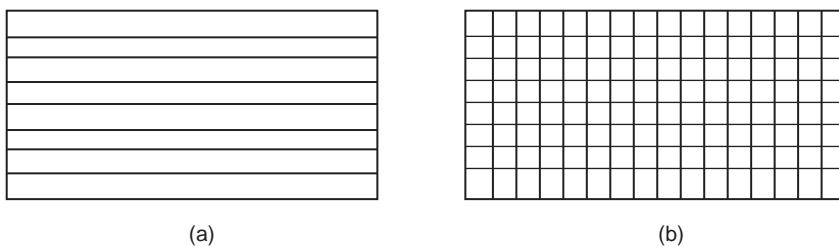
$$\sigma_f \text{ (Kevlar)} = 145,000 \times 3.0 \times 10^{-3} = 435.0 \text{ N/mm}^2$$



**FIGURE 26.5** Cross-section of the Cable of Example 26.2

## 26.2 STRESS-STRAIN RELATIONSHIPS FOR AN ORTHOTROPIC PLY (MACRO APPROACH)

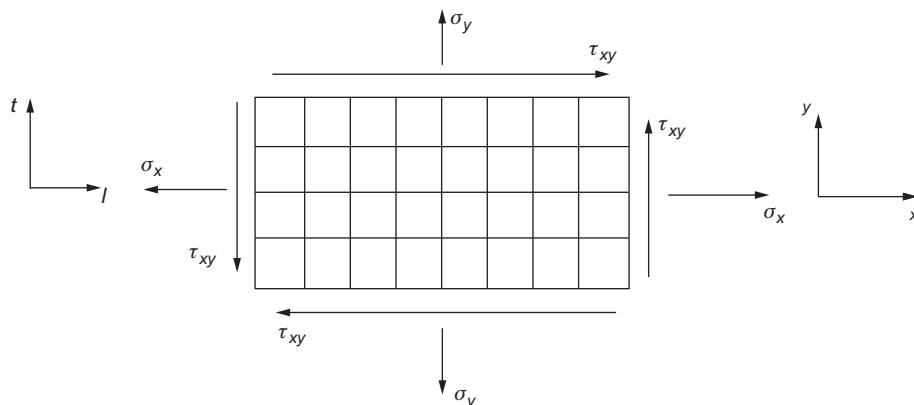
A single sheet of composite material in which the sheet has been preimpregnated with resin (a prepreg) and with the fibers aligned in one particular direction is called a *unidirectional ply* or *lamina* (Fig. 26.6(a)). On the other hand, a woven ply has the fibers placed in two perpendicular directions (Fig. 26.6(b)); generally, the fiber reinforcement is the same in both directions. In plies where this is not the case, so that the material properties are different in the two mutually perpendicular directions, the ply is said to be *orthotropic* (see Section 11.7). Two cases of orthotropic plies arise. In the first, the directions of the applied loads coincide with directions of the plies; these are known as *specially orthotropic plies*. In the second, the applied loads are applied in any direction; these are termed *generally orthotropic plies*.



**FIGURE 26.6 Types of Ply: (a) Unidirectional Ply; (b) Woven Ply**

### 26.2.1 Specially orthotropic ply

Figure 26.7 shows an element of a specially orthotropic ply. The ply reference axes are the same as in Section 26.1, that is, longitudinal (subscript  $l$ ) and transverse (subscript  $t$ ). Of course, these axes do not have the same significance for a woven ply as they do for a unidirectional ply, but reference axes must



**FIGURE 26.7 Reference Axes for a Specially Orthotropic Ply**

be specified and these are as convenient as any. We also specify loading axes,  $x$  and  $y$ , which, for a specially orthotropic ply, coincide with the ply reference axes.

Suppose that the ply is subjected to direct stresses  $\sigma_x$  and  $\sigma_y$ , shear and complementary shear stresses  $\tau_{xy}$  and that the elastic constants for the ply are  $E_l$ ,  $E_t$ ,  $G_{lt}$  ( $= G_{tl}$ ),  $v_{lt}$ , and  $v_{tl}$  (see Eqs. (26.5)–(26.10)). Note that, unlike an isotropic material, the shear modulus  $G_{lt}$  is not related to the other elastic constants. From Section 1.15 (see Eqs. (1.52)), the strains in the longitudinal and transverse directions are given by

$$\left. \begin{aligned} \varepsilon_l &= \frac{\sigma_x}{E_l} - \frac{v_{tl}\sigma_y}{E_t} \\ \varepsilon_t &= \frac{\sigma_y}{E_t} - \frac{v_{lt}\sigma_x}{E_l} \end{aligned} \right\} \quad (26.11)$$

Equations (26.11) may be written in matrix form; that is,

$$\begin{Bmatrix} \varepsilon_l \\ \varepsilon_t \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_l} & -\frac{v_{tl}}{E_t} \\ \frac{v_{lt}}{E_l} & \frac{1}{E_t} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} \quad (26.12)$$

or, in general terms,

$$[\varepsilon] = [S][\sigma] \quad (26.13)$$

in which  $[S]$  is frequently termed *the compliance matrix*. It may be shown, using an energy approach, that the compliance matrix  $[S]$  must be symmetric about the leading diagonal. Therefore,

$$-\frac{v_{tl}}{E_t} = -\frac{v_{lt}}{E_l}$$

giving

$$\frac{v_{tl}}{E_t} = \frac{v_{lt}}{E_l} \quad (26.14)$$

so that, of the four elastic constants  $E_l$ ,  $E_t$ ,  $v_{lt}$ , and  $v_{tl}$ , only three are independent.

Equations (26.11) may be transposed (as in Section 1.15) to give stress-strain relationships. Then,

$$\left. \begin{aligned} \sigma_x &= \frac{E_l}{1-v_{lt}v_{tl}}\varepsilon_l + \frac{v_{tl}E_l}{1-v_{lt}v_{tl}}\varepsilon_t \\ \sigma_y &= \frac{E_t}{1-v_{lt}v_{tl}}\varepsilon_t + \frac{v_{lt}E_t}{1-v_{lt}v_{tl}}\varepsilon_l \end{aligned} \right\} \quad (26.15)$$

From the last of Eqs. (1.52),

$$\gamma_{lt} = \frac{\tau_{xy}}{G_{lt}} \quad (26.16)$$

or

$$\tau_{xy} = \gamma_{lt}G_{lt}$$

Equations (26.15) and (26.16) may be written in matrix form, that is,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{E_l}{1 - v_{lt}v_{tl}} & \frac{v_{lt}E_l}{1 - v_{lt}v_{tl}} & 0 \\ \frac{v_{lt}E_t}{1 - v_{lt}v_{tl}} & \frac{E_t}{1 - v_{lt}v_{tl}} & 0 \\ 0 & 0 & G_{lt} \end{bmatrix} \begin{Bmatrix} \epsilon_l \\ \epsilon_t \\ \gamma_{lt} \end{Bmatrix} \quad (26.17)$$

### Example 26.3

A single sheet of woven ply is subjected to longitudinal and transverse direct stresses of 50 and 25 N/mm<sup>2</sup>, respectively, together with a shear stress of 40 N/mm<sup>2</sup>. The elastic constants for the ply are  $E_l=120,000$  N/mm<sup>2</sup>,  $E_t=80,000$  N/mm<sup>2</sup>,  $G_{lt}=5000$  N/mm<sup>2</sup>, and  $v_{lt}=0.3$ . Calculate the direct strains in the longitudinal and transverse directions and the shear strain in the ply.

The value of the minor Poisson's ratio,  $v_{lt}$ , is not given and must be calculated first. From Eq. (26.14),

$$v_{lt} = v_{lt} \frac{E_t}{E_l} = 0.3 \times \frac{80,000}{120,000} = 0.2$$

Therefore, from Eqs. (26.11),

$$\begin{aligned} \epsilon_l &= \frac{50}{120,000} - \frac{0.2 \times 25}{80,000} = 3.54 \times 10^{-4} \\ \epsilon_t &= \frac{25}{80,000} - \frac{0.3 \times 50}{120,000} = 1.88 \times 10^{-4} \end{aligned}$$

and, from Eq. (26.16),

$$\gamma_{lt} = \frac{40}{5,000} = 80.0 \times 10^{-4}$$

### 26.2.2 Generally orthotropic ply

In Fig. 26.8, the direction of the fibers in the ply does not coincide with the loading axes  $x$  and  $y$ . We specify that the longitudinal fibers of the ply are inclined at an angle  $\theta$  to the  $x$  axis;  $\theta$  is positive when the fibers are rotated in a counterclockwise sense from the  $x$  axis.

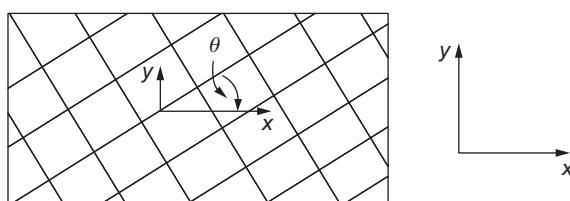


FIGURE 26.8 Generally Orthotropic Ply

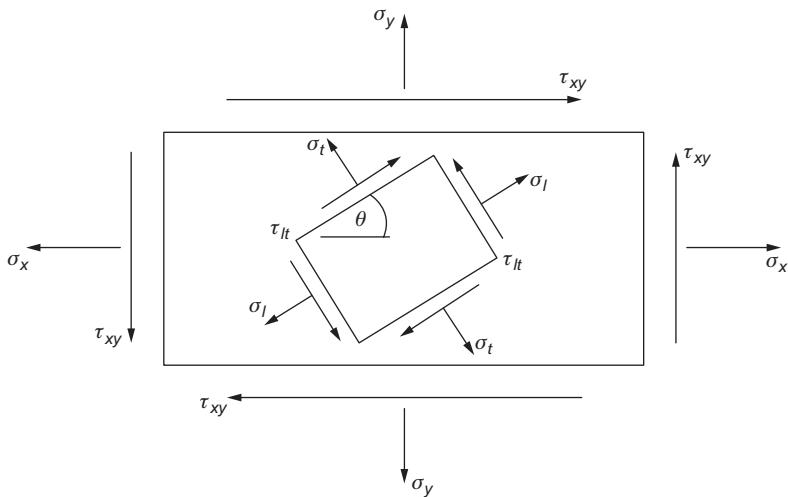


FIGURE 26.9 Stresses in a Generally Orthotropic Ply

Suppose that an element of the ply is subjected to stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$ , as shown in Fig. 26.9. The stresses on an element of the ply in the directions of the fibers may be found in terms of the applied stresses using the method described in Section 1.6. Therefore, by comparison with Eq. (1.8),

$$\sigma_l = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \cos \theta \sin \theta \quad (26.18)$$

Similarly,

$$\sigma_t = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2\tau_{xy} \cos \theta \sin \theta \quad (26.19)$$

and, by comparison with Eq. (1.9) but noting that  $\tau_{lt}$  is in the opposite sense to  $\tau$ ,

$$\tau_{lt} = -\sigma_x \cos \theta \sin \theta + \sigma_y \cos \theta \sin \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \quad (26.20)$$

If we write  $m = \cos \theta$  and  $n = \sin \theta$ , Eqs. (26.18) through (26.20) become

$$\sigma_l = m^2 \sigma_x + n^2 \sigma_y + 2mn \tau_{xy} \quad (26.21)$$

$$\sigma_t = n^2 \sigma_x + m^2 \sigma_y - 2mn \tau_{xy} \quad (26.22)$$

$$\tau_{lt} = -mn \sigma_x + mn \sigma_y + (m^2 - n^2) \tau_{xy} \quad (26.23)$$

Writing Eqs. (26.21)–(26.23) in matrix form, we have

$$\begin{Bmatrix} \sigma_l \\ \sigma_t \\ \tau_{lt} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (26.24)$$

Similarly, from Eqs. (1.31) and (1.34),

$$\begin{Bmatrix} \varepsilon_l \\ \varepsilon_t \\ \gamma_{lt} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (26.25)$$

Equations (26.24) may be transposed so that the applied stresses are expressed in terms of the ply stresses:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_l \\ \sigma_t \\ \tau_{lt} \end{Bmatrix} \quad (26.26)$$

In Eqs. (26.17), for a specially orthotropic ply, the ply stresses and loading stresses are identical, so that we may use this equation to relate the ply stresses in a generally orthotropic ply to the ply strains. Then,

$$\begin{Bmatrix} \sigma_l \\ \sigma_t \\ \tau_{lt} \end{Bmatrix} = \begin{bmatrix} \frac{E_l}{1 - v_{lt}v_{tl}} & \frac{v_{lt}E_l}{1 - v_{lt}v_{tl}} & 0 \\ \frac{v_{lt}E_t}{1 - v_{lt}v_{tl}} & \frac{E_t}{1 - v_{lt}v_{tl}} & 0 \\ 0 & 0 & G_{lt} \end{bmatrix} \begin{Bmatrix} \varepsilon_l \\ \varepsilon_t \\ \gamma_{lt} \end{Bmatrix} \quad (26.27)$$

Substituting for the ply stresses in Eqs. (26.26) from Eqs. (26.27), we express the applied stresses in terms of the ply strains; that is,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \frac{E_l}{1 - v_{lt}v_{tl}} & \frac{v_{lt}E_l}{1 - v_{lt}v_{tl}} & 0 \\ \frac{v_{lt}E_t}{1 - v_{lt}v_{tl}} & \frac{E_t}{1 - v_{lt}v_{tl}} & 0 \\ 0 & 0 & G_{lt} \end{bmatrix} \begin{Bmatrix} \varepsilon_l \\ \varepsilon_t \\ \gamma_{lt} \end{Bmatrix} \quad (26.28)$$

Finally, by substituting for the ply strains in Eqs. (26.28) from Eqs. (26.25), we obtain the applied stresses in terms of the strains referred to the  $xy$  axes; that is,

$$\begin{aligned} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} &= \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \frac{E_l}{1 - v_{lt}v_{tl}} & \frac{v_{lt}E_l}{1 - v_{lt}v_{tl}} & 0 \\ \frac{v_{lt}E_t}{1 - v_{lt}v_{tl}} & \frac{E_t}{1 - v_{lt}v_{tl}} & 0 \\ 0 & 0 & G_{lt} \end{bmatrix} \begin{Bmatrix} \varepsilon_l \\ \varepsilon_t \\ \gamma_{lt} \end{Bmatrix} \\ &\times \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \end{aligned} \quad (26.29)$$

Writing the individual terms of the central matrix as

$$k_{11} = \frac{E_l}{1 - v_{lt}v_{ll}}, \quad k_{12} = \frac{v_{lt}E_l}{1 - v_{lt}v_{ll}} = k_{21} \quad (\text{from Eq. (26.17)})$$

$$k_{22} = \frac{E_t}{1 - v_{lt}v_{ll}}, \quad k_{33} = G_{lt}$$

$k_{11}, k_{12}$ , etc., are sometimes referred to as *the reduced stiffnesses*. Eqs. (26.29) become

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{12} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \times \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (26.30)$$

Carrying out the matrix multiplication in Eqs. (26.30), we obtain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} m^4 k_{11} + m^2 n^2 (2k_{12} \\ + 4k_{33}) + n^4 k_{22} \\ m^2 n^2 (k_{11} + k_{22} - 4k_{33}) \\ + (m^4 + n^4) k_{12} \\ m^3 n (k_{11} - k_{12} - 2k_{33}) \\ + mn^3 (k_{12} + k_{22} + 2k_{33}) \end{bmatrix} \begin{bmatrix} m^2 n^2 (k_{11} + k_{22} - 4k_{33}) \\ + (m^4 + n^4) k_{12} \\ n^4 k_{11} + m^2 n^2 (2k_{12} \\ + 4k_{33}) + m^4 k_{22} \\ mn^3 (k_{11} - k_{12} - 2k_{33}) \\ + m^3 n (k_{12} - k_{22} + 2k_{33}) \end{bmatrix} \begin{bmatrix} m^3 n (k_{11} - k_{12} - 2k_{33}) \\ + mn^3 (k_{12} - k_{22} + 2k_{33}) \\ mn^3 (k_{11} - k_{12} - 2k_{33}) \\ + m^3 n (k_{12} - k_{22} + 2k_{33}) \\ m^2 n^2 (k_{11} + k_{22} - 2k_{12} \\ - 2k_{33}) + (m^4 + n^4) k_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (26.31)$$

It can be seen that, for a specially orthotropic ply where  $\theta = 0$ , Eqs. (26.31) reduce to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{12} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (26.32)$$

which are identical to Eqs. (26.17). Eqs (26.31) and (26.32) may be written in general terms as

$$[\sigma] = [K][\epsilon]$$

in which  $[K]$  is *the reduced stiffness matrix*.

Having expressed the applied stresses in terms of the  $xy$  strains, Eqs. (26.31) may be transposed to obtain the  $xy$  strains in terms of the applied stresses. This may be shown to be

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} m^4 s_{11} + n^4 s_{22} \\ + 2m^2 n^2 s_{12} + m^2 n^2 s_{33} \\ m^2 n^2 s_{11} + m^2 n^2 s_{22} \\ + (m^4 + n^4) s_{12} \\ - m^2 n^2 s_{33} \\ 2m^3 n s_{11} - 2m n^3 s_{22} \\ + 2(m n^3 - m^3 n) s_{12} \\ + (m n^3 - m^3 n) s_{33} \end{bmatrix} \begin{bmatrix} m^2 n^2 s_{11} + m^2 n^2 s_{22} \\ + (m^4 + n^4) s_{12} \\ - m^2 n^2 s_{33} \\ n^4 s_{11} + m^4 s_{22} \\ + 2m^2 n^2 s_{12} + m^2 n^2 s_{33} \\ 2m n^3 s_{11} - 2m^3 n s_{22} \\ + 2(m^3 n - m n^3) s_{12} \\ + (m^3 n - m n^3) s_{33} \end{bmatrix} \begin{bmatrix} 2m^3 n s_{11} - 2m n^3 s_{22} \\ + 2(m n^3 - m^3 n) s_{12} \\ + (m n^3 - m^3 n) s_{33} \\ 2m^2 n^2 s_{11} + 4m^2 n^2 s_{22} \\ - 8m^2 n^2 s_{12} \\ + (m^2 - n^2)^2 s_{33} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (26.33)$$

in which

$$s_{11} = 1/E_l, \quad s_{12} = -v_{tl}/E_t, \quad s_{22} = 1/E_t, \quad s_{33} = 1/G_{lt}$$

For a specially orthotropic ply in which only direct stresses  $\sigma_x$  and  $\sigma_y$  are applied, Eqs. (26.33) reduce to

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix}$$

that is,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_l} & -\frac{v_{tl}}{E_t} \\ -\frac{v_{tl}}{E_l} & \frac{1}{E_t} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix}$$

which are identical to Eqs. (26.12).

### Example 26.4

A thin ply consists of an isotropic material having a Young's modulus of 80,000 N/mm<sup>2</sup> and a Poisson's ratio of 0.3. Determine the terms in the reduced stiffness and compliance matrices.

The reduced stiffnesses are

$$k_{11} = k_{22} = E/(1 - v^2) = 80,000/(1 - 0.3^2) = 87,912 \text{ N/mm}^2$$

$$k_{12} = vE/(1 - v^2) = 0.3 \times 80,000/(1 - 0.3^2) = 26,374 \text{ N/mm}^2$$

Also  $k_{33} = G$ , but since the ply is isotropic, the shear modulus may be expressed in terms of Young's modulus and Poisson's ratio, i.e.,

$$G = E/2(1 + v) \quad (\text{see Eq. (1.50)})$$

so that

$$k_{33} = 80,000/2(1 + 0.3) = 30,769 \text{ N/mm}^2$$

The reduced compliances are

$$s_{11} = s_{22} = 1/E = 1/80,000 = 12.5 \times 10^{-6}$$

$$s_{12} = s_{21} = -v/E = -0.3/80,000 = -3.75 \times 10^{-6}$$

$$s_{33} = 1/G = 1/30,769 = 32.5 \times 10^{-6}$$

### Example 26.5

If the reduced stiffnesses in a unidirectional ply are  $k_{11} = 40,000 \text{ N/mm}^2$ ,  $k_{12} = k_{21} = 2000 \text{ N/mm}^2$ ,  $k_{22} = 8000 \text{ N/mm}^2$  and  $k_{33} = 5000 \text{ N/mm}^2$ , determine the elastic constants of the ply.

From Eqs (26.27) and (26.32),

$$k_{11} = E_l/(1 - v_{tl}v_{lt}) \quad (\text{i})$$

$$k_{12} = v_{tl}E_l/(1 - v_{tl}v_{lt}) \quad (\text{ii})$$

Dividing Eq. (ii) by Eq. (i),

$$v_{lt} = k_{12}/k_{11} = 2000/40,000 = 0.05$$

Also

$$k_{21} = v_{lt}E_t/(1 - v_{lt}v_{lt}) \quad (\text{iii})$$

$$k_{22} = E_t/(1 - v_{lt}v_{lt}) \quad (\text{iv})$$

Dividing Eq. (iii) by Eq. (iv),

$$v_{lt} = k_{21}/k_{22} = 2000/8000 = 0.25$$

Substituting in Eq. (i) for  $v_{lt}$  and  $v_{lt}$ ,

$$40,000 = E_l/(1 - 0.25 \times 0.05)$$

which gives

$$E_l = 39,500 \text{ N/mm}^2$$

From Eq. (iii),

$$E_t = 2000(1 - 0.25 \times 0.05)/0.25 = 7900 \text{ N/mm}^2$$

Also

$$G = k_{33} = 5000 \text{ N/mm}^2.$$

### Example 26.6

A generally orthotropic ply is subjected to direct stresses of  $60 \text{ N/mm}^2$  parallel to the  $x$  reference axis and  $40 \text{ N/mm}^2$  perpendicular to the  $x$  reference axis. If the longitudinal plies are inclined at an angle of  $45^\circ$  to the  $x$  axis and the elastic constants are  $E_l = 150,000 \text{ N/mm}^2$ ,  $E_t = 90,000 \text{ N/mm}^2$ ,  $G_{lt} = 5000 \text{ N/mm}^2$ , and  $v_{lt} = 0.3$ , calculate the direct strains parallel to the  $x$  and  $y$  directions and the shear strain referred to the  $xy$  axes.

We note that there is no applied shear stress, so it is unnecessary to calculate the terms in the third column of the matrix of Eqs. (26.33). Then,

$$s_{11} = \frac{1}{E_l} = \frac{1}{150,000} = 6.7 \times 10^{-6}$$

$$s_{22} = \frac{1}{E_t} = \frac{1}{90,000} = 11.1 \times 10^{-6}$$

$$s_{12} = -\frac{v_{lt}}{E_l} = -\frac{0.3}{150,000} = -2.0 \times 10^{-6}$$

$$s_{33} = \frac{1}{G_{lt}} = \frac{1}{5000} = 200 \times 10^{-6}$$

Also,

$$\cos \theta = \sin \theta = \cos 45^\circ = 1/\sqrt{2}$$

so that

$$m^2 = 0.5 = n^2, \quad m^4 = n^4 = 0.25, \quad m^2 n^2 = 0.25, \text{ and so forth}$$

Substituting these values in Eqs. (26.33), we have

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 53.45 & -46.55 & - \\ -46.55 & 53.45 & - \\ -2.2 & -2.2 & - \end{bmatrix} \begin{Bmatrix} 60 \\ 40 \\ 0 \end{Bmatrix}$$

which gives

$$\varepsilon_x = 1345 \times 10^{-6}$$

$$\varepsilon_y = -655 \times 10^{-6}$$

$$\gamma_{xy} = -220 \times 10^{-6}$$

### Example 26.6 MATLAB

Use MATLAB to repeat Example 26.6.

The direct and shear strains are obtained through the following MATLAB file:

```
% Declare any needed variables
sig_x=60;
sig_y=40;
tau_xy=0;
theta=45*pi/180;
E_l=150e3;
E_t=90e3;
G_lt=5e3;
v_lt=0.3;

% Calculate variables needed to obtain the desired strains
m=cos(theta);
n=sin(theta);
s_11=round(1/E_l*10^7)/(10^7);
s_12=round(-v_lt/E_l*10^7)/(10^7);
s_22=round(1/E_t*10^7)/(10^7);
s_33=round(1/G_lt*10^7)/(10^7);
a_11=m^4*s_11+n^4*s_22+2*m^2*n^2*s_12+m^2*n^2*s_33;
a_12=m^2*n^2*(s_11+s_22-s_33)+(m^4+n^4)*s_12;
a_13=2*m^3*n*s_11-2*m*n^3*s_22+2*(m*n^3-m^3*n)*s_12+(m*n^3-m^3*n)*s_33;
a_21=a_12;
a_22=n^4*s_11+m^4*s_22+2*m^2*n^2*s_12+m^2*n^2*s_33;
a_23=2*m*n^3*s_11-2*m^3*n*s_22+2*(m^3*n-m*n^3)*s_12+(m^3*n-m*n^3)*s_33;
a_31=a_13;
```

```

a_32=a_23;
a_33=4*m^2*n^2*(s_11+s_22)-8*m^2*n^2*s_12+(m^2-n^2)*s_33;

% Substitute these variables into Eq. (26.33)
A=[a_11 a_12 a_13;
   a_21 a_22 a_23;
   a_31 a_32 a_33];
b=[sig_x sig_y tau_xy]';
strains=A*b;

% Output the direct and shear strains to the Command Window
disp(['e_x=' num2str(strains(1))])
disp(['e_y=' num2str(strains(2))])
disp(['gamma_xy=' num2str(strains(3))])

```

The Command Window outputs resulting from this MATLAB file are as follows:

```

e_x=0.001345
e_y=-0.000655
gamma_xy=-0.00022

```

### Example 26.7

A generally orthotropic ply is subjected to stresses of  $\sigma_x = 120 \text{ N/mm}^2$ ,  $\sigma_y = 45 \text{ N/mm}^2$ , and  $\tau_{xy} = 70 \text{ N/mm}^2$  referred to an  $xy$  system of loading axes. If the ply angle is  $45^\circ$ , determine the equivalent stresses referred to the material axes.

Since  $\cos \theta = \sin \theta = \cos 45^\circ = 1/\sqrt{2}$ ,

$$m^2 = n^2 = mn = 0.5$$

Substituting in Eqs (26.24),

$$\begin{Bmatrix} \sigma_l \\ \sigma_t \\ \tau_{lt} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & -1 \\ -0.5 & 0.5 & 0 \end{bmatrix} \begin{Bmatrix} 120 \\ 45 \\ 70 \end{Bmatrix}$$

which gives  $\sigma_l = 152.5 \text{ N/mm}^2$ ,  $\sigma_t = 12.5 \text{ N/mm}^2$ ,  $\tau_{lt} = -37.5 \text{ N/mm}^2$ .

## 26.3 LAMINATES

In Sections 26.1 and 26.2 we have been concerned with the properties of a single ply (lamina), which may be unidirectional (Fig. 26.6(a)), specially orthotropic (Fig. 26.7), or generally orthotropic (Fig. 26.8). Clearly these plies are limited in the magnitude and type of loading they are able to support, so that generally, in practice, two or more plies are bonded together, one on top of the other (i.e., are *stacked*), to form a single element that is then called a *layered laminate*.

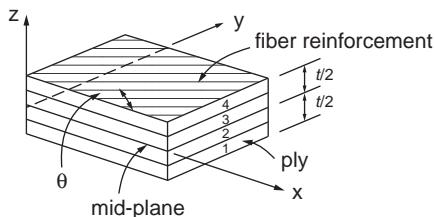
The individual plies in a laminate may be arranged at different angles relative to the laminate reference axes so that the required strength and stiffness of the laminate in specific directions are achieved. These properties of a laminate depend on the individual ply properties and the order in which they are stacked.

### 26.3.1 Assumptions and notation

In addition to the assumptions made for a single ply the following apply to laminates.

- (i) The laminate obeys Hooke's law.
- (ii) Each ply may be isotropic or orthotropic.
- (iii) The laminate is thin so that shear stresses and strains across the thickness are negligible.
- (iv) The bond between the plies is perfect and infinitesimally thin so that there is no slippage between plies, and strains and displacements across the bond are continuous.
- (v) Displacements and deformations are small.

The method of specifying the constituent plies in a laminate is illustrated by reference to Fig. 26.10 where the plies are listed in sequence from one face to the other. We shall adopt the same axes system and designation of ply orientation as shown in Fig. 26.8 with the addition of the  $z$  axis perpendicular to the plane of the laminate. Generally the  $xy$  plane will coincide with the mid-plane of the laminate.



**FIGURE 26.10 Axes System for a Laminate**

Suppose in Fig. 26.10 that for ply 1,  $\theta = -45^\circ$ , for plies 2 and 3,  $\theta = 0^\circ$ , and for ply 4,  $\theta = +45^\circ$ . The laminate plies would then be designated  $[-45/0/0/+45]$ . In some cases of multi-ply laminates this method of ply designation may be abbreviated to take advantage of symmetry or repetition. For example, if, in Fig. 26.10,  $\theta = +45^\circ$  for ply 1,  $\theta = 0^\circ$  for plies 2 and 3, and  $\theta = +45^\circ$  for ply 4, then the designation would be  $[+45/0]_S$  where the suffix  $s$  denotes symmetry. Further, if, in Fig. 26.10,  $\theta = 0^\circ$ ,  $0^\circ$ ,  $+45^\circ$ ,  $+45^\circ$  for plies 1–4, respectively, the designation would be  $[0_2/+45_2]$ . For an eight-ply laminate, say,  $[(0/45_2)/\pm60_2]$  stands for  $[0/45/0/45/\pm60/\pm60/-60/-60]$ . Symmetry and repetition can be combined so that, for example,  $[(0/45)_2]_S$  or  $[0/45]_{2S}$  stands for  $[0/45/0/45/45/0/45/0]$ .

Particular cases of laminates are given special names. If a laminate comprises plies for which  $\theta = 0^\circ$  and  $90^\circ$ , it is called a *cross-ply laminate*. A laminate consisting of only  $+\theta$  and  $-\theta$  plies is called an *angle-ply laminate*; one in which there is a  $-\theta$  ply for every  $+\theta$  ply is a *balanced laminate*.

### 26.3.2 Laminate under in-plane loading

Laminates, particularly in aircraft construction, are thin and generally subjected to in-plane loading. We shall therefore restrict the following to plane stress systems. The bending and torsion of laminates is presented in Ref. (2).

Figure 26.11 shows a small element  $\delta x \times \delta y$  at a point in a laminate subjected to in-plane direct and shear force intensities,  $N_x$ ,  $N_y$ , and  $N_{xy}$ ; these are forces per unit length of the laminate sides and are positive in the directions shown. The thickness of the laminate is  $t$ , and again we take the plane of the  $xy$  axes to coincide with the mid-plane of the laminate.

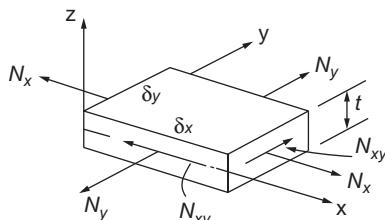


FIGURE 26.11 In-Plane Forces Acting at a Point in a Laminate

The load intensities will produce direct and shear stresses,  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$ , which, although we have assumed perfectly bonded plies, thereby ensuring continuity of strain across the laminate thickness, can differ from ply to ply due to the possibly different properties of the plies. We shall now express the laminate load intensities in terms of the ply stresses.

Figure 26.12 shows part of the cross-section of a multi-ply laminate in the  $xz$  plane. Suppose that there are  $N$  plies in the laminate and that they are numbered from bottom to top. Since we are concerned with force intensities, i.e., the force per unit width, the actual width of the laminate cross-section does not enter into our equations.

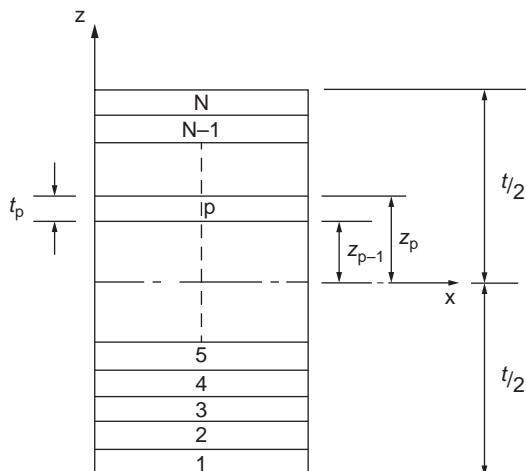


FIGURE 26.12 Portion of a Layered Laminate

Consider the  $p$ th ply. The ordinate of its lower surface is  $z_{p-1}$  while that of its upper surface is  $z_p$ . The normal force per unit width in the  $x$  direction on this ply is given by

$$N_{x,p} = \int_{z_{p-1}}^{z_p} \sigma_{x,p} dz \quad (\text{acting perpendicularly to the plane of the page in Fig. 26.12}) \quad (26.34)$$

The total normal force intensity on the laminate in the  $x$  direction is the sum of the normal force intensities on the individual plies. That is,

$$N_x = \sum_{p=1}^N N_{x,p} = \sum_{p=1}^N \int_{z_{p-1}}^{z_p} \sigma_{x,p} dz \quad (26.35)$$

Similarly,

$$N_y = \sum_{p=1}^N \int_{z_{p-1}}^{z_p} \sigma_{y,p} dz \quad (26.36)$$

and

$$N_{xy} = \sum_{p=1}^N \int_{z_{p-1}}^{z_p} \tau_{xy,p} dz \quad (26.37)$$

We have noted that although the stresses in the different plies in a laminate can differ from ply to ply, the strains are constant across the laminate thickness. We now, therefore, express Eqs. (26.35)–(26.37) in terms of the laminate strains.

Equation (26.31) relating ply stresses to ply strains is rather cumbersome, so we rewrite it in short-hand form as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \bar{k}_{12} & \bar{k}_{22} & \bar{k}_{23} \\ \bar{k}_{13} & \bar{k}_{23} & \bar{k}_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (26.38)$$

in which, for example,  $\bar{k}_{11} = m^4 k_{11} + m^2 n^2 (2k_{12} + 4k_{33}) + n^4 k_{22}$  and so on. Substituting for  $\sigma_{x,p}$  in Eq. (26.35) from Eq. (26.38), we have

$$N_x = \sum_{p=1}^N \int_{z_{p-1}}^{z_p} [\bar{k}_{11} \quad \bar{k}_{12} \quad \bar{k}_{13}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} dz \quad (26.39)$$

Since the strains are constant through the thickness of the laminate, they may be taken outside the integration in Eq. (26.39). Then

$$N_x = \sum_{p=1}^N [\bar{k}_{11} \quad \bar{k}_{12} \quad \bar{k}_{13}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \int_{z_{p-1}}^{z_p} dz$$

which gives

$$N_x = \sum_{p=1}^N [\bar{k}_{11} \quad \bar{k}_{12} \quad \bar{k}_{13}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} (z_p - z_{p-1}) \quad (26.40)$$

Similarly,

$$N_y = \sum_{p=1}^N [\bar{k}_{12} \quad \bar{k}_{22} \quad \bar{k}_{23}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} (z_p - z_{p-1}) \quad (26.41)$$

and

$$N_{xy} = \sum_{p=1}^N [\bar{k}_{13} \quad \bar{k}_{23} \quad \bar{k}_{33}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} (z_p - z_{p-1}) \quad (26.42)$$

Rewriting Eqs. (26.40)–(26.42) in single-matrix format,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{p=1}^N (z_p - z_{p-1}) \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \bar{k}_{12} & \bar{k}_{22} & \bar{k}_{23} \\ \bar{k}_{13} & \bar{k}_{23} & \bar{k}_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (26.43)$$

Again, Eq. (26.43) is rather lengthy to manipulate, so we rewrite it in shorthand form as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (26.44)$$

in which

$$A_{11} = \sum_{p=1}^N (z_p - z_{p-1}) \bar{k}_{11} \quad (26.45)$$

$$A_{12} = \sum_{p=1}^N (z_p - z_{p-1}) \bar{k}_{12} \quad (26.46)$$

and so on.  $A_{11}, A_{12}, \dots, A_{33}$  are called the *extensional stiffnesses* of the laminate and relate the in-plane load intensities to the strains in the laminate mid-plane; the strains, as we have already seen, are assumed constant through the thickness.

The extensional stiffnesses  $A_{11}$  etc. are dependent on the reduced stiffness of a ply ( $k_{11} = E_l/(1 - v_u/v_t)$  etc. (see Eq. (26.30)) and the ordinate term  $(z_p - z_{p-1})$ , which is, in fact, the thickness of a ply,  $t_p$ . Then

$$A_{11} = \sum_{p=1}^N t_p \bar{k}_{11} \text{ etc.}$$

In general terms,

$$A_{ij} = \sum_{p=1}^N t_p (\bar{k}_{ij})_p \quad (i = 1, 2, 3, \dots, j = 1, 2, 3, \dots) \quad (26.47)$$

In Eq. (26.44) the extensional stiffness  $A_{11}$  relates the load intensity,  $N_x$ , to the direct strain,  $\varepsilon_x$ , which is, in effect, a Hooke's law relationship. Further, the extensional stiffness  $A_{12}$  relates  $N_x$ , that is the load intensity in the  $x$ -direction, to the lateral strain  $\varepsilon_y$ , in other words gives the Poisson effect. Also it is possible in some laminates for  $N_x$ , say, to produce a shear strain  $\gamma_{xy}$ ; these are linked by the term,  $A_{13}$ .

### 26.3.3 Equivalent elastic constants

In some instances, particularly in an initial design stage, it may be convenient to express the load intensity-strain relationships (Eq. (26.44)) as strain-load intensity relationships. Rewriting Eq. (26.44) in simplified form as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (26.48)$$

we can express the strains as functions of the applied load intensities by

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [A^{-1}] \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \quad (26.49)$$

where  $[A^{-1}]$  is the inverse of the matrix  $[A]$ .

For a symmetric laminate it may be shown (Ref. 2) that the in-plane load intensities do not produce out-of-plane curvature deformations. We can therefore define average direct stresses across the thickness of the laminate (note that stresses can vary from ply to ply) as

$$\bar{\sigma}_x = \frac{N_x}{t} \quad \bar{\sigma}_y = \frac{N_y}{t} \quad \bar{\tau}_{xy} = \frac{N_{xy}}{t}$$

Equation (26.49) then becomes

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = t[A^{-1}] \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{Bmatrix} \quad (26.50)$$

The inversion of the  $[A]$  matrix may be carried out using standard matrix algebra; the result is quoted here. Let the terms in  $[A^{-1}]$  be as follows:

$$[A^{-1}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad (26.51)$$

in which

$$\begin{aligned} a_{11} &= (A_{22}A_{33} - A_{23}^2)/AA \\ a_{22} &= (A_{11}A_{33} - A_{13}^2)/AA \\ a_{33} &= (A_{11}A_{22} - A_{12}^2)/AA \\ a_{12} &= (A_{13}A_{23} - A_{12}A_{33})/AA \\ a_{13} &= (A_{12}A_{23} - A_{22}A_{13})/AA \\ a_{23} &= (A_{12}A_{13} - A_{11}A_{23})/AA \end{aligned} \quad (26.52)$$

where

$$AA = A_{11}A_{22}A_{33} + 2A_{12}A_{23}A_{13} - A_{22}A_{13}^2 - A_{33}A_{12}^2 - A_{11}A_{23}^2 \quad (26.53)$$

In some cases, for example a specially orthotropic ply (see Eq. (26.32)), the terms  $A_{13}$  and  $A_{23}$  are zero. Then

$$AA = (A_{11}A_{22} - A_{12}^2)A_{33} \quad (26.54)$$

and

$$\begin{aligned} a_{11} &= (A_{22})/(A_{11}A_{22} - A_{12}^2) \\ a_{22} &= (A_{11})/(A_{11}A_{22} - A_{12}^2) \\ a_{33} &= 1/A_{33} \\ a_{12} &= -(A_{12})/(A_{11}A_{22} - A_{12}^2) \\ a_{13} &= 0 \\ a_{23} &= 0 \end{aligned} \quad (26.55)$$

Suppose now that a symmetric laminate is subjected to a single load intensity,  $N_x$ . Then

$$\bar{\sigma}_x = \frac{N_x}{t}, \quad \bar{\sigma}_y = \bar{\tau}_{xy} = 0$$

so that, from Eqs. (26.50) and (26.51),

$$\varepsilon_x = ta_{11}\bar{\sigma}_x$$

$$\text{or } \frac{\bar{\sigma}_x}{\varepsilon_x} = \frac{1}{ta_{11}}$$

From Hooke's law,  $\bar{\sigma}_x/\varepsilon_x = E_x$ , which is Young's modulus for the laminate in the  $x$  direction. Therefore,

$$E_x = \frac{1}{ta_{11}} \quad (26.56)$$

Further,  $\varepsilon_y = ta_{12}\bar{\sigma}_x = ta_{12}\varepsilon_x/ta_{11} = (a_{12}/a_{11})\varepsilon_x$ , which gives the strain in the  $y$  direction due to a load in the  $x$  direction, that is the Poisson effect. Then, Poisson's ratio in the  $xy$  plane of the laminate is given by

$$\nu_{xy} = \frac{-\varepsilon_y}{\varepsilon_x} = \frac{-a_{12}}{a_{11}} \quad (26.57)$$

It is possible, as can be seen from Eq. (26.50), that the single load intensity,  $N_x$ , can produce a shear strain,  $\gamma_{xy}$ . That is,

$$\gamma_{xy} = ta_{13}\bar{\sigma}_x = ta_{13}\varepsilon_x/ta_{11} = (a_{13}/a_{11})\varepsilon_x$$

Then

$$\frac{\gamma_{xy}}{\varepsilon_x} = \frac{a_{13}}{a_{11}} = -m_x \quad (26.58)$$

in which  $m_x$  is termed the *shear coupling coefficient* and is analogous to Poisson's ratio.

The above arguments may be applied to a symmetric laminate subjected to a single load intensity,  $N_y$ . Then

$$E_y = \frac{1}{ta_{22}} \quad (26.59)$$

$$v_{yx} = \frac{-a_{12}}{a_{22}} \quad (26.60)$$

and

$$m_y = \frac{-a_{23}}{a_{22}} \quad (26.61)$$

We now consider the case of a symmetric laminate subjected to a shear force intensity,  $N_{xy}$ , only. From Eqs. (26.50) and (26.51),

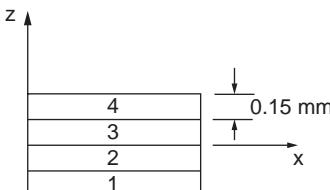
$$\frac{\bar{\tau}_{xy}}{\gamma_{xy}} = \frac{1}{ta_{33}} = G_{xy} \quad (26.62)$$

the shear modulus of the laminate in the  $xy$  plane.

Eqs. (26.56)–(26.62) define the *equivalent elastic constants* of a laminate in the  $xy$  plane. Note that these relationships are derived for a laminate subjected to in-plane load intensities only. Generally the equivalent elastic constants, when the laminate is subjected to bending will be different to those derived for in-plane loading. However, in some special cases, for example a symmetric singly orientated ply laminate, the two sets of equivalent elastic constants are identical [Ref. 2].

### Example 26.8

Figure 26.13 shows the cross-section in the  $xz$  plane of a symmetric laminate comprising four plies that are singly orientated and isotropic; all the plies have the same thickness of 0.15 mm. If Young's modulus for the plies is 70000 N/mm<sup>2</sup> and Poisson's ratio is 0.3, calculate the equivalent elastic constants of the laminate for the case of in-plane loading.



**FIGURE 26.13 Laminate of Example 26.8**

Since the laminate is symmetric and isotropic, the elastic constants for each ply are

$$E_l = E_t = E = 70,000 \text{ N/mm}^2$$

$$v_{lt} = v_{tl} = v = 0.3$$

Also, for an isotropic laminate (see Eq. (1.50)),

$$G_h = E/2(1 + v) = 70,000/2(1 + 0.3) = 26,923 \text{ N/mm}^2$$

For an isotropic ply, which is a special case of an orthotropic ply, the reduced stiffness terms  $k_{13}$  and  $k_{23}$  in Eq. (26.31) are zero. Also, from Eq. (26.17),

$$\begin{aligned} k_{11} &= E/(1 - v^2) = 70,000/(1 - 0.3^2) = 76,923 \text{ N/mm}^2 \\ k_{12} &= vE/(1 - v^2) = 0.3 \times 70,000(1 - 0.3^2) = 23,077 \text{ N/mm}^2 \\ k_{22} &= E/(1 - v^2) = 76,923 \text{ N/mm}^2 \\ k_{33} &= G = 26,923 \text{ N/mm}^2 \end{aligned}$$

For an isotropic ply there is no ply angle so that in effect,  $\theta = 0^\circ$ , and in Eq. (26.38),

$$\bar{k}_{11} = k_{11}, \quad \bar{k}_{12} = k_{12}, \quad \bar{k}_{22} = k_{22}$$

From Eq. (26.45) etc. and since there are four identical plies each 0.15 mm thick,

$$\begin{aligned} A_{11} &= 4 \times 0.15 \times 76,923 = 46,154 \text{ N/mm} \\ A_{12} &= 4 \times 0.15 \times 23,077 = 13,846 \text{ N/mm} \\ A_{22} &= 4 \times 0.15 \times 76,923 = 46,154 \text{ N/mm} \\ A_{33} &= 4 \times 0.15 \times 26,923 = 16,154 \text{ N/mm} \end{aligned}$$

Then, from Eqs. (26.55), the values of the terms in the inverse of the [A] matrix are

$$a_{11} = 46,154 / (46,154 \times 46,154 - 13,846^2) = 23.8 \times 10^{-6}$$

Similarly,

$$\begin{aligned} a_{22} &= 23.8 \times 10^{-6} \\ a_{33} &= 61.9 \times 10^{-6} \\ a_{12} &= -7.1 \times 10^{-6} \end{aligned}$$

From Eq. (26.56), the equivalent Young's modulus is given by

$$E_x = \frac{1}{4 \times 0.15 \times 23.8 \times 10^{-6}} = 70,000 \text{ N/mm}^2$$

and from Eq. (26.57), Poisson's ratio is

$$v_{xy} = \frac{-(-7.1 \times 10^{-6})}{23.8 \times 10^{-6}} = 0.3$$

Lastly, from Eq. (26.62),

$$G_{xy} = \frac{1}{4 \times 0.15 \times 61.9 \times 10^{-6}} = 26,923 \text{ N/mm}^2$$

The shear coupling coefficient,  $m_x$  is, from Eq. (26.58), zero.

Note that the equivalent elastic constants are identical in value to the given elastic constants for the ply. This could have been deduced initially, since, for an isotropic ply, the elastic constants are independent of laminate thickness. However, the example serves to illustrate the method.

**Example 26.9**

If the laminate in [Example 26.8](#) comprises plies of the same thickness but which are orthotropic, not isotropic, determine the equivalent elastic constants. The ply angle  $\theta=0^\circ$ , and the ply elastic constants are  $E_l=140,000 \text{ N/mm}^2$ ,  $E_t=10,000 \text{ N/mm}^2$ ,  $G_{lt}=5000 \text{ N/mm}^2$  and  $v_{lt}=0.3$ .

Initially we calculate the minor Poisson's ratio  $v_{tl}$ . From Eq. (26.14),

$$v_{tl} = \frac{E_t}{E_l} v_{lt} = \frac{10,000}{140,000} \times 0.3 = 0.021$$

Again, for a symmetric orthotropic ply, the reduced stiffness terms,  $k_{13}$  and  $k_{23}$ , are zero [Ref. 2]. Then, from Eq. (26.17),

$$k_{11} = E_l/(1 - v_{lt}v_{tl}) = 140,000/(1 - 0.3 \times 0.021) = 140,888 \text{ N/mm}^2$$

Similarly,  $k_{12}=2958 \text{ N/mm}^2$ ,  $k_{22}=10,060 \text{ N/mm}^2$ , and  $k_{33}=5000 \text{ N/mm}^2$ .

The ply angle  $\theta=0^\circ$  so that in Eq. (26.31),  $m=\cos\theta=1$  and  $\sin\theta=0$ . Then

$$\bar{k}_{11} = k_{11}, \quad \bar{k}_{12} = k_{12}, \quad \bar{k}_{13} = 0, \quad \bar{k}_{22} = k_{22}, \quad \bar{k}_{23} = 0, \quad \bar{k}_{33} = k_{33}$$

There are four identical plies in the laminate each 0.15 mm thick. Therefore, from Eq. (26.45) etc.,

$$A_{11} = 4 \times 0.15 \times 140,888 = 84,533 \text{ N/mm}$$

$$A_{12} = 4 \times 0.15 \times 2958 = 1775 \text{ N/mm}$$

$$A_{22} = 4 \times 0.15 \times 10,060 = 6036 \text{ N/mm}$$

$$A_{33} = 4 \times 0.15 \times 5000 = 3000 \text{ N/mm}$$

The terms in the inverse of the  $[A]$  matrix are, then, from Eqs. (26.55)

$$a_{11} = 6036/(84,533 \times 6036 - 1775^2) = 1.19 \times 10^{-5}$$

Similarly,  $a_{22}=16.65 \times 10^{-5}$ ,  $a_{33}=33.33 \times 10^{-5}$ ,  $a_{12}=-0.35 \times 10^{-5}$

From Eq. (26.56),

$$E_z = \frac{1}{4 \times 0.15 \times 1.19 \times 10^{-5}} = 140,000 \text{ N/mm}^2$$

From Eq. (26.59),

$$E_y = \frac{1}{4 \times 0.15 \times 16.65 \times 10^{-5}} = 10,000 \text{ N/mm}^2$$

From Eq. (26.57),

$$v_{xy} = \frac{-(-0.35 \times 10^{-5})}{1.19 \times 10^{-5}} = 0.3$$

From Eq. (26.60),

$$v_{yx} = \frac{0.35 \times 10^{-5}}{16.65 \times 10^{-5}} = 0.021$$

Also, from Eqs. (26.58) and (26.61),  $m_x = m_y = 0$ .

Again, as in Example 26.8, the equivalent elastic constants for the laminate are the same as the elastic constants of the plies. This is to be expected, since the four plies are identical and orientated in the same way.

### Example 26.10

If the ply angle in the laminate of Example 26.9 is  $45^\circ$ , calculate the equivalent elastic constants.

The minor Poisson's ratio has the same value, 0.021, as in Example 26.9. Also the reduced stiffness terms are the same as in Example 26.9, i.e.,  $k_{13} = k_{23} = 0$ ,  $k_{11} = 140,888 \text{ N/mm}^2$ ,  $k_{12} = 2958 \text{ N/mm}^2$ ,  $k_{22} = 10,060 \text{ N/mm}^2$ , and  $k_{33} = 5000 \text{ N/mm}^2$ .

Now, however,  $\theta = 45^\circ$  so that  $m = \cos 45^\circ = 1/\sqrt{2}$  and  $n = \sin 45^\circ = 1/\sqrt{2}$ . Then, in Eq. (26.31),

$$\bar{k}_{11} = m^4 k_{11} + m^2 n^2 (2k_{12} + 4k_{33}) + n^4 k_{22} \quad (\text{i})$$

Substituting the above values of  $k_{11}$  etc. in Eq. (i) gives

$$\bar{k}_{11} = 44,216 \text{ N/mm}^2$$

Similarly,

$$\begin{aligned}\bar{k}_{12} &= 34,216 \text{ N/mm}^2 \\ \bar{k}_{13} &= 32,707 \text{ N/mm}^2 \\ \bar{k}_{22} &= 44,216 \text{ N/mm}^2 \\ \bar{k}_{23} &= 32,707 \text{ N/mm}^2 \\ \bar{k}_{33} &= 36,258 \text{ N/mm}^2\end{aligned}$$

Then, from Eq. (26.45),

$$A_{11} = 4 \times 0.15 \times 44,216 = 25,296 \text{ N/mm}$$

Similarly,

$$\begin{aligned}A_{12} &= 20,530 \text{ N/mm} \\ A_{13} &= 19,624 \text{ N/mm} \\ A_{22} &= 25,296 \text{ N/mm} \\ A_{23} &= 19,624 \text{ N/mm} \\ A_{33} &= 21,755 \text{ N/mm}\end{aligned}$$

From Eq. (26.53)

$$\begin{aligned}AA &= 25,296 \times 25,296 \times 21,755 + 2 \times 20,530 \times 19,624 \times 19,624 - 25,296 \times 19,624^2 - 21,755 \\ &\quad \times 20,530^2 - 25,296 \times 19,624^2\end{aligned}$$

$$\text{i.e., } AA = 1.08 \times 10^{12}$$

Then, from Eqs. (26.52),

$$a_{11} = (25,296 \times 21,755 - 19,624^2) / 1.08 \times 10^{12} = 1.53 \times 10^{-4}$$

Similarly,

$$\begin{aligned}a_{22} &= 1.53 \times 10^{-4} \\a_{33} &= 2.02 \times 10^{-4} \\a_{12} &= -0.57 \times 10^{-4} \\a_{13} &= -0.87 \times 10^{-4} \\a_{23} &= -0.87 \times 10^{-4}\end{aligned}$$

From Eq. (26.56),

$$E_x = \frac{1}{4 \times 0.15 \times 1.53 \times 10^{-4}} = 10,893 \text{ N/mm}^2$$

From Eq. (26.59),

$$E_y = \frac{1}{4 \times 0.15 \times 1.53 \times 10^{-4}} = 10,893 \text{ N/mm}^2$$

From Eq. (26.62),

$$G_{xy} = \frac{1}{4 \times 0.15 \times 2.02 \times 10^{-4}} = 8251 \text{ N/mm}^2$$

From Eq. (26.57),

$$v_{xy} = \frac{-(-0.57 \times 10^{-4})}{1.53 \times 10^{-4}} = 0.37$$

From Eq. (26.60),

$$v_{yx} = \frac{-(-0.57 \times 10^{-4})}{1.53 \times 10^{-4}} = 0.37$$

From Eq. (26.58),

$$m_x = \frac{-(-0.87 \times 10^{-4})}{1.53 \times 10^{-4}} = 0.57$$

From Eq. (26.61),

$$m_y = \frac{-(-0.87 \times 10^{-4})}{1.53 \times 10^{-4}} = 0.57$$

Since the shear coupling coefficients have values, it follows that, for this laminate, direct loads will produce shear strains and shear loads will produce direct strains. However, in many laminate configurations the ply angles are chosen such that the shear coupling effect is eliminated.

The method of determining the equivalent elastic constants of a laminate illustrated in Examples 26.8–26.10 is applicable to all symmetric laminates. Clearly, however, considerable simplification occurs when the plies are identical in terms of material and ply angles. This, obviously, is not always the case.

**Example 26.11**

The symmetric four-ply laminate whose cross-section is shown in Fig. 26.14 comprises plies of the same thickness but of two different materials with ply angles of  $30^\circ$  and  $45^\circ$ . For plies 1 and 4,  $E_l=200,000 \text{ N/mm}^2$ ,  $E_t=15,000 \text{ N/mm}^2$ ,  $G_{lt}=10,000 \text{ N/mm}^2$ , and  $v_{tl}=0.3$ . For plies 2 and 3,  $E_l=140,000 \text{ N/mm}^2$ ,  $E_t=10,000 \text{ N/mm}^2$ ,  $G_{lt}=5000 \text{ N/mm}^2$ , and  $v_{tl}=0.3$ . Calculate the equivalent elastic constants for the laminate.

Consider plies 1 and 4. The minor Poisson's ratio is found using Eq. (26.14), i.e.,

$$v_{tl} = \frac{15,000}{200,000} \times 0.3 = 0.023$$

As in Example 26.9, the reduced stiffness terms  $k_{13}$  and  $k_{23}$  are zero. Then, from Eq. (26.17),

$$k_{11} = 200,000 / (1 - 0.3 \times 0.023) = 201,410 \text{ N/mm}^2$$

Similarly,

$$\begin{aligned} k_{12} &= 4632 \text{ N/mm}^2 \\ k_{22} &= 15,106 \text{ N/mm}^2 \\ k_{33} &= 10,000 \text{ N/mm}^2 \end{aligned}$$

For plies 1 and 4,  $\theta=30^\circ$  so that  $m=\cos 30^\circ=0.866$  and  $n=\sin 30^\circ=0.5$ . Then, from Eq. (26.31) (see Eq. (i) of Example 26.10),

$$\begin{aligned} \bar{k}_{11} &= 123,583 \text{ N/mm}^2 \\ \bar{k}_{12} &= 35,908 \text{ N/mm}^2 \\ \bar{k}_{13} &= 58,436 \text{ N/mm}^2 \\ \bar{k}_{22} &= 30,431 \text{ N/mm}^2 \\ \bar{k}_{23} &= 22,232 \text{ N/mm}^2 \\ \bar{k}_{33} &= 41,266 \text{ N/mm}^2 \end{aligned}$$

The above procedure is repeated for plies 2 and 3 ( $\theta=45^\circ$ ) and gives the following results.

$$\begin{aligned} v_{tl} &= 0.021 \\ k_{11} &= 140,888 \text{ N/mm}^2 \\ k_{12} &= 2958 \text{ N/mm}^2 \\ k_{22} &= 10,060 \text{ N/mm}^2 \\ k_{33} &= 5000 \text{ N/mm}^2 \end{aligned}$$

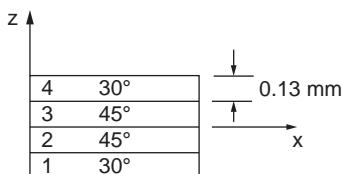


FIGURE 26.14 Laminate of Example 26.11

Then,

$$\begin{aligned}\bar{k}_{11} &= 44,216 \text{ N/mm}^2 \\ \bar{k}_{12} &= 34,216 \text{ N/mm}^2 \\ \bar{k}_{13} &= 32,707 \text{ N/mm}^2 \\ \bar{k}_{22} &= 44,216 \text{ N/mm}^2 \\ \bar{k}_{23} &= 32,707 \text{ N/mm}^2 \\ \bar{k}_{33} &= 36,258 \text{ N/mm}^2\end{aligned}$$

The above results are now combined to produce the  $A$  terms in the  $[A]$  matrix for the complete laminate.

From Eq. (26.45),

$$\begin{aligned}A_{11} &= 2 \times 0.13(123,583 + 44,216) = 43,628 \text{ N/mm} \\ A_{12} &= 2 \times 0.13(35,908 + 34,216) = 18,232 \text{ N/mm} \\ A_{13} &= 2 \times 0.13(58,436 + 32,707) = 23,697 \text{ N/mm} \\ A_{22} &= 2 \times 0.13(30,431 + 44,216) = 19,408 \text{ N/mm} \\ A_{23} &= 2 \times 0.13(22,232 + 32,707) = 14,284 \text{ N/mm} \\ A_{33} &= 2 \times 0.13(41,266 + 36,258) = 20,156 \text{ N/mm}\end{aligned}$$

Then, from Eq. (26.53),

$$\begin{aligned}AA &= 43,628 \times 19,408 \times 20,156 + 2 \times 18,232 \times 14,284 \times 23,697 - 19,408 \times 23,697^2 - 20,156 \\ &\quad \times 18,232^2 - 43,628 \times 14,284^2 \\ \text{i.e., } AA &= 2.91 \times 10^{12}\end{aligned}$$

Substituting the above in Eqs. (26.52) gives

$$\begin{aligned}a_{11} &= 6.43 \times 10^{-5} \\ a_{22} &= 10.92 \times 10^{-5} \\ a_{33} &= 17.67 \times 10^{-5} \\ a_{12} &= -0.10 \times 10^{-5} \\ a_{13} &= -6.85 \times 10^{-5} \\ a_{23} &= -6.57 \times 10^{-5}\end{aligned}$$

Now, from Eq. (26.56),

$$E_x = \frac{1}{4 \times 0.13 \times 6.43 \times 10^{-5}} = 29,908 \text{ N/mm}^2$$

From Eq. (26.59),

$$E_y = \frac{1}{4 \times 0.13 \times 10.92 \times 10^{-5}} = 17,611 \text{ N/mm}^2$$

From Eq. (26.62),

$$G_{xy} = \frac{1}{4 \times 0.13 \times 17.67 \times 10^{-5}} = 10,883 \text{ N/mm}^2$$

From Eq. (26.57),

$$v_{xy} = \frac{-(-0.10 \times 10^{-5})}{6.43 \times 10^{-5}} = 0.016$$

From Eq. (26.60),

$$v_{yx} = \frac{-(-0.10 \times 10^{-5})}{10.92 \times 10^{-5}} = 0.009$$

From Eq. (26.58),

$$m_x = \frac{-(-6.85 \times 10^{-5})}{6.43 \times 10^{-5}} = 1.07$$

From Eq. (26.61),

$$m_y = \frac{-(6.85 \times 10^{-5})}{10.92 \times 10^{-5}} = 0.64$$

### 26.3.4 Laminate strengths

Before the strength of a laminate can be evaluated, the individual ply stresses must be calculated. We have noted that although all plies in a laminate suffer the same values of strain, the stresses can vary from ply to ply depending on their orientation and material properties. Having calculated the stresses for each ply, the strength of the laminate can be investigated using any of the six commonly used failure criteria. Of these the simplest to apply is the *maximum stress theory*, which states:

Ply failure will occur if any stress value in the material axes direction exceeds their respective ultimate strengths:

$$\text{for tensile stresses, } \sigma_l/X_T \leq 1 \\ \sigma_l/Y_T \leq 1$$

$$\text{for compressive stresses, } |\sigma_l/X_C| \leq 1 \\ |\sigma_l/Y_C| \leq 1$$

$$\text{for shear stresses, } |\tau_l/S| \leq 1$$

where

$X_T$  is the ultimate tensile strength in the longitudinal direction

$X_C$  is the ultimate compressive strength in the longitudinal direction

$Y_T$  is the ultimate tensile strength in the transverse direction

$Y_C$  is the ultimate compressive strength in the transverse direction

$S$  is the ultimate in-plane shear strength.

The procedure is best illustrated by an example.

#### Example 26.12

If the laminate in Example 26.11 is subjected to a single load intensity  $N_x=100 \text{ N/mm}$ , calculate the ply stresses and investigate the strength of the laminate given that  $X_T=1500 \text{ N/mm}^2$ ,  $X_C=1200 \text{ N/mm}^2$ ,  $Y_T=50 \text{ N/mm}^2$ ,  $Y_C=250 \text{ N/mm}^2$ , and  $S=70 \text{ N/mm}^2$ .

The ply stresses are linked to the laminate strains through Eq. (26.38), i.e.,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \bar{k}_{12} & \bar{k}_{22} & \bar{k}_{23} \\ \bar{k}_{13} & \bar{k}_{23} & \bar{k}_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (\text{i})$$

Note that the ply and laminate strains are identical. From Eq. (26.49) we obtain the ply strains in terms of the applied loading:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [A^{-1}] \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \quad (\text{ii})$$

Then, from Eqs. (i) and (ii),

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \bar{k}_{12} & \bar{k}_{22} & \bar{k}_{23} \\ \bar{k}_{13} & \bar{k}_{23} & \bar{k}_{33} \end{bmatrix} [A^{-1}] \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \quad (\text{iii})$$

or, from Eq. (26.51),

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \bar{k}_{12} & \bar{k}_{22} & \bar{k}_{23} \\ \bar{k}_{13} & \bar{k}_{23} & \bar{k}_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \quad (\text{iv})$$

In Eq. (iv) the  $[\bar{k}_{ij}]$  matrix relates to a particular ply while the  $[a_{ij}]$  matrix relates to the laminate. The terms in the former matrix have been calculated for the different plies in Example 26.11, as have the  $a_{ij}$  terms for the laminate. Also  $N_x = 100 \text{ N/mm}$ ,  $N_y = N_{xy} = 0$ . Substituting the relevant values in Eq. (iv), we have, for plies 1 and 4,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 123,583 & 35,908 & 58,436 \\ 35,908 & 30,431 & 22,232 \\ 58,436 & 22,232 & 41,266 \end{bmatrix} \begin{bmatrix} 6.43 & -0.10 & -6.85 \\ -0.10 & 10.92 & -6.5 \\ -6.85 & -6.5 & 17.67 \end{bmatrix} \begin{Bmatrix} 100 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-5} \quad (\text{v})$$

which gives

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 123,583 & 35,908 & 58,436 \\ 35,908 & 30,431 & 22,232 \\ 58,436 & 22,232 & 41,266 \end{bmatrix} \begin{bmatrix} 643 \\ -10 \\ -685 \end{bmatrix} \times 10^{-5} \quad (\text{vi})$$

Then,

$$\begin{aligned} \sigma_{x,1,4} &= 390.8 \text{ N/mm}^2 \\ \sigma_{y,1,4} &= 75.6 \text{ N/mm}^2 \\ \tau_{xy,1,4} &= 90.8 \text{ N/mm}^2 \end{aligned}$$

The ply stresses can now be obtained using Eq. (26.24) in which  $m = \cos 30^\circ = 0.866$  and  $n = \sin 30^\circ = 0.5$ . Then

$$\begin{Bmatrix} \sigma_l \\ \sigma_t \\ \tau_{lt} \end{Bmatrix} = \begin{bmatrix} 0.75 & 0.25 & 0.866 \\ 0.25 & 0.75 & -0.866 \\ -0.433 & 0.433 & 0.5 \end{bmatrix} \begin{Bmatrix} 390.8 \\ 75.6 \\ 90.8 \end{Bmatrix}$$

which gives

$$\sigma_{l,1,4} = 390.6 \text{ N/mm}^2$$

$$\sigma_{t,1,4} = 75.8 \text{ N/mm}^2$$

$$\tau_{lt,1,4} = -91.1 \text{ N/mm}^2$$

For plies 2 and 3, the  $a_{ij}$  terms are identical to those in Eq. (v). Then

$$\begin{Bmatrix} \sigma_l \\ \sigma_t \\ \tau_{lt} \end{Bmatrix} = \begin{bmatrix} 44,216 & 34,216 & 32,707 \\ 34,216 & 44,216 & 32,707 \\ 32,707 & 32,707 & 36,258 \end{bmatrix} \begin{Bmatrix} 643 \\ -10 \\ -685 \end{Bmatrix} \times 10^{-5}$$

which gives

$$\sigma_{x,2,3} = 56.8 \text{ N/mm}^2$$

$$\sigma_{y,2,3} = -8.5 \text{ N/mm}^2$$

$$\tau_{xy,2,3} = -41.3 \text{ N/mm}^2$$

Again, from Eq. (26.24) in which  $m = \cos 45^\circ = 1/\sqrt{2}$  and  $n = \sin 45^\circ = 1/\sqrt{2}$ ,

$$\begin{Bmatrix} \sigma_l \\ \sigma_t \\ \tau_{lt} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & -1 \\ -0.5 & 0.5 & 0 \end{bmatrix} \begin{Bmatrix} 56.8 \\ -8.5 \\ -41.3 \end{Bmatrix}$$

which gives

$$\sigma_{l,2,3} = -17.2 \text{ N/mm}^2$$

$$\sigma_{t,2,3} = 65.4 \text{ N/mm}^2$$

$$\tau_{lt,2,3} = -32.7 \text{ N/mm}^2$$

The distribution of ply stresses is shown in Fig. 26.15.

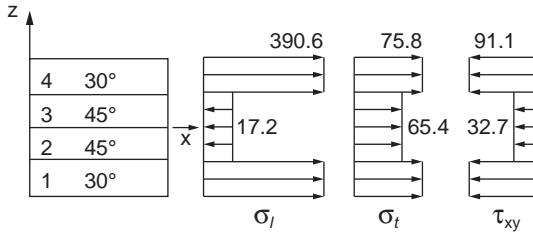


FIGURE 26.15 Distribution of Ply Stresses ( $\text{N/mm}^2$ ) in Laminate of Example 26.12

Having obtained the individual ply stresses, we can now investigate the strength of the laminate. For plies 1 and 4,

$$\sigma_l/1500 = 390.6/1500 = 0.26$$

$$\sigma_t/50 = 75.8/50 = 1.52$$

$$\tau_{lt}/70 = 91.1/70 = 1.30$$

For plies 2 and 3,

$$\sigma_t/1500 = 17.2/1500 = 0.01$$

$$\sigma_t/50 = 65.4/50 = 1.37$$

$$\tau_h/70 = 32.7/70 = 0.45$$

Clearly there are direct stress failures in the transverse direction in all four plies while there is a shear stress failure in plies 1 and 4.

## 26.4 THIN-WALLED COMPOSITE BEAMS

We noted, in [Chapter 11](#), that some structural components in many modern aircraft are fabricated from composite materials. These components are generally in the form of laminates, which are stacks of plies bonded together. The orientation of each ply may be different to that of its immediate neighbors so that the required strength and stiffness in a particular direction is obtained. In Section 26.3 we determined the equivalent elastic constants of a laminate; these will now be used in the analysis of composite structural members.

In [Chapters 16 through 18](#), we determined stresses and displacements in open and closed section thin-walled beams subjected to bending, shear, and torsional loads; the effect of axial load was considered in [Chapter 1](#). We now reexamine these cases to determine the effect of composite construction.

[Figure 26.16](#) shows a thin-walled beam that may be of either open or closed section and is fabricated from laminates ①, ②, ③, . . . The dimensions of each laminate are different, as are their elastic properties.

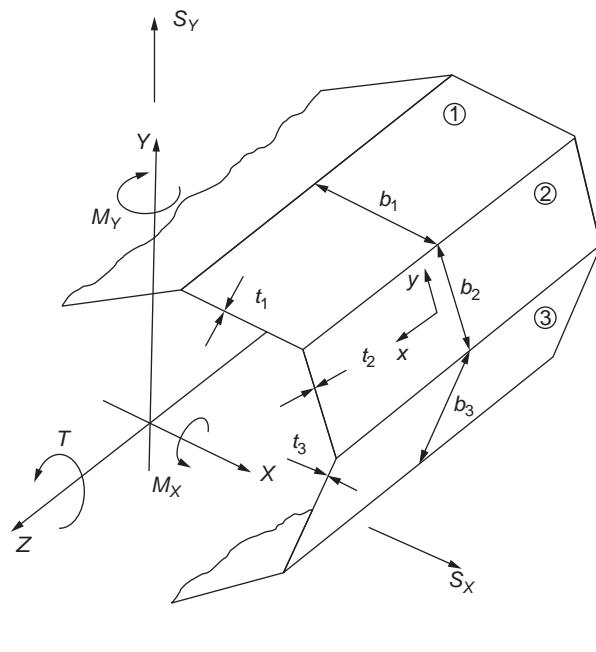


FIGURE 26.16 Composite Thin-Walled Section

The beam is subjected to axial, bending, shear, and torsional loads that are positive in the directions shown (see also Fig 16.9). The beam axes  $XYZ$  are now in uppercase letters to avoid confusion with the laminate axes  $xy$ .

### 26.4.1 Axial load

Suppose that the portion of the axial load  $P$  taken by the  $i$ th laminate is  $P_i$ . The longitudinal strain  $\varepsilon_{x,i}$  in the laminate is equal to the longitudinal strain  $\varepsilon_z$  in the beam, since one of the basic assumptions of our analysis, except in the case of torsion, is that plane sections remain plane after the load is applied. Then, from Eq. (1.40),

$$\frac{P_i}{b_i t_i} = \varepsilon_{x,i} E_{x,i}$$

Therefore,

$$P_i = b_i t_i \varepsilon_{x,i} E_{x,i}$$

that is,

$$P_i = \varepsilon_z b_i t_i E_{x,i} \quad (26.63)$$

The total axial load on the beam is then given by

$$P = \varepsilon_z \sum_{i=1}^n b_i t_i E_{x,i} \quad (26.64)$$

Note that, in Eq. (26.64),  $\varepsilon_z$  is the longitudinal strain in the beam section and is therefore the same for every laminate; it may therefore be taken outside the summation. Further, the value of Young's modulus for a particular laminate is the same whether referred to the laminate  $x$  axis or the beam  $Z$  axis; we therefore refer it to the beam  $Z$  axis. Equation (26.64) may therefore be written

$$P = \varepsilon_z \sum_{i=1}^n b_i t_i E_{Z,i} \quad (26.65)$$

from which

$$\varepsilon_z = \frac{P}{\sum_{i=1}^n b_i t_i E_{Z,i}} \quad (26.66)$$

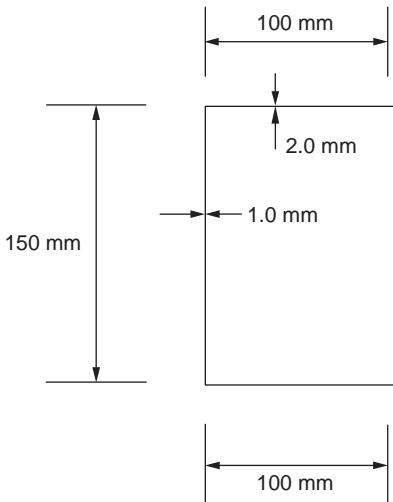
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#### Example 26.13

A beam has the singly symmetrical composite section shown in Fig. 26.17. The flange laminates are identical and have a Young's modulus,  $E_Z$ , of 60,000 N/mm<sup>2</sup> while the vertical web has a Young's modulus,  $E_Z$ , of 20,000 N/mm<sup>2</sup>. If the beam is subjected to an axial load of 40 kN, determine the axial load in each laminate.

For each flange,

$$b_i t_i E_{Z,i} = 100 \times 2.0 \times 60,000 = 12 \times 10^6$$



**FIGURE 26.17 Beam Section of Example 26.13**

and, for the web,

$$b_i t_i E_{Z,i} = 150 \times 1.0 \times 20,000 = 3 \times 10^6$$

Therefore,

$$\sum_{i=1}^n b_i t_i E_{Z,i} = 2 \times 12 \times 10^6 + 3 \times 10^6 = 27 \times 10^6$$

Then, from Eq. (26.66),

$$\varepsilon_Z = \frac{40 \times 10^3}{27 \times 10^6} = 1.48 \times 10^{-3}$$

Therefore, from Eq. (26.63),

$$\begin{aligned} P(\text{flanges}) &= 1.48 \times 10^{-3} \times 12 \times 10^6 = 17,760 \text{ N} = 17.76 \text{ kN} \\ P(\text{web}) &= 1.48 \times 10^{-3} \times 3 \times 10^6 = 4440 \text{ N} = 4.44 \text{ kN} \end{aligned}$$

Note that  $2 \times 17.76 + 4.44 = 39.96$  kN: the discrepancy, 0.04 kN, is due to rounding-off errors.

### Example 26.14

In the box section beam shown in Fig. 26.18 the covers 1 and 3 are identical laminates, each comprising four orthotropic plies, 0.15 mm thick, and for which the ply angle is  $45^\circ$ . The ply elastic constants are  $E_l = 140,000 \text{ N/mm}^2$ ,  $E_t = 10,000 \text{ N/mm}^2$ ,  $G_{lt} = 5000 \text{ N/mm}^2$ , and  $\nu_{lt} = 0.3$ . The webs 2 and 4 are also identical laminates each comprising four plies 0.13 mm thick but in which the outer ply angle is  $30^\circ$  while the angle of the two inner plies is  $45^\circ$ . For the outer plies  $E_l = 200,000 \text{ N/mm}^2$ ,  $E_t = 15,000 \text{ N/mm}^2$ ,  $G_{lt} = 10,000 \text{ N/mm}^2$ , and  $\nu_{lt} = 0.3$ , and for the two inner plies

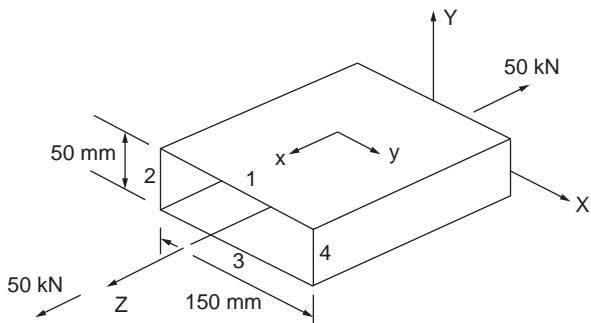


FIGURE 26.18 Beam of Example 26.14

$E_l = 140,000 \text{ N/mm}^2$ ,  $E_t = 10,000 \text{ N/mm}^2$ ,  $G_{lt} = 5000 \text{ N/mm}^2$ , and  $v_{lt} = 0.3$ . If the beam is subjected to an axial tensile load of 50 kN, determine the share of the load carried by the covers and the webs.

The equivalent elastic constants for the covers have been calculated in Ex. 26.10 and are

$$E_Z = E_x = 10,893 \text{ N/mm}^2$$

For the webs, from Example 26.11,

$$E_Z = E_x = 29,908 \text{ N/mm}^2$$

Then, for each cover,

$$b_i t_i E_{Z,i} = 150 \times 4 \times 0.15 \times 10,893 = 9.8 \times 10^5 \text{ N}$$

For each web,

$$b_i t_i E_{Z,i} = 50 \times 4 \times 0.13 \times 29,908 = 7.8 \times 10^5 \text{ N}$$

Then

$$\sum_{i=1}^n b_i t_i E_{Z,i} = 2 \times 9.8 \times 10^5 + 2 \times 7.8 \times 10^5 = 35.2 \times 10^5 \text{ N}$$

From Eq. (26.66),

$$\varepsilon_Z = \frac{50 \times 10^3}{35.2 \times 10^5} = 1.42 \times 10^{-2}$$

Therefore, from Eq. (26.65),

$$P(\text{covers}) = 1.42 \times 10^{-2} \times 9.8 \times 10^5 = 13,916 \text{ N}$$

$$P(\text{webs}) = 1.42 \times 10^{-2} \times 7.8 \times 10^5 = 11,076 \text{ N}$$

Check: Total load =  $2(13,916 + 11,076) = 49,984 \text{ N}$ , which is approximately 50 kN, the discrepancy being due to rounding-off errors.

Having obtained the axial force in each cover and web, the axial force intensities can be calculated. The strength of the box beam can then be investigated as illustrated in Example 26.12.

### 26.4.2 Bending

In Section 16.2, we derived an expression for the direct stress distribution in a beam of unsymmetrical cross-section (Eq. (16.17) or (16.18)). In this derivation, the direct stress on an element of the beam cross-section was expressed in terms of Young's modulus, the radius of curvature of the beam, the coordinates of the element, and the inclination of the neutral axis to the section  $x$  axis (see Eq. (16.15)). The beam was assumed to be composed of homogenous material so that Young's modulus is a constant. This, as we have seen, is not necessarily the case for a composite beam where  $E$  can vary from laminate to laminate. We therefore rewrite Eqs. (16.16) in the form

$$M_x = \int_A \frac{E_{Z,i}}{\rho} (x \sin \alpha + y \cos \alpha) y \, dA, \quad M_y = \int_A \frac{E_{Z,i}}{\rho} (x \sin \alpha + y \cos \alpha) x \, dA,$$

or

$$\begin{aligned} M_x &= \frac{\sin \alpha}{\rho} \int_A E_{Z,i} xy \, dA + \frac{\cos \alpha}{\rho} \int_A E_{Z,i} y^2 \, dA, \\ M_y &= \frac{\sin \alpha}{\rho} \int_A E_{Z,i} x^2 \, dA + \frac{\cos \alpha}{\rho} \int_A E_{Z,i} xy \, dA. \end{aligned}$$

We therefore define modified second moments of area that include the laminate value of Young's modulus,  $E_{Z,i}$ , and that are referred to the  $XYZ$  axes of Fig. 26.16. Then,

$$I'_{XX} = \int_A E_{Z,i} Y^2 \, dA, \quad I'_{YY} = \int_A E_{Z,i} X^2 \, dA, \quad I'_{XY} = \int_A E_{Z,i} XY \, dA \quad (26.67)$$

so that

$$\begin{aligned} M_X &= \frac{\sin \alpha}{\rho} I'_{XY} + \frac{\cos \alpha}{\rho} I'_{XX} \\ M_Y &= \frac{\sin \alpha}{\rho} I'_{YY} + \frac{\cos \alpha}{\rho} I'_{XY} \end{aligned}$$

Solving, we obtain

$$\begin{aligned} \frac{\sin \alpha}{\rho} &= \frac{M_Y I'_{XX} - M_X I'_{XY}}{I'_{XX} I'_{YY} - I'_{XY}^2} \\ \frac{\cos \alpha}{\rho} &= \frac{M_X I'_{YY} - M_Y I'_{XY}}{I'_{XX} I'_{YY} - I'_{XY}^2} \end{aligned}$$

Then, from Eq. (16.15),

$$\sigma_Z = E_{Z,i} \left[ \left( \frac{M_Y I'_{XX} - M_X I'_{XY}}{I'_{XX} I'_{YY} - I'_{XY}^2} \right) X + \left( \frac{M_X I'_{YY} - M_Y I'_{XY}}{I'_{XX} I'_{YY} - I'_{XY}^2} \right) Y \right] \quad (26.68)$$

Note that the preceding applies equally to open or closed section thin-walled beams.

**Example 26.15**

A thin-walled beam has the composite cross-section shown in Fig. 26.19 and is subjected to a bending moment of 1 kNm applied in a vertical plane. If the values of Young's modulus for the flange laminates are each 50,000 N/mm<sup>2</sup> and that of the web is 15,000 N/mm<sup>2</sup>, determine the maximum value of direct stress in the cross-section of the beam.

From Section 16.4.5 and Eqs. (26.67),

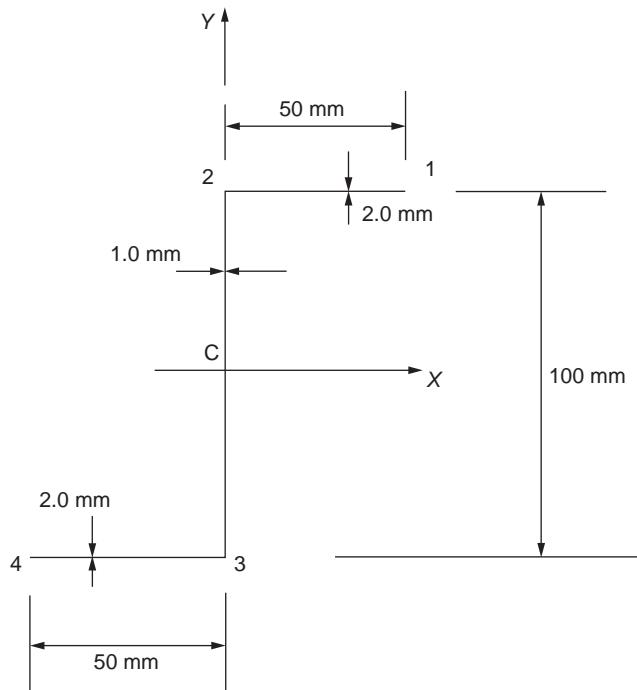
$$I'_{XX} = 2 \times 50,000 \times 50 \times 2.0 \times 50^2 + 15,000 \times 1.0 \times \frac{100^3}{12} = 2.63 \times 10^{10} \text{ Nmm}^2$$

$$I'_{YY} = 50,000 \times 2.0 \times \frac{100^3}{12} = 0.83 \times 10^{10} \text{ Nmm}^2$$

$$I'_{XY} = 50,000 \times 50 \times 2(+25)(+50) + 50,000 \times 50 \times 2(-25)(-50) = 1.25 \times 10^{10} \text{ mm}^4$$

Also, since  $M_x = 1 \text{ kNm}$  and  $M_y = 0$ , Eq. (26.68) becomes

$$\sigma_z = E_{z,i} \left[ \frac{-1 \times 10^6 \times 1.25 \times 10^{10}}{(2.63 \times 0.83 - 1.25^2) 10^{20}} X + \frac{1 \times 10^6 \times 0.83 \times 10^{10}}{(2.63 \times 0.83 - 1.25^2) 10^{20}} Y \right]$$



**FIGURE 26.19 Beam Section of Example 26.15**

which reduces to

$$\sigma_z = E_{z,i}(-2.02 \times 10^{-4}X + 1.34 \times 10^{-4}Y) \quad (\text{i})$$

On the flange 12,  $E_{z,i} = 50,000 \text{ N/mm}^2$  and  $Y = 50 \text{ mm}$ , so that Eq. (i) becomes

$$\sigma_z = -10.1X + 335$$

Then  $\sigma_{z,1} = -10.1 \times 50 + 335 = -170 \text{ N/mm}^2$

$$\sigma_{z,2} = +335 \text{ N/mm}^2$$

In the web 23,  $E_{z,i} = 15,000 \text{ N/mm}^2$  and  $X = 0$ . Eq. (i) then becomes

$$\sigma_z = 15,000 \times 1.34 \times 10^{-4}Y = 2.01Y$$

Then  $\sigma_{z,2} = +100.5 \text{ N/mm}^2$

The remaining distribution follows from symmetry so that the maximum direct stress in the beam cross-section is  $\pm 335 \text{ N/mm}^2$ .

### Example 26.16

If, in the box section beam of Example 26.14, the axial load is replaced by a bending moment  $M_X = 0.5 \text{ kNm}$ , calculate the distribution of direct stress in the covers and in the webs.

Again, as in Example 26.14,  $E_Z(\text{covers}) = 10,893 \text{ N/mm}^2$  and  $E_Z(\text{webs}) = 29,908 \text{ N/mm}^2$ . Then, since the cross-section is doubly symmetrical,  $I_{XY}' = 0$ , and since  $M_Y = 0$ , Eq. (26.68) reduces to

$$\sigma_Z = E_{Z,i} \frac{M_X}{I_{XX}'} Y \quad (\text{i})$$

where

$$I_{XX}' = 2 \times 10,893 \times 150 \times 0.6 \times 25^2 + 2 \times 29,908 \times 0.52 \times \frac{50^3}{12} = 1.55 \times 10^9 \text{ mm}^2$$

Eq. (i) then becomes

$$\sigma_Z = E_{Z,i} \frac{0.5 \times 10^6}{1.55 \times 10^9} Y = E_{Z,i} \times 0.32 \times 10^{-3} Y \quad (\text{ii})$$

For the covers  $E_{Z,i} = 10,893 \text{ N/mm}^2$  and  $Y = \pm 25 \text{ mm}$ . Therefore, from Eq. (ii),

$$\sigma_Z(\text{covers}) = 10,893 \times 0.32 \times 10^{-3} \times (\pm 25) = \pm 87.4 \text{ N/mm}^2$$

At the extremities of the webs  $Y = \pm 25 \text{ mm}$  and for each web  $E_{Z,i} = 29,908 \text{ N/mm}^2$ .

Then, from Eq. (ii),

$$\sigma_Z(\text{web, top}) = 29,908 \times 0.32 \times 10^{-3} \times (+25) = 239.3 \text{ N/mm}^2$$

$$\sigma_Z(\text{web, bottom}) = 29,908 \times 0.32 \times 10^{-3} \times (-25) = -239.3 \text{ N/mm}^2$$

### 26.4.3 Shear

#### *Open section beams*

In Section 17.2, we derived an expression for the shear flow distribution in an open section thin-walled beam subjected to shear loads (Eq. (17.14)). This is related to the direct stress distribution in the section (Eq. (17.2)), so that the arguments applied to composite section beams subjected to bending apply to the case of composite beams subjected to shear. Equation (17.14) then becomes

$$q_s = -E_{Z,i} \left[ \left( \frac{S_X I'_{XX} - S_Y I'_{XY}}{I'_{XX} I'_{YY} - I'^2_{XY}} \right) \int_0^s t_i X \, ds + \left( \frac{S_Y I'_{YY} - S_X I'_{XY}}{I'_{XX} I'_{YY} - I'^2_{XY}} \right) \int_0^s t_i Y \, ds \right] \quad (26.69)$$

Note that, in Eq. (26.69),  $s$  is measured from an open edge in the beam section and the second moments of area are those defined in Eq. (26.67).

#### Example 26.17

Determine the position of the shear center of the channel section beam of Example 26.13.

As in Example 17.2, the shear center lies on the horizontal axis of symmetry so that we only need to apply an arbitrary shear load,  $S_Y$ , through the shear center and calculate the shear flow distribution in, say, the bottom flange.

In this case,  $S_X = 0$ ,  $I'_{XY} = 0$ , and Eq. (26.40) reduces to

$$q_s = -E_{Z,i} (S_Y / I'_{XX}) \int_0^s t_i Y \, ds \quad (i)$$

Referring to Fig. 26.17 and Example 26.13,

$$I'_{XX} = 2 \times 60,000 \times 100 \times 2 \times 75^2 + 20,000 (1.0 \times 150^3 / 12) = 14.1 \times 10^{10} \text{ N/mm}^2$$

On the bottom flange  $Y = -75 \text{ mm}$  and  $t_i = 2.0 \text{ mm}$  so that Eq.(i) becomes

$$q_s = 60,000 (S_Y / 14.1 \times 10^{10}) \int_0^s 2 \times 75 \, ds$$

where  $s$  is measured from the free edge of the flange. Then

$$q_s = 64 \times 10^{-5} S_Y s \quad (ii)$$

Suppose that the shear center is a distance  $\xi_S$  to the left of the vertical web. Then, taking moments about the mid-point of the web,

$$S_Y \xi_S = 2S_Y \int_0^{100} 6.4 \times 10^{-5} \times 75s \, ds$$

which gives

$$\xi_S = 48.0 \text{ mm}$$

### Closed section beams

Again, the same arguments apply to the composite case as before, and Eq. (17.15) becomes

$$q_s = -E_{Z,i} \left[ \left( \frac{S_X I'_{XX} - S_Y I'_{XY}}{I'_{XX} I'_{YY} - I'^2_{XY}} \right) \int_0^s t_i X \, ds + \left( \frac{S_Y I'_{YY} - S_X I'_{XY}}{I'_{XX} I'_{YY} - I'^2_{XY}} \right) \int_0^s t_i Y \, ds \right] + q_{s,0} \quad (26.70)$$

In Eq. (26.70), the value of the shear flow,  $q_{s,0}$ , at the origin for  $s$  is found using either Eq. (17.17) or (17.18).

### Example 26.18

The composite triangular section thin-walled beam shown in Fig. 26.20 carries a vertical shear load of 2 kN applied at the apex. If the walls 12 and 13 have a laminate Young's modulus of 45,000 N/mm<sup>2</sup> while that of the vertical web 23 is 20,000 N/mm<sup>2</sup>, determine the shear flow distribution in the section.

The  $X$  axis is an axis of symmetry, so that  $I'_{XY}=0$ , and since  $S_X=0$ , Eq. (26.70) reduces to

$$q_s = -E_{Z,i} \frac{S_Y}{I'_{XX}} \int_0^s t Y \, ds + q_{s,0} \quad (i)$$

From Section 16.4.5 and Eq. (26.67),

$$\begin{aligned} I'_{XX} &= \frac{2 \times 45,000 \times 2.0 \times 250^3 (150/250)^2}{12} + \frac{20,000 \times 1.5 \times 300^3}{12} \\ &= 15.2 \times 10^{10} \text{ N mm}^2 \end{aligned}$$

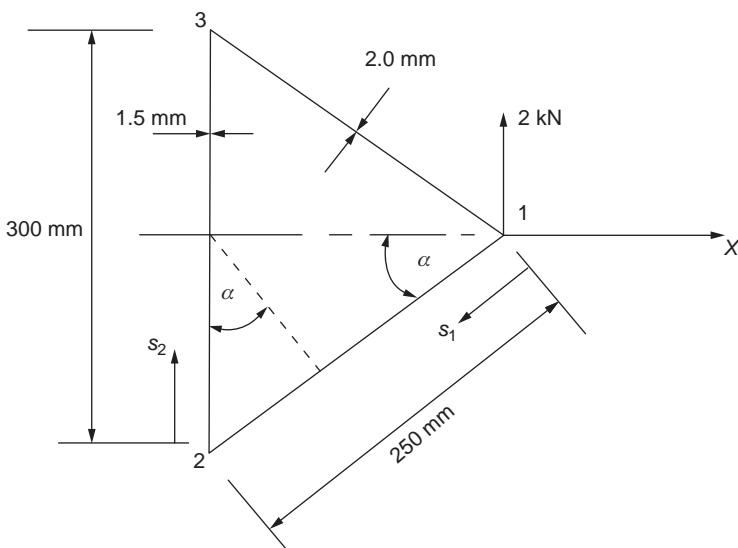


FIGURE 26.20 Beam Section of Example 26.18

“Cut” the section at 1. Then, from the first term on the right-hand side of Eq. (i),

$$q_{b,12} = -\frac{45,000 \times 2 \times 10^3}{15.2 \times 10^{10}} \int_0^{s_1} 2.0(-s_1 \sin \alpha) ds_1$$

in which  $\sin \alpha = 150/250 = 0.6$ . Therefore,

$$q_{b,12} = 3.6 \times 10^{-4} s_1^2 \quad (\text{ii})$$

so that

$$q_{b,2} = 22.2 \text{ N/mm}$$

Also,

$$q_{b,23} = -\frac{20,000 \times 2 \times 10^3}{15.2 \times 10^{10}} \int_0^{s_2} 1.5(-150 + s_2) ds_2 + 22.2$$

from which

$$q_{b,23} = 0.06s_2 - 1.95 \times 10^{-4}s_2^2 + 22.2 \quad (\text{iii})$$

Taking moments about the mid-point of the wall 23 (or about point 1), we have

$$2 \times 10^3 \times 250 \cos \alpha = -2 \int_0^{250} q_{b,12} 150 \cos \alpha ds_2 + 2 \times \frac{300}{2} \times (250 \cos \alpha) q_{s,0}$$

which gives

$$q_{s,0} = 14.2 \text{ N/mm} \text{ (in a counterclockwise sense)}$$

The shear flow distribution is then

$$\begin{aligned} q_{12} &= 3.6 \times 10^{-4} s_1^2 - 14.2 \\ q_{23} &= -1.95 \times 10^{-4} s_2^2 + 0.06s_2 + 8.0 \end{aligned}$$

■

#### 26.4.4 Torsion

##### **Closed section beams**

We consider composite closed section beams first, since, as we saw in Chapters 17 and 18, the strain-displacement relationships derived for open and closed section beams subjected to shear loads apply to the torsion of closed section beams, so that the analysis follows logically on.

The shear flow distribution in a closed section thin-walled beam subjected to a torque in which the warping is unrestrained is given by Eq. (18.1); that is,

$$T = 2Aq$$

or

$$q = \frac{T}{2A} \quad (26.71)$$

The derivation of Eq. (26.71) is based purely on equilibrium considerations and does not, therefore, rely on the elastic properties of the beam section. Equation (26.71) therefore applies equally to composite and isotropic beam sections.

The rate of twist of a closed section beam subjected to a torque is given by Eq. (18.4); that is,

$$\frac{d\theta}{dz} = \frac{T}{4A^2} \oint \frac{ds}{G}$$

This expression also applies to a composite closed section beam, provided that the shear modulus  $G$  remains within the integration and that the laminate shear modulus  $G_{XY,i}$  is used as appropriate. Equation (18.4) then becomes

$$\frac{d\theta}{dz} = \frac{T}{4A^2} \oint \frac{ds}{G_{XY,i} t_i} \quad (26.72)$$

Rearranging,

$$T = \frac{4A^2}{\oint \frac{ds}{G_{XY,i} t_i}} d\theta \quad (26.73)$$

We saw, in Chapter 3, Eq. (3.12), that the torque and rate of twist in a beam are related by the torsional stiffness  $GJ$ . Therefore, from Eq. (26.73), we see that the torsional stiffness of a composite closed section beam is given by

$$GJ = \frac{4A^2}{\oint \frac{ds}{G_{XY,i} t_i}} \quad (26.74)$$

These arguments apply to the determination of the warping distribution in a closed section composite beam. This is then given by (see the derivation of Eq. (18.5))

$$W_s - W_0 = q \int_0^s \frac{ds}{G_{XY,i} t_i} - \frac{A_{0s}}{A} q \oint \frac{ds}{G_{XY,i} t_i} \quad (26.75)$$

or, from Eq. (26.71) in terms of the applied torque,

$$W_s - W_0 = \frac{T}{2A} \left( \int_0^s \frac{ds}{G_{XY,i} t_i} - \frac{A_{0s}}{A} \oint \frac{ds}{G_{XY,i} t_i} \right) \quad (26.76)$$

### Example 26.19

The rectangular section, thin-walled, composite beam shown in Fig. 26.21 is subjected to a torque of 10 kNm. If the laminate shear modulus of the covers is 20,000 N/mm<sup>2</sup> and that of the webs is 35,000 N/mm<sup>2</sup>, determine the shear flow distribution in the section and the distribution of warping.

The shear flow distribution is obtained from Eq. (26.71) and is

$$q = \frac{10 \times 10^6}{2 \times 200 \times 100} = 250 \text{ N/mm}$$

The warping distribution is given by Eq. (26.76), in which

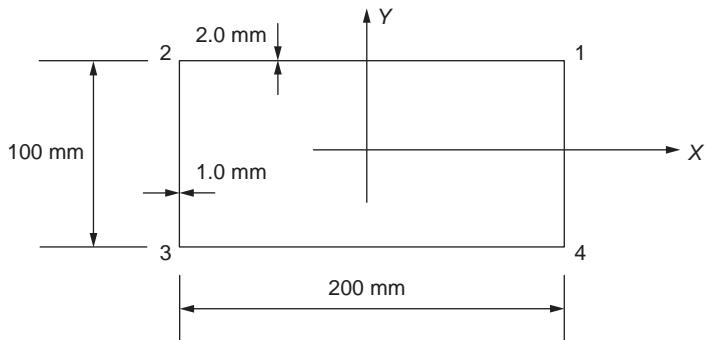


FIGURE 26.21 Beam Section of Example 26.19

$$\oint \frac{ds}{G_{XY,i} t_i} = \frac{2 \times 200}{20,000 \times 2.0} + \frac{2 \times 100}{35,000 \times 1.0} = 0.0157$$

Equation (26.76) then becomes

$$W_s - W_0 = 250 \left( \int_0^s \frac{ds}{G_{XY,i} t_i} - \frac{A_{0s}}{200 \times 100} \times 0.0157 \right)$$

or

$$W_s - W_0 = 250 \left( \int_0^s \frac{ds}{G_{XY,i} t_i} - 0.785 \times 10^{-6} A_{0s} \right) \quad (i)$$

We saw, in Example 18.2, that the warping distribution in a rectangular section thin-walled beam is linear with zero values at the mid-points of the webs and covers. The same situation applies in this example, so that it is necessary to calculate the value of warping only at, say, corner 1. Then, from Eq. (i),

$$W_1 = 250 \left( \frac{50}{35,000 \times 1.0} - 0.785 \times 10^{-6} \times 100 \times \frac{50}{2} \right)$$

which gives

$$W_1 = -0.13 \text{ mm}$$

The remaining distribution follows from symmetry. ■

### Open section beams

The torsional stiffness of an open section thin-walled beam is, as for a closed section beam,  $GJ$ , but in which the torsion constant,  $J$ , is given by either of Eqs. (18.11). However, for a composite beam section, the shear modulus must be taken inside the summation or integral and is the laminate shear modulus  $G_{XY,i}$ . Then,

$$GJ = \sum_{i=1}^n G_{XY,i} \frac{st_i^3}{3} \quad \text{or} \quad GJ = \frac{1}{3} \int_{\text{sect}} G_{XY,i} t_i^3 ds \quad (26.77)$$

The rate of twist of a beam is related to the applied torque by Eq. (3.12). For a composite open section beam, the relationship holds but the torsional stiffness is given by either of Eqs. (26.77); that is,

$$T = \left( \sum_{i=1}^n G_{XY,i} \frac{st_i^3}{3} \right) \frac{d\theta}{dZ} \quad \text{or} \quad T = \left( \frac{1}{3} \int_{\text{sect}} G_{XY,i} t_i^3 ds \right) \frac{d\theta}{dZ} \quad (26.78)$$

Having obtained the rate of twist, Eq. (18.9) gives the shear stress distribution across the thickness at any point round the beam section; that is,

$$\tau = 2G_{XY,i} n \frac{d\theta}{dZ} \quad (26.79)$$

Again, the maximum shear stress occurs at the surface of the beam section, where  $n = \pm t/2$ .

The primary warping distribution follows from Eq. (18.19), in which the rate of twist is found from either of Eqs. (26.78).

### Example 26.20

A composite channel section has the dimensions shown in Fig. 18.12 and is subjected to a torque of 10 Nm. If the flanges have a laminate shear modulus of 20,000 N/mm<sup>2</sup> and that of the web is 15,000 N/mm<sup>2</sup>, determine the maximum shear stress in the beam section and the distribution of warping, assuming that the beam is constrained to twist about an axis through the mid-point of the web.

The torsional stiffness of the section is obtained from the first of Eqs. (26.77) and is

$$GJ = 2 \times 20,000 \times 25 \times \frac{1.5^3}{3} + 15,000 \times 50 \times \frac{2.5^3}{3} = 5.03 \times 10^6 \text{ Nmm}^2$$

Then, from Eq. (26.78),

$$\frac{d\theta}{dZ} = \frac{10 \times 10^3}{5.03 \times 10^6} = 1.99 \times 10^{-3}$$

and, from Eq. (26.79),

$$\begin{aligned} \tau_{\max}(12) &= 2 \times 20,000 \times (1.5/2) \times 1.99 \times 10^{-3} = 59.7 \text{ N/mm}^2 \\ \tau_{\max}(23) &= 2 \times 15,000 \times (2.5/2) \times 1.99 \times 10^{-3} = 74.6 \text{ N/mm}^2 \end{aligned}$$

The maximum therefore occurs in the web and is 74.6 N/mm<sup>2</sup>.

The section is constrained to twist about an axis through the mid-point of the web, so that  $W$  is zero everywhere in the web. Then, from Eq. (18.19),

$$W_1 = -2 \times \frac{1}{2} \times 25 \times 25 \times 1.99 \times 10^{-3} = -1.24 \text{ mm}$$

The warping is linear along the flange 12; the warping along the flange 34 follows from symmetry.

Note that if the axis of twist is not specified, the position of the shear center of the section has to be found using the method previously described.

### 26.4.5 Idealization of composite beams

In Chapter 20, we saw that beam sections could be idealized into a combination of direct stress carrying booms and shear stress carrying panels provided that the ratio of the direct stresses at the extremities of the panel is known or could be assumed (Eqs. (20.1) and (20.2)). Having obtained the boom areas, the second moments of area, based on the boom areas, of the beam section could be calculated.

A difficulty arises, however, in idealizing a composite beam section, since the possibly different laminates will have differing values of Young's modulus. The modified second moments of area then contain the laminate value of Young's modulus as defined in Eqs. (26.67). However, it can be seen in Eqs. (26.67) that, in effect, the contribution to the second moments of area of a beam section would be *boom area*  $\times$   $E_{z,I}$   $\times$  *distance*<sup>2</sup>. Applying this argument to the beam section of [Example 26.15](#) and referring additionally to [Example 20.3](#), the “effective” boom areas are, for  $I'_{XX}$ ,

$$B'_1 = \frac{50 \times 2}{6} (2+1)50,000 = 2.5 \times 10^6 \quad (= B'_4)$$

$$B'_2 = 2.5 \times 10^6 + \frac{100 \times 1}{6} (2-1)15,000 = 2.75 \times 10^6$$

Then

$$I'_{XX} = 2 \times 2.5 \times 10^6 \times 50^2 + 2 \times 2.75 \times 10^6 \times 50^2 = 2.63 \times 10^{10} \text{ Nmm}^2$$

For  $I'_{YY}$ ,

$$B'_1 = \frac{100 \times 2}{6} (2-1)50,000 = 1.67 \times 10^6 \quad (= B'_4)$$

so that

$$I'_{YY} = 2 \times 1.67 \times 10^6 \times 50^2 = 0.83 \times 10^{10} \text{ Nmm}^2$$

Note that  $B'_2$  (and  $B'_3$ ) does not contribute to  $I'_{YY}$ .

Finally, since the actual loading is  $M_x$ ,

$$I'_{XY} = 2.5 \times 10^6 (+50)(+50) + 2.5 \times 10^6 (-50)(-50) = 1.25 \times 10^{10} \text{ Nmm}^2$$

which results are identical to those for the unidealized beam section in the solution of [Example 26.15](#).

## References

- [1] Calcote LR. The analysis of laminated composite structures. New York: Van Nostrand Reinhold; 1969.
- [2] Datoor MH. Mechanics of fibrous composites. London: Elsevier Applied Science; 1991.

## PROBLEMS

**P.26.1.** A bar, whose cross-section is shown in [Fig. P.26.1](#), is composed of a polyester matrix and Kevlar filaments; the respective moduli are 3000 and 140,000 N/mm<sup>2</sup> with corresponding Poisson's ratios of 0.16 and 0.28. If the bar is 1 m long and is subjected to a compressive axial load of 500 kN,

determine the shortening of the bar, the increase in its thickness, and the stresses in the polyester and Kevlar.

Answer: 3.26 mm, 0.032 mm, 9.78 N/mm<sup>2</sup>, 456.4 N/mm<sup>2</sup>

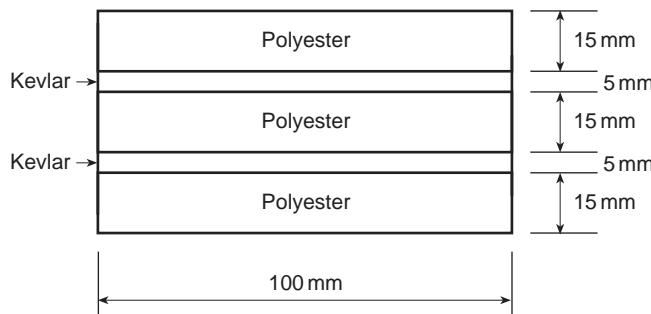


FIGURE P.26.1

**P.26.1. MATLAB** Use MATLAB to repeat Problem P.26.1 for compressive axial loads ( $P$ ) ranging from 300 to 700 kN in increments of 50 kN.

Answer:	$P$	$\Delta_l$	$\Delta_t$	$\sigma_m$ (polyester)	$\sigma_f$ (Kevlar)
(i)	300 kN	1.95 mm	0.019 mm	5.85 N/mm <sup>2</sup>	273 N/mm <sup>2</sup>
(ii)	350 kN	2.28 mm	0.023 mm	6.84 N/mm <sup>2</sup>	319.2 N/mm <sup>2</sup>
(iii)	400 kN	2.61 mm	0.026 mm	7.83 N/mm <sup>2</sup>	365.4 N/mm <sup>2</sup>
(iv)	450 kN	2.93 mm	0.029 mm	8.79 N/mm <sup>2</sup>	410.2 N/mm <sup>2</sup>
(v)	500 kN	3.26 mm	0.032 mm	9.78 N/mm <sup>2</sup>	456.4 N/mm <sup>2</sup>
(vi)	550 kN	3.58 mm	0.035 mm	10.74 N/mm <sup>2</sup>	501.2 N/mm <sup>2</sup>
(vii)	600 kN	3.91 mm	0.039 mm	11.73 N/mm <sup>2</sup>	547.4 N/mm <sup>2</sup>
(viii)	650 kN	4.23 mm	0.042 mm	12.69 N/mm <sup>2</sup>	592.2 N/mm <sup>2</sup>
(ix)	700 kN	4.56 mm	0.045 mm	13.68 N/mm <sup>2</sup>	638.4 N/mm <sup>2</sup>

**P.26.2.** If a thin isotropic ply has a Young's modulus of 60,000 N/mm<sup>2</sup> and a Poisson's ratio of 0.25, determine the terms in the reduced stiffness and compliance matrices.

Answer:  $k_{11} = k_{22} = 64,000 \text{ N/mm}^2$ ,  $k_{12} = 16,000 \text{ N/mm}^2$ ,  $k_{33} = 24,000 \text{ N/mm}^2$ ,  $s_{11} = s_{22} = 1.67 \times 10^{-5}$ ,  $s_{12} = s_{21} = -0.42 \times 10^{-5}$ ,  $s_{33} = 4.17 \times 10^{-5}$ .

**P.26.3.** The reduced stiffnesses in a unidirectional ply are  $k_{11} = 50,000 \text{ N/mm}^2$ ,  $k_{12} = k_{21} = 4000 \text{ N/mm}^2$ ,  $k_{22} = 15,000 \text{ N/mm}^2$ , and  $k_{33} = 6000 \text{ N/mm}^2$ . Calculate the elastic constants of the ply and also the reduced compliances.

Answer:  $v_{lt} = 0.08$ ,  $v_{lt} = 0.27$ ,  $E_l = 48,920 \text{ N/mm}^2$ ,  $E_t = 14,495 \text{ N/mm}^2$ ,  $G = 6000 \text{ N/mm}^2$ ,  $s_{11} = 2.04 \times 10^{-5}$ ,  $s_{12} = -0.55 \times 10^{-5} = s_{21}$ ,  $s_{22} = 6.90 \times 10^{-5}$ ,  $s_{33} = 1.67 \times 10^{-4}$ .

**P.26.4.** A generally orthotropic ply is subjected to direct stresses of  $120 \text{ N/mm}^2$  and  $60 \text{ N/mm}^2$  parallel to the  $x$  and  $y$  reference axes, respectively, together with a shear stress of  $80 \text{ N/mm}^2$ . If the ply angle is  $45^\circ$ , determine the direct and shear stresses referred to the material axes.

Answer:  $\sigma_l = 170 \text{ N/mm}^2$ ,  $\sigma_t = 10 \text{ N/mm}^2$ ,  $\tau_{lt} = -30 \text{ N/mm}^2$ .

**P.26.5.** The strains at a point in a generally orthotropic ply are  $\varepsilon_x = 0.005$ ,  $\varepsilon_y = 0.002$ , and  $\gamma_{xy} = 0.0002$  referred to the  $xy$  reference axis system. If the ply angle is  $45^\circ$ , calculate the strains in the directions of the material axes.

Answer:  $\varepsilon_l = 3.6 \times 10^{-3}$ ,  $\varepsilon_t = 3.4 \times 10^{-3}$ ,  $\gamma_{lt} = -3.0 \times 10^{-3}$ .

**P.26.6.** A generally orthotropic ply is subjected to direct stresses of  $100 \text{ N/mm}^2$  and  $50 \text{ N/mm}^2$  parallel to the  $x$  and  $y$  reference axis system, respectively. If the ply also carries a shear stress of  $75 \text{ N/mm}^2$  referred to the  $xy$  axis system and the ply angle is  $45^\circ$ , calculate the direct and shear stresses referred to the material axes and the corresponding strains. The elastic constants are  $E_l = 150,000 \text{ N/mm}^2$ ,  $E_t = 90,000 \text{ N/mm}^2$ ,  $G_{lt} = 5000 \text{ N/mm}^2$ ,  $v_{lt} = 0.3$ , and  $v_{tl} = 0.1$ .

Answer:  $\sigma_l = 150 \text{ N/mm}^2$ ,  $\sigma_t = 0$ ,  $\tau_{lt} = -25 \text{ N/mm}^2$ ,  $\varepsilon_l = 10.0 \times 10^{-4}$ ,  $\varepsilon_t = -3.03 \times 10^{-4}$ ,  $\gamma_{lt} = -50.0 \times 10^{-4}$ .

**P.26.7.** Write down in full the ply orientation for laminates designated as

- (a)  $[+30_2/60_2]$
- (b)  $[(0/45)_3/\pm 60_2]$
- (c)  $[60/-30/45]_S$
- (d)  $[45/(0/60)_2]_S$

Answer: See Solutions Manual.

**P.26.8.** A symmetric laminate comprises four isotropic plies each  $0.125 \text{ mm}$  thick and for which Young's modulus is  $140,000 \text{ N/mm}^2$ . If Poisson's ratio  $v=0.3$ , calculate the reduced stiffnesses for the laminate and hence verify that the equivalent elastic constants are identical to the given elastic constants for the plies.

Answer:  $k_{13}=k_{23}=0$ ,  $k_{11}=k_{22}=153,846 \text{ N/mm}^2$ ,  $k_{12}=46,154 \text{ N/mm}^2$   
 $k_{33}=53,846 \text{ N/mm}^2$

**P.26.9.** A symmetric laminate consists of four generally orthotropic plies in which the ply angle for the two inner plies is  $45^\circ$  while that for the two outer plies is  $30^\circ$ . If the thickness of each ply is  $0.15 \text{ mm}$  and the ply elastic constants are  $E_l=140,000 \text{ N/mm}^2$ ,  $E_t=10,000 \text{ N/mm}^2$ ,  $G_{lt}=5000 \text{ N/mm}^2$ , and  $v=0.3$ , determine the equivalent elastic constants for the laminate.

Answer:  $E_x=25,253 \text{ N/mm}^2$ ,  $E_y=16,835 \text{ N/mm}^2$ ,  $G_{lt}=10,352 \text{ N/mm}^2$   
 $v_{xy}=0.21$ ,  $v_{yx}=0.14$ ,  $m_x=1.0$ ,  $m_y=0.53$ .

**P.26.10.** A symmetric single-ply laminate comprises four isotropic plies of which the two outer plies are each  $0.15 \text{ mm}$  thick and are of aluminium alloy material while the two inner plies are each  $0.15 \text{ mm}$  thick but are of mild steel material. The ply angle for the outer plies is  $+45^\circ$  while that for the inner plies is  $-45^\circ$ . If, for the aluminium alloy  $E_{al}=70,000 \text{ N/mm}^2$  and  $v_{al}=0.3$  while for mild steel  $E_{st}=200,000 \text{ N/mm}^2$  and  $v_{st}=0.3$ , calculate the equivalent elastic constants for the laminate.

Answer:  $E_x=135,501 \text{ N/mm}^2$ ,  $E_y=135,501 \text{ N/mm}^2$ ,  $G_{xy}=51,921 \text{ N/mm}^2$   
 $v_{xy}=0.3$ ,  $v_{yx}=0.3$ ,  $m_x=m_y=0$ .

**P.26.11.** If the laminate of P.26.10 is subjected to a membrane load intensity  $N_x = 100 \text{ N/mm}$ , investigate the strength of the laminate given that  $X_T = 1200 \text{ N/mm}^2$ ,  $X_C = 1000 \text{ N/mm}^2$ ,  $Y_T = 75 \text{ N/mm}^2$ ,  $Y_C = 200 \text{ N/mm}^2$ , and  $S = 70 \text{ N/mm}^2$ .

Answer:  $\sigma_l = 90.4 \text{ N/mm}^2$ ,  $\sigma_t = 93.4 \text{ N/mm}^2$ ,  $\tau_{lt} = -105 \text{ N/mm}^2$ .

Therefore a direct stress failure of the plies occurs in the transverse direction together with a failure in shear.

**P.26.12.** Investigate the strength of the laminate of P.26.11 if it is subjected to a compressive load intensity of  $-100 \text{ N/mm}$ . The strength criteria are the same as those of P.26.11.

Answer: A failure in shear only.

**P.26.13.** A box beam has the thin-walled composite cross-section shown in Fig. P.26.13. The cover laminates are identical and have a Young's modulus of  $20,000 \text{ N/mm}^2$  while that of the vertical webs is  $60,000 \text{ N/mm}^2$ . If the beam is subjected to an axial load of  $40 \text{ kN}$ , determine the axial force in each laminate.

Answer: Covers,  $4 \text{ kN}$ ; webs,  $16 \text{ kN}$

**P.26.14.** Determine the axial force in the flanges and web of the channel section shown in Fig. P.26.14 when it is subjected to a total axial force of  $40 \text{ kN}$ . The ply properties are  $E_l = 140,000 \text{ N/mm}^2$ ,  $E_t = 10,000 \text{ N/mm}^2$ ,  $G_{lt} = 5000 \text{ N/mm}^2$ , and  $v_{lt} = 0.3$ .

Answer: Flange;  $17.2 \text{ kN}$ , Web;  $5.6 \text{ kN}$ .

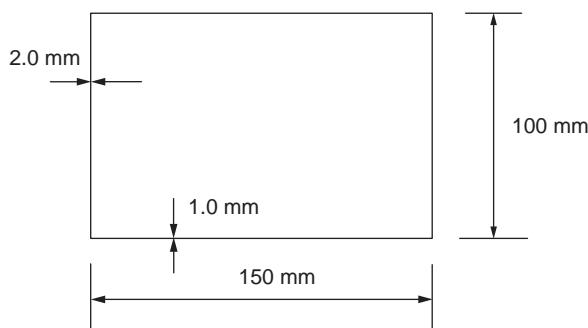


FIGURE P.26.13

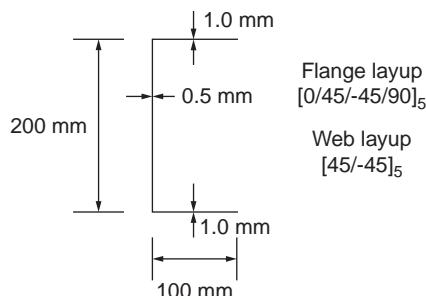


FIGURE P.26.14

**P.26.15.** If the thin-walled box beam of Fig. P.26.13 carries a bending moment of 1 kNm applied in a vertical plane, determine the maximum direct stress in the cross-section of the beam.

Answer: 85.8 N/mm<sup>2</sup>

**P.26.16.** If the thin-walled composite beam of Example 26.15 is subjected to a bending moment of 0.5 kNm applied in a horizontal plane, calculate the maximum value of direct stress in the beam section.

Answer: 277.5 N/mm<sup>2</sup>

**P.26.17.** The I-section beam shown in Fig. P.26.17 carries bending moments  $M_x=0.2$  kNm and  $M_y=0.2$  kNm. If for the flanges  $E_x=54,100$  N/mm<sup>2</sup> and for the web  $E_x=17,700$  N/mm<sup>2</sup>, determine the distribution of axial force intensity.

Answer: At 1: 136 N/mm. At 2: 194 N/mm (flange), 32 N/mm (web). At 3: 254 N/mm.

At 4: -136 N/mm. At 5: -194 N/mm (flange), -32 N/mm (web). At 6: -254 N/mm.

**P.26.18.** The Z-section beam shown in Fig. P.26.18 carries a bending moment of 0.2 kNm about its X axis. If, for the flanges  $E_x=54,100$  N/mm<sup>2</sup> and for the web  $E_x=17,700$  N/mm<sup>2</sup>, calculate the maximum load intensity in the section.

Answer:  $\pm 136$  N/mm at 2 and 3 in the flanges, respectively.

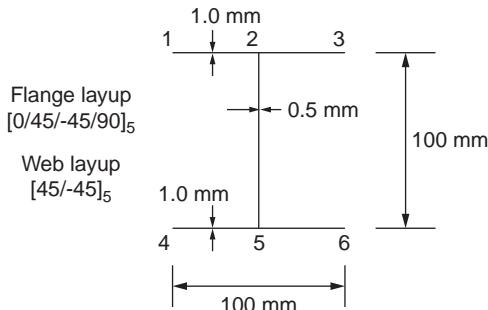


FIGURE P.26.17

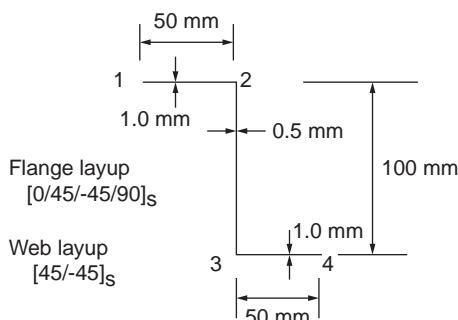


FIGURE P.26.18

**P.26.19.** The thin-walled composite beam section of Example 26.15 carries a vertical shear load of 2kN applied in the plane of the web. Determine the shear flow distribution.

Answer:  $q_{12} = -0.02s_1^2 - 1.99s_1$   
 $q_{23} = -0.40s_2 + 4.02 \times 10^{-3}s_2^2 - 149.8$

**P.26.20.** The closed, composite section, thin-walled beam shown in Fig. P.26.20 is subjected to a vertical shear load of 20 kN applied through its center of symmetry. If the laminate elastic properties are, for the covers,  $E_{Z,i}=54,100 \text{ N/mm}^2$  and for the webs,  $E_{Z,i}=17,700 \text{ N/mm}^2$ , determine the distribution of shear flow around the cross-section.

Answer:  $q_{01} = -1.98s_1$ ,  $q_{12} = 6.5 \times 10^{-3}s_2^2 - 0.325s_2 - 198$ .

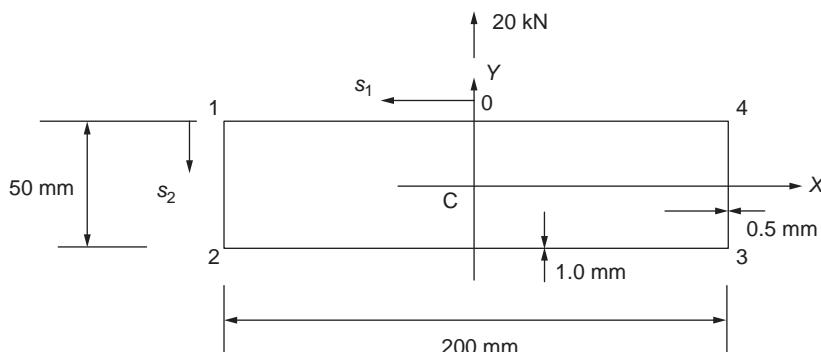


FIGURE P.26.20

**P.26.21.** The beam section shown in Fig. P.26.20 is subjected to a counterclockwise torque of 1 kNm. If the laminate shear modulus of the covers is  $20,700 \text{ N/mm}^2$  and that of the webs is  $36,400 \text{ N/mm}^2$ , determine the maximum shear stress in the section, its rate of twist, and the distribution of warping.

Answer:  $100 \text{ N/mm}^2$ ,  $6.25 \times 10^{-5} \text{ rad/mm}$ ,  $-0.086 \text{ mm}$  (at 4, zero at 0)

**P.26.21. MATLAB** Assuming the dimension for the web height shown in Fig. P.26.20 is labeled  $h$ , use MATLAB to repeat Problem P.26.21 for values of  $h$  ranging from 20 to 80 mm in increments of 10 mm.

Answer:	$h$	$\tau_{\max}$	$d\theta/dz$	$W$
(x)	20 mm	$250 \text{ N/mm}^2$	$33.33 \times 10^{-5} \text{ rad/mm}$	0.268 mm (at 4, zero at 0)
(xi)	30 mm	$166.7 \text{ N/mm}^2$	$16.67 \times 10^{-5} \text{ rad/mm}$	0.167 mm (at 4, zero at 0)
(xii)	40 mm	$125 \text{ N/mm}^2$	$9.09 \times 10^{-5} \text{ rad/mm}$	0.117 mm (at 4, zero at 0)
(xiii)	50 mm	$100 \text{ N/mm}^2$	$6.25 \times 10^{-5} \text{ rad/mm}$	0.086 mm (at 4, zero at 0)
(xiv)	60 mm	$83.3 \text{ N/mm}^2$	$4.55 \times 10^{-5} \text{ rad/mm}$	0.066 mm (at 4, zero at 0)
(xv)	70 mm	$71.4 \text{ N/mm}^2$	$3.45 \times 10^{-5} \text{ rad/mm}$	0.052 mm (at 4, zero at 0)
(xvi)	80 mm	$62.5 \text{ N/mm}^2$	$2.78 \times 10^{-5} \text{ rad/mm}$	0.041 mm (at 4, zero at 0)

**P.26.22.** The thin-walled, composite beam section shown in Fig. P.26.22 has laminate shear moduli of 16,300 N/mm<sup>2</sup> for the flanges and 20,900 N/mm<sup>2</sup> for the web. If the beam is subjected to a torque of 0.5 kNmm, determine the rate of twist in the section, the maximum shear stress, and the value of warping at the point 1.

Answer:  $0.8 \times 10^{-3}$  rad/mm,  $\pm 13$  N/mm<sup>2</sup> (in flanges), 2.0 mm

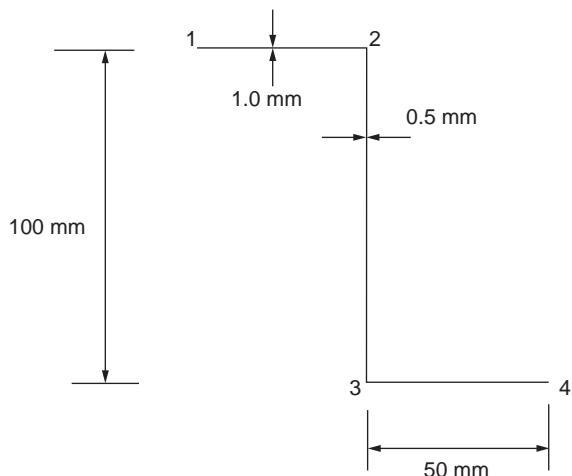


FIGURE P.26.22

# Closed section beams

# 27

The analysis presented in [Chapters 16 through 20](#) relies on elementary theory for the determination of stresses and displacements produced by axial loads, shear forces and bending moments, and torsion. No allowance is made for the effects of restrained warping produced by structural or loading discontinuities in the torsion of open or closed section beams or for the effects of shear strains on the calculation of direct and shear stresses in beams subjected to bending and shear.

In this chapter, we examine some relatively simple examples of these effects; more complex cases require analysis by computer-based techniques, such as the finite element method.

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## 27.1 GENERAL ASPECTS

Structural constraint stresses in either closed or open beams result from a restriction on the freedom of any section of the beam to assume its normal displaced shape under load. Such a restriction arises when one end of the beam is built in, although the same effect may be produced practically, in a variety of ways. For example, the root section of a beam subjected to torsion is completely restrained from warping into the displaced shape indicated by [Eq. \(18.5\)](#), and a longitudinal stress system is induced, which, in a special case discussed later, is proportional to the free warping of the beam.

A slightly different situation arises when the beam supports shear loads. The stress system predicted by elementary bending theory relies on the basic assumption of plane sections remaining plane after bending. However, for a box beam comprising thin skins and booms, the shear strains in the skins are of sufficient magnitude to cause a measurable redistribution of the direct load in the booms, and hence previously plane sections warp. We discuss the phenomenon of load redistribution resulting from shear, known as *shear lag*, in detail later in the chapter. The prevention of this warping by some form of axial constraint modifies the stress system still further.

The most comprehensive analysis yet published of multicell and single-cell beams under arbitrary loading and support conditions is that by Argyris and Dunne.<sup>1</sup> Their work mainly concentrates on beams of idealized cross-section, and while the theory they present is in advance of that required here, it is beneficial to examine some of the results of their analysis. We limit the present discussion to closed beams of idealized cross-section.

The problem of axial constraint may be conveniently divided into two parts. In the first, the shear stress distribution due to an arbitrary loading is calculated exclusively at the built-in end of the beam. In the second, the stress (and/or load) distributions are calculated along the length of the beam for the separate loading cases of torsion and shear. Obviously, the shear stress systems predicted by each portion of theory must be compatible at the built-in end.

Argyris and Dunne showed that the calculation of the shear stress distribution at a built-in end is a relatively simple problem, the solution being obtained for any loading and beam cross-section by

statics. More complex is the determination of the stress distributions at sections along the beam. These stresses, for the torsion case, are shown to be the sum of the stresses predicted by elementary theory and stresses caused by systems of self-equilibrating end loads. For a beam supporting shear loads, the total stresses are again the sum of those corresponding to elementary bending theory and stresses due to systems of self-equilibrating end loads.

For an  $n$ -boom, idealized beam, Argyris and Dunne found that there are  $n - 3$  self-equilibrating end load, or *eigenload*, systems required to nullify  $n - 3$  possible modes of warping displacement. These eigenloads are analogous to, say, the buckling loads corresponding to the different buckled shapes of an elastic strut. The fact that, generally, a number of warping displacements are possible in an idealized beam invalidates the use of the shear center or flexural axis as a means of separating torsion and shear loads, because associated with each warping displacement is an axis of twist that is different for each warping mode. In practice, a good approximation is obtained if the torsion loads are referred to the axis of twist corresponding to the lowest eigenload. Transverse loads through this axis, the *zero warping axis*, produce no warping due to twist, although axial constraint stresses due to shear are still present.

In the special case of a doubly symmetrical section, the problem of separating the torsion and bending loads does not arise, since it is obvious that the torsion loads may be referred to the axis of symmetry. Double symmetry has the further effect of dividing the eigenloads into four separate groups corresponding to  $(n/4) - 1$  pure flexural modes in each of the  $xz$  and  $yz$  planes,  $(n/4)$  pure twisting modes about the center of symmetry, and  $(n/4) - 1$  pure warping modes that involve neither flexure nor twisting. A doubly symmetrical six boom beam supporting a single shear load has therefore just one eigenload system if the center boom in the top and bottom panels is regarded as being divided equally on either side of the axis of symmetry, thereby converting it, in effect, into an eight-boom beam.

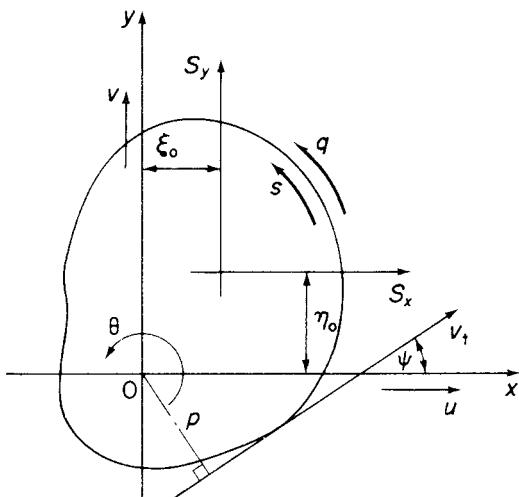
It is obvious from this that, generally, the self-equilibrating stress systems cannot be proportional to the free warping of the beam unless the free warping can be nullified by just one eigenload system. This is true only for the four boom beam that, from the preceding, has one possible warping displacement. If, in addition, the beam is doubly symmetrical, then its axis of twist passes through the center of symmetry. We note that only in cases of doubly symmetrical beams do the zero warping and flexural axes coincide.

A further special case arises when the beam possesses the properties of a Neuber beam ([Section 18.1.2](#)), which does not warp under torsion. The stresses in this case are the elementary torsion theory stresses, since no constraint effects are present. When bending loads predominate, however, it is generally impossible to design an efficient structure that does not warp.

In this chapter, the calculation of spanwise stress distributions in closed section beams is limited to simple cases of beams having doubly symmetrical cross-sections. It should be noted that simplifications of this type can be misleading, in that some of the essential characteristics of beam analysis – for example, the existence of the  $n - 3$  self-equilibrating end load systems – vanish.

## 27.2 SHEAR STRESS DISTRIBUTION AT A BUILT-IN END OF A CLOSED SECTION BEAM

This special case of structural constraint is of interest because the shear stress distribution at the built-in end of a closed section beam is statically determinate. [Figure 27.1](#) represents the cross-section of a thin-walled closed section beam at its built-in end. It is immaterial for this analysis whether or not the



**FIGURE 27.1** Cross-Section of a Thin-Walled Beam at the Built-in End

section is idealized, since the expression for shear flow in Eq. (17.19), on which the solution is based, is applicable to either case. The beam supports shear loads  $S_x$  and  $S_y$ , which generally produce torsion in addition to shear. We again assume that the cross-section of the beam remains undistorted by the applied loads, so that the displacement of the beam cross-section is completely defined by the displacements  $u$ ,  $v$ ,  $w$ , and the rotation  $\theta$  referred to an arbitrary system of axes  $Oxy$ . The shear flow  $q$  at any section of the beam is then given by Eq. (17.20); that is,

$$q = Gt \left( p \frac{d\theta}{dz} + \frac{du}{dz} \cos \psi + \frac{dv}{dz} \sin \psi + \frac{\partial w}{\partial s} \right)$$

At the built-in end,  $\partial w/\partial s$  is zero, and hence

$$q = Gt \left( p \frac{d\theta}{dz} + \frac{du}{dz} \cos \psi + \frac{dv}{dz} \sin \psi \right) \quad (27.1)$$

in which  $d\theta/dz$ ,  $du/dz$ , and  $dv/dz$  are the unknowns, the remaining terms being functions of the section geometry.

The resultants of the internal shear flows  $q$  must be statically equivalent to the applied loading, so that

$$\left. \begin{cases} \int q \cos \psi ds = S_x \\ \int q \sin \psi ds = S_y \\ \int qp ds = S_y \xi_0 - S_x \eta_0 \end{cases} \right\} \quad (27.2)$$

Substitution for  $q$  from Eq. (27.1) in Eqs. (27.2) yields

$$\left. \begin{aligned} \frac{d\theta}{dz} \int tp \cos \psi ds + \frac{du}{dz} \int t \cos^2 \psi ds + \frac{dv}{dz} \int t \cos \psi \sin \psi ds &= \frac{S_x}{G} \\ \frac{d\theta}{dz} \int tp \sin \psi ds + \frac{du}{dz} \int t \sin \psi \cos \psi ds + \frac{dv}{dz} \int t \sin^2 \psi ds &= \frac{S_y}{G} \\ \frac{d\theta}{dz} \int tp^2 ds + \frac{du}{dz} \int tp \cos \psi ds + \frac{dv}{dz} \int tp \sin \psi ds &= \frac{(S_y \xi_0 - S_x \eta_0)}{G} \end{aligned} \right\} \quad (27.3)$$

Equations (27.3) are solved simultaneously for  $d\theta/dz$ ,  $du/dz$ , and  $dv/dz$ . These values are then substituted in Eq. (27.1) to obtain the shear flow and hence the shear stress distribution.

Attention must be paid to the signs of  $\psi$ ,  $p$ , and  $q$  in Eqs. (27.3). Positive directions for each parameter are suggested in Fig. 27.1, although alternative conventions may be adopted. In general, however, there are rules that must be obeyed, these having special importance in the solution of multicell beams. Briefly, these are as follows. The positive directions of  $q$  and  $s$  are the same but may be assigned arbitrarily in each wall. Then,  $p$  is positive if movement of the foot of the perpendicular along the positive direction of the tangent leads to a counterclockwise rotation of  $p$  about O. The term  $\psi$  is the clockwise rotation of the tangent vector necessary to bring it into coincidence with the positive direction of the  $x$  axis.

### Example 27.1

Calculate the shear stress distribution at the built-in end of the beam shown in Fig. 27.2(a) when, at this section, it carries a shear load of 22,000 N acting at a distance of 100 mm from and parallel to side 12. The modulus of rigidity  $G$  is constant throughout the section:

Wall	12	34	23
Length (mm)	375	125	500

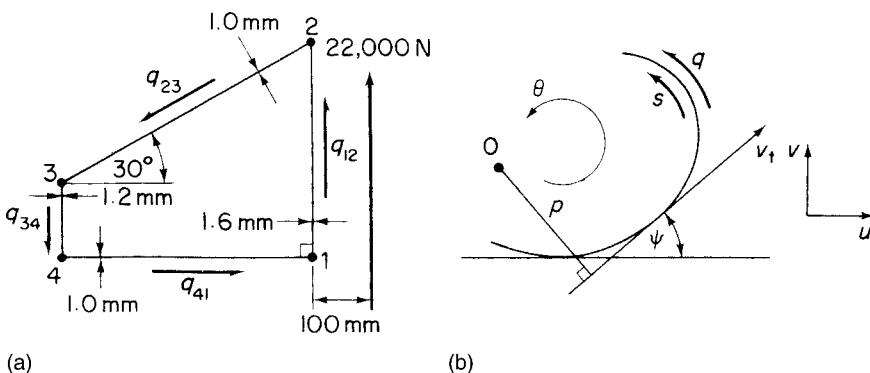


FIGURE 27.2 (a) Beam Cross-Section at Built-in End; (b) Notation and Sign Convention

It is helpful at the start of the problem to sketch the notation and sign convention, as shown in Fig. 27.2(b). The walls of the beam are flat, and therefore  $p$  and  $\psi$  are constant along each wall. Also, the thickness of each wall is constant, so that the shear flow  $q$  is independent of  $s$  in each wall. Let point 1 be the origin of the axes; then, writing  $\theta' = d\theta/dz$ ,  $u' = du/dz$ , and  $v' = dv/dz$ , we obtain, from Eq. (27.1),

$$q_{12} = 1.6Gv' \quad (\text{i})$$

$$q_{23} = 1.0G(375 \times 0.886\theta' - 0.886u' - 0.5v') \quad (\text{ii})$$

$$q_{34} = 1.2G(500 \times 0.866\theta' - v') \quad (\text{iii})$$

$$q_{41} = 1.0Gu' \quad (\text{iv})$$

For horizontal equilibrium,

$$500 \times 0.886q_{41} - 500 \times 0.886q_{23} = 0$$

giving

$$q_{41} = q_{23} \quad (\text{v})$$

For vertical equilibrium,

$$375q_{12} - 125q_{34} - 250q_{23} = 22,000 \quad (\text{vi})$$

For moment equilibrium about point 1,

$$500 \times 375 \times 0.886q_{23} + 125 \times 500 \times 0.886q_{34} = 22,000 \times 100$$

or

$$3q_{23} + q_{34} = 40.6 \quad (\text{vii})$$

Substituting for  $q_{12}$  and the like from Eqs. (i), (ii), (iii), and (iv) into Eqs. (v), (vi), and (vii) and solving for  $\theta'$ ,  $u'$ , and  $v'$  gives  $\theta' = 0.122/G$ ,  $u' = 9.71/G$ ,  $v' = 42.9/G$ . The values of  $\theta'$ ,  $u'$ , and  $v'$  are now inserted in Eqs. (i), (ii), (iii), and (iv), giving  $q_{12} = 68.5 \text{ N/mm}$ ,  $q_{23} = 9.8 \text{ N/mm}$ ,  $q_{34} = 11.9 \text{ N/mm}$ ,  $q_{41} = 9.8 \text{ N/mm}$ , from which

$$\tau_{12} = 42.8 \text{ N/mm}^2, \quad \tau_{23} = \tau_{41} = 9.8 \text{ N/mm}^2, \quad \tau_{34} = 9.9 \text{ N/mm}^2$$

We note, in Example 27.1, that there is a discontinuity of shear flow at each of the corners of the beam. This implies the existence of axial loads at the corners, which would, in practice, be resisted by booms if stress concentrations are to be avoided. We see also that in a beam having straight walls the shear flows are constant along each wall, so that, from Eq. (17.2), the direct stress gradient  $\partial\sigma_z/\partial z = 0$  in the walls at the built-in end, although not necessarily in the booms. Finally, the center of twist of the beam section at the built-in end may be found using Eq. (17.11), that is,

$$x_R = -\frac{v'}{\theta'}, \quad y_R = \frac{u'}{\theta'}$$

which, from the results of Example 27.1, give  $x_R = -351.6 \text{ mm}$ ,  $y_R = 79.6 \text{ mm}$ . Thus, the center of twist is 351.6 mm to the left of and 79.6 mm above corner 1 of the section and does not, as we note in Section 27.1, coincide with the shear center of the section.

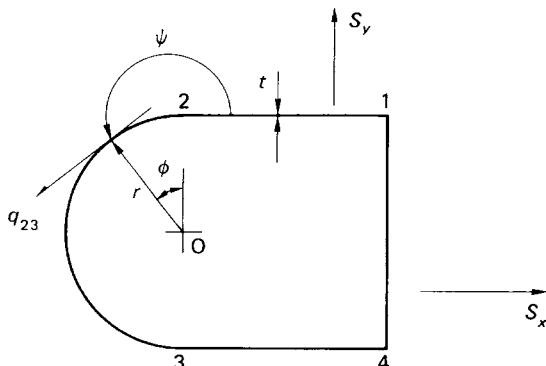


FIGURE 27.3 Built-in End of a Beam Section Having a Curved Wall

The method of analysis of beam sections having curved walls is similar to that of [Example 27.1](#), except that, in the curved walls, the shear flow is not constant, since both  $p$  and  $\psi$  in [Eq. \(27.1\)](#) generally vary. Consider the beam section shown in [Fig. 27.3](#), in which the curved wall 23 is semi-circular and of radius  $r$ . In the wall 23,  $p = r$  and  $\psi = 180 + \phi$ , so that [Eq. \(27.1\)](#) gives

$$q_{23} = Gt(r\theta' - u' \cos \phi - v' \sin \phi)$$

The resultants of  $q_{23}$  are then

$$\text{Horizontally: } \int_0^\pi q_{23} \cos \phi \, r d\phi$$

$$\text{Vertically: } \int_0^\pi q_{23} \sin \phi \, r d\phi$$

$$\text{Moment (about 0): } \int_0^\pi q_{23} r^2 \, d\phi$$

The shear flows in the remaining walls are constant and the solution proceeds as before.

### Example 27.2

Determine the shear flow distribution at the built-in end of a beam whose cross-section is shown in [Fig. 27.4](#). All walls have the same thickness  $t$  and shear modulus  $G$ ;  $R = 200$  mm.

In general, at a built-in end (see [Eq. \(27.1\)](#)),

$$q = Gt \left( p \frac{d\theta}{dz} + \frac{du}{dz} \cos \psi + \frac{dv}{dz} \sin \psi \right)$$

Therefore, taking O as the origin and writing  $\theta' = d\theta/dz$ ,  $u' = du/dz$ , and  $v' = dv/dz$ ,

$$q_{41} = Gt(-2R\theta' + v') \quad (\text{i})$$

$$q_{12} = Gt(-R\theta' + u') \quad (\text{ii})$$

$$q_{34} = Gt(-R\theta' - u') \quad (\text{iii})$$

$$q_{23} = Gt(-R\theta' + u' \cos \phi - v' \sin \phi) \quad (\text{iv})$$

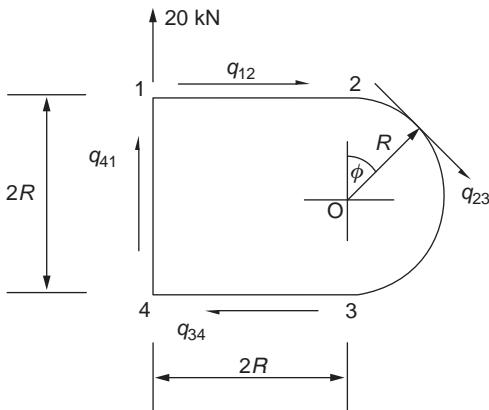


FIGURE 27.4 Beam Section of Example 27.2

From symmetry,

$$q_{12} = q_{34}$$

that is,

$$Gt(-R\theta' + u') = Gt(-R\theta' - u')$$

Therefore,

$$u' = 0$$

Resolving vertically,

$$q_{41}2R - \int_0^\pi q_{23} \sin \phi R d\phi = 20 \times 10^3$$

that is,

$$q_{41} - \frac{1}{2} \int_0^\pi q_{23} \sin \phi d\phi = \frac{10,000}{R}$$

Substituting from Eqs. (i) and (iv) gives

$$-R\theta' + 1.79v' = \frac{10,000}{GtR} \quad (v)$$

Now, taking moments about O,

$$q_{41} 2R 2R + q_{12} 2RR + q_{34} 2RR + \int_0^\pi q_{23} R^2 d\phi = 20,000 \times 2R$$

which gives

$$2q_{41} + q_{12} + q_{34} + \frac{1}{2} \int_0^\pi q_{23} d\phi = \frac{20,000}{R}$$

Substituting from Eqs. (i), (ii), (iii), and (iv),

$$2Gt(-2R\theta' + v') - 2GtR\theta' + \frac{Gt}{2} \int_0^\pi (-R\theta' - v' \sin \phi) d\phi = \frac{20,000}{R}$$

from which

$$R\theta' - 0.13v' = -\frac{2641.7}{GtR} \quad (\text{vi})$$

Solving Eqs. (v) and (vi),

$$v' = \frac{4432.7}{GtR}, \quad R\theta' = -\frac{2065.4}{GtR}$$

Therefore,

$$q_{41} = Gt \left( 2 \times \frac{2065.4}{200Gt} + \frac{4432.7}{200Gt} \right) = 42.8 \text{ N/mm}$$

Similarly,

$$q_{12} = q_{34} = 10.3 \text{ N/mm}$$

Finally,

$$q_{23} = 10.3 - 22.2 \sin \phi \text{ N/mm}$$



### 27.3 THIN-WALLED RECTANGULAR SECTION BEAM SUBJECTED TO TORSION

In Example 18.2, we determined the warping distribution in a thin-walled rectangular section beam that was not subjected to structural constraint. This free warping distribution ( $w_0$ ) was found to be linear around a cross-section and uniform along the length of the beam, having values at the corners of

$$w_0 = \pm \frac{T}{8abG} \left( \frac{b}{t_b} - \frac{a}{t_a} \right)$$

The effect of structural constraint, such as building one end of the beam in, is to reduce this free warping to zero at the built-in section, so that direct stresses are induced that subsequently modify the shear stresses predicted by elementary torsion theory. These direct stresses must be self-equilibrating, since the applied load is a pure torque.

The analysis of a rectangular section beam built in at one end and subjected to a pure torque at the other is simplified if the section is idealized into one comprising four corner booms that are assumed to carry all the direct stresses together with shear-stress-only carrying walls. The assumption on which the idealization is based is that the direct stress distribution at any cross-section is directly proportional to the warping that has been suppressed. Therefore, the distribution of direct stress is linear around any cross-section and has values equal in magnitude but opposite in sign at opposite corners of a wall.

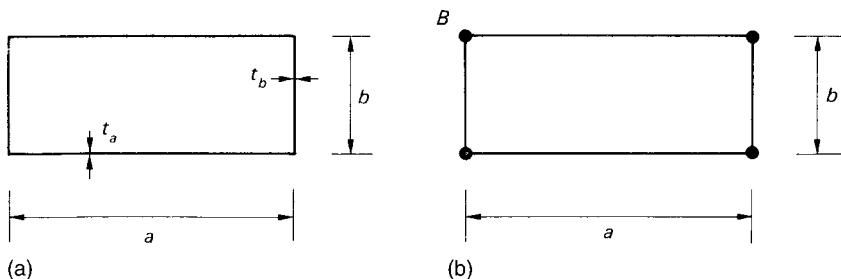


FIGURE 27.5 Idealization of a Rectangular Section Beam Subjected to Torsion: (a) Actual; (b) Idealized

This applies at all cross-sections, since the free warping is suppressed to some extent along the complete length of the beam. In Fig. 27.5(b), all the booms have the same cross-sectional area from antisymmetry and, from Eq. (20.1) or (20.2),

$$B = \frac{at_a}{6}(2-1) + \frac{bt_b}{6}(2-1) = \frac{1}{6}(at_a + bt_b)$$

To the boom area  $B$  is added existing concentrations of area, such as the connecting angle sections at the corners. The contributions of stringers may be included by allowing for their direct stress-carrying capacity by increasing the actual wall thickness by an amount equal to the total stringer area on one wall before idealizing the section.

We saw, in Chapter 20, that the effect of structural idealization is to reduce the shear flow in the walls of a beam to a constant value between adjacent booms. In Fig. 27.6, suppose that the shear flows in the covers and webs at any section are  $q_a$  and  $q_b$ , respectively; from antisymmetry, the shear flows in both covers are  $q_a$  and in both webs  $q_b$ . The resultant of these shear flows is equivalent to the applied torque, so that

$$T = \oint qp \, ds = 2q_a a \frac{b}{2} + 2q_b b \frac{a}{2}$$

or

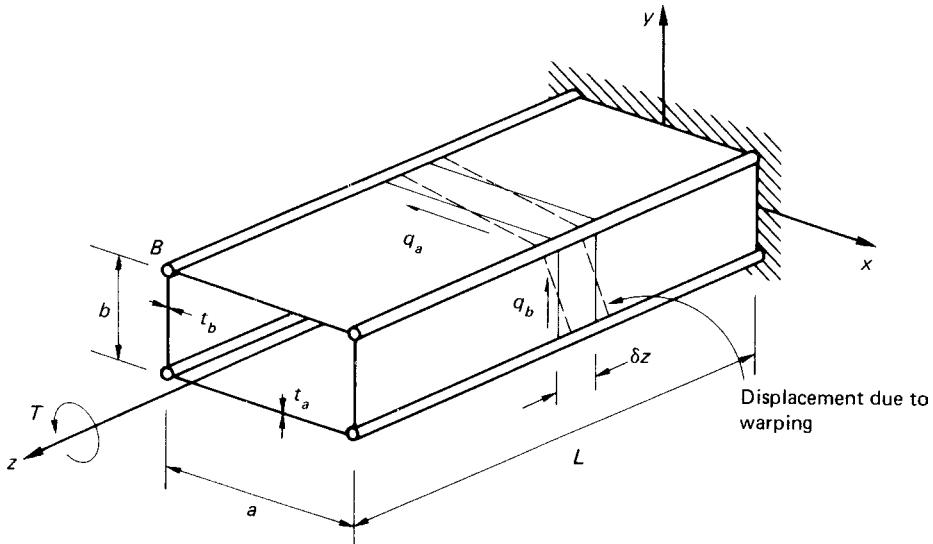
$$T = ab(q_a + q_b) \quad (27.4)$$

We now use Eq. (17.19), that is,

$$q = Gt \left( \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \right)$$

to determine  $q_a$  and  $q_b$ . Since the beam cross-section is doubly symmetrical, the axis of twist passes through the center of symmetry at any section, so that, from Eq. (17.8),

$$\frac{\partial v_t}{\partial z} = p_R \frac{d\theta}{dz} \quad (27.5)$$



**FIGURE 27.6 Idealized Rectangular Section Beam Built in at One End and Subjected to Torque at the Other**

Therefore, for the covers of the beam,

$$\frac{\partial v_t}{\partial z} = \frac{b}{2} \frac{d\theta}{dz} \quad (27.6)$$

and, for the webs,

$$\frac{\partial v_t}{\partial z} = \frac{a}{2} \frac{d\theta}{dz} \quad (27.7)$$

The elements of length  $\delta z$  of the covers and webs of the beam warp into the shapes shown in Fig. 27.6 if  $T$  is positive (counterclockwise) and  $b/t_b > a/t_a$ . Clearly, there must be compatibility of displacement at adjacent edges of the elements. From Fig. 27.7(a),

$$\frac{\partial w}{\partial s} = \frac{-w}{a/2} \quad (27.8)$$

and, from Fig. 27.7(b),

$$\frac{\partial w}{\partial s} = \frac{w}{b/2} \quad (27.9)$$

Substituting for  $\partial w/\partial s$  and  $\partial v_t/\partial z$  in Eq. (17.19) separately for the covers and webs, we obtain

$$q_a = Gt_a \left( \frac{-2w}{a} + \frac{b}{2} \frac{d\theta}{dz} \right), \quad q_b = Gt_b \left( \frac{2w}{b} + \frac{a}{2} \frac{d\theta}{dz} \right) \quad (27.10)$$

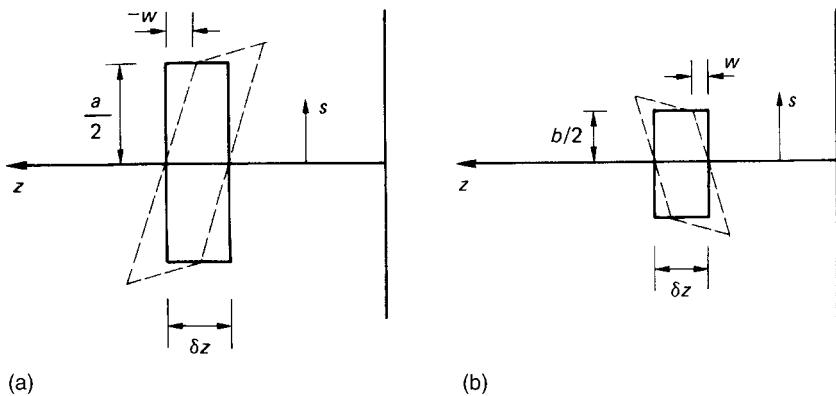


FIGURE 27.7 Shear Distortion of (a) an Element of the Top Cover; (b) an Element of the Right-Hand Web

Now, substituting for  $q_a$  and  $q_b$  in Eq. (27.4), we have

$$T = abG \left[ t_a \left( \frac{-2w}{a} + \frac{b}{2} \frac{d\theta}{dz} \right) + t_b \left( \frac{2w}{b} + \frac{a}{2} \frac{d\theta}{dz} \right) \right]$$

Rearranging,

$$\frac{d\theta}{dz} = \frac{4w(bt_a - at_b)}{ab(bt_a + at_b)} + \frac{2T}{abG(bt_a + at_b)} \quad (27.11)$$

If we now substitute for  $d\theta/dz$  from Eq. (27.11) into Eqs. (27.10), we have

$$q_a = \frac{-4wGt_b t_a}{bt_a + at_b} + \frac{Tt_a}{a(bt_a + at_b)}, \quad q_b = \frac{-4wGt_b t_a}{bt_a + at_b} + \frac{Tt_b}{b(bt_a + at_b)} \quad (27.12)$$

Equations (27.11) and (27.12) give the rate of twist and the shear flows (and hence shear stresses) in the beam in terms of the warping  $w$  and the applied torque  $T$ . Their derivation is based on the compatibility of the displacement that exists at the cover-boom-web junctions. We shall now use the further condition of equilibrium between the shears in the covers and webs and the direct load in the booms to obtain expressions for the warping displacement and the distributions of boom stress and load. Thus, for the equilibrium of an element of the top right-hand boom, shown in Fig. 27.8,

$$\left( \sigma_z + \frac{\partial \sigma_z}{\partial z} \delta z \right) B - \sigma_z B + q_a \delta z - q_b \delta z = 0$$

that is,

$$B \frac{\partial \sigma_z}{\partial z} + q_a - q_b = 0 \quad (27.13)$$

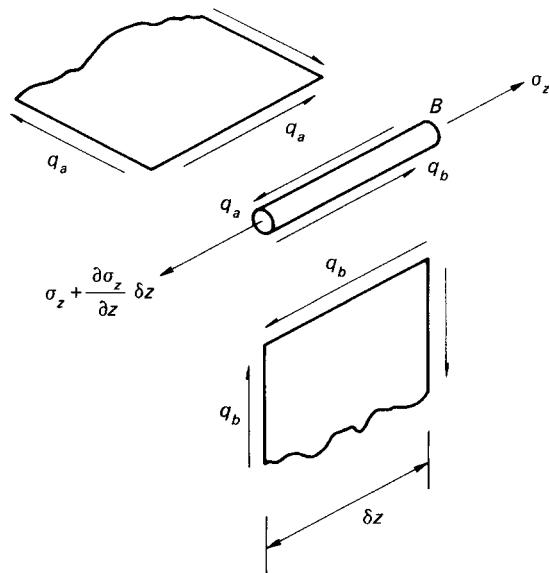


FIGURE 27.8 Equilibrium of Boom Element

Now,

$$\sigma_z = E \frac{\partial w}{\partial z} \quad (\text{see Chapter 1})$$

Substituting for  $\sigma_z$  in Eq. (27.13), we obtain

$$BE \frac{\partial^2 w}{\partial z^2} + q_a - q_b = 0 \quad (27.14)$$

Replacing  $q_a$  and  $q_b$  from Eqs. (27.12) gives

$$BE \frac{\partial^2 w}{\partial z^2} - \frac{8Gt_b t_a}{bt_a + at_b} w = -\frac{T}{ab} \frac{(bt_a - at_b)}{(bt_a + at_b)}$$

or

$$\frac{\partial^2 w}{\partial z^2} - \mu^2 w = -\frac{T}{abBE} \frac{(bt_a - at_b)}{(bt_a + at_b)} \quad (27.15)$$

where

$$\mu^2 = \frac{8Gt_b t_a}{BE(bt_a + at_b)}$$

The differential equation (27.15) is of standard form and its solution is

$$w = C \cosh \mu z + D \sinh \mu z + \frac{T}{8abG} \left( \frac{b}{t_b} - \frac{a}{t_a} \right) \quad (27.16)$$

in which the last term is seen to be the free warping displacement  $w_0$  of the top right-hand corner boom. The constants  $C$  and  $D$  in Eq. (27.16) are found from the boundary conditions of the beam. In this particular case, the warping  $w = 0$  at the built-in end and the direct strain  $\partial w / \partial z = 0$  at the free end, where there is no direct load. From the first of these,

$$C = -\frac{T}{8abG} \left( \frac{b}{t_b} - \frac{a}{t_a} \right) = -w_0$$

and, from the second,

$$D = w_0 \tanh \mu L$$

Then,

$$w = w_0 (1 - \cosh \mu z + \tanh \mu L \sinh \mu z) \quad (27.17)$$

or, rearranging,

$$w = w_0 \left[ 1 - \frac{\cosh \mu(L-z)}{\cosh \mu L} \right] \quad (27.18)$$

The variation of direct stress in the boom is obtained from  $\sigma_z = E \partial w / \partial z$  and Eq. (27.18); that is,

$$\sigma_z = \mu E w_0 \frac{\sinh \mu(L-z)}{\cosh \mu L} \quad (27.19)$$

and the variation of boom load  $P$  is then

$$P = B \sigma_z = B \mu E w_0 \frac{\sinh \mu(L-z)}{\cosh \mu L} \quad (27.20)$$

Substituting for  $w$  in Eqs. (27.12) and rearranging, we obtain the shear stress distribution in the covers and webs:

$$\tau_a = \frac{q_a}{t_a} = \frac{T}{2abt_a} \left[ 1 + \frac{(bt_a - at_b)}{(bt_a + at_b)} \frac{\cosh \mu(L-z)}{\cosh \mu L} \right] \quad (27.21)$$

$$\tau_b = \frac{q_b}{t_b} = \frac{T}{2abt_b} \left[ 1 - \frac{(bt_a - at_b)}{(bt_a + at_b)} \frac{\cosh \mu(L-z)}{\cosh \mu L} \right] \quad (27.22)$$

Inspection of Eqs. (27.21) and (27.22) shows that the shear stress distributions each comprise two parts. The first terms,  $T/2abt_a$  and  $T/2abt_b$ , are the shear stresses predicted by elementary theory (see Section 18.1), while the hyperbolic second terms represent the effects of the warping restraint. Clearly, for a counterclockwise torque and  $bt_a > at_b$ , the effect of this constraint is to increase the shear stress in the covers over that predicted by elementary theory and decrease the shear stress in the webs. It may

also be noted that, for  $bt_a$  to be greater than  $at_b$  for the beam of Fig. 27.6, in which  $a > b$ ,  $t_a$  must be appreciably greater than  $t_b$ , so that  $T/2abt_a < T/2abt_b$ . Also, at the built-in end ( $z = 0$ ), Eqs. (27.21) and (27.22) reduce to  $\tau_a = T/a(bt_a + at_b)$  and  $\tau_b = T/b(bt_a + at_b)$ , so that, even though  $\tau_b$  is reduced by the axial constraint and  $\tau_a$  increased,  $\tau_b$  is still greater than  $\tau_a$ . It should also be noted that these values of  $\tau_a$  and  $\tau_b$  at the built-in end may be obtained using the method of Section 27.2, and that these are the values of shear stress irrespective of whether the section has been idealized or not. In other words, the presence of intermediate stringers or direct stress-carrying walls does not affect the shear flows at the built-in end, since the direct stress gradient at this section is zero (see Section 27.2 and (Eq. 17.2)) except in the corner booms. Finally, when both  $z$  and  $L$  become large, that is, at the free end of a long, slender beam,

$$\tau_a \rightarrow \frac{T}{2abt_a} \quad \text{and} \quad \tau_b \rightarrow \frac{T}{2abt_b}$$

This situation is shown in Fig. 27.9.

In the particular case when  $bt_a = at_b$ , we see that the second terms on the right-hand side of Eqs. (27.21) and (27.22) disappear and no constraint effects are present; the direct stress of Eqs. (27.19) is also zero, since  $w_0 = 0$  (see Example 18.2).

The rate of twist is obtained by substituting for  $w$  from Eq. (27.18) in Eq. (27.11). Thus,

$$\frac{d\theta}{dz} = \frac{T}{2a^2b^2G} \left( \frac{b}{t_b} + \frac{a}{t_a} \right) \left[ 1 - \left( \frac{bt_a - at_b}{bt_a + at_b} \right)^2 \frac{\cosh \mu(L-z)}{\cosh \mu L} \right] \quad (27.23)$$

in which we see that again the expression on the right-hand side comprises the rate of twist given by elementary theory,  $T(b/t_b + a/t_a)/2a^2b^2G$  (see Section 18.1), together with a correction due to the warping restraint. Clearly, the rate of twist is always reduced by the constraint, since  $(bt_a - at_b)^2$  is always positive. Integration of Eq. (27.23) gives the distribution of angle of twist along the length of the beam, the boundary condition in this case being  $\theta = 0$  at  $z = 0$ .

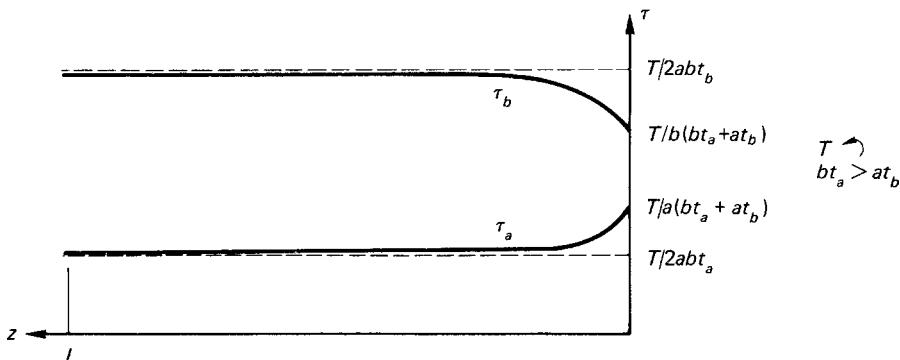


FIGURE 27.9 Shear Stress Distribution Along the Beam of Fig. 27.6

**Example 27.3**

A uniform four boom box of span 5 m is 500 mm wide by 200 mm deep and has four corner booms, each of cross-sectional area 800 mm<sup>2</sup>; its wall thickness is 1.0 mm. If the box is subjected to a uniformly distributed torque loading of 20 Nm/mm along its length and it is supported at each end such that complete freedom of warping exists at the end cross-sections, calculate the angle of twist at the mid-span section. Take  $G = 20,000 \text{ N/mm}^2$  and  $G/E = 0.36$ .

The reactive torques at each support are  $= 20 \times 5,000/2 = 50,000 \text{ Nm}$ .

Taking the origin for  $z$  at the mid-span of the beam, the torque at any section is given by

$$T(z) = 20(2,500 - z) - 50,000 = -20z \text{ Nm}$$

Substituting in Eq. (27.16), we obtain

$$w = C \cosh \mu z + D \sinh \mu z - \frac{20z \times 10^3(b - a)}{8abGt}$$

The boundary conditions are

$$w = 0, \text{ when } z = 0 \text{ from symmetry, and } \partial w / \partial z = 0, \text{ when } z = L (L = 2,500 \text{ mm})$$

From the first of these,  $C = 0$ , while from the second,

$$D = \frac{20 \times 10^3(b - a)}{8\mu abGt \cosh \mu L}$$

Therefore,

$$w = \frac{20(b - a) \times 10^3}{8abGt} \left( \frac{\sinh \mu z}{\mu \cosh \mu L} - z \right) \quad (\text{i})$$

Further,

$$\mu^2 = \frac{8Gt}{AE(b + a)} = \frac{8 \times 0.36 \times 1.0}{800(200 + 500)} = 5.14 \times 10^{-6}$$

so that Eq. (i) becomes

$$w = -3.75 \times 10^{-4}(3.04 \sinh \mu z - z) \quad (\text{ii})$$

Substituting for  $w$  and the like in Eq. (27.11),

$$\frac{d\theta}{dz} = 10^{-8}(1.95 \sinh \mu z - 3.49z)$$

Hence,

$$\theta = 10^{-8} \left( \frac{1.95}{\mu} \cosh \mu z - 1.75z^2 + F \right) \quad (\text{iii})$$

When  $z = L$  (2,500 mm),  $\theta = 0$ . Then, from Eq. (iii),

$$F = 10.8 \times 10^6$$

so that

$$\theta = 10^{-8} (859 \cosh \mu z - 1.75z^2 + 10.8 \times 10^6) \quad (\text{iv})$$

At mid-span, where  $z = 0$ , from Eq. (iv),

$$\theta = 0.108 \text{ rad} \quad \text{or} \quad \theta = 6.2^\circ$$

## 27.4 SHEAR LAG

A problem closely related to the restrained torsion of rectangular section beams is generally known as *shear lag*. We saw, in Chapter 18, that torsion induces shear stresses in the walls of beams and these cause shear strains, which produce warping of the cross-section. When this warping is restrained, direct stresses are set up that modify the shear stresses. In a similar manner, the shear strains in the thin walls of beams subjected to shear loads cause cross-sections to distort or warp, so that the basic assumption of elementary bending theory of plane sections remaining plane is no longer valid. The direct and shear stress distributions predicted by elementary theory therefore become significantly inaccurate. Further modifications arise when any form of structural constraint prevents the free displacement of the cross-sections of a beam. Generally, shear lag becomes a problem in wide, relatively shallow, thin-walled beams, such as wings, in which the shear distortion of the thin upper and lower surface skins causes redistribution of stress in the stringers and spar caps, while the thicker and shallower spar webs experience little effect.

Consider the box beam shown in Fig. 27.10. Elementary bending theory predicts that the direct stress at any section AA is uniform across the width of the covers, so that all the stringers and web flanges are subjected to the same stress. However, the shear strains at the section cause the distortion shown, so that the intermediate stringers carry lower stresses than the web flanges. Since the resultant of the direct stresses must be equivalent to the applied bending moment, this means that the direct stresses in the web flanges must be greater than those predicted by elementary bending theory. Our

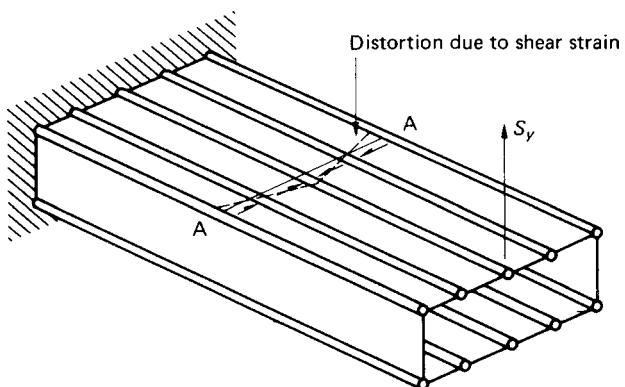
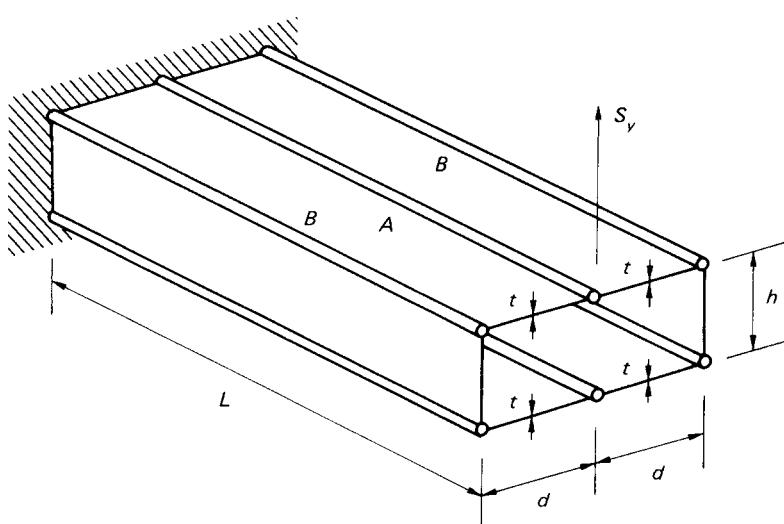


FIGURE 27.10 Shear Distortion in the Covers of a Box Beam



**FIGURE 27.11 Six-Boom Beam Subjected to a Shear Load**

investigation of the shear lag problem is restricted to idealized six- and eight-boom doubly symmetrical rectangular section beams subjected to shear loads acting in the plane of symmetry and in which the axis of twist, the flexural axis, and the zero warping axis coincide; the shear loads therefore produce no twist and hence no warping due to twist. In the analysis, we assume that the cross-sections of beams remain undistorted in their own plane.

Figure 27.11 shows an idealized six-boom beam built in at one end and carrying a shear load at the other; the corner booms have a cross-sectional area  $B$  while the central booms have a cross-sectional area  $A$ . At any section, the vertical shear load is shared equally by the two webs. Also, since the beam has been idealized, the shear flow at any section is constant between the booms, so that, for a web, the situation is that shown in the free body diagram of Fig. 27.12; in addition, the corner booms are subjected to equal and opposite loads  $P_B$ . The complementary shear flows  $S_y/2h$  are applied to the corner booms as shown so that the top cover, say, is subjected to loads as shown in Fig. 27.13. We assume that suitable edge members are present at the free end of the cover to equilibrate the shear flows; we also assume that strains in the transverse direction are negligible.

It is advantageous to adopt a methodical approach in the analysis. Thus, use may be made of the symmetry of the cover, so that only one edge boom, one panel, and the central boom need be considered as long as the symmetry is allowed for in the assumed directions of the panel shear flows  $q$ , as shown in Fig. 27.13. Further, the origin for  $z$  may be taken to be at either the free or built-in end. A marginally simpler solution is obtained if the origin is taken to be at the free end, in which case the solution represents that for an infinitely long panel. Considering the equilibrium of an element of an edge boom (Fig. 27.14), in which we assume that the boom load is positive (tension) and increases with increasing  $z$ , we have

$$P_B + \frac{\partial P_B}{\partial z} \delta z - P_B - q \delta z + \frac{S_y}{2h} \delta z = 0$$

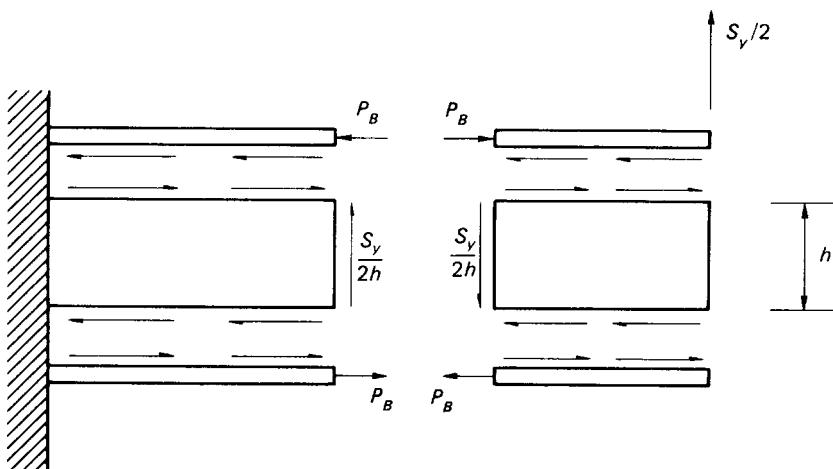


FIGURE 27.12 Loads on Webs and Corner Booms of the Beam of Fig. 27.11

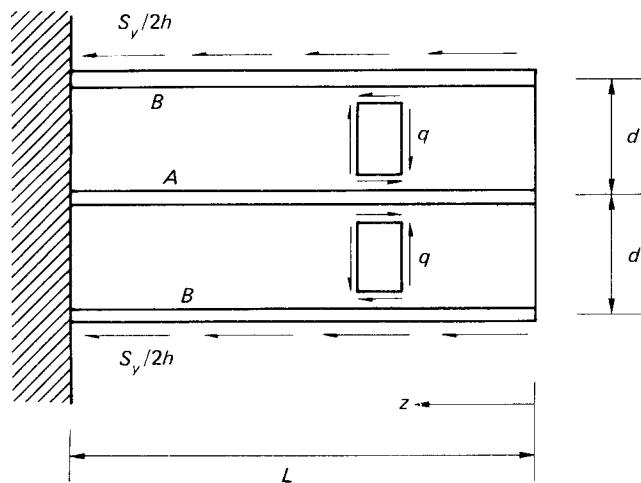


FIGURE 27.13 Top Cover of the Beam of Fig. 27.11

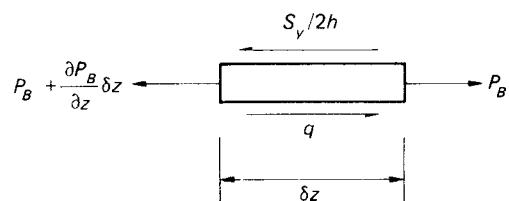
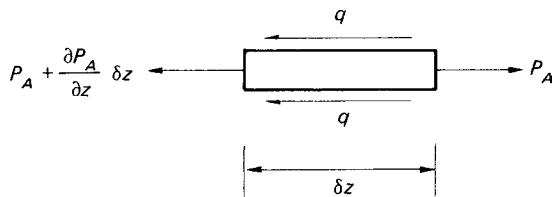


FIGURE 27.14 Equilibrium of Boom Element



**FIGURE 27.15 Equilibrium of Element of Central Boom**

or

$$\frac{\partial P_B}{\partial z} - q + \frac{S_y}{2h} = 0 \quad (27.24)$$

Similarly, for an element of the central boom (Fig. 27.15),

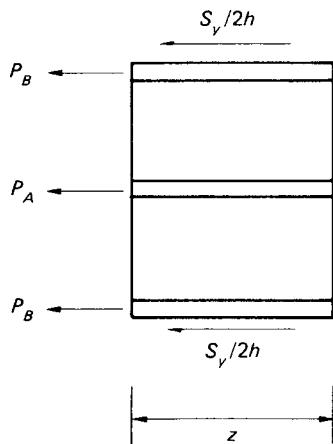
$$\frac{\partial P_A}{\partial z} + 2q = 0 \quad (27.25)$$

Now, considering the overall equilibrium of a length  $z$  of the cover (Fig. 27.16), we have

$$2P_B + P_A + \frac{S_y}{h}z = 0 \quad (27.26)$$

We now consider the compatibility condition that exists in the displacements of elements of the booms and adjacent elements of the panels. Figure 27.17(a) shows the displacements of the cover and an element of a panel and the adjacent elements of the boom. Note that the element of the panel is distorted in a manner that agrees with the assumed directions of the shear flows in Fig. 27.13 and that the shear strain increases with  $z$ . From Fig. 27.17(b),

$$(1 + \varepsilon_B)\delta z = (1 + \varepsilon_A)\delta z + d \frac{\partial \gamma}{\partial z} \delta z$$



**FIGURE 27.16 Equilibrium of a Length  $z$  of Cover**

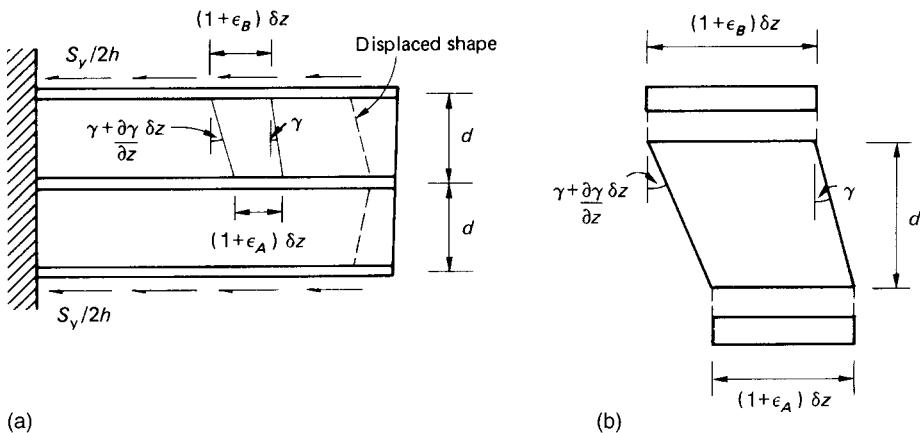


FIGURE 27.17 Compatibility Condition

in which  $\varepsilon_B$  and  $\varepsilon_A$  are the direct strains in the elements of boom. Then, rearranging and noting that  $\gamma$  is a function of  $z$  only when the section is completely idealized, we have

$$\frac{d\gamma}{dz} = \frac{1}{d}(\varepsilon_B - \varepsilon_A) \quad (27.27)$$

Now,

$$\varepsilon_B = \frac{P_B}{BE}, \quad \varepsilon_A = \frac{P_A}{AE}, \quad \gamma = \frac{q}{Gt}$$

so that Eq. (27.27) becomes

$$\frac{dq}{dz} = \frac{Gt}{dE} \left( \frac{P_B}{B} - \frac{P_A}{A} \right) \quad (27.28)$$

We now select the unknown to be determined initially. Generally, it is simpler mathematically to determine either of the boom load distributions,  $P_B$  or  $P_A$ , rather than the shear flow  $q$ . Thus, choosing  $P_A$ , say, as the unknown, we substitute in Eq. (27.28) for  $q$  from Eq. (27.25) and for  $P_B$  from Eq. (27.26). Hence,

$$-\frac{1}{2} \frac{\partial^2 P_A}{\partial z^2} = \frac{Gt}{dE} \left( -\frac{P_A}{2B} - \frac{S_y z}{2Bh} - \frac{P_A}{A} \right)$$

Rearranging, we obtain

$$\frac{\partial^2 P_A}{\partial z^2} - \frac{Gt(2B+A)}{dEBh} P_A = \frac{GtS_y z}{dEBh}$$

or

$$\frac{\partial^2 P_A}{\partial z^2} - \lambda^2 P_A = \frac{GtS_y z}{dEBh} \quad (27.29)$$

in which  $\lambda^2 = Gt(2B + A)/dEAB$ . The solution of Eq. (27.29) is of standard form and is

$$P_A = C \cosh \lambda z + D \sinh \lambda z - \frac{S_y A}{h(2B + A)} z$$

The constants  $C$  and  $D$  are determined from the boundary conditions of the cover of the beam, namely  $P_A = 0$  when  $z = 0$  and  $\gamma = q/Gt = -(\partial P_A / \partial z)/2Gt = 0$  when  $z = L$  (see Eq. (27.25)). From the first of these,  $C = 0$ , and from the second,

$$D = \frac{S_y A}{\lambda h(2B + A) \cosh \lambda L}$$

Thus,

$$P_A = -\frac{S_y A}{h(2B + A)} \left( z - \frac{\sinh \lambda z}{\lambda \cosh \lambda L} \right) \quad (27.30)$$

The direct stress distribution  $\sigma_A (= P_A/A)$  follows:

$$\sigma_A = -\frac{S_y}{h(2B + A)} \left( z - \frac{\sinh \lambda z}{\lambda \cosh \lambda L} \right) \quad (27.31)$$

The distribution of load in the edge booms is obtained by substituting for  $P_A$  from Eq. (27.30) in Eq. (27.26), thus,

$$P_B = -\frac{S_y B}{h(2B + A)} \left( z + \frac{A}{2B\lambda} \frac{\sinh \lambda z}{\cosh \lambda L} \right) \quad (27.32)$$

from which

$$\sigma_B = -\frac{S_y}{h(2B + A)} \left( z + \frac{A}{2B\lambda} \frac{\sinh \lambda z}{\cosh \lambda L} \right) \quad (27.33)$$

Finally, from either pairs of Eqs. (27.25) and (27.30) or (27.24) and (27.32),

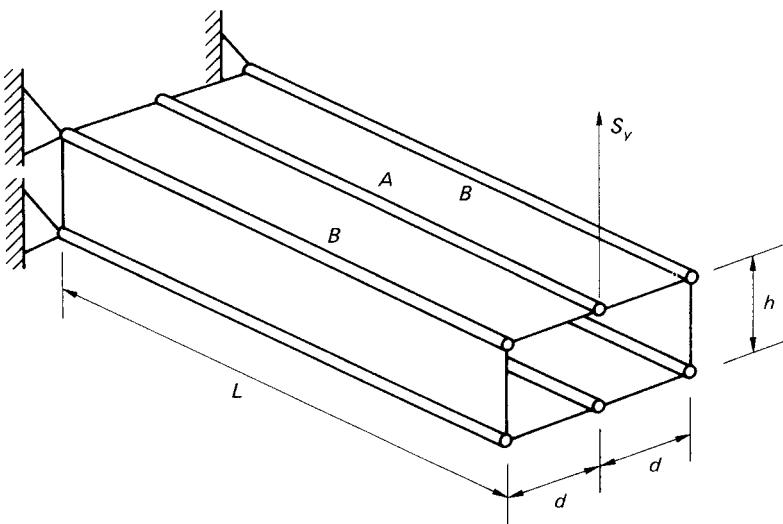
$$q = \frac{S_y A}{2h(2B + A)} \left( 1 - \frac{\cosh \lambda z}{\cosh \lambda L} \right) \quad (27.34)$$

so that the shear stress distribution  $\tau (= q/t)$  is

$$\tau = \frac{S_y A}{2ht(2B + A)} \left( 1 - \frac{\cosh \lambda z}{\cosh \lambda L} \right) \quad (27.35)$$

Elementary theory gives

$$\sigma_A = \sigma_B = -\frac{S_y z}{h(2B + A)} \quad (\text{see Eq. (16.9)})$$



**FIGURE 27.18 Rectangular Section Beam Supported at Corner Booms Only**

and

$$q = \frac{S_y A}{2h(2B + A)}$$

so that, as in the case of the torsion of a four-boom rectangular section beam, the solution comprises terms corresponding to elementary theory together with terms representing the effects of shear lag and structural constraint.

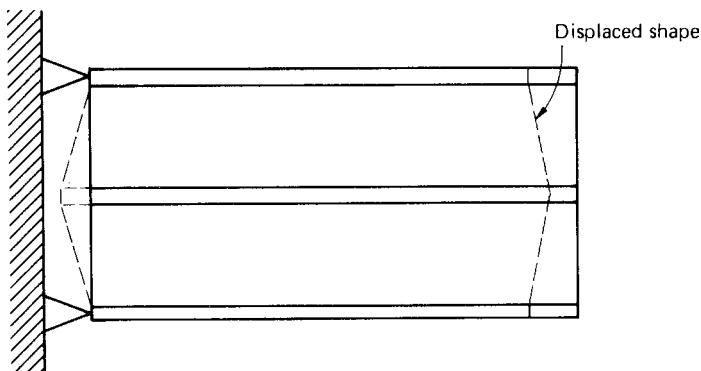
Many wing structures are spliced only at the spars, so that the intermediate stringers are not subjected to bending stresses at the splice. The situation for a six-boom rectangular section beam is then as shown in Fig. 27.18. The analysis is carried out in an identical manner to that in the previous case except that the boundary conditions for the central stringer are  $P_A = 0$  when  $z = 0$  and  $z = L$ . The solution is

$$P_A = -\frac{S_y A}{h(2B + A)} \left( z - L \frac{\sinh \lambda z}{\sinh \lambda L} \right) \quad (27.36)$$

$$P_B = -\frac{S_y B}{h(2B + A)} \left( z + \frac{AL}{2B} \frac{\sinh \lambda z}{\sinh \lambda L} \right) \quad (27.37)$$

$$q = \frac{S_y A}{2h(2B + A)} \left( 1 - \lambda L \frac{\cosh \lambda z}{\sinh \lambda L} \right) \quad (\text{see Ex. 22.2}) \quad (27.38)$$

where  $\lambda^2 = Gt(2B + A)/dEAB$ . Examination of Eq. (27.38) shows that  $q$  changes sign when  $\cosh \lambda z = (\sinh \lambda L)/\lambda L$ , the solution of which gives a value of  $z$  less than  $L$ , that is,  $q$  changes sign at some point along the length of the beam. The displaced shape of the top cover is therefore as shown in Fig. 27.19. Clearly, the final length of the central stringer is greater than in the previous case and appreciably



**FIGURE 27.19 Displaced Shape of top Cover of Box Beam of Fig. 27.18**

greater than the final length of the spar flanges. The shear lag effect is therefore greater than before. In some instances, this may be beneficial, since a larger portion of the applied bending moment is resisted by the heavier section spar flanges. These are also restrained against buckling in two directions by the webs and covers while the lighter section stringers are restrained in one direction only. The beam is therefore able to withstand higher bending moments than those calculated from elementary theory.

#### Example 27.4

A shallow box section beam whose cross-section is shown in Fig. 27.20 is simply supported over a span of 2 m and carries a vertically downward load of 20 kN at mid-span. Idealize the section into one suitable for shear lag analysis, comprising eight booms, and hence determine the distribution of direct stress along the top right-hand corner of the beam. Take  $G/E = 0.36$ .

The idealized section is shown in Fig. 27.21.

Using either Eq. (20.1) or (20.2),

$$B_1 = B_4 = B_8 = B_5 = \frac{100 \times 3}{6}(2 - 1) + \frac{200 \times 2}{6}(2 + 1) = 250 \text{ mm}^2$$

$$B_2 = B_3 = B_6 = B_7 = \frac{200 \times 2}{6}(2 + 1) \times 2 = 400 \text{ mm}^2$$



**FIGURE 27.20 Beam Section of Example 27.4**

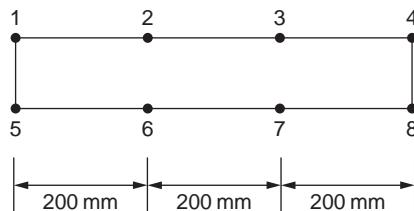


FIGURE 27.21 Idealized Beam Section of Example 27.4

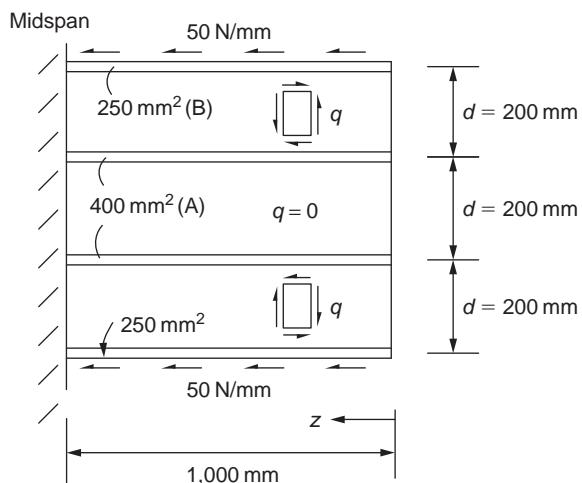


FIGURE 27.22 Shear Flows Acting on Top Cover of Idealized Beam Section of Example 27.4

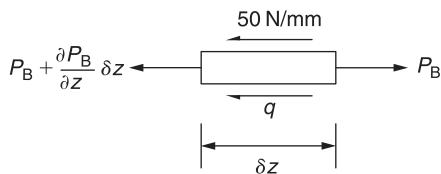


FIGURE 27.23 Element of Boom B

The support reactions of 10 kN produce loads of 5 kN on each vertical web. These, in turn, produce shear flows of 50 N/mm along each corner boom, as shown in Fig. 27.22 for the top cover of the beam.

Considering the equilibrium of elements of the booms, we have, for the top boom (Fig. 27.23),

$$P_B + \frac{\partial P_B}{\partial z} \delta z - P_B + q\delta z + 50\delta z = 0$$

which gives

$$\frac{\partial P_B}{\partial z} = -q - 50 \quad (\text{i})$$

Similarly, for an element of boom A,

$$\frac{\partial P_A}{\partial z} = q \quad (\text{ii})$$

Overall equilibrium of a length  $z$  of the panel gives

$$2P_B + 2P_A + 2 \times 50z = 0$$

that is,

$$P_B + P_A + 50z = 0 \quad (\text{iii})$$

The compatibility of displacement between elements of the boom and adjacent panel (Fig. 27.24) gives

$$\frac{\partial \gamma}{\partial z} = \frac{1}{d}(\varepsilon_A - \varepsilon_B) \quad (\text{iv})$$

But

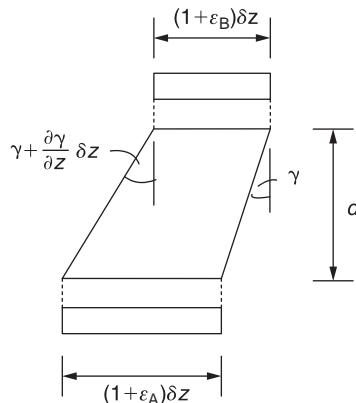
$$\varepsilon_A = P_A/AE_A, \quad \varepsilon_B = P_B/BE_B, \quad \gamma = q/Gt$$

Substituting in Eq. (iv), we obtain

$$\frac{\partial q}{\partial z} = \frac{Gt}{dE} \left( \frac{P_A}{A} - \frac{P_B}{B} \right) \quad (\text{v})$$

From Eq. (iii),

$$P_A = -P_B - 50z$$



**FIGURE 27.24** Compatibility Condition for the Top Cover of the Beam of Example 27.4

From Eq. (i),

$$\frac{\partial q}{\partial z} = -\frac{\partial^2 P_B}{\partial z^2}$$

Substituting in Eq. (v),

$$\frac{\partial^2 P_B}{\partial z^2} - \mu^2 P_B = \frac{50Gt}{dEA} z \quad (\text{vi})$$

in which

$$\mu^2 = \frac{Gt}{dE} \left( \frac{A+B}{AB} \right)$$

The solution to Eq. (vi) is

$$P_B = C \cosh \mu z + D \sinh \mu z - \frac{50B}{A+B} z$$

The boundary conditions are, when  $z = 0, P_B = 0$  and when  $z = 1000 \text{ mm}, \frac{\partial P_B}{\partial z} = -50$  (from Eq. (i), since  $q = 0$  at  $z = 1000 \text{ mm}$ )

From the first of these,  $C = 0$ , while from the second,

$$D = \frac{-50A}{(A+B)\mu \cosh 1000\mu}$$

Therefore,

$$\sigma_B = \frac{P_B}{B} = \frac{-50A}{B(A+B)\mu \cosh 1000\mu} \sinh \mu z - \frac{50}{A+B} z$$

Substituting the boom areas and the like gives

$$\sigma_B = -0.4 \sinh \mu z - 0.08z$$

in which

$$\mu = \left[ \frac{0.36 \times 2}{200} \frac{(400+250)}{(400 \times 250)} \right]^{1/2} = 4.84 \times 10^{-3}$$

In certain situations beams, or parts of beams, carry loads that cause in-plane bending of the covers. An example is shown in Fig. 27.25, where the loads  $P$  cause bending in addition to axial effects. Shear lag modifies the stresses predicted by elementary theory in a manner similar to the previous cases. From symmetry, we can consider either the top or bottom cover in isolation, as shown in Fig. 27.26(a). In this case, the load  $P$  causes bending as well as extension of the cover, so that, at any section  $z$ , the beam has a slope  $\partial v / \partial z$  (Fig. 27.26(b)). We again assume that transverse strains are negligible and that the booms carry all the direct load.

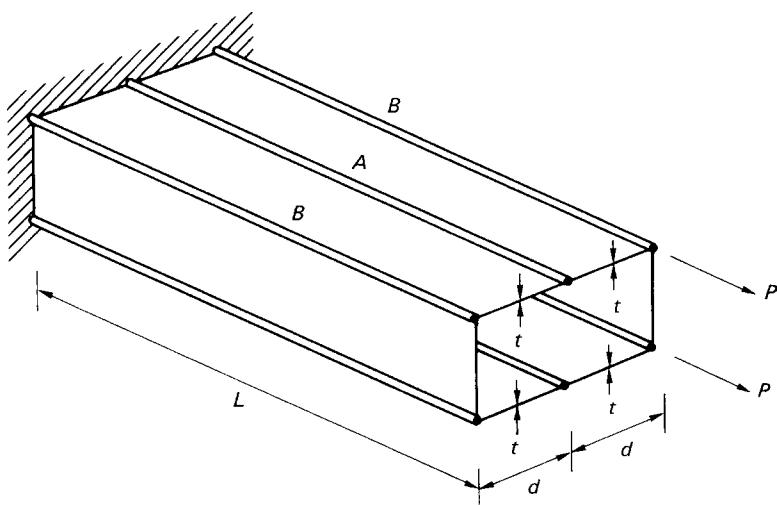


FIGURE 27.25 Beam Subjected to Combined Bending and Axial Load

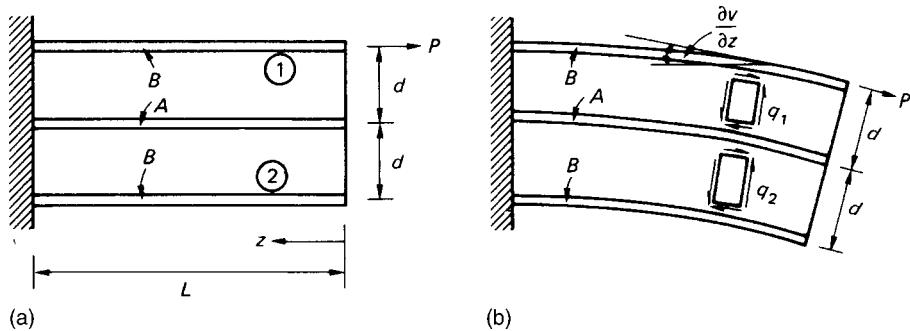


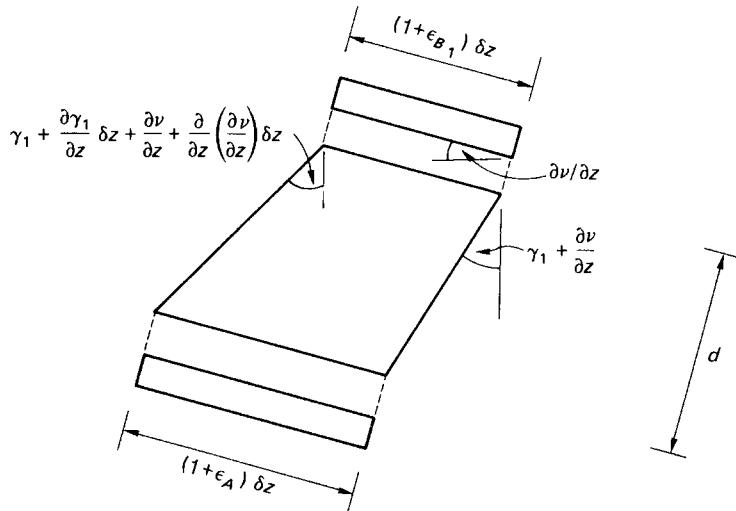
FIGURE 27.26 Cover of Beam of Fig. 27.25

Initially, as before, we choose directions for the shear flows in the top and bottom panels. Any directions may be chosen, since the question of symmetry does not arise. The equilibrium of an element  $\delta z$  of each boom is first considered, giving

$$\frac{\partial P_{B1}}{\partial z} = -q_1, \quad \frac{\partial P_A}{\partial z} = q_1 - q_2, \quad \frac{\partial P_{B2}}{\partial z} = q_2 \quad (27.39)$$

where  $P_{B1}$  is the load in boom 1 and  $P_{B2}$  is the load in boom 2. Longitudinal and moment equilibrium about boom 2 of a length  $z$  of the cover give, respectively,

$$P_{B1} + P_{B2} + P_A = P, \quad P_{B1} 2d + P_A d = P 2d \quad (27.40)$$



**FIGURE 27.27 Compatibility Condition for Combined Bending and Axial Load**

The compatibility condition now includes the effect of bending in addition to extension, as shown in Fig. 27.27. Note that the panel is distorted in a manner that agrees with the assumed direction of shear flow, and that  $\gamma_1$  and  $\partial v / \partial z$  increase with  $z$ . Thus,

$$(1 + \varepsilon_A)\delta z = (1 + \varepsilon_{B1})\delta z + d \left( \frac{d\gamma_1}{dz} + \frac{d^2 v}{dz^2} \right) \delta z$$

where  $\gamma_1$  and  $v$  are functions of  $z$  only. Thus,

$$\frac{d\gamma_1}{dz} = \frac{1}{d}(\varepsilon_A - \varepsilon_{B1}) - \frac{d^2 v}{dz^2} \quad (27.41)$$

Similarly, for an element of the lower panel,

$$\frac{d\gamma_2}{dz} = \frac{1}{d}(\varepsilon_{B2} - \varepsilon_A) - \frac{d^2 v}{dz^2} \quad (27.42)$$

Subtraction of Eq. (27.42) from Eq. (27.41) eliminates  $d^2 v / dz^2$ ; that is,

$$\frac{d\gamma_1}{dz} - \frac{d\gamma_2}{dz} = \frac{1}{d}(2\varepsilon_A - \varepsilon_{B1} - \varepsilon_{B2})$$

or, as before,

$$\frac{dq_1}{dz} - \frac{dq_2}{dz} = \frac{Gt}{dE} \left( \frac{2P_A}{A} - \frac{P_{B1}}{B} - \frac{P_{B2}}{B} \right) \quad (27.43)$$

In this particular problem, the simplest method of solution is to choose  $P_A$  as the unknown, since, from Eqs. (27.39),

$$\frac{dq_1}{dz} - \frac{dq_2}{dz} = \frac{\partial^2 P_A}{\partial z^2}$$

Also, substituting for  $P_{B1}$  and  $P_{B2}$  from Eq. (27.40), we obtain

$$\frac{\partial^2 P_A}{\partial z^2} - \frac{Gt}{dE} \left( \frac{2B + A}{AB} \right) P_A = -\frac{PGt}{dEB}$$

or

$$\frac{\partial^2 P_A}{\partial z^2} - \lambda^2 P_A = -\frac{PGt}{dEB} \quad (27.44)$$

where  $\lambda^2 = Gt(2B + A)/dEAB$ . The solution of Eq. (27.44) is of standard form and is

$$P_A = C \cosh \lambda z + D \sinh \lambda z + \frac{PA}{2B + A} \quad (27.45)$$

The boundary conditions are  $P_A = 0$  when  $z = 0$  and  $q_1 = q_2 = 0 = \partial P_A / \partial z$  at the built-in end (no shear loads are applied). Hence,

$$P_A = \frac{PA}{2B + A} (1 - \cosh \lambda z + \tanh \lambda L \sinh \lambda z)$$

or, rearranging,

$$P_A = \frac{PA}{2B + A} \left[ 1 - \frac{\cosh \lambda(L - z)}{\cosh \lambda L} \right] \quad (27.46)$$

Hence,

$$\sigma_A = \frac{P}{2B + A} \left[ 1 - \frac{\cosh \lambda(L - z)}{\cosh \lambda L} \right] \quad (27.47)$$

Substituting for  $P_A$  in the second of Eqs. (27.40), we have

$$P_{B1} = \frac{PA}{2(2B + A)} \left[ \frac{4B + A}{A} + \frac{\cosh \lambda(L - z)}{\cosh \lambda L} \right] \quad (27.48)$$

from which

$$\sigma_{B1} = \frac{PA}{2B(2B + A)} \left[ \frac{4B + A}{A} + \frac{\cosh \lambda(L - z)}{\cosh \lambda L} \right] \quad (27.49)$$

Also, from Eqs. (27.40),

$$P_{B2} = -\frac{P_A}{2}$$

so that

$$P_{B2} = \frac{-PA}{2(2B + A)} \left[ 1 - \frac{\cosh \lambda(L - z)}{\cosh \lambda L} \right] \quad (27.50)$$

and

$$\sigma_{B2} = \frac{-PA}{2B(2B + A)} \left[ 1 - \frac{\cosh \lambda(L - z)}{\cosh \lambda L} \right] \quad (27.51)$$

Finally, the shear flow distributions are obtained from Eqs. (27.39), thus,

$$q_1 = \frac{-\partial P_{B1}}{\partial z} = \frac{PA\lambda}{2(2B+A)} \frac{\sinh \lambda(L-z)}{\cosh \lambda L} \quad (27.52)$$

$$q_2 = \frac{\partial P_{B2}}{\partial z} = \frac{-PA\lambda}{2(2B+A)} \frac{\sinh \lambda(L-z)}{\cosh \lambda L} \quad (27.53)$$

Again, we see that each expression for direct stress, Eqs. (27.47), (27.49), and (27.51), comprises a term that gives the solution from elementary theory together with a correction for the shear lag effect.

For example, the loading on the cover is equivalent to an axial load  $P$  together with a bending moment  $Pd$ . The direct stress in boom 1, say, is then, from elementary theory (Ref. [2]),

$$\sigma_{z(1)} = \frac{P}{A+2B} + \frac{Pd(d)}{I}$$

where  $I = 2Bd^2$  (about an axis into the plane of the paper). Therefore,

$$\sigma_{z(1)} = \frac{P}{A+2B} + \frac{Pd^2}{2Bd^2} = \frac{P(4B+A)}{2B(2B+A)}$$

The shear flows  $q_1$  and  $q_2$  are self-equilibrating, as can be seen from Eqs. (27.52) and (27.53), and are entirely produced by the shear lag effect ( $q_1$  and  $q_2$  must be self-equilibrating, since no shear loads are applied).

### Example 27.5

The unsymmetrical panel shown in Fig. 27.28 comprises three direct stress carrying booms and two shear stress carrying panels. If the panel supports a load  $P$  at its free end and is pinned to supports at the ends of its outer booms,

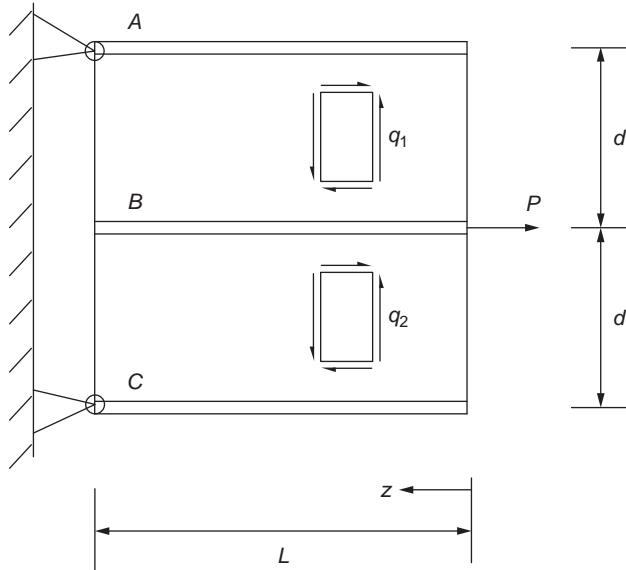


FIGURE 27.28 Panel of Example 27.5

determine the distribution of direct load in the central boom. Determine also the load in the central boom when  $A = B = C$  and shear lag effects are absent.

As before, we consider the equilibrium of elements of the booms, say,  $A$  and  $B$ . This gives

$$\frac{\partial P_A}{\partial z} = -q_1 \quad (\text{i})$$

and

$$\frac{\partial P_B}{\partial z} = q_1 - q_2 \quad (\text{ii})$$

For overall equilibrium of a length  $z$  of the panel,

$$P_A + P_B + P_C = P \quad (\text{iii})$$

and, taking moments about boom  $C$ ,

$$2P_A + P_B = P \quad (\text{iv})$$

The compatibility condition is shown in Fig. 27.29 and gives

$$\frac{\partial \gamma_1}{\partial z} = \frac{1}{d}(\varepsilon_A - \varepsilon_B) - \frac{\partial^2 v}{\partial z^2} \quad (\text{v})$$

Similarly, for elements of the booms  $B$  and  $C$ ,

$$\frac{\partial \gamma_2}{\partial z} = \frac{1}{d}(\varepsilon_C - \varepsilon_B) - \frac{\partial^2 v}{\partial z^2} \quad (\text{vi})$$

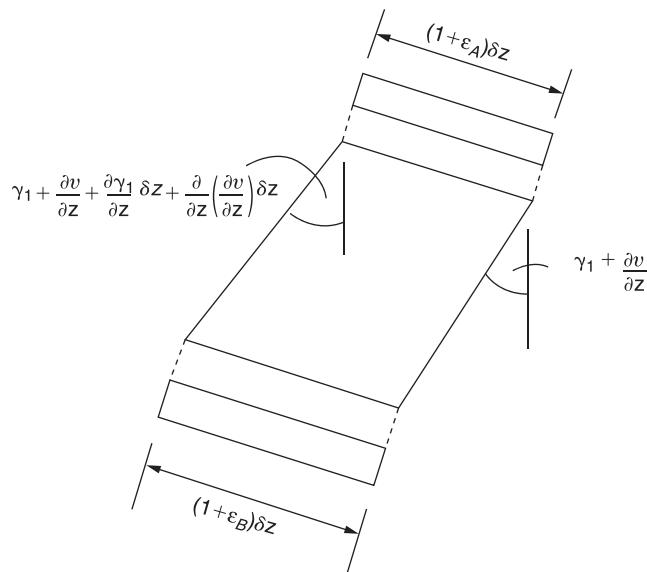


FIGURE 27.29 Compatibility Condition for the Panel of Example 27.5

Subtracting Eq. (vi) from (v) gives

$$\frac{\partial \gamma_1}{\partial z} - \frac{\partial \gamma_2}{\partial z} = \frac{1}{d}(2\varepsilon_B - \varepsilon_A - \varepsilon_C) \quad (\text{vii})$$

Also,

$$\gamma_1 = \frac{q_1}{Gt}, \quad \gamma_2 = \frac{q_2}{Gt}, \quad \varepsilon_A = \frac{P_A}{AE}, \quad \varepsilon_B = \frac{P_B}{BE}, \quad \text{and} \quad \varepsilon_C = \frac{P_C}{CE}$$

Substituting these expressions in Eq. (vii) gives

$$\frac{\partial q_1}{\partial z} - \frac{\partial q_2}{\partial z} = \frac{Gt}{dE} \left( \frac{2P_B}{B} - \frac{P_A}{A} - \frac{P_C}{C} \right) \quad (\text{viii})$$

From Eqs. (iv) and (iii),

$$P_A = \frac{1}{2}(P - P_B), \quad P_C = \frac{1}{2}(P - P_B)$$

Substituting in Eq. (viii), using Eq. (ii), and rearranging, we have

$$\frac{\partial^2 P_B}{\partial z^2} - \frac{Gt}{dE} \left( \frac{4AC + BC + AB}{2ABC} \right) P_B = -\frac{GtP}{2dE} \left( \frac{A + C}{AC} \right)$$

the solution of which is

$$P_B = D \cosh \mu z + F \sinh \mu z + \frac{B(A + C)P}{(4AC + BC + AB)}$$

where

$$\mu^2 = \frac{Gt}{dE} \left( \frac{4AC + BC + AB}{2ABC} \right)$$

The boundary conditions are when  $z = 0, P_B = P$ , and when  $z = L, P_B = 0$ . From the first of these,

$$D = \frac{4AC}{4AC + BC + AB} P$$

while from the second,

$$F = -\frac{P}{\sinh \mu L} \left[ \frac{4AC}{4AC + BC + AB} \cosh \mu L + \frac{B(A + C)}{4AC + BC + AB} \right]$$

The expression for the load in the central boom is then

$$P_B = \frac{P}{4AC + BC + AB} \left[ 4AC \cosh \mu z - \left( \frac{4AC \cosh \mu L + AB + BC}{\sinh \mu L} \right) \times \sinh \mu z + B(A + C) \right]$$

If there is no shear lag, the hyperbolic terms disappear and, when  $A = B = C$ ,

$$P_B = P/3$$

## Reference

- [1] Argyris JH, Dunne PC. The general theory of cylindrical and conical tubes under torsion and bending loads. J Roy Aero Soc Parts I–IV, February 1947; Part V, September–November 1947; Part VI, May–June 1949.  
[2] Megson THG. Structural and stress analysis. 4th ed. Oxford: Elsevier; 2019.

## PROBLEMS

**P.27.1** A thin-walled beam with the singly symmetrical cross-section, shown in Fig. P.27.1, is built in at one end, where the shear force  $S_y = 111,250 \text{ N}$  is applied through web 25. Assuming the cross-section remains undistorted by the loading, determine the shear flow and the position of the center of twist at the built in end. The shear modulus  $G$  is the same for all walls.

Answer:  $q_{12} = q_{56} = 46.6 \text{ N/mm}$ ,  $q_{52} = 180.8 \text{ N/mm}$   
 $q_{32} = q_{54} = 1.4 \text{ N/mm}$ ,  $q_{43} = 74.6 \text{ N/mm}$   
 $x_R = -630.1 \text{ mm}$ ,  $y_R = 0$  (relative to the mid-point of 52)

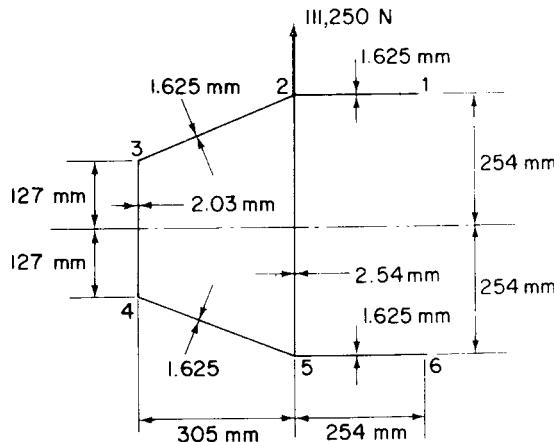


FIGURE P.27.1

**P.27.2** A thin-walled two-cell beam with the singly symmetrical cross-section shown in Fig. P.27.2 is built-in at one end, where the torque is 11,000 Nm. Assuming the cross-section remains undistorted by the loading, determine the distribution of shear flow and the position of the center of twist at the built-in end. The shear modulus  $G$  is the same for all walls.

Answer:  $q_{12} = q_{45} = 44.1 \text{ N/mm}$ ,  $q_{23} = q_{34} = 42.9 \text{ N/mm}$   
 $q_{51} = 80.2 \text{ N/mm}$ ,  $q_{24} = 37.4 \text{ N/mm}$   
 $x_R = -79.5 \text{ mm}$ ,  $y_R = 0$  (referred to the mid-point of web 24)

**P.27.3** A singly symmetrical, thin-walled, closed section beam is built in at one end, where a shear load of 10,000 N is applied as shown in Fig. P.27.3. Calculate the resulting shear flow distribution at the

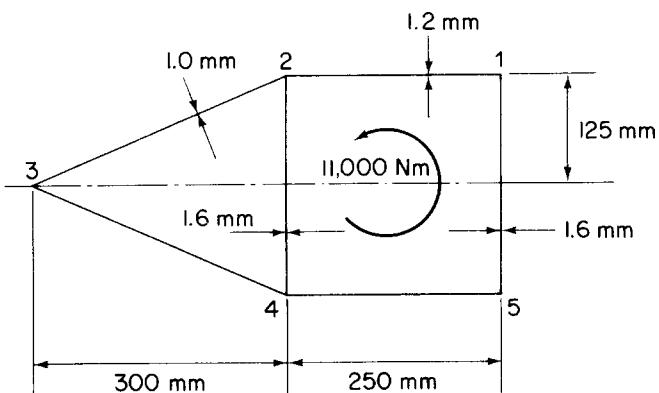


FIGURE P.27.2

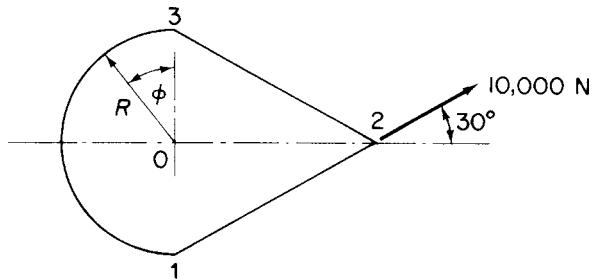


FIGURE P.27.3

built-in end if the cross-section of the beam remains undistorted by the loading and the shear modulus  $G$  and wall thickness  $t$  are each constant throughout the section.

$$\text{Answer: } q_{12} = 3992.9/R \text{ N/mm}, q_{23} = 711.3/R \text{ N/mm}$$

$$q_{31} = (1502.4 - 1894.7 \cos \phi - 2102.1 \sin \phi)/R \text{ N/mm}$$

**P.27.4** A uniform, four-boom beam, built in at one end, has the rectangular cross-section shown in Fig. P.27.4. The walls are assumed to be effective only in shear, the thickness and shear modulus being the same for all walls, while the booms, which are of equal area, carry only direct stresses. Assuming that the cross-section remains undistorted by the loading, calculate the twist at the free end due to a uniformly distributed torque loading  $T = 20 \text{ Nm/mm}$  along its entire length. Take  $G = 20,000 \text{ N/mm}^2$  and  $G/E = 0.36$ .

Answer:  $5.9^\circ$  counterclockwise

**P.27.5** Figure P.27.5 shows the doubly symmetrical idealized cross-section of a uniform box beam of length  $l$ . Each of the four corner booms has area  $B$  and Young's modulus  $E$ , and they constitute the

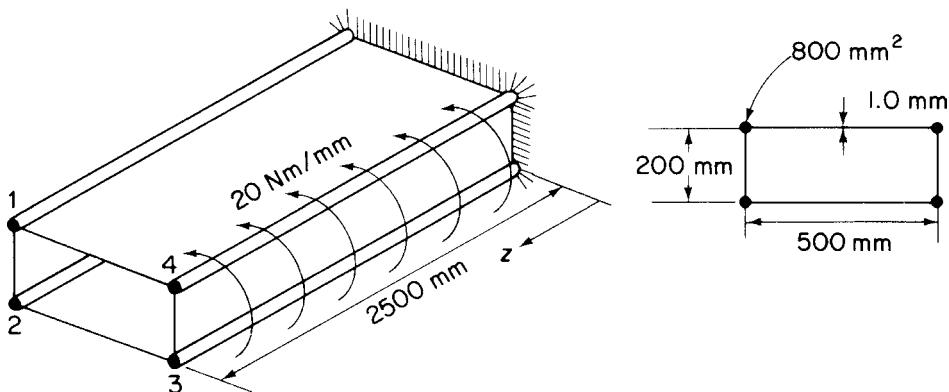


FIGURE P.27.4

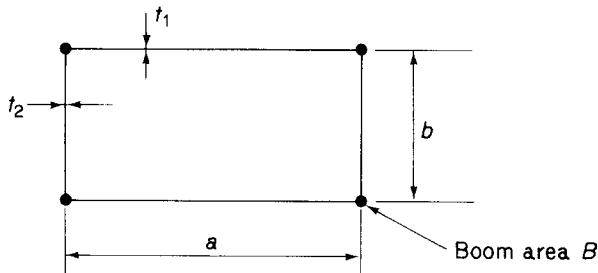


FIGURE P.27.5

entire direct stress carrying area. All the thin walls have the same shear modulus  $G$ . The beam transmits a torque  $T$  from one end to the other, and at each end, warping is completely suppressed. Between the ends, the shape of the cross-section is maintained without further restriction of warping. Obtain an expression for the distribution of the end load along the length of one of the corner booms. Assuming  $bt_1 > at_2$ , indicate graphically the relation between torque direction and tension and compression in the boom end loads.

$$\text{Answer: } P = \frac{\mu BET}{8abGt_1t_2} (bt_1 - at_2) \left[ -\sinh \mu z + \frac{(\cosh \mu l - 1)}{\sinh \mu l} \cosh \mu z \right]$$

where

$$\mu^2 = 8Gt_1t_2/B(E(at_2 + bt_1)).$$

**P.27.5 MATLAB** Using the results of Problem P.27.5 and assuming  $bt_1 > at_2$ , use MATLAB to graphically show the relationship between the torque direction and tension and compression in the boom end

loads using the variable values that follow. Also, calculate the value and location of the maximum and minimum end load for  $\pm T$ .

$$G = 36,400 \text{ N/mm}^2, \quad t_1 = 2 \text{ mm}, \quad t_2 = 1 \text{ mm}, \quad B = 1290 \text{ mm}^2, \quad E = 200,000 \text{ N/mm}^2 \\ a = 20 \text{ mm}, \quad b = 30 \text{ mm}, \quad l = 1 \text{ m}, \quad T = \pm 1 \text{ Nmm}$$

Answer:  $T < 0: P_{\max} = 0.0496 \text{ N}(z = 1 \text{ m}), \quad P_{\min} = -0.0496 \text{ N}(z = 0 \text{ m})$   
 $T > 0: P_{\max} = 0.0496 \text{ N}(z = 0 \text{ m}), \quad P_{\min} = -0.0496 \text{ N}(z = 1 \text{ m})$

**P.27.6** The idealized cross-section of a beam is shown in Fig. P.27.6. The beam is of length  $L$  and is attached to a flexible support at one end, which only partially prevents warping of the cross-section; at its free end, the beam carries a concentrated torque  $T$ . Assuming that the warping at the built-in end is directly proportional to the free warping, that is,  $w = kw_0$ , derive an expression for the distribution of direct stress along the top right-hand corner boom. State the conditions corresponding to the values  $k = 0$  and  $k = 1$ .

Answer:  $\sigma = -\mu E w_0 (k-1) \frac{\sinh \mu(L-z)}{\cosh \mu L}, \quad \mu^2 = \frac{8Gt_b t_a}{BE(bt_a + at_b)}$

when  $k = 0$ ,  $\sigma = \mu E w_0 \frac{\sinh \mu(L-z)}{\cosh \mu L}$  (i.e., a rigid foundation)

when  $k = 1$ ,  $\sigma = 0$  (i.e., free warping)

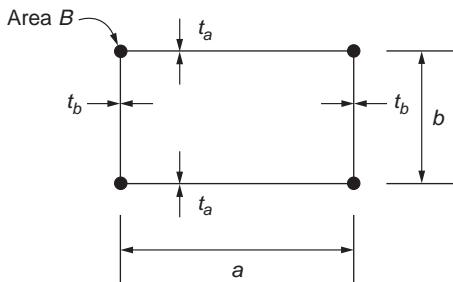


FIGURE P.27.6

**P.27.7** In the panel shown in Fig. P.27.7 the area,  $A_s$ , of the central stringer is to be designed so that the stress in it is 80 percent of the constant stress,  $\sigma_e$ , in the edge members, each of area  $B$ . Assuming that the sheet, which is of constant thickness,  $t$ , carries only shear stress and that transverse strains are prevented, derive expressions for  $A_s$  and  $B$  in terms of the applied loads and the appropriate elastic moduli,  $E$  for the longitudinal members and  $G$  for the sheet. Evaluate these expressions in the case where  $P = 450,000 \text{ N}$ ;  $P_s = 145,000 \text{ N}$ ;  $S = 350 \text{ N/mm}$ ;  $\sigma_e = 275 \text{ N/mm}^2$ ;  $l = 1,250 \text{ mm}$ ;  $b = 250 \text{ mm}$ ;  $t = 2.5 \text{ mm}$ ; and  $G = 0.38E$ . Find the fraction of the total tension at the abutment carried by the stringer.

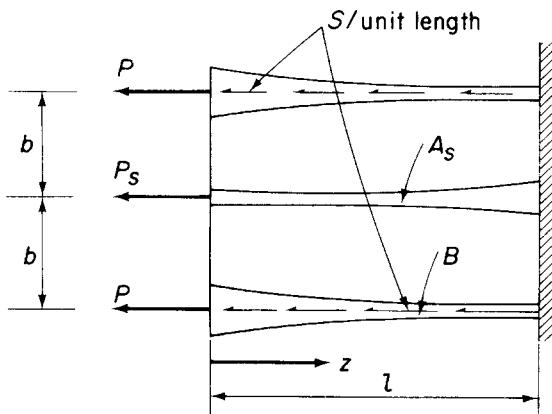


FIGURE P.27.7

$$\text{Answer: } A_s = \frac{Gt}{2Eb} \left( lz - \frac{z^2}{2} \right) + \frac{1.25P_s}{\sigma_e}$$

$$B = \frac{0.1Gt}{Eb} z^2 + \frac{1}{\sigma_e} \left[ \left( S - \frac{0.2Gt\sigma_e l}{bE} \right) z + P \right], \quad 0.25$$

**P.27.8** A symmetrical panel has the form shown in Fig. P.27.8. The longerons are of constant area,  $B_1$  for the edge members and  $B_2$  for the central member, and the sheet is of uniform thickness  $t$ . The panel is assembled without stress. Obtain an expression for the distribution of end load in the central longeron if it is then raised to a temperature  $T$  (constant along its length) above the edge members. Also, give the longitudinal displacement, at one end of the panel, of the central longeron relative to the edge members. Assume that end loads are carried only by the longerons, that the sheet carries only shear, and that transverse members are provided to prevent transverse straining and to ensure shear effectiveness of the sheet at the ends of the panel.

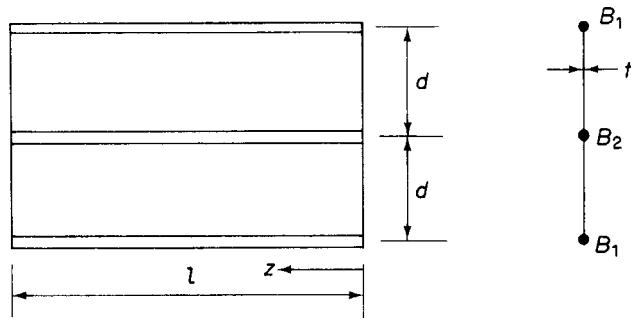


FIGURE P.27.8

$$\text{Answer: } P_2 = E\alpha T \left( \cosh \mu z - \tanh \frac{\mu l}{2} \sinh \mu z - 1 \right) \left/ \left( \frac{1}{2B_1} + \frac{1}{B_2} \right) \right.$$

$$\text{Displacement} = \frac{\alpha T}{\mu} \tanh \frac{\mu l}{2}$$

where

$$\mu^2 = \frac{2Gt}{dE} \left( \frac{1}{2B_1} + \frac{1}{B_2} \right).$$

**P.27.9** The flat panel shown in Fig. P.27.9 comprises a sheet of uniform thickness  $t$ , a central stringer of constant area  $A$ , and edge members of varying area. The panel is supported on pinned supports and is subjected to externally applied shear flows  $S_1$  and  $S_2$ , together with end loads  $P_{1,0}$  and  $P_{2,0}$ , as shown. The areas of the edge members vary such that the direct stresses  $\sigma_1$  and  $\sigma_2$  in the edge members are constant. Assuming that transverse strains are prevented, that the sheet transmits shear stress only, and that each part has suitable end members to take the complementary shear stresses, derive expressions for the variation of direct stress  $\sigma_3$  in the stringer and for the variation of shear flow in the upper panel in terms of the dimensions given and the elastic moduli  $E$  and  $G$  for the material.

$$\text{Answer: } \sigma_3 = \left( \frac{\sigma_1 + \sigma_2}{2} \right) \left[ 1 - \cosh \mu z - \frac{\sinh \mu z}{\sinh \mu l} (1 - \cosh \mu l) \right]$$

$$q_1 = A \left( \frac{\sigma_1 + \sigma_2}{4} \right) \mu \left[ \sinh \mu z + \frac{\cosh \mu z}{\sinh \mu l} (1 - \cosh \mu l) \right]$$

where

$$\mu^2 = 2Gt/bAE$$

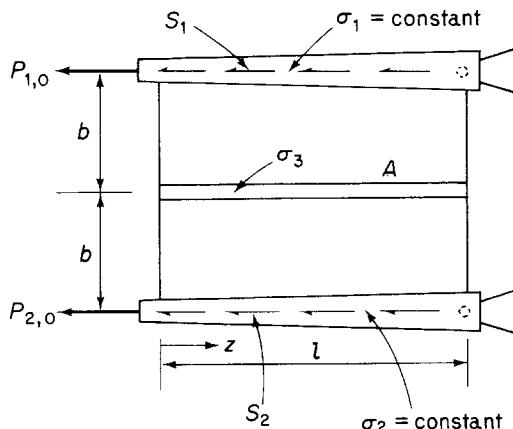
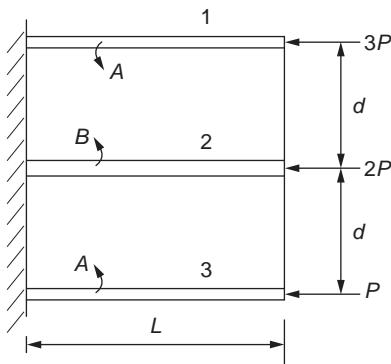


FIGURE P.27.9

**FIGURE P.27.10**

**P.27.10** The panel shown in Fig. P.27.10 is idealized into a combination of direct stress carrying booms and shear stress carrying plates; the boom areas are shown and the plate thickness is  $t$ . Derive expressions for the distribution of a direct load in each boom and state how the load distributions are affected when  $A = B$ .

$$\text{Answer: } P_1 = \frac{6P}{2A+B} \left[ -\left( \frac{B+8A}{6} \right) - \left( \frac{B-A}{3} \right) \frac{\cosh \mu(L-z)}{\cosh \mu L} \right]$$

$$P_2 = \frac{6P}{2A+B} \left[ -B + \frac{2}{3}(B-A) \frac{\cosh \mu(L-z)}{\cosh \mu L} \right]$$

$$P_3 = \frac{6P}{2A+B} \left[ -\left( \frac{4A-B}{6} \right) - \left( \frac{B-A}{3} \right) \frac{\cosh \mu(L-z)}{\cosh \mu L} \right]$$

when  $A = B$ ,  $P_1 = -3P$ ,  $P_2 = -2P$ ,  $P_3 = -P$ , that is, no shear lag.

**P.27.11** A uniform cantilever of length  $l$  has the doubly symmetrical cross-section shown in Fig. P.27.11. The section shape remains undistorted in its own plane after loading. Direct stresses in the cross-section are carried only in the concentrated longeron areas shown, and the wall thickness dimensions given relate only to shearing effects. All longerons have the same Young's modulus  $E$  and all walls the same effective shear modulus  $G$ . The root of the cantilever is built in, warping being completely suppressed there, and a shearing force  $S$  is applied at the tip in the position indicated. Derive an expression for the resultant end load in a corner longeron. Also calculate the resultant deflection of the tip, including the effects of both direct and shear strains.

$$\text{Answer: } P = -\frac{S}{8h} \left( \frac{\sinh \mu z}{\mu \cosh \mu l} + 3z \right)$$

where  $\mu^2 = 4Gt/3dBE$  (top right hand) (origin for  $z$  at free end)

$$\text{Deflection} = \frac{Sl}{12h} \left( \frac{11}{4Gt} + \frac{l^2}{EBh} \right)$$

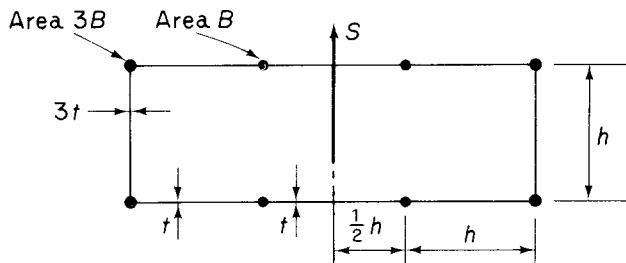


FIGURE P.27.11

**P.27.12** The idealized cantilever beam shown in Fig. P.27.12 carries a uniformly distributed load of intensity  $w$ . Assuming that all direct stresses are carried by the booms while the panels are effective only in shear, determine the distribution of direct load in the central boom in the top cover. Young's modulus for the booms is  $E$ , and the shear modulus of the walls is  $G$ .

$$\text{Answer: } P_A = -\frac{wA}{h(2B+A)} \left[ \frac{\cosh \mu z}{\mu^2} + \left( \frac{\mu L - \sinh \mu L}{\mu^2 \cosh \mu L} \right) \sinh \mu z - \frac{1}{\mu^2} - \frac{z^2}{2} \right]$$

where

$$\mu^2 = \frac{Gt(2B+A)}{dEAB}$$

**P.27.12 MATLAB** Use MATLAB to graphically illustrate the distribution of  $P_A$  using the variable values that follow and the results of Problem P.27.12. Also, calculate the value and location of the maximum and minimum  $P_A$ .

$$G = 36,400 \text{ N/mm}^2, \quad t = 1.5 \text{ mm}, \quad A = 900 \text{ mm}^2, \quad B = 870 \text{ mm}^2, \quad E = 200,000 \text{ N/mm}^2 \\ h = 50 \text{ mm}, \quad d = 60 \text{ mm}, \quad L = 0.5 \text{ m}, \quad w = 50 \text{ N/mm}$$

$$\text{Answer: } P_{A,\max} = 40,088.18 \text{ N (} z = 0.5 \text{ m}), \quad P_{A,\min} = 0 \text{ N (} z = 0 \text{ m)}$$

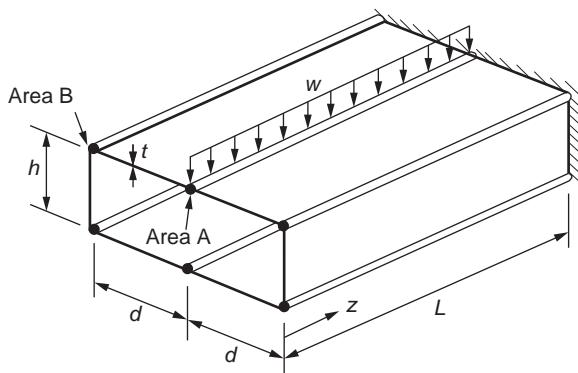


FIGURE P.27.12

**P.27.13** Figure P.27.13 shows a portion of the undersurface of a wing comprising two outer longerons, two central stringers, and skin panels. The wing includes a cut-out for the undercarriage bay and may be assumed to be attached to a rigid bulkhead at its left-hand end. Aerodynamic calculations show that for a particular loading case, the end loads in the outer longerons are equal to  $P_0$ , as is the end load on the central stringer. If the wing skin is effective only in shear and has a shear modulus  $G$  while the direct stresses are carried by the longerons and stringer, each having a Young's modulus  $E$ , determine the distribution of load in the stringer 9–10. What are the values of the loads in portions 32 and 67 of the longerons?

Answer:

$$P_{(stringer\ 9-10)} = \frac{P_0}{2A+B} \left\{ 2(A-B) \cosh \mu z - [2(A-B) \cosh \mu L + 3B] \frac{\sinh \mu z}{\sinh \mu L} + 3B \right\}$$

$$P_{(longerons\ 23\ and\ 67)} = 1.5P_0$$

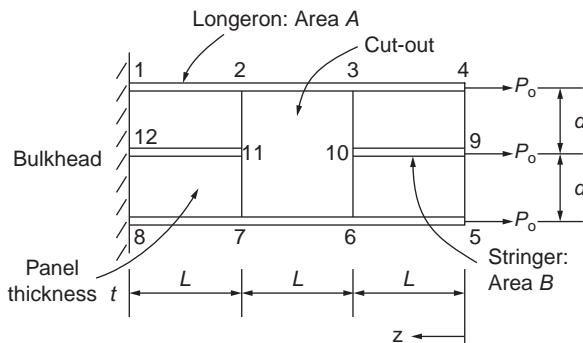


FIGURE P.27.13

# Open section beams

# 28

Instances of open section beams occurring in isolation are infrequent in aircraft structures. The majority of wing structures, however, contain cut-outs for undercarriages, inspection panels, and the like, so that, at these sections, the wing is virtually an open section beam. We saw in [Chapter 23](#) that one method of analysis for such cases is to regard the applied torque as being resisted by the differential bending of the front and rear spars in the cut-out bay. An alternative approach is to consider the cut-out bay as an open section beam built-in at each end and subjected to a torque. We now investigate the method of analysis of such beams.

## 28.1 I-SECTION BEAM SUBJECTED TO TORSION

If such a beam is axially unconstrained and loaded by a pure torque  $T$ , the rate of twist is constant along the beam and is given by

$$T = GJ \frac{d\theta}{dz} \quad (\text{from Eq. (18.12)})$$

We also showed, in [Section 18.2](#), that the shear stress varies linearly across the thickness of the beam wall and is zero at the middle plane ([Fig. 28.1](#)). It follows that, although the beam and the middle plane warp (we are concerned here with primary warping), there is no shear distortion of the middle plane. The mechanics of this warping are more easily understood by reference to the thin-walled I-section beam of [Fig. 28.2\(a\)](#). A plane view of the beam ([Fig. 28.2\(b\)](#)) reveals that the middle plane of each flange remains rectangular, although twisted, after torsion. We now observe the effect of applying a restraint to one end of the beam. The flanges are no longer free to warp and bend in their own planes into the shape shown in plane in [Fig. 28.3](#). Obviously, the beam still twists along its length, but the rate of twist is no longer constant, and the resistance to torsion is provided by the St. Venant shear stresses (unrestrained warping) plus the resistance of the flanges to bending. The total torque may therefore be written  $T = T_J + T_F$ , where  $T_J = GJ d\theta/dz$  from the unconstrained torsion of open sections but in which  $d\theta/dz$  is not constant, and  $T_F$  is obtained from a consideration of the bending of the flanges. It is instructive to derive an expression for  $T_F$  for the I-section beam of [Fig. 28.4](#) before we turn our attention to the case of a beam of arbitrary section.

Suppose that, at any section  $z$ , the angle of twist of the I-beam is  $\theta$ . Then the lateral displacement  $u$  of the lower flange is

$$u = \theta \frac{h}{2}$$

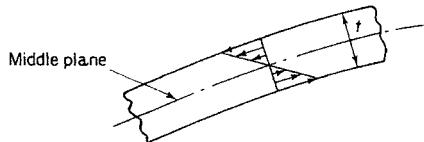


FIGURE 28.1 Shear Stress Distribution Across the Wall of an Open Section Beam Subjected to Torsion

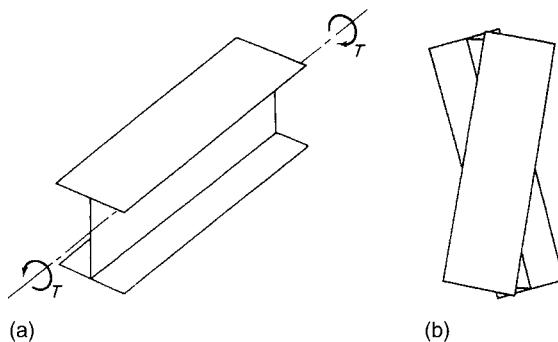


FIGURE 28.2 (a) Torsion of I-Section Beam; (b) Plan View of Beam Showing Undistorted Shape of Flanges

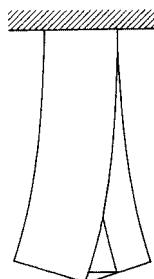
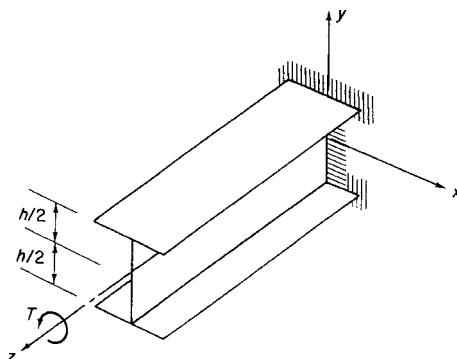


FIGURE 28.3 Bending Effect of Axial Constraint on Flanges of I-Section Beam Subjected to Torsion

and the bending moment  $M_F$  in the plane of the flange is given by

$$M_F = -EI_F \frac{d^2u}{dz^2}$$

where  $I_F$  is the second moment of area of the *flange* cross-section about the  $y$  axis. It is assumed here that displacements produced by shear are negligible, so that the lateral deflection of the flange is completely due to the self-equilibrating direct stress system  $\sigma_T$  set up by the bending of the flange.



**FIGURE 28.4 Torsion of I-Section Beam Fully Built in at One End**

We shall not, however, assume that the shear stresses in the flange are negligible. The shear  $S_F$  in the flange is then

$$S_F = \frac{dM_F}{dz} = -EI_F \frac{d^3u}{dz^3}$$

or, substituting for  $u$  in terms of  $\theta$  and  $h$ ,

$$S_F = -EI_F \frac{h d^3\theta}{2 dz^3}$$

Similarly, there is a shear force in the top flange of the same magnitude but opposite in direction. Together they form a couple that represents the second part  $T_F$  of the total torque, thus,

$$T_F = S_F h = -EI_F \frac{h^2 d^3\theta}{2 dz^3}$$

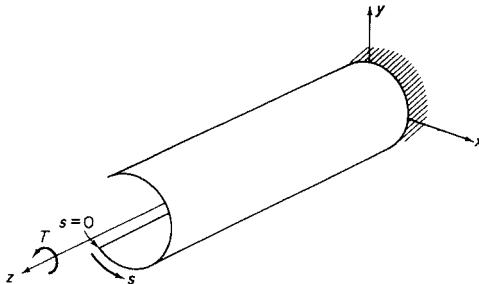
and the expression for the total torque may be written

$$T = GJ \frac{d\theta}{dz} - EI_F \frac{h^2 d^3\theta}{2 dz^3}$$

## 28.2 TORSION OF AN ARBITRARY SECTION BEAM

The insight into the physical aspects of the problem gained in the preceding will be found helpful in the development of the general theory for the arbitrary section beam shown in Fig. 28.5.

The theory, originally developed by Wagner and Kappus, is most generally known as the *Wagner torsion bending theory*. It assumes that the beam is long compared with its cross-sectional dimensions, that the cross-section remains undistorted by the loading, and that the shear strain  $\gamma_{zs}$  of the middle plane of the beam is negligible, although the stresses producing the shear strain are



**FIGURE 28.5 Torsion of an Open Section Beam Fully Built in at One End**

not. From similar assumptions is derived, in [Section 18.2.1](#), an expression for the primary warping  $w$  of the beam, namely

$$w = -2A_R \frac{d\theta}{dz} \quad (\text{Eq. (18.19)})$$

In the presence of axial constraint,  $d\theta/dz$  is no longer constant, so that the longitudinal strain  $\partial w/\partial z$  is not zero and direct (also shear) stresses are induced. Then,

$$\sigma_T = E \frac{\partial w}{\partial z} = -2A_R E \frac{d^2\theta}{dz^2} \quad (28.1)$$

The  $\sigma_T$  stress system must be self-equilibrating, since the applied load is a pure torque. Therefore, at any section, the resultant end load is zero and

$$\int_c \sigma_T t \, ds = 0$$

( $\int_c$  denotes integration around the beam section), or, from [Eq. \(28.1\)](#) and observing that  $d^2\theta/dz^2$  is a function of  $z$  only,

$$\int_c 2A_R t \, ds = 0 \quad (28.2)$$

The limits of integration of [Eq. \(28.2\)](#) present some difficulty, in that  $A_R$  is zero when  $w$  is zero at an unknown value of  $s$ . Let

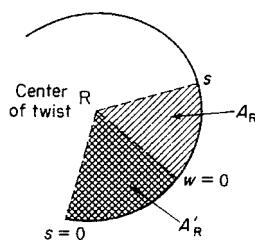
$$2A_R = 2A_{R,0} - 2A'_R$$

where  $A_{R,0}$  is the area swept out from  $s = 0$  and  $A'_R$  is the value of  $A_{R,0}$  at  $w = 0$  (see [Fig. 28.6](#)). Then, in [Eq. \(28.2\)](#),

$$\int_c 2A_{R,0} t \, ds - 2A'_R \int_c t \, ds = 0$$

and

$$2A'_R = \frac{\int_c 2A_{R,0} t \, ds}{\int_c t \, ds}$$

FIGURE 28.6 Computation of Swept Area  $A_R$ 

giving

$$2A_R = 2A_{R,0} - \frac{\int_c 2A_{R,0}t \, ds}{\int_c t \, ds} \quad (28.3)$$

The axial constraint shear flow system,  $q_\Gamma$ , is in equilibrium with the self-equilibrating direct stress system. Thus, from Eq. (17.2),

$$\frac{\partial q_\Gamma}{\partial s} + t \frac{\partial \sigma_\Gamma}{\partial z} = 0$$

Hence,

$$\frac{\partial q_\Gamma}{\partial s} = -t \frac{\partial \sigma_\Gamma}{\partial z}$$

Substituting for  $\sigma_\Gamma$  from Eq. (28.1) and noting that  $q_\Gamma = 0$  when  $s = 0$ , we have

$$q_\Gamma = \int_0^s 2A_R Et \frac{d^3\theta}{dz^3} \, ds$$

or

$$q_\Gamma = E \frac{d^3\theta}{dz^3} \int_0^s 2A_R t \, ds \quad (28.4)$$

Now,

$$T_\Gamma = \int_c p_R q_\Gamma \, ds \quad (\text{see Figs. 18.2 and 28.7})$$

or, from Eq. (28.4),

$$T_\Gamma = E \frac{d^3\theta}{dz^3} \int_c p_R \left( \int_0^s 2A_R t \, ds \right) \, ds$$

The integral in this equation is evaluated by substituting  $p_R = (d/ds)(2A_R)$  and integrating by parts. Thus,

$$\int_c \frac{d}{ds} (2A_R) \left( \int_0^s 2A_R t \, ds \right) \, ds = \left[ 2A_R \int_0^s 2A_R t \, ds \right]_c - \int_c 4A_R^2 t \, ds$$

At each open edge of the beam  $q_T$ , and therefore  $\int_0^s 2A_R t \, ds$ , is zero so that the integral reduces to  $-\int_c 4A_R^2 t \, ds$ , giving

$$T_T = -E\Gamma_R \frac{d^3\theta}{dz^3} \quad (28.5)$$

where  $\Gamma_R = \int_c 4A_R^2 t \, ds$ , the *torsion-bending constant*, and is purely a function of the geometry of the cross-section. The total torque  $T$ , which is the sum of the St. Venant torque and the Wagner torsion bending torque, is then written

$$T = GJ \frac{d\theta}{dz} - E\Gamma_R \frac{d^3\theta}{dz^3} \quad (28.6)$$

(Note: Compare Eq. (28.6) with the expression derived for the I-section beam.)

In the expression for  $\Gamma_R$ , the thickness  $t$  is actually the direct stress carrying thickness  $t_D$  of the beam wall, so that  $\Gamma_R$ , for a beam with  $n$  booms, may be generally written

$$\Gamma_R = \int_c 4A_R^2 t_D \, ds + \sum_{r=1}^n (2A_{R,r})^2 B_r$$

where  $B_r$  is the cross-sectional area of the  $r$ th boom. The calculation of  $\Gamma_R$  enables the second-order differential equation in  $d\theta/dz$  (Eq. (28.6)) to be solved. The constraint shear flows,  $q_T$ , follow from Eqs. (28.4) and (28.3) and the longitudinal constraint stresses from Eq. (28.1). However, before illustrating the complete method of solution with examples, we examine the calculation of  $\Gamma_R$ .

So far, we referred the swept area  $A_R$ , and hence  $\Gamma_R$ , to the center of twist of the beam without locating its position. This may be accomplished as follows. At any section of the beam, the resultant of the  $q_T$  shear flows is a pure torque (as is the resultant of the St. Venant shear stresses), so that in Fig. 28.7,

$$\int_c q_T \sin \psi \, ds = S_y = 0$$

Therefore, from Eq. (28.4),

$$E \frac{d^3\theta}{dz^3} \int_c \left( \int_0^s 2A_R t \, ds \right) \sin \psi \, ds = 0$$

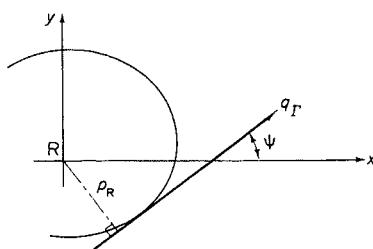


FIGURE 28.7 Determination of the Position of the Center of the Twist

Now,

$$\sin \psi = \frac{dy}{ds}, \quad \frac{d}{ds}(2A_R) = p_R$$

and this expression may be integrated by parts, thus,

$$\int_c \frac{dy}{ds} \left( \int_0^s 2A_R t \, ds \right) \, ds = \left[ y \int_0^s 2A_R t \, ds \right] - \int_c y 2A_R t \, ds = 0$$

The first term on the right-hand side vanishes, as  $\int_0^s 2A_R t \, ds$  is zero at each open edge of the beam, leaving

$$\int_c y 2A_R t \, ds = 0$$

Again, integrating by parts,

$$\int_c y 2A_R t \, ds = \left[ 2A_R \int_0^s yt \, ds \right]_c - \int_c p_R \left( \int_0^s yt \, ds \right) \, ds = 0$$

The integral in the first term on the right-hand side of this equation may be recognized, from [Chapter 17](#), as being directly proportional to the shear flow produced in a singly symmetrical open section beam supporting a shear load  $S_y$ . Its value is therefore zero at each open edge of the beam. Hence,

$$\int_c p_R \left( \int_0^s yt \, ds \right) \, ds = 0 \quad (28.7)$$

Similarly, for the horizontal component  $S_x$  to be zero,

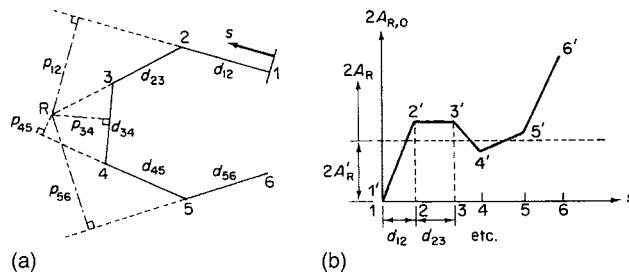
$$\int_c p_R \left( \int_0^s xt \, ds \right) \, ds = 0 \quad (28.8)$$

[Equations \(28.7\) and \(28.8\)](#) hold if the center of twist coincides with the shear center of the cross section. To summarize, the center of twist of a section of an open section beam carrying a pure torque is the shear center of the section.

We are now in a position to calculate  $\Gamma_R$ . This may be done by evaluating  $\int_c 4A_R^2 t \, ds$ , in which  $2A_R$  is given by [Eq. \(28.3\)](#). In general, the calculation may be lengthy, unless the section has flat sides, in which case a convenient analogy shortens the work considerably. For the flat-sided section in [Fig. 28.8\(a\)](#), we first plot the area  $2A_{R,0}$  swept out from the point 1, where we choose  $s = 0$  ([Fig. 28.8\(b\)](#)). The swept area  $A_{R,0}$  increases linearly from zero at 1 to  $(1/2)p_{12}d_{12}$  at 2 and so on. Note that movement along side 23 produces no increment of  $2A_{R,0}$ , as  $p_{23} = 0$ . Further, we adopt a sign convention for  $p$  such that  $p$  is positive if movement in the positive  $s$  direction of the foot of  $p$  along the tangent causes counterclockwise rotation about R. The increment of  $2A_{R,0}$  from side 34 is therefore negative.

In the derivation of [Eq. \(28.3\)](#), we showed that

$$2A'_R = \frac{\int_c 2A_{R,0}t \, ds}{\int_c t \, ds}$$



**FIGURE 28.8 Computation of Torsion Bending Constant  $\Gamma_R$ : (a) Dimensions of the Flat-Sided Open Section Beam; (b) Variation of  $2A_{R,0}$  Around the Beam Section**

Suppose now that the line 1'2'3'...6' is a wire of varying density such that the weight of each element  $\delta s'$  is  $t \delta s$ . Thus, the weight of length 1'2' is  $td_{12}$  and so forth. The y coordinate of the center of gravity of the wire is then

$$\bar{y} = \frac{\int y t \, ds}{\int t \, ds}$$

Comparing this expression with the previous one for  $2A'_R$ ,  $y$  and  $\bar{y}$  are clearly analogous to  $2A_{R,0}$  and  $2A'_R$ , respectively. Further,

$$\Gamma_R = \int_c (2A_R)^2 t \, ds = \int_c (2A_{R,0} - 2A'_R)^2 t \, ds$$

Expanding and substituting,

$$2A'_R \int_c t \, ds \quad \text{for} \quad \int_c 2A_{R,0} t \, ds$$

gives

$$\Gamma_R = \int_c (2A_{R,0})^2 t \, ds - (2A'_R)^2 \int_c t \, ds \quad (28.9)$$

Therefore, in Eq. (28.9),  $\Gamma_R$  is analogous to the moment of inertia of the wire about an axis through its center of gravity parallel to the  $s$  axis.

### Example 28.1

An open section beam of length  $L$  has the section shown in Fig. 28.9. The beam is firmly built in at one end and carries a pure torque  $T$ . Derive expressions for the direct stress and shear flow distributions produced by the axial constraint (the  $\sigma_\Gamma$  and  $q_\Gamma$  systems) and the rate of twist of the beam.

The beam is loaded by a pure torque, so that the axis of twist passes through the shear center S(R) of each section. We take the origin for  $s$  at the point 1 and initially plot  $2A_{R,0}$  against  $s$  to determine  $\Gamma_R$  (see Fig. 28.10). Note that there is no addition to  $2A_{R,0}$  as  $s$  moves down the web 23 and that in the flange 34 the contribution is negative. The position of the center of gravity,  $(2A'_R)$ , of the wire 1'2'3'4' is found by taking moments about the  $s$  axis. Then,

$$t(2d + h)2A'_R = td\left(\frac{hd}{4}\right) + th\left(\frac{hd}{2}\right) + td\left(\frac{hd}{4}\right)$$

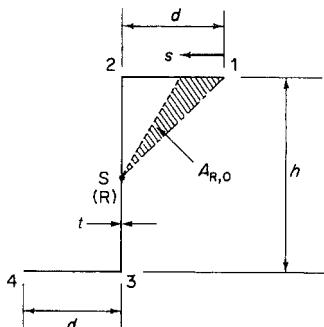
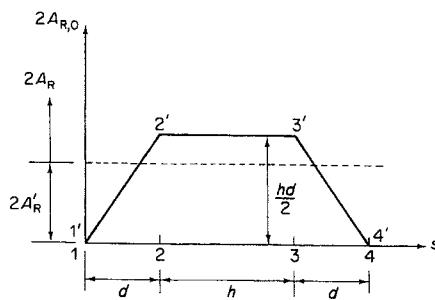


FIGURE 28.9 Section of Axially Constrained Open Section Beam Under Torsion

FIGURE 28.10 Calculation of  $\Gamma_R$  for the Section of Example 28.1.

from which

$$2A'_R = \frac{hd(h+d)}{2(h+2d)} \quad (i)$$

$\Gamma_R$  follows from the moment of inertia of the wire about an axis through its center of gravity. Hence,

$$\Gamma_R = 2td \frac{1}{3} \left( \frac{hd}{2} \right)^2 + th \left( \frac{hd}{2} \right)^2 - \left[ \frac{hd(h+d)}{2(h+2d)} \right]^2 t(h+2d)$$

which simplifies to

$$\Gamma_R = \frac{t d^3 h^2}{12} \left( \frac{2h+d}{h+2d} \right) \quad (ii)$$

Equation (28.6), that is,

$$T = GJ \frac{d\theta}{dz} - E\Gamma_R \frac{d^3\theta}{dz^3}$$

may now be solved for  $d\theta/dz$ . Rearranging and writing  $\mu^2 = GJ/E\Gamma_R$ , we have

$$\frac{d^3\theta}{dz^3} - \mu^2 \frac{d\theta}{dz} = -\mu^2 \frac{T}{GJ} \quad (iii)$$

The solution to Eq. (iii) is of standard form; that is,

$$\frac{d\theta}{dz} = \frac{T}{GJ} + A \cosh \mu z + B \sinh \mu z$$

The constants  $A$  and  $B$  are found from the boundary conditions:

1. At the built-in end, the warping  $w = 0$ , and since  $w = -2A_R d\theta/dz$ ,  $d\theta/dz = 0$  at the built-in end.
2. At the free end,  $\sigma_T = 0$ , as there is no constraint and no externally applied direct load. Therefore, from Eq. (28.1),  $d^2\theta/dz^2 = 0$  at the free end.

From (1),

$$A = -T/GJ$$

From (2),

$$B = (T/GJ) \tanh \mu L$$

so that

$$\frac{d\theta}{dz} = \frac{T}{GJ} (1 - \cosh \mu z + \tanh \mu L \sinh \mu z)$$

or

$$\frac{d\theta}{dz} = \frac{T}{GJ} \left[ 1 - \frac{\cosh \mu(L-z)}{\cosh \mu L} \right] \quad (\text{iv})$$

The first term in Eq. (iv) is seen to be the rate of twist derived from the St. Venant torsion theory. The hyperbolic second term is therefore the modification introduced by the axial constraint. Equation (iv) may be integrated to find the distribution of angle of twist  $\theta$ , the appropriate boundary condition being  $\theta = 0$  at the built-in end, that is,

$$\theta = \frac{T}{GJ} \left[ z + \frac{\sinh \mu(L-z)}{\mu \cosh \mu L} - \frac{\sinh \mu L}{\mu \cosh \mu L} \right] \quad (\text{v})$$

and the angle of twist,  $\theta_{F,E}$ , at the free end of the beam is

$$\theta_{F,E} = \frac{TL}{GJ} \left( 1 - \frac{\tanh \mu L}{\mu L} \right) \quad (\text{vi})$$

Plotting  $\theta$  against  $z$  (Fig. 28.11) illustrates the stiffening effect of axial constraint on the beam.

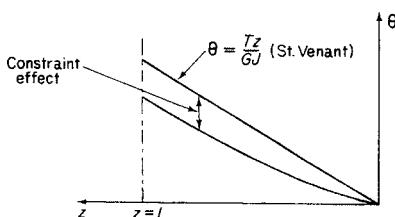


FIGURE 28.11 Stiffening Effect of the Axial Constraint

The decrease in the effect of axial constraint toward the free end of the beam is shown by an examination of the variation of the St. Venant ( $T_J$ ) and Wagner ( $T_F$ ) torques along the beam. From Eq. (iv),

$$T_J = GJ \frac{d\theta}{dz} = T \left[ 1 - \frac{\cosh \mu(L-z)}{\cosh \mu L} \right] \quad (\text{vii})$$

and

$$T_F = -E\Gamma_R \frac{d^3\theta}{dz^3} = T \frac{\cosh \mu(L-z)}{\cosh \mu L} \quad (\text{viii})$$

$T_J$  and  $T_F$  are now plotted against  $z$  as fractions of the total torque  $T$  (Fig. 28.12). At the built-in end, the entire torque is carried by the Wagner stresses, but although the constraint effect diminishes toward the free end, it does not disappear entirely. This is because the axial constraint shear flow,  $q_F$ , does not vanish at  $z=L$ , for at this section (and all other sections),  $d^3\theta/dz^3$  is not zero.

Equations (iii)–(viii) are, of course, valid for open section beams of any cross-section. Their application in a particular case is governed by the value of the torsion bending constant  $\Gamma_R$  and the St. Venant torsion constant  $J [= (h+2d)t^3/3]$ , for this example]. With this in mind, we can proceed, as required by the example, to derive the direct stress and shear flow distributions. The former is obtained from Eqs. (28.1) and (iv); that is,

$$\sigma_F = -2A_R E \frac{T}{GJ} \mu \frac{\sinh \mu(L-z)}{\cosh \mu L}$$

or, writing  $\mu^2 = GJ/E\Gamma_R$  and rearranging,

$$\sigma_F = -\sqrt{\frac{E}{GJ\Gamma_R}} T 2A_R \frac{\sinh \mu(L-z)}{\cosh \mu L} \quad (\text{ix})$$

In Eq. (ix)  $E$ ,  $G$ ,  $J$ , and  $\Gamma_R$  are constants for a particular beam;  $T$  is the applied torque;  $A_R$  is a function of  $s$ , and the hyperbolic term is a function of  $z$ . It follows that, at a given section of the beam, the direct stress is proportional to  $-2A_R$ , and for the beam of this example, the direct stress distribution has, from Fig. 28.10, the form shown in Figs 28.13(a) and (b). In addition, the value of  $\sigma_F$  at a particular value of  $s$  varies along the beam in the manner shown in Fig. 28.14.

Finally, the axial constraint shear flow,  $q_F$ , is obtained from Eq. (28.4):

$$q_F = E \frac{d^3\theta}{dz^3} \int_0^s 2A_R t \, ds$$

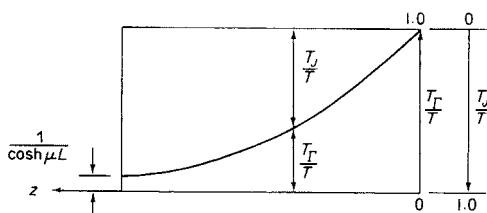


FIGURE 28.12 Distribution of St. Venant and Torsion-Bending Torques Along the Length of the Open Section Beam Shown in Fig. 28.9

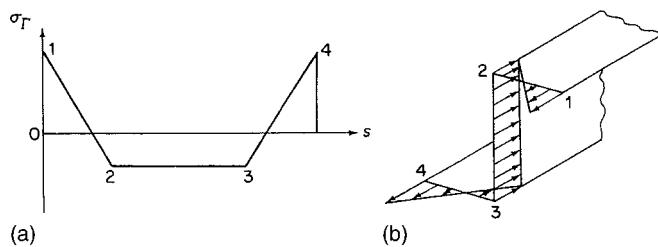


FIGURE 28.13 Distribution of Axial Constraint Direct Stress Around the Section

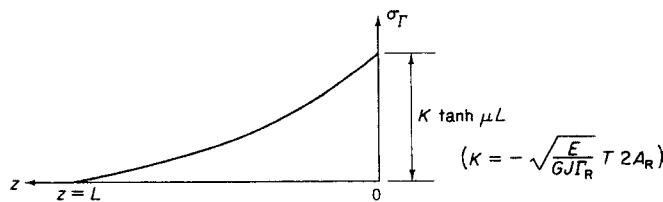


FIGURE 28.14 Spanwise Distribution of Axial Constraint Direct Stress

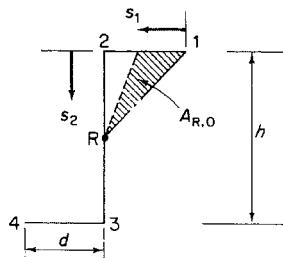


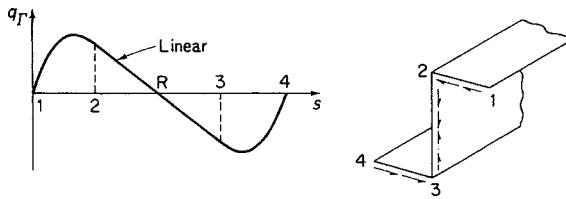
FIGURE 28.15 Calculation of Axial Constraint Shear Flows

At any section  $z$ ,  $q_T$  is proportional to  $\int_0^s 2A_R t \, ds$  and is computed as follows. Referring to Fig. 28.15,  $2A_R = 2A_{R,0} - 2A'_{R,0}$ , so that, in flange 12,

$$2A_R = \frac{hs_1}{2} - \frac{hd}{2} \left( \frac{h+d}{h+2d} \right)$$

Hence,

$$\int_0^s 2A_R t \, ds = t \left[ \frac{hs_1^2}{4} - \frac{hd}{2} \left( \frac{h+d}{h+2d} \right) s_1 \right]$$



**FIGURE 28.16 Distribution of Axial Constraint Shear Flows**

so that

$$q_{\Gamma,1} = 0 \quad \text{and} \quad q_{\Gamma,2} = -E \frac{d^3\theta}{dz^3} \frac{h^2 d^2 t}{4(h+2d)}$$

Similarly,

$$q_{\Gamma,23} = E \frac{d^3\theta}{dz^3} \left[ \frac{hd^2t}{2(h+2d)} s_2 - \frac{h^2d^2t}{4(h+2d)} \right]$$

from which

$$q_{\Gamma,2} = -E \frac{d^3\theta}{dz^3} \frac{h^2 d^2 t}{4(h+2d)}, \quad q_{\Gamma,3} = E \frac{d^3\theta}{dz^3} \frac{h^2 d^2 t}{4(h+2d)}$$

Note that, in the above,  $d^3\theta/dz^3$  is negative (Eq. (viii)). Also, at the mid-point of the web, where  $s_2 = h/2$ ,  $q_{\Gamma} = 0$ . The distribution on the lower flange follows from antisymmetry, and the distribution of  $q_{\Gamma}$  around the section is of the form shown in Fig. 28.16.

The spanwise variation of  $q_{\Gamma}$  has the same form as the variation of  $T_{\Gamma}$ , since

$$T_{\Gamma} = -E\Gamma_R \frac{d^3\theta}{dz^3}$$

giving

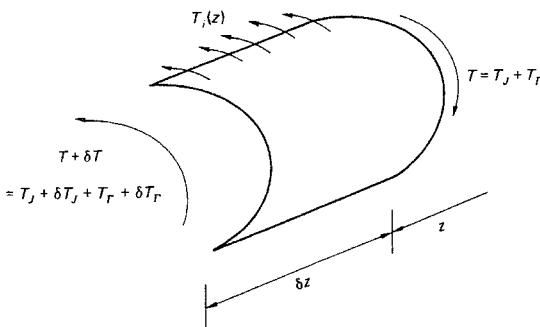
$$q_{\Gamma} = \frac{T_{\Gamma}}{\Gamma_R} \int_0^s 2A_R t \, ds$$

from Eq. (28.4). Hence, for a given value of  $s$ , ( $\int_0^s 2A_R t \, ds$ ),  $q_{\Gamma}$  is proportional to  $T_{\Gamma}$  (see Fig. 28.12).

## 28.3 DISTRIBUTED TORQUE LOADING

We now consider the more general case of a beam carrying a distributed torque loading. In Fig. 28.17, an element of a beam carries a distributed torque of intensity  $T_i(z)$ , that is, a torque per unit length. At the section  $z$ , the torque comprises the St. Venant torque  $T_J$  plus the torque due to axial constraint  $T_{\Gamma}$ . At the section  $z + \delta z$ , the torque increases to  $T + \delta T (= T_J + \delta T_J + T_{\Gamma} + \delta T_{\Gamma})$ , so that for equilibrium of the beam element,

$$T_J + \delta T_J + T_{\Gamma} + \delta T_{\Gamma} + T_i(z)\delta z - T_J - T_{\Gamma} = 0$$



**FIGURE 28.17 Beam Carrying a Distributed Torque Loading**

or

$$-T_i(z)\delta z = \delta T_J + \delta T_R = \delta T$$

Hence,

$$\frac{dT}{dz} = -T_i(z) = \frac{dT_J}{dz} + \frac{dT_R}{dz} \quad (28.10)$$

Now,

$$TJ = GJ \frac{d\theta}{dz} \quad (\text{Eq.(18.12)})$$

and

$$T_R = -EG \frac{d^3\theta}{dz^3} \quad (\text{Eq.(28.5)})$$

so that [Eq. \(28.10\)](#) becomes

$$EG \frac{d^4\theta}{dz^4} - GJ \frac{d^2\theta}{dz^2} = T_i(z) \quad (28.11)$$

The solution of [Eq. \(28.11\)](#) is again of standard form, in which the constants of integration are found from the boundary conditions of the particular beam under consideration. For example, for a cantilever beam of length  $L$  in which the origin for  $z$  is at the built-in end and which is subjected to a uniform torque loading, the boundary conditions are

when  $z = L, d^2\theta/dz^2 = 0$  (from [Eq. \(28.1\)](#))

when  $z = 0, d\theta/dz = 0$  (since the warping is zero at the built-in end, see [Eq. \(18.19\)](#))

when  $z = L, d^3\theta/dz^3 = 0$  (since  $T_R = T_J = T = 0$  at the free end, see [Eq. \(28.5\)](#))

when  $z = 0, \theta = 0$  (there is no rotation at the built-in end)

## 28.4 EXTENSION OF THE THEORY TO ALLOW FOR GENERAL SYSTEMS OF LOADING

So far, we have been concerned with open section beams subjected to torsion in which, due to constraint effects, axial stresses are induced. Since pure torsion can generate axial stresses, it is logical to suppose that certain distributions of axial stress applied as external loads will cause twisting. The problem is to determine which component of an applied direct stress system causes twisting.

Figure 28.18 shows the profile of a thin-walled open section beam subjected to a general system of loads, which produce longitudinal, transverse, and rotational displacements of its cross-section. In the analysis, we assume that the cross-section of the beam is not distorted by the loading and that displacements corresponding to the shear strains are negligible. In Fig. 28.18, the tangential displacement  $v_t$  is given by Eq. (17.7); that is,

$$v_t = p_R \theta + u \cos \psi + v \sin \psi \quad (28.12)$$

Also, since shear strains are assumed to be negligible, Eq. (17.6) becomes

$$\gamma = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} = 0 \quad (28.13)$$

Substituting for  $v_t$  in Eq. (28.13) from (28.12) and integrating from the origin for  $s$  to any point  $s$  around the cross-section, we have

$$w_s - w_0 = -\frac{d\theta}{dz} 2A_{R,0} - \frac{du}{dz}(x - x_0) - \frac{dv}{dz}(y - y_0) \quad (28.14)$$

where  $2A_{R,0} = \int_0^s p_R \, ds$ . The direct stress at any point in the wall of the beam is given by

$$\sigma_z = E \frac{\partial w_s}{\partial z}$$

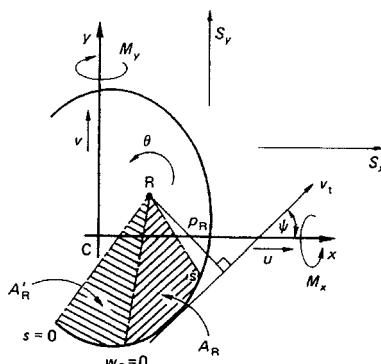


FIGURE 28.18 Cross-section of an Open Section Beam Subjected to a General System of Loads

Therefore, from Eq. (28.14),

$$\sigma_z = E \left[ \frac{\partial w_0}{\partial z} - \frac{d^2\theta}{dz^2} 2A_{R,0} - \frac{d^2u}{dz^2}(x - x_0) - \frac{d^2v}{dz^2}(y - y_0) \right] \quad (28.15)$$

Now,  $A_{R,0} = A'_R + A_R$  (Fig. 28.18), so that Eq. (28.15) may be rewritten

$$\sigma_z = f_1(z) - E \frac{d^2\theta}{dz^2} 2A_R - E \frac{d^2u}{dz^2} x - E \frac{d^2v}{dz^2} y \quad (28.16)$$

in which

$$f_1(z) = E \left( \frac{\partial w_0}{\partial z} - \frac{d^2\theta}{dz^2} 2A'_R + \frac{d^2u}{dz^2} x_0 + \frac{d^2v}{dz^2} y_0 \right)$$

The axial load  $P$  on the section is given by

$$P = \int_c \sigma_z t \, ds = f_1(z) \int_c t \, ds - E \frac{d^2\theta}{dz^2} \int_c 2A_R t \, ds - E \frac{d^2u}{dz^2} \int_c tx \, ds - E \frac{d^2v}{dz^2} \int_c ty \, ds$$

where  $\int_c$  denotes integration taken completely around the section. From Eq. (28.2), we see that  $\int_c 2A_R t \, ds = 0$ . Also, if the origin of axes coincides with the centroid of the section,  $\int_c tx \, ds = \int_c ty \, ds = 0$  and  $\int_c ty \, ds = 0$ , so that

$$P = \int_c \sigma_z t \, ds = f_1(z)A \quad (28.17)$$

in which  $A$  is the cross-sectional area of the material in the wall of the beam.

The component of bending moment,  $M_x$ , about the  $x$  axis is given by

$$M_x = \int_c \sigma_z ty \, ds$$

Substituting for  $\sigma_z$  from Eq. (28.16), we have

$$M_x = f_1(z) \int_c ty \, ds - E \frac{d^2\theta}{dz^2} \int_c 2A_R ty \, ds - E \frac{d^2u}{dz^2} \int_c txy \, ds - E \frac{d^2v}{dz^2} \int_c ty^2 \, ds$$

We saw, in the derivation of Eqs. (28.7) and (28.8), that  $\int_c 2A_R ty \, ds = 0$ . Also, since

$$\int_c ty \, ds = 0, \quad \int_c txy \, ds = I_{xy}, \quad \int_c ty^2 \, ds = I_{xx}$$

$$M_x = -E \frac{d^2u}{dz^2} I_{xy} - E \frac{d^2v}{dz^2} I_{xx} \quad (28.18)$$

Similarly,

$$M_y = \int_c \sigma_z tx \, ds = -E \frac{d^2u}{dz^2} I_{yy} - E \frac{d^2v}{dz^2} I_{xy} \quad (28.19)$$

Equations (28.18) and (28.19) are identical to Eqs. (16.30), so that, from Eqs. (16.28),

$$E \frac{d^2u}{dz^2} = \frac{M_x I_{xy} - M_y I_{xx}}{I_{xx} I_{yy} - I_{xy}^2}, \quad E \frac{d^2v}{dz^2} = \frac{-M_x I_{yy} + M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \quad (28.20)$$

The first differential,  $d^2\theta/dz^2$ , of the rate of twist in Eq. (28.16), may be isolated by multiplying throughout by  $2A_R t$  and integrating around the section. Thus,

$$\begin{aligned} \int_c \sigma_z 2A_R t \, ds &= f_1(z) \int_c 2A_R t \, ds - E \frac{d^2\theta}{dz^2} \int_c (2A_R)^2 t \, ds - E \frac{d^2u}{dz^2} \int_c 2A_R t x \, ds \\ &\quad - E \frac{d^2v}{dz^2} \int_c 2A_R t y \, ds \end{aligned}$$

As before,

$$\int_c 2A_R t \, ds = 0, \quad \int_c 2A_R t x \, ds = \int_c 2A_R t y \, ds = 0$$

and

$$\int_c (2A_R)^2 t \, ds = \Gamma_R$$

so that

$$\int_c \sigma_z 2A_R t \, ds = -E \Gamma_R \frac{d^2\theta}{dz^2}$$

or

$$\frac{d^2\theta}{dz^2} = - \int_c \frac{\sigma_z 2A_R t \, ds}{E \Gamma_R} \quad (28.21)$$

Substituting in Eq. (28.16) from Eqs. (28.17), (28.20), and (28.21), we obtain

$$\sigma_z = \frac{P}{A} + \left( \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) x + \left( \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) y + \frac{2A_R \int_c \sigma_z 2A_R t \, ds}{\Gamma_R} \quad (28.22)$$

The second two terms on the right-hand side of Eq. (28.22) give the direct stress due to bending as predicted by elementary beam theory; note that this approach provides an alternative method of derivation of Eq. (16.17).

Comparing the last term on the right-hand side of Eq. (28.22) with Eq. (28.1), we see that

$$\frac{2A_R \int_c \sigma_z 2A_R t \, ds}{\Gamma_R} = \sigma_\Gamma$$

It follows therefore that the external application of a direct stress system  $\sigma_z$  induces a self-equilibrating direct stress system  $\sigma_\Gamma$ . Also, the first differential of the rate of twist ( $d^2\theta/dz^2$ ) is related to the applied

$\sigma_z$  stress system through the term  $\int_c \sigma_z 2A_R t \, ds$ . Therefore, if  $\int_c \sigma_z 2A_R t \, ds$  is interpreted in terms of the applied loads at a particular section, then a boundary condition exists (for  $d^2\theta/dz^2$ ), which determines one of the constants in the solution of either Eq. (28.6) or (28.11).

## 28.5 MOMENT COUPLE (BIMOMENT)

The units of  $\int_c \sigma_z 2A_R t \, ds$  are *force*  $\times$  (*distance*)<sup>2</sup> or *moment*  $\times$  *distance*. A simple physical representation of this expression thus consists of two equal and opposite moments applied in parallel planes some distance apart. This combination has been termed a *moment couple*<sup>1</sup> or a *bimoment*<sup>2</sup> and is given the symbol  $M_\Gamma$  or  $B_\omega$ . Equation (28.22) is then written as

$$\sigma_z = \frac{P}{A} + \left( \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) x + \left( \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) y + \frac{M_\Gamma 2A_R}{I_R} \quad (28.23)$$

As a simple example of the determination of  $M_\Gamma$ , consider the open section beam shown in Fig. 28.19, which is subjected to a series of concentrated loads  $P_1, P_2, \dots, P_k, \dots, P_n$  parallel to its longitudinal axis. The term  $\sigma_z t \, ds$  in  $\int_c \sigma_z 2A_R t \, ds$  may be regarded as a concentrated load acting at a point in the wall of the beam. Thus,  $\int_c \sigma_z 2A_R t \, ds$  becomes  $\sum_{k=1}^n P_k 2A_{Rk}$ , and hence

$$M_\Gamma = \sum_{k=1}^n P_k 2A_{Rk} \quad (28.24)$$

$M_\Gamma$  is determined for a range of other loading systems in Vlasov<sup>2</sup>.

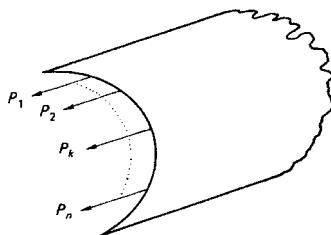


FIGURE 28.19 Open Section Beam Subjected to Concentrated Loads Parallel to Its Longitudinal Axis

### Example 28.2

The column shown in Fig. 28.20(a) carries a vertical load of 100 kN. Calculate the angle of twist at the top of the column and the distribution of direct stress at its base.  $E = 200,000 \text{ N/mm}^2$  and  $G/E = 0.36$ .

The center of twist R of the column cross-section coincides with its shear center at the midpoint of web 23. The distribution of  $2A_R$  is obtained by the method detailed in Example 28.1 and is shown in Fig. 28.21. The torsion bending constant  $I_R$  is given by Eq. (ii) of Example 28.1 and has the value  $2.08 \times 10^{10} \text{ mm}^6$ . The St. Venant

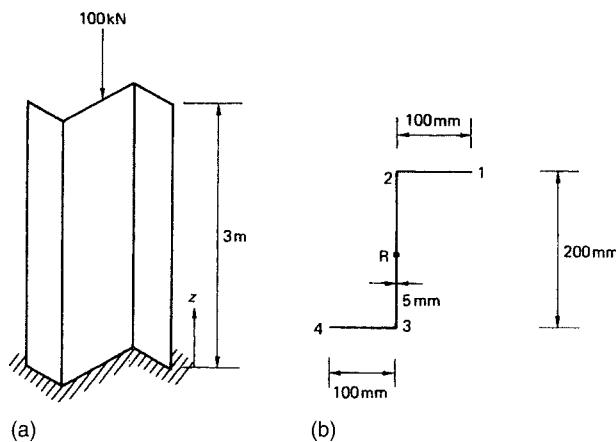
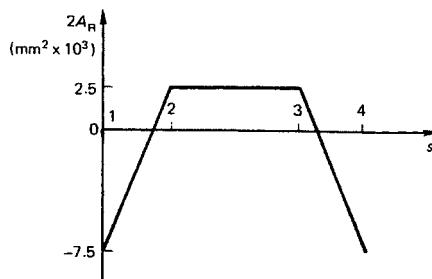


FIGURE 28.20 Column of Example 28.2

FIGURE 28.21 Distribution of Area  $2A_R$  in the Column of Example 28.2

torsion constant  $J = \sum st^3/3 = 0.17 \times 10^5 \text{ mm}^4$  so that  $\sqrt{GJ/EI_R}$  ( $= \mu$  in Eq. (iii) of Example 28.1)  $= 0.54 \times 10^{-3}$ . Since no torque is applied to the column, the solution of Eq. (iii) in Example 28.1 is

$$\frac{d\theta}{dz} = C \cosh \mu z + D \sinh \mu z \quad (i)$$

At the base of the column, warping of the cross-section is suppressed, so that, from Eq. (18.19),  $d\theta/dz = 0$  when  $z = 0$ . Substituting in Eq. (i) gives  $C = 0$ . The moment couple at the top of the column is obtained from Eq. (28.24) and is

$$M_F = P2A_R = -100 \times 2.5 \times 10^3 = -25 \times 10^5 \text{ kNm}^2$$

Therefore, from Eq. (28.21) and noting that  $\int_c \sigma_z 2A_R t \, ds = M_F$ , we have

$$\frac{d^2\theta}{dz^2} = \frac{2.5 \times 10^5 \times 10^3}{200,000 \times 2.08 \times 10^{10}} = 0.06 \times 10^{-6}/\text{mm}^2$$

at  $z = 3000$  mm. Substitution in the differential of Eq. (i) gives  $D = 0.04 \times 10^{-3}$ , so that Eq. (i) becomes

$$\frac{d\theta}{dz} = 0.04 \times 10^{-3} \sinh 0.54 \times 10^{-3}z \quad (\text{ii})$$

Integration of Eq. (ii) gives

$$\theta = 0.08 \cosh 0.54 \times 10^{-3}z + F$$

At the built-in end ( $z = 0$ ),  $\theta = 0$ , so that  $F = -0.08$ . Hence,

$$\theta = 0.08(\cosh 0.54 \times 10^{-3}z - 1) \quad (\text{iii})$$

At the top of the column ( $z = 3000$  mm), the angle of twist is then

$$\theta(\text{top}) = 0.08 \cosh 0.54 \times 10^{-3} \times 3,000 = 0.21 \text{ rad } (12.01^\circ)$$

The axial load is applied through the centroid of the cross-section, so that no bending occurs and Eq. (28.23) reduces to

$$\sigma_z = \frac{P}{A} + \frac{M_\Gamma 2A_R}{I_R} \quad (\text{iv})$$

At the base of the column,

$$(M_\Gamma)_{z=0} = -E I_R \left( \frac{d^2\theta}{dz^2} \right)_{z=0} \quad (\text{see Eq. (28.21)})$$

Therefore, from Eq. (ii),

$$(M_\Gamma)_{z=0} = -200,000 \times 2.08 \times 10^{10} \times 0.02 \times 10^{-6} = -83.2 \times 10^6 \text{ Nmm}^2$$

The direct stress distribution at the base of the column is then, from Eq. (iv),

$$\sigma_z = -\frac{100 \times 10^3}{400 \times 5} - \frac{83.2 \times 10^6}{2.08 \times 10^{10}} 2A_R$$

or

$$\sigma_z = -50 - 4.0 \times 10^{-3} 2A_R$$

The direct stress distribution is therefore linear around the base of the column (see Fig. 28.21) with

$$\sigma_{z1} = \sigma_{z4} = 20.0 \text{ N/mm}^2$$

$$\sigma_{z2} = \sigma_{z3} = -68.0 \text{ N/mm}^2$$



### 28.5.1 Shear flow due to $M_\Gamma$

The self-equilibrating shear flow distribution,  $q_\Gamma$ , produced by axial constraint is given by

$$\frac{\partial q_\Gamma}{\partial s} = -t \frac{\partial \sigma_\Gamma}{\partial z} \quad (\text{see derivation of Eq. (28.4)})$$

From the last term on the right-hand side of Eqs. (28.23),

$$\frac{\partial \sigma_T}{\partial z} = \frac{\partial M_T}{\partial z} \frac{2A_R}{\Gamma_R}$$

From Eq. (28.21),

$$M_T = -E\Gamma_R \frac{d^2\theta}{dz^2}$$

so that

$$\frac{\partial M_T}{\partial z} = -E\Gamma_R \frac{d^3\theta}{dz^3} = T_T \quad (\text{see Eq. (28.5)})$$

Hence,

$$\frac{\partial q_T}{\partial s} = -T_T \frac{2A_R t}{\Gamma_R}$$

and

$$q_T = -\frac{T_T}{\Gamma_R} \int_0^s 2A_R t \, ds$$

as before.

## References

- [1] Megson THG. Extension of the Wagner torsion bending theory to allow for general systems of loading. Aeronautical Quarterly August 1975;XXVI.
- [2] Vlasov VZ. Thin-walled elastic beams. Jerusalem: Israel Program for Scientific Translations; 1961.

## PROBLEMS

**P.28.1** An axially symmetric beam has the thin-walled cross-section shown in Fig. P.28.1. If the thickness  $t$  is constant throughout and making the usual assumptions for a thin-walled cross-section, show that the torsion bending constant  $\Gamma_R$  calculated about the shear center S is

$$\Gamma_R = \frac{13}{12} d^5 t$$

**P.28.2** A uniform beam has the point-symmetric cross-section shown in Fig. P.28.2. Making the usual assumptions for a thin-walled cross-section, show that the torsion-bending constant  $\Gamma$  calculated about the shear center S is  $\Gamma = \frac{8}{3} d^5 t \sin^2 2\alpha$ . The thickness  $t$  is constant throughout.

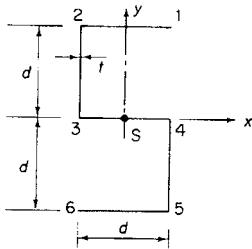


FIGURE P.28.1

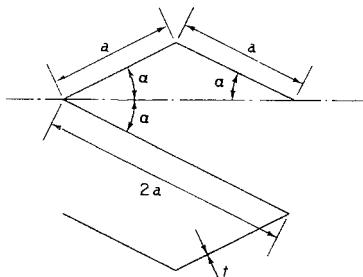


FIGURE P.28.2

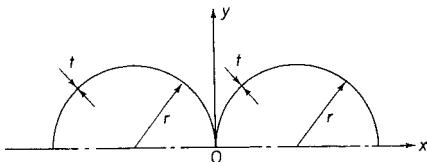


FIGURE P.28.3

**P.28.3** The thin-walled section shown in Fig. P.28.3 consists of two semi-circular arcs of constant thickness  $t$ . Show that the torsion bending constant about the shear center S is

$$\Gamma = \pi^2 r^5 t \left( \frac{\pi}{3} - \frac{3}{\pi} \right)$$

**P.28.4** A thin-walled, I-section beam, of constant wall thickness  $t$ , is mounted as a cantilever with its web horizontal. At the tip, a downward force is applied in the plane of one of the flanges, as shown in Fig. P.28.4. Assuming the necessary results of the elementary theory of bending, the St. Venant theory of torsion, and the Wagner torsion-bending theory, determine the distribution of direct stress over the cross-section at the supported end. Take

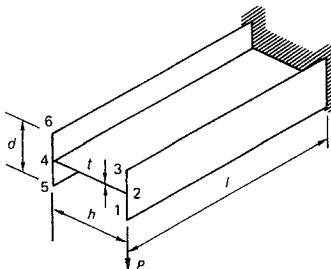


FIGURE P.28.4

$$\begin{aligned}E/G &= 2.6, \quad P = 200 \text{ N} \\h &= 75 \text{ mm}, \quad d = 37.5 \text{ mm} \\t &= 2.5 \text{ mm}, \quad l = 375 \text{ mm}\end{aligned}$$

Answer:  $-\sigma_1 = \sigma_3 = 108.9 \text{ N/mm}^2$ ,  $\sigma_6 = -\sigma_5 = 18.9 \text{ N/mm}^2$ ,  $\sigma_2 = \sigma_4 = \sigma_{24} = 0$

**P.28.5** An open section beam of length  $2l$ , whose ends are free to warp, consists of two uniform portions of equal length  $l$ , as shown in Fig. P.28.5. The cross-sections of the two halves are identical except that the thickness in one half is  $t$  and in the other  $2t$ . If the St. Venant torsion constant and the torsion-bending constant for the portion of thickness  $t$  are  $J$  and  $\Gamma$ , respectively, show that, when the beam is loaded by a constant torque  $T$ , the relative twist between the free ends is given by

$$\theta = \frac{Tl}{8GJ} \left[ 9 - \frac{49 \sinh 2\mu l}{2\mu l(10 \cosh^2 \mu l - 1)} \right]$$

where

$$\mu^2 = GJ/E\Gamma \text{ and } G = \text{shear modulus (constant throughout)}$$

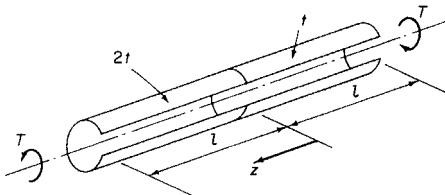
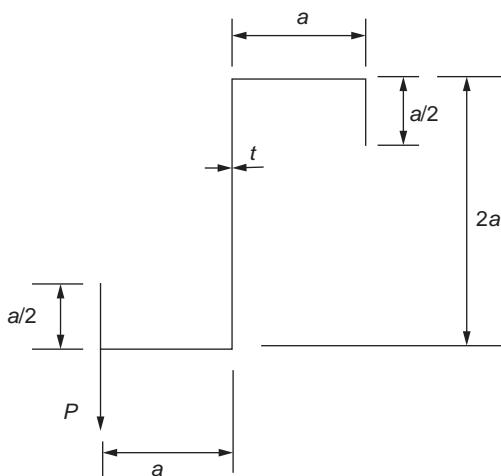


FIGURE P.28.5

**P.28.6** A thin-walled cantilever beam of length  $L$  has the cross-section shown in Fig. P.28.6 and carries a load  $P$  positioned as shown at its free end. Determine the torsion bending constant for the beam section and derive an expression for the angle of twist  $\theta_T$  at the free end of the beam. Calculate the value of this angle for  $P = 100 \text{ N}$ ,  $a = 30 \text{ mm}$ ,  $L = 1000 \text{ mm}$ ,  $t = 2.0 \text{ mm}$ ,  $E = 70,000 \text{ N/mm}^2$ , and  $G = 25,000 \text{ N/mm}^2$ .

**FIGURE P.28.6**

Answer:  $\Gamma = 1.25a^5t$ ,  $\theta_T = 6.93^\circ$

$$\theta_\Gamma = \frac{TL}{GJ} \left[ 1 - \frac{\tanh \mu L}{\mu L} \right]$$

**P.28.6 MATLAB** Using MATLAB and the functions for  $\Gamma$  and  $\theta_\Gamma$  derived in Problem P.28.6, calculate the values of  $L$  that result in values of  $\theta_\Gamma$  ranging from  $5.5^\circ$  to  $6.5^\circ$  in increments of  $0.1^\circ$ . Assume that  $G = 25,000 \text{ N/mm}^2$ ,  $t = 2 \text{ mm}$ ,  $a = 30 \text{ mm}$ ,  $T = 3000 \text{ Nmm}$ ,  $E = 70,000 \text{ N/mm}^2$ , and  $J = 400 \text{ mm}^4$

Answer:	$\theta_\Gamma$	$L$
(i)	$5.5^\circ$	892.86 mm
(ii)	$5.6^\circ$	900.38 mm
(iii)	$5.7^\circ$	907.85 mm
(iv)	$5.8^\circ$	915.28 mm
(v)	$5.9^\circ$	922.67 mm
(vi)	$6.0^\circ$	930.02 mm
(vii)	$6.1^\circ$	937.33 mm
(viii)	$6.2^\circ$	944.60 mm
(ix)	$6.3^\circ$	951.84 mm
(x)	$6.4^\circ$	959.05 mm
(xi)	$6.5^\circ$	966.22 mm

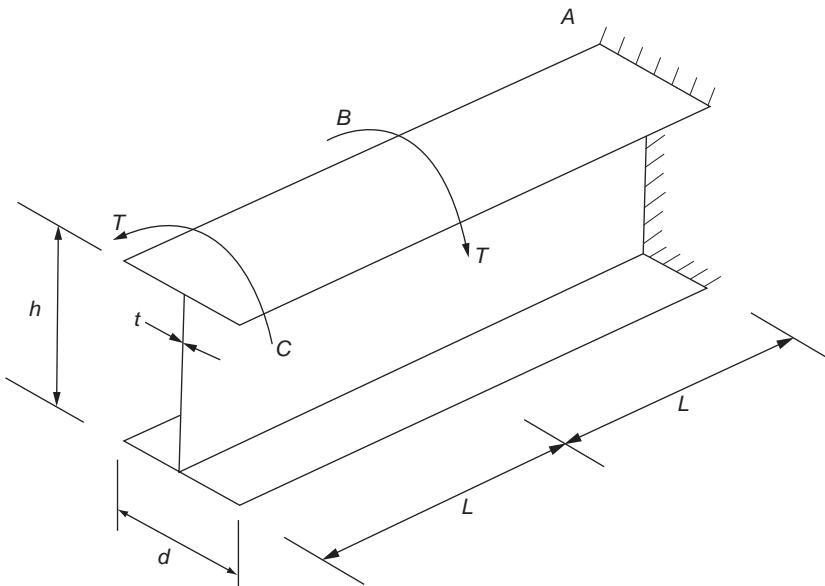


FIGURE P.28.7

**P.28.7** Determine the torsion bending constant for the thin-walled beam shown in Fig. P.28.7 and also derive an expression for the angle of twist at its free end.

$$\text{Answer: } \Gamma = th^2d^3/24, \quad \theta_T = \frac{T}{GJ} \left( L - \frac{\sinh \mu L}{\mu L \cosh 2\mu L} \right)$$

**P.28.8** A thin-walled cantilever beam of length  $L$  has the cross-section shown in Fig. P.28.8 and carries a counterclockwise torque  $T$  at its free end. Determine the torsion bending constant for the beam section and derive an expression for the rate of twist along the length of the beam. In a practical case, the beam supports a shear load of 150 N at its free end applied vertically upward in the plane of the web. If  $L = 500$  mm,  $a = 20$  mm,  $t = 1.0$  mm, and  $G/E = 0.3$ , calculate the value of direct stress at point 2, including both axial constraint and elementary bending stresses.

$$\text{Answer: } \Gamma = 7a^5t/24, \quad \frac{d\theta}{dz} = \frac{T}{GJ} \left( 1 - \frac{\cosh \mu(L-z)}{\cosh \mu L} \right)$$

125.7 N/mm<sup>2</sup> (compression)

**P.28.9** Calculate the direct stress distribution (including both axial constraint and elementary bending stresses) at the built-in end of the cantilever beam shown in Fig. P.28.9 for the case when  $w = 0.5$  N/mm,  $L = 1500$  mm,  $h = 200$  mm,  $d = 50$  mm,  $t = 5$  mm, and  $E/G = 3.0$ .

$$\text{Answer: } \sigma_1 = -\sigma_3 = 197.5 \text{ N/mm}^2, \quad \sigma_2 = \sigma_5 = 0, \quad \sigma_4 = -\sigma_6 = -72.5 \text{ N/mm}^2$$

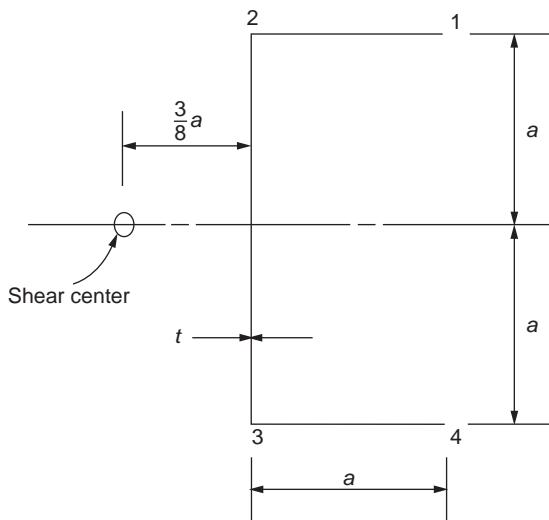


FIGURE P.28.8

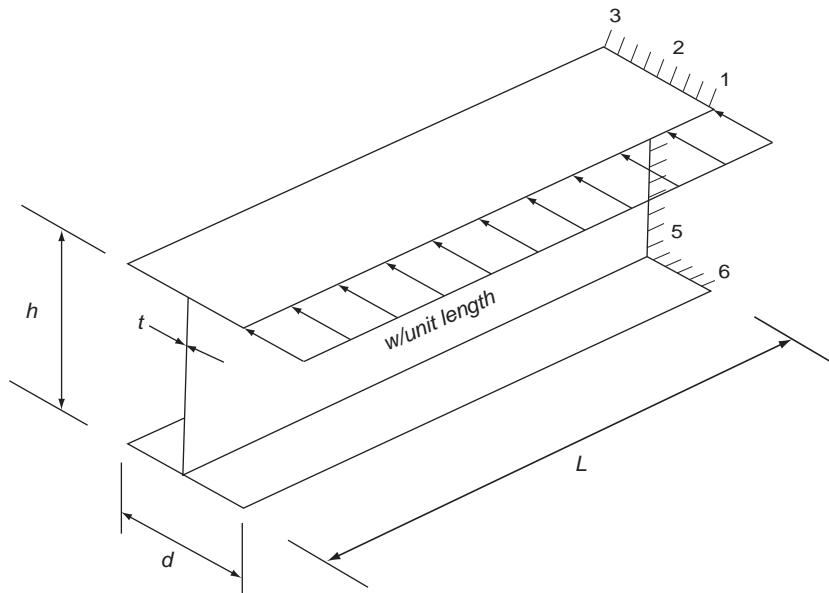


FIGURE P.28.9

# Wing problems

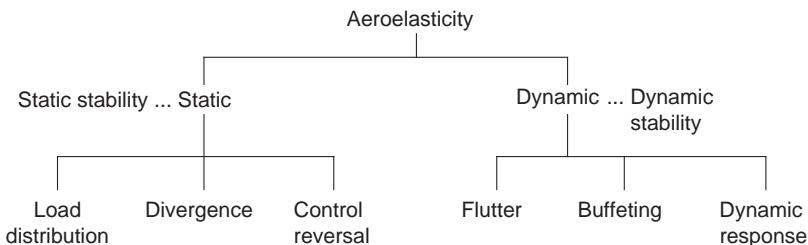
# 29

Aircraft structures, being extremely flexible, are prone to distortion under load. When these loads are caused by aerodynamic forces, which themselves depend on the geometry of the structure and the orientation of the various structural components to the surrounding airflow, structural distortion results in changes in aerodynamic load, leading to further distortion and so on. The interaction of aerodynamic and elastic forces is known as *aeroelasticity*.

## 29.1 TYPES OF PROBLEM

Two distinct types of aeroelastic problem occur. One involves the interaction of aerodynamic and elastic forces of the type just described. Such interactions may exhibit divergent tendencies in a too flexible structure, leading to failure or, in an adequately stiff structure, convergence until a condition of stable equilibrium is reached. In this type of problem, *static* or *steady state* systems of aerodynamic and elastic forces produce such aeroelastic phenomena as *divergence* and *control reversal*. The second class of problem involves the inertia of the structure as well as aerodynamic and elastic forces. Dynamic loading systems, of which gusts are of primary importance, induce oscillations of structural components. If the natural or resonant frequency of the component is in the region of the frequency of the applied loads, then the amplitude of the oscillations may diverge, causing failure. Also, as we observed in Chapter 15, the presence of fluctuating loads is a fatigue hazard. For obvious reasons, we refer to these problems as *dynamic*. Included in this group are flutter, buffeting, and dynamic response.

The various aeroelastic problems may be conveniently summarized in the form of a “tree,” as follows:



In this chapter, we concentrate on the purely structural aspects of aeroelasticity; its effect on aircraft static and dynamic stability is treated in books devoted primarily to aircraft stability and control.<sup>1,2</sup>

## 29.2 LOAD DISTRIBUTION AND DIVERGENCE

Redistribution of aerodynamic loads and divergence are closely related aeroelastic phenomena; we therefore consider them simultaneously. It is essential in the design of structural components that the aerodynamic load distribution on the component is known. Wing distortion, for example, may produce significant changes in lift distribution from that calculated on the assumption of a rigid wing, especially in instances of high wing loadings, such as those experienced in maneuvers and gusts. To estimate actual lift distributions, the aerodynamicist needs to know the incidence of the wing at all stations along its span. Obviously, this is affected by any twisting of the wing that may be present.

Let us consider the case of a simple straight wing with the center of twist behind the aerodynamic center (see Fig. 29.1). The moment of the lift vector about the center of twist causes an increase in wing incidence, which produces a further increase in lift, leading to another increase in incidence, and so on. At speeds below a critical value, called the *divergence speed*, the increments in lift converge to a condition of stable equilibrium, in which the torsional moment of the aerodynamic forces about the center of twist is balanced by the torsional rigidity of the wing. The calculation of lift distribution then proceeds from a knowledge of the distribution of twist along the wing. For a straight wing, the redistribution of lift usually causes an outward spanwise movement of the center of pressure, resulting in greater bending moments at the wing root. In the case of a swept wing, a reduction in streamwise incidence of the outboard sections due to bending deflections causes a movement of the center of pressure toward the wing root.

All aerodynamic surfaces of the aircraft suffer similar load redistribution due to distortion.

### 29.2.1 Wing torsional divergence (two-dimensional case)

The most common divergence problem is the torsional divergence of a wing. It is useful, initially, to consider the case of a wing of area  $S$  without ailerons and in a two-dimensional flow, as shown in Fig. 29.2. The torsional stiffness of the wing, which we represent by a spring of stiffness,  $K$ , resists the moment of the lift vector,  $L$ , and the wing pitching moment,  $M_0$ , acting at the aerodynamic center of the wing section. For moment equilibrium of the wing section about the aerodynamic center, we have

$$M_0 + Lec = K\theta \quad (29.1)$$

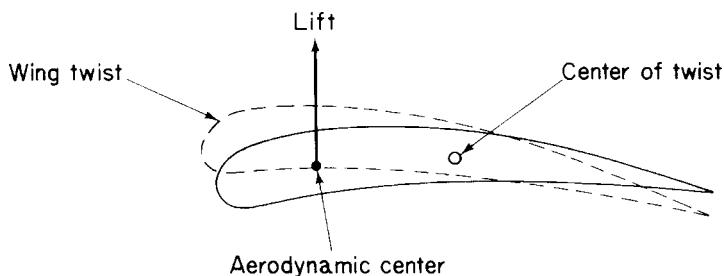
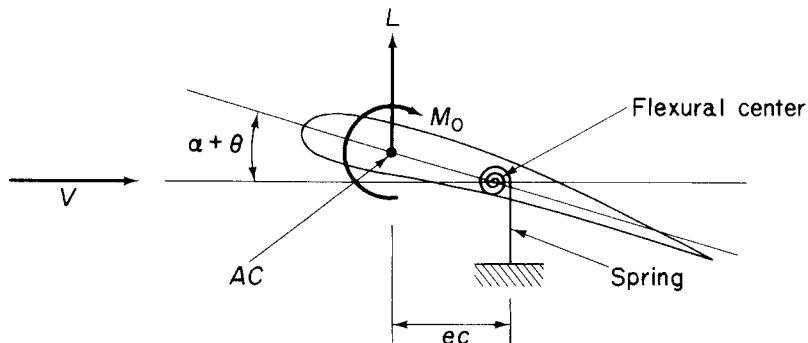


FIGURE 29.1 Increase in Wing Incidence Due to Wing Twist



**FIGURE 29.2 Determination of Wing Divergence Speed (Two-Dimensional Case)**

where  $ec$  is the distance of the aerodynamic center forward of the flexural center expressed in terms of the wing chord,  $c$ , and  $\theta$  is the elastic twist of the wing. From aerodynamic theory,

$$M_0 = \frac{1}{2} \rho V^2 S c C_{M,0}, \quad L = \frac{1}{2} \rho V^2 S C_L$$

Substituting in Eq. (29.1) yields

$$\frac{1}{2} \rho V^2 S (c C_{M,0} + ec C_L) = K \theta$$

or, since

$$C_L = C_{L,0} + \frac{\partial C_L}{\partial \alpha} (\alpha + \theta)$$

in which  $\alpha$  is the initial wing incidence or, in other words, the incidence corresponding to given flight conditions, assuming that the wing is rigid and  $C_{L,0}$  is the wing lift coefficient at zero incidence, then,

$$\frac{1}{2} \rho V^2 S \left[ c C_{M,0} + ec C_{L,0} + ec \frac{\partial C_L}{\partial \alpha} (\alpha + \theta) \right] = K \theta$$

where  $\partial C_L / \partial \alpha$  is the wing lift-curve slope. Rearranging gives

$$\theta \left( K - \frac{1}{2} \rho V^2 S c \frac{\partial C_L}{\partial \alpha} \right) = \frac{1}{2} \rho V^2 S c \left( C_{M,0} + e C_{L,0} + e \frac{\partial C_L}{\partial \alpha} \alpha \right)$$

or

$$\theta = \frac{\frac{1}{2} \rho V^2 S c [C_{M,0} + e C_{L,0} + e (\partial C_L / \partial \alpha) \alpha]}{K - \frac{1}{2} \rho V^2 S c (\partial C_L / \partial \alpha)} \quad (29.2)$$

Equation (29.2) shows that divergence occurs (i.e.,  $\theta$  becomes infinite) when

$$K = \frac{1}{2} \rho V^2 S c \frac{\partial C_L}{\partial \alpha}$$

The divergence speed  $V_d$  is then

$$V_d = \sqrt{\frac{2K}{\rho S c (\partial C_L / \partial \alpha)}} \quad (29.3)$$

We see, from Eq. (29.3), that  $V_d$  may be increased either by stiffening the wing (increasing  $K$ ) or by reducing the distance  $ec$  between the aerodynamic and flexural centers. The former approach involves weight and cost penalties, so that designers usually prefer to design a wing structure with the flexural center as far forward as possible. If the aerodynamic center coincides with or is aft of the flexural center, then the wing is stable at all speeds.

### Example 29.1

An aircraft wing has an area of  $30 \text{ m}^2$  and an aerodynamic mean chord of  $3.0 \text{ m}$ . If the aerodynamic center of the wing is one-quarter of the wing chord forward of its flexural center, its lift-curve slope is  $3.5$ , and the design diving speed of the aircraft at sea level is  $200 \text{ m/s}$ , calculate the minimum required torsional stiffness of the wing. Assume two-dimensional flow and take the air density at sea level to be  $1.226 \text{ kg/m}^3$ .

The minimum required torsional stiffness will occur when the wing divergence speed is equal to the design diving speed. Therefore, from Eq. (29.3),

$$K = (1/2)\rho V_d^2 S c (\partial C_L / \partial \alpha)$$

i.e.,

$$K = (1/2)1.226 \times 200^2 \times 30 \times 0.25 \times 3.0 \times 3.5$$

which gives

$$K = 1.93 \times 10^6 \text{ Nm}^2$$

### 29.2.2 Wing torsional divergence (finite wing)

We consider the simple case of a straight wing having its flexural axis nearly perpendicular to the aircraft's plane of symmetry (Fig. 29.3(a)). We also assume that wing cross-sections remain undistorted under the loading. Applying strip theory in the usual manner—that is, we regard a small element of chord  $c$  and spanwise width  $\delta z$  as acting independently of the remainder of the wing and consider its equilibrium—we have, from Fig. 29.3(b), neglecting wing weight,

$$\left( T + \frac{dT}{dz} \delta z \right) - T + \Delta L e c + \Delta M_0 = 0 \quad (29.4)$$

where  $T$  is the applied torque at any spanwise section  $z$  and  $\Delta L$  and  $\Delta M_0$  are the lift and pitching moment on the elemental strip acting at its aerodynamic center, respectively. As  $\delta z$  approaches zero, Eq. (29.4) becomes

$$\frac{dT}{dz} + ec \frac{dL}{dz} + \frac{dM_0}{dz} = 0 \quad (29.5)$$

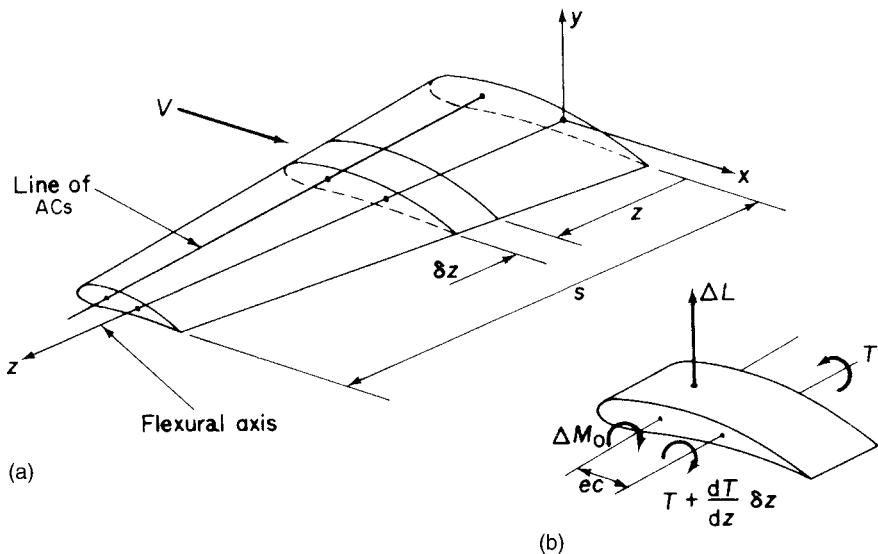


FIGURE 29.3 Determination of Wing Divergence Speed (Three-Dimensional Case)

In Eq. (29.4),

$$\Delta L = \frac{1}{2} \rho V^2 c \delta z \frac{\partial c_1}{\partial \alpha} (\alpha + \theta)$$

where  $\partial c_1 / \partial \alpha$  is the local two-dimensional lift-curve slope and

$$\Delta M_0 = \frac{1}{2} \rho V^2 c^2 \delta z c_{m,0}$$

in which  $c_{m,0}$  is the local pitching moment coefficient about the aerodynamic center. Also, from torsion theory (see Chapter 3),  $T = GJ d\theta/dz$ . Substituting for  $L$ ,  $M_0$ , and  $T$  in Eq. (29.5) gives

$$\frac{d^2\theta}{dz^2} + \frac{\frac{1}{2} \rho V^2 e c^2 (\partial c_1 / \partial \alpha) \theta}{GJ} = \frac{-\frac{1}{2} \rho V^2 e c^2 (\partial c_1 / \partial \alpha) \alpha}{GJ} - \frac{\frac{1}{2} \rho V^2 c^2 c_{m,0}}{GJ} \quad (29.6)$$

Equation (29.6) is a second-order differential equation in  $\theta$ , having a solution of the standard form

$$\theta = A \sin \lambda z + B \cos \lambda z - \left[ \frac{c_{m,0}}{e(\partial c_1 / \partial \alpha)} + \alpha \right] \quad (29.7)$$

where

$$\lambda^2 = \frac{\frac{1}{2} \rho V^2 e c^2 (\partial c_1 / \partial \alpha)}{GJ}$$

and  $A$  and  $B$  are unknown constants obtained from the boundary conditions, namely  $\theta = 0$  when  $z = 0$  at the wing root and  $d\theta/dz = 0$  at  $z = s$ , since the torque is zero at the wing tip. From the first of these,

$$B = \left[ \frac{c_{m,0}}{e(\partial c_1 / \partial \alpha)} + \alpha \right]$$

and, from the second,

$$A = \left[ \frac{c_{m,0}}{e(\partial c_1 / \partial \alpha)} + \alpha \right] \tan \lambda s$$

Hence,

$$\theta = \left[ \frac{c_{m,0}}{e(\partial c_1 / \partial \alpha)} + \alpha \right] (\tan \lambda s \sin \lambda z + \cos \lambda z - 1) \quad (29.8)$$

or, rearranging,

$$\theta = \left[ \frac{c_{m,0}}{e(\partial c_1 / \partial \alpha)} + \alpha \right] \left[ \frac{\cos \lambda(s-z)}{\cos \lambda s} - 1 \right] \quad (29.9)$$

Therefore, at divergence when the elastic twist,  $\theta$ , becomes infinite,

$$\cos \lambda s = 0$$

so that

$$\lambda s = (2n+1) \frac{\pi}{2} \quad \text{for } n = 0, 1, 2, \dots \infty \quad (29.10)$$

The smallest value corresponding to the divergence speed  $V_d$  occurs when  $n = 0$ , therefore,

$$\lambda s = \pi/2$$

or

$$\lambda^2 = \pi^2/4s^2$$

from which

$$V_d = \sqrt{\frac{\pi^2 G J}{2 \rho e c^2 s^2 (\partial c_1 / \partial \alpha)}} \quad (29.11)$$

Mathematical solutions of the type given in Eq. (29.10) rarely apply with any accuracy to actual wing or tail surfaces. However, they do give an indication of the order of the divergence speed,  $V_d$ . In fact, when the two-dimensional lift-curve slope,  $\partial c_1 / \partial \alpha$ , is used, they lead to conservative estimates of  $V_d$ . It has been shown that when  $\partial c_1 / \partial \alpha$  is replaced by the three-dimensional lift-curve slope of the finite wing, values of  $V_d$  become very close to those determined from more sophisticated aerodynamic and aeroelastic theory.

The lift distribution on a straight wing, accounting for the elastic twist, is found by introducing a relationship between incidence and lift distribution from aerodynamic theory. In the case of simple strip theory, the local wing lift coefficient,  $c_1$ , is given by

$$c_1 = \frac{\partial c_1}{\partial \alpha} (\alpha + \theta)$$

in which the distribution of elastic twist  $\theta$  is known from Eq. (29.9).

**Example 29.2**

The wing of Example 29.1 has a span of 9.5 m. If the lift-curve slope of the finite wing may be assumed to be the same as that in Example 29.1, compare the required torsional stiffnesses of the two cases.

From Eq. (29.11),

$$GJ = 2\rho c^2 s^2 (\partial C_L / \partial \alpha) V_d / \pi^2$$

i.e.,

$$GJ = 2 \times 1.226 \times 0.25 \times 3.0^2 \times 9.5^2 \times 3.5 \times 200^2 / \pi^2$$

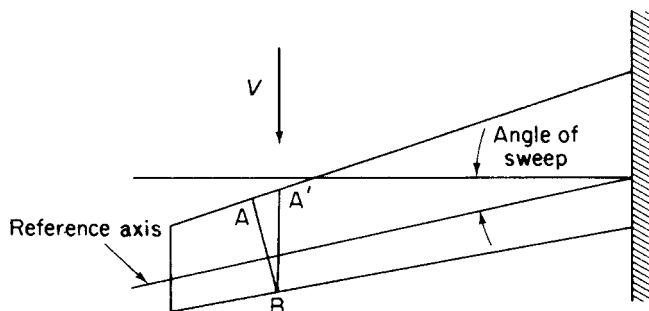
which gives

$$GJ = 7.06 \times 10^6 \text{ Nm}^2$$

Therefore the required calculated wing torsional stiffness for the finite wing is  $(7.06/1.93) = 3.7$  times that, assuming two-dimensional flow.

**29.2.3 Swept wing divergence**

In the calculation of divergence speeds of straight wings, the flexural axis was taken to be nearly perpendicular to the aircraft's plane of symmetry. The bending of such wings has no influence on divergence, this being entirely dependent on the twisting of the wing about its flexural axis. This is no longer the case for a swept wing, where the spanwise axes are inclined to the aircraft's plane of symmetry. Let us consider the swept wing of Fig. 29.4. The wing lift distribution causes the wing to bend in an upward direction. Points A and B on a line perpendicular to the reference axis deflect by approximately the same amount, but this is greater than the deflection of A', which means that bending reduces the streamwise incidence of the wing. The corresponding negative increment of lift opposes the elastic twist, thereby reducing the possibility of wing divergence. In fact, the divergence speed of swept wings is so high that it poses no problems for the designer. Diederich and Budiansky, in 1948, showed that wings



**FIGURE 29.4 Effect of Wing Sweep on Wing Divergence Speed**

with moderate or large sweepback cannot diverge. The opposite of course is true for swept-forward wings, where bending deflections have a destabilizing effect and divergence speeds are extremely low. The determination of lift distributions and divergence speeds for swept-forward wings is presented in Bisplinghoff, Ashley, and Halfman.<sup>3</sup>

## 29.3 CONTROL EFFECTIVENESS AND REVERSAL

The flexibility of the major aerodynamic surfaces (wings, vertical and horizontal tails) adversely affects the effectiveness of the corresponding control surfaces (ailerons, rudder, and elevators). For example, the downward deflection of an aileron causes a nose-down twisting of the wing, which consequently reduces the aileron incidence. Thus, the wing twist tends to reduce the increase in lift produced by the aileron deflection and, thereby, the rolling moment to a value less than that for a rigid wing. The aerodynamic twisting moment on the wing due to aileron deflection increases as the square of the speed, but the elastic restoring moment is constant, since it depends on the torsional stiffness of the wing structure. Therefore, ailerons become markedly less effective as the speed increases until, at a particular speed, the *aileron reversal speed*, aileron deflection produces no rolling moment at all. At higher speeds, reversed aileron movements are necessary, in that a positive increment of wing lift requires an upward aileron deflection and vice versa.

Similar, less critical problems arise in the loss of effectiveness and reversal of the rudder and elevator controls. They are complicated by the additional deformations of the fuselage and tailplane-fuselage attachment points, which may be as important as the deformations of the tailplane itself. We concentrate in this section on the problem of aileron effectiveness and reversal.

### 29.3.1 Aileron effectiveness and reversal (two-dimensional case)

We illustrate the problem by investigating, as in Section 29.1, the case of a wing-aileron combination in a two-dimensional flow. In Fig. 29.5, an aileron deflection  $\xi$  produces changes  $\Delta L$  and  $\Delta M_0$  in the wing lift,  $L$ , and wing pitching moment,  $M_0$ ; these in turn cause an elastic twist,  $\theta$ , of the wing. Thus,

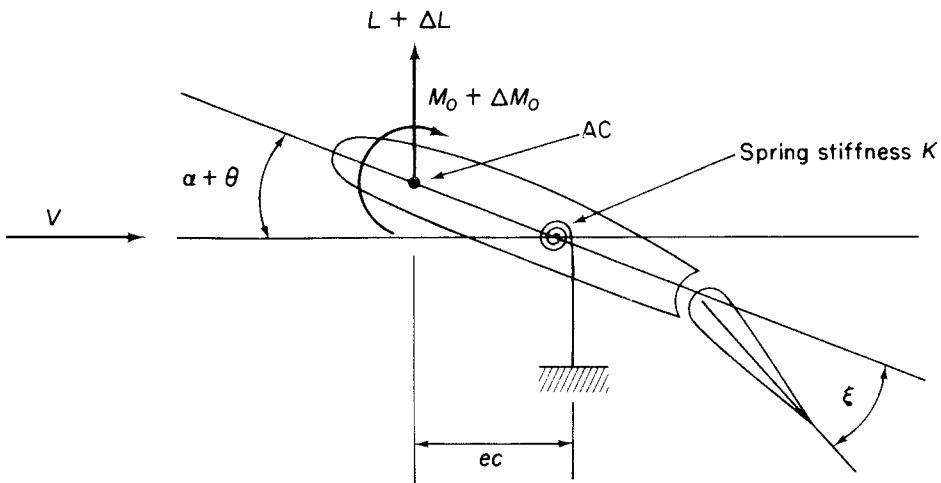
$$\Delta L = \left( \frac{\partial C_L}{\partial \alpha} \theta + \frac{\partial C_L}{\partial \xi} \xi \right) \frac{1}{2} \rho V^2 S \quad (29.12)$$

where  $\partial C_L / \partial \alpha$  has been previously defined and  $\partial C_L / \partial \xi$  is the rate of change of lift coefficient with aileron angle. Also,

$$\Delta M_0 = -\frac{\partial C_{M,0}}{\partial \xi} \xi \frac{1}{2} \rho V^2 S c \quad (29.13)$$

in which  $\partial C_{M,0} / \partial \xi$  is the rate of change of wing pitching moment coefficient with aileron deflection. The moment produced by these increments in lift and pitching moment is equilibrated by an increment of torque  $\Delta T$  about the flexural axis. Hence,

$$\Delta T = K\theta = \frac{1}{2} \rho V^2 S c \left[ \left( \frac{\partial C_L}{\partial \alpha} \theta + \frac{\partial C_L}{\partial \xi} \xi \right) e + \frac{\partial C_{M,0}}{\partial \xi} \xi \right] \quad (29.14)$$



**FIGURE 29.5 Aileron Effectiveness and Reversal Speed (Two-Dimensional Case)**

Isolating  $\theta$  from Eq. (29.14) gives

$$\theta = \frac{\frac{1}{2}\rho V^2 Sc[(\partial C_L / \partial \xi)e + \partial C_{M,0} / \partial \xi]\xi}{K - \frac{1}{2}\rho V^2 Sce(\partial C_L / \partial \alpha)} \quad (29.15)$$

Substituting for  $\theta$  in Eq. (29.12), we have

$$\Delta L = \frac{1}{2}\rho V^2 S \left[ \frac{\frac{1}{2}\rho V^2 Sc\{(\partial C_L / \partial \xi)e + \partial C_{M,0} / \partial \xi\} \partial C_L}{K - \frac{1}{2}\rho V^2 Sce(\partial C_L / \partial \alpha)} + \frac{\partial C_L}{\partial \xi} \right] \xi$$

which simplifies to

$$\Delta L = \frac{1}{2}\rho V^2 S \left[ \frac{\frac{1}{2}\rho V^2 Sc(\partial C_{M,0} / \partial \xi)(\partial C_L / \partial \alpha) + K(\partial C_L / \partial \xi)}{K - \frac{1}{2}\rho V^2 Sce(\partial C_L / \partial \alpha)} \right] \xi \quad (29.16)$$

The increment of wing lift is therefore a linear function of aileron deflection and becomes zero—that is, aileron reversal occurs—when

$$\frac{1}{2}\rho V^2 Sc \frac{\partial C_{M,0}}{\partial \xi} \frac{\partial C_L}{\partial \alpha} + K \frac{\partial C_L}{\partial \xi} = 0 \quad (29.17)$$

Hence, the aileron reversal speed,  $V_r$ , is, from Eq. (29.17),

$$V_r = \sqrt{\frac{-K(\partial C_L / \partial \xi)}{\frac{1}{2}\rho Sc(\partial C_{M,0} / \partial \xi)(\partial C_L / \partial \alpha)}} \quad (29.18)$$

We may define aileron effectiveness at speeds below the reversal speed in terms of the lift  $\Delta L_R$  produced by an aileron deflection on a rigid wing. Thus,

$$\text{Aileron effectiveness} = \Delta L / \Delta L_R \quad (29.19)$$

where

$$\Delta L_R = \frac{\partial C_L}{\partial \xi} \xi \frac{1}{2} \rho V^2 S \quad (29.20)$$

Hence, substituting in Eq. (29.19) for  $\Delta L$  from Eq. (29.16) and  $\Delta L_R$  from Eq. (29.20), we have

$$\text{Aileron effectiveness} = \frac{\frac{1}{2} \rho V^2 Sc (\partial C_{M,0} / \partial \xi) (\partial C_L / \partial \alpha) + K (\partial C_L / \partial \xi)}{[K - \frac{1}{2} \rho V^2 Sce (\partial C_L / \partial \alpha)] \partial C_L / \partial \xi} \quad (29.21)$$

Equation (29.21) may be expressed in terms of the wing divergence speed  $V_d$  and aileron reversal speed  $V_r$ , using Eqs. (29.3) and (29.18), respectively; hence,

$$\text{Aileron effectiveness} = \frac{1 - V^2/V_r^2}{1 - V^2/V_d^2} \quad (29.22)$$

We see that, when  $V_d = V_r$ , which occurs when  $\partial C_L / \partial \xi = -(\partial C_{M,0} / \partial \xi)/e$ , the aileron is completely effective at all speeds. Such a situation arises because the nose-down wing twist caused by aileron deflection is cancelled by the nose-up twist produced by the increase in wing lift.

Although the analysis just described is based on a two-dimensional case, it is sometimes used in practice to give approximate answers for finite wings. The method is to apply the theory to a representative wing cross-section at an arbitrary spanwise station and use the local wing section properties in the formulae.

### Example 29.3

Calculate the aileron reversal speed corresponding to the minimum required torsional stiffness for the wing of Example 29.1 and determine the aileron effectiveness at a speed of 150 m/s. Take the rate of change of lift coefficient with aileron angle as 0.8 and the rate of change of pitching moment coefficient with aileron angle as -0.25.

From Example 29.1,  $K = 1.93 \times 10^6 \text{ Nm}^2$ . Then, from Eq. (29.18),

$$V_r = \sqrt{[(1.93 \times 10^6 \times 0.8) / (1.226 \times 30 \times 3.0 \times 0.25 \times 3.5 / 2)]}$$

which gives

$$V_r = 178.8 \text{ m/s.}$$

From Example 29.1,  $V_d = 200 \text{ m/s}$ . Then, from Eq. (29.22),

$$\text{Aileron effectiveness} = (1 - 150^2/178.8^2) / (1 - 150^2/200^2) = 0.68$$

Therefore the aileron is 68% effective.

### 29.3.2 Aileron effectiveness and reversal (finite wing)

We again apply strip theory to investigate the aeroelastic effects of aileron deflection on a finite wing. In Fig. 29.6(a), the deflection of the aileron through an angle  $\xi$  produces a rolling velocity  $p$  rad/s, having the sense shown. The wing incidence at any section  $z$  is thus reduced due to  $p$  by an amount  $pz/V$ . The downward aileron deflection shown here coincides with an upward deflection on the opposite wing, thereby contributing to the rolling velocity  $p$ . The incidence of the opposite wing is therefore increased by this direction of roll. Since we are concerned with aileron effects, we consider the

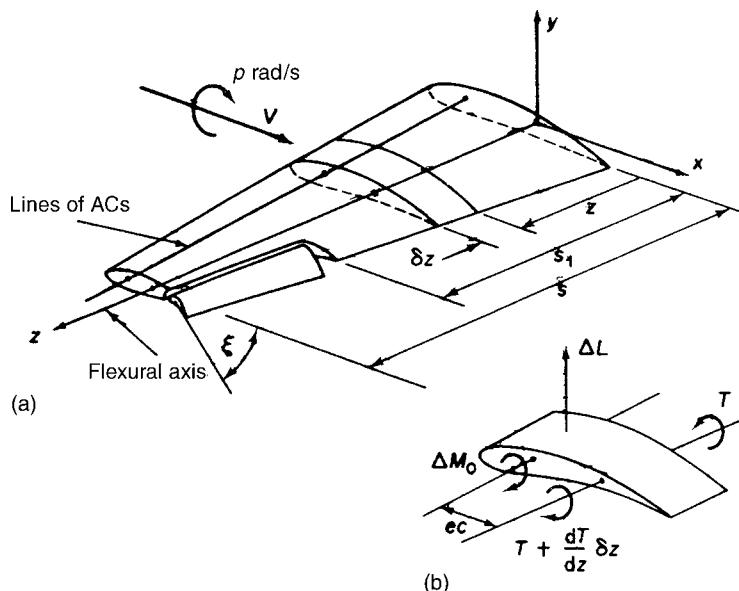


FIGURE 29.6 Aileron Effectiveness and Reversal Speed (Finite Wing)

antisymmetric lift and pitching moment produced by aileron deflection. Thus, in Fig. 29.6(b), the forces and moments are *changes* from the level flight condition.

The lift  $\Delta L$  on the strip shown in Fig. 29.6(b) is given by

$$\Delta L = \frac{1}{2} \rho V^2 c \delta z \left[ \frac{\partial c_1}{\partial \alpha} \left( \theta - \frac{p z}{V} \right) + \frac{\partial c_1}{\partial \xi} f_a(z) \xi \right] \quad (29.23)$$

where  $\partial c_1 / \partial \alpha$  has been previously defined and  $\partial c_1 / \partial \xi$  is the rate of change of local wing lift coefficient with aileron angle. The function  $f_a(z)$  represents aileron forces and moments along the span; for  $0 \leq z \leq s_1$ ,  $f_a(z) = 0$  and for  $s_1 \leq z \leq s$ ,  $f_a(z) = 1$ . The pitching moment  $\Delta M_0$  on the elemental strip is given by

$$\Delta M_0 = \frac{1}{2} \rho V^2 c^2 \delta z \frac{\partial c_{m,0}}{\partial \xi} f_a(z) \xi \quad (29.24)$$

in which  $\partial c_{m,0} / \partial \xi$  is the rate of change of the local pitching moment coefficient with aileron angle.

Considering the moment equilibrium of the elemental strip of Fig. 29.6(b), we obtain, neglecting wing weight,

$$\frac{dT}{dz} \delta z + \Delta L e c + \Delta M_0 = 0 \quad (29.25)$$

or, substituting for  $\Delta L$  and  $\Delta M_0$  from Eqs. (29.23) and (29.24),

$$\frac{dT}{dz} + \frac{1}{2} \rho V^2 e c^2 \left[ \frac{\partial c_1}{\partial \alpha} \left( \theta - \frac{p z}{V} \right) + \frac{\partial c_1}{\partial \xi} f_a(z) \xi \right] + \frac{1}{2} \rho V^2 c^2 \frac{\partial c_{m,0}}{\partial \xi} f_a(z) \xi = 0 \quad (29.26)$$

Substituting for  $T$  in Eq. (29.26) from torsion theory ( $T = GJ \frac{d\theta}{dz}$ ) and rearranging, we have

$$\frac{d^2\theta}{dz^2} + \frac{\frac{1}{2}\rho V^2 ec^2 (\partial c_1 / \partial \alpha)}{GJ} \theta = \frac{\frac{1}{2}\rho V^2 c^2}{GJ} \left[ e \frac{\partial c_1}{\partial \alpha} \frac{pz}{V} - e \frac{\partial c_1}{\partial \xi} f_a(z) \xi - \frac{\partial c_{m,0}}{\partial \xi} f_a(z) \xi \right] \quad (29.27)$$

Writing

$$\frac{\frac{1}{2}\rho V^2 ec^2 (\partial c_1 / \partial \alpha)}{GJ} = \lambda^2$$

we obtain

$$\frac{d^2\theta}{dz^2} + \lambda^2 \theta = \lambda^2 \frac{pz}{V} - \frac{\lambda^2}{\partial c_1 / \partial \alpha} \left( \frac{\partial c_1}{\partial \xi} + \frac{1}{e} \frac{\partial c_{m,0}}{\partial \xi} \right) f_a(z) \xi \quad (29.28)$$

It may be shown that the solution to Eq. (29.28), satisfying the boundary conditions

$$\theta = 0 \text{ at } z = 0 \text{ and } d\theta/dz = 0 \text{ at } z = s$$

is

$$\begin{aligned} \theta = & \frac{p}{V} \left( z - \frac{\sin \lambda z}{\lambda \cos \lambda s} \right) - \frac{1}{\partial c_1 / \partial \alpha} \left( \frac{\partial c_1}{\partial \xi} + \frac{1}{e} \frac{\partial c_{m,0}}{\partial \xi} \right) \\ & \times \left[ f_a(z) \left\{ 1 - \cos \lambda(z - s_1) \right\} - \frac{\sin \lambda(s - s_1)}{\cos \lambda s} \sin \lambda z \right] \xi \end{aligned} \quad (29.29)$$

where

$$\cos \lambda(z - s_1) = 0, \text{ when } z < s_1$$

The spanwise variation of total local wing lift coefficient is given by strip theory as

$$c_1 = \frac{\partial c_1}{\partial \alpha} \left( \alpha + \theta - \frac{pz}{V} \right) + \frac{\partial c_1}{\partial \xi} f_a(z) \xi \quad (29.30)$$

where  $\theta$  is known from Eq. (29.29) and  $\alpha$  is the steady flight wing incidence.

The aileron effectiveness is often measured in terms of the wing-tip helix angle ( $ps/V$ ) per unit aileron displacement during a steady roll. In this condition, the rolling moments due to a given aileron deflection,  $\xi$ , wing twist, and aerodynamic damping are in equilibrium, so that from Fig. 29.6(a) and Eq. (29.23) and noting that ailerons on opposite wings both contribute to the rolling, we have

$$2 \int_0^s \frac{1}{2} \rho V^2 c \left[ \frac{\partial c_1}{\partial \alpha} \left( \theta - \frac{pz}{V} \right) + \frac{\partial c_1}{\partial \xi} f_a(z) \xi \right] z dz = 0 \quad (29.31)$$

from which

$$\int_0^s \frac{\partial c_1}{\partial \alpha} \left( \theta - \frac{pz}{V} \right) z dz = -\xi \int_0^s \frac{\partial c_1}{\partial \xi} f_a(z) z dz \quad (29.32)$$

Substituting for  $\theta$  from Eq. (29.29) into Eq. (29.32) gives

$$\begin{aligned} & \int_0^s \frac{\partial c_1}{\partial \alpha} \left\{ \frac{ps}{V} \frac{\sin \lambda z}{\lambda s \cos \lambda s} - \frac{1}{\partial c_1 / \partial \alpha} \left( \frac{\partial c_1}{\partial \xi} + \frac{1}{e} \frac{\partial c_{m,0}}{\partial \xi} \right) \right. \\ & \quad \times \left. \left[ f_a(z) \{1 - \cos \lambda(z - s_1)\} - \frac{\sin \lambda(s - s_1)}{\cos \lambda s} \sin \lambda z \right] \xi \right\} z \, dz \\ & = -\xi \int_0^s \frac{\partial c_1}{\partial \xi} f_a(z) z \, dz \end{aligned}$$

Hence,

$$\begin{aligned} \xi \int_0^s \left\{ \left( \frac{\partial c_1}{\partial \xi} + \frac{1}{e} \frac{\partial c_{m,0}}{\partial \xi} \right) \left[ f_a(z) \{1 - \cos \lambda(z - s_1)\} - \frac{\sin \lambda(s - s_1)}{\cos \lambda s} \sin \lambda z \right] \right. \\ \left. - \frac{\partial c_1}{\partial \xi} f_a(z) \right\} z \, dz = \frac{ps}{V} \int_0^s \frac{\partial c_1}{\partial \alpha} \frac{\sin \lambda z}{\lambda s \cos \lambda s} z \, dz \end{aligned} \quad (29.33)$$

Therefore, aileron effectiveness  $(ps/V)/\xi$  is given by

$$\begin{aligned} \frac{(ps/V)}{\xi} &= \frac{\int_0^s \left\{ -\frac{\partial c_1}{\partial \xi} f_a(z) + \left( \frac{\partial c_1}{\partial \xi} + \frac{1}{e} \frac{\partial c_{m,0}}{\partial \xi} \right) \right.}{\int_0^s \frac{\partial c_1}{\partial \alpha} \frac{\sin \lambda z}{\lambda s \cos \lambda s} z \, dz} \\ &\quad \left. \times \left[ f_a(z) \{1 - \cos \lambda(z - s_1)\} - \frac{\sin \lambda(s - s_1)}{\cos \lambda s} \sin \lambda z \right] \right\} z \, dz \end{aligned}$$

Integration of the right-hand side of this equation gives

$$\frac{(ps/V)}{\xi} = \frac{\left( \frac{\cos \lambda s_1}{\cos \lambda s} - 1 \right) \frac{1}{\partial c_1 / \partial \alpha} \frac{\partial c_1}{\partial \xi} + \left( \frac{\cos \lambda s_1}{\cos \lambda s} - 1 - \lambda^2 \frac{s^2 - s_1^2}{2} \right) \frac{1}{e(\partial c_1 / \partial \alpha)} \frac{\partial c_{m,0}}{\partial \xi}}{\left( \frac{\tan \lambda s}{\lambda s} - 1 \right)} \quad (29.34)$$

The aileron reversal speed occurs when the aileron effectiveness is zero. Thus, equating the numerator of Eq. (29.34) to zero, we obtain the transcendental equation:

$$\left( \frac{\partial c_1}{\partial \xi} + \frac{1}{e} \frac{\partial c_{m,0}}{\partial \xi} \right) (\cos \lambda s - \cos \lambda s_1) + \left( \lambda^2 \frac{s^2 - s_1^2}{2} \frac{1}{e} \frac{\partial c_{m,0}}{\partial \xi} \right) \cos \lambda s = 0 \quad (29.35)$$

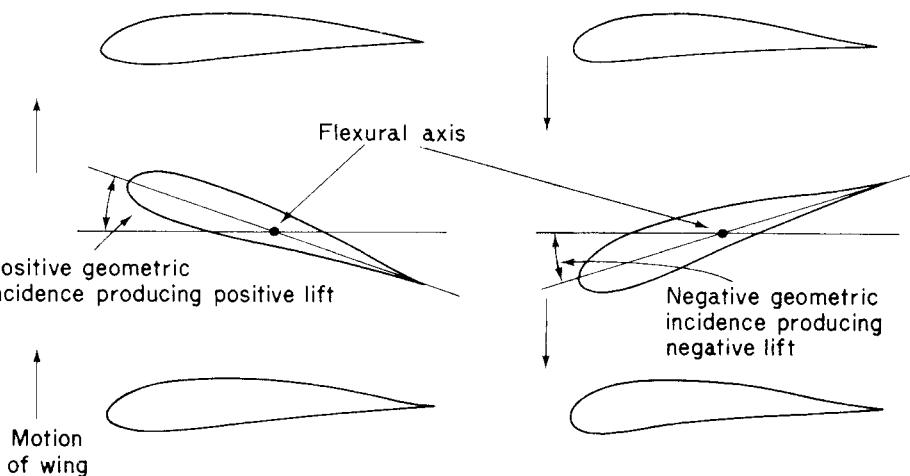
Alternative methods of obtaining divergence and control reversal speeds employ matrix or energy procedures. Details of such treatments may be found in Bisplinghoff, Ashley, and Halfman.<sup>3</sup>

## 29.4 INTRODUCTION TO “FLUTTER”

We previously defined “flutter” as the dynamic instability of an elastic body in an airstream. It is found most frequently in aircraft structures subjected to large aerodynamic loads, such as wings, tail units, and control surfaces. Flutter occurs at a critical or flutter speed  $V_f$ , which in turn is defined as the lowest airspeed at which a given structure oscillates with sustained simple harmonic motion. Flight at speeds below and above the flutter speed represents conditions of stable and unstable (that is, divergent) structural oscillation, respectively.

Generally, an elastic system having just one degree of freedom cannot be unstable unless some peculiar mechanical characteristic exists, such as a negative spring force or a negative damping force. However, it is possible for systems with two or more degrees of freedom to be unstable without possessing unusual characteristics. The forces associated with each degree of freedom can interact, causing divergent oscillations for certain phase differences. The flutter of a wing in which the flexural and torsional modes are coupled is an important example of this type of instability. Some indication of the physical nature of *wing-bending–torsion–flutter* may be had from an examination of aerodynamic and inertia forces during a combined bending and torsional oscillation, in which the individual motions are  $90^\circ$  out of phase. In a pure bending or pure torsional oscillation, the aerodynamic forces produced by the effective wing incidence oppose the motion; the geometric incidence in pure bending remains constant and therefore does not affect the aerodynamic damping force, while in pure torsion, the geometric incidence produces aerodynamic forces that oppose the motion during one half of the cycle but assist it during the other half, so that the overall effect is nil. Thus, pure bending or pure torsional oscillations are quickly damped out. This is not the case in the combined oscillation when the maximum twist occurs at zero bending and vice versa; that is, a  $90^\circ$  phase difference.

Consider the wing shown in Fig. 29.7 in various stages of a bending–torsion oscillation. At the position of zero bending, the twisting of the wing causes a positive geometric incidence and therefore an aerodynamic force in the same direction as the motion of the wing. A similar but reversed situation



**FIGURE 29.7 Coupling of Bending and Torsional Oscillations and Destabilizing Effect of Geometric Incidence**

exists as the wing moves in a downward direction; the negative geometric incidence due to wing twist causes a downward aerodynamic force. It follows that, although the effective wing incidence produces aerodynamic forces that oppose the motion at all stages, the aerodynamic forces associated with the geometric incidence have a destabilizing effect. At a certain speed—the flutter speed  $V_f$ —this destabilization action becomes greater than the stabilizing forces, and the oscillations diverge. In practical cases, the bending and torsional oscillations are not as much as  $90^\circ$  out of phase; however, the same basic principles apply.

The type of flutter just described, in which two distinctly different types of oscillating motion interact such that the resultant motion is divergent is known as *classical flutter*. Other types of flutter, *nonclassical flutter*, may involve only one type of motion.

For example, *stalling flutter* of a wing occurs at a high incidence, where, for particular positions of the spanwise axis of twist, self-excited twisting oscillations occur that, above a critical speed, diverge.

Another nonclassical form of flutter, *aileron buzz*, occurs at high subsonic speeds and is associated with the shock wave on the wing forward of the aileron. If the aileron oscillates downward, the flow over the upper surface of the wing accelerates, intensifying the shock and resulting in a reduction in pressure in the boundary layer behind the shock. The aileron, therefore, tends to be sucked back to its neutral position. When the aileron rises the shock intensity reduces and the pressure in the boundary layer increases, tending to push the aileron back to its neutral position. At low frequencies, these pressure changes are approximately  $180^\circ$  out of phase with the aileron deflection and therefore become aerodynamic damping forces. At higher frequencies, a component of pressure appears in phase with the aileron velocity, which excites the oscillation. If this is greater than all other damping actions on the aileron, a high-frequency oscillation results, in which only one type of motion, rotation of the aileron about its hinge, is present—that is, aileron buzz. Aileron buzz may be prevented by employing control jacks of sufficient stiffness to ensure that the natural frequency of aileron rotation is high.

*Buffeting* is produced most commonly in a tailplane by eddies caused by poor airflow in the wing wake striking the tailplane at a frequency equal to its natural frequency; a resonant oscillation having one degree of freedom could then occur. The problem may be alleviated by proper positioning of the tailplane and clean aerodynamic design.

### 29.4.1 Coupling

We saw that the classical flutter of an aircraft wing involves the interaction of flexural and torsional motions. Separately neither motion causes flutter, but together, at critical values of amplitude and phase angle, the forces produced by one motion excite the other; the two types of motion are then said to be *coupled*. Various forms of coupling occur: inertial, aerodynamic, and elastic.

The cross-section of a small length of wing is shown in Fig. 29.8. Its center of gravity is a distance  $gc$  ahead of its flexural axis,  $c$  is the wing section chord and the mass of the small length of wing is  $m$ . If the length of wing is subjected to an upward acceleration  $\ddot{y}$  an accompanying inertia force,  $m\ddot{y}$  acts at its center of gravity in a downward direction, thereby producing a nose-down torque about the flexural axis of  $m\ddot{y}gc$ , causing the wing to twist. The vertical motion therefore induces a twisting motion by virtue of the inertia forces present, that is, *inertial coupling*. Conversely, an angular acceleration  $\ddot{\alpha}$  about the flexural axis causes a linear acceleration of  $gc\ddot{\alpha}$  at the center of gravity with a corresponding inertia force of  $mgc\ddot{\alpha}$ . Thus, angular acceleration generates a force producing translation, again inertial

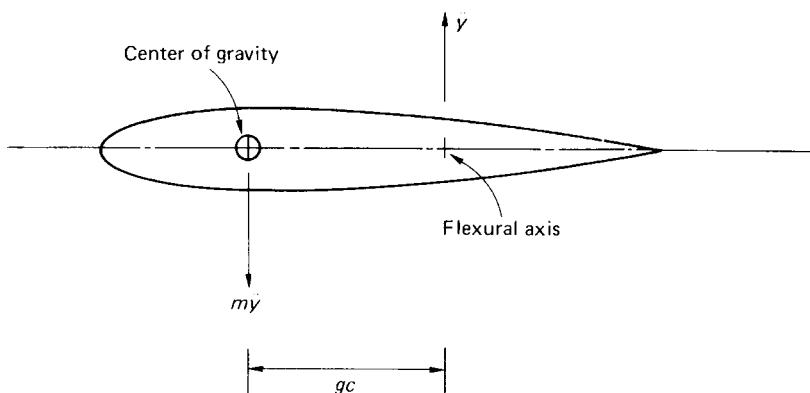


FIGURE 29.8 Inertial Coupling of a Wing

coupling. Note that the inertia torque due to unit linear acceleration ( $mgc$ ) is equal to the inertia force due to unit angular acceleration ( $mgc$ ); the inertial coupling therefore possesses symmetry.

*Aerodynamic coupling* is associated with changes of lift produced by wing rotation or translation. A change of wing incidence—that is, a rotation of the wing—induces a change of lift, which causes translation while a translation of velocity  $\dot{y}$ , say, results in an effective change in incidence, thereby yielding a lift which causes rotation. These aerodynamic forces, which oscillate in a flutter condition, act through a center analogous to the aerodynamic center of a wing in steady motion; this center is known as the *center of independence*.

Consider now the wing section shown in Fig. 29.9 and suppose that the wing stiffness is represented by a spring of stiffness  $k$  positioned at its flexural axis. Suppose also that the displacement of the wing is defined by the vertical deflection  $y$  of an arbitrary point O (Fig. 29.9(a)) and a rotation  $\alpha$  about O (Fig. 29.9(b)). In Fig. 29.9(a), the vertical displacement produces a spring force that causes a clockwise torque ( $kyd$ ) on the wing section about O, resulting in an increase in wing incidence  $\alpha$ . In Fig. 29.9(b), the clockwise rotation  $\alpha$  about O results in a spring force  $kdx$  acting in an upward direction on the wing section, thereby producing translations in the positive  $y$  direction. Thus, translation and rotation are

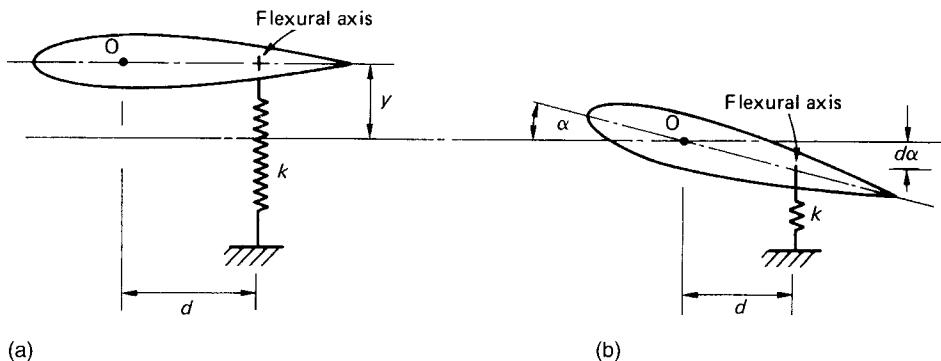


FIGURE 29.9 Elastic Coupling of a Wing

coupled by virtue of the elastic stiffness of the wing, hence *elastic coupling*. We note that, as in the case of inertial coupling, elastic coupling possesses symmetry, since the moment due to unit displacement ( $kd$ ) is equal to the force produced by the unit rotation ( $kd$ ). Also, if the arbitrarily chosen point O is made to coincide with the flexural axis,  $d = 0$ , and the coupling disappears.

From the preceding, it can be seen that flutter is prevented by uncoupling the two constituent motions. Thus, inertial coupling is prevented if the center of gravity coincides with the flexural axis, while aerodynamic coupling is eliminated when the center of independence coincides with the flexural axis. This, in fact, also eliminates elastic coupling, since O in Fig. 29.9 generally is the center of independence. Unfortunately, in practical situations, the center of independence is usually forward of the flexural axis, while the center of gravity is behind it, giving conditions that promote flutter.

### 29.4.2 Determination of critical flutter speed

Consider a wing section of chord  $c$  oscillating harmonically in an airflow of velocity  $V$  and density  $\rho$  and having instantaneous displacements, velocities, and accelerations of, rotationally,  $\alpha, \dot{\alpha}, \ddot{\alpha}$ , and, translationally,  $y, \dot{y}, \ddot{y}$ . The oscillation causes a reduction in lift from the steady state lift,<sup>4</sup> so that, in effect, the lift due to the oscillation acts downward. The downward lift corresponding to  $\alpha, \dot{\alpha}$ , and  $\ddot{\alpha}$  is, respectively,

$$\begin{aligned} l_\alpha \rho c V^2 \alpha &= L_\alpha \alpha \\ l_{\dot{\alpha}} \rho c^2 V \dot{\alpha} &= L_{\dot{\alpha}} \dot{\alpha} \\ l_{\ddot{\alpha}} \rho c^3 \ddot{\alpha} &= L_{\ddot{\alpha}} \ddot{\alpha} \end{aligned}$$

in which  $l_\alpha, l_{\dot{\alpha}}, l_{\ddot{\alpha}}$  are nondimensional coefficients analogous to the lift-curve slopes in steady motion. Similarly, downward forces due to the translation of the wing section occur and are

$$\begin{aligned} l_y \rho c V^2 y/c &= L_y y \\ l_{\dot{y}} \rho c^2 V \dot{y}/c &= L_{\dot{y}} \dot{y} \\ l_{\ddot{y}} \rho c^3 \ddot{y}/c &= L_{\ddot{y}} \ddot{y} \end{aligned}$$

Thus, the total aerodynamic lift on the wing section due to the oscillating motion is given by

$$L = L_y y + L_{\dot{y}} \dot{y} + L_{\ddot{y}} \ddot{y} + L_\alpha \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_{\ddot{\alpha}} \ddot{\alpha} \quad (29.36)$$

We previously saw that rotational and translational displacements produce moments about any chosen center. Thus, the total nose-up moment on the wing section is

$$M = M_y y + M_{\dot{y}} \dot{y} + M_{\ddot{y}} \ddot{y} + M_\alpha \alpha + M_{\dot{\alpha}} \dot{\alpha} + M_{\ddot{\alpha}} \ddot{\alpha} \quad (29.37)$$

where

$$\begin{aligned} M_y y &= l_y \rho c^2 V^2 y/c \\ M_{\dot{y}} \dot{y} &= l_{\dot{y}} \rho c^3 V \dot{y}/c \\ M_{\ddot{y}} \ddot{y} &= l_{\ddot{y}} \rho c^4 \ddot{y}/c \\ M_\alpha \alpha &= m_\alpha \rho c^2 V^2 \alpha/c \\ M_{\dot{\alpha}} \dot{\alpha} &= m_{\dot{\alpha}} \rho c^3 V \dot{\alpha}/c \\ M_{\ddot{\alpha}} \ddot{\alpha} &= m_{\ddot{\alpha}} \rho c^4 \ddot{\alpha}/c \end{aligned}$$

in which  $m_\alpha$  and the like are analogous to the steady motion local pitching moment coefficients.

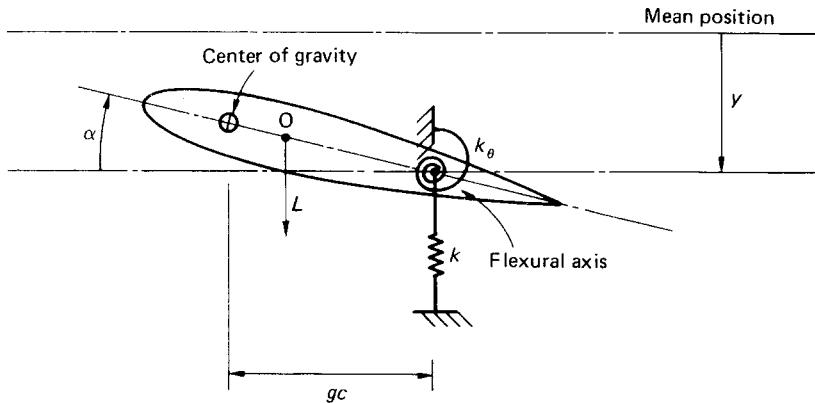


FIGURE 29.10 Flutter of a Wing Section

Now consider the wing section shown in Fig. 29.10. The wing section is oscillating about a mean position, and its flexural and torsional stiffnesses are represented by springs of stiffness  $k$  and  $k_0$ , respectively. Suppose that its instantaneous displacement from the mean position is  $y$ , which is now taken as positive downward. In addition to the aerodynamic lift and moment forces of Eqs. (29.36) and (29.37), the wing section experiences inertial and elastic forces and moments. Thus, if the mass of the wing section is  $m$  and  $I_O$  is its moment of inertia about  $O$ , instantaneous equations of vertical force and moment equilibrium may be written as follows. For vertical force equilibrium,

$$L - m\ddot{y} + mgc\ddot{\alpha} - ky = 0 \quad (29.38)$$

and, for moment equilibrium about  $O$ ,

$$M - I_O\ddot{\alpha} + mgc\ddot{y} - k_0\alpha = 0 \quad (29.39)$$

Substituting for  $L$  and  $M$  from Eqs. (29.36) and (29.37), we obtain

$$(m - L_{\ddot{y}})\ddot{y} - L_{\dot{y}}\dot{y} + (k - L_y)y - (mgc + L_{\ddot{\alpha}})\ddot{\alpha} - L_{\dot{\alpha}}\dot{\alpha} - L_{\alpha}\alpha = 0 \quad (29.40)$$

$$-(mgc - M_{\ddot{y}})\ddot{y} - M_{\dot{y}}\dot{y} - M_yy + (I_O - M_{\ddot{\alpha}})\ddot{\alpha} - M_{\dot{\alpha}}\dot{\alpha} + (k_0 - M_{\alpha})\alpha = 0 \quad (29.41)$$

The terms involving  $y$  in the force equation and  $\alpha$  in the moment equation are known as *direct* terms, while those containing  $\alpha$  in the force equation and  $y$  in the moment equation are known as *coupling* terms.

The critical flutter speed  $V_f$  is contained in Eqs. (29.40) and (29.41) within the terms  $L_y, L_{\dot{y}}, L_{\ddot{y}}, L_{\alpha}, L_{\dot{\alpha}}, M_y, M_{\dot{y}}, M_{\ddot{y}}$ , and  $M_{\alpha}$ . Its value corresponds to the condition that these equations represent simple harmonic motion. Above this critical value, the equations represent divergent oscillatory motion, while at lower speeds, they represent damped oscillatory motion. For simple harmonic motion,

$$y = y_0 e^{i\omega t}, \quad \alpha = \alpha_0 e^{i\omega t}$$

Substituting in Eqs. (29.40) and (29.41) and rewriting in matrix form, we obtain

$$\begin{bmatrix} -\omega^2(m - L_{\ddot{y}}) - i\omega L_{\dot{y}} + k - L_y & \omega^2(mgc + L_{\ddot{\alpha}}) - i\omega L_{\dot{\alpha}} - L_{\alpha} \\ \omega^2(mgc + M_{\ddot{y}}) - i\omega M_{\dot{y}} - M_y & -\omega^2(I_O - M_{\ddot{\alpha}}) - i\omega M_{\dot{\alpha}} + k_0 - M_{\alpha} \end{bmatrix} \begin{Bmatrix} y_0 \\ \alpha_0 \end{Bmatrix} = 0 \quad (29.42)$$

The solution of Eq. (29.42) is most readily obtained by computer<sup>4</sup> for which several methods are available. One method represents the motion of the system at a general speed  $V$  by

$$y = y_0 e^{(\delta+i\omega)t}, \quad \alpha = \alpha_0 e^{(\delta+i\omega)t}$$

in which  $\delta + i\omega$  is one of the complex roots of the determinant of Eq. (29.42). For any speed  $V$ , the imaginary part  $\omega$  gives the frequency of the oscillating system while  $\delta$  represents the exponential growth rate. At low speeds, the oscillation decays ( $\delta$  is negative) and, at high speeds, it diverges ( $\delta$  is positive). Zero growth rate corresponds to the critical flutter speed  $V_f$ , which may therefore be obtained by calculating  $\delta$  for a range of speeds and determining the value of  $V_f$  for  $\delta = 0$ .

### 29.4.3 Prevention of flutter

We previously saw that flutter can be prevented by eliminating inertial, aerodynamic, and elastic coupling by arranging for the center of gravity, the center of independence, and the flexural axis of the wing section to coincide. The means by which this may be achieved are indicated in the coupling terms in Eqs. (29.40) and (29.41).

In Eq. (29.41), the inertial coupling term is  $mgc + M_{\dot{y}}$ , in which  $M_{\dot{y}}$  is usually very much smaller than  $mgc$ . Thus, inertial coupling may be virtually eliminated by adjusting the position of the center of gravity of the wing section through *mass balancing* so that it coincides with the flexural axis; that is,  $gc = 0$ . The aerodynamic coupling term  $M_y\dot{y}$  vanishes, as we saw, when the center of independence coincides with the flexural axis. Further, the terms  $M_{\dot{y}}y$  and  $L_{\dot{\alpha}}\dot{\alpha}$  are very small and may be neglected so that Eqs. (29.40) and (29.41) now reduce to

$$(m - L_{\ddot{y}})\ddot{y} - L_y\dot{y} + (k - L_y)y - L_{\dot{\alpha}}\alpha = 0 \quad (29.43)$$

and

$$(I_O - M_{\ddot{\alpha}})\ddot{\alpha} - M_{\dot{\alpha}}\dot{\alpha} + (k_0 - M_{\alpha})\alpha = 0 \quad (29.44)$$

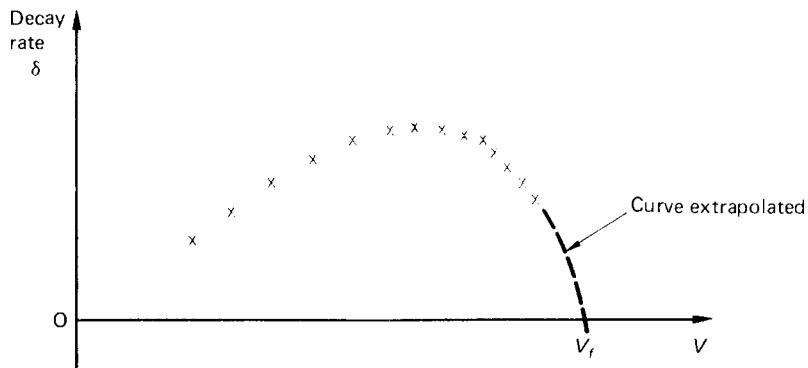
The remaining coupling term  $L_{\dot{\alpha}}\alpha$  cannot be eliminated, since the vertical force required to maintain flight is produced by wing incidence.

Equation (29.44) governs the torsional motion of the wing section and contains no coupling terms, so that, since all the coefficients are positive at speeds below the wing section torsional divergence speed, any torsional oscillation produced, say, by a gust decays. Also, from Eq. (29.43), it would appear that a vertical oscillation could be maintained by the incidence term  $L_{\dot{\alpha}}\alpha$ . However, rotational oscillations, as we saw from Eq. (29.44), decay, so that the lift force  $L_{\dot{\alpha}}\alpha$  is a decaying force and cannot maintain any vertical oscillation.

In practice, it is not always possible to prevent flutter by eliminating coupling terms. However, increasing structural stiffness, although carrying the penalty of increased weight, can raise the value of  $V_f$  above the operating speed range. Further, arranging for the center of gravity of the wing section to be as close as possible to and forward of the flexural axis is beneficial. Thus, wing-mounted jet engines are housed in pods well ahead of the flexural axis of the wing.

### 29.4.4 Experimental determination of flutter speed

The previous analysis has been concerned with the flutter of a simple two degrees of freedom model. In practice, the structure of an aircraft can oscillate in many different ways. For example, a wing has fundamental bending and torsional modes of oscillation on which secondary or overtone modes of



**FIGURE 29.11 Experimental Determination of Flutter Speed**

oscillation are superimposed. Also, it is possible for fuselage bending oscillations to produce changes in wing camber, thereby affecting wing lift, and for control surfaces oscillating about their hinges to produce aerodynamic forces on the main surfaces.

The equations of motion for an actual aircraft are therefore complex with a number  $N$ , say, of different motions being represented ( $N$  can be as high as 12). There are, therefore,  $N$  equations of motion that are aerodynamically coupled. At a given speed, solution of these  $N$  equations yields  $N$  different values of  $\delta + i\omega$  corresponding to the  $N$  modes of oscillation. Again, as in the simple two degrees of freedom case, the critical flutter speed for each mode may be found by calculating  $\delta$  for a range of speeds and determining the value of speed at which  $\delta = 0$ .

A similar approach is used experimentally on actual aircraft. The aircraft is flown at a given steady speed and caused to oscillate either by exploding a small detonator on the wing or control surface or by a sudden control jerk. The resulting oscillations are recorded and analyzed to determine the decay rate. The procedure is repeated at increasing speeds with smaller increments being used at higher speeds. The measured decay rates are plotted against speed, producing a curve such as that shown in Fig. 29.11. This curve is then extrapolated to the zero decay point that corresponds to  $V_f$ . Clearly, this approach requires as accurate as possible a preliminary estimation of flutter speed, since induced oscillations above the flutter speed diverge, leading to possibly catastrophic results.

Other experimental work involves wind tunnel tests on flutter models, the results being used to check theoretical calculations.<sup>3</sup>

#### 29.4.5 Control surface flutter

If a control surface oscillates about its hinge, oscillating forces are induced on the main surface. For example, if a wing oscillates in bending at the same time as the aileron oscillates about its hinge, flutter can occur provided there is a phase difference between the two motions. In similar ways, elevator and rudder flutter can occur as the fuselage oscillates in bending. Other forms of control surface flutter involve more than two different types of motion. Included in this category are wing bending/aileron rotation/tab rotation and elevator rotation/fuselage bending/rigid body pitching and translation of the complete aircraft.

It can be shown<sup>4</sup> that control surface flutter can be prevented by eliminating the inertial coupling between the control rotation and the motion of the main surface. This may be achieved by mass balancing the control surface whereby weights are attached to the control surface forward of the hinge line.

All newly designed aircraft are subjected early in the life of a prototype to a *ground resonance test* to determine actual normal modes and frequencies. The primary objectives of such tests are to check the accuracy of the calculated normal modes on which the flutter predictions are based and to show up any unanticipated peculiarities in the vibrational behavior of the aircraft. Usually, the aircraft rests on some low-frequency support system or even on its deflated tires. Electrodynamic exciters are mounted in pairs on the wings and tail with accelerometers as the measuring devices. The test procedure is generally first to discover the resonant frequencies by recording the amplitude and phase of a selected number of accelerometers over a given frequency range. Having obtained the resonant frequencies, the aircraft is then excited at each of these frequencies in turn and all accelerometer records taken simultaneously.

## References

- [1] Babister AW. Aircraft stability and control. London: Pergamon Press; 1961.
- [2] Duncan WJ. The principles of the control and stability of aircraft. Cambridge: Cambridge University Press; 1959.
- [3] Bisplinghoff RL, Ashley H, Halfman RL. Aero elasticity. Cambridge, MA: Addison-Wesley; 1955.
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## PROBLEMS

**P.29.1.** The wing of Example 29.1 is required to have a minimum divergence speed of 350 m/s. If the torsional stiffness of the wing is to remain unchanged, calculate the required movement aft of the aerodynamic center.

Answer: 0.51 m.

**P.29.2.** A finite wing has a torsional stiffness of  $12 \times 10^6 \text{ Nm}^2$ , a span of 8.0 m, and a mean chord of 2.8 m. If the two-dimensional lift-curve slope is 3.1 and its aerodynamic center is 1.2 m forward of its flexural axis, calculate the wing divergence speed at sea level.

Answer: 269.2 m/s.

**P.29.3.** Calculate the aileron reversal speed for the wing of P.29.2 if the rate of change of lift coefficient with aileron angle is 0.15 and the rate of change of pitching moment coefficient with aileron angle is -0.3. What is the aileron effectiveness at an aircraft speed of 160 m/s?

Answer: 224.4 m/s. 76%.

**P.29.4.** An initially untwisted rectangular wing of semi-span  $s$  and chord  $c$  has its flexural axis normal to the plane of symmetry and is of constant cross-section with torsional rigidity  $GJ$ . The aerodynamic center is  $ec$  ahead of the flexural axis, the lift-coefficient slope is  $a$ , and the pitching moment coefficient at zero lift is  $C_{m,0}$ . At speed  $V$  in air of density  $\rho$ , the wing-root incidence from zero lift is  $\alpha_0$ . Using simple strip theory—that is, ignoring the downwash effects—show that the incidence at a section distant  $y$  from the plane of symmetry is given by

$$\alpha_0 + \theta = \left( \frac{C_{m,0}}{ea} + \alpha_0 \right) \frac{\cos \lambda(s-y)}{\cos \lambda s} - \frac{C_{m,0}}{ea}$$

where

$$\lambda^2 = \frac{ea\frac{1}{2}\rho V^2 c^2}{GJ}$$

Hence, assuming  $C_{m,0}$  to be negative, find the condition giving the speed at which the lift would be reduced to zero.

Answer:  $V_d = \sqrt{\frac{\pi^2 GJ}{2\rho e c^2 s^2 \alpha}}$

**P.29.4. MATLAB** Using MATLAB and the function for  $V_d$  derived in Problem P.29.1, calculate the lift coefficient slope ( $a$ ) assuming that  $G = 25,000 \text{ N/mm}^2$ ,  $\rho = 1.2 \text{ kg/m}^3$ ,  $ec = 25 \text{ mm}$ ,  $s = 500 \text{ mm}$ , and  $V_d$  ranges from 3 to 12 m/s in increments of 1 m/s.

Answer:	$V_d$	$a$
(i)	3 m/s	7.3108
(ii)	3 m/s	4.1123
(iii)	3 m/s	2.6319
(iv)	3 m/s	1.8277
(v)	3 m/s	1.3428
(vi)	3 m/s	1.0281
(vii)	3 m/s	0.81231
(viii)	3 m/s	0.65797
(ix)	3 m/s	0.54378
(x)	3 m/s	0.45693

**P.29.5.** The rectangular wing shown in Fig. P.29.5 has a constant torsional rigidity  $GJ$  and an aileron of constant chord. The aerodynamic center of the wing is at a constant distance  $ec$  ahead of the flexural axis, while the additional lift due to operation of the aileron acts along a line a distance  $hc$  aft of the

flexural axis; the local, two-dimensional lift-curve slopes are  $a_1$  for the wing and  $a_2$  for aileron deflection. Using strip theory and considering only the lift due to the change of incidence arising from aileron movement, show that the aileron reversal speed is given by

$$\tan \lambda ks \int_0^{ks} y \sin \lambda y \, dy - \tan \lambda s \int_0^s y \sin \lambda y \, dy - \int_{ks}^s y \cos \lambda y \, dy = \frac{(e + h)}{2h \cos \lambda ks} [(ks)^2 - s^2]$$

where

$$\lambda^2 = \frac{1}{2} \rho V^2 a_1 e c^2 / GJ$$

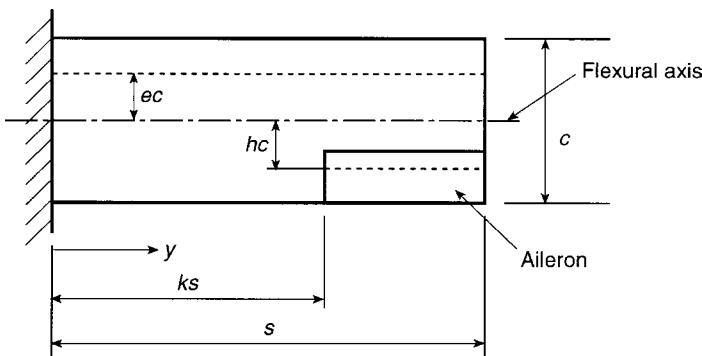


FIGURE P.29.5

# Appendix: Design of a rear fuselage

Figure A.1 shows the elevation of a two-seater trainer/semi-aerobatic aircraft. It is required to carry out the detailed structural design of the portion of the rear fuselage between the sections AA and BB.

---

## A.1 SPECIFICATION

The required flight envelope for this particular aircraft is shown in Fig. A.2 (refer also to Fig. 13.1), where

$$n_1 = 6.28, \quad V_D(\text{design diving speed}) = 183.8 \text{ m/s}$$

Also,

$$V_C = 0.8V_D = 147.0 \text{ m/s}$$

$$n_2 = 0.75n_1 = 4.71$$

$$n_3 = 0.5n_1 = 3.14$$

Note also that airworthiness requirements specify that, since  $n_1 > 3$ , the point D<sub>2</sub> lies on the  $n = 0$  axis.  
Further requirements are that

1. At any point in the flight envelope, an additional pitching acceleration given by

$$\left(20 + \frac{475}{W}\right) \frac{n}{V} \text{ rad/s}^2 \quad (\text{A.1})$$

be applied where  $W$  is the total weight of the aircraft in kN and  $V$  is the velocity of the aircraft in m/s.

2. For asymmetric flight, an angle of yaw given by

$$\psi = 0.7n_1 + \frac{457.2}{V_D} \text{ degrees} \quad (\text{A.2})$$

must be allowed for; the angle of yaw increases the overall pitching moment coefficient of the aircraft by  $-0.0015/\text{degree of yaw}$ .

---

## A.2 DATA

Preliminary design work produced the following data.

### *Aircraft*

Fully loaded weight = 37.43 kN.

Moment of inertia of fully loaded aircraft about the center of gravity (G in Fig. A.1)

$$= 22,235 \text{ kgm}^2$$

Position of G and the body drag centers, engine on and off, are shown in Fig. A.1.

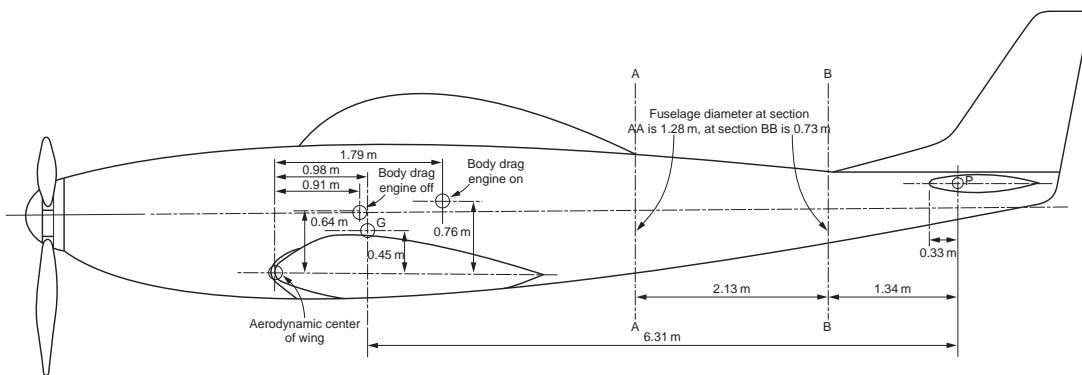


FIGURE A.1 Aircraft of A.1 Specification

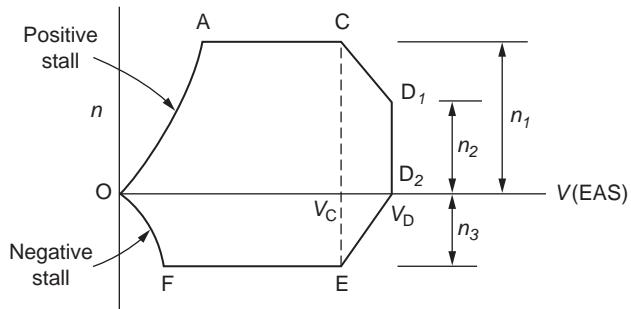


FIGURE A.2 Flight Envelope

The body drag coefficients are

$$C_{D,B} \text{ (engine on)} = 0.01583$$

$$C_{D,B} \text{ (engine off)} = 0.0576$$

The engine has a maximum horse power of 905 and the propeller efficiency is 90 percent.

### **Wing**

The wing has a span of 14.07 m and gross area of  $29.64 \text{ m}^2$ . Its aerodynamic mean chord,  $c = 2.82 \text{ m}$  and the variations of lift and drag coefficients with incidence are shown in Fig. A.3.

Also, the pitching moment coefficient is given by

$$C_M = -0.238C_L$$

and, due to a rigger's incidence of  $-1.5^\circ$ , there is an additional pitching moment coefficient equal to  $-0.036$ .

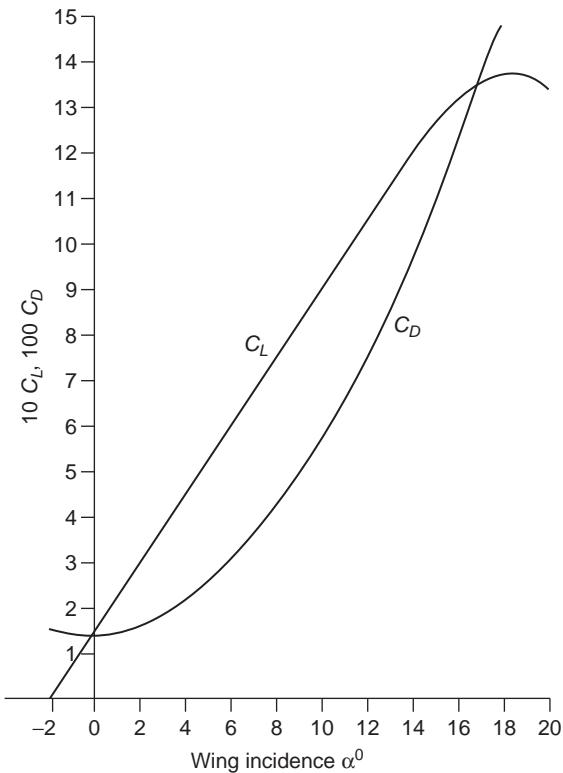


FIGURE A.3 Wing Characteristics

### Tailplane

The tailplane has a span of 6.55 m and a gross area of  $8.59 \text{ m}^2$ ; the position of the aerodynamic center, P, of the tailplane is shown in Fig. A.1.

Due to the asymmetry of the slipstream induced by yaw, asymmetric loads are induced on the tailplane. These loads, upward on one side and downward on the other, result in a torque given by

$$\frac{0.00125}{\sqrt{1 - M^2}} \rho V^2 S_t b_t \psi \quad (\text{A.3})$$

where  $M$  is the Mach number,  $S_t$  the tailplane area, and  $b_t$  the tailplane span.

### Fin

The fin has a height of 1.65 m, an area of  $1.80 \text{ m}^2$ , and an aspect ratio of 1.5. Also, it may be shown that the lift-curve slope,  $a_1$ , of the fin is given by

$$a_1 = \frac{5.5A}{A + 2} \quad (\text{A.4})$$

in which  $A$  is the aspect ratio of a wing, which is equivalent to two fins.

In yawed flight, the incidence of the fin to the air flow is  $\psi$ , so that a fin load equal to  $\frac{1}{2} \rho V^2 S_F a_1 \psi$  is generated where  $V$  is the aircraft speed and  $S_F$  the fin area.

The position of the center of pressure of the fin depends upon the geometry of the pressure distribution. Calculations show that the center of pressure is 1.13 m above the axis of the rear fuselage and a distance of 3.7 m aft of the section AA.

### A.3 INITIAL CALCULATIONS

#### *Flight envelope*

The positive stall curve in the flight envelope of Fig. A.2 is found from basic aerodynamic wing theory and is given by

$$C_{L,\max} = \frac{nW}{\frac{1}{2} \rho V_s^2 S}$$

where  $V_s$  is the stalling speed and  $S$  the wing area. Then,

$$V_s = \left( \frac{2nW}{\rho S C_{L,\max}} \right)^{1/2} \quad (\text{A.5})$$

Substituting the values given in the preceding data and taking  $\rho$ , the air density at sea level, as 1.226 kg/m<sup>3</sup>,

$$V_s = \left( \frac{2 \times 37.43 \times 10^3}{1.226 \times 29.64 \times 1.38} \right)^{1/2} (n)^{1/2}$$

that is,

$$V_s = 38.6(n)^{1/2} \quad (\text{A.6})$$

The positive stall curve is found by assigning a series of values to  $n$  and calculating the corresponding stalling speeds. For  $n = n_1 = 6.28$ ,

$$V_s = 38.6(6.28)^{1/2} = 96.7 \text{ m/s (A on flight envelope)}$$

#### *Fin lift-curve slope*

From Eq. (A.4),

$$a_1 = \frac{5.5 \times 3.0}{3.0 + 2.0} = 3.3$$

#### *Speed of sound*

At sea level at a temperature of 15 °C, the speed of sound is 340.8 m/s.

## A.4 BALANCING OUT CALCULATIONS

The tailplane and fin loads corresponding to the various critical points in the flight envelope are now calculated, so that, subsequently, values of shear force, bending moment, and torque acting on the rear fuselage may be determined. The cases to be investigated are

- Case A (point A on the flight envelope, engine on)
- Case A' (point A on the flight envelope, engine off)
- Case C (point C on the flight envelope, engine off)
- Case D<sub>1</sub> (point D<sub>1</sub> on the flight envelope, engine off)
- Case D<sub>2</sub> (point D<sub>2</sub> on the flight envelope, engine off)

### Case A

From the flight envelope  $n = 6.28$ ,  $V = 96.7 \text{ m/s}$  and, from Fig. A.3, the wing incidence  $\alpha$  corresponding to  $C_{L,\max} = 1.38$  is  $18^\circ$ . The forces acting on the aircraft and their lines of action are shown in Fig. A.4, the dimensions may be scaled from an actual drawing (the simplest approach) or calculated.

Since  $1 \text{ hp} = 746 \text{ W} = 746 \text{ mN/s}$ , the thrust  $T$  of the engine is given by

$$T = \frac{\eta \times \text{hp} \times 746}{V}$$

that is,

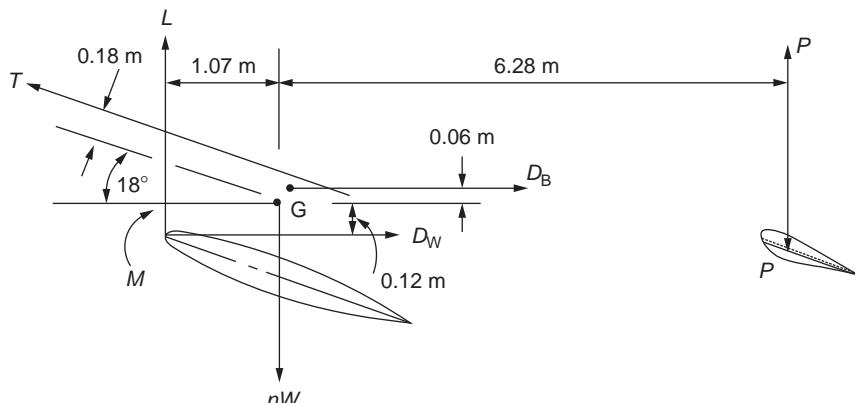
$$T = \frac{0.9 \times 905 \times 746}{96.7}$$

so that

$$T = 6,284 \text{ N}$$

Also,

$$nW = 6.28 \times 37.43 \times 10^3 = 235,060 \text{ N}$$



**FIGURE A.4** Balancing out Calculations

A first approximation for the wing lift,  $L$ , is obtained by neglecting the tailplane lift,  $P$ , that is,

$$L = nW - T \sin(18^\circ - 1.5^\circ)$$

which gives

$$L = 235,060 - 6,284 \sin 16.5^\circ = 233,275 \text{ N}$$

From Fig. A.3, the wing drag coefficient,  $C_{D,W}$ , is 0.149, so that the wing drag, which is given by

$$D_W = C_{D,W} \frac{1}{2} \rho V^2 S$$

is

$$D_W = 0.149 \times 1.226 \times 96.7^2 \times 29.64/2 = 25,315 \text{ N}$$

The body drag coefficient is 0.01583, so that

$$D_B = 0.01583 \times 1.226 \times 96.7^2 \times 29.64/2 = 2,690 \text{ N}$$

The angle of yaw is given by Eq. (A.2), that is,

$$\psi = 0.7 \times 6.28 + \frac{457.2}{183.8} = 6.9^\circ$$

The total pitching moment coefficient is then

$$C_M = -0.238 \times 1.38 - 0.036 - 0.0015 \times 6.9 = -0.375$$

so that

$$M = C_M \frac{1}{2} \rho V^2 Sc = -0.375 \times 1.226 \times 96.7^2 \times 29.64 \times 2.82/2 = -179,669 \text{ Nm}$$

The additional pitching moment acceleration is, from Eq. (A.1),

$$\left(20 + \frac{475}{37.43}\right) \frac{6.28}{96.7} = 2.12 \text{ rad/s}^2$$

Then, taking moments about G (refer to Fig A.4),

$$1.07L - 0.18T + 0.06D_B - 0.12D_W - 6.28P - 179,669 = 22,235 \times 2.12$$

that is,

$$1.07L - 0.18 \times 6,284 + 0.06 \times 2,690 - 0.12 \times 25,315 - 6.28P - 179,669 = 22,235 \times 2.12$$

which simplifies to

$$5.78P = L - 215,715$$

First approximation:  $L = 233,275 \text{ N}$  gives  $P = 2,991 \text{ N}$

Second approximation:  $L = 233,275 - 2,991 = 230,284 \text{ N}$  gives  $P = 2,482 \text{ N}$

Third approximation:  $L = 233,275 - 2,482 = 230,793 \text{ N}$  gives  $P = 2,569 \text{ N}$

Fourth approximation:  $L = 233,275 - 2,569 = 230,706 \text{ N}$  gives  $P = 2,554 \text{ N}$

Fifth approximation:  $L = 233,275 - 2,554 = 230,721 \text{ N}$  gives  $P = 2,556 \text{ N}$

Therefore, the tail load:  $P = 2556 \text{ N}$

The torque produced by the asymmetric loading on the tailplane is given by Eq. (A.3), that is,

$$\text{Tailplane torque} = \frac{0.00125}{\sqrt{1 - (96.7/340.8)^2}} \times 1.226 \times 96.7^2 \times 8.59 \times 6.55 \times 6.9 = 5,802 \text{ Nm}$$

The load on the fin caused by the yawed flight is given by  $\frac{1}{2}\rho V^2 S \psi a_1$ , that is,

$$\text{Fin load} = 1.226 \times 96.7^2 \times 1.8(6.9 \times \pi/180) \times 3.3/2 = 4,100 \text{ N}$$

The torque produced on the fuselage by this fin load is  $4,100 \times 1.13 = 4,633 \text{ Nm}$ . The total torque on the rear fuselage is therefore given by

$$\text{Total torque (rear fuselage)} = 5,802 + 4,633 = 10,435 \text{ Nm}$$

The tail and fin loads and the rear fuselage torque corresponding to the remaining flight envelope cases are calculated in an identical manner and are listed in Table A.1.

**Table A.1** Remaining Flight Envelope Cases

Case	Tail load (N) (+ ↑)	Fin load (N) (+ →)	Fuselage torque (Nm) (+ ↓)
A	2,556	4,100	10,435
A'	2,292	4,100	10,435
C	596	9,501	24,957
D <sub>1</sub>	-4,997	12,460	34,031
D <sub>2</sub>	-9,412	5,340	14,635

## A.5 FUSELAGE LOADS

The dimensions of the portion of the rear fuselage to be designed are given in Fig. A.1.

### Fuselage section

The construction of structural components was discussed in Chapter 12, where it was seen that fuselages generally comprise arrangements of stringers, frames, and skin. For this particular aircraft, the fuselage is unpressurized, so that the frames do not support significant loads. However, they are required to maintain the fuselage shape but may therefore be nominal in size, suitable frame sections are suggested later. The combination of stringers and skin resists the shear forces, bending moments, and torques produced by self-weight and aerodynamic loads. For this purpose a circular cross-section meets the design requirements of the aircraft and is simple to fabricate and design.

Figure A.5 shows a possible section. Twenty-four stringers arranged symmetrically, each having the same cross-sectional area, are spaced at approximately 168 mm at the section AA and at 96 mm at the section BB.

### Material

An aluminum alloy will be used for both stringers and skin and has the following properties:

$$\begin{aligned}0.1\% \text{ Proof stress} &= 186 \text{ N/mm}^2 \\ \text{Shear strength} &= 117 \text{ N/mm}^2\end{aligned}$$

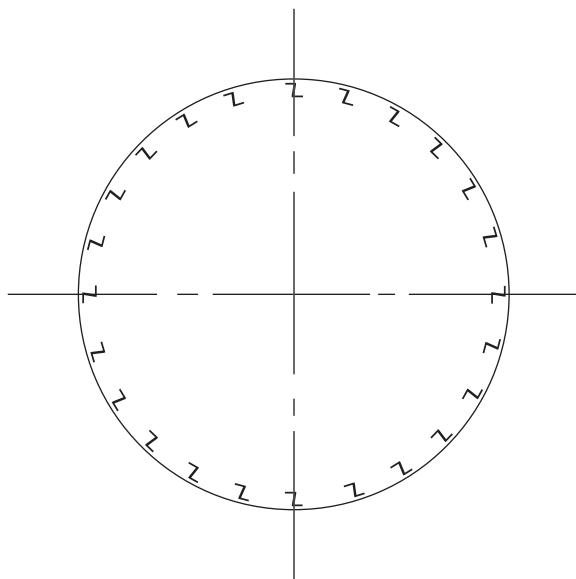


FIGURE A.5 Stringer Arrangement in Rear Fuselage

### **Self-weight**

In a conventional single-engine aircraft of the type shown in Fig. A.1, it is usual to assume that the fuselage weight is from 4.8 to 8.0 percent of the total weight and that the weight of the tailplane/fin assembly is from 1.2 to 2.5 percent of the total weight. It is further assumed in this case that half of the fuselage weight is aft of the section AA and that the weight distribution varies directly as the skin surface area. Therefore, taking average values,

$$\text{Weight of rear fuselage} = \frac{37.43 \times 10^3 \times 6.4}{2 \times 100} = 1,198 \text{ N}$$

$$\text{Weight of tailplane/fin} = \frac{37.43 \times 10^3 \times 1.8}{100} = 674 \text{ N}$$

For ease of calculation, the rear fuselage is assumed to taper uniformly, as shown in Fig. A.6; CC is a section midway between AA and BB. The total skin area is given by

$$\text{Skin area} = \pi(1.28 + 0.1) \times 4.57/2 = 9.91 \text{ m}^2$$

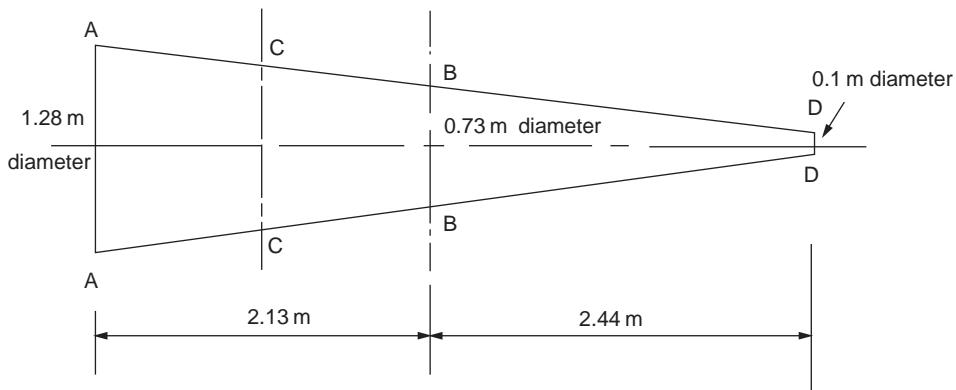
$$\text{At the section AA, the weight/m of fuselage} = 1,198 \times \pi \times 1.28/9.91 = 486.1 \text{ N/m}$$

$$\text{At the section CC, the weight/m} = 1,198 \times \pi \times 1.01/9.91 = 383.6 \text{ N/m}$$

$$\text{At the section BB, the weight/m} = 1,198 \times \pi \times 0.73/9.91 = 277.2 \text{ N/m}$$

$$\text{At the section DD, the weight/m} = 1,198 \times \pi \times 0.1/9.91 = 38.0 \text{ N/m}$$

Also, the center of gravity of the tailplane/fin assembly has been estimated to be 4.06 m from the section AA on a line parallel to the fuselage center line.



**FIGURE A.6 Rear Fuselage Sections**

### ***Shear forces and bending moments due to self-weight***

At the section AA,

$$SF = (1,198 + 674)n = 1,872n \text{ N} \quad (\text{A.7})$$

$$\begin{aligned} BM &= [(38.0 \times 4.57^2/2) + (448.1 \times 4.57^2/2 \times 3) + 674 \times 4.06]n \cos \alpha \\ &= 4,693n \cos \alpha \text{ Nm} \end{aligned} \quad (\text{A.8})$$

where  $n$  is the normal acceleration coefficient and  $\alpha$  the wing incidence.

At the section CC,

$$SF = [1,872 - (486.1 + 383.6) \times (2.13/2 \times 2)]n = 1,409n \text{ N} \quad (\text{A.9})$$

$$\begin{aligned} BM &= [(38.0 \times 3.51^2/2) + (345.6 \times 3.51^2/2 \times 3) + 674 \times 2.99]n \cos \alpha \\ &= 2,959n \cos \alpha \text{ Nm} \end{aligned} \quad (\text{A.10})$$

At the section BB,

$$SF = [1,872 - (486.1 + 277.2) \times (2.13/2)]n = 1,059n \text{ N} \quad (\text{A.11})$$

$$\begin{aligned} BM &= [(38.0 \times 2.44^2/2) + (239.2 \times 2.44^2/2 \times 3) + 674 \times 1.93]n \cos \alpha \\ &= 1,651n \cos \alpha \text{ Nm} \end{aligned} \quad (\text{A.12})$$

### ***Total shear forces, bending moments, and torques***

The values of shear force, bending moment, and torque at the sections AA, BB, and CC are now calculated for the flight envelope cases listed in [Section A.4](#).

Case A ( $n = 6.28$ ,  $\alpha = 18^\circ$ )

For Section AA, the shear force due to the self-weight and tail load is, from [Eq. \(A.7\)](#) and [Table A.1](#),

$$SF (S_y) = 1,872 \times 6.28 - 2,556 = 9,200 \text{ N} \text{ (acting vertically downward)}$$

The shear force due to the fin load is, from [Table A.1](#),

$$SF (S_x) = 4,100 \text{ N} \text{ (acting horizontally to the right)}$$

The bending moment due to the self-weight and tail load is, from Eq. (A.8) and Table A.1 (see also Fig. A.1),

$$\text{BM} (M_x) = 4,693 \times 6.28 \cos 18^\circ - 2,556 \times 3.47 = 19,160 \text{ Nm}$$

The bending moment due to the fin load is, from Table A.1,

$$\text{BM} (M_y) = 4,100 \times 3.7 = 15,170 \text{ Nm}$$

The torque due to asymmetric flight and the fin load is, from Table A.1,

$$T = 10,435 \text{ Nm}$$

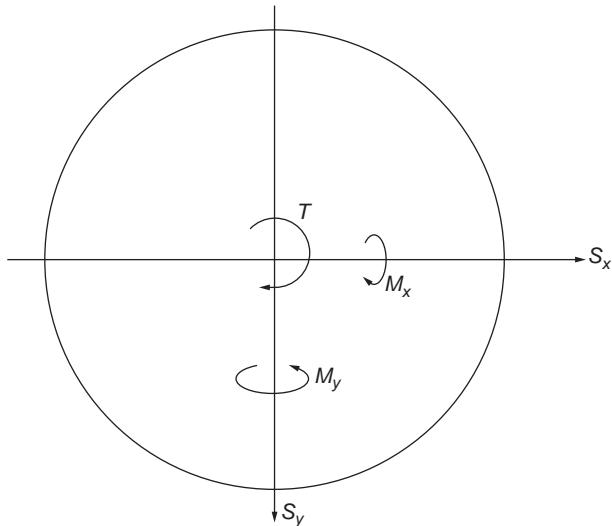
The values of shear force, bending moment, and torque at the section AA due to the remaining flight envelope cases are calculated in an identical manner. The complete procedure is then repeated for the sections CC and BB. The results are listed in Table A.2 with the positive directions and senses of the forces, moments and torques shown in Fig. A.7; these are as specified in Section 16.2.1 for an internal section when viewed in the direction Oz (see Fig. 16.9) except for torsion, where it is assumed that a positive fin load produces a positive torque.

**Table A.2** Shear Force, Bending Moment, and Torque

Section	Case	$S_x$ (N)	$S_y$ (N)	$M_x$ (Nm)	$M_y$ (Nm)	$T$ (Nm)
AA	A	4,100	9,200	19,160	15,170	10,435
	A'	4,100	9,434	19,938	15,170	10,435
	C	9,501	11,125	27,534	28,958	24,957
	D <sub>1</sub>	12,460	13,350	39,470	37,978	34,031
	D <sub>2</sub>	5,340	9,412	32,688	16,276	14,635
CC	A	4,100	6,675	10,308	8,301	10,435
	A'	4,100	7,120	11,393	8,301	10,435
	C	9,501	8,811	16,276	18,813	24,957
	D <sub>1</sub>	12,460	11,837	26,490	24,686	34,031
	D <sub>2</sub>	5,340	9,412	22,651	10,580	14,635
BB	A	4,100	4,673	4,747	3,824	10,435
	A'	4,100	5,118	5,358	3,824	10,435
	C	9,501	6,809	8,003	8,687	24,957
	D <sub>1</sub>	12,460	10,547	13,347	11,393	34,031
	D <sub>2</sub>	5,340	9,412	12,614	4,883	14,635

## A.6 FUSELAGE DESIGN CALCULATIONS

Two approaches to the actual design are possible. *Elastic design* uses *allowable* or *working* stresses, obtained from, say, the 0.1 percent proof stress, by incorporating a factor of safety, usually 1.5; these stresses are then combined with the actual loads to produce skin and stringer sizes. Alternatively, *ultimate load design* is based on the actual loads multiplied by an ultimate load factor (see Section 13.1), which then produces failure loads; the stresses involved are therefore the ultimate stresses. For linear



**FIGURE A.7 Positive Directions for Internal Forces and Moments**

systems, the methods produce identical results, so that, in this case, since the 0.1 percent proof stress is given, elastic design is used. The working, or allowable, stresses are then

$$\begin{aligned} \text{Direct} &= 186/1.5 = 124\text{N/mm}^2 \\ \text{Shear} &= 117/1.5 = 78\text{N/mm}^2 \end{aligned}$$

The proposed fuselage section is circular, as previously shown in Fig. A.5. The design process is required to produce suitable stringer sections and a skin thickness. Suppose that each stringer (or boom) has a cross-sectional area  $B \text{ mm}^2$  and that the skin thickness is  $t \text{ mm}$ . The idealized fuselage section (see Section 20.2) is shown in Fig A.8.

### Stringer sections

The direct stress in each stringer produced by bending moments  $M_x$  and  $M_y$  is given by Eq. (16.19), that is,

$$\sigma_z = \frac{M_x}{I_{xx}}y + \frac{M_y}{I_{yy}}x \quad (\text{A.13})$$

where

$$I_{xx} = I_{yy} = 4BD^2(0.1294^2 + 0.25^2 + 0.353^2 + 0.433^2 + 0.483^2 + 0.5^2/2)$$

that is,

$$I_{xx} = I_{yy} = 3.0BD^2 \text{ mm}^4$$

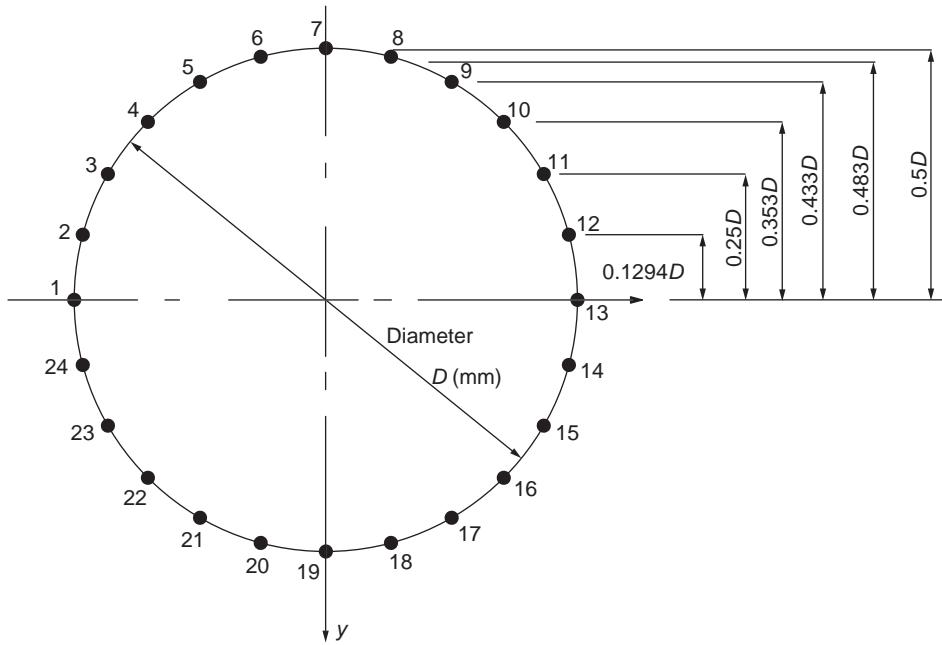


FIGURE A.8 Idealized Fuselage Cross-Section

A positive value of  $M_x$  causes tensile stresses in stringers 2 to 12 (there is no direct stress in stringers 1 and 13) and compressive stresses in stringers 14 to 24. A positive value of  $M_y$  produces tensile stresses in stringers 8 to 18 and compressive stresses in stringers 1 to 6 and 20 to 24 (no stress in stringers 7 and 19). Therefore,  $M_x$  and  $M_y$  both produce tensile stresses in stringers 7 to 13 and compressive stresses in stringers 19 to 1; in the remaining stringers, the stresses due to  $M_x$  and  $M_y$  are of opposite sign.

Inspection of Table A.2 shows that  $M_x$  and  $M_y$  reach their greatest values at each fuselage section in Case D<sub>1</sub>. For

Section AA (diameter  $D = 1.28$  m),

$$M_x = 39,470 \text{ Nm}, \quad M_y = 37,978 \text{ Nm}$$

Equation (A.13) becomes

$$\sigma = \frac{39,470 \times 10^3}{3.0B \times 1.28^2 \times 10^6} y + \frac{37,978 \times 10^3}{3.0B \times 1.28^2 \times 10^6} x$$

that is,

$$\sigma = (8.03y + 7.73x)/B$$

At stringer 7,  $x = 0$ ,  $y = 0.64$  m. Then,

$$\sigma_7 = 8.03 \times 0.64 \times 10^3 / B$$

that is,

$$\sigma_7 = 5,139/B \text{ N/mm}^2$$

Similarly,

$$\begin{aligned}\sigma_8 &= 6,245/B \text{ N/mm}^2 \\ \sigma_9 &= 6,924/B \text{ N/mm}^2 \\ \sigma_{10} &= 7,121/B \text{ N/mm}^2 \\ \sigma_{11} &= 6,854/B \text{ N/mm}^2 \\ \sigma_{12} &= 6,109/B \text{ N/mm}^2 \\ \sigma_{13} &= 4,947/B \text{ N/mm}^2\end{aligned}$$

Section CC (diameter  $D = 1.01 \text{ m}$ ),

$$M_x = 26,490 \text{ Nm}, \quad M_y = 24,686 \text{ Nm}$$

Equation (A.13) becomes

$$\sigma = (8.66y + 8.07x)/B$$

Then,

$$\begin{aligned}\sigma_7 &= 4,373/B \text{ N/mm}^2 \\ \sigma_8 &= 5,279/B \text{ N/mm}^2 \\ \sigma_9 &= 5,825/B \text{ N/mm}^2 \\ \sigma_{10} &= 5,965/B \text{ N/mm}^2 \\ \sigma_{11} &= 5,716/B \text{ N/mm}^2 \\ \sigma_{12} &= 5,069/B \text{ N/mm}^2 \\ \sigma_{13} &= 4,075/B \text{ N/mm}^2\end{aligned}$$

Section BB (diameter  $D = 0.73 \text{ m}$ ),

$$M_x = 13,347 \text{ Nm}, \quad M_y = 11,393 \text{ Nm}$$

Equation (A.13) becomes

$$\sigma = (8.35y + 7.13x)/B$$

Then,

$$\begin{aligned}\sigma_7 &= 3,048/B \text{ N/mm}^2 \\ \sigma_8 &= 3,618/B \text{ N/mm}^2 \\ \sigma_9 &= 3,941/B \text{ N/mm}^2 \\ \sigma_{10} &= 3,989/B \text{ N/mm}^2 \\ \sigma_{11} &= 3,834/B \text{ N/mm}^2 \\ \sigma_{12} &= 3,303/B \text{ N/mm}^2 \\ \sigma_{13} &= 2,602/B \text{ N/mm}^2\end{aligned}$$

From the above, it can be seen that the maximum direct stress at each fuselage section occurs in stringer 10. Also, the stress in stringer 10 (and all other stringers) is lower at section CC than at section AA and lower at section BB than at section CC. Therefore, if fuselage frames are positioned at each of these sections, lighter stringers may be used between CC and BB than between AA and CC. An additional frame will be positioned midway between AA and CC and between CC and BB and will be slotted to allow the stringers to pass through. The arrangement is shown diagrammatically in Fig. A.9 and in detail in Fig. A.13.

The allowable direct stress in a stringer is  $124 \text{ N/mm}^2$ . The maximum direct stress in stringer 10 at the section AA is  $7,121/B \text{ N/mm}^2$ . The required stringer area of cross-section is then given by

$$7,121/B = 124$$

that is,

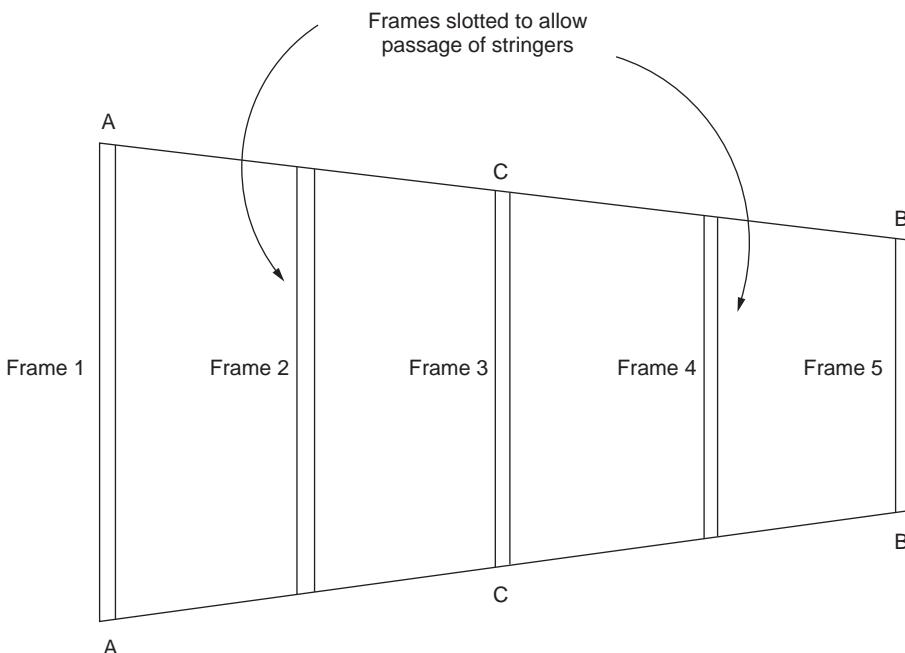
$$B = 57.4 \text{ mm}^2$$

The Z-section stringer, shown in Fig. A.10, has a cross-sectional area =  $58.1 \text{ mm}^2$  and therefore is satisfactory.

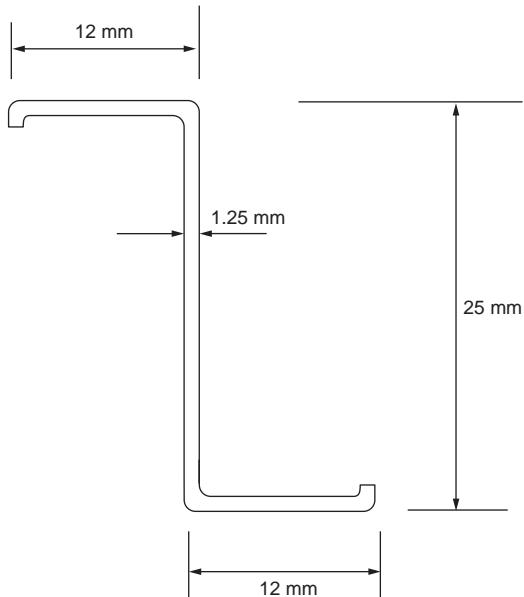
The maximum direct stress in stringer 10 at the section CC is  $5,965/B \text{ N/mm}^2$ .

The required stringer area is then given by

$$5,965/B = 124$$



**FIGURE A.9 Arrangement of Fuselage Frames**



**FIGURE A.10 Stringer Section, AA to CC (Type A)**

that is,

$$B = 48.1 \text{ mm}^2$$

The cross-section shown in Fig. A.11 has a cross-sectional area of  $51.9 \text{ mm}^2$  and is therefore satisfactory.

Although the fuselage frames are not load bearing, the frames at AA, CC, and BB must be of sufficient size to allow the ends of the stringers to be connected to them via brackets, while intermediate frames must be of sufficient size to allow slots to be cut, so that the stringers can pass through them. The frame sections to be used are shown in Fig. A.12.

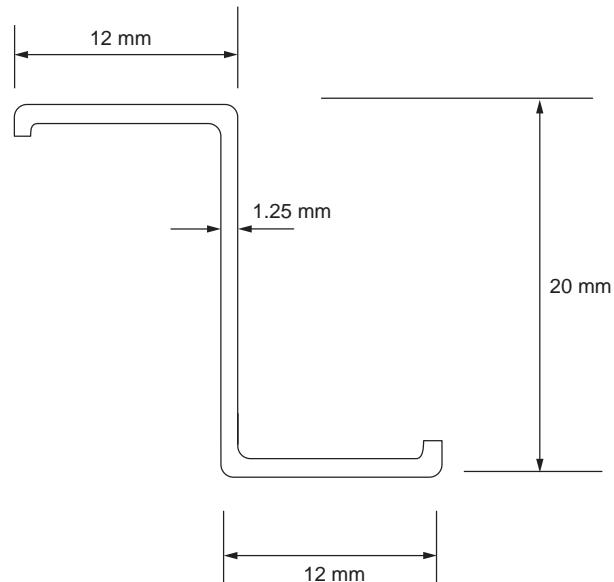
### Skin thickness

The fuselage cross-section is subjected to shear loads  $S_x$  and  $S_y$  along two perpendicular axes of symmetry. Equation (20.5) gives the change in shear flow as a boom, that is, a stringer, is crossed and, due to symmetry, reduces to

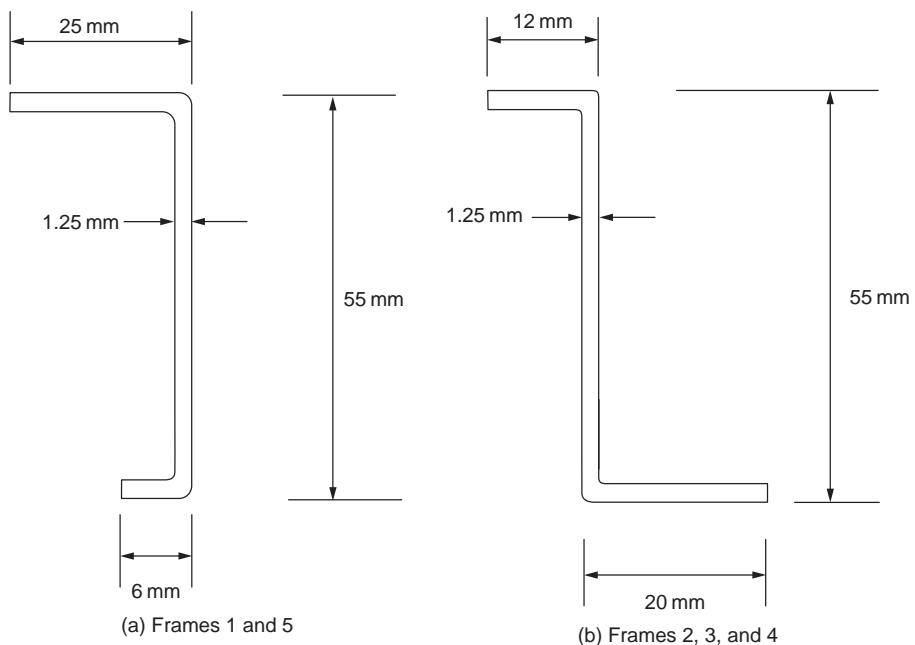
$$q_2 - q_1 = -\frac{S_x}{I_{yy}} B_r x_r - \frac{S_y}{I_{xx}} B_r y_r \quad (\text{A.14})$$

Then, since,  $B_r (= B)$  is constant around the fuselage section and  $I_{xx} = I_{yy} = 3.0BD^2$ , Eq. (A.14) reduces to

$$q_2 - q_1 = -\frac{S_x}{3.0D^2} x_r - \frac{S_y}{3.0D^2} y_r$$



**FIGURE A.11 Stringer Section, CC to BB (Type B)**



**FIGURE A.12 Fuselage Frame Sections**

Consider the action of  $S_y$  (or  $S_x$ ) only. Then,

$$q_2 - q_1 = -\frac{S_y}{3.0D^2}y_r \quad (\text{A.15})$$

Referring now to Fig. A.8,

$$q_{23} = q_{12} + \frac{S_y}{3.0D^2} \times 0.1294D = q_{12} + 0.043S_y/D$$

$$q_{34} = q_{23} + \frac{S_y}{3.0D^2} \times 0.25D = q_{12} + 0.126S_y/D$$

$$q_{45} = q_{34} + \frac{S_y}{3.0D^2} \times 0.353D = q_{12} + 0.244S_y/D$$

$$q_{56} = q_{45} + \frac{S_y}{3.0D^2} \times 0.433D = q_{12} + 0.388S_y/D$$

$$q_{67} = q_{56} + \frac{S_y}{3.0D^2} \times 0.483D = q_{12} + 0.549S_y/D$$

$$q_{78} = q_{67} + \frac{S_y}{3.0D^2} \times 0.5D = q_{12} + 0.716S_y/D$$

From symmetry  $q_{78} = -q_{67}$ , so that

$$q_{12} + \frac{0.716S_y}{D} = -q_{12} - \frac{0.549S_y}{D}$$

giving

$$q_{12} = -\frac{0.633S_y}{D}$$

Then,

$$q_{23} = (-0.633 + 0.043)S_y/D = -0.59S_y/D$$

Similarly,

$$q_{34} = -0.507S_y/D$$

$$q_{45} = -0.389S_y/D$$

$$q_{56} = -0.245S_y/D$$

$$q_{67} = -0.084S_y/D$$

Consider now the action of  $S_x$  only. Equation (A.14) becomes

$$q_2 - q_1 = -\frac{S_x}{3.0D^2}x_r$$

Again, referring to Fig. A.8,

$$q_{65} = q_{76} - \frac{S_x}{3.0D^2} \times 0.1294D = q_{76} - 0.043 S_x/D$$

$$q_{54} = q_{65} - \frac{S_x}{3.0D^2} \times 0.25D = q_{76} - 0.126 S_x/D$$

$$q_{43} = q_{54} - \frac{S_x}{3.0D^2} \times 0.353D = q_{76} - 0.244 S_x/D$$

$$q_{32} = q_{43} - \frac{S_x}{3.0D^2} \times 0.433D = q_{76} - 0.388 S_x/D$$

$$q_{21} = q_{32} - \frac{S_x}{3.0D^2} \times 0.483D = q_{76} - 0.549 S_x/D$$

$$q_{1\ 24} = q_{21} - \frac{S_x}{3.0D^2} \times 0.5D = q_{76} - 0.716 S_x/D$$

But  $q_{21} = -q_{1\ 24}$  from symmetry, so that

$$q_{76} - 0.549 S_x/D = -q_{76} + 0.716 S_x/D$$

that is,

$$q_{76} = \frac{0.633 S_x}{D}$$

Then,

$$q_{65} = 0.59 S_x/D$$

$$q_{54} = 0.507 S_x/D$$

$$q_{43} = 0.389 S_x/D$$

$$q_{32} = 0.245 S_x/D$$

$$q_{21} = 0.084 S_x/D$$

Note that the shear flows due to  $S_x$  and  $S_y$  in skin panels 7 to 6 and 2 to 1 inclusive are in the same direction. An identical situation arises in panels 19 to 18 and 14 to 13, but in the remaining panels, the shear flows are opposed.

The shear flow produced by the applied torque is given by Eq. (18.1), that is,

$$q = \frac{T}{2A}$$

where  $A$  is the area enclosed by the fuselage skin. Then,

$$q = \frac{T}{2(\pi D^2/4)}$$

or

$$q = 0.637 T/D^2 \quad (\text{A.16})$$

It can be seen from [Table A.2](#) that all the applied torques are positive, that is, clockwise. The shear flow is then in the same sense in skin panels 7 6 to 2 1 as the shear flows due to  $S_x$  and  $S_y$ ; these panels are therefore subjected to the greatest shear stresses.

The total shear flow in each of the panels 7 6 to 2 1 is then

$$\begin{aligned} q_{76} &= 0.084S_y/D + 0.633S_x/D + 0.637T/D^2 \\ q_{65} &= 0.245S_y/D + 0.590S_x/D + 0.637T/D^2 \\ q_{54} &= 0.389S_y/D + 0.507S_x/D + 0.637T/D^2 \\ q_{43} &= 0.507S_y/D + 0.389S_x/D + 0.637T/D^2 \\ q_{32} &= 0.590S_y/D + 0.245S_x/D + 0.637T/D^2 \\ q_{21} &= 0.633S_y/D + 0.084S_x/D + 0.637T/D^2 \end{aligned} \quad (\text{A.17})$$

From [Table A.2](#), the maximum values of  $S_y$ ,  $S_x$ , and  $T$  at each section are produced by Case D<sub>1</sub> in the flight envelope.

Section AA (diameter  $D = 1.28$  m),

$$S_x = 12,460\text{N}, \quad S_y = 13,350\text{N}, \quad T = 34,031\text{Nm}$$

Then, from [Eqs \(A.17\)](#),

$$\begin{aligned} q_{76} &= 0.084 \times 13,350/(1.28 \times 10^3) + 0.633 \times 12,460/(1.28 \times 10^3) \\ &\quad + 0.637 \times 34,031 \times 10^3/(1.28 \times 10^3)^2 \end{aligned}$$

that is,

$$q_{76} = 20.3\text{ N/mm}$$

Similarly,

$$\begin{aligned} q_{65} &= 21.5\text{ N/mm} \\ q_{54} &= 22.2\text{ N/mm} \\ q_{43} &= 22.3\text{ N/mm} \\ q_{32} &= 21.8\text{ N/mm} \\ q_{21} &= 20.7\text{ N/mm} \end{aligned}$$

Section CC (diameter  $D = 1.01$  m),

$$S_x = 12,460\text{N}, \quad S_y = 11,837\text{N}, \quad T = 34,031\text{Nm}$$

Then, from [Eqs. \(A.17\)](#),

$$q_{76} = \frac{0.084 \times 11,837}{1.01 \times 10^3} + \frac{0.633 \times 12,460}{1.01 \times 10^3} + \frac{0.637 \times 34,031 \times 10^3}{(1.01 \times 10^3)^2}$$

that is,

$$q_{76} = 30.0\text{ N/mm}$$

Similarly,

$$q_{65} = 31.4 \text{ N/mm}$$

$$q_{54} = 33.1 \text{ N/mm}$$

$$q_{43} = 32.0 \text{ N/mm}$$

$$q_{21} = 29.7 \text{ N/mm}$$

Section BB (diameter  $D = 0.73 \text{ m}$ ),

$$S_x = 12,460 \text{ N}, \quad S_y = 10,547 \text{ N}, \quad T = 34,031 \text{ Nm}$$

From Eqs. (A.17),

$$q_{76} = \frac{0.084 \times 10,547}{0.73 \times 10^3} + \frac{0.633 \times 12,460}{0.73 \times 10^3} + \frac{0.637 \times 34,031 \times 10^3}{(0.73 \times 10^3)^2}$$

that is,

$$q_{76} = 52.7 \text{ N/mm}$$

Similarly,

$$q_{65} = 54.3 \text{ N/mm}$$

$$q_{54} = 55.0 \text{ N/mm}$$

$$q_{43} = 54.6 \text{ N/mm}$$

$$q_{32} = 53.4 \text{ N/mm}$$

$$q_{21} = 51.3 \text{ N/mm}$$

The skin will be of constant thickness, so that the maximum shear stress in the skin will occur in the panel in which the shear flow is a maximum. This, from the preceding, is 55.0 N/mm in panel 54 at section BB. From Section A.5, the maximum allowable shear stress is 78 N/mm<sup>2</sup>, therefore,

$$\frac{55.0}{t} = 78$$

which gives

$$t = 0.71 \text{ mm}$$

A skin thickness of, say, 0.75 mm does not meet the requirements of a minimum thickness for rivet diameters equal to or greater than 2.5 mm (the probable rivet diameter but determined later). A skin thickness of 1.0 mm will therefore be used.

### Rivet size

Skin/stringer rivets

The change in end load over a unit length of stringer can be found using the method of Section 20.3.4. This change in end load is then the shear force on the stringer/skin connection, that is, the rivets. Using this approach, the bending moment due to  $S_x$  at a section 1 mm (say) from the section in which  $S_x$  is applied is  $S_x \times 1 \text{ Nmm}$ . The direct stress in the  $r$ th stringer produced by the bending moment is given by the second of Eqs. (16.20), that is,

$$\sigma_z = \frac{S_x \times 1}{I_{yy}} x_r$$

The end load in the stringer is then

$$P_r = \sigma_z B_r = \frac{S_x B_r x_r}{I_{yy}}$$

Similarly, due to  $S_y$ ,

$$P_r = \frac{S_y B_r y_r}{I_{xx}}$$

Since  $I_{xx} = I_{yy} = 3.0BD^2$ , the total change in end load over the 1 mm length of stringer is given by

$$\text{Total change in end load } P_r = \frac{S_x}{3.0D^2} x_r + \frac{S_y}{3.0D^2} y_r \quad (\text{A.18})$$

Clearly the change in end load will be a maximum when  $S_x$  and  $S_y$  have the same sign and  $x_r$  and  $y_r$  have the same sign; this occurs in stringers 7 to 13 and 19 to 1. In the former case, the change in end load is tensile while in the latter it is compressive. Further, the maximum values of  $S_x$  and  $S_y$  at sections AA, CC, and BB all occur for Case D<sub>1</sub> (see Table A.2); these cases are now investigated.

Section AA (diameter  $D = 1.28$  m),

$$S_x = 12,460\text{ N}, \quad S_y = 13,350\text{ N}$$

Stringer 7,

$$P_7 = \frac{12,460}{3.0(1.28 \times 10^3)^2}(0) + \frac{13,350}{3.0(1.28 \times 10^3)^2}(0.5D)$$

that is,

$$P_7 = 1.74\text{ N/mm}$$

Similarly,

$$\begin{aligned} P_8 &= 2.10\text{ N/mm} \\ P_9 &= 2.32\text{ N/mm} \\ P_{10} &= 2.37\text{ N/mm} \\ P_{11} &= 2.27\text{ N/mm} \\ P_{12} &= 2.02\text{ N/mm} \\ P_{13} &= 1.62\text{ N/mm} \end{aligned}$$

Section CC (Diameter  $D = 1.01$  m),

$$S_x = 12,460\text{ N}, \quad S_y = 11,837\text{ N}$$

$$\begin{aligned} P_7 &= 1.95\text{ N/mm} \\ P_8 &= 2.42\text{ N/mm} \\ P_9 &= 2.72\text{ N/mm} \\ P_{10} &= 2.83\text{ N/mm} \\ P_{11} &= 2.76\text{ N/mm} \\ P_{12} &= 2.49\text{ N/mm} \\ P_{13} &= 2.06\text{ N/mm} \end{aligned}$$

Section BB (diameter  $D = 0.73$  m),

$$S_x = 12,460 \text{ N}, \quad S_y = 10,547 \text{ N}$$

$$P_7 = 2.41 \text{ N/mm}$$

$$P_8 = 3.06 \text{ N/mm}$$

$$P_9 = 3.51 \text{ N/mm}$$

$$P_{10} = 3.71 \text{ N/mm}$$

$$P_{11} = 3.67 \text{ N/mm}$$

$$P_{12} = 3.37 \text{ N/mm}$$

$$P_{13} = 2.84 \text{ N/mm}$$

From the above, it can be seen that the maximum load on the rivets occurs at section BB in stringer 10 and is 3.71 N/mm. Assuming 2.5 mm diameter countersunk rivets, which have, in a skin thickness of 1.0 mm, an allowable load in shear of 668 N, the number of rivets/m is given by

$$n = \frac{3.71 \times 10^3}{668} = 5.6$$

say, six rivets per metre. However, this would give a rivet pitch of approximately 167 mm, which is not sufficient to ensure a rigid structure. Therefore, 2.5 mm diameter rivets are used at a pitch of 25 mm.

#### Frame/stringer rivets

The maximum stringer load at section AA is 7,121 N and this is resisted by the rivets connecting the skin to the frame over a length equal to the stringer spacing of 167.6 mm. Therefore, the number of 2.5 mm diameter rivets required is  $7,121/668 = 10.7$ , say 11. This gives a rivet pitch of  $167.6/11 \approx 15$  mm.

At section BB, the maximum stringer load is 3,989 N, so that the number of rivets required is  $3,989/668 \approx 6$ . This gives a rivet pitch of  $0.73 \times 10^3 \times \pi/(24 \times 6) \approx 16$  mm.

At section CC, the maximum stringer load is 5,965 N, so that the number of rivets required is  $5,965/668 \approx 9$ . The required rivet pitch is then  $1.01 \times 10^3 \times \pi/(24 \times 9) \approx 14$  mm.

Therefore, for all frames, a rivet pitch of 12.5 mm will be used.

The layout of a quarter of the rear fuselage is shown in Fig. A.13, with the detail design shown in Figs. A.14(a)–(e).

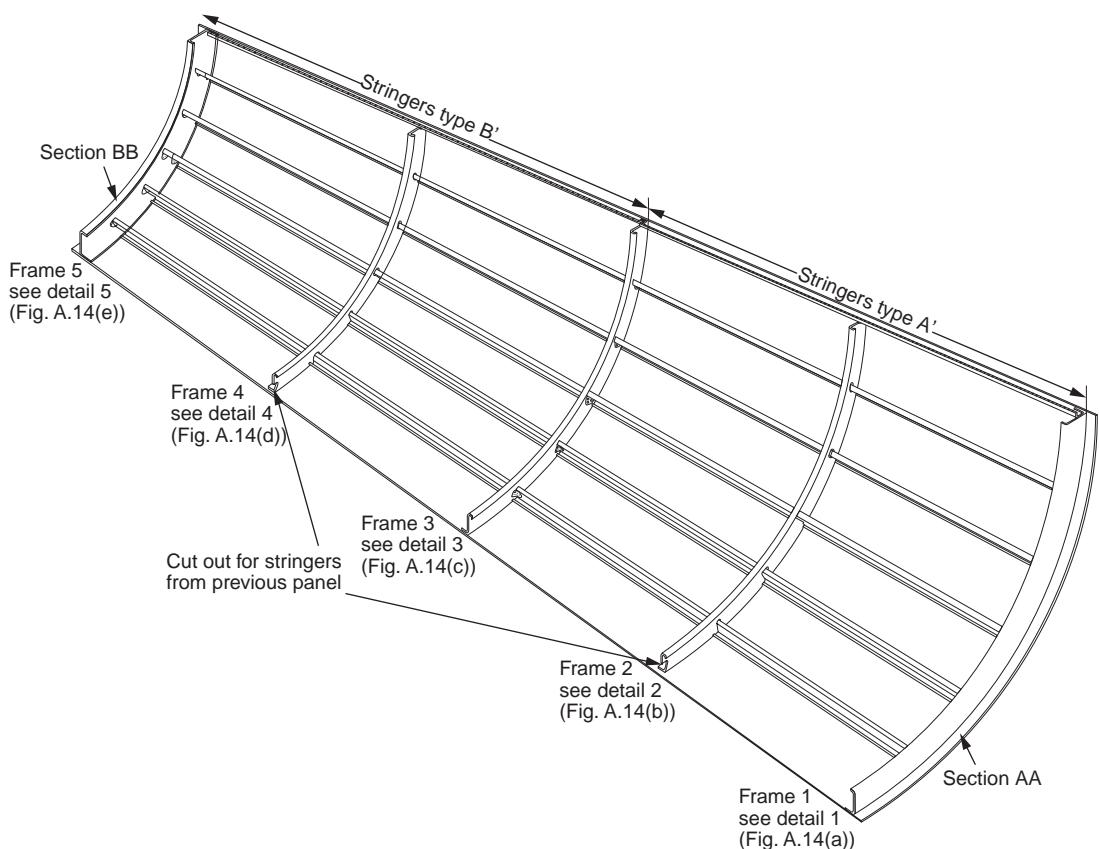
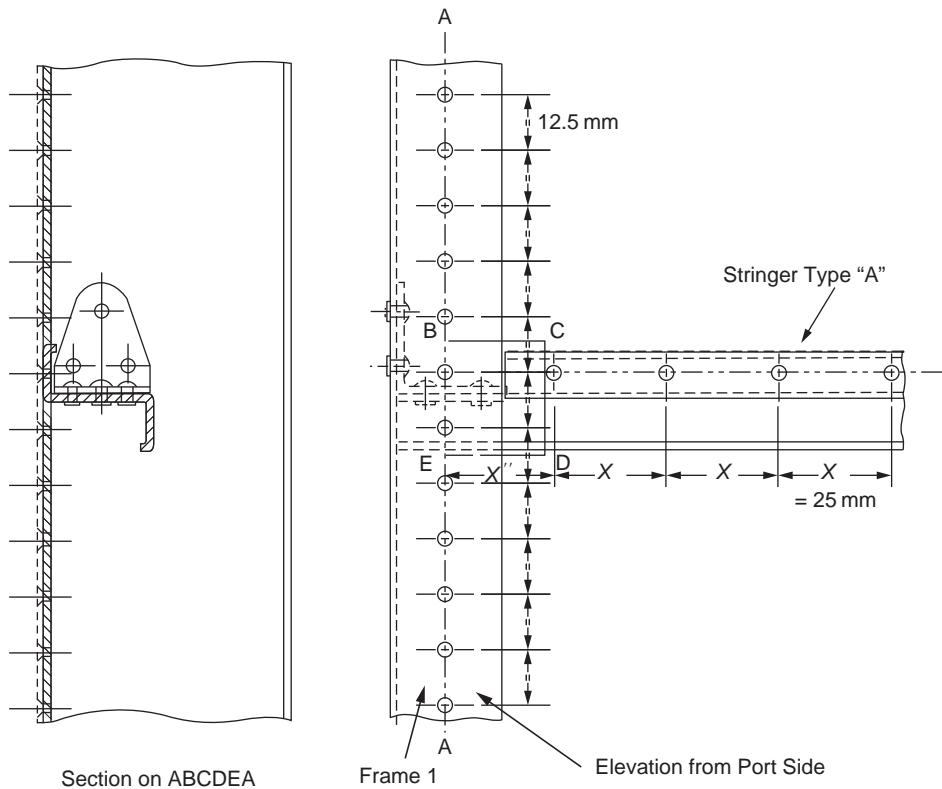
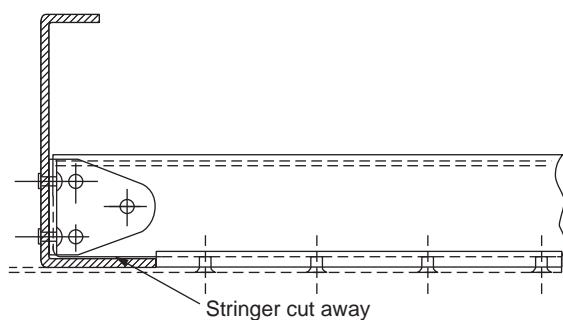


FIGURE A.13 Layout of quarter fuselage.



For Details of Bracket  
see Fig. A.14 (c)  
Skins from Previous  
Sections Overlap where  
Necessary.  
All Rivets 2.5 mm Countersunk  
Except for Bracket.



**FIGURE A.14(a) Detail, Frame 1.**

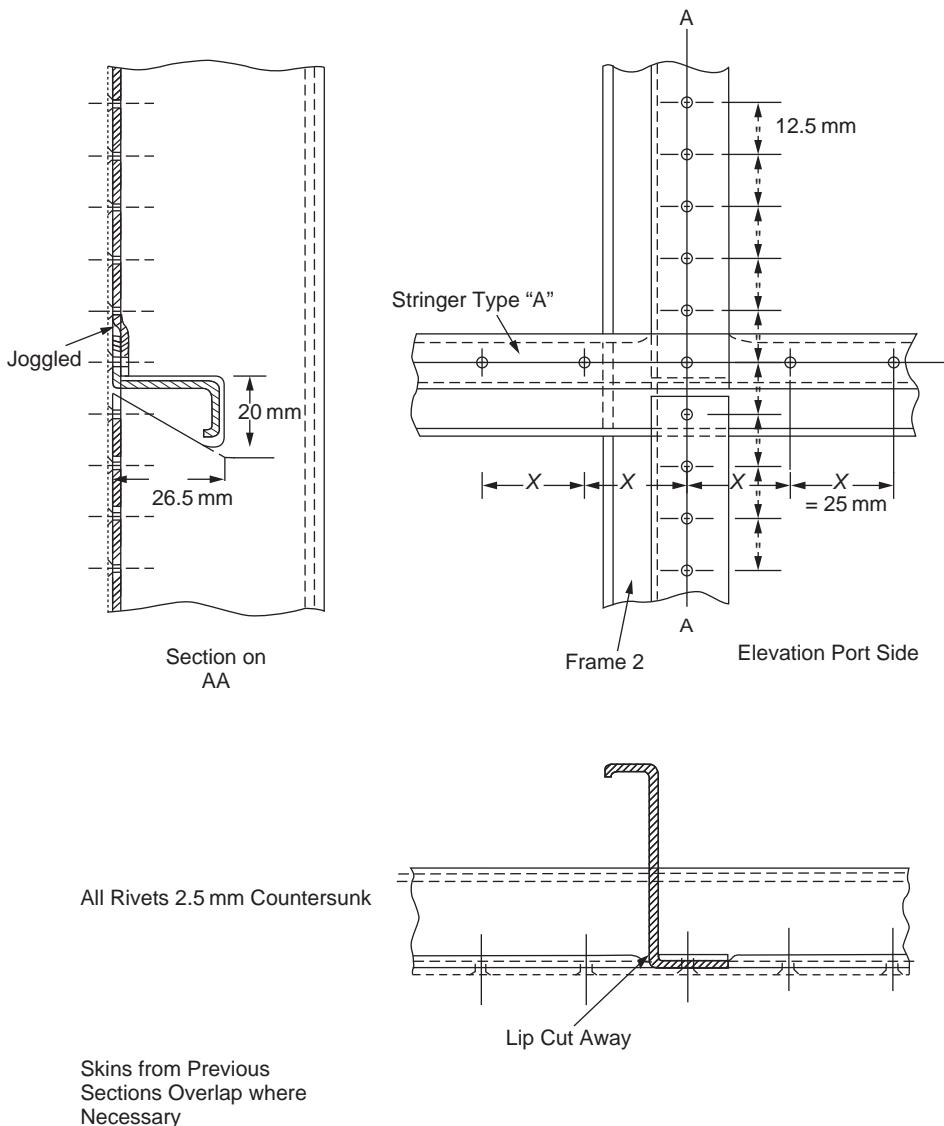
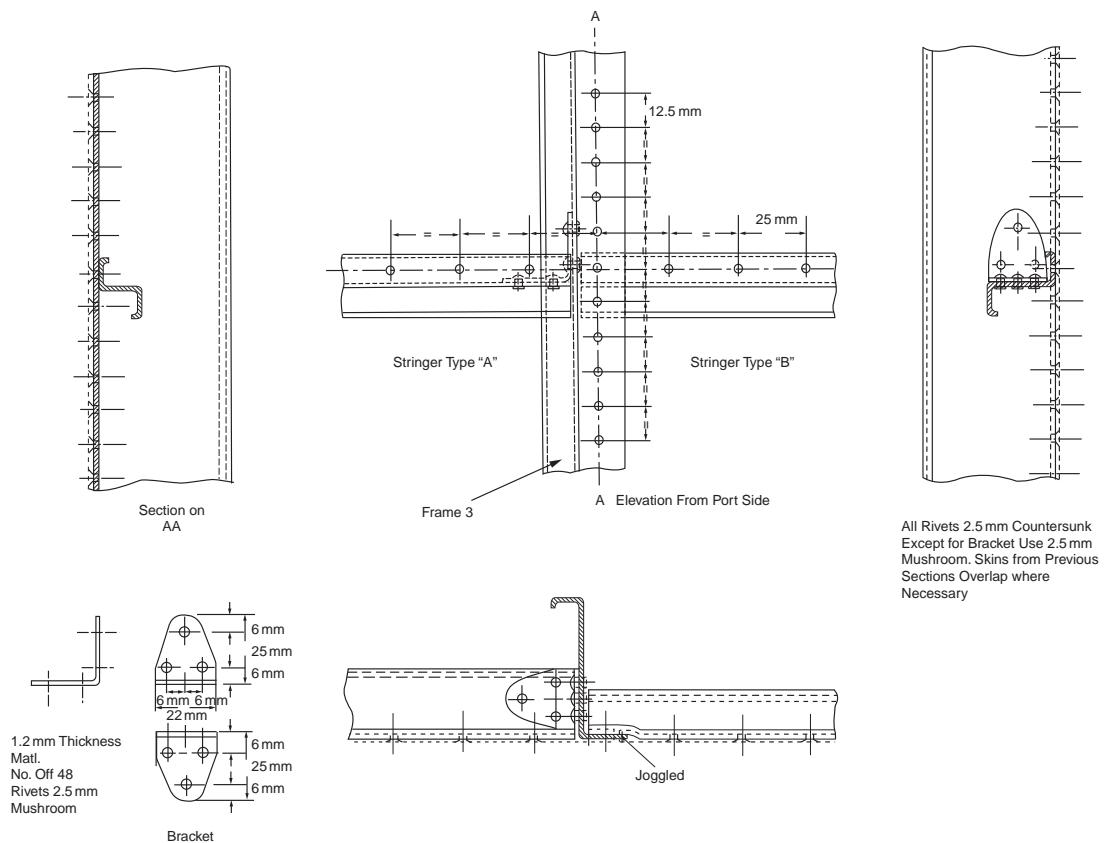


FIGURE A.14(b) Detail 2, Frame 2.



**FIGURE A.14(c) Detail 3, Frame 3.**

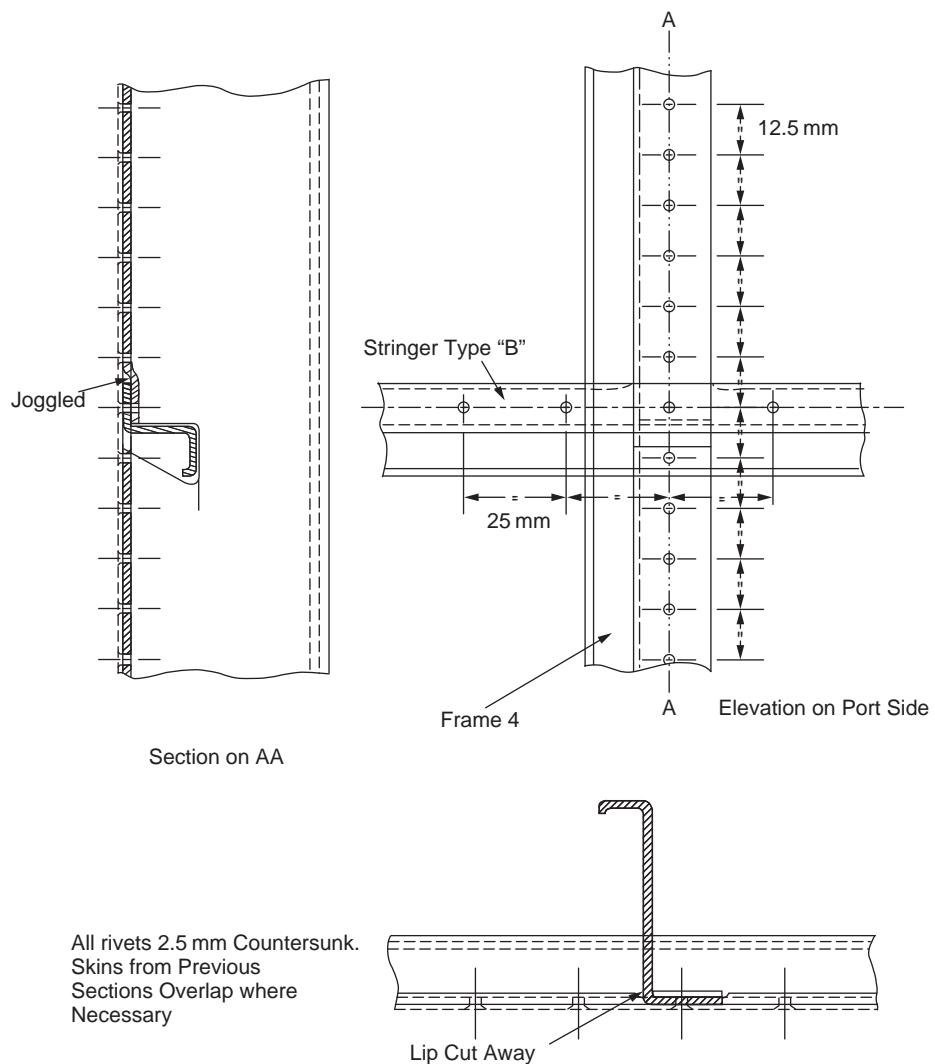
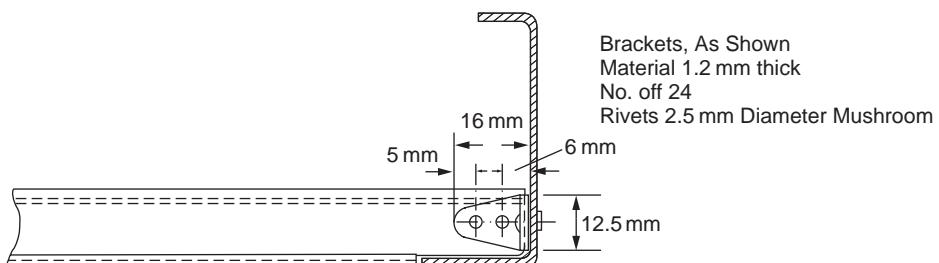
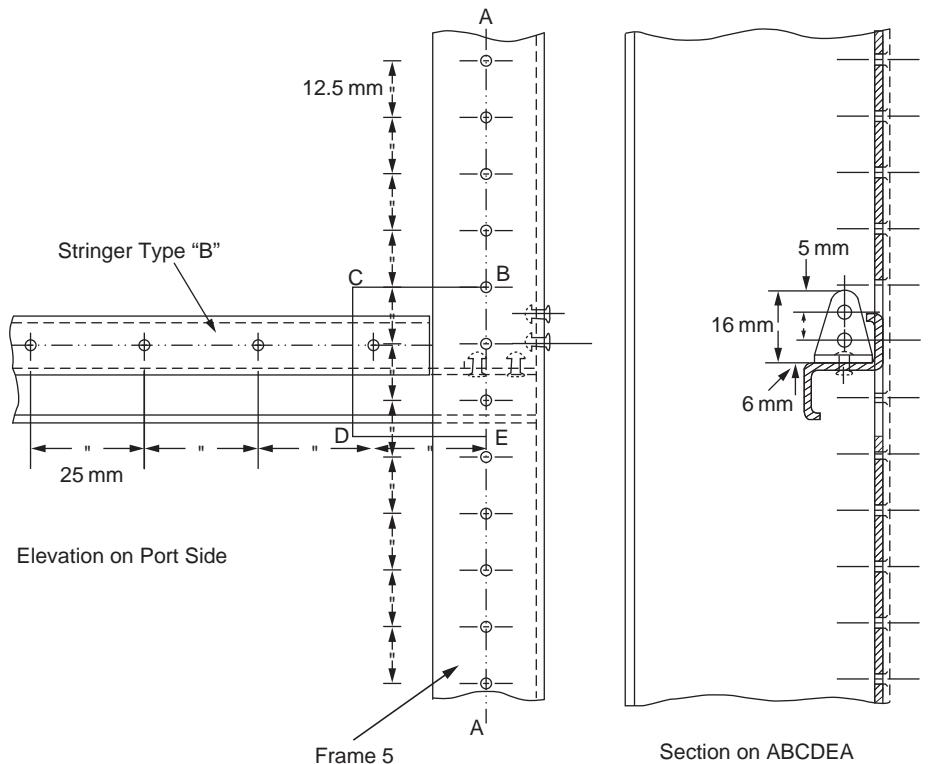


FIGURE A.14(d) Detail 4, Frame 4.



Skins from Previous  
Sections Overlap  
where Necessary  
Rivets 2.5 mm Diameter Countersunk.

FIGURE A.14(e) Detail 5, Frame 5.

# Index

Note: Page numbers followed by *f* indicate figures and *b* indicate boxes.

## A

Aeroelasticity, wing problems, 885–910  
control effectiveness and reversal, 894–899  
    aileron effectiveness and reversal  
        (finite wing), 896–899  
    aileron effectiveness and reversal  
        (two-dimensional case), 894–896  
    aileron reversal speed, 894  
flutter, 900–909  
    aerodynamic coupling, 902  
    aileron buzz, 901  
    buffeting, 901  
    center of independence, 902  
    classical flutter, 901  
    control surface flutter, 906–909  
    coupling, 901–903  
    determination of critical flutter speed,  
        903–905  
    elastic coupling, 902–903  
experimental determination of flutter  
    speed, 905–906  
ground resonance test, 907–909  
inertial coupling, 901–902, 902*f*  
nonclassical flutter, 901  
prevention of flutter, 905  
stalling flutter, 901  
wing-bending-torsion-flutter, 900  
load distribution and divergence, 888–894  
swept wing divergence, 893–894  
wing torsional divergence (finite wing),  
    890–893  
wing torsional divergence (two-  
    dimensional case), 888–890  
types of problem, 887  
Airframe loads, 455–484  
aircraft inertia loads, 455–461  
design loads, spacecraft, 478–484  
gust loads, 470–478  
    1 - cosine gust, 471*f*, 472  
    graded gust, 471–472, 471*f*, 474–475  
    gust alleviation factor, 474–475  
    gust envelope, 475–478  
    power spectral analysis, 472  
    sharp-edged gust, 471–474, 471*f*, 473*f*  
normal accelerations associated with  
    various types of maneuver, 466–470  
    correctly banked turn, 468–470  
    steady pull-out, 467–468  
symmetric maneuver loads, 461–466  
    general case, 463–466  
    level flight, 462–463  
Airworthiness, 447–454

factors of safety, flight envelope, 449–450

    fatigue (*see* Fatigue)

    flight envelope, 449–450, 450*f*

    limit load, 449–451

    load factor determination, 451–453

    proof factor, 449

    proof load, 449

    ultimate factor, 449

    ultimate load, 449

    uncertainties in design/structural  
        deterioration, 451

    variation in structural strength, 451–452

spacecraft, 453–454

    factors of safety, 454

Airy stress function, 56

Anticlastic bending, 520–521

Anticlastic surface, 248

Axial stress, spacecraft, 758–759

## B

Basic elasticity, 1–52

Beam columns, 300–304

Bending of an end-loaded cantilever,  
    64–73

    effect of shear strains, 68–69

Bending of open and closed section  
    thin-walled beams, 509–572

anticlastic bending, 520–521

applicability of bending theory, 559

calculation of section properties, 549

    approximations for thin-walled sections,  
        554–559

    parallel axes theorem, 549

product second moment of area,  
    553–554

second moments of area of standard  
    sections, 550–553

theorem of perpendicular axes, 550

deflections due to bending, 529–543

    moment-area method for symmetrical  
        bending, 543–549

    singularity functions (Macauley's  
        method), 535–543

load intensity, shear force and bending  
    moment relationships, 528

symmetrical bending, 543–549

    assumptions, 513

    centre, radius of curvature, 514

    direct stress distribution, 523–526

    examples of symmetrical sections, 512*f*

    neutral axis, 512

    neutral plane, 511

temperature effects, 559–571

unsymmetrical bending, 521–528

    direct stress distribution, 523–526

    position of neutral axis, 526–527

    resolution of bending moments, 522–523

    sign conventions and notation, 521

Bending of thin plates. *See* Plates, thin

Bending rigidity of a beam, 139

Bifurcation point, 286–291

Biharmonic equation, 56

Bimoment (moment couple), 876–884

Body forces, 9

    boundary conditions, 11–12

    compatibility equations, 28–29

    equations of equilibrium, 9–10

Bredt-Batho theory. *See* Torsion of closed  
    section beams

Buckling

    columns (*see* Columns)

    plates (*see* Plates, thin)

Bulk modulus, 36

## C

Columns, 281–330

    bifurcation point, 286–291

    buckling load for a pin ended column, 284*f*

    modes of buckling, 285

    critical stress, 283, 285–286, 291

    definition of buckling load for a perfect

    column, 284*f*

    effective length, 286

    effective lengths of columns having

    varying end conditions, 286

    effect of initial imperfections, 296–300

    Southwell plot, 298–300

    eigenfunctions, eigenvalues, 286

    energy (Rayleigh-Ritz) method for the  
        calculation of buckling loads,  
            304–307

    Euler buckling, 283–291

    flexural-torsional buckling of thin-walled  
        columns, 308–329

    inelastic buckling, 291–296

    reduced elastic modulus, 292*f*

    reduced modulus theory, 294

    tangent modulus, 291

    tangent modulus theory, 294

    primary instability, 283

    secondary instability, 283

    slenderness ratio, 285–286, 291

    stability of beams under transverse and  
        axial loads (beam-columns), 300–304

- Combined open and closed section beams, 629–640  
bending, 629–630  
shear, 631–633  
torsion, 634–639
- Compatibility equations, 28–29
- Complementary energy. *See* Energy methods
- Complementary shear stress, 10
- Components of stress, 8/*f*
- Composite materials, 402–403  
carbon-fiber-reinforced plastics (CFRP), 402  
glass reinforced plastics (GRP), 402
- Composite structures. *See* Laminated composite structures
- Connections. *See* Structural components of aircraft
- Control effectiveness and reversal.  
*See* Aeroelasticity, wing problems
- Crack propagation. *See* Fatigue
- D**
- Deflection of thin plates. *See* Plates, thin
- Deflection of thin-walled beams due to bending, shear and torsion, 659–667
- Deflections of beams due to bending, 529–543
- Determination of strains on inclined planes, 29–31
- Determination of stresses on inclined planes, 12–16
- Diagonal tension. *See* Plates, thin
- E**
- Effective length of a column. *See* Columns
- Eigenloads, 818
- Elasticity, 1–96  
basic elasticity, 1–52  
torsion of solid sections, 75–96  
two-dimensional problems, 53–74
- Energy methods, 131–192, 304–307  
bending of thin plates, 267–279  
energy method for the calculation of buckling loads  
in columns, 304–307  
in plates, 331–335
- flexibility method, 164–169  
influence or flexibility coefficient, 174–175
- principle of superposition, 174
- principle of the stationary value of the total complementary energy, 133–134
- application to deflection problems, 134–143
- application to the solution of statically indeterminate systems, 144–161
- fictitious or dummy load method, 135–136
- reciprocal theorem, 174–178
- self-straining trusses, 168–169
- strain energy and complementary energy, 131–133
- temperature effects, 178–192
- total potential energy, 169–170
- principle of the stationary value of the total potential energy, 133–134
- unit load method, 162–164
- Euler buckling, 283–291. *See also* Columns
- Experimental determination of critical load for a thin plate, 337
- Experimental measurement of surface strains, 42–52  
strain gauge rosette, 42–43, 42/*f*
- F**
- Fabrication of structural components.  
*See* Structural components of aircraft
- Factors of safety, flight envelope.  
*See* Airworthiness
- Fail-safe structures. *See* Fatigue
- Failure stress in plates and stiffened panels.  
*See* Plates, thin
- Fatigue, 485–508  
corrosion fatigue, 485  
crack propagation, 499–508  
crack propagation rates, 503–508  
crack tip plasticity, 502–503  
fracture toughness, 501–503  
modes of crack growth, 499, 499/*f*  
stress concentration factor, 499–502  
stress field in vicinity of a crack, 500/*f*  
stress intensity factor, 499  
cycle fatigue, 485  
designing against fatigue, 486–487  
fatigue load spectrum, 486  
gust exceedance, 487  
gust frequency curves, 487
- endurance limit  
Miner's cumulative damage theory, 490–491  
*S-N* curve, 452, 487–488
- fatigue strength, 417
- fatigue strength of components, 487–493  
confidence limits, 487–488  
Goodman diagram, 488–489  
scatter factor, 493
- fretting fatigue, 485
- prediction of aircraft fatigue life, 493–498  
ground-air-ground cycle, 493–494  
gust damage, 495–498
- safe life and fail-safe structures, 485–486
- stress concentrations, 486
- thermal fatigue, 485
- Fiber metal laminate (FML), 403
- Fictitious or dummy load method, 135–136
- Finite element method. *See* Matrix methods
- Flexibility method, 164–169  
flexibility (influence) coefficient, 174–175
- Flexural axis, 818
- Flexural rigidity of a beam, 139
- Flexural-torsional buckling of thin-walled columns, 308–329
- Flutter. *See* Aeroelasticity, wing problems
- Force, 7–9  
body forces, 9  
notation, 7–9  
surface forces, 9
- Friction stir welding, 442
- Function of structural components.  
*See* Structural components of aircraft
- Fuselage frames and wing ribs, analysis of, 745–756
- Fuselages, analysis of, 689–709
- G**
- General stress, strain and displacement relationships. *See* Shear of beams
- Glass. *See* Materials of aircraft construction
- Glass reinforced plastics (GRP).  
*See* Materials of aircraft construction
- Goodman diagram, 488–489
- Ground resonance test, 907–909
- Gust loads. *See* Airframe loads; Fatigue
- H**
- Hooke's law, 33
- I**
- Inelastic buckling, 291–296  
columns, 291–296  
reduced elastic modulus, 292/*f*  
reduced modulus theory, 294  
tangent modulus, 291, 294  
tangent modulus theory, 294
- thin plates, 335–336
- Instability of stiffened panels, 338–340  
failure stress, 338, 340–346
- interrivet buckling, 340
- wrinkling, 340
- Inverse and semi-inverse methods for elasticity problems, 56–63
- L**
- Laminated composite structures, 763–814  
elastic constants for a simple lamina, 763–769  
law of mixtures, 764
- laminates, 778–795  
angle-ply laminate, 779

- assumptions and notation, 779  
 balanced laminate, 779  
 cross-ply laminate, 779  
 equivalent elastic constants, 783–792  
 extensional stiffnesses, 782–783  
 in-plane loading, 780–783  
 layered laminate, 778  
 maximum stress theory, 792  
 shear coupling coefficient, 784–785  
 strengths, 792–795  
 stress-strain relationships for an orthotropic  
 ply, 769–778  
 thin-walled composite beams, 795–814  
 axial load, 796–799  
 bending, 799–802  
 idealization of composite beams,  
   808–814  
 shear, 802–804  
 torsion, 804–807  
 types of ply, 769–778, 769f  
   generally orthotropic ply, 771–778, 771f  
   specially orthotropic ply, 769–771, 769f
- Laplacian operator, 77  
 Laser beam welding, 442  
 Load intensity, shear force and bending  
   moment relationships for a beam, 528  
 Loads on structural components.  
   *See* Structural components of aircraft  
 Local instability in plates, 337–338  
 Lumped mass concept. *See* Structural  
   vibration
- M**  
 Macauley's method (singularity functions),  
   535–543  
 Materials of aircraft construction, 393–420  
   aluminium alloys, 397–399  
   ceramics, 401  
   composite materials (*see* Composite  
     materials)  
   creep and relaxation, 415  
   fatigue (*see* Fatigue)  
   glass, 401  
   maraging steels, 399–400  
   plastics, 401  
   polymers, 401  
   properties of materials (*see* Properties of  
     materials)  
   steel, 399–400  
   strain hardening, 415  
   stress-strain curves, 409–414  
   testing of engineering materials  
     (*see* Testing of engineering materials)  
   titanium, 400–401  
 Matrix methods, 193–242  
   application to statically indeterminate  
     frameworks, 207
- finite element method, 216–241  
 stiffness matrix for a beam element,  
   217–221  
 stiffness matrix for a quadrilateral  
   element, 228–241  
 stiffness matrix for a triangular element,  
   221–228  
 flexibility (force) method, 193  
 matrix analysis of pin-jointed frameworks,  
   199–207  
 notation, 194–195  
 stiffness matrix, 195–196  
   for an elastic spring, 195–196  
   for two elastic springs in line, 196–199  
 stiffness matrix for a uniform beam,  
   209–216  
   idealization into beam elements, 212f  
 stiffness (displacement) method, 193  
 Membrane analogy, 89–91  
 Modulus of elasticity (Young's modulus), 33  
 Modulus of rigidity (shear modulus), 35  
 Mohr's circle of strain, 32  
 Mohr's circle of stress, 18–24  
 Moment-area method for symmetrical  
   bending, 543–549  
 Moment couple (bimoment), 876–884
- N**  
 Natural frequencies, spacecraft, 757–758  
 Neuber beam, 614–615  
 Neutral axis, 512  
 Neutral plane  
   of a beam, 511–512  
   of a plate, 245
- O**  
 Oscillation of beams. *See* Structural vibration
- P**  
 Parallel axes theorem, 549  
 Perpendicular axes theorem, 550  
 Plane strain, 29  
 Plane stress, 11  
 Plates, thin, 243–280  
   bending and twisting of, 248–252  
   principal curvatures, 249–250  
   principal moments, 249–250  
   bending of plates having a small initial  
     curvature, 267  
   buckling of plates, 331–335  
     buckling coefficient, 333  
   combined bending and in-plane loading of  
     a rectangular plate, 263–267  
     governing differential equation,  
       265–267  
     energy method, 267–279
- potential energy of a transverse load,  
   269  
 potential energy of in-plane loads,  
   270–279  
 Rayleigh-Ritz method, 261, 267–268,  
   273  
 strain energy due to bending and  
   twisting, 268–269  
 experimental determination of critical load  
   (Southwell plot), 337  
 failure stress in plates and stiffened panels,  
   338, 340–346  
 inelastic buckling of plates, 335–336  
   buckling coefficients, 336f  
 instability of stiffened panels  
   interrivet buckling, 340  
   wrinkling, 340  
 local instability, 337–338  
 pure bending, 245–248  
   anticlastic surface, 248  
   flexural rigidity, 247  
   neutral plane, 245  
   synclastic surface, 248  
 subjected to a distributed transverse load,  
   252–263  
   built-in edge, 256  
   differential equation for deflection, 257  
 Fourier series for deflections and loads,  
   257–258  
   free edge, 256–263  
   Laplace operator, 254–255  
   simply supported edge, 255  
 tension field beams, 347–367  
   complete diagonal tension, 347–354  
   diagonal tension factor, 354–357  
   incomplete diagonal tension, 354–357  
   loading or buckling stress ratio, 355  
   postbuckling behaviour, 357–367  
 Point of zero warping in an open section  
   beam, 617–628
- Poisson's ratio, 33  
 Potential energy. *See* Energy methods  
 Prandtl stress function solution, 75–88  
 Laplacian operator, 77  
 lines of shear stress, 81  
 polar second moment of area, 83–84  
 torsional rigidity, 80–81  
 torsion constant, 80–81  
 torsion of a circular section bar, 82–84b  
 torsion of an elliptical section bar,  
   84–86b  
 warping displacement, 80  
 Pressurized fuselages, 695–702  
 Primary instability, 283  
 Primary warping in an open section beam,  
   617  
 Principal strains, 32  
 Principal stresses, planes, 16–18

- Principle of superposition, 174
- Principle of the stationary value of the total complementary energy, 133–134
- application to deflection problems, 134–143
- application to the solution of statically indeterminate systems, 144–161
- Principle of the stationary value of the total potential energy, 170–173
- Principle of virtual work, 100–114
- Principles of stressed skin construction, 393–446
- materials (*see* Materials of aircraft construction)
- Product second moment of area, 553–554
- Properties of materials
- anisotropic materials, 404
  - brittleness, 404
  - creep and relaxation, 415
  - ductility, 404
  - elastic materials, 404
  - fatigue (*see* Fatigue)
  - isotropic materials, 404
  - orthotropic materials, 404
  - plasticity, 404
  - strain hardening, 415
- Q**
- Quasistatic loads, 758
- R**
- Rayleigh, 386
- Rayleigh-Ritz method, 261, 267–268, 273, 306, 386
- Reciprocal theorem, 174–178
- Reduced modulus theory, 294
- reduced elastic modulus, 292f
- Resistance welding, 442
- S**
- Safe life structures. *See* Fatigue
- Secondary instability, 283
- Secondary warping in an open section beam, 617
- Second moments of area, 550–553
- Self-straining trusses, 168–169
- Shear center. *See* Shear of beams
- Shear flow. *See* Shear of beams; Torsion of beams
- Shear lag. *See* Structural and loading discontinuities
- Shear of beams, 573–604
- center of twist, 576–577, 576f
  - general stress, strain and displacement relationships, 573–577
  - shear flow, 573–574, 577–586, 580–581f
- shear of closed section beams, 586–604
- shear center, 592–604
  - shear flow, 586–604
  - twist and warping, 590–592
- shear of open section beams, 577–586
- shear center, 577, 581–586
  - twist and warping, 590–592
- Shear stress distribution at a built-in end, 818–824
- Singularity functions (Macauley's method), 535–543
- Slenderness ratio for a column. *See* Columns
- Southwell plot, 298–300, 337
- Spacecraft, 757–762
- axial stress, 758–759
  - natural frequencies, 757–758
  - stability, 759–761
- Stability of beams under transverse and axial loads (beam-columns), 300–304
- Stability, spacecraft, 759–761
- Statically indeterminate systems, 33
- Strain, 24–27
- determination of strains on inclined planes, 29–31
  - longitudinal (direct) strain, 24
  - Mohr's circle of strain, 32
  - plane strain, 29
  - principal strains, 32
  - shear strain, 24
- Strain energy, 132f
- in simple tension, 131–133
- Strain gauge rosette, 42–43, 42f
- Stress, 5–6
- complementary shear stress, 10
  - components at a point, 8f
  - definition, 5–6
  - determination of stresses on inclined planes, 12–16
  - direct (normal) stress, 5–6
  - maximum shear stress at a point, 18
  - Mohr's circle of stress, 18–24
  - notation for stresses, 7–9, 8f
  - plane stress, 11
  - principal stresses, planes, 16–18
  - resultant stress, 6
  - shear stress, definition, 5–6
- Stress analysis of aircraft components
- fuselage frames and wing ribs, 745–756
  - fuselage frames, 750–751
  - principles of stiffener/web construction, 745–750
  - wing ribs, 751–756
- fuselages, 689–709
- in bending, 689–691
  - effect of cut-outs, 702–709
  - pressurized, 695–702
  - in shear, 691–693
  - in torsion, 693–695
- laminated composite structures
- (*see* Laminated composite structures)
- wings, 711–744
- bending of, 712–714
  - cut-outs in wings, 729–744
  - deflections, 728–729
  - shear, 718–724
  - shear center, 725
  - tapered wings, 725–728
  - three-boom shell, 711–712
  - torsion, 714–717
- wing spars and tapered box beams, 671–687
- beams having variable stringer areas, 680–687
  - open and closed section beams, 675–680
  - tapered wing spar, 671–675
- Stress functions, 55–56
- Stress-strain relationships, 32–42
- Structural and loading discontinuities
- closed section beams, 817–857
  - eigenloads, 818
  - flexural axis, 818
  - general aspects, 817–818
  - rectangular section beam subjected to torsion, 824–832
  - shear lag, 832–857
  - shear stress distribution at a built-in end, 818–824
  - zero warping axis, 818
- open section beams, 859–884
- distributed torque loading, 871–872
  - extension of theory to allow for general systems of loading, 873–876
- I-section beam subjected to torsion, 859–861
- moment couple (bimoment), 876–884
- shear flow due to a moment couple, 878–884
- torsion bending constant, 863–864
- torsion of an arbitrary section beam, 861–871
- Wagner torsion bending theory, 861–863
- “wire” analogy, 865–871
- Structural components of aircraft, 421–446
- connections, 436–443
- eccentrically loaded riveted joints, 439–441
  - group riveted joints, 439
  - joint efficiency, 439
  - riveted connections, 436–439
  - simple lap joint, 436
  - use of adhesives, 442–443
  - welded connections, 441–442
- fabrication of structural components, 428–433
- integral construction, 431–433

- sandwich panels, 431, 432f  
 sub-assemblies, 429, 430f  
 function of structural components,  
   424–428  
 fuselages, 428  
 monococoque structures, 424  
 semi-monocoque structures, 424  
 tailplanes, 428  
 wings, 423–424
- loads on components, 421–423  
 aerodynamic center, 422–423  
 body forces, 421–422  
 center of pressure, 422–423  
 drag, 422–423  
 ground loads, 421, 423  
 pitching moment, 422f  
 surface forces, 421–422  
 wing lift, 421  
 yawing moment, 422f
- Structural components of spacecraft,  
   433–436  
 adaptors, 435  
 loads on, 436  
 payload fairings, 435  
 stage structures, 435  
 thrust structure, 433–434
- Structural idealization, 641–644, 646–650,  
   659–667  
 deflections of open and closed section  
   beams, 659–667  
 effect of idealization on the analysis of  
   open and closed section beams,  
   646–650  
 alternative method for shear flow  
   distribution, 658–659  
 bending of open and closed section  
   beams, 647–648  
 deflections of open and closed section  
   beams, 659–667  
 shear loading of closed section beams,  
   656–658  
 shear of open section beams, 650–656  
 torsion of open and closed section  
   beams, 659
- idealization of a panel, 642–644  
 unsymmetrical sections, 644–646  
 principle, 641
- Structural instability, 281–368  
 columns (*see* Columns)  
 thin plates (*see* Plates, thin)
- Structural vibration, 369–392  
 approximate methods for natural  
   frequencies (Rayleigh, Rayleigh–  
   Ritz), 386  
 flexibility method, 373  
 lumped mass concept, 371  
 mass/spring systems, 371–381  
 normal modes of vibration, 371
- oscillation of beams, 381–386  
 stiffness method, 373  
 St.Venant's principle, 63  
 St Venant warping function solution, 88–89  
 torsion constant, 89  
 warping function, 88
- Surface forces, 9
- Symmetrical bending. *See* Bending of open  
   and closed section thin-walled beams
- Symmetric maneuver loads. *See* Airframe  
   loads
- Synclastic surface, 248
- T**
- Tangent modulus theory, 294  
 tangent modulus, 291
- Temperature effects, 38–42  
 in beams, 559–571  
 mechanical strain, 38–39  
 thermal strain, 38–39  
 total strain, 38–39
- Tension field beams. *See* Plates, thin
- Testing of engineering materials, 404–409  
 bending tests, 406  
   modulus of rupture, 406  
 compression tests, 406  
 hardness tests, 408–409  
 impact tests, 409  
 shear tests, 406–408  
 stress-strain curves, 409–414  
   aluminum, 412–413  
   brittle materials, 404  
   composites, 414  
   mild steel, 410–412  
 tensile tests, 405–406  
   actual stress, 405  
   nominal stress, 405
- Torsion bending constant, 863–864
- Torsion of an L-section beam built-in at one  
 end, 859–861
- Torsion of an open, arbitrary section beam  
 subjected to axial constraint, 861–871
- Torsion of a rectangular section beam  
 subjected to axial constraint,  
   824–832
- Torsion of beams, 605–628  
 torsion of closed section beams, 605–615  
   condition for zero warping (Neuber  
     beam), 614–615  
   displacements, 607–614  
   mechanics of warping, 612  
   rate of twist, 607–608  
   shear flow (Bredt-Batho theory),  
     607–614  
   warping, 609–612, 611–612f, 614  
   warping in a rectangular section beam,  
     609–614b
- torsion of open section beams, 615–628  
 point of zero warping, 617–628  
 primary warping, 617  
 rate of twist, 616  
 secondary warping, 617  
 shear lines, 615  
 shear stress distribution, 616  
 torsion constant, 616  
 warping of cross-section, 616–628
- Torsion of closed section beams, 607–614
- Torsion of solid sections, 75–96  
 membrane analogy, 89–91  
   torsion of a narrow rectangular strip,  
     91–96  
   warping of a thin rectangular strip, 93f
- Total potential energy, 169–170
- Twist and warping in closed section beams,  
   590–592
- Twist and warping in open section beams,  
   615–628
- Two-dimensional problems in elasticity,  
   53–74  
 bending of an end-loaded cantilever, 64–73  
 biharmonic equation, 56  
 displacements, 64  
 inverse and semi-inverse methods, 56–63  
 stress functions, 55–56  
 St Venant's principle, 63
- U**
- Unit load method, 162–164
- Unsymmetrical bending. *See* Bending of open  
   and closed section thin-walled beams
- V**
- Virtual work, 97–130  
 applications of principle, 114–129  
 in a deformable body, 107–108  
 principle of virtual work, 100–114  
   for a particle, 101–102  
   for a rigid body, 102–107  
 use of virtual force systems, 114  
 virtual work in a deformable body,  
   107–108
- work, definition, 99–100
- work done by external force systems,  
   113–114
- work done by internal force systems,  
   108–112  
   axial force, 108–110  
   bending moment, 111–112  
   hinges, 112  
   shear force, 110–111  
   sign of, 112  
   torsion, 112

**W**

Wagner torsion bending theory, 861–863  
Warping  
    in an open section beam, 616–628  
    in a closed section beam, 590–592  
    in a solid section beam, 80–81  
St Venant's warping function, 88–89  
of a thin rectangular strip, 93f

Wings, analysis of, 711–744

Wing spars and box beams, analysis of,  
    671–687

Wing torsional divergence.

*See* Aeroelasticity, wing problems

“Wire” analogy for the calculation of the  
    torsion bending constant, 865–871

Work, definition, 99–100

**Y**

Young's modulus, 33

**Z**

Zero warping axis, 818

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