SOLUTIONS MANUAL

Aerodynamics for Engineering Students

Seventh Edition

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Aerodynamics for engineering students, seventh edition

E. L. Houghton, P. W. Carpenter, Steven H. Collicott
and
Daniel T. Valentine

Chapter 5 solutions

1 Solutions to Chapter 5 problems

Problem 5.1: Define vorticity in a fluid and obtain an expression for vorticity at a point with polar coordinates (r, θ), the motion being assumed two-dimensional. From the definition of a line vortex as irrotational flow in concentric circles determine the variation of velocity with radius, hence obtain the stream function (ψ), and the velocity potential (ϕ), for the line vortex. (*Hint:* This is described in Section 5.3.2 in the text.) Note that the vorticity in (x, y) is

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

In (r, θ) this can be written as follows:

$$\zeta = \frac{1}{r}\frac{\partial r u_\theta}{\partial r} - \frac{\partial u_r}{\partial \theta}.$$

In this problem $\zeta = 0$ (irrotational flow) and $(u_r, u_\theta) = (0, u_\theta(r))$, i.e., the tangential velocity (tangent to the concentric circles) is a function of r only and the radial component of the velocity is zero. Thus, the second equation reduces to

$$\frac{1}{r}\frac{\partial r u_\theta}{\partial r} = 0$$

Integrating this equation, we get

$$u_\theta = \frac{\Gamma}{2\pi r},$$

where $\Gamma/(2\pi)$ is the constant of integration (note that we used the definition of the circulation Γ to write the constant in this form).

To find the formula for the stream function and the potential, you need to solve the following equations for ψ and ϕ after you substitute the velocity field just found.

$$u_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}.$$

Show that

$$\psi = -\frac{\Gamma}{2\pi}\ln\frac{r}{r_o}, \quad \phi = \frac{\Gamma}{2\pi}\theta.$$

Convert the stream function and potential from cylindrical polar to Cartesian coordinates. Note that $x = r\cos\theta$ and $y = r\sin\theta$.

Problem 5.2: Execute the MATLAB m-file `Soln_P5_2` supplied with the solutions manual. It is as follows:

```
%
% Problem 5.2 solution via MATLAB
% by Daniel T. Valentine, 2012/2016.
%
% The problem asks for the length diivided by the width
% ratio for a Rankine oval in a uniform stream. This
% problem is described in Section 5.3.7 in the text.
clear;clc
% Given information (input):
m = 120;% m^2/s;
xso = -1;% m
yso = 0;
```

```
xsi = 1;% m
ysi = 0;
U = 30; % m/s
c = 1;% half-distance between source and sink.
%
% SOLUTION METHOD 1: Substitution into formulas for bo and to.
% Equation (5.36) is the formula for the half chord. It is
bo = sqrt((m/pi/U)*c + c^2);
% Equation (5.35) is a formula for the half thickness. It can be
% solved interatively by successive approximation as follows:
to = .5; % Initial guess
for it = 1:1000 % 1000 iterations
to = (m/2/pi/U)*atan2(2*c*to,to^2-c^2); % This converges easily.
end
% Results: Output to command window
disp(' ')
disp(' Fineness ratio is equal to bo/to ')
disp('      to       bo       bo/to ')
disp([ to bo bo/to ])
% RESULTS PRINTED TO COMMAND WINDOW
% Fineness ratio is equal to bo/to
%       to       bo       bo/to
%    1.0000    1.5077    1.5077
%
% SOLUTION METHOD 2: GRAPHICAL SOLUTION that applies the
% formulas in Section 5.3.7 to find phis (velocity potential)
% and psis (stream function) for the Rankine oval applied to
% solve 5.2 in the text:
x = -3:.02:3;
y = -2:.02:2;
for mm = 1:length(x)
    for nn = 1:length(y)
        xx(mm,nn) = x(mm); yy(mm,nn) = y(nn);
        phis(mm,nn) = U * x(mm) ...
            + (m/4/pi) * log((x(mm)-xso)^2+y(nn)^2)...
            - (m/4/pi) * log((x(mm)-xsi)^2+ y(nn)^2);
        psis(mm,nn) = U * y(nn) ...
            + (m/2/pi) * atan2(y(nn),x(mm)-xso) ...
            - (m/2/pi) * atan2(y(nn),x(mm)-xsi);
        psi2(mm,nn) = U * y(nn) ...
            + (m/2/pi)*atan2(-2*y(nn)*c,x(mm)^2+y(nn)^2-c^2);
    end
end
contour(xx,yy,psis,[0 0],'k'),grid minor
title('Illustration of Rankine oval')
xlabel('x'),ylabel('y')
hold on
contour(xx,yy,psis,20)
contour(xx,yy,phis)
legend('oval','\psi', '\phi')
hold off
figure(2)
contour(xx,yy,psi2)
% Remark: On the graph of the oval to = 1 and bo = 1.51. Hence,
% the graph of the surface of the oval checks with the fineness
% ratio given in the text and computed by method 1.
```

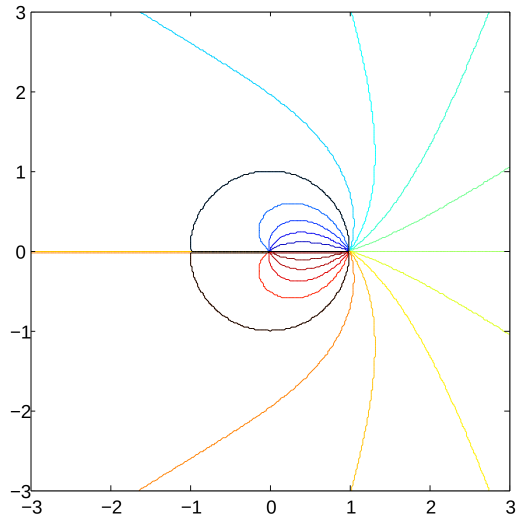

Figure 1: Illustration of source/sink flows examined in Problems 3.2 & 3.3.

Problems 5.3 & 5.4: Problems 5.3 and 5.4 are essentially the same. *Solution:* The problems ask us to examine the flow due to a source of strength m and a sink of strength $2m$ a distance c apart. Locate any stagnation points. Draw the streamlines. For this flow field we can place the source and sink on the x axis; thus,

$$\psi = \frac{m}{2\pi}\tan^{-1}\frac{y}{x} - \frac{m}{\pi}\tan^{-1}\frac{y}{x-c}$$

The flow field is illustrated in Fig. 1. It was drawn as follows:

```
m = 1;
U = 1;
c = 1;
x = -3:.02:3;
y = -3:.02:3;
for mm = 1:length(x)
    for nn = 1:length(y)
        xx(mm,nn) = x(mm); yy(mm,nn) = y(nn);
        psi(mm,nn) = (m/2/pi)*atan2(y(nn),x(mm))...
            - (m/pi)*atan2(y(nn),x(mm) - c);
    end
end
contour(xx,yy,psi,[-1:.1:1]),hold on
contour(xx,yy,psi,[m/2 -m/2],'k'),axis image
```

By zooming into the stagnation point on the figure in MATLAB it was determined that it is located at $(x, y) = (-1, 0)$. In addition, there is only one stagnation point in this flow field.

Problems 3.5 & 3.7: Both problems are on the flow around a circular cylinder. The first question is: What is the stream function for a uniform flow around a circular cylinder? The second is: At what angular locations on the cylinder is the pressure equal to the pressure in the undisturbed free stream?

Solution: The stream function is the sum of the stream functions for a free stream and a doublet, e.g.,

$$\psi = Uy - \frac{\mu}{2\pi}\frac{y}{x^2+y^2},$$

where $\mu = 2\pi R^2 U$ for a cylinder of radius R in a uniform stream $U\mathbf{i}$. The velocity field is, thus,

$$u = \frac{\partial\psi}{\partial y} = U + \frac{\mu}{2\pi}\frac{y^2-x^2}{(x^2+y^2)^2},$$

and

$$v = -\frac{\partial\psi}{\partial x} = -\frac{\mu}{2\pi}\frac{2yx}{(x^2+y^2)^2}.$$

From the Bernoulli equation we get the pressure distribution

$$C_p = 1 - \left(\frac{u^2}{U^2} + \frac{v^2}{U^2}\right).$$

To find where $C_p = 0$ on the cylinder it is more convenient to investigate the flow in polar coordinates. Recall, the flow field is also described by the velocity potential

$$\phi = Ux + \frac{\mu x}{x^2+y^2}.$$

For $U = \mu = 1$ the flow is around a circular cylinder with radius $R = 1$. In polar coordinates this formula is

$$\phi = r\cos\theta + \frac{\cos\theta}{r}.$$

Thus, the velocity field is

$$u_r = \frac{\partial\phi}{\partial r} = \cos\theta - \frac{\cos\theta}{r^2}.$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\sin\theta - \frac{1}{r^2}\sin\theta.$$

We know that $r = R = 1$ is a closed streamline because $u_r = 0$ on this surface. On the surface of the cylinder, $r = R = 1$, the total velocity is the tangential component, i.e., it is

$$u_\theta = -2\sin\theta.$$

Substituting this into the Bernoulli equation, we get

$$p = p_\infty + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right),$$

where the far field static pressure is equal to p_∞. Therefore, the dimensionless pressure coefficient on a cylinder in a cross flow is as follows (Remark: This is the solution to Problem 7 in Chapter 3 of the text):

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta.$$

Its maximum value is $C_p = 1$ at $\theta = 0$ and $\theta = \pi$, which are the trailing and leading edges, respectively. The minimum value is $C_p = -3$. It occurs at $\theta = \pi/2$ and $\theta = 3\pi/2$, i.e., at top and bottom dead center. At these locations the tangential velocity is twice the upstream value. It is the maximum velocity in this flow field. Finally, to answer the second question: The "generators" or the angular locations where $p = p_\infty$ are the angles that satisfy the following equation:

$$\theta = \sin^{-1} 0.5.$$

Table 1: Listing of MATLAB CircleCircV.m

```
% Example: Flow over a cylinder with circulation
clear;clc
V = 1; mu = 1; Gamma = -0; R = sqrt(mu/V);
N = 360; dthe = 2*pi/N;
the = 0:dthe:2*pi;
for n = 1:length(the)-1
    them(n) = the(n) + dthe/2;
    x(n) = R*cos(them(n)); y(n) = R*sin(them(n));
    u(n) = V + V*R^2*(y(n)^2 - x(n)^2)/(x(n)^2 + y(n)^2)^2  ...
         - Gamma/2/pi*y(n)/(x(n)^2+y(n)^2);
    v(n) = -2*V*R^2*y(n)*x(n)/(x(n)^2 + y(n)^2)^2 ...
         + Gamma/2/pi*x(n)/(x(n)^2+y(n)^2);
    ut(n) = -u(n)*sin(them(n)) + v(n)*cos(them(n));
    ur(n) = u(n)*cos(them(n)) + v(n)*sin(them(n));
    Cp(n) = 1 - (u(n)^2+v(n)^2)/V^2;
end
 plot(x,y,'k',[-1 1],[0 0],'k'),axis image, hold on
 plot(x,Cp),axis([-2 2 -5 1]), hold off
 CL = -sum(Cp.*sin(them))*dthe
 CLexact = -2*Gamma/R/V
 for mm=1:length(the)-1
     them(mm) = the(mm) + dthe/2;
     uthe(mm) = -2*sin(them(mm)) + Gamma/2/pi/R/V;
    Cp2(mm) = 1 - uthe(mm)^2;
 end
 CL2 = -sum(Cp2.*sin(them))*dthe
```

The angles are $\theta = \pm 30^o$ and $\theta = \pm 150^o$. There are four locations on the cylinder where the pressure is equal to the far field pressure. On page 132 in the text there is a summary on the problem of the flow around a cylinder.

This problem can be done by executing the MATLAB code CircleCircV.m with gam = 0; see also Table 1 below. Subsequently, execute

```
plot(them*180/pi,Cp2,'-o'),grid minor
```

Finally, use the magnifying tool to read the angular locations on the cylinder where the pressure coefficient crosses zero. You should be able to easily see that the angles given as answers in the text are correct.

Problem 5.6: The solution to this problem is obtained by executing the MATLAB script `Soln_P5_6.m`. It is as follows:

```
% Solution to Problem 5.6
% The psi equations below are identical
clear;clc
xx = -.8+.001:.002:.8;
yy = -1+.001:.002:0;
%yy = +.05:.1:2;
m = 1; V = 1; c = 1;
for nx = 1:length(xx)
    for ny = 1:length(yy)
        x(ny,nx) = xx(nx);
        y(ny,nx) = yy(ny);
%         ps2(ny,nx) = V*y(ny,nx) - (m/2/pi)* ...
%             atan2(2*c*y(ny,nx),(x(ny,nx)^2+y(ny,nx)^2-c^2));
%         ps1(ny,nx) = V*y(ny,nx) + (m/2/pi)*atan2(y(ny,nx),x(ny,nx)+c) ...
%             - (m/2/pi)*atan2(y(ny,nx),x(ny,nx)-c);
%         ps1(ny,nx) = (m/2/pi)*atan2(y(ny,nx),x(ny,nx)+c) ...
%             - (m/2/pi)*atan2(y(ny,nx),x(ny,nx)-c);
        ps1(ny,nx) = (m/2/pi)*atan2(y(ny,nx)+2*c*cos(30*pi/180),x(ny,nx)) ...
            + (m/2/pi)*atan2(y(ny,nx),x(ny,nx)+c) ...
            + (m/2/pi)*atan2(y(ny,nx),x(ny,nx)-c);
    end
end
contour(x,y,ps1,50),hold on, contour(x,y,ps1,[-.25],'k')
% cL = min(min(ps1))/2;
% cH = max(max(ps1))/2;
% contour(x,y,ps1,[cL cH],'k'),axis image
% contour(x,y,ps1)
% hold on
% contour(x,y,ps2,'m')
```

Problem 5.8: The pressures at the top and the bottom of a "spinning" cylinder are given by Equations (5.52) and (5.53), respectively. The difference in pressure divided by the dynamic pressure ($\rho U^2/2$) is thus

$$\frac{p_T - p_B}{\frac{1}{2}\rho U^2} = -\frac{4\Gamma}{\pi a U},$$

where a is the radius of the cylinder (and, hence, half the given diameter of the cylinder). The tangential velocity on the surface of the spinning cylinder is $u_\theta = a\omega$. Thus, the circulation is

$$\Gamma = \int_0^{2\pi} a\omega a d\theta = 2\pi\omega a^2.$$

Table 2: **Problem 8** MATLAB script for flow over a cylinder with circulation

```
 %
 %  Uniform stream + doublet + vortex
 %
 clear;clc
 mu = 1; % Doublet strength
 gam = -2; % Vortex strength
 V=1; % Free stream speed
 x = -3:.02:3;
 y = -2:.02:2;
for m = 1:length(x)
   for n = 1:length(y)
   xx(m,n) = x(m); yy(m,n) = y(n);
   psis(m,n) = V * y(n) - mu * y(n)/(x(m)^2+(y(n)+.01)^2) ...
       - (gam/4/pi)*log(x(m)^2+(y(n)+.01)^2); % Stream function
   end
end
contour(xx,yy,psis,[-3:.3:3],'k')
axis image
```

Hence,

$$\frac{p_T - p_B}{\frac{1}{2}\rho U^2} = -\frac{4\Gamma}{\pi a U} = \frac{8a\omega}{U}.$$

This is the answer given in the text; there is a typographical error in the problem statement (diameter is $2a$ and not a as stated). In Fig. 2 the stagnation points move downward from the horizontal axis in $\Gamma = -2$; hence, the induced velocity u_θ is in the negative angular direction. The script used to plot this figure is given in Table 2. Finally, further discussion of the movement of the stagnation points with Γ is given on pages 135 and 136 in the text.

Problem 5.9: The following script is supplied with this solutions manual. It is `Soln_P5_9.m`.

```
% Problem 5.9 can be solved with this code by
% changing m and V; xstg is distance upstream of source and
```

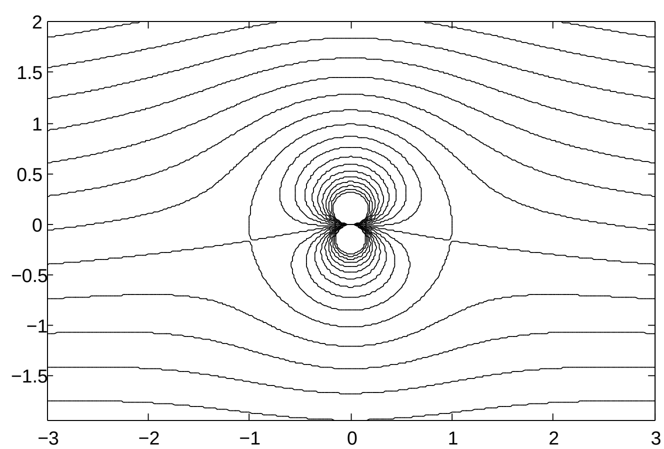

Figure 2: Problem 8 illustration of stagnation points for $\Gamma = -2$.

```
% t = m/V. If m = 0.9 and V = 6, |xstg| = 0.0239 and t = 0.15
% (these results are consistent with the answers in the text).
% The Rankine nose (or leading edge)
% Daniel T. Valentine, Spring 2012/Summer 2016
% Source in a uniform stream: A 2D potential flow
%
clear;clc
disp(' Example: Rankine nose')
m = 1; % Source strength for source at (x,y) = (0,0).
V = 1; % Free stream velocity in the x-direction
disp('     V     m ')
disp([V m])
disp(' Velocity potential:')
disp('   phi = V*x + (m/4/pi)*log(x^2+y^2)')
disp('Stream function:')
disp('   psi = V*y + (m/2/pi)*atan2(y,x)')
disp('The (x,y) componnents of velocity (u,v):')
disp('    u = V + m/2/pi * xc/(x^2+y^2)')
disp('    v = m/2/pi * y/(x^2+y^2)')
%
xstg = - m/2/pi/V; ystg = 0; % Location of stagnation point.
%
N = 1000;
xinf = 3;
xd = xstg:xinf/N:xinf;
for n = 1:length(xd)
    if n==1
        yd(1) = 0;
    else
    yd(n) = m/2/V;
    for it = 1:2000
        yd(n) = (m/2/V)*( 1 - 1/pi*atan2(yd(n),xd(n)) );
    end
    end
end
xL(1) = xd(end); yL = -yd(end);
for nn = 2:length(xd)-1
xL(nn) = xd(end-nn); yL(nn) = -yd(end-nn);end
plot([xd xL],[yd yL],'k',[-1 3],[0 0],'k'),axis([-1 3 -1 1])
 u = V + m/2/pi * xd./(xd.^2+yd.^2);
 v = m/2/pi * yd./(xd.^2+yd.^2);
 Cp = 1 - (u.^2+v.^2)/V^2;
 hold on
 plot(xd,Cp),axis([-1 3 -1 1])
plot(0,m/V/4,'o')
plot(xstg,ystg,'o')
plot([1 3],[m/2/V m/2/V],'--k')
[Cpmin ixd] = min(Cp);
xmin = xd(ixd);
ymin = yd(ixd);
plot(xmin,ymin,'+r')
Cpmin
% Computation of normal and tangential velocity on (xd,yd):
phi = V*xd + (m/4/pi).*log(xd.^2+yd.^2);
dx = diff(xd); dy = diff(yd); ds = sqrt(dx.^2 + dy.^2);
dph = diff(phi); ut = dph./ds; xm = xd(1:end-1) + dx/2;
```

```
psi = V*yd + (m/2/pi).*atan2(yd,xd);
%figure(2)
plot(xm,1-ut.^2/V^2,'r')
%
% Check on shape equation
%
th = 0:pi/25:2*pi;
r = (m/2/pi/V)*(pi - th)./sin(th);
xb = r.*cos(th);
yb = r.*sin(th);
plot(xb,yb,'om')
%
% Exact location of minimum pressure
thm = 0;
for nit = 1:1000
    thm = atan2(pi-thm,pi-thm-1);
end
thdegrees = thm*180/pi
rm = (m/2/pi/V)*(pi - thm)/sin(thm)
xm = rm*cos(thm);
ym = rm*sin(thm);
plot(xm,ym,'dk')
um = V + m/2/pi * xmin/(xmin^2+ymin^2);
 vm = m/2/pi * ymin/(xmin^2+ymin^2);
 Cpm = 1 - (um^2+vm^2)/V^2
```

Problem 5.10: This problem can be solved by determining the pressure distribution around the cylinder with $\Gamma = 0$ by executing the script in Table 1 (given in the solution of Problem 5 above). This gives the drag coefficient on the cylinder of $C_{D_{front}} = -2/3$. Thus,

$$D = -\frac{1}{3}\rho U^2 a.$$

Note that this can also be found by integrating the pressure distribution for this problem given in Problem 7 in the text (also given in the solution of Problem 5 above). This can be done in MATLAB symbolically by executing the following command:

```
2*int('(1 - 4*sin(x)^2)*cos(x)','x',0,pi/2)
```

which gives $-2/3$ as the answer. This agrees with the above result as well as the answer given in the text.

Problem 5.11: The solution was found by executing the MATLAB file `Soln_P5_11.m`. It is supplied with this solutions manual. It is

```
%
% Hints and solution to Problem 5.11
% February 2012/ August 2016
 clear;clc
%
% SYMBOLICS USED TO DETERMINE p-lines, ps-lines and ph-lines,
% i.e., isobars, streamlines and equipotentials, respectively.
format compact
syms x y a b c H
ps = x^2/2 + b*x*y - y^2/2
u =   diff(ps,'y')
```

```
v = - diff(ps,'x')
% diff(v,'x') - diff(u,'y')
% RELATIONSHIP FOR IRROTATIONAL FLOW:
% % This difference is -a+c ==> a = c for irrotational flows.
% diff(u,'x') + diff(v,'y')
%
% This sum is zero; hence, flow is incompressible.
p = H - (u^2+v^2)/2
% ph1 = int(u,'x')
% ph2 = int(v,'y')
phi = 1/2*b*x^2 - y*x-1/2*b*y^2
uu = diff(phi,'x')
vv = diff(phi,'y')
format
clear
figure(1)
% Results of symbolics applied computationally to show that
% isobars, streamlines and equipotentials do not coincide.
[x,y] = meshgrid(-5:.125:5,-3:.125:3);
xx = -5:.125:5;
yy = -3:.125:3;
b = 10; H = 1;
for n = 1:length(xx)
    for m = 1:length(yy)
        ps(m,n) = x(m,n)^2/2 + b*x(m,n)*y(m,n) - y(m,n)^2/2;
        u(m,n) = b*x(m,n) - y(m,n);
        v(m,n) = -x(m,n) - b*y(m,n);
        p(m,n) = 1 - (u(m,n)^2+v(m,n)^2)/2;
        phi(m,n) = 1/2*b*x(m,n)^2 - y(m,n)*x(m,n)-1/2*b*y(m,n)^2;
    end
end
contour(x,y,ps,'b'),hold on
contour(x,y,p,'k')
contour(x,y,phi,'r')
% % CONCLUSION: Constant lines of ps, p and phi do not coincide (as
% % shown graphically by the contour plots). You could also show that
% % the gradients of p, ps and phi are not coincident vectors.
clear
figure(2)
%
% Rotational flows
% In rotational flow curl u ~= 0 and, hence, a velocitty potential
% cannot be defined. The streamline, on the other hand, certainly
% can be defined.
% format compact
% syms x y a b c po
% ps = a*x^2/2 + b*x*y - c*y^2/2
% u =   diff(ps,'y')
% v = - diff(ps,'x')
%  diff(v,'x') - diff(u,'y')
% % % This difference is -a+c ==> a = c for irrotational flows.
%  diff(u,'x') + diff(v,'y')
% % This sum is zero; hence, flow is incompressible.
% p =  - (u^2+v^2)/2
% format
[x,y] = meshgrid(-5:.125:5,-3:.125:3);
```

Table 3: MATLAB script: Streamlines of vortex & source "doublets"

```
      sig = 50;
      V = 1;
      x = -2:.02:2;
      y = -2:.02:2;
      gam = -10;
    for m = 1:length(x)
     for n = 1:length(y)
        xx(m,n) = x(m); yy(m,n) = y(n);
        psiV(m,n) = V * y(n) + gam/4/pi*log(x(m)^2+(y(n)+.05)^2) ...
            - gam/4/pi*log(x(m)^2+(y(n)-.05)^2) ;
        psiS(m,n) =  V*y(n) + (sig/2/pi) * atan2(y(n),x(m)+0.01) ...
            - (sig/2/pi) * atan2(y(n),(x(m)-.01));
     end
    end
    contour(xx,yy,psiV,41,'k'),axis image
    figure(2)
    contour(xx,yy,psiS,41,'k'),hold on,
    contour(xx,yy,psiS,[0 0],'k'),axis image
```

```
xx = -5:.125:5;
yy = -3:.125:3;
b = 1; c = 1; a = -1;
for n = 1:length(xx)
    for m = 1:length(yy)
        ps(m,n) = a*x(m,n)^2/2 + b*x(m,n)*y(m,n) - c*y(m,n)^2/2;
        u(m,n) = b*x(m,n) - c*y(m,n);
        v(m,n) = -a*x(m,n) - b*y(m,n);
        p(m,n) =  1 - (u(m,n)^2+v(m,n)^2)/2;
    end
end
contour(x,y,ps,'b'),hold on
contour(x,y,p,'+k')
%
% Yes. The streamlines and isobars can coincide if flow is rotational.
%
```

Problem 5.12: We will graphically show that a pair of vortices and a source/sink pair appropriately placed very close together approaches the flow field of a doublet. We will do this by adding a uniform stream to each of the pair of singularities and illustrate that the flow field obtained approaches a flow around a circle. Since this is what is obtained when a doublet is placed in a uniform stream, the questions raised in the problem statement are answered in Fig. 3. The MATLAB script applied to produce Fig. 3 is given in Table 3. It is worth noting that the strengths of the sources and the strengths of the vortices need to be increased substantially from unity to produce a circle of radius of order 1. (To show mathematically in the limit of the separation distance that the two pairs of singular solutions approach a doublet as their strengths approach infinity in such a way that the strength of the vortex doublet or source doublet approaches the finite doublet strength in the formula for the doublet. This mathematical demonstration is beyond the scope of this course.)

Problem 5.13: This problem requires the student to go back to Section 2.6 and briefly para-

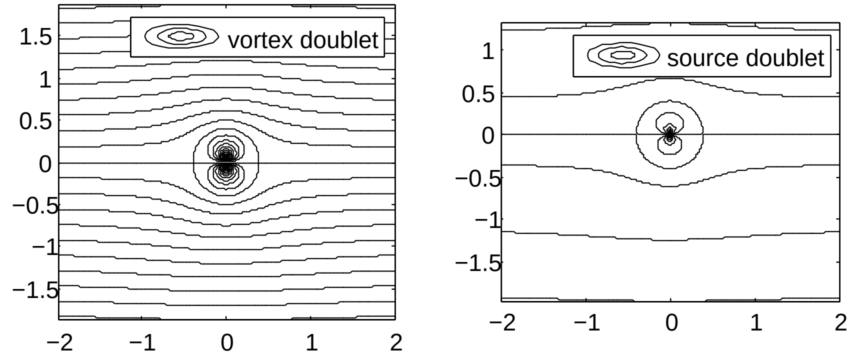

Figure 3: Problem 12, a comparison of vortex & source doublets.

phrase the discussion given on the stream function. The illustration of the fact that any number of sources located on a circle leads to a streamline that is a circle that passes through the sources is easily done with MATLAB. You need to code the sources located on a circle $r = \sqrt{x^2 + y^2}$. Start with one source located at $x = y = 1$, i.e., on the unit circle. Execute a contour plot to illustrate the fact that it is located on a unit circle. Place other sources on the same circle to find what happens. then select the sum of the strengths to be equal to zero. *Hint:* Start with a source/sink pair with equal and opposite strengths.

Problem 5.14: This is a problem that applies the method of images to create a wall next to the vortex for which we want to determine the interaction force acting on the wall due to its proximity to the wall. To construct a wall we need an image vortex placed the same distance on the opposite side of $x = 0$ with opposite sign but equal magnitude in strength; the streamlines for this flow field are illustrated in Fig. 4. This figure was produced by executing the MATLAB script given in Table 4. Since the force acting on the wall must be equal in magnitude and opposite in direction of the force acting on the vortex, let us apply the Kutta-Joukowski theorem to calculate this interaction force. The velocity induced by the image vortex on the primary vortex is equal to $V = \Gamma/(2\pi 2c)$ and it is in the y direction because both vortices are on the x axis a distance c from $x = 0$. Applying the law of Kutta-Zhukovsky gives us the force in the x direction such that the force on the vortex is towards the wall and equal in magnitude to $F_x = \rho V \Gamma = \rho \Gamma^2/(4\pi c)$. An integration of the pressure difference acting on the wall will show that this is the force acting on the wall and it is in the direction perpendicular to the wall and in it acts in the direction toward the vortex. This can be done numerically and or mathematically. In engineering practice, of course, it must be done both ways to check the force given.

Let us examine the pressure distribution along the wall (the y axis). From the solution to Problem 17 given in this document we find the formula

$$v_w = \frac{\Gamma}{\pi} \frac{c}{c^2 + y^2},$$

where v_w is the speed at each point on the y axis. From the Bernoulli theorem, the pressure difference along the wall at $x = 0$ for this flow field is,

$$\frac{p - p_o}{\rho} = -\frac{1}{2} v_w^2$$

Substituting for the speed, we get

$$\frac{p - p_o}{\rho} = -\frac{1}{2} \left(\frac{\Gamma}{\pi} \frac{c}{c^2 + y^2} \right)^2$$

Table 4: MATLAB script: Streamlines of vortex and its image

```
c = 1.01;
x = -2:.02:2;
y = -2:.02:2;
gam = -1;
for m = 1:length(x)
 for n = 1:length(y)
    xx(m,n) = x(m); yy(m,n) = y(n);
    psiV(m,n) = gam/4/pi*log((x(m)-c)^2+y(n)^2) ...
        - gam/4/pi*log((x(m)+c)^2+y(n)^2) ;
 end
end
   contour(xx,yy,psiV,41,'k'),axis image
```

The horizontal force on the wall is

$$F_x = -\int_{-\infty}^{\infty} (p - p_o)\; dy.$$

Thus,

$$F_x = \int_{-\infty}^{\infty} \frac{\rho}{2}\left(\frac{\Gamma}{\pi}\frac{c}{c^2+y^2}\right)^2 dy.$$

$$F_x = \frac{\rho}{2}\frac{c^2\Gamma^2}{c^3\pi^2}\int_{-\infty}^{\infty}\frac{1}{\left[1+(y/c)^2\right]^2}\, d(y/c).$$

Hence,

$$F_x = \frac{\rho}{2}\frac{\Gamma^2}{c\pi^2}\left[\frac{\pi}{2}\right],$$

where the following script was executed to solve the integral:

```
int('1/(1+x^2)^2',x,-Inf,Inf)
```

Rearranging, we get

$$F_x = \rho\Gamma^2/(4\pi c).$$

This result is the same as what we already found applying the Kutta-Zhukovsky theorem. Since it is positive it is towards the source which is located on the right side of the wall located at $x = c$. Hence, the force on the wall acts in the direction of the location of the vortex. Note that to model the flow of a vortex near a wall we needed to include its image at $x = -c$.

Problem 3.15: To determine the lift on a cylinder of radius a the pressure distribution is integrated as follows:

$$L = -\int_0^{2\pi} p\sin\theta\, a d\theta.$$

The pressure distribution on the cylinder is given in Equation (3.49) on page 133. It can be rearranged to define a pressure coefficient corresponding to the pressure around a as follows:

$$C_p = \frac{p - p_o}{\frac{1}{2}\rho U^2} = 1 - \left(2\sin\theta + \frac{\Gamma}{2\pi U a}\right)^2.$$

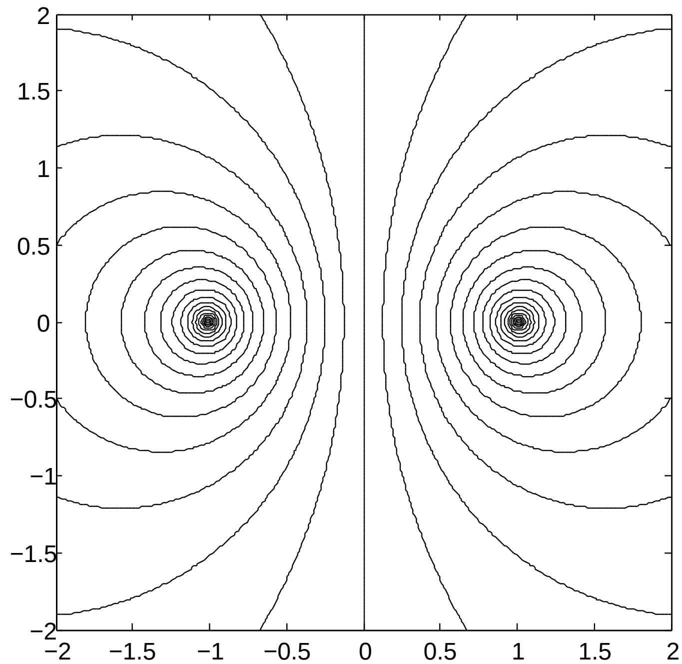

Figure 4: Problem 14, vortex near wall at $x = 0$ and its image.

With this formula we can rewrite the lift formula as

$$L = \frac{1}{2}\rho U^2 \int_0^{2\pi} C_p \sin\theta \, a d\theta$$

because the integral of p_o around the cylinder is zero. The integral was solved by executing the following MATLAB command,

```
>> int(' ( 1 - (2*sin(x) + Gamma/2/pi/U/a )^2 ) *a*sin(x)',x,0,2*pi)
ans =
    -(2*Gamma)/U
```

With this result, we get

$$L = \frac{1}{2}\rho U^2 \frac{2 * \Gamma}{U} = \rho U \Gamma,$$

which is what we wanted to show.

A plausible argument that shows that this theorem applies to any shaped object is provided in Section 6.1.3. It is this argument that needs to be studied. It is an argument that applies control-volume analysis to demonstrate that the Kutta-Zhukovsky theorem also applies to airfoils.

Problem 5.16: The vorticity in two-dimensional flow is

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

At $y = 0$, for the velocity profile given in the problem statement, the vorticity is $\zeta|_{y=0} = -\frac{\partial u}{\partial y}|_{y=0} = -(3/2)(u_o/\delta)$.

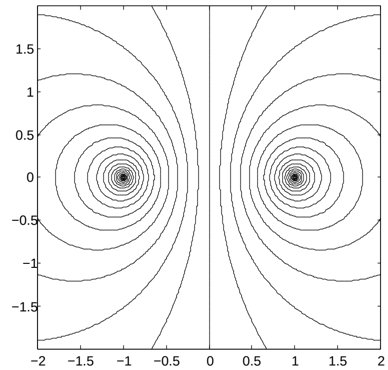

Figure 5: Problem 17, vortex near wall at $x = 0$ and its image.

Problem 5.17: The streamline pattern is essentially the same as the streamline pattern illustrated graphically in Problem 14. The velocity potential for this flow is

$$\phi = -\frac{\Gamma}{2\pi}\tan^1\left(\frac{y}{x-c}\right) + \frac{\Gamma}{2\pi}\tan^1\left(\frac{y}{x+c}\right)$$

where the wall is at $x = 0$, the vortex is at $(x, y) = (c, 0)$ and its image is at $(x, y) = (c, 0)$. The stream function is

$$\psi = \frac{\Gamma}{4\pi}\ln\left[(x-c)^2 + y^2\right] - \frac{\Gamma}{4\pi}\ln\left[(x+c)^2 + y^2\right].$$

Figure 5 is a plot of the streamline pattern for $c = 1$ and $\Gamma = \pi$. Because the streamline at $x = 0$ is the y axis the velocity along the wall is $(u, v) = (0, v_w)$. The y component of the velocity vector is

$$v = -\frac{\partial\psi}{\partial x} = -\frac{\Gamma}{4\pi}\left(\frac{2\,(x-c)}{(x-c)^2 + y^2} - \frac{2\,(x+c)}{(x+c)^2 + y^2}\right).$$

At $x = 0$ this reduces to

$$v_w = -\frac{\Gamma}{2\pi}\left(\frac{-c}{c^2 + y^2} - \frac{c}{c^2 + y^2}\right) = \frac{\Gamma}{\pi}\frac{c}{c^2 + y^2}.$$

For $c = 1$ and $\Gamma = \pi$ this reduces to

$$v_w = \frac{1}{1 + y^2}.$$

This velocity distribution is plotted in Figure 6. This was plotted with the following MATLAB script:

```
y = -10:.01:10; vw = 1./(1+y.^2);plot(y,vw)
```

The maximum velocity is at $y = 0$ and it is equal to unity.

Problem 5.18: The stream function is defined in terms of u and v in two-dimensional flow as follows:

$$u = 2y + \frac{y}{(x^2 + y^2)^{1/2}} = \frac{\partial\psi}{\partial y}$$

$$v = -2x - \frac{x}{(x^2 + y^2)^{1/2}} = -\frac{\partial\psi}{\partial x}$$

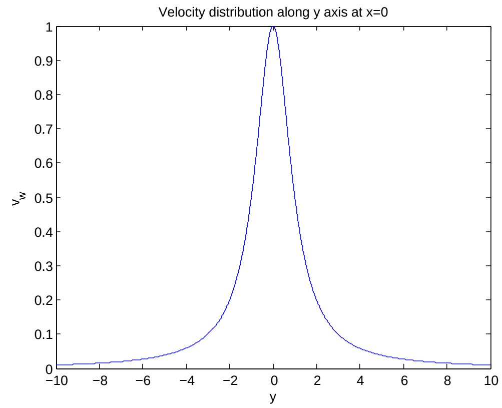

Figure 6: Problem 17, velocity distribution along y axis at $x = 0$.

Thus,

$$\psi = \int yudy = -\int vdx$$

Thus, using the symbolic toolbox of MATLAB, viz., the script

```
syms x y
u = 2*y + y/(x^2+y^2)^(1/2); % =  d psi / dy
v = -2*x -x/(x^2+y^2)^(1/2); % = -d psi / dx
psi1 = int(u,'y')
psi2 = -int(v,'x')
```

we get

$$\psi = x^2 + y^2 + \sqrt{x^2 + y^2}$$

The only requirement for the stream function to exist is the continuity equation; hence, the vorticity can be finite. In fact, in this problem it is. It is found as follows.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Hence, doing the differentiation with MATLAB, i.e., executing the script

```
syms x y
u = 2*y + y/(x^2+y^2)^(1/2); % =  d psi / dy
v = -2*x -x/(x^2+y^2)^(1/2); % = -d psi / dx
zeta = diff(v,'x') - diff(u,'y');
simple(zeta)
latex(ans)
```

we get

$$\zeta = -\frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}} - \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}} - 4$$

This is the answer given.

The following script can be applied to plot the streamlines for this flow field.

```
[x,y] = meshgrid(-1:.05:1);
psi = x.^2 + y.^2 + sqrt(x.^2 + y.^2);
contour(x,y,psi),axis image
```

The plot that you get, circular streamlines. This is not surprising since ψ is a function of $(x * 2 + y^2)$.

Problem 5.19: The velocity potential for a point source in a uniform stream is

$$\phi = UR\cos\varphi - \frac{Q}{4\pi R}$$

in the spherical coordinates (R, φ). In terms of the cylindrical coordinates (r, z), $R = \sqrt{r^2 + z^2}$, $r = R\sin\varphi$ and $z = R\cos\varphi$. Thus, the velocity potential of this source in a uniform stream can be written as follows:

$$\phi = Uz - \frac{Q}{4\pi\sqrt{r^2 + z^2}}$$

For this potential the horizontal component of the velocity is

$$u = \frac{\partial\phi}{\partial z} = U + \frac{Q\,z}{4\,\pi\,(r^2 + z^2)^{\frac{3}{2}}}$$

The radial component of the velocity is

$$v = \frac{Q\,r}{4\,\pi\,(r^2 + z^2)^{\frac{3}{2}}}$$

(a) The question raised in this part of the problem statement was: At what distance upstream of the source is the stagnation point located? It is at the location where $(u, v) = (0, 0)$, i.e., at $r = 0$ (along which $v = 0$ and, hence, by setting $u = 0$ in the formula for u, we get

$$z = -\sqrt{\frac{Q}{4\pi U}}$$

(b) For the case of the stream function for a point source in a uniform stream is

$$\psi = -\frac{Q}{4\,\pi}\frac{z}{\sqrt{r^2 + z^2}} + \frac{1}{2}Ur^2$$

At $(r, z) = (0, -\sqrt{Q/(4\pi U)}$ the dividing streamline is $\psi = \psi_d = -Q/(4\pi)$. It is this streamline that defines the shape. The stream function given above was verified by executing the following script:

```
syms U R r z Q
 phi = U*z - Q/(4*pi*sqrt(r^2+z^2));
 u = diff(phi,'z');
 v = diff(phi,'r');
psi = U*r^2/2 - Q/(4*pi)*(z/sqrt(r^2+z^2));
us = diff(psi,'r')/r;
us = simple(us);
vs = -diff(psi,'z')/r;
vs = simple(vs)
format compact
u
us
v
vs
format
```

In spherical coordinates the stream function is

$$\psi = -\frac{Q}{4\,\pi}\cos\varphi + \frac{1}{2}UR^2\sin^2\varphi$$

Thus, the formula for the dividing streamline is

$$-\frac{Q}{4\pi} = -\frac{Q}{4\,\pi}\cos\varphi + \frac{1}{2}UR_b^2\sin^2\varphi$$

Thus,

$$R_b^2 = \frac{Q}{4\pi}\frac{1+\cos\varphi}{\frac{1}{2}U\sin^2\varphi}$$

Let $a = \sqrt{Q/(4\pi U)}$, as given in the problem, we can write this formula as

$$R_b^2 = \frac{2a^2\,(1+\cos\varphi)}{\sin^2\varphi}$$

This is the formula given in the problem statement that we sought to verify.

The cross sections of the body are circles in the $r,\theta)$ planes. The shaped in (r,z) was found by executing the script

```
z = -2:.1:2;
for ip=1:length(z)
    r = 0.1;
    for it = 1:2000
        r = sqrt(2*z(ip)/sqrt(z(ip)^2+r^2) + 2);
    end
    rb(ip) = r;
end
plot(z,rb,'r')
```

Fig.7 illustrates the shape.

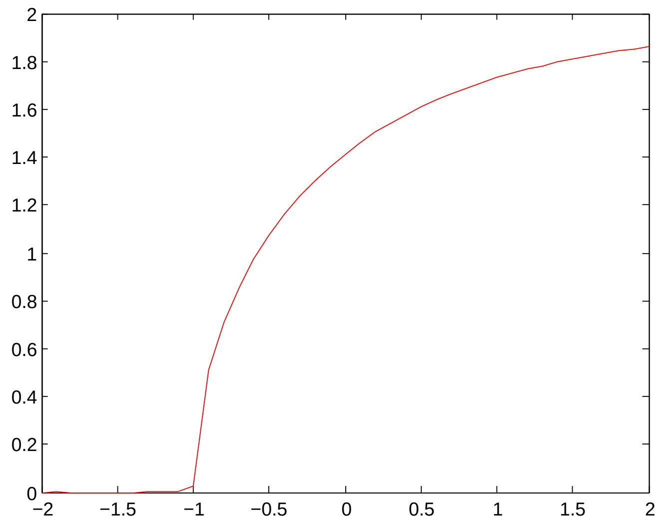

Figure 7: Illustration of the streamlines around a cylinder in a cross flow.

2 Additional examples of Chapter 5 type problems

Example 5.1: (a) The potential for a source of strength m upstream of a sink of strength $-m$ in a uniform stream U flowing in the x direction is

$$\phi = Ux + \frac{m}{4\pi} \ln \left[(x+c)^2 + y^2 \right] - \frac{m}{4\pi} \ln \left[(x-c)^2 + y^2 \right]$$

This is the velocity potential for the Rankine oval. It is a *linear superposition* of three elemental solutions of Laplace's equation. The last two are *singular* solutions because they "*blow up*" at the location where they are placed. Show that this formula can be re-written as follows:

$$\phi = Ux + \frac{m}{4\pi} \ln \left(\frac{x^2 + y^2 + c^2 + 2xc}{x^2 + y^2 + c^2 - 2xc} \right)$$

What rule of the arithmetic of logarithms did you have to apply to show this? *Solution:* **To put the first formula into the form above the following rule of logarithms was applied:** $\ln(a/b) = \ln a - \ln b$**. The sums and differences that are squared in the first equation were expanded. The two algebraic operations led to the second equation given above.**

(b) The stream function for this flow field is

$$\psi = Uy + \frac{m}{2\pi} \tan^{-1} \left(\frac{y}{x+c} \right) - \frac{m}{2\pi} \tan^{-1} \left(\frac{y}{x-c} \right)$$

Show that this can be re-written as follows:

$$\psi = Uy + \frac{m}{2\pi} \tan^{-1} \left(\frac{-2cy}{x^2 + y^2 - c^2} \right)$$

Hint: Consider applying the following identity:

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \, \tan\beta}$$

Solution: Rearrange the formula for the stream function as follows:

$$\frac{2\pi}{m}\left(\psi - Uy\right) = \tan^{-1}\left(\frac{y}{x+c}\right) - \tan^{-1}\left(\frac{y}{x-c}\right)$$

Next, take the tangent of both sides, i.e.,:

$$\tan\left[\frac{2\pi}{m}\left(\psi - Uy\right)\right] = \tan\left[\tan^{-1}\left(\frac{y}{x+c}\right) - \tan^{-1}\left(\frac{y}{x-c}\right)\right]$$

Applying the identity in the hint to the right hand side, we get

$$\tan\left[\frac{2\pi}{m}\left(\psi - Uy\right)\right] = \frac{\tan\left(\tan^{-1}\frac{y}{x+c}\right) - \tan\left(\tan^{-1}\frac{y}{x-c}\right)}{1+\tan\left(\tan^{-1}\frac{y}{x+c}\right)\tan\left(\tan^{-1}\frac{y}{x-c}\right)}$$

Thus,

$$\tan\left[\frac{2\pi}{m}\left(\psi - Uy\right)\right] = \frac{\frac{y}{x+c} - \frac{y}{x-c}}{1+\frac{y}{x+c}\frac{y}{x-c}}$$

Multiplying the numerator and denominator of the right hand side by $(x+c)(x-c)$, we get

$$\tan\left[\frac{2\pi}{m}\left(\psi - Uy\right)\right] = \frac{y\left(x-c\right) - y\left(x+c\right)}{\left(x+c\right)\left(x-c\right)+y^2}$$

Thus,

$$\tan\left[\frac{2\pi}{m}\left(\psi - Uy\right)\right] = \frac{-2yc}{x^2+y^2-c^2}$$

Taking the arctangent of both sides, we get

$$\frac{2\pi}{m}\left(\psi - Uy\right) = \tan^{-1}\frac{-2yc}{x^2+y^2-c^2}$$

Hence, we can write

$$\psi = Uy + \frac{m}{2\pi}\tan^{-1}\frac{-2yc}{x^2+y^2-c^2}.$$

This is the formula we sought. **Remark: The minus sign in the numerator is correct. This was checked by plotting the contours of ψ using both formulas.** There was a typographical error of an implied plus sign in the numerator in the original problem statement of this homework assignment.

Example 5.2: (a) In Section 5.3.8 the x-directed doublet potential is presented. It is described by applying a limit process to derive the doublet potential from the potential of a source-sink pair a zero distance apart. The doublet potential examined in this problem is for the $-x$-directed doublet. Start with a source located at $(x,y) = (-c,0)$ and a sink located at $(x,y) = (c,0)$; this orientation is opposite of the orientation examined in the text. Hence, for this potential you need to start with is

$$\phi = \frac{m}{4\pi}\ln\left(\frac{x^2+y^2+c^2+2xc}{x^2+y^2+c^2-2xc}\right)$$

Show that for $c \to 0$ with $\mu = 2mc$ the limiting construct is

$$\phi = \frac{\mu}{2\pi}\frac{x}{x^2+y^2}$$

This is the velocity potential sought. It is for a doublet located at $(x, y) = (0, 0)$. *Solution:* Let us rearrange the equation for the source/sink pair in the following way:

$$\phi = \frac{m}{4\pi} \ln \left(\frac{x^2 + y^2 + c^2 - 2xc + 4xc}{x^2 + y^2 + c^2 - 2xc} \right) = \frac{m}{4\pi} \ln \left(1 + \frac{4xc}{x^2 + y^2 + c^2 - 2xc} \right)$$

Since the second term in the argument of the logarithm is small, we can apply the following expansion

$$\ln (1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots$$

Applying this formula and retaining the first term only, we get

$$\phi = \frac{m}{4\pi} \frac{4xc}{x^2 + y^2 + c^2 - 2xc} + \dots$$

Substituting $\mu = 2mc$, we get

$$\phi = \frac{\mu}{2\pi} \frac{x}{x^2 + y^2 + c^2 - 2xc}$$

Taking the limit of this formula as $c \to 0$, we get

$$\phi = \frac{\mu}{2\pi} \frac{x}{x^2 + y^2}$$

which is the equation we sought to find.

(b) The corresponding formula for the stream function is

$$\psi = -\frac{\mu}{2\pi} \frac{y}{x^2 + y^2}$$

Show that this is the correct stream function by determining the velocity components from ϕ and, subsequently, from ψ to demonstrate that the same results are obtained. *Solution:* One way to demonstrate that this is the correct formula for ψ is to find the formulas for the velocity components using the formulas for ϕ and ψ to show that they are the same. This can be easily done by applying the MATLAB Symbolics toolbox as follows:

```
% Problem 2
syms x y mu
phi = (mu/2/pi) * x/(x^2 + y^2)
psi = -(mu/2/pi) * y/(x^2 + y^2)
u1 =   diff(phi,x)
v1 =   diff(phi,y)
u2 =   diff(psi,y)
v2 = - diff(psi,x)
u1 = simple(u1)
u2 = simple(u2)
u1 == u2
v1 == v2
% Executing this script, the following was obtained"
% u1 = -(mu*(x^2 - y^2))/(2*pi*(x^2 + y^2)^2)
% v1 = -(mu*x*y)/(pi*(x^2 + y^2)^2)
% u2 = -(mu*(x^2 - y^2))/(2*pi*(x^2 + y^2)^2)
% v2 = -(mu*x*y)/(pi*(x^2 + y^2)^2)
% u1 == u2 ==> 1 (true)
% v1 == v2 ==> 1 (true)
```

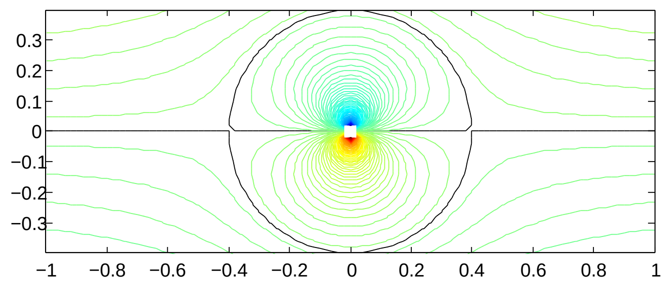

Figure 8: Illustration of the streamlines around a cylinder in a cross flow.

This is what we wanted to demonstrate.

Example 5.3: Use MATLAB to draw the streamlines for the following flow field:

$$\phi = Ux + \frac{\mu}{2\pi}\frac{x}{x^2 + y^2}$$

$$\psi = Uy - \frac{\mu}{2\pi}\frac{y}{x^2 + y^2}$$

Note that at $y = 0$, $\psi = 0$; hence, the dividing and closed streamline that divides the inner and outer flows is part of the $\psi = 0$ streamline. What is the shape of the object (demonstrate that it is a circle)? What are the components of the velocity for this flow? Show that the radius of the cylinder is $R = \sqrt{\mu/(2\pi U)}$. What is the formula for the pressure field? *Hint:* Apply the Bernoulli equation to answer the last question. [Although this question does *not* ask you to derive the formula for the pressure distribution on the surface of the cylinder, note that the pressure distribution on the surface of the cylinder is given by Equation (5.45) in the text. The derivation of this formula is outlined in Section 3.3.9.] *Solution:* The plot is given in Fig. 8. The shape of the object in the figure is a circular cylinder. This will be demonstrated next. To show that the shape of the object is a circle let us determine the velocity vectors in (r, θ) polar coordinates. Thus, let is write the potential as follows:

$$\phi = Ur\cos\theta + \frac{\mu}{2\pi}\frac{r\cos\theta}{r^2}$$

where $x = r\cos\theta$ and $r^2 = x^2 + y^2$ were used. The radial and tangential components of the velocity for this velocity potential are

$$u_r = \frac{\partial \phi}{\partial r} = U\cos\theta - \frac{\mu}{2\pi r^2}\cos\theta$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -Ur\sin\theta - \frac{\mu}{2\pi r}\sin\theta$$

The radial velocity component is equal to zero for $r = R$; thus,

$$R = \sqrt{\frac{\mu}{2\pi U}}$$

is the radius of a circle; it is therefore the closed streamline that models a circular cylinder in a cross flow. On the cylinder the only finite component of the velocity vector is

$$u_\theta|_{r=R} = -2U\sin\theta$$

where the formula for R in terms of μ was used. The formula for the pressure field is the Bernoulli theorem in the following form:

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(u_r^2 + u_\theta^2\right)$$

On the circle the pressure distribution is

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(-2U\sin\theta\right)^2$$

On the cylinder we get for the pressure coefficient the following formula

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta$$

This concludes the development of the formulas for the flow around a circular cylinder in a cross flow.