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ON THE SET COVERING POLYTOPE: I. ALL THE FACETS WITH COEFFICIENTS IN {0, 1, 2}

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While the set packing polytope, through its connection with vertex packing, has lent itself to fruitful investigations, little is known about the set covering polytope. We characterize the class of valid inequalities for the set covering polytope with coefficients equal to 0, 1 or 2, and give necessary and sufficient conditions for such an inequality to be minimal and to be facet defining. We show that all inequalities in the above class are contained in the elementary closure of the constraint set, and that 2 is the largest value of k such that all valid inequalities for the set covering polytope with integer coefficients no greater than k are contained in the elementary closure. We point out a connection between minimal inequalities in the class investigated and certain circulant submatrices of the coefficient matrix. Finally, we discuss conditions for an inequality to cut off a fractional solution to the linear programming relaxation of the set covering problem and to improve the lower bound given by a feasible solution to the dual of the linear programming relaxation.

Key words: Set covering, facets, polyhedral combinatorics, integer programming.

1. Introduction

The set covering problem can be stated as

$$\text{(SC)}\qquad \min\{cx \mid Ax \geq 1, x \in \{0,1\}^n\},$$

where $A=(a_{i,j})$ is an $m \times n$ matrix with $a_{ij} \in \{0,1\}$, $\forall i, j$, and 1 is the m-vector of 1's. If $Ax \geq 1$ is replaced by $Ax = 1$, the problem is called set partitioning. Both models have applications to crew scheduling, facility location, vehicle routing and a host of other areas (see Appendix to Balas and Padberg [3] for a bibliography of applications).

If we reverse the inequality in the definition of (SC), we obtain the set packing problem

$$\text{(SP)}\qquad \max\{cx \mid Ax \leq 1, x \in \{0,1\}^n\}$$

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which is known to be equivalent to the vertex packing problem on the intersection graph G_A of A. Both (SC) and (SP) are NP-complete problems for a general 0–1 matrix A. As far as structural properties go, because of the connection between (SP) and vertex packing, the properties of the set packing polytope (the convex hull of points satisfying the constraints of (SP)) have been thoroughly studied. In particular, many classes of facets of this polytope have been identified (see, for instance, [6, 7]), as well as families of matrices A for which the corresponding polytope is given by the linear inequalities $Ax \leqslant 1$, $0 \leqslant x \leqslant 1$. The same cannot be said about the set covering polytope

$$P_I(A) := \operatorname{conv}\{x \in R^n \mid Ax \geqslant 1, 0 \leqslant x \leqslant 1, x \text{ integer}\},$$

about which much less is known. In the following, we will denote

$$P(A) := \{x \in R^n \mid Ax \geqslant 1, 0 \leqslant x \leqslant 1\}.$$

Let M and N be the row and column index sets, respectively, of A. For any $R \subseteq M$ and $S \subseteq N$, we will write A_R^S for the submatrix of A whose rows and columns are indexed by R and S, respectively. Also, we will denote $A_R = A_R^N$ and $A^S = A_M^S$. Finally, for $i \in M$, we will denote $N^i := \{j \in N \mid a_{ij} = 1\}$.

A polyhedron is the intersection of a finite number of halfspaces. A polytope is a bounded polyhedron. A face of a polyhedron is the intersection of the polyhedron with some of its boundary planes. For an n-dimensional polyhedron, the 0-dimensional faces are its vertices, the $(n-1)$-dimensional faces are its facets. An inequality is valid for a polyhedron P if it is satisfied by all $x \in P$. An inequality $\alpha x \geqslant \alpha_0$ is dominated by, or is a weakening of, the inequality $\beta x \geqslant \alpha_0$, if $\alpha \geqslant \beta$. If in addition $\alpha_j > \beta_j$ for some j, then $\alpha x \geqslant \alpha_0$ is strictly dominated by $\beta x \geqslant \alpha_0$. A coefficient α_j of a valid inequality $\alpha x \geqslant \alpha_0$ is minimal if $\alpha x \geqslant \alpha_0$ becomes invalid when α_j is decreased (without changing other coefficients). A valid inequality whose coefficients are all minimal is called minimal. Thus a minimal inequality is one not strictly dominated by any valid inequality. An inequality $\alpha x \geqslant \alpha_0$, valid for a polyhedron P, defines (or induces) a facet of P if and only if $\alpha x = \alpha_0$ for n ($=\dim P$) affinely independent points $x \in P$. Valid inequalities that are facet defining are minimal, but the converse is not true.

Among the few facts known about the polyhedron $P_I(A)$ are the following. We assume throughout that A has no zero columns or zero rows.

1. *$P_I(A)$ is full dimensional if and only if $|N^i| \geqslant 2$ for all $i \in M$.*

In the following we assume that $P_I(A)$ is full dimensional.

2. *The inequality $x_j \geqslant 0$ defines a facet of $P_I(A)$ if and only if $|N^i \setminus \{j\}| \geqslant 2$ for all $i \in M$.*

3. *All inequalities $x_j \leqslant 1$ define facets of $P_I(A)$.*

4. *All facet defining inequalities $\alpha x \geqslant \alpha_0$ for $P_I(A)$ have $\alpha \geqslant 0$ if $\alpha_0 > 0$.*

5. *The inequality*

$$\sum (x_j : j \in N^i) \geqslant 1 \tag{1.1}$$

defines a facet of $P_I(A)$ *if and only if* (i) *there exists no* $k \in M$ *with* $N^k \subsetneq N^i$; *and* (ii) *for each* $k \in N \setminus N^i$, *there exists* $j(k) \in N^i$ *such that* $a_{hj(k)} = 1$ *for all* $h \in M^0(k)$, *where* $M^0(k) := \{h \in M \mid a_{hk} = 1$ *and* $a_{hj} = 0, \forall j \in N \setminus N^i \cup \{k\}\}$.

6. *The only minimal valid inequalities (hence the only facet defining inequalities) for* $P_I(A)$ *with integer coefficients and righthand side equal to* 1 *are those of the system* $Ax \geqslant 1$.

Statements 1 through 4 are easily seen to be true.

Proof of 5. Corollary 2 to Proposition 14 of [5] states (in the context of the polytope obtained from $P_I(A)$ by complementing the variables) that (1.1) defines a facet of $P_I(A)$ if and only if (i) and the following condition holds: (ii′) for $k \in N \setminus N^i$, there exists $j(k) \in N^i$ such that $x^k \in P_I(A)$, where x^k is given by

$$x_l^k = \begin{cases} 0, & l \in \{k\} \cup N^i \setminus \{j(k)\}, \\ 1, & \text{otherwise.} \end{cases} \tag{1.2}$$

But x^k defined by (1.2) satisfies $Ax \geqslant 1$ if and only if $a_{hj(k)} = 1$ for all $h \in M^0(k)$. Hence (ii) is equivalent to (ii′). □

Proof of 6. Let $\pi x \geqslant 1$ be any inequality with $\pi \geqslant 0$, and let $S := \{j \in N \mid \pi_j > 0\}$. If there exists $i \in M$ with $N^i \subseteq S$, then $\pi x \geqslant 1$ is either not minimal, or identical to some inequality of $Ax \geqslant 1$. Otherwise $N^i \setminus S \neq \emptyset$, $i \in M$, and hence $\bar{x}$ defined by $\bar{x}_j = 0$, $j \in S$, $\bar{x}_j = 1$, $j \in N \setminus S$, satisfies $Ax \geqslant 1$, but $\pi \bar{x} = 0 < 1$; thus $\pi x \geqslant 1$ is not a valid inequality for $P_I(A)$. □

Valid inequalities for a polyhedron related to $P_I(A)$, namely the convex hull of those $x \in P_I(A)$ that satisfy a given inequality $cx \leqslant z_U - 1$, where z_U is the value of a known solution, have been studied in [1]. The inequalities derived there have been successfully used as cutting planes, as reported in [2].

2. A class of facets of $P_I(A)$

We will be studying inequalities of the form $\alpha x \geqslant 2$, with $\alpha_j = 0$, 1 or 2, $j \in N$. As before, we let M and N be the row and column index sets, respectively, of A. For each such inequality, we denote

$$J_t(\alpha) = \{j \in N \mid \alpha_j = t\}, \quad t = 0, 1, 2,$$

and simply write J_t for $J_t(\alpha)$ whenever the meaning is clear from the context.

With each nonempty subset $S \subseteq M$ we associate the inequality $\alpha^S x \geq 2$, where

$$\alpha_j^S = \begin{cases} 0 & \text{if } a_{ij} = 0 \text{ for all } i \in S, \\ 2 & \text{if } a_{ij} = 1 \text{ for all } i \in S, \\ 1 & \text{otherwise.} \end{cases} \tag{2.1}$$

Notice that if $|S| = 1$, say $S = \{i\}$, the inequality $\alpha^S x \geq 2$ is just $a^i x \geq 1$ multiplied by 2.

Let C denote the class of inequalities $\alpha^S x \geq 2$ for all $S \subseteq M$. It is easy to see that C is in fact the class of inequalities obtainable from the system $Ax \geq 1$ by the following procedure, which we will also call C:

Procedure C. (i) Add the inequalities $a^i x \geq 1$, $i \in S$;

(ii) divide the resulting inequality by $|S| - \varepsilon$, $0.5 < \varepsilon < 1$; and

(iii) round up all coefficients to the nearest integer.

Thus for any $S \subseteq M$, $\alpha^S x \geq 2$ is a valid inequality for $P_I(A)$. To show that the converse is also true, we define for every $Q \subseteq N$,

$$M(Q) := \{i \in M \mid a_{ij} = 0, \forall j \in Q\},$$

with $M(\emptyset) := M$.

Theorem 2.1. *Every valid inequality $\beta x \geq 2$ for $P_I(A)$, with β_j integer, $j \in N$, is dominated by the inequality $\alpha^S x \geq 2$, where $S = M(J_0(\beta))$.*

Proof. By contradiction. W.l.o.g., we may assume that $\beta_j \in \{0, 1, 2\}$. The inequality $\alpha^S x \geq 2$ is well defined; for otherwise, i.e., if $M(J_0(\beta)) = \emptyset$, then $\bar{x}$ defined by $\bar{x}_j = 1$, $j \in J_0(\beta)$, $\bar{x}_j = 0$ otherwise, satisfies $Ax \geq 1$ and violates $\beta x \geq 2$, a contradiction.

If $\beta x \geq 2$ is not dominated by $\alpha^S x \geq 2$, then $\beta_{j_*} < \alpha_{j_*}^S$ for some $j_* \in N$. From the definition of α^S with $S = M(J_0(\beta))$, $\beta_j = 0$ implies $\alpha_j^S = 0$; hence $\beta_{j_*} = 1$ and $\alpha_{j_*}^S = 2$. This in turn implies that $a_{ij_*} = 1$ for all $i \in M(J_0(\beta))$. Therefore $\bar{x}$ defined by $\bar{x}_j = 1$ for $j \in J_0 \cup \{j_*\}$, $\bar{x}_j = 0$ otherwise, satisfies $Ax \geq 1$, $0 \leq x \leq 1$, but violates $\beta x \geq 2$, a contradiction. $\square$

Procedure C is easily seen to be a specialized version of Chvatal's procedure [4], which consists of recursively applying to the constraint set (I) of an integer program (with integer coefficients) the following operation: generate all positive linear combinations of inequalities of (I) that yield inequalities not dominated by those already generated, round up to the nearest integer the coefficients of the resulting inequalities, and redefine (I) by adding to it the new inequalities. This procedure is known to yield the convex hull of integer points satisfying the initial set (I) after a finite number of applications of the recursive step. The number of times the recursion needs to be applied to obtain a certain inequality is called the *rank* of that inequality. The original system together with the rank 1 inequalities forms the *elementary closure* of the system. From Theorem 2.1 we then have the following

Corollary 2.2. *Every valid inequality for $P_I(A)$ with coefficients in $\{0, 1, 2\}$ belongs to the elementary closure of the system $Ax \geqslant 1$, $x \geqslant 0$.*

A question that arises naturally in this context is whether the property of belonging to the elementary closure of the system $Ax \geqslant 1$, $x \geqslant 0$ (amended perhaps with $x \leqslant 1$) extends to some larger class of valid inequalities. Our next remark answers this question in the negative.

Remark 2.3. *For every $k \geqslant 3$, there exists a 0–1 matrix A and a valid inequality $\beta x \geqslant k$ for $P_I(A)$ with $\beta_j \in \{0, 1\}$, $j \in N$, that is not contained in the elementary closure of the system $Ax \geqslant 1$, $0 \leqslant x \leqslant 1$.*

Proof. Let $k \geqslant 3$ and let A be the edge-vertex incidence matrix of K_{k+1}, the complete graph on $k+1$ vertices. Then the inequality

$$\sum (x_j : j = 1, \ldots, k+1) \geqslant k$$

is valid for $P_I(A)$. Yet it is well known and easy to show that this inequality does not belong to the elementary closure of $Ax \geqslant 1$, $0 \leqslant x \leqslant 1$. □

Next we identify those members of the class C that are not strictly dominated by other members. Given any pair of inequalities $\alpha^S x \geqslant 2$ and $\alpha^T x \geqslant 2$ in C, such that $J_0(\alpha^S) = J_0(\alpha^T)$ and $T \subset S$, it is clear from the definitions that $\alpha^S x \geqslant 2$ dominates $\alpha^T x \geqslant 2$. Hence among all inequalities $\alpha^S x \geqslant 2$ with a fixed J_0, it is sufficient to consider those with $S = M(J_0)$.

Further, given any inequality $\alpha^S x \geqslant 2$ with $S = M(J_0)$, we will say that the set J_0 is *maximal* if for every $j \in J_1$ there exists $k \in J_1 \backslash \{j\}$ such that $a_{ik} = 1$ for all $i \in M(J_0 \cup \{j\})$. In other words, J_0 is maximal if transferring any column from J_1 to J_0 requires the transfer of some column from J_1 to J_2. This concept plays an important role in the sequel.

Theorem 2.4. *The inequality $\alpha^S x \geqslant 2$, where $S = M(J_0)$, is minimal if and only if J_0 is maximal.*

Proof. *Necessity.* If J_0 is not maximal, then there exists $j \in J_1$ such that for every $k \in J_1 \backslash \{j\}$, $a_{ik} = 0$ for some $i \in M(J_0 \cup \{j\})$. But then the inequality $\alpha^T x \geqslant 2$, where $T = M(J_0 \cup \{j\})$, strictly dominates $\alpha^S x \geqslant 2$.

Sufficiency. Suppose $\alpha^S x \geqslant 2$ is not minimal. Then there exists $\alpha^T x \geqslant 2$ in C such that $\alpha^T \leqslant \alpha^S$ and $\alpha^T_{j_*} < \alpha^S_{j_*}$ for some $j_* \in N \backslash J_0(\alpha^S)$. Since $S = M(J_0(\alpha^S))$, $T \subsetneq S$ and therefore $J_2(\alpha^S) \subseteq J_2(\alpha^T)$. Hence $j_* \in J_1(\alpha^S)$. But then $J_0(\alpha^S) \cup \{j_*\} \subseteq J_0(\alpha^T)$, $T \subseteq M(J_0(\alpha^S) \cup \{j_*\})$ and for all $k \in J_1(\alpha^S) \backslash \{j_*\}$ there exists some $i \in M(J_0(\alpha^S) \cup \{j_*\})$ such that $a_{ik} = 0$ (since $k \in J_0(\alpha^T) \cup J_1(\alpha^T)$); i.e., $J_0(\alpha^S)$ is not maximal. □

Example 2.1. Consider the set covering polytope defined by the matrix

$$A=\begin{pmatrix}1&1&1&0&0&0&0\\1&1&0&1&0&0&0\\1&0&1&1&0&0&0\\0&1&1&1&1&0&0\\1&0&1&0&1&1&1\\0&1&1&1&0&1&0\end{pmatrix}.$$

The inequality $\sum (x_j := j=1,\ldots,6) \geqslant 2$, obtained by applying Procedure C to the subsystem consisting of rows 1, 2, 3, 4, 6, is not minimal, since $J_0=\{7\}$ is not maximal: column 6 can be added to J_0 without having to transfer any column from J_1 to J_2. Thus $\sum (x_j : j=1,\ldots,5) \geqslant 2$ is a valid inequality. It is also minimal, since $J_0=\{6,7\}$ is maximal. Another valid inequality in class C is $2x_1+x_2+x_3+x_4 \geqslant 2$. The associated set $J_0=\{5,6,7\}$ is maximal, so this inequality is also minimal.

An alternative condition for the minimality of an inequality in the class C is along the following lines. For any valid inequality $\alpha^S x \geqslant 2$ in C, a pair $j, h \in J_1$ will be called a *2-cover* of $A^{J_1}_{M(J_0)}$, the submatrix of A with row set $M(J_0)$ and column set J_1, if

$$a_{ij} + a_{ih} \geqslant 1 \quad \text{for all } i \in M(J_0). \tag{2.2}$$

Corollary 2.5. *The inequality $\alpha^S x \geqslant 2$, where $S = M(J_0)$, is minimal if and only if every $j \in J_1$ belongs to some 2-cover of $A^{J_1}_{M(J_0)}$.*

Proof. We show that J_0 is maximal if and only if every $j \in J_1$ belongs to some 2-cover of $A^{J_1}_{M(J_0)}$. From the definition, J_0 is maximal if and only if for every $j \in J_1$ there exists $h \in J_1 \backslash \{j\}$ with $a_{ih}=1$ for all $i \in M(J_0)$ such that $a_{ij}=0$. But this condition is satisfied if and only if every $j \in J_1$ belongs to some 2-cover (j, h) of $A^{J_1}_{M(J_0)}$. $\square$

Next we address the question, which inequalities of the class C are facet inducing for $P_I(A)$. In stating the conditions for this we will assume that $P_I(A)$ is full dimensional. This is the case if and only if

$$\sum (a_{ij} : j \in N) \geqslant 2, \quad i \in M. \tag{2.3}$$

Assuming that (2.3) holds involves no loss of generality; for if not, then either $P_I(A)=\emptyset$, or else there exists some $F \subset N$, $F \neq \emptyset$, such that $x \in P_I(A)$ implies $x_j = 1$ for all $j \in F$. In the former case the inequality is obviously not facet defining; whereas in the latter case setting $x_j=1$, $j \in F$, and removing the inequalities satisfied by this assignment, produces a set covering polytope for which (2.3) is satisfied.

We define the *2-cover graph of $A^{J_1}_{M(J_0)}$* as the graph that has a vertex for every $j \in J_1$ and an edge for every 2-cover of $A^{J_1}_{M(J_0)}$. Further, for every $k \in J_0$, we define $T(k)$ as the set of rows such that k is the only column in J_0 to cover $T(k)$; i.e.,

$$T(k) = \{i \in M \mid a_{ik}=1,\ a_{ij}=0 \text{ for all } j \in J_0 \backslash \{k\}\}.$$

Theorem 2.6. *Let $P_I(A)$ be full dimensional and let $\alpha^S x \geqslant 2$ be a valid inequality for $P_I(A)$, with $S = M(J_0)$. Then $\alpha^S x \geqslant 2$ defines a facet of $P_I(A)$ if and only if*

(i) *every component of the 2-cover graph of $A^{J_1}_{M(J_0)}$ has an odd cycle;*

(ii) *for every $k \in J_0$ such that $T(k) \neq \emptyset$ there exists either*

(a) *some $j(k) \in J_2$ such that $a_{ij(k)} = 1$ for all $i \in T(k)$; or*

(b) *some pair $j(k)$, $h(k) \in J_1$ such that $a_{ij(k)} + a_{ih(k)} \geqslant 1$ for all $i \in T(k) \cup M(J_0)$.*

Proof. *Necessity.* Suppose $\alpha^S x \geqslant 2$ defines a facet of $P_I(A)$. Then there exists a collection of n affinely independent points $x^i \in P_I(A)$, usch that $\alpha^S x^i = 2$ for $i = 1, \ldots, n$. Let X be the $n \times n$ matrix whose rows are the vectors x^i; then X is of the form (modulo row and column permutations)

$$X = \begin{pmatrix} X_1 & X_2 & 0 \\ X_3 & 0 & X_4 \end{pmatrix},$$

where the columns of X_1, X_2, X_3 and X_4 are indexed by J_0, J_1, J_0 and J_2, respectively, X_4 is the identity matrix of order $|J_2|$, and every row of X_2 is a row of the edge-vertex incidence matrix of the 2-cover graph G of $A^{J_1}_{M(J_0)}$. Since X is nonsingular, X_2 is of full column rank, and hence every component of G has an odd cycle. Thus (i) holds.

To show that (ii) also holds, suppose there exists $k \in J_0$ and $T(k)$ for which neither (a) nor (b) is satisfied. Then $x_k = 1$ for every $x \in P_I(A)$ such that $\alpha^S x = 2$, which contradicts the fact that $\alpha x \geqslant 2$ is facet defining.

Sufficiency. Suppose conditions (i) and (ii) are satisfied. We exhibit a set of n affinely independent points $x^k \in P_I(A)$ such that $\alpha^S x^k = 2$, $k = 1, \ldots, n$.

For $t = 0, 1, 2$, let e^t and 0^t denote the $|J_t|$-vector whose components are all 1 and all 0, respectively. For $t = 0$, 1, 2, let e^t_j be the jth unit vector with $|J_t|$ components.

Our first $|J_0|$ vectors x^k, $k \in J_0$, are defined as

$$x^k = \begin{cases} (e^0 - e^0_k, 0^1, e^2_j) & \text{for some } j \in J_2 \\ & \text{if } T(k) = \emptyset \text{ and } J_2 \neq \emptyset \\ (e^0 - e^0_k, e^1_j + e^1_h, 0^2) & \text{for some 2-cover } (j, h) \\ & \text{if } T(k) = \emptyset = J_2 \\ (e^0 - e^0_k, 0^1, e^2_{j(k)}) & \text{if } T(k) \neq \emptyset \text{ and (a) holds} \\ & \text{(with } j(k) \text{ as in (a))} \\ (e^0 - e^0_k, e^1_{j(k)} + e^1_{h(k)}, 0^2) & \text{if } T(k) \neq \emptyset \text{ and not (a) but (b) holds} \\ & \text{(with } j(k), h(k) \text{ as in (b))} \end{cases}$$

By property (ii), these vectors exist and belong to $P_I(A)$.

Our next $|J_1|$ vectors are of the form

$$x^k = (e^0, e^1_{j(k)} + e^1_{h(k)}, 0^2), \quad k \in J_1.$$

where the pair $j(k)$, $h(k) \in J_1$ satisfies (2.2), and the vectors $e^1_{j(k)} + e^1_{h(k)}$ are linearly independent. By property (i), there exists a set of $|J_1|$ vectors $x^k \in P_I(A)$ satisfying these conditions.

Finally, the last $|J_2|$ vectors are of the form

$$x^k = (e^0, 0^1, e_k^2), \quad k \in J_2.$$

Here the vectors e_k^2 form the identity matrix of order $|J_2|$. The existence of these vectors $x^k \in P_I(A)$ follows from the definition of J_2.

It is now easy to see that the matrix X whose rows are the n vectors x^k, $k \in J_0 \cup J_1 \cup J_2$, is nonsingular. Also, every x^k satisfies $\alpha^S x^k = 2$. Hence $\alpha^S x \geq 2$ defines a facet of $P_I(A)$. □

Example 2.2. Consider the matrix A of Example 2.1 and the valid inequality $\sum (x_j: j = 1, \ldots, 5) \geq 2$ for $P_I(A)$, which was shown to be minimal. We have $J_1 = \{1, \ldots, 5\}$, $J_0 = \{6, 7\}$ and $M(J_0) = \{1, 2, 3, 4\}$. The two-cover graph of $A^{J_1}_{M(J_0)}$, shown in Fig. 1, is connected and has odd cycles; thus condition (i) of Theorem 2.3 is satisfied. The only $k \in J_0$ such that $T(k) \neq \emptyset$ is 6, with $T(6) = \{6\}$; and any of the pairs $h, j \in J_1 \setminus \{5\}$ satisfies $a_{ih} + a_{ij} \geq 1$ for all $i \in M(J_0) \cup T(6) = \{1, 2, 3, 4, 6\}$. Hence $\sum (x_j: j = 1, \ldots, 5) \geq 2$ induces a facet of $P_I(A)$.

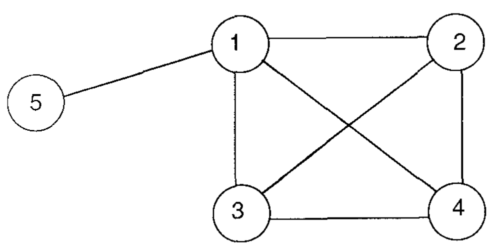

Fig. 1.

3. Generating minimal inequalities in C

Before addressing the question of how to generate minimal inequalities in C with some desirable properties, we examine the relationship between such inequalities and circulant submatrices of A with exactly one zero in every row and column. Such a matrix, if of order k, will be denoted C_k^{k-1} and called the *full circulant* of order k. Since any square matrix with $k-1$ ones and one zero in every row and column is equal to C_k^{k-1} up to row and column permutations, we will refer to such a matrix as a C_k^{k-1} without invoking every time the row and column permutations.

Consider the inequality $\alpha^S x \geq 2$, where $S = M(J_0)$. Often there exist proper subsets $T \subset S$ such that $\alpha^T x \geq 2$ is the same inequality as $\alpha^S x \geq 2$. Such a subset $T \subset S$ will be called *C-equivalent to S*. If T is C-equivalent to S and no proper subset of T has this property, we say that T is a *minimal C-equivalent subset* of S. A set $S = M(J_0)$ may have several minimal C-equivalent subsets.

Theorem 3.1. *For every minimal C-equivalent subset T of $M(J_0)$, the matrix $A_T^{J_1}$ contains as a submatrix C_t^{t-1}, the full circulant of order $t=|T|$.*

Proof. Since T is minimal with respect to the property that $\alpha^T=\alpha^S$, and $\alpha^S x \geqslant 2$ is a minimal inequality, it follows that for any $Q \subsetneq T$, $\alpha_{j_*}^Q > \alpha_{j_*}^S$ for some $j_* \in N$. Since $J_0(\alpha^S) \subseteq J_0(\alpha^Q)$, $j_* \notin J_0(\alpha^S)$. Also, $j_* \notin J_2(\alpha^S)$. Thus $j_* \in J_1(\alpha^S)$, and $a_{ij_*}=1$ for all $i \in Q$, $a_{ij_*}=0$ for some $i \in T \setminus Q$. Since this is true for any proper subset of T and in particular for every subset of the form $Q=T\setminus\{i\}$ for some $i \in T$, it follows that for every row $i \in T$ there exists a column $j(i) \in J_1$ such that $a_{hj(i)}=0$ for $h=i$ and $a_{hj(i)}=1$ for all $h \in T\setminus\{i\}$. Clearly, the t columns $j(i)$, $i \in T$, must be distinct since every column has exactly one zero in position i. But the submatrix of $A_T^{J_1}$ consisting of these t columns is precisely C_t^{t-1}. □

Example 3.1. Consider the matrix

$$A=\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The inequality of C associated with the row set $S=\{2,3,4,5\}$ is

$$x_2+x_3+x_4+x_5+x_6+x_7 \geqslant 2,$$

and it is minimal. However, the set S is not minimal with respect to this inequality. Removing any one of the four rows of S produces a minimal C-equivalent subset, and each such subset contains a C_3^2.

The correspondence between the minimal inequalities of the class C and the full circulant submatrices of A is not one to one, since for any given minimal inequality $\alpha^S x \geqslant 2$ of C, there may be several minimal sets T_i C-equivalent to S, each one containing one or several full circulants of order $|T_i|$. Nevertheless the full circulants of A can be used to list all the minimal inequalities in the class C: every minimal inequality in C can be obtained from some $S \subseteq M$ such that (i) A_S contains a full circulant of order $|S|$; and (ii) there exists no $T \supsetneq S$ such that $J_0(\alpha^T) \supseteq J_0(\alpha^S)$ and A_T contains a full circulant or order $|T|$.

The correspondence between full circulants and minimal members of the class C is also helpful in counting the latter, viz., in bounding their number.

Remark 3.2. *The number of minimal inequalities in C is $O(m^k)$, where $m=|M|$ and k is the cardinality of the largest full circulant submatrix of A.*

Proof. From Theorem 3.1 the number of distinct minimal members of C is bounded by the number of row sets T such that A_T contains C_t^{t-1} with $t=|T|$. Since A has

$\binom{m}{i}$ row sets of size i, if k is the order of the largest full circulant submatrix of A, then the number of row sets T such that A_T contains a full circulant of order $|T|$ is bounded by $\sum_{i\leq k}\binom{m}{i}<m^k$. $\square$

It is a well known result in polyhedral combinatorics (see, for instance, [6, 7, 8]) that minimal inequalities for a polyhedron $P_I(A)$ can often be obtained by lifting minimal inequalities for a polyhedron $P_I(A^V)$ for some $V\subset N$. The above Theorem suggests that the minimal inequalities of the class C for $P_I(A)$ might be obtainable by lifting the minimal inequalities of C for some polyhedra of the form $P_I(A^K)$, where K is the column index set of a full circulant. This, however, is not true, since often $\alpha^S x\geq 2$ is a minimal inequality for $P_I(A)$, but the corresponding inequality

$$\sum (x_j: j\in K)\geq 2$$

is not minimal for $P_I(A^K)$. Obtaining the inequalities in C by lifting is a complex problem, which will be dealt with in another paper.

We now turn to the problem of generating inequalities in C with some desirable property.

First, we note that based on Theorem 3.1, all members of C can be generated by listing the full circulants of A satisfying conditions (i) and (ii) listed above, and applying procedure C to each subsystem corresponding to the row set of a full circulant. Of course, this procedure may generate some inequalities repeatedly.

A frequent situation encountered in practice is the one where a fractional solution to the current problem is available, and one is interested in generating an inequality in C that cuts it off. Let $\bar{x}$ be a fractional solution to $Ax\geq 1$, $0\leq x\leq 1$, with

$$I=\{j\in N\,|\,\bar{x}_j=1\},\qquad F=\{j\in N\,|\,0<\bar{x}_j<1\},$$

and let

$$M(I):=\{i\in M\,|\,a_{ij}=0,\ \forall j\in I\}.$$

Theorem 3.3. *Let $\alpha^S x\geq 2$ be an inequality in C that cuts off $\bar{x}$. Then $\alpha_j^S=0$ for all $j\in I$; i.e., $S\subseteq M(I)$.*

Proof. By contradiction. Suppose $\alpha_{j_*}^S\geq 1$ for some $j_*\in I$. If $\alpha_{j_*}^S=2$, then $\alpha^S\bar{x}\geq \alpha_{j_*}^S\bar{x}_{j_*}\geq 2$, contrary to the assumption that $\alpha^S x\geq 2$ cuts off $\bar{x}$. If $\alpha_{j_*}^S=1$, then there exists $i_*\in M(I)$ such that $a_{i_*j_*}=0$. Then substituting $\bar{x}_{j_*}=1$ into the inequalities $a^{i_*}x\geq 1$ and $\alpha x\geq 2$ yields

$$\sum (a_{i_*j}\bar{x}_j : j\in N\backslash\{j_*\})\geq 1$$

and

$$\sum (\alpha_j\bar{x}_j : j\in N\backslash\{j_*\})\geq 1$$

respectively, with $a_{i_*j}\leq \alpha_j$ for all $j\in N\backslash\{j_*\}$. Since $\bar{x}$ satisfies the first of these inequalities, it cannot violate the second one. This proves that $\alpha_j^S=0$ for all $j\in I$.

From the definition of the class C, this implies that $S \subseteq M(I)$. $\square$

Thus in order to generate an inequality in C that cuts off $\bar{x}$, one can restrict the examination of full circulants with the desired property to those contained in the row set $M(I)$.

Some of the more recent methods for solving set covering problems never solve the linear programming relaxation of the problem and thus never generate fractional solutions to be cut off. These methods (see, for example, [2]) use instead subgradient optimization or other techniques to find an approximate (feasible) solution to the dual of the linear programming relaxation, whose objective function value provides a lower bound on the value of an optimal cover. To use the inequalities of the class C in this context, one has to be able to answer the following question: given a feasible solution u to the dual of the linear relaxation of the set covering problem, is there an inequality in C whose addition to the constraint set would make it possible to strengthen the lower bound associated with u. Our next theorem addresses this question.

Theorem 3.4. *Let $\alpha x \geqslant 2$ be a minimal valid inequality for $P_I(A)$, and let T be any C-equivalent subset of $M(J_0(\alpha))$. Further, let $u \in \mathbb{R}^m$ satisfy $u \geqslant 0$, $uA \leqslant c$, and define*

$$\delta(T)_k := \min\{c_j - \sum (u_i a_{ij}\colon i \in M \setminus T)\colon j \in J_k(\alpha)\}, \quad k = 1, 2,$$

with

$$\delta(T) := \min\{\delta(T)_1, \tfrac{1}{2}\delta(T)_2\}$$

Then

$$\sum (u_i\colon i \in M \setminus T) + 2\delta(T) \leqslant cx$$

for all $x \in \{0, 1\}^n$ satisfying $Ax \geqslant 1$.

Proof. Define $\bar{u} \in \mathbb{R}^{m+1}$ by

$$\bar{u}_i = \begin{cases} 0, & i \in T, \\ u_i, & i \in M \setminus T, \\ \delta(T), & i = |M| + 1. \end{cases} \tag{3.1}$$

Then $\bar{u} \geqslant 0$ and

$$c_j - \sum (\bar{u}_i a_{ij}\colon i \in M) - \bar{u}_{|M|+1}\alpha_j = c_j - \sum (u_i a_{ij}\colon i \in M \setminus T) - \delta(T)\alpha_j \geqslant 0, \quad j \in N,$$

i.e., $\bar{u}$ is a feasible solution to the linear program dual to

$$\min\{cx\colon Ax \geqslant 1, \alpha x \geqslant 2, x \geqslant 0\}.$$

Therefore

$$\begin{aligned} \sum (\bar{u}_i\colon i = 1, \ldots, |M|) + 2\bar{u}_{|M|+1} &= \sum (u_i\colon i \in M \setminus T) + 2\delta(T) \\ &\leqslant cx \end{aligned}$$

for any x satisfying $Ax \geqslant 1$, $\alpha x \geqslant 2$, $x \geqslant 0$, hence for any $x \in \{0, 1\}^n$ satisfying $Ax \geqslant 1$. $\square$

Corollary 3.5. *Adding $\alpha x \geqslant 2$ to the constraint set $Ax \geqslant 1$ strengthens the lower bound on cx provided by u if*

$$\delta(T) > \tfrac{1}{2} \sum (u_i\colon i \in T). \tag{3.2}$$

If (3.2) *holds and, in addition, u is an optimal solution to the dual of*

$$\min\{cx\colon Ax \geqslant 1, x \geqslant 0\}, \tag{3.3}$$

then the inequality $\alpha x \geqslant 2$ cuts off all optimal solutions to (3.3).

Proof. The difference between the lower bounds provided by $\bar{u}$ defined in (3.1) and u (i.e., the difference due to $\alpha x \geqslant 2$) is

$$\sum (u_i\colon i \in M \backslash T) + 2\delta(T) - \sum (u_i\colon i \in M) = 2\delta(T) - \sum (u_i\colon i \in T),$$

which proves the first statement.

If this difference is positive and u is an optimal solution to the dual of (3.3), then for any optimal solution $\hat{x}$ to (3.3),

$$\begin{aligned} c\hat{x} &= \sum (u_i\colon i \in M) \\ &< \sum (\bar{u}_i\colon i \in M \backslash T) + 2\delta(T) \leqslant cx \end{aligned}$$

for any x satisfying $Ax \geqslant 1$, $\alpha x \geqslant 2$, $x \geqslant 0$. Hence the inequality $\alpha x \geqslant 2$ cuts off $\hat{x}$. $\square$

Note that a straightforward modification of Theorem 3.4 and Corollary 3.5 holds for the case when the constraint set $Ax \geqslant 1$ is amended by $\alpha^i x \geqslant 2$, $i \in M'$, i.e., the dual constraint set $uA \leqslant c$ is replaced by

$$uA + \sum (u_i \alpha^i\colon i \in M') \leqslant c.$$

In other words, inequalities in C that improve the lower bound can be generated recursively.

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