# On the Set-Covering Polytope: All the facets with coefficients in $\{0, 1, 2, 3\}$

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**Abstract**

Balas and Ng [3] characterized the class of facets of the set-covering polytope defined by inequalities with coefficients in $\{0, 1, 2\}$. We extend their work to the class of valid inequalities with coefficients in $\{0, 1, 2, 3\}$. In particular, we give necessary and sufficient conditions for such an inequality to be minimal and to be facet-defining. We address the problem of generating all minimal valid inequalities with integral coefficients and right hand side of 3 which are support-equivalent to and dominate a given inequality with integral coefficients and right hand side of 3. In particular, we exhibit a precise one-one correspondence between every such inequality and independent dominating sets of a certain hypergraph, called the *generator hypergraph*. We exploit this correspondence in devising a general procedure which can generate all minimal valid inequalities (and hence facets) with integral coefficients and right hand side of 3. Various manifestations of the general procedure are analysed and connections to a similar procedure by Sànchez et al [12] are established. The separation problem associated with inequalities in this class is addressed and related complexity results are established. Finally, we discuss conditions for an inequality in this class to cut off a fractional solution to the linear programming relaxation of the set covering problem.

*Key words*: Set-Covering, facets, polyhedral combinatorics, integer programming, generator hypergraphs, kernel hypergraphs.

# 1 Introduction

The set covering problem can be stated as,

$$(SC) \quad min \{cx | Ax \geq \mathbf{1}, x \in \{0, 1\}^n\}, \tag{1.1}$$

where $A = (A_{ij})$ is an $m \times n$ matrix with $A_{ij} \in \{0, 1\}, \forall i, j$, and $\mathbf{1}$ is the $m$-vector of ones. If $Ax \geq \mathbf{1}$ is replaced by $Ax = \mathbf{1}$, the problem is called set-partitioning. Both models have applications to crew scheduling, facility location, vehicle routing and a host of other areas (see Appendix to Balas and Padberg [2] for a bibliography of applications).

The convex hull of $0 \setminus 1$ solutions to $(SC)$ is called the set-covering polytope, $P_I(A)$. This paper focusses on the facets of the set-covering polytope with coefficients in $\{0, 1, 2, 3\}$. We denote the family of valid inequalities of the set-covering polytope with coefficients in $\{0, 1, 2, 3\}$ by $\mathcal{I}_3^A$. The development of ideas in this paper is motivated by earlier work by Balas and Ng [3, 4]. Balas and Ng [3] gave a characterization of the facets of the set-covering polytope with coefficients in $\{0, 1, 2\}$, which was subsequently connected to the lifting theory [4]. This paper and a companion paper [11] extends their work to facets with coefficients in $\{0, 1, 2, 3\}$. In this paper, we study theoretical properties of facets in $\mathcal{I}_3^A$ and give various procedures to generate them. In particular, necessary and sufficient conditions for an inequality in $\mathcal{I}_3^A$ to be a minimal valid inequality, to be a facet-defining inequality are established. We introduce a notion of *cover hypergraphs*, which generalizes the notion of 2-cover introduced by Balas and Ng [3]. Cover hypergraphs are used as fundamental structures to formulate and illustrate most of the results in this paper. We give a general procedure to generate minimal valid inequalities in $\mathcal{I}_3^A$ and study two manifestations of the general procedure. We address the separation problem associated with inequalities in $\mathcal{I}_3^A$ and establish complexity results about the same. Finally we discuss conditions for an inequality to cut off a fractional solution to the linear programming relaxation of the set covering problem. All the results are illustrated by examples and are followed by a brief discussion, to put them in context with earlier known results. The companion paper [11] connects the theoretical framework developed in this paper to the theory of facet-lifting.

Facets of the set-covering polytope with small coefficients has been studied in literature [3, 4, 5, 9, 12]. The work of Sànchez, Sobròn, Vitoriano [12] is the one which closely relates to that of ours. Sànchez et al [12] also studied valid inequalities with coefficients in $\{0, 1, 2, 3\}$ and gave procedures to generate the same. However there is a fundamental difference between our approach and that of theirs, namely in the way *minimal* valid inequalities are defined. Whereas our treatment of minimal valid inequalities is identical to that of Balas and Ng [3], Sànchez et al [12] define minimal valid inequalities in terms of certain cardinality conditions. Detailed comparison of the two approaches is presented in this paper and we show that their procedure can be viewed as an interesting (special) case of our general procedure. An approach which treats $A$ as the adjacency matrix of a bipartite graph has been (in combination with lifting) successfully used by Sassano [9] and Cornuéjols and Sassano [5] to investigate the class of facets of $P_I(A)$ with left hand side coefficients equal to 0 or 1. While this class of facets overlaps with that of ours, none of the two classes contains the other, since one of them allows for arbitrary positive right hand sides but restricts the left hand side coefficients to 0 or 1, whereas the other allows for coefficients equal to $0, 1, 2$ or 3 on both sides.

*Section 1.1* defines the notations and terminology used in this paper. In *Section 1.2* we discuss the main results of this paper and give an outline for the remaining sections.

## 1.1 Definitions and Notations

The set-covering problem can be stated as:

$$(SC) \quad min\{cx | Ax \geq \mathbf{1}, x \in \{0, 1\}^n\}$$

where $A = (A_{ij})$ is an $m \times n$ matrix with $A_{ij} \in \{0, 1\}\ \forall i, j$, and $\mathbf{1}$ is the $m$-vector of ones. The convex hull of feasible solutions to the above integer program is called the *set-covering* polytope associated with $A$ and is denoted by $P_I(A)$. The polytope associated with the corresponding linear programming relaxation is denoted by $P(A)$.

$$P(A) = \{x \in \mathbb{R}^n | Ax \geq \mathbf{1}, \mathbf{0} \leq x \leq \mathbf{1}\}$$

We denote the boolean set $\{0, 1\}$ by $\mathbb{B}$. Let $M$ and $N$ be the row and column index sets, respectively, of $A$. Let $n = |N|$ and $m = |M|$. For any $R \subseteq M$ and $S \subseteq N$, we will write $A_R^S$ for the submatrix of $A$ whose rows and columns are indexed by $R$ and $S$, respectively. Also, we denote $A_R = A_R^N$ and $A^S = A_M^S$. Finally for $i \in M$, we will denote $N^i = \{j \in N | A_{ij} = 1\}$ and for $j \in N$, we will denote $M^j = \{i \in M | A_{ij} = 1\}$. For every $Q \subseteq N$, we define,

$$M(Q) = \{i \in M | A_{ij} = 0, \forall j \in Q\}$$

with $M(\phi) = M$. The $j^{th}$ column of $A$ is denoted by $A_j$ and the $i^{th}$ row of $A$ would be denoted by $a^i$. For $j \in N$, if $i \in M^j$, then the column $A_j$ is said to the cover row $a^i$. Without loss of generality we assume that $A$ has no zero rows or zero columns. The minimum number of columns required to cover every row of $A$ is called the *covering number* of $A$ and is denoted by $\beta(A)$. Trivially valid inequality $\sum_{j \in N} x_j \geq \beta(A)$ is called the rank inequality of $P_I(A)$. Let $G_A$ be the graph defined on the column index set $N$ of $A$, such that two columns in $N$ are adjacent in $G$ if and only if they cover a common row of $A$. $G_A$, as defined here, is called the column intersection graph of $A$. Suppose $S$ is a set, then $S \setminus \{i\}$ and $S \cup \{i\}$ are represented as $S - i$ and $S + i$ for sake of notational simplicity. If $\alpha \in \mathbb{R}^N$ and $S \subseteq N$, we denote $\sum_{j \in S} \alpha_j$ by $\alpha(S)$.

A polyhedron is the intersection of a finite number of halfspaces. A polytope is a bounded polyhedron. A face of a polyhedron is the intersection of the polyhedron with some of its boundary planes. For an $n$-dimensional polyhedron, the 0-dimensional faces are its vertices and the $(n-1)$-dimensional faces are its facets. An inequality is valid for a polyhedron $P$ if it is satisfied for all $x \in P$. An inequality $\alpha x \geq \alpha_0$ is dominated by, or is a weakening of, the inequality $\beta x \geq \alpha_0$, if $\alpha \geq \beta$. If in addition, $\alpha_j > \beta_j$ for some $j$, then $\alpha x \geq \alpha_0$ is strictly dominated by $\beta x \geq \alpha_0$. A coefficient $\alpha_j$ of $\alpha$ is minimal if $\alpha x \geq \alpha_0$ becomes invalid if $\alpha_j$ is decreased (without changing other coefficients).

A valid inequality whose coefficients are all minimal is called minimal. Thus a minimal inequality is one not strictly dominated by any other valid inequality. An inequality $\alpha x \geq \alpha_0$, valid for a polyhedron $P$, defines (or induces) a facet of $P$ if and only if $\alpha x = \alpha_0$ for $dim(P)$ affinely independent points $x \in P$. Valid inequalities that are facet defining are minimal, but the converse is not true. Let $\alpha x \geq \alpha_0$ be a valid inequality for a polyhedron $P$. The support of $\alpha x \geq \alpha_0$, denoted by $Sup(\alpha)$, is defined as, $Sup(\alpha) = \{j \in N | \alpha_j \neq 0\}$. Two valid inequalities of the set-covering polytope which have the same support are called *support-equivalent* inequalities. Throughout this paper, we assume that *the inequalities under consideration have integral coefficients, unless otherwise stated.* Furthermore, for a positive integer $k \geq 2$, we define sets $\mathcal{I}_k^A$ and $\mathcal{M}_k^A$ as follows:

$$\mathcal{I}_k^A = \{\alpha \in \{0, 1 \dots k\}^n | \alpha x \geq k \ is \ a \ valid \ inequality \ for \ P_I(A)\}$$

$$\mathcal{M}_k^A = \left\{\alpha \in \mathcal{I}_k^A | \alpha x \geq k \ is \ a \ minimal \ valid \ inequality \ for \ P_I(A)\right\}$$

If $J_0 \subseteq N$, we define $\mathcal{M}_3^A(J_0) = \{\beta \in \mathcal{M}_3^A | J_0^\beta = J_0\}$. Thus $\mathcal{M}_3^A(J_0)$ represents all those minimal valid inequalities with coefficients in $\{0, 1, 2, 3\}$ whose support is equal to $N \setminus J_0$. Throughout this paper, we do not make any distinction between vector $\alpha \in \mathcal{I}_k^A$ and the associated inequality $\alpha x \geq 3$.

Let $G = (V, E)$ be a graph on vertex set $V$ and edge-set $E$. Throughout this paper, we assume that all graphs considered are simple graphs with no loops or parallel edges. An edge joining vertices $u$ and $v$ is denoted by $uv$. For $v \in V$, $N_G(v) = \{u \in V | uv \in E\}$ and $N_G[v] = N(v) \cup \{v\}$ are called the open and closed neighbourhoods of the vertex $v$, respectively. The degree of a vertex $v \in V$ is defined as $d_G(v) = |N_G(v)|$. Also $\delta_G = min_{v \in V} d_G(v)$, $\Delta_G = max_{v \in V} d_G(v)$. If $\delta_G = \Delta_G = r$, then $G$ is said to be $r$-regular. In particular, 3-regular graphs are called *cubic* graphs. The subscript $G$ is dropped, whenever the graph under consideration is clear from context. A subset $S \subseteq V$ is called an independent set of $G$ if for all $u, v \in S$, $uv$ is not an edge of $G$. A subset $S \subseteq V$ is called a dominating set of $G$ if $S$ intersects the closed neighbourhood of every vertex of $G$. A subset of $V$ which is both an independent set and a dominating set is called an independent dominating set of $G$. A dominating set of $G$ is called a connected dominating set if the subgraph of $G$ induced by the vertices in the set is connected. A dominating set of $G$ is called a weakly connected dominating set if the subgraph of $G$ induced by the edges which have at least one end-point in the set is connected. The minimum cardinality of a dominating set of $G$ is called the *domination number* of $G$ and is denoted by $\gamma_G$. The convex hull of the characteristic vectors of dominating sets of $G$ is termed as the dominating set polytope of $G$ and is denoted by $P_{DS}^G$. Suppose $V = \{v_1, v_2 \dots v_n\}$. An $n \times n$ $0 \setminus 1$ matrix $A$, such that $A_{ij} = 1$ if and only if $v_i$ is adjacent to $v_j$ in $G$, is termed as the adjacency matrix of $G$. The matrix obtained from the adjacency matrix by replacing the zeros on the diagonal by ones is termed as the neighbourhood matrix of $G$.

Let $H = (V_H, E_H)$ be a hypergraph defined on vertex set $V_H$ and edge set $E_H \subseteq 2^{V_H}$, such that $\forall e \in E_H, |e| \geq 2$. We say that $e$ is an $k$-edge of $H$ if $e \in E_H$ and $|e| = k$. If every edge of $H$ is a $k$-edge for some positive integer $k \geq 2$, then $H$ is called a $k$-uniform hypergraph. We follow the following convention to represent hypergraphs. 2-edge of the hypergraph will be drawn as a line joining the vertices in the edge. $k$-edge ($k \geq 3$) of the hypergraph will be represented by a star centered at a point representing the edge and arcs emanating from that point to the vertices in the edge. For example a hypergraph defined on the vertex set $\{1 \dots 10\}$ and edge set $(1, 5), (1, 6), (2, 6, 10), (2, 9, 10), (3, 6, 7), (4, 6), (6, 8), (6, 7, 10)$ is represented in *Figure 1.* Let $u, v \in V_H$ be two vertices in $H$, then $u$ is said to be connected to $v$ if there exists a sequence of edges $e_0, e_1 \dots e_p$ such that $u \in e_0$, $v \in e_p$ and if $p > 0$, then $\forall i = 1, 2 \dots p$, $e_i \cap e_{i-1} \neq \phi$. A hypergraph in which every pair of vertices are connected is called a connected hypergraph. By connected components of a hypergraph $H$ we mean maximally connected components (inclusion wise) of $H$. We define a concept of *strong connectivity* of hypergraphs. Given a subset $U \subseteq V_H$, say $U = \{v_1 \dots v_k\}$, an ordering of vertices in $V_H \setminus \{v_1 \dots v_k\}$, say $v_{k+1}, v_{k+2} \dots v_n$ is said to be *U-connected* if $\forall i = k+1 \dots n$, there exists an edge $e_i$ of $H$ such that $v_i \in e_i$ and $e_i \subseteq \{v_1, v_2 \dots v_i\}$. For sake of notational simplicity, we also the call the following sequence of vertices: $v_1, \dots v_k, v_{k+1}, \dots v_n$ a $U$-connected sequence if $v_{k+1}, \dots v_n$ is a $U$-connected sequence in the sense defined above. $H$ is said to be strongly connected w.r.t a subset $U \subseteq V_H$, if there exists an $U$-connected ordering of vertices in $V_H \setminus U$. In particular, $H$ is said to be strongly connected w.r.t an edge $e \in E_H$, if there exists an $e$-connected ordering of vertices in $V_H \setminus e$. A hypergraph is strongly connected if it is strongly connected w.r.t at least one of its edges. Note that every strongly connected hypergraph is connected. However, the converse is not true. It is easy to generate examples of hypergraphs which are connected but not strongly connected. We remark that it is the strong connectivity (and not just connectivity) of certain *cover hypergraphs* (defined below) that allow us to derive rather strong results as presented in the companion paper [11]. A subset $S_H \subseteq V_H$, is called an independent set in $H$, if for all edges $e \in E_H$, $e \setminus S_H \neq \phi$. Similarly, a subset $S_H \subseteq V_H$ is called a dominating set of $H$, if $\forall v \in V_H \setminus S_H$, there exists an edge $e$ incident to $v$ such that $e \subseteq S_H \cup \{v\}$. A subset of vertices which is both an independent and a dominating set is called an independent dominating set of $H$.

Let $\alpha x \geq k$ be a valid inequality for the set-covering polytope associated with a $0 \setminus 1$ matrix $A$, such that all the components of $\alpha$, and $k$ are non-negative integers. Note that if some components of $\alpha$ are more than $k$, then we can reduce them to $k$ without violating the validity of the inequality. Hence we assume that all components of $\alpha$ are in the set $\{0, 1 \ldots k\}$. For $t \in \{0 \ldots k\}$, we define $J_t^\alpha$ as, $J_t^\alpha = \{j \in N | \alpha_j = t\}$. The superscript in $J_t^\alpha$ will be dropped whenever the inequality under consideration is clear from context. The cover hypergraph of $\alpha x \geq k$, $H^\alpha$ is a hypergraph defined on vertex set $V_\alpha = N \setminus (J_0^\alpha \cup J_k^\alpha)$, such that $S \subseteq V_\alpha$ forms an edge of $H^\alpha$ if and only if columns in $S$ cover all rows in $M(J_0^\alpha)$, and $\alpha(S) = \sum_{j \in S} \alpha_j = k$. For the special case when $k = 3$, we define a 2-cover graph, $G_2^\alpha$ of $\alpha x \geq 3$ as the graph on vertex set $J_2$, such that two columns in $J_2^\alpha$ are adjacent if and only if they cover all rows in $M(J_0^\alpha)$.

Let $\alpha$ be an $n$-dimensional vector with non-negative integral components. Let $B$ be a submatrix of $A$ with row index set $R$ and column index set $S$. The covering number of $B$ w.r.t $\alpha$, denoted by $\beta(B, \alpha)$, is defined as

$$min \left\{ \sum_{j \in S} \alpha_j x_j | Bx \geq \mathbf{1}, x \in \{0,1\}^S \right\}$$

$B$ is called a $k$-critical matrix w.r.t $\alpha$ if $\beta(B, \alpha) = k$ and $\beta(B_{R-i}, \alpha) < k$ for any $i \in R$. For the special case when $\alpha$ is the characteristic vector of $S$ i.e $\alpha_j = 1$ iff $j \in S$, a $k$-critical matrix w.r.t $\alpha$, would be referred to as, simply a $k$-critical matrix. A $k$-critical matrix $B = A_R^S$ is said to be column minimal if $B^{S-j}$ is not $k$-critical for any $j \in S$. Column minimal 3-critical matrices play an important role in the subsequent development of theory of facets of the set-covering polytope with coefficients in $\{0, 1, 2, 3\}$. Note that the only column minimal 2-critical submatrices of $A$ are the circulant submatrices with exactly one zero in every row and column. 2-critical matrices played a central role in the work of Balas and Ng[3, 4] related to theory of facets of the set-covering polytope with coefficients in $\{0, 1, 2\}$.

## 1.2 Main Results

We use the same notation as introduced in *Section 1.1*. Let $A$ be an $m \times n$ matrix as defined in *Section 1.1*.

In *Section 2* we derive necessary and sufficient condition for a valid inequality, $\alpha x \geq 3$ ($\alpha \in \mathcal{I}_3^A$) to be a minimal valid inequality and to be facet-defining. We introduce a concept of *kernel hypergraphs* which generalize to hypergraphs the concept of odd cycles, defined for graphs. Kernel hypergraphs are used to formulate our main result (*Theorem 2.6*) in *Section 2*. We compare our results and their consequences with earlier work of Balas and Ng [3] and Sànchez et al [12]. Two methods for constructing large kernel hypergraphs are presented. The first method constructs kernel hypergraphs from neighbourhood matrices of cycles of length $k$, where $k$ is not a multiple of 3. The second construction is based on some properties of cubic det-extremal bipartite graphs [7, 10].

Given $\alpha \in \mathcal{I}_3^A$, let $\mathcal{F}^\alpha = \{\beta \in \mathcal{M}_3^A | \beta \leq \alpha, J_0^\beta = J_0^\alpha\}$. There is a one-one map from $\mathcal{F}^\alpha$ to minimal valid inequalities dominating $\alpha x \geq 3$ and having the same support as that of $\alpha x \geq 3$. *Section 3* discusses a systematic method to derive all inequalities in $\mathcal{F}^\alpha$. The central result of that section establishes a precise one-one correspondence between independent dominating sets of a certain hypergraph, called the *generator hypergraph* and inequalities in $\mathcal{F}^\alpha$. This precise correspondence, is later used in *Section 4* to devise a procedure to derive all inequalities in $\mathcal{M}_3^A$. We show that generator hypergraph associated with $\alpha x \geq 3$ can be constructed in $O(mn^3)$ time and discuss methods which can generate inequalities in $\mathcal{I}_3^A$ by constructing only a part of the generator hypergraph. This technique of embedding a family of inequalities as special structures (independent dominating sets in our case) in graphs/hypergraphs has surfaced in some recent researches. For instance, Easton, Hooker and Lee [1] establish relationship between some classes of facet-defining inequalities of the independent set polytope and hypercliques of a certain *conflict hypergraph*. An effective use of this technique in deriving a procedure to generate all the sequential and simultaneously lifted facets from certain 3-critical submatrices can be found in our companion paper [11].

*Section 4* uses the framework developed in *Section 3* to give a procedure, which can derive all the minimal valid inequalities of $P_I(A)$ with coefficients in $\{0, 1, 2, 3\}$. We discuss a manifestation of the general procedure, which can be carried out more efficiently at the cost of producing valid inequalities which are not necessarily minimal. We call these inequalities as *residual* minimal valid inequalities and discuss some of their properties. In particular, it is shown that there is a one-one correspondence between residual minimal valid inequalities in $\mathcal{F}^\alpha$ and independent dominating sets of a certain graph, called the *residual graph.* Furthermore, we show that the inequalities generated by the procedure of Sànchez et al [12] are precisely those residual minimal valid inequalities which can be derived from maximum cardinality independent dominating sets of the residual graph. This observation is further exploited to suggest more tractable generalizations of the procedure devised by Sànchez et al [12].

*Section 5* discusses application of the theoretical framework developed in *Section 3* in cutting off fractional solutions to the linear programming relaxations of the set-covering polytope. In particular, it is shown that problem of finding the most violated inequality in $\mathcal{F}^\alpha$ can be formulated as a maximum weight (constrained) independent dominating set problem on the generator hypergraph. Given a connected graph $G = (V, E)$ we define an associated matrix $A^G$, called the *residual matrix* of the graph, which has the same number of columns as the number of nodes in $G$ and has the property that two columns form a cover of the rows of $A^G$ if and only if the corresponding vertices of the graph are adjacent. We use the device of residual matrices to prove $NP$-Completeness of the separation problem for inequalities in $\mathcal{F}^\alpha$. We discuss conditions for an inequality to cut off a fractional solution to the linear programming relaxations of the set-covering polytope.

# 2 Facets and Minimal Valid Inequalities with coefficients in $\{0, 1, 2, 3\}$

Let $A$ be a $0 \setminus 1$ matrix as described in *Section 1.1* and let $P$ be the associated set-covering polytope. Without loss of generality we assume that every row of $A$ has at least two non-zero entries and hence $P$ is full-dimensional. We next give a necessary and sufficient condition for $\alpha x \geq 3$ to be a minimal valid inequality.

**Theorem 2.1** *Let $\alpha x \geq 3$ be a valid inequality for the set-covering polytope, $P_I(A)$ such the components of $\alpha$ are non-negative integers. $\alpha x \geq 3$ is a minimal valid inequality if and only if the coefficients of $\alpha$ are contained in $\{0, 1, 2, 3\}$ and the following conditions are satisfied:*

1. $\forall j \in J_3^\alpha \ \forall i \in M(J_0^\alpha) \ A_{ij} = 1$
2. *The cover hypergraph of $\alpha x \geq 3$ has no isolated vertex.*

**Proof:** Follows directly from the fact that a valid inequality $\pi x \geq \pi_0$ for a $0 \setminus 1$ polytope $T$ is minimal if and only if for every $j \in N$, there exists $x \in T$, such that $x_j = 1$ and $\pi x = \pi_0$. ■

Consider the matrix shown below.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

It is easy to verify that the following inequality is a valid inequality for the set-covering polytope associated with the above matrix (infact it defines a facet of the same).

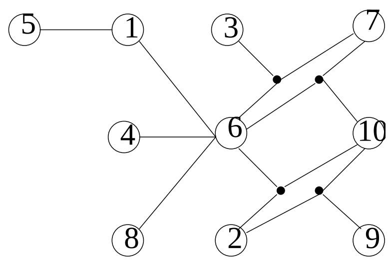



Figure 1: Cover Sub-hypergraph

$$2x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7 + 2x_8 + x_9 + x_{10} \geq 3 \tag{2.3}$$

Let $\alpha x \geq 3$ be the *inequality (2.3)*. Note that $J_3^\alpha = \phi$. A spanning sub-hypergraph of the cover-hypergraph of $\alpha x \geq 3$ is shown in *Figure 1*. Note that the spanning sub-hypergraph has no isolated vertex, which implies the cover hypergraph of $\alpha x \geq 3$ has no isolated vertex and hence *inequality (2.3)* is a minimal valid inequality.

**Remark 2.2** Balas and Ng [3] proved that if the support of the inequality is fixed, then there is a unique minimal valid inequality with integral coefficients and right hand side of 2, on the given support (Theorem 2.1 on page 60 of [3]). However this property *does not* hold true for inequalities with right hand side of 3. For example, following inequalities are facet-defining (and hence minimal valid inequalities) of the set-covering polytope associated with matrix (2.2) and have the same support as that of the facet given by *inequality (2.3)*.

$$2x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 + x_7 + 2x_8 + x_9 + x_{10} \geq 3 \tag{2.4}$$

$$x_1 + x_2 + 2x_3 + x_4 + 2x_5 + 2x_6 + 2x_7 + x_8 + x_9 + 2x_{10} \geq 3 \tag{2.5}$$

This distinction between inequalities with right hand side of 2 and 3, puts forth an interesting question. Given a fractional solution $\hat{x}$, to the set-covering problem and a valid inequality $\alpha x \geq k$, is it possible to efficiently determine the most violated inequality (if any) from the family of minimal valid inequalities $\beta x \geq k$ such that $J_0^\beta = J_0^\alpha$ (i.e $\beta x \geq k$ has the same support as $\alpha x \geq k$) and $\beta \leq \alpha$ ?. Note that by result of Balas and Ng [3], above problem can be solved in linear time for $k = 2$. We prove in *Section 5*, that the above problem is, however, $NP-Complete$ for $k = 3$.

**Remark 2.3** Minimal valid inequalities with coefficients in $\{0, 1, 2, 3\}$ were also studied by Sànchez et al[12]. However there is a fundamental difference between our approach and that of theirs, in the way *dominated inequalities* are defined. According to their definition, (definition 2.1 on page 346 of [12]), a valid inequality $\beta x \geq 3$ for $P_I(A)$, is *dominated* by $\gamma x \geq 3$ if and only if

$$J_3^\gamma \subseteq J_3^\beta,\ J_0^\beta \subseteq J_0^\gamma,\ |J_0^\gamma \cup J_1^\gamma| \geq |J_0^\beta \cup J_1^\beta|$$

We say that an inequality with right hand side of 3 is $V$-dominated by another inequality, if it is dominated in the above sense. Similarly we call a valid inequality $V$-minimal, if it is not $V$-dominated by any other valid inequality. Note that if the above definition is used, then it is possible to have facets of the set-covering polytope with coefficients in $\{0, 1, 2, 3\}$, which are $V$-dominated by other valid inequalities. For example, *inequality (2.5)* defines a facet of the set-covering polytope associated with matrix (2.2), even though it is $V$-dominated by the valid *inequality (2.4)*. This however is not case if our definition of *dominated* inequality, as given in *Section 1.1*, is used. A procedure to generate valid inequalities with coefficients in $\{0, 1, 2, 3\}$, which are not $V$-dominated by other inequalities is given in [12]. Note that their procedure will never generate the facet-defining *inequality (2.5)*, of the set-covering polytope associated with matrix (2.2), since it is $V$-dominated by the *inequality (2.4)*. In *Section 4* we define a notion of *residual minimal* inequalities, which are not necessarily minimal valid inequalities, and exhibit a one-one correspondence between residual minimal valid inequalities and independent dominating sets of a certain graph, called the *residual graph*. Therein we show that the procedure of Sànchez et al [12] generates precisely those residual minimal valid inequalities, which correspond to the maximum cardinality independent dominating sets of the residual graph.

We next derive necessary and sufficient conditions for a valid inequality $\alpha x \geq 3$ ($\alpha \in \mathcal{I}_3^A$) to define a facet of $P_I(A)$. We would need some preparatory results to arrive at the main result.

**Definition 2.4** A hypergraph $H = (V, E)$ is called a *kernel hypergraph* if the vertex-edge incidence matrix of $H$ is a square non-singular matrix with at least two ones in every row.

Note that the only connected 2-uniform kernel hypergraphs are the odd cycles. Thus kernel hypergraphs generalize to hypergraphs the concept of odd cycles, defined for graphs.

**Theorem 2.5** *Let $H = (V, E)$ be a hypergraph on $n = |V|$ nodes. Then $H$ has $n$ edges with linearly independent incident vectors if and only if there exists a kernel subhypergraph of $H$, say $\hat{H}$ such that $H$ is strongly connected w.r.t $V(\hat{H})$.*

**Proof:** *Sufficiency*: Suppose $H$ contains a kernel subhypergraph, say $\hat{H}$ such that $H$ is strongly connected w.r.t $V(\hat{H})$. Let $k = |V(\hat{H})|$, $V(\hat{H}) = \{v_1, \dots v_k\}$ and let $e^1, e^2 \dots e^k$ be the $k$ edges of $\hat{H}$. By definition of strong connectivity, there exists an ordering of $V \setminus V(\hat{H})$, say $(v_{k+1}, \dots v_n)$, such that for every $i = k+1, \dots n$, there exists an edge, say $e^i$ such that $v_i \in e^i$ and $e^i \subseteq \{v_1 \dots v_i\}$. It is easy to verify that the incident vectors of $n$ edges $\{e^i | i = 1 \dots n\}$ are linearly independent.

*Necessity:* We need to prove that if $H = (V, E)$ is a hypergraph on $n = |V|$ nodes and has $n$ edges with linearly independent incident vectors then there exists a kernel subhypergraph of $H$, say $\hat{H}$ such that $H$ is strongly connected w.r.t $V(\hat{H})$. We prove the same by induction on the number of nodes, $n$. The result is easily seen to be true for $n = 4, 5$. Suppose, the result holds true for every hypergraph with less than $n$ nodes. Let $H = (V, E)$ be a hypergraph on $n$ nodes and let $H$ have has $n$ edges, say $e^1, \dots e^n$, with linearly independent incident vectors. Let $\tilde{H}$ be the subhypergraph of $H$ defined by the edges $e^1, \dots e^n$. Note that in order to prove that there exists a kernel subhypergraph of $H$, say $\hat{H}$ such that $H$ is strongly connected w.r.t $V(\hat{H})$ it suffices to prove the same for $\tilde{H}$. If $\tilde{H}$ is a kernel hypergraph, then the result follow trivially. Otherwise, there exists a vertex of $\tilde{H}$, say $v_n$, which is incident to exactly one edge, say $e^n$. Consider the hypergraph, say $H^\star$ obtained from $\tilde{H}$ by removing vertex $v_n$ and the incident edge $e^n$. It is easy to see that $H^\star$ has $(n-1)$ nodes and $(n-1)$ edges with linearly independent incident vectors. By induction hypothesis $H^\star$ contains a kernel subhypergraph, say $\hat{H}^\star$, such that $H^\star$ is strongly connected w.r.t $V(\hat{H}^\star)$, which immediately implies that $\tilde{H}$ is strongly connected w.r.t $V(\hat{H}^\star)$, as expected. ■

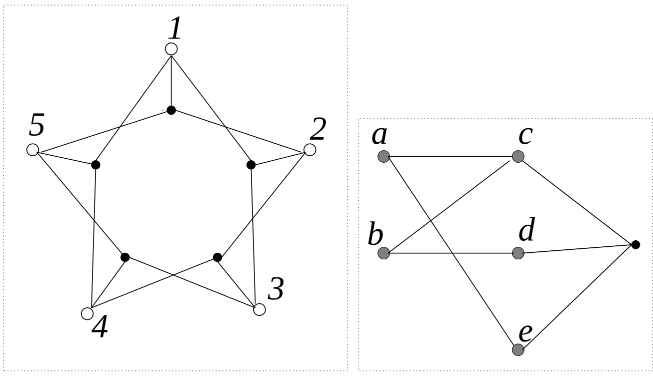
*Figure 2(a)* and *Figure 2(b)* give examples of kernel hypergraphs.

Next theorem derives necessary and sufficient condition for $\alpha x \geq 3$ to define a facet of the set-covering polytope.

**Theorem 2.6** *Let $P_I(A)$ be full dimensional and let $\alpha \in \mathcal{I}_3^A$, with $S = M(J_0)$. Let $k \in J_0$, we define $T(k)$ as the set of rows such that $k$ is the only column in $J_0$ to cover $T(K)$ i.e $T(k) = \{i \in M | A_{ik} = 1, A_{ij} = 0, \forall j \in J_0 \setminus \{k\}\}$. Then $\alpha x \geq 3$ defines a facet of $P_I(A)$ if and only it is a minimal valid inequality and satisfies the following:*

1. *Every component (say $H_i$) of the* cover *hypergraph of $\alpha x \geq 3$, has a kernel subhypergraph, say $\hat{H}_i$, such that $H_i$ is strongly connected w.r.t $V(\hat{H}_i)$.*
2. *For every $k \in J_0$, such that $T(k) \neq \phi$, there exists either*
   (a) *some $j(k) \in J_3$ such that $A_{ij(k)} = 1$ for all $i \in T(K)$; or*
   (b) *some edge of the cover hypergraph of $\alpha x \geq 3$, such that the columns incident to the corresponding edge, cover all the rows in $T(k)$.*

**Proof:** *Necessity:* Suppose $\alpha x \geq 3$ defines a facet of the set-covering polytope, then there exists $n$ affinely independent points $x^i \in P_I(A)$, such that $\alpha x^i = 3$ for $i \in \{1, \dots n\}$. Let $X$ be the $n \times n$ matrix whose rows are the vectors $x^i$; then $X$ is of the form (modulo row and column permutation)




(a) $H_1$ (b) $H_2$

Figure 2: Examples of Kernel Hypergraphs

$$X = \begin{pmatrix} X_1 & \mathbf{0} & \mathbf{0} & X_2 \\ \mathbf{0} & X_3 & X_4 & X_5 \\ \mathbf{0} & \mathbf{0} & X_6 & X_7 \end{pmatrix}$$

where the columns of $X_1, X_2, X_3, X_4, X_5, X_6$ and $X_7$ are indexed by $J_3, J_0, J_2, J_1, J_0, J_1$ and $J_0$ respectively, $\mathbf{0}$ represent *zero* matrices of conformable dimensions, every row of $X_1$ and $X_3$ has exactly one 1 and every row of $X_6$ has exactly three ones. Since $X$ is of full rank, the following matrix is of full column rank.

$$X_{12} = \begin{pmatrix} X_3 & X_4 \\ \mathbf{0} & X_6 \end{pmatrix}$$

This implies that $X_{12}$ has $|J_2|+|J_1|$ linear independent rows. It is easy to see that this implies that every component (say $H_i$) of the *cover* hypergraph of $\alpha x \geq 3$ has $n_i$ edges with linearly independent incident vectors, where $n_i$ is the number of nodes in $H_i$. By *Theorem 2.5*, this implies that every component (say $H_i$) of the *cover* hypergraph of $\alpha x \geq 3$, has a kernel subhypergraph, say $\hat{H}_i$, such that $H_i$ is strongly connected w.r.t $V(\hat{H}_i)$.

To show that condition 2 also holds, suppose there exists $k \in J_0$ and $T(k)$ for which neither 2a nor 2b is satisfied . Then $x_k = 1$ for every $x \in P_I(A) \cap \mathbb{B}^n$ such that $\alpha x = 3$, which contradicts the fact that $\alpha x \geq 3$ is facet-defining.

*Sufficiency:* By *Theorem 2.5*, condition (1) of the theorem implies that every component (say $H_i$) of the *cover* hypergraph of $\alpha x \geq 3$ has $n_i$ edges with linearly independent incident vectors, where $n_i$ is the number of nodes in $H_i$. Suppose the conditions of the above theorem hold true, then we exhibit $n$ linear independent $0 \setminus 1$ vectors in $P_I(A)$ which satisfy $\alpha x \geq 3$ at equality. Consider the following set of points:

1. By condition 2, for every $k \in J_0$ there exists a $0 \setminus 1$ point $x^k \in P_I(A)$ such that $\alpha x^k = 3$, $x^k_j = 1$ for all $j \in J_0 \setminus \{k\}$ and $x^k_k = 0$.
2. For all $k \in J_3$, let $x^k \in P_I(A)$ be defined as $x^k_k = 1$, $x^k_j = 1$ for all $j \in J_0$ and $x^k_j = 0$ for $j \in J_1 \cup J_2$.
3. Let $\{e_k | k \in J_1 \cup J_2\}$, be $|J_1| + |J_2|$ edges of the cover hypergraph with linearly independent incident vectors, where the edges have been arbitrarily mapped to vertices in $J_1 \cup J_2$ to create a one-one mapping. By condition 1, there exists a set of $|J_1| + |J_2|$ edges satisfying these conditions. For $k \in J_1 \cup J_2$, let $x^k$ be defined as,

$$x^k_j = \left\{ \begin{array}{ll} 1 & j \in e_k \\ 1 & j \in J_0 \\ 0 & otherwise \end{array} \right\}$$

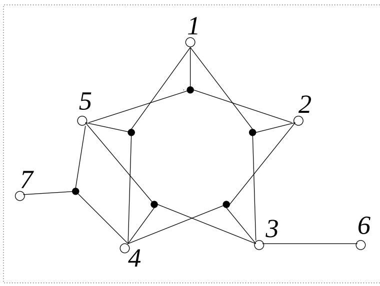



Figure 3: Cover Hypergraph

It is easy to verify that the above vectors are affinely independent, are contained in $P_I(A)$ and satisfy $\alpha x \geq 3$ at equality. ■

**Remark 2.7** The above theorem can be considered as a generalization of Theorem 2.6 on Page 63 of [3], which shows that an inequality with right hand side of 2 defines a facet if and only if every component of the 2-cover graph (as defined in [3]) has an odd cycle and a condition very similar to condition 2 given in the above theorem holds. Note that odd cycles are the only connected 2-uniform kernel hypergraphs. *Theorem 2.6* can be easily generalized to general valid inequalities which have integer coefficients, not necessarily restricted to $\{0, 1, 2, 3\}$. A different version of the above theorem was independently obtained by Sànchez in [12] (Theorem 3.3 on Page 353 of [12]).

The next corollary follows immediately from condition 1 in the above theorem.

**Corollary 2.8** *Let $\alpha x \geq 3$ be a facet-defining inequality of $P_I(A)$, then $|J_1| \geq 3$ and the cover hypergraph has at least one 3-edge.*

**Proof:** By *Theorem 2.5*, condition (1) of the theorem implies that the cover hypergraph of $\alpha x \geq 3$ has $|J_1| + |J_2|$ edges with linearly independent incident vectors. If $|J_1| \leq 2$, then the cover hypergraph is a 2-uniform hypergraph. Furthermore, all edges have one end-point in $J_2$ and other end-point in $J_1$, which implies that the cover hypergraph is infact a bipartite graph. Hence the cover hypergraph does not contain $|J_1| + |J_2|$ edges with linearly independent incident vectors. Similarly, if the cover hypergraph has no 3-edge then it cannot contain $|J_1| + |J_2|$ edges with linearly independent incident vectors. ■

Next we illustrate *Theorem 2.6* on a small example. Consider the matrix (2.6), say $A$.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \tag{2.6}$$

Consider the following inequality, say $\alpha x \geq 3$, obtained by adding the first 5 rows of $Ax \geq \mathbf{1}$, dividing by 2.4 and performing integer rounding: $x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 + x_7 \geq 3$. Clearly $\alpha \in \mathcal{I}_3^A$. The cover hypergraph, say $H$, of $\alpha x \geq 3$ is shown in *Figure 3*, which is easily seen to be $U$-connected for $U = \{1, 2, 3, 4, 5\}$. Furthermore, the vertex-incidence matrix of the subhypergraph of $H$, say $\hat{H}$, induced by vertices of $U$ is easily seen to be the neighbourhood matrix of a cycle on 5 nodes. Hence by *Prop 2.9* (proved in *Section 2.1*) $\hat{H}$ is a kernel hypergraph. Hence condition (1) of *Theorem 2.6* is satisfied. $J_0^\alpha = \{8, 9\}$, $T(8) = \phi$, $T(9) = \{7\}$, and columns of the 3-edge $\{1, 2, 3\}$ cover all rows in $T(9)$. Hence condition (2) of *Theorem 2.6* is also satisfied and $\alpha x \geq 3$ defines a facet of $P_I(A)$.

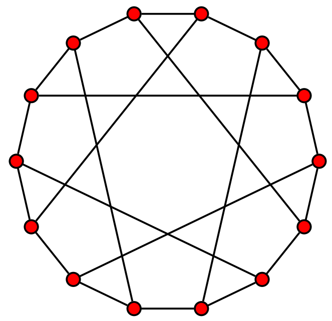


Figure 4: *Heawood* graph

## 2.1 Kernel Hypergraphs

In this section, we briefly discuss some methods to construct large kernel hypergraphs. The emphasis in this section, however, would be to show that unlike the case of 2-uniform kernel hypergraphs (which are disjoint unions of odd cycles), kernel hypergraphs which contain $k$-edges ($k \geq 3$) have a more complex and nontrivial structure. Detailed investigation of kernel hypergraphs is beyond the scope of this paper and would be disussed in a separate paper.

Next proposition characterizes a large family of kernel hypergraphs.

**Proposition 2.9** *Let $k$ be a positive integer, which is not an exact multiple of 3 and let $A^k$ be the neighbourhood matrix of a cycle of length $k$. If $A^k$ is the vertex-edge incidence matrix of a hypergraph $H$, then $H$ is a kernel hypergraph.*

**Proof:** Follows directly from the fact that $A^k$ is a square non-singular matrix (since $k$ is not a multiple of 3) with exactly three ones in every row and every column. ■

We would need some notions to derive the next construction. Let $G$ be a bipartite graph with partite sets $X$ and $Y$ of equal cardinality (say $n$). Let $A$ be a $\{0, 1\}$ $n \times n$ square matrix, such that $A_{ij} = 1$ if and only if the $i^{th}$ vertex of $X$ is connected to the $j^{th}$ vertex of $Y$. $A$ is called the reduced adjacency matrix of $G$. $G$ is said to be *det-extremal* if $per(A) = |det(A)|$ where $per(A)$ is the permanent of matrix $A$ and $det(A)$ is the determinant of the matrix $A$. Next proposition, gives a method of constructing kernel hypergraphs from det-extremal bipartite graphs.

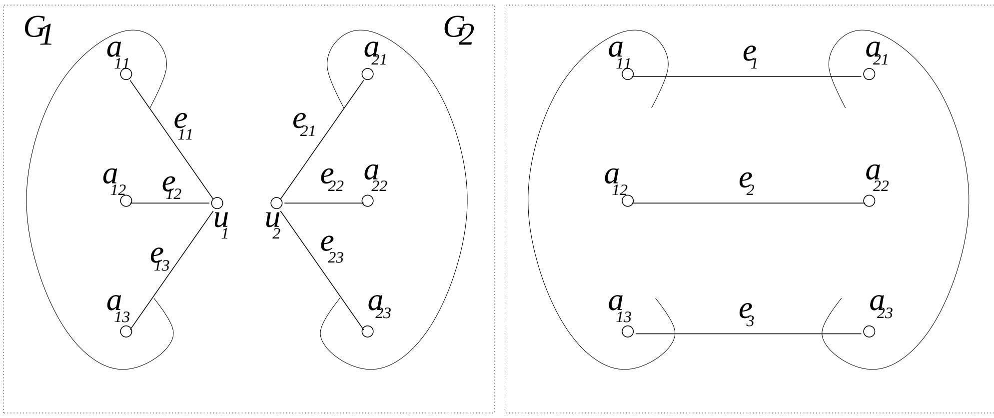
**Theorem 2.10** *Let $G$ be a connected det-extremal bipartite graph, such that $\delta_G \geq 2$. Let $H$ be a hypergraph whose vertex-edge incidence matrix is isomorphic to the reduced adjacency matrix of $G$. If $G$ has a perfect matching, then $H$ is a kernel hypergraph.*

**Proof:** Let $A$ be the reduced adjacency matrix of $G$. Since $per(A)$ is equal to the number of perfect matchings of $G$ and $G$ has a perfect matching, $per(A) \geq 1$. Furthermore since $G$ is det-extremal we have that $|det(A)| \geq 1$, which implies that $A$ is a non-singular square matrix, such that every row has at least two non-zero entries ($\delta_G \geq 2$). Hence by definition $H$, as defined in the statement of the theorem, is a kernel hypergraph. ■

Since every regular bipartite graph has a perfect matching [13], we have the following corollary.

**Corollary 2.11** *If $G$ is a cubic det-extremal bipartite graph and $H$ is a hypergraph whose vertex-edge incidence matrix is isomorphic to the reduced adjacency matrix of $G$, then $H$ is a kernel hypergraph.* ■

Cubic det-extremal bipartite graphs have been studied in literature [10, 7] and various sub-famlies have been characterized. *Heawood graph* shown in *Figure 4* is a known to be a cubic det-extremal bipartite graph. Next we give a method to produce larger cubic det-extremal bipartite graphs from the heawood graph by the operation of *star-product*. *Star-product* is a method of constructing a cubic bipartite graph from two cubic bipartite graphs. Let $G_i$




(a) $G_1$ and $G_2$

(b) $G_1 \oplus G_2$

Figure 5: Star Product of Cubic Bipartite Graphs

$(i = 1, 2)$ be two cubic bipartite graphs and let $u_1$ and $u_2$ be vertices of $G_1$ and $G_2$ respectively. Let $\{a_{i1}, a_{i2}, a_{i3}\}$ be degree 2 vertices in $G_i - u_i$ for $i = 1, 2$. Let $e_j = (a_{1j}, a_{2j})$ for $j \in \{1, 2, 3\}$, $E_{cut} = \{e_1, e_2, e_3\}$ and $E_i = E(G_i - u_i)$ for $i = 1, 2$. The graph $G = (V(G_1 - u_1) \cup V(G_2 - u_2), E_1 \cup E_2 \cup E_{cut})$ is said to be obtained by *star-product* of $G_1$ and $G_2$ by eliminating vertices $u_1$ and $u_2$. This construction is illustrated is *Figure 5.* Next theorem shows a way to generate larger cubic det-extremal graphs from Heawood graph.

**Theorem 2.12** *[10] Let $\mathcal{F}$ be a family of graphs defined as follows:*

1. *$\mathcal{F}$ contains the Heawood graph.*
2. *If $G_1, G_2 \in F$, then the star-product of $G_1$ and $G_2$ is also contained in $\mathcal{F}$*

*Then every graph in $\mathcal{F}$ is a cubic det-extremal bipartite graph.* ■

Thus we can construct arbitrarily large kernel hypergraphs by applying *Corollary 2.11* to graphs in family $\mathcal{F}$, constructed in *Theorem 2.12.*

# 3 Support Equivalent Minimal Valid Inequalities

Let $A$ be a $0 \setminus 1$ matrix, as described in *Section 1.1.* Let $\alpha x \geq 3$ be a valid inequality of $P_I(A)$, such that $\alpha \in \mathcal{I}_3^A$. We define the following set:

$$\mathcal{F}^\alpha = \{\beta \in \mathcal{M}_3^A | \beta \leq \alpha, J_0^\beta = J_0^\alpha\}$$

There is a one-one map from $\mathcal{F}^\alpha$ to minimal valid inequalities dominating $\alpha x \geq 3$ and having the same support as that of $\alpha x \geq 3$. Let $I$ be the set of isolated vertices of the cover hypergraph of $\alpha x \geq 3$.

**Definition 3.1** The subgraph of the 2-cover graph of $\alpha x \geq 3$, induced by vertices in $I \cap J_2^\alpha$ is called the *residual graph* of $\alpha x \geq 3$.

**Lemma 3.2** *Let $\alpha \in \mathcal{I}_3^A$ and let $\tilde{J} = \{j \in J_3^\alpha | \exists i \in M(J_0^\alpha), A_{ij} = 0\}$. Let $\tilde{\alpha} x \geq 3$ represent the following inequality: $\sum_{j \in N \setminus \tilde{J}} \alpha_j x_j + \sum_{j \in \tilde{J}} 2x_j \geq 3$. Then $\tilde{\alpha} \in \mathcal{I}_3^A$ and $\mathcal{F}^{\tilde{\alpha}} = \mathcal{F}^\alpha$.*

**Proof:** Suppose $\tilde{\alpha} \notin \mathcal{I}_3^A$, then there exists a $0 \setminus 1$ point $\tilde{x} \in P_I(A)$, such that $\tilde{\alpha}\tilde{x} \le 2$. Since $\alpha \in \mathcal{I}_3^A$, this implies that there exists a $k \in \tilde{J}$ such that $x_k = 1$ and $x_j = 0$ for all $j \in J_3^{\tilde{\alpha}} \cup J_2^{\tilde{\alpha}} \cup J_1^{\tilde{\alpha}} \setminus \{k\}$. By definition of $\tilde{J}$, this implies that at least one row of $M(J_0^{\tilde{\alpha}})$ is not covered by $\tilde{x}$, contradicting $\tilde{x} \in P_I(A)$. Hence $\tilde{\alpha} \in \mathcal{I}_3^A$. It is easy to see that $\mathcal{F}^{\tilde{\alpha}} \subseteq \mathcal{F}^{\alpha}$. If $\mathcal{F}^{\alpha} = \phi$, then trivially $\mathcal{F}^{\tilde{\alpha}} = \mathcal{F}^{\alpha}$. Otherwise suppose $\beta \in \mathcal{F}^{\alpha} \setminus \mathcal{F}^{\tilde{\alpha}}$. Since $J_0^{\alpha} = J_0^{\tilde{\alpha}}$, this implies that $\beta \nleq \tilde{\alpha}$. By definition of $\tilde{\alpha}$, $\exists k \in \tilde{J}$ such that $\beta_k = 3$. Since $\beta \in \mathcal{M}_3^A$, by *Theorem 2.1*, $\forall i \in M(J_0^{\beta}) = M(J_0^{\alpha})$, $A_{ik} = 1$, contradicting $k \in \tilde{J}$. Hence $F^{\tilde{\alpha}} = \mathcal{F}^{\alpha}$. ■

If $\tilde{\alpha} \in \mathcal{I}_3^A$ such that $\forall j \in J_3^{\tilde{\alpha}}, \forall i \in M(J_0^{\tilde{\alpha}}), A_{ij} = 1$, then $\tilde{\alpha}x \ge 3$ is called a $J_3$-minimal inequality. In the following discussion we denote the $J_3$-minimal inequality obtained from $\alpha x \ge 3$ by an application of *Lemma 3.2*, by $\tilde{\alpha}x \ge 3$.

**Lemma 3.3** *Let $\alpha \in \mathcal{I}_3^A$ and let $\tilde{\alpha}x \ge 3$ be the $J_3$-minimal inequality associated with $\alpha x \ge 3$. Let $\tilde{I}$ be the set of isolated vertices of the residual graph of $\tilde{\alpha}x \ge 3$. Let $\hat{\alpha}x \ge 3$ denote the following inequality: $\sum_{j \in N \setminus \tilde{I}} \tilde{\alpha}_j x_j + \sum_{j \in \tilde{I}} x_j \ge 3$. Then $\hat{\alpha} \in \mathcal{I}_3^A$ and $\mathcal{F}^{\hat{\alpha}} = \mathcal{F}^{\alpha}$. Furthermore $\hat{\alpha}x \ge 3$ is a $J_3$-minimal inequality and the residual graph of $\hat{\alpha}x \ge 3$ has no isolated vertices.*

**Proof:** Suppose $\hat{\alpha} \notin \mathcal{I}_3^A$, which implies $\exists \hat{x} \in P_I(A) \cap \mathbb{B}^n$, such that $\hat{\alpha}\hat{x} \le 2$. Since $\tilde{\alpha} \in \mathcal{I}_3^A$ and $\tilde{\alpha}\hat{x} \ge 3$ there exists $k \in \tilde{I}$ such that $\hat{x}_k = 1$. Furthermore since $k \notin J_3^{\tilde{\alpha}}$, there exists $j \in J_1^{\hat{\alpha}}$ such that $\hat{x}_j = 1$ and $\hat{x}_{\bar{j}} = 0, \forall \bar{j} \in J_3^{\hat{\alpha}} \cup J_2^{\hat{\alpha}} \cup J_1^{\hat{\alpha}} \setminus \{j,k\}$, which implies that columns $A_j$ and $A_k$ cover all rows in $M(J_0^{\tilde{\alpha}})$. Hence $\{j,k\}$ is either an edge of the cover hypergraph of $\tilde{\alpha}x \ge 3$ (i.e $j \in J_1^{\tilde{\alpha}}$) or an edge of the 2-cover graph of $\tilde{\alpha}x \ge 3$ (i.e $j \in J_2^{\tilde{\alpha}}$). In either case, $k \notin \tilde{I}$, a contradiction. Hence $\hat{\alpha} \in \mathcal{I}_3^A$. Since $\mathcal{F}^{\tilde{\alpha}} = \mathcal{F}^{\alpha}$, it suffices to prove $\mathcal{F}^{\tilde{\alpha}} = \mathcal{F}^{\hat{\alpha}}$. It is easy to see that $\mathcal{F}^{\hat{\alpha}} \subseteq \mathcal{F}^{\tilde{\alpha}}$. If $\mathcal{F}^{\tilde{\alpha}} = \phi$, then trivially $\mathcal{F}^{\tilde{\alpha}} = \mathcal{F}^{\hat{\alpha}}$. Otherwise suppose there exists $\beta \in \mathcal{F}^{\tilde{\alpha}} \setminus \mathcal{F}^{\hat{\alpha}}$. Since $J_0^{\tilde{\alpha}} = J_0^{\hat{\alpha}}$, this implies that $\exists k \in \tilde{I}$ such that $\beta_k = 2$ and $\hat{\alpha}_k = 1$. Since $\beta \in \mathcal{M}_3^A$, by *Theorem 2.1*, $\exists \hat{x} \in P_I(A) \cap \mathbb{B}^n$ such that $\beta\hat{x} = 3$ and $\hat{x}_k = 1$, which implies that there exists $j \in J_1^{\beta}$ such that $\hat{x}_j = 1$ and $\hat{x}_{\bar{j}} = 0, \forall \bar{j} \in J_3^{\beta} \cup J_2^{\beta} \cup J_1^{\beta} \setminus \{j,k\}$, which implies that columns $A_j$ and $A_k$ cover all rows in $M(J_0^{\tilde{\alpha}})$. Hence $\{j,k\}$ is either an edge of the cover hypergraph of $\tilde{\alpha}x \ge 3$ (i.e $j \in J_1^{\tilde{\alpha}}$) or an edge of the 2-cover graph of $\tilde{\alpha}x \ge 3$ (i.e $j \in J_2^{\tilde{\alpha}}$). In either case, $k \notin \tilde{I}$, a contradiction. Hence $\mathcal{F}^{\hat{\alpha}} = \mathcal{F}^{\tilde{\alpha}}$.

Since $\hat{\alpha} \le \tilde{\alpha}$ and $\tilde{\alpha}x \ge 3$ is a $J_3$-minimal inequality, it easily follows that $\hat{\alpha}x \ge 3$ is also a $J_3$-minimal inequality. It is easy to see that the residual graph of $\hat{\alpha}x \ge 3$ is obtained by removing isolated vertices from the residual graph of $\tilde{\alpha}x \ge 3$. Hence the residual graph of $\hat{\alpha}x \ge 3$ has no isolated vertices. ■

An inequality $\hat{\alpha}x \ge 3$ is said to be *refined* if it is $J_3$-minimal and the residual graph of $\hat{\alpha}x \ge 3$ has no isolated vertices. We denote the *refined* inequality obtained from $\alpha x \ge 3$ ($\alpha \in \mathcal{I}_3^A$) by application of *Lemma 3.3* by $\hat{\alpha}x \ge 3$ and call it the *refinement* of the inequality $\alpha x \ge 3$. Let $I^{\hat{\alpha}}$ be the set of isolated vertices of the cover-hypergraph of $\hat{\alpha}x \ge 3$ and let $I_1^{\hat{\alpha}} = I^{\hat{\alpha}} \cap J_1^{\hat{\alpha}}$ and $I_2^{\hat{\alpha}} = I^{\hat{\alpha}} \cap J_2^{\hat{\alpha}}$. We define a *generator hypergraph* $\mathcal{G}^{\hat{\alpha}}$ on vertex set $I^{\hat{\alpha}}$ and edge-set $E(\mathcal{G}^{\hat{\alpha}})$ defined as follows:

1. $E(\mathcal{G}^{\hat{\alpha}})$ has only 2-edges and 3-edges.
2. Every edge of the residual graph of $\hat{\alpha}x \ge 3$ is a 2-edge of $\mathcal{G}^{\hat{\alpha}}$.
3. If $j \in I_1^{\hat{\alpha}}$ and $k \in I_2^{\hat{\alpha}}$, such that $\exists l \in J_1^{\hat{\alpha}}$, and columns $A_j, A_k$ and $A_l$ cover all rows in $M(J_0^{\hat{\alpha}})$, then $\{j,k\}$ is a 2-edge of the cover hypergraph.
4. If $j \in I_1^{\hat{\alpha}}$ and $k,l \in I_2^{\hat{\alpha}}$ are non-adjacent vertices of the residual graph of $\hat{\alpha}x \ge 3$, such that columns $A_j, A_k$ and $A_l$ cover all rows in $M(J_0^{\hat{\alpha}})$, then $\{j,k,l\}$ is a 3-edge of the cover hypergraph.
5. $\mathcal{G}^{\hat{\alpha}}$ has no other edges.

The *generator hypergraph* of an inequality $\alpha x \ge 3$ for $\alpha \in \mathcal{I}_3^A$ is defined to be the generator hypergraph associated with inequality obtained as the refinement of $\alpha x \ge 3$.

We illustrate the construction of generator hypergraph on a small example. Consider the matrix (2.2) (say $A$). Consider the following valid inequality, obtained by adding all rows of $Ax \ge 1$, dividing by $4.5 = \frac{10-1}{2}$ and performing integer rounding.

$$2x_1 + 2x_2 + 2x_3 + x_4 + 2x_5 + 2x_6 + 2x_7 + 2x_8 + x_9 + 2x_{10} \geq 3 \tag{3.7}$$

Let the above inequality be $\alpha x \geq 3$. Note that $J_0^\alpha = \phi$ and $M(J_0^\alpha) = M$, the row-index set of $A$. Since the above inequality has no coefficient with value 3, the above inequality is trivially $J_3$-minimal. It is easy to verify that the cover hypergraph of $\alpha x \geq 3$ has a single 2-edge joining $v_4$ and $v_6$ (where $v_j$ is the node of the hypergraph corresponding to column $j$). The residual graph of $\alpha x \geq 3$ is shown in *Figure 6(a)*. Since the residual graph has an isolated vertex, namely $v_2$, the inequality is not a refined valid inequality. The refinement of $\alpha x \geq 3$ yields the following inequality by an application of *Lemma 3.3*. Note that coefficient of $x_2$ has been reduced to 1.

$$2x_1 + x_2 + 2x_3 + x_4 + 2x_5 + 2x_6 + 2x_7 + 2x_8 + x_9 + 2x_{10} \geq 3 \tag{3.8}$$

The residual graph of the above inequality, as suggested by *Lemma 3.3* is obtained by removing isolated vertices from the residual graph of $\alpha x \geq 3$, shown in *Figure 6(a)*. We denote *inequality (3.8)* by $\hat{\alpha} x \geq 3$. The generator hypergraph of $\hat{\alpha} x \geq 3$ is shown in *Figure 6(b)*. The edges of $\mathcal{G}^\alpha$ are obtained as follows: ($I_1 = \{2, 9\}$ and $I_2 = \{1, 3, 5, 7, 8, 10\}$)

1. Since every edge of the residual graph of $\hat{\alpha} x \geq 3$ is an edge of generator hypergraph, $\{v_1, v_5\}$, $\{v_5, v_8\}$, $\{v_8, v_3\}$, $\{v_8, v_{10}\}$, $\{v_8, v_7\}$ are 2-edges of $\mathcal{G}^\alpha$.
2. (a) Columns $A_2, A_9$ and $A_1$ cover rows in $M(J_0^\alpha)$, hence vertices $v_2$ and $v_1$, $v_9$ and $v_1$ are connected in $\mathcal{G}^\alpha$.
   (b) Similarly, columns $A_2, A_9$ and $A_{10}$ cover rows in $M(J_0^\alpha)$, hence vertices $v_2$ and $v_{10}$, $v_9$ and $v_{10}$ are connected in $\mathcal{G}^\alpha$.
3. (a) $v_1$ and $v_8$ are non-adjacent vertices in the 2-cover graph of *inequality (3.8)* and columns $A_1, A_2$ and $A_8$ cover all rows in $M(J_0^\alpha)$, hence $\{v_1, v_2, v_8\}$ is an edge of $\mathcal{G}^\alpha$. Similarly $\{v_2, v_5, v_{10}\}$ is an edge of $\mathcal{G}^\alpha$.
   (b) Similarly, $\{v_1, v_3, v_9\}$, $\{v_1, v_7, v_9\}$, $\{v_1, v_9, v_{10}\}$, $\{v_7, v_9, v_{10}\}$ and $\{v_3, v_9, v_{10}\}$ are edges of the generator hypergraph.

The next theorem is the main result of this section.

**Theorem 3.4** *Let $\alpha \in \mathcal{I}_3^A$ and let $\hat{\alpha} x \geq 3$ be the refinement of $\alpha x \geq 3$. Let $\mathcal{G}^\alpha$ be the generator hypergraph of $\alpha x \geq 3$ (and $\hat{\alpha} x \geq 3$). Let $I_2^{\hat{\alpha}}$ (possibly empty) be the vertex set of the residual graph of $\hat{\alpha} x \geq 3$. Then, $\mathcal{F}^\alpha$ is non-empty if and only if $\mathcal{G}^\alpha$ has an independent dominating set contained in $I_2^{\hat{\alpha}}$. Furthermore, if $\mathcal{F}^\alpha \neq \phi$, then $\beta \in \mathcal{F}^\alpha$ if and only if there exists an independent dominating set $S^\beta$ of $\mathcal{G}^{\hat{\alpha}}$ such that $S^\beta = J_1^\beta \cap I_2^{\hat{\alpha}}$.*

**Proof:** If $\mathcal{F}^\alpha \neq \phi$ and $\beta \in \mathcal{F}^\alpha$, then it is easy to see that $J_3^\beta = J_3^{\hat{\alpha}}$, $J_0^\beta = J_0^{\hat{\alpha}}$, $J_1^{\hat{\alpha}} \subseteq J_1^\beta \subseteq J_1^{\hat{\alpha}} \cup I_2^{\hat{\alpha}}$, and $J_2^{\hat{\alpha}} \setminus I_2^{\hat{\alpha}} \subseteq J_2^\beta \subseteq J_2^{\hat{\alpha}}$. Hence if $I_2^{\hat{\alpha}} = \phi$, then $\beta = \hat{\alpha}$. Thus, if $I_2^{\hat{\alpha}} = \phi$ then $\mathcal{F}^\alpha \neq \phi$ if and only if $\hat{\alpha} x \geq 3$ is a minimal valid inequality, which is equivalent to $I_1^{\hat{\alpha}} = \phi$. Hence the above theorem trivially hold true if $I_2^{\hat{\alpha}} = \phi$. Next we assume that $I_2^{\hat{\alpha}} \neq \phi$.

**Claim 3.5** *Suppose $S \subseteq I_2^{\hat{\alpha}}$ is an independent dominating set of $\mathcal{G}^\alpha$. Consider the following inequality, say $\beta^S x \geq 3$: $\sum_{j \in N \setminus S} \hat{\alpha}_j x_j + \sum_{j \in S} x_j \geq 3$. Then $\beta^S \in \mathcal{F}^\alpha$.*

**Proof:** Let $\beta = \beta^S$. Since $\mathcal{F}^\alpha = \mathcal{F}^{\hat{\alpha}}$ by *Lemma 3.3*, it suffices to prove that $\beta \in \mathcal{F}^{\hat{\alpha}}$. It is easy to see that $\beta \leq \hat{\alpha}$ and $J_0^\beta = J_0^{\hat{\alpha}}$. Hence we just need to prove that $\beta \in \mathcal{M}_3^A$. Suppose $\beta \notin \mathcal{I}_3^A$. This implies that there exists a $0\backslash 1$ point $\hat{x} \in P_I(A)$ such that $\beta \hat{x} \leq 2$. As $\hat{\alpha} x \geq 3$ is a $J_3$-minimal valid inequality, $\{j \in N | A_{ij} = 1 \; \forall i \in M(J_0^{\hat{\alpha}})\} = J_3^{\hat{\alpha}} = J_3^\beta$, and it follows that $\beta \hat{x} = 2$. Since $\hat{\alpha} \in \mathcal{I}_3^A$ by *Lemma 3.3*, $\hat{\alpha} \hat{x} \geq 3$ and there exists $k \in S$ such that $\hat{x}_k = 1$. This implies that there exists $j \in J_1^\beta - \{k\}$ such that $\hat{x}_j = 1$ and $\hat{x}_{\bar{j}} = 0$ for all $\bar{j} \in J_3^\beta \cup J_2^\beta \cup J_1^\beta \setminus \{j, k\}$. Note that columns $A_j, A_k$ cover all rows in $M(J_0^{\hat{\alpha}})$. Since $k \in I_2^{\hat{\alpha}}$, this implies that $j \notin J_1^{\hat{\alpha}}$. To see this, if $j \in J_1^{\hat{\alpha}}$, then $\{j, k\}$ would be an edge of the cover hypergraph of $\hat{\alpha} x \geq 3$, contradicting the fact that $k$ is a vertex of the residual graph. Hence $j \in J_2^{\hat{\alpha}}$, which along with $\beta_j = 1$ implies that $j \in S$. But then columns $A_j$ and $A_k$ would be adjacent in the 2-cover graph (hence residual graph as $j, k \in I_2^{\hat{\alpha}}$) of $\hat{\alpha} x \geq 3$, which implies by definition of $\mathcal{G}^\alpha$ that $\{j, k\}$ is an edge of $\mathcal{G}^\alpha$, contradicting the fact that $S$ is an independent set of $\mathcal{G}^\alpha$. Hence $\beta \in \mathcal{I}_3^A$.

Next we prove that $\beta x \geq 3$ is a minimal inequality. By *Theorem 2.1*, $\beta x \geq 3$ is a minimal inequality if and only if $\beta x \geq 3$ is a $J_3$-minimal inequality and the cover hypergraph of $\beta x \geq 3$ has no isolated vertex. Since $\beta \leq \hat{\alpha}$ and $\hat{\alpha}$ is a $J_3$-minimal inequality by *Lemma 3.3*, it easily follows that $\beta x \geq 3$ is a $J_3$-minimal inequality. Hence we just need to prove that the cover hypergraph $H^\beta$, of $\beta x \geq 3$ has no isolated vertices. We prove the same by producing a subhypergraph of $H^\beta$ which has no isolated vertices. By definition of $\beta$ it follows immediately that isolated vertices of $H^\beta$ (if any) are contained in $I^\alpha$. To see this, note that every edge of the cover hypergraph of $\hat{\alpha}x \geq 3$ is also an edge of $H^\beta$, since $\beta_j = \hat{\alpha}_j, \forall j \in N \setminus S$. Suppose $H^\beta$ has an isolated vertex $j \in I^{\hat{\alpha}}$. There are three possible cases:

1. $j \in S$: Note that the residual graph of $\hat{\alpha}x \geq 3$ has no isolated vertices by *Lemma 3.3*, hence $j$ has at least one neighbour, say $k$, in the residual graph of $\hat{\alpha}x \geq 3$, which implies that $\{j,k\}$ is an edge of $\mathcal{G}^\alpha$, $k \notin S$ (since $S$ is an independent set of $\mathcal{G}^\alpha$) and columns $A_j, A_k$ cover all rows in $M(J_0^{\hat{\alpha}})$. Hence the $0 \setminus 1$ incident vector of $\{j,k\} \cup J_0^{\hat{\alpha}}$ is present in $P_I(A) \cap \{x | \beta x = 3\}$, which implies that $\{j,k\}$ is also an edge of $H^\beta$ contradicting the fact that $j$ is isolated in $H^\beta$.

2. $j \in I_2^{\hat{\alpha}} \setminus S$: Since $S$ is a dominating set of $\mathcal{G}^\alpha$, there exists $k \in S$, such that $\{j,k\}$ is an edge of $\mathcal{G}^\alpha$, which implies that columns $A_j, A_k$ cover all rows in $M(J_0^{\hat{\alpha}})$. Hence the $0 \setminus 1$ incident vector of $\{j,k\} \cup J_0^{\hat{\alpha}}$ is present in $P_I(A) \cap \{x | \beta x = 3\}$, which implies that $\{j,k\}$ is also an edge of $H^\beta$ contradicting the fact that $j$ is isolated in $H^\beta$.

3. $j \in I_1^{\hat{\alpha}}$: Since $S$ is a dominating set of $\mathcal{G}^\alpha$, by definition of $E(\mathcal{G}^\alpha)$, there exists an edge, say $e$, incident to $j$ such that $e \subseteq S \cup \{j\}$. Suppose $e = \{j,k\}$ where $k \in S$ and there exists $l \in J_1^{\hat{\alpha}}$ such that columns $A_j, A_k$ and $A_l$ cover all rows in $M(J_0^{\hat{\alpha}})$. Then the $0 \setminus 1$ incident vector of $\{j,k,l\} \cup J_0^{\hat{\alpha}}$ is present in $P_I(A) \cap \{x | \beta x = 3\}$, which implies that $\{j,k,l\}$ is an edge of $H^\beta$, contradicting the fact that $j$ is an isolated vertex of $H^\beta$. Otherwise $e = \{j,k,l\}$, such that $k,l \in S$, which implies that columns $A_j, A_k$ and $A_l$ cover all rows in $M(J_0^{\hat{\alpha}})$, the $0 \setminus 1$ incident vector of $\{j,k,l\} \cup J_0^{\hat{\alpha}}$ is present in $P_I(A) \cap \{x | \beta x = 3\}$ and $\{j,k,l\}$ is an edge of $H^\beta$, contradicting the fact that $j$ is an isolated vertex of $H^\beta$.

■

By *Claim 3.5* if $\mathcal{G}^\alpha$ has an independent dominating set $S \subseteq I_2^{\hat{\alpha}}$, then $\mathcal{F}^\alpha \neq \phi$ and $\beta^S$, as constructed in *Claim 3.5*, is contained in $\mathcal{F}^\alpha$. Also, note that $S = J_1^{\beta^S} \cap I_2^{\hat{\alpha}}$.

**Claim 3.6** *Suppose $\beta \in \mathcal{F}^\alpha$ and $S^\beta = J_1^\beta \cap I_2^{\hat{\alpha}}$. Then $S^\beta$ is an independent dominating set of $\mathcal{G}^\alpha$.*

**Proof:** Clearly $S^\beta \subseteq I_2^{\hat{\alpha}}$. Suppose $S^\beta$ is not an independent set of $\mathcal{G}^\alpha$, then exists an edge $e$ of $\mathcal{G}^\alpha$ such that $e \subseteq S^\beta$. Since $S^\beta \subseteq I_2^{\hat{\alpha}}$ and $\mathcal{G}^\alpha[I_2^{\hat{\alpha}}]$ has only 2-edges, this implies that there exist $j,k \in S^\beta$ such that columns $A_j, A_k$ cover all rows in $M(J_0^{\hat{\alpha}})$. Hence the $0 \setminus 1$ incident vector $\hat{x}$, of the set $\{j,k\} \cup J_0^{\hat{\alpha}}$ is in $P_I(A)$. But $\beta\hat{x} = 2 < 3$, which implies that $\beta \notin \mathcal{I}_3^A$, a contradiction. Hence $S^\beta$ is an independent set of $\mathcal{G}^\alpha$. Next suppose, $S^\beta$ is not a dominating set of $\mathcal{G}^\alpha$. This implies that there exists a vertex $k \in I^{\hat{\alpha}}$, such that $k$ is not dominated by $S^\beta$. Since $\beta \in \mathcal{M}_3^A$, there exists a $0 \setminus 1$ vector $\hat{x} \in P_I(A) \cap \{x | \beta x = 3\}$, such that $\hat{x}_k = 1$. Note that $k \notin S^\beta$. Let $H^{\hat{\alpha}}$ be the cover hypergraph of $\hat{\alpha}x \geq 3$. We consider two cases.

1. $k \in I_2^{\hat{\alpha}}$: Since $\beta\hat{x} = 3$, there exists $j \in J_1^\beta$, such that $\hat{x}_j = 1$ and columns $A_j, A_k$ cover all rows of $M(J_0^\beta)$. If $j \in J_1^{\hat{\alpha}}$, then $\{j,k\}$ is an edge of $H^{\hat{\alpha}}$, contradicting $k \in I^{\hat{\alpha}}$. Hence $j \in J_2^{\hat{\alpha}}$, which along with $j \in J_1^\beta$ implies that $j \in S^\beta$. Since columns $A_j, A_k$ cover all rows of $M(J_0^\beta) = M(J_0^{\hat{\alpha}})$, $\{j,k\}$ is an edge of $\mathcal{G}^{\hat{\alpha}}$ and $k$ is dominated by $j \in S^\beta$, a contradiction.

2. $k \in I_1^{\hat{\alpha}}$: If there exists $j \in J_2^\beta$ such that $\hat{x}_j = 1$, then columns $A_j, A_k$ cover all rows in $M(J_0^\beta)$ and $\{j,k\}$ is an edge of $H^{\hat{\alpha}}$, contradicting $k \in I_1^{\hat{\alpha}}$. Otherwise, there exist $j,l \in J_1^\beta$ such that $\hat{x}_j = \hat{x}_l = 1$ such that columns $A_j, A_k$ and $A_l$ cover all rows in $M(J_0^\beta)$. If $\{j,l\} \subseteq J_1^{\hat{\alpha}}$, then $\{j,k,l\}$ is an edge of $H^{\hat{\alpha}}$, contradicting $k \in I_1^{\hat{\alpha}}$. Hence $\{j,l\} \cap J_2^{\hat{\alpha}} \neq \phi$. Note that $\{j,l\} \cap (J_2^{\hat{\alpha}} \setminus I_2^{\hat{\alpha}}) = \phi$, since $\beta \in \mathcal{I}_3^A$ and every vertex in $J_2^{\hat{\alpha}} \setminus I_2^{\hat{\alpha}}$ is incident to at least one edge of $H^{\hat{\alpha}}$. Hence $\{j,l\} \cap I_2^{\hat{\alpha}} \neq \phi$. Without loss of generality $j \in I_2^{\hat{\alpha}}$, which along with $j \in J_1^\beta$ implies that $j \in S^\beta$. If $l \in J_1^{\hat{\alpha}}$, then $\{j,k\}$ is an edge of $\mathcal{G}^{\hat{\alpha}}$, contradicting the fact that $k$ is not dominated by

$S^\beta$. Hence $l \in J_2^{\hat{\alpha}}$, which along with $\{j, l\} \cap (J_2^{\hat{\alpha}} \setminus I_2^{\hat{\alpha}}) = \phi$ implies that $l \in I_2^{\hat{\alpha}}$ and hence $l \in S^\beta$. By definition of $\mathcal{G}^\alpha$, we have that $\{j, k, l\}$ is an edge of $\mathcal{G}^\alpha$, $\{j, l\} \subseteq S^\beta$ and $k$ is dominated by $S^\beta$, a contradiction.

■

Thus if $\mathcal{F}^\alpha$ is non-empty, say $\beta \in \mathcal{F}^\alpha$, then by *Claim 3.6* it follows that there exists an independent dominating set $S^\beta$ of $\mathcal{G}^\alpha$, as defined in *Claim 3.6* such that $S^\beta = J_1^\beta \cap I_2^{\hat{\alpha}}$. ■

**Remark 3.7** Note that the generator hypergraph can be constructed in $O(mn^3)$ time. Furthermore, the number of inequalities in $\mathcal{M}_3^A(J_0)$ is exactly equal to the number of independent dominating sets of the generator hypergraph contained in $I_2^{\hat{\alpha}}$.

Let $\gamma \in \mathcal{I}_3^A$. $\gamma x \geq 3$ is called a *residual minimal* inequality if the $\gamma x \geq 3$ is $J_3$-minimal and residual graph of the same is an empty graph. Hence the isolated vertices of the cover hypergraph of $\gamma x \geq 3$ are contained in $J_1^\gamma$.

**Lemma 3.8** *Let $\alpha \in \mathcal{I}_3^A$ and let $\hat{\alpha}x \geq 3$ be the inequality obtained by refinement of $\alpha x \geq 3$. Let $\mathcal{G}^\alpha$ be the generator hypergraph of $\alpha x \geq 3$ (and $\hat{\alpha}x \geq 3$) and let $I_2^{\hat{\alpha}}$ be the vertex set of the residual graph of $\hat{\alpha}x \geq 3$. If $\gamma x \geq 3$ is a residual minimal inequality, such that $\gamma \leq \alpha$ and $J_0^\gamma = J_0^\alpha$, then $J_1^\gamma \cap I_2^{\hat{\alpha}}$ is an independent dominating set of $\mathcal{G}^\alpha[I_2^{\hat{\alpha}}]$, the subhypergraph of $\mathcal{G}^\alpha$ induced by vertices in $I_2^{\hat{\alpha}}$. Furthermore, if $S \subseteq I_2^{\hat{\alpha}}$ is an independent dominating set of $\mathcal{G}^\alpha[I_2^{\hat{\alpha}}]$, then the following inequality is a residual minimal valid inequality $\sum_{j \in N \setminus S} \hat{\alpha}_j x_j + \sum_{j \in S} x_j \geq 3$.*

**Proof:** It can be proved using the same arguments as used to prove *Theorem 3.4*. ■

**Remark 3.9** Since $\hat{\alpha}x \geq 3$ is a refined valid inequality, $\mathcal{G}^\alpha[I_2^{\hat{\alpha}}]$ is identical to the residual graph of $\hat{\alpha}x \geq 3$. Hence residual minimal inequalities can be obtained from independent dominating sets of residual graph of $\hat{\alpha}x \geq 3$, which is typically a much smaller graph than the corresponding generator hypergraph. Furthermore, independent dominating sets in a graph can be generated in linear time by generating maximal independent sets by any greedy algorithm. Thus in our framework, computationally, it is significantly easier to generate residual minimal valid inequalities than minimal valid inequalities. Sànchez et al [12] gave a procedure to generate valid inequalities with coefficients in $\{0, 1, 2, 3\}$ and report computational experience with their procedure. In *Section 4*, we will show that the inequalities generated by their procedure are precisely those residual minimal valid inequalities which correspond to maximum cardinality independent dominating set of the residual graph.

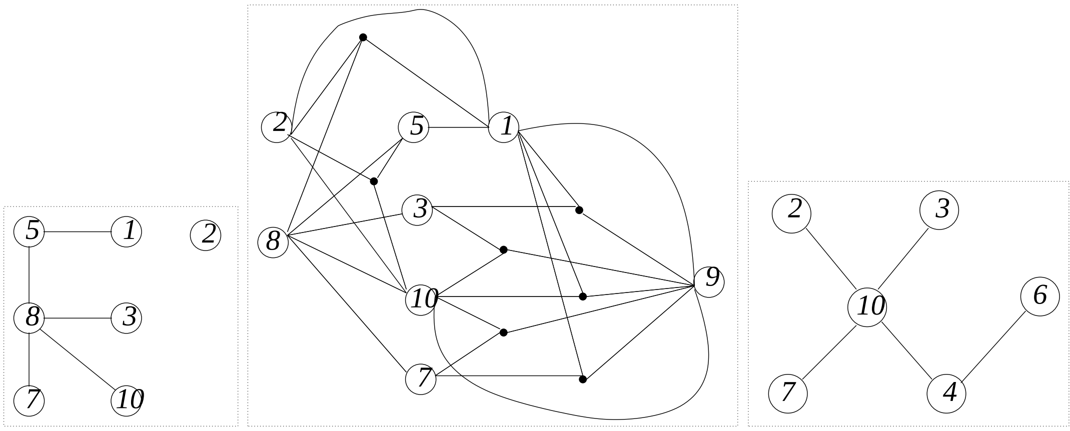
**Remark 3.10** Suppose $\gamma x \geq 3$ is a residual minimal inequality obtained from $\alpha x \geq 3$, $\alpha \in \mathcal{I}_3^A$. Let $S^\gamma \subseteq I_2^{\hat{\alpha}}$ be the independent dominating set of $\mathcal{G}^\alpha[I_2^{\hat{\alpha}}]$, which yields $\gamma x \geq 3$ by an application of *Lemma 3.8*. It is easy to prove that the isolated vertices in the cover hypergraph of $\gamma x \geq 3$ are precisely those vertices in $\mathcal{I}_1^{\hat{\alpha}}$ which are not dominated by vertices in $S^\gamma$. Hence, if $\beta x \geq 3$ is any minimal valid inequality which dominates $\gamma x \geq 3$, then the set $\{j \in N | \beta_j < \gamma_j\}$ is contained in the subset of $I_1^{\hat{\alpha}}$ formed by vertices which are not dominated by $S^\gamma$.

Next we illustrate the *Theorem 3.4* on a small example. Consider the matrix (2.2) (say $A$) and the valid *inequality (3.7)* (say $\alpha x \geq 3$) for $P_I(A)$. It was shown above that the refinement of $\alpha x \geq 3$ is given by *inequality (3.8)* (say $\hat{\alpha}x \geq 3$). The generator hypergraph of $\hat{\alpha}x \geq 3$ is shown in *Figure 6(b)*. The independent dominating sets of the generator graph contained in $I_2^{\hat{\alpha}}$ are $\{v_1, v_8\}$, $\{v_3, v_5, v_7, v_{10}\}$ and $\{v_1, v_3, v_7, v_{10}\}$. By applying *Theorem 3.4* we get the following three minimal valid inequalities:

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 + 2x_5 + 2x_6 + 2x_7 + x_8 + x_9 + 2x_{10} &\geq 3 \\ 2x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 + x_7 + 2x_8 + x_9 + x_{10} &\geq 3 \\ x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 + x_7 + 2x_8 + x_9 + x_{10} &\geq 3 \end{aligned}$$

It can be verified that the first two inequalities define facets of the set-covering polytope associated with matrix (2.2). The third inequality can obtained as sum of the $9^{th}$ row of the matrix and the facet defining inequality given by $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_{10} \geq 2$.

Next consider the following inequality obtained by adding first four rows of the matrix (2.2), dividing by $1.5 = \frac{4-1}{2}$ and performing integer rounding:




(a) Residual Graph of *inequality (3.7)*

(b) Generator Hypergraph of *inequality (3.7)*

(c) Generator Hypergraph of *inequality (3.9)*

Figure 6: Illustration of *Theorem 3.4*

$$3x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_6 + 2x_7 + 3x_8 + 2x_{10} \geq 3 \tag{3.9}$$

The residual graph, which also happens to be the generator hypergraph of the above inequality is shown in *Figure 6(c)*. The inequalities obtained from the independent dominating sets of the hypergraph are shown below along with the respective independent dominating sets.

$$\begin{array}{rcll} 3x_1 + x_2 + x_3 + x_4 + 2x_6 + x_7 + 3x_8 + 2x_{10} & \geq & 3 & \{2,3,4,7\} \\ 3x_1 + x_2 + x_3 + 2x_4 + x_6 + x_7 + 3x_8 + 2x_{10} & \geq & 3 & \{2,3,6,7\} \\ 3x_1 + 2x_2 + 2x_3 + 2x_4 + x_6 + 2x_7 + 3x_8 + x_{10} & \geq & 3 & \{6,10\} \end{array}$$

All of these inequalities are minimal valid inequalities, and the inequality corresponding to independent dominating set $\{2,3,4,7\}$ defines a facet of $P_I(A)$.

# 4 Procedure to generate all minimal valid inequalities with coefficients in $\{0,1,2,3\}$

In this section, we give a procedure which can generate all minimal valid inequalities (and hence facets) with coefficients in $\{0,1,2,3\}$. Given $J_0 \subseteq N$, let $\mathcal{M}_3^A(J_0) = \{\beta \in \mathcal{M}_3^A | J_0^\beta = J_0\}$, as defined in *Section 1.1*. Thus $\mathcal{M}_3^A(J_0)$ represents all those minimal valid inequalities with coefficients in $\{0,1,2,3\}$ whose support is equal to $N \setminus J_0$. A subset $J \subseteq N$ is called *maximal* if $\forall j \in N \setminus J, \exists i \in M(J)$ such that $A_{ij} = 1$.

**Lemma 4.1** *Let $J_0 \subseteq N$, then $\mathcal{M}_3^A(J_0) \neq \phi$, only if $J_0$ is a maximal. Hence if $\alpha \in \mathcal{M}_3^A$, then $J_0^\alpha$ is maximal.*

**Proof:** Suppose, $J_0 \subseteq N$ is not maximal. Then there $\exists j \in N \setminus J_0$ such that $\forall i \in M(J_0)$ $A_{ij} = 0$. Suppose $\beta \in \mathcal{M}_3^A(J_0)$, which implies that $\beta_j \geq 1$. Since $\beta x \geq 3$ is a minimal inequality, there exists a $0 \setminus 1$ point $\hat{x} \in P_I(A)$ such that $\beta\hat{x} = 3$ and $\hat{x}_j = 1$. Let $\tilde{x}$ be defined as $\tilde{x}_k = \hat{x}_k, \forall k \in N \setminus (J_0 \cup \{j\})$, $\tilde{x}_j = 0$ and $\tilde{x}_k = 1, \forall k \in J_0$. It is easy to see that $\tilde{x} \in P_I(A)$ and $\beta\tilde{x} \leq 2$, contradicting the fact that $\beta x \geq 3$ is a valid inequality for $P_I(A)$. Hence if $J_0$ is not maximal then, $M_3^A(J_0) = \phi$. ■

**Lemma 4.2** *Let $S \subseteq M$ and let $J_0$ be defined as $J_0 = \{j \in N | A_{ij} = 0, \forall i \in S\}$. Then $J_0$ is maximal.*

**Proof:** (by contradiction) Suppose, $J_0$ is not maximal, then there $\exists j \in N \setminus J_0$ such that $\forall i \in M(J_0), A_{ij} = 0$. Since $S \subseteq M(J_0)$, this implies that $\forall i \in SA_{ij} = 0$, which by definition of $J_0$ implies that $j \in J_0$, contradicting the fact that $j \in N \setminus J_0$. ■

Next we give a procedure, which can generate any minimal valid inequality with coefficients in $\{0, 1, 2, 3\}$.

**Procedure** *GenIneq*

1. Choose a maximal subset $J_0$ (possibly empty) of $N$ (*Lemma 4.2* gives one way of finding a maximal subset of $N$). Let $J_3 = \{j \in N | A_{ij} = 1, \forall i \in M(J_0)\}$ and $J_2 = N \setminus (J_0 \cup J_3)$.
2. Let $\alpha x \geq 3$ be the following inequality: $\sum_{j \in J_3} 3x_j + \sum_{j \in J_2} 2x_j \geq 3$. Let $\hat{\alpha} x \geq 3$ be the refinement of $\alpha x \geq 3$.
3. Generate the generator hypergraph of $\mathcal{G}^\alpha$ of $\alpha x \geq 3$. Let $I_2^{\hat{\alpha}}$ be the vertex set of the residual graph of $\hat{\alpha} x \geq 3$.
4. Let $S \subseteq I_2^{\hat{\alpha}}$ be an independent dominating set of $\mathcal{G}^\alpha$.
5. Output $\sum_{j \in N \setminus S} \hat{\alpha}_j x_j + \sum_{j \in S} x_j \geq 3$ as a minimal valid inequality.

The next lemma shows that the above procedure generates minimal valid inequalities with coefficients in $\{0, 1, 2, 3\}$.

**Lemma 4.3** *The procedure GenIneq generates minimal valid inequality for $P_I(A)$. with coefficients in $\{0, 1, 2, 3\}$. For a given choice of a maximal $J_0$, the procedure generates support equivalent minimal valid inequalities whose common support is $N \setminus J_0$.*

**Proof:** Suppose $J_0 \subseteq N$ is chosen in step 1 and let $\alpha x \geq 3$ be the inequality described in step 2 of the above procedure. Note that $\alpha^S x \geq 3$ is dominated by the valid inequality obtained by adding the rows of $Ax \geq 1$ corresponding to indices in $M(J_0)$, dividing the resulting inequality by $\frac{|M(J_0)|-1}{2}$ and performing integer rounding. Hence $\alpha \in \mathcal{I}_3^A$. The remaining part of the theorem follows from *Theorem 3.4*. ■

**Lemma 4.4** *Every minimal valid inequality of $P_I(A)$ with coefficients in $\{0, 1, 2, 3\}$ can be generated by the procedure GenIneq by appropriate choice of $J_0$ and $S$ in steps 1 and 4, respectively.*

**Proof:** Let $\beta x \geq 3$ be a minimal valid inequality of $P_I(A)$ such that $\beta \in \mathcal{M}_3^A$. Since $\beta \in \mathcal{M}_3^A$, by *Lemma 4.1* $J_0^\beta$ is maximal. Let $J_0$ be chosen as $J_0^\beta$ in step 1 of the above procedure and let $\alpha x \geq 3$ be the inequality described in step 2 of the procedure. It is easy to verify that $\beta \in \mathcal{F}^\alpha$. Hence, by *Theorem 3.4* the set $J_1^\beta \cap I_2^{\hat{\alpha}}$ is an independent dominating set of $\mathcal{G}^\alpha$. Hence $\beta x \geq 3$ can be derived by choosing $S = J_1^\beta \cap I_2^{\hat{\alpha}}$ in step 4 of the procedure. ■

**Remark 4.5** Let $\alpha x \geq 3$ and $\hat{\alpha} x \geq 3$ be the inequalities generated in step 2 of the above procedure. Let $G^\alpha$ and $G^{\hat{\alpha}}$ be the residual graph of $\alpha x \geq 3$ and $\hat{\alpha} x \geq 3$, respectively. Let $\tilde{I}$ be the set of isolated vertices of $G^\alpha$. Then $G^{\hat{\alpha}}$ is obtained from $G^\alpha$ by removing the isolated vertices, namely $\tilde{I}$. Furthermore, $I_2^{\hat{\alpha}} = J_2^\alpha \setminus \tilde{I}$, and the inequality generated in step 5 by choosing set $S \subseteq I_2^{\hat{\alpha}}$ in step 4 is given by $\sum_{j \in J_3^\alpha} 3x_j + \sum_{j \in J_2^\alpha \setminus (S \cup \tilde{I})} 2x_j + \sum_{j \in (S \cup \tilde{I})} x_j \geq 3$.

Next we describe some variants of the procedure *GenIneq* which are easier and more efficient to implement, but they yield residual minimal valid inequalities instead of minimal valid inequalities. Consider the procedure obtained by replacing step 4 of the procedure *GenIneq* by the following:

**Procedure** *GenIneq*$^\star$

*Step 4$'$:* Let $S \subseteq I_2^{\hat{\alpha}}$ be an independent dominating set of $\mathcal{G}^\alpha[I_2^{\hat{\alpha}}]$, the subhypergraph of $\mathcal{G}^\alpha$ induced by vertices in $I_2^{\hat{\alpha}}$.

It follows immediately from *Lemma 3.8* that the modified procedure *GenIneq*$^\star$ generates residual minimal valid inequalities of $P_I(A)$.

Next we describe the procedure of Sànchez et al [12] to generate $V$-minimal (defined in remark 2.3) valid inequalities with coefficients in $\{0, 1, 2, 3\}$.

**Procedure** *GenSSV*

1. Choose a nonempty subset $\tilde{S} \subseteq M$.
2. Define $N_3, N_0$ and $\bar{N}$ as: $N_3 = \{j \in N | A_{ij} = 1 \ \forall i \in \tilde{S}\}$, $N_0 = \{j \in N | A_{ij} = 0 \ \forall i \in \tilde{S}\}$ and $\bar{N} = N \setminus (N_0 \cup N_3)$.
3. Let $N_1$ be a maximum cardinality set in $\bar{N}$ such that $\forall j, h \in N_1, \exists i \in \tilde{S}$ such that $A_{ij} + A_{ih} = 0$.
4. Output the inequality given by: $\sum_{j \in N_3} 3x_j + \sum_{j \in \bar{N} \setminus N_1} 2x_j + \sum_{j \in N_1} x_j \geq 3$.

Sànchez et al (Theorem 2.1 on page 346 of [12]) proved that the above procedure generates valid inequalities for $P_I(A)$. We prove that their procedure can be obtained as a special case of *GenIneq*$^\star$. Consider the following modification of *GenIneq*$^\star$ obtained by replacing step $4'$ by:

**Procedure** *GenIneq*$^{\#}$

*Step $4''$:* Let $S \subseteq I_2^{\hat{\alpha}}$ be a maximum cardinality independent dominating set of $\mathcal{G}^\alpha[I_2^{\hat{\alpha}}]$, the subhypergraph of $\mathcal{G}^\alpha$ induced by vertices in $I_2^{\hat{\alpha}}$.

**Lemma 4.6** *Suppose the inequality $\gamma x \geq 3$ is generated by procedure GenSSV by choosing set $\tilde{S} \subseteq M$ in step 1 and set $N_1$ in step 3. Let $N_0, N_3$ and $\bar{N}$ be defined as in the procedure. Then $\gamma x \geq 3$ can be generated by procedure GenIneq$^{\#}$ applied to the matrix $\bar{A} = A_{M \setminus \bar{S}}$, where $\bar{S} = \{i \in M \setminus \tilde{S} | A_{ij} = 0, \forall j \in N_0\} = M(N_0) \setminus \tilde{S}$.*

**Proof:** We use the same notation as used in the statement of the lemma. All the cover hypergraphs and residual graphs in the following proof are defined relative to matrix $\bar{A}$ (and not matrix $A$). Consider the following instantiation of procedure *GenIneq*$^{\#}$, applied to $\bar{A} = A_{M \setminus \bar{S}}$.

$N_0$ is chosen as the set $J_0$ is step 1 of procedure *GenIneq*$^{\#}$. Note that by definition, $N_0 = \{j \in N | A_{ij} = 0 \ \forall i \in \tilde{S}\}$. Hence by *Lemma 4.2* $N_0$ is a maximal set w.r.t $\bar{A}$. Furthermore, it is easy to see that $J_3 = N_3$, $J_2 = \bar{N}$ and $M(J_0) = \tilde{S}$ w.r.t matrix $\bar{A}$. (Note that if the original matrix $A$ was used instead of $\bar{A}$ then the condition $M(J_0) = \tilde{S}$ is not guaranteed to be satisfied). Let $\alpha x \geq 3$ be the inequality generated in step 2 of procedure *GenIneq* and $\hat{\alpha} x \geq 3$ be the refinement of $\alpha x \geq 3$. Let $G^\alpha$ and $G^{\hat{\alpha}}$ be the residual graphs of $\alpha x \geq 3$ and $\hat{\alpha} x \geq 3$, respectively. Let $\tilde{I}$ be the set of isolated vertices of $G^\alpha$, then $\hat{\alpha} x \geq 3$ is given by $\sum_{j \in J_3} 3x_j + \sum_{j \in J_2 \setminus \tilde{I}} 2x_j + \sum_{j \in \tilde{I}} x_j \geq 3$.

**Claim 4.7** *Let $\hat{N}_1 \subseteq \bar{N}$. Then $\forall j, h \in \hat{N}_1, \exists i \in \tilde{S}$ such that $A_{ij} + A_{ih} = 0$ if and only if $\hat{N}_1$ is an independent set of the $G^\alpha$.*

**Proof:** Note that $J_2^\alpha = \bar{N}$. Furthermore, since $J_1^\alpha = \phi$, $J_2^\alpha = \bar{N}$ is also the vertex set of $G^\alpha$. $\hat{N}_1 \subseteq \bar{N}$ is an independent set of $G^\alpha$ if and only if for any $j, h \in \hat{N}_1$, columns $A_j, A_h$ do not cover all rows in $M(J_0^\alpha) = \tilde{S}$ (as shown above), which is equivalent to $\forall j, h \in \hat{N}_1, \exists i \in \tilde{S}$ such that $A_{ij} + A_{ih} = 0$. ∎

By the above claim it follows that $N_1$ chosen in step 3 of *GenSSV* is a maximum cardinality independent set of $G^\alpha$. Furthermore, since the $G^{\hat{\alpha}}$ is obtained from $G^\alpha$ by removing the set of isolated vertices of $G^\alpha$, it follows that $N_1 = \tilde{I} \cup \hat{N}_1$, where $\hat{N}_1 = N_1 \setminus \tilde{I}$ is a maximum cardinality independent set of $G^{\hat{\alpha}}$. Note that a maximum cardinality independent set is also a maximum cardinality independent dominating set. Hence we can choose $S = \hat{N}_1$ in step $4''$ of the procedure *GenIneq*$^{\#}$, such that the inequality generated in step 4 of the procedure *GenIneq*$^{\#}$ is given by $\sum_{j \in J_3} 3x_j + \sum_{j \in J_2 \setminus (\hat{N}_1 \cup \tilde{I})} 2x_j + \sum_{j \in (\hat{N}_1 \cup \tilde{I})} x_j \geq 3$, which is equivalent to $\sum_{j \in N_3} 3x_j + \sum_{j \in \bar{N} \setminus N_1} 2x_j + \sum_{j \in N_1} x_j \geq 3$, since $J_3 = N_3, J_2 = \bar{N}$ and $N_1 = \hat{N}_1 \cup \tilde{I}$. $\sum_{j \in N_3} 3x_j + \sum_{j \in \bar{N} \setminus N_1} 2x_j + \sum_{j \in N_1} x_j \geq 3$ is easily seen to be identical to $\gamma x \geq 3$. ∎

**Lemma 4.8** *Every inequality which is generated by the procedure GenIneq$^{\#}$ can be generated by the procedure GenSSV.*

**Proof:** Let $\beta x \geq 3$ be an inequality generated by procedure *GenIneq*$^{\#}$ by choosing set $J_0$ in step 1 of the procedure and choosing $S$ in step $4^{''}$. Let $\tilde{I}$ be the set of isolated vertices of the residual graph of the $J_3$-minimal inequality $\alpha x \geq 3$ generated in step 2 of the procedure *GenIneq*$^{\#}$. Consider the following instantiation of procedure *GenSSV*.

Choose set $M(J_0^\beta)$ as $\tilde{S}$ in step 1 of procedure *GenSSV*. It is easy to see that $N_3$ and $N_0$ defined in step 2 of the procedure *GenSSV*, are identical to $J_3^\beta$ and $J_0^\beta$ respectively, and $\bar{N} = J_1^\beta \cup J_2^\beta$. Let $G^\alpha$ and $G^{\hat{\alpha}}$ be the residual graphs of $\alpha x \geq 3$ and $\hat{\alpha} x \geq 3$ respectively. Note that $G^{\hat{\alpha}}$ is obtained from $G^\alpha$ by removing the isolated vertices, $\tilde{I}$. Following claim can be proved along the lines of *Claim 4.7*.

**Claim 4.9** *Let $\hat{N}_1 \subseteq \bar{N}$. Then $\forall j, h \in \hat{N}_1, \exists i \in \tilde{S}$ such that $A_{ij} + A_{ih} = 0$ if and only if $\hat{N}_1$ is an independent set of $G^\alpha$.* ■

Hence a maximum cardinality independent set, say $N_1$, of $G^\alpha$, is also a maximum cardinality set in $\bar{N}$ such that $\forall j, h \in N_1, \exists i \in \tilde{S}$ such that $A_{ij} + A_{ih} = 0$. Since $S$ is a maximum cardinality independent dominating set of $G^{\hat{\alpha}}$, $S$ is also a maximum cardinality independent set of $G^{\hat{\alpha}}$, which implies that $S \cup \tilde{I}$ is a maximum cardinality independent set of $G^\alpha$. Thus we can choose $N_1 = S \cup \tilde{I}$ in step 3 of the procedure *GenSSV*, so that inequality generated in step 4 of the procedure is given by $\sum_{j \in N_3} 3x_j + \sum_{j \in \bar{N} \setminus N_1} 2x_j + \sum_{j \in N_1} x_j \geq 3$, which is equivalent to $\sum_{j \in J_3^\beta} 3x_j + \sum_{j \in (J_1^\beta \cup J_2^\beta) \setminus (S \cup \tilde{I})} 2x_j + \sum_{j \in (S \cup \tilde{I})} x_j \geq 3$, since $N_3 = J_3^\beta, \bar{N} = J_1^\beta \cup J_2^\beta$ and $N_1 = S \cup \tilde{I}$. $\sum_{j \in J_3^\beta} 3x_j + \sum_{j \in (J_1^\beta \cup J_2^\beta) \setminus (S \cup \tilde{I})} 2x_j + \sum_{j \in (S \cup \tilde{I})} x_j \geq 3$ is easily seen to be identical to $\beta x \geq 3$ by *Remark 4.5.* ■

**Remark 4.10** *Lemma 4.6* and *Lemma 4.8* show that the procedure *GenSSV* generates precisely those inequalities which are generated by the procedure *GenIneq*$^{\#}$. Hence the procedure of Sànchez et al [12] can be thought to be a special version of *GenIneq*$^\star$ which generates only those residual minimal valid inequalities which correspond to maximum cardinality independent dominating sets in the corresponding residual graphs. Furthermore, note that finding maximum cardinality independent dominating sets in a general graph is an $NP-Complete$ problem [6], whereas an independent dominating set in a graph can be found by any greedy algorithm. In this sense procedure *GenIneq*$^\star$ is more tractable than procedure *GenSSV*, in the absence of any further structural knowledge of the residual graph under consideration. Infact in *Section 5* we prove that given any general graph $G$, we can generate a $0 \setminus 1$ matrix $A^G$ and a refined valid inequality $\beta^G x \geq 3$ of $P_I(A^G)$ with coefficients in $\{0, 1, 2, 3\}$, such that the residual graph of $\beta^G x \geq 3$ is isomorphic to $G$.

# 5 Separation Problem

In this section, we illustrate an application of the generator hypergraphs in developing algorithms and heuristics to separate fractional solutions to the set-covering problem. Let $A$ be a $0 \setminus 1$ matrix as described in *Section 1.1.* Let $\hat{x}$ be a fractional solution which results by solving the linear programming relaxation of the following problem, obtained by removing the integrality constraints.

$$\begin{array}{lccc} max & cx & & \\ & Ax & \geq & \mathbf{1} \\ & x & \geq & \mathbf{0} \\ & x & \leq & \mathbf{1} \\ & x & \in & \mathbb{B}^n \end{array} \tag{5.10}$$

Consider the following problem:

*SepProb* $\{\hat{x}, A, \eta, J_0, c\}$: Given a solution $\hat{x}$ to the linear programming relaxation of the set-covering problem (5.10) associated with a $0 \setminus 1$ matrix $A$, a rational number $\eta$ and a set $J_0 \subseteq N$, does there exist $\beta \in \mathcal{M}_3^A(J_0)$ such that $\beta \hat{x} - 3 \leq \eta$.

An optimisation version of the above problem would be to find the inequality $\beta x \geq 3, \beta \in \mathcal{M}_3^A(J_0)$ which is most violated by $\hat{x}$, or prove that none exists.

**Lemma 5.1** *Let $\hat{x}, J_0, \eta$ be as described in SepProb. Let $\mathcal{G}^\alpha$ be the generator hypergraph obtained in step 3 of the procedure GenIneq by choosing the given set $J_0$ in the first step of the procedure, and let $J_3, \alpha, \hat{\alpha}$ be as described in* *procedure GenIneq. Let $f : I_2^{\hat{\alpha}} \longrightarrow \mathbb{R}$ be a weight function defined on the set $I_2^{\hat{\alpha}}$ such that $f(j) = \hat{x}_j, \forall j \in I_2^{\hat{\alpha}}$. Then there exists $\beta \in \mathcal{M}_3^A(J_0)$ such that $\beta\hat{x} - 3 \leq \eta$ if and only if $\mathcal{G}^\alpha$ has an independent dominating set $S \subseteq I_2^{\hat{\alpha}}$ such that $\sigma - 3 - \eta \leq f(S) = \sum_{j\in S} f(j)$, where $\sigma = \sum_{j\in J_3} 3\hat{x}_j + \sum_{j\in N\setminus(J_0\cup J_3)} 2\hat{x}_j - \sum_{j\in J_1^{\hat{\alpha}}} \hat{x}_j$. Hence SepProb can be solved by finding a maximum weight independent dominating set in $\mathcal{G}^\alpha$ contained in $I_2^{\hat{\alpha}}$.*

**Proof:** Note that there is a one-one correspondence between $\beta \in \mathcal{M}_3^A(J_0)$ and independent dominating sets of $\mathcal{G}^\alpha$ contained in $I_2^{\hat{\alpha}}$. Furthermore, if $S \subseteq I_2^{\hat{\alpha}}$ is an independent dominating set of $\mathcal{G}^\alpha$, then the associated minimal valid inequality, $\beta^S x \geq 3$, is given by $\sum_{j\in N\setminus S} \hat{\alpha}_j x_j + \sum_{j\in S} x_j \geq 3$, which is same as $\sum_{j\in J_3} 3\hat{x}_j + \sum_{j\in N\setminus(J_0\cup J_3)} 2\hat{x}_j - \sum_{j\in J_1^{\hat{\alpha}}} \hat{x}_j - \sum_{j\in S} x_j \geq 3$. Hence $\beta^S\hat{x} - 3 \leq \eta$ is equivalent to $\sum_{j\in J_3} 3\hat{x}_j + \sum_{j\in N\setminus(J_0\cup J_3)} 2\hat{x}_j - \sum_{j\in J_1^{\hat{\alpha}}} \hat{x}_j - \sum_{j\in S} x_j - 3 \leq \eta$, which can be restated as $\sigma - 3 - \eta \leq f(S)$, using the notation defined above. The result then follows from the one-one correspondence between $\beta \in \mathcal{M}_3^A(J_0)$ and independent dominating sets of $\mathcal{G}^\alpha$ contained in $I_2^{\hat{\alpha}}$. ■

**Remark 5.2** The above lemma illustrates a way to exploit the compact representation of all the inequalities in $\mathcal{M}_3^A(J_0)$ by way of generator hypergraphs. Note that, in general, families of minimal valid inequalities have extremely complex structure, with each inequality arising in a complicated manner such as by lifting another inequality, or by strengthening coefficients of another known valid inequality etc. Hence, it is not immediately clear, how to tackle the problem of finding an inequality among the family which is most violated by a given fractional solution. The above lemma, proves that for inequalities in $\mathcal{M}_3^A(J_0)$, the separation problem can be formulated as a discrete optimisation problem, which makes it easier to devise heuristics to solve the separation problem.

**Remark 5.3** The problem of finding a maximum weight independent dominating set in a graph is known to be $NP-Complete$ [6]. Also, given a graph $G = (V, E)$ and a subset $S \subseteq V$, finding whether there exists $\hat{S} \subseteq S$, such that $\hat{S}$ is an independent dominating set of $G$ is $NP-Complete$ [8]. Hence *Lemma 5.1* reduces the problem of finding most violated inequality in $\mathcal{M}_3^A(J_0)$ to an $NP-Complete$ problem. Nevertheless, one of the variant of procedure *GenIneq* such as *GenIneq*$^\star$ can be used to derive residual minimal inequalities (if any) which are violated by the fractional solution.

We next prove that *SepProb* is $NP-Complete$. We reduce the problem of finding a maximum weight independent dominating set in a graph to *SepProb*. We would need the following construction for the proof.

**Residual Matrices of Graphs**

Given a connected graph $G = (V, E)$ on $n$ nodes and $m$ edges, such that $|V| = n \geq 2$, consider a $0 \setminus 1$ matrix $A^G$ on $\left(\frac{n(n-1)}{2} - m\right)$ rows and $n$ columns, defined as follows (the columns of the matrix $A^G$ are identified with the vertices of the graph $G$):

1. If $n = 2$ and $V = \{v_1, v_2\}$, then $A^G$ has one row and two columns and is given by [0 1], where the first column corresponds to $v_1$ and the second column corresponds to $v_2$.
2. If $n > 2$, let $v \in G$ such that $G - v$ (i.e graph obtained by removing vertex $v$ and all edges incident to $v$), is connected. Note that every connected graph has at least two vertices which satisfy this property. Without loss of generality, we assume that columns corresponding to $d_G(v)$ neighbours of $v$ are the first $d_G(v)$ columns of $A^{G-v}$. If $v$ is connected to every vertex in $G - v$, then $A^G$ is obtained by appending a column of ones to $A^{G-v}$. Otherwise, if $V \setminus N_G[v] \neq \phi$, then let $k = |V \setminus N_G[v]|$. $A^G$ is defined as follows ($B = A^{G-v}, U = N_G(v), \bar{U} = V \setminus N_G[v]$):

$$\begin{bmatrix} B^U & B^{\bar{U}} & 1 \\ 1_k^{|U|} & C_k^{k-1} & 0^k \end{bmatrix} \tag{5.11}$$

where 1 is a column vector of ones of conformable dimension and and $0^k$ is a column vector of zeros of dimension $k$, $1_k^{|U|}$ is a $k \times |U|$ matrix of ones, $C_k^{k-1}$ is a $k \times k$ $0 \setminus 1$ matrix with exactly one zero in every row and every column. It is easy to verify that if $A^{G-v}$ had $\left(\frac{(n-1)(n-2)}{2} - \hat{m}\right)$ rows, where $\hat{m}$ is the number of edges in $G - v$, then $A^G$ as defined above has $\left(\frac{n(n-1)}{2} - m\right)$ rows.

The matrix $A^G$ defined above is called the *residual matrix* of $G$. The next lemma proves an interesting property of the matrix $A^G$.

**Lemma 5.4** *Let $G = (V, E)$ be a connected graph ($|V| \geq 2$) and let $A^G$ be the residual matrix of $G$. Then columns $j_1$ and $j_2$ of $A^G$ cover all the rows of $A^G$ if and only if $j_1$ and $j_2$ are adjacent in $G$.*

**Proof:** We prove the result by induction on $n = |V|$. If $n = 2$, then above statement follows trivially from step 1 of the definition of residual matrices. Next, suppose the above statement holds true for all connected graphs with exactly $(n-1)$ nodes. Let $G = (V, E)$ be a connected graph on $n$ nodes. Let $v \in G$ such that $G - v$ i.e graph obtained by removing vertex $v$ and all edges incident to it, is connected. Without loss of generality, we assume that columns corresponding to $d_G(v)$ neighbours of $v$ are the first $d_G(v)$ columns of $A^{G-v}$. We use the same notation as used in step 2 in the definition of residual matrices.

If $v$ is connected to every vertex in $G - v$, then $A^G$ is obtained by appending a column of ones to $A^{G-v}$. In this case, since the number of rows in $A^G$ is same as the number of rows in $A^{G-v}$, two columns in $V \setminus \{v\}$ cover all rows of $A^G$ if and only if they cover all rows of $A^{G-v}$, which implies they are adjacent in $G$ iff they are adjacent in $G - v$. Furthermore, since column corresponding to node $v$ covers all rows of $A^G$, it is adjacent to every vertex in $V - v$, as required.

Otherwise $V \setminus N_G[v] \neq \phi$. Consider the matrix (5.11). Note that by construction, if $j_1, j_2 \in V - v$, the columns $j_1$ and $j_2$ of $A^G$ cover all rows of $A^G$ if and only if they cover all rows of $A^{G-v}$, which implies they are adjacent in $G$ iff they are adjacent in $G - v$. The column $v$ forms a cover of rows of $A^G$ with any column in $N_G(v) = U$, by construction, since column $v$ covers all the rows of $A^{G-v}$ and any column in $N_G(v)$ covers all additional rows. Furthermore, if $u \notin N_G[v]$ i.e $u \in \bar{U}$, then there exists a row of the circulant matrix $C_k^{k-1}$, where $C_k^{k-1}$ has a zero in column corresponding to node $u$. This implies that both columns $u$ and $v$ do not cover the corresponding row of $A^G$, which implies that they do not form a cover, as expected. ■

**Lemma 5.5** *Let $G = (V, E)$ be a connected graph on $n \geq 3$ nodes, such that $\gamma_G \geq 2$. Let $P_I(A^G)$ be the set-covering polytope associated with the residual matrix of $G$. Then*

1. *$P_I(A^G)$ is full-dimensional.*
2. *$\sum_{j \in V} 2x_j \geq 3$ is a $J_3$-minimal refined valid inequality of $P_I(A^G)$.*
3. *The residual graph and the generator hypergraph of $\sum_{j \in V} 2x_j \geq 3$ is isomorphic to $G$.*

**Proof:** **Proof of Statement 1**: It suffices to prove that every row of $A^G$ has at least two ones. By *Lemma 5.4*, two nodes in $V$ are adjacent if and only if the corresponding columns of $A^G$ cover all rows of $A^G$. Since $G$ is a connected graph on $n \geq 3$ nodes, if $A^G$ has a zero row, then by *Lemma 5.4* $E = \phi$, a contradiction. Furthermore, if $A^G$ has a row with single one, say in the $j^{th}$ column, then every edge of $G$ is incident to node $j$, contradicting the fact that $\gamma_G \geq 2$. Hence every row of $A^G$ has at least two ones.

**Proof of Statement 2 and 3**: Let $\alpha x \geq 3$ be the inequality $\sum_{j \in V} 2x_j \geq 3$. Suppose there exists a $0 \setminus 1$ vector $\hat{x} \in P_I(A^G)$ such that $\alpha\hat{x} \leq 2$, this implies that there exists $v \in V$, such that $\hat{x}_v = 1$ and $\hat{x}_u = 0 \; \forall u \in V - v$. Hence column $v$ covers all rows of $A^G$, which implies by *Lemma 5.4* that node $v$ is adjacent to every other node in the graph, contradicting the fact $\gamma_G \geq 2$. Hence $\alpha x \geq 3$ is a valid inequality of $P_I(A^G)$. Furthermore, $J_3^\alpha = \phi$, which implies trivially that $\alpha x \geq 3$ is a $J_3$-minimal inequality. Also, the cover hypergraph of $\alpha x \geq 3$ consists of $n$ isolated vertices and by *Lemma 5.4*, the 2-cover graph of the same is isomorphic to $G$, which implies that the residual graph of $\alpha x \geq 3$ is isomorphic to $G$ and has no isolated vertices. Hence $\alpha x \geq 3$ is a refined valid inequality and generator hypergraph of the same is isomorphic to $G$. ■

**Theorem 5.6** *SepProb is $NP - Complete$.*

**Proof:** Given an instance $\{\hat{x}, A, \eta, J_0, c\}$ of *SepProb* and a vector $\beta \in \mathbb{R}^n$, by *Theorem 3.4* it can be verified in polynomial time whether $\beta \in \mathcal{M}_3^A(J_0)$ and $\beta\hat{x} - 3 \leq \eta$. Hence *SepProb* is in $NP$. We reduce the following problem, which is known to be $NP - Complete$ [6], to *SepProb*.

$IDS$: Given a connected graph $G = (V, E)$ such that $\gamma_G \geq 2$ and a positive number $k$, does there exists $S \subseteq V$ such that $S$ is an independent dominating set of $G$ and $|S| \geq k$.

Let $A = A^G$, the residual matrix associated with graph $G$, $J_0 = \phi$ and let $c$ be the $n$-dimensional vector of ones. Let $n_1$ be the minimum number of ones in any row of $A^G$ and let $\eta = \frac{2n}{n_1} - 3 - \frac{k}{n_1}$. Furthermore, let $\hat{x}$ be the unique $n$-dimensional vector with all components equal to $\frac{1}{n_1}$. Note that $n_1 \geq 2$ by statement 1 of *Lemma 5.5*. It is easy to verify that $\hat{x}$ is a solution to the linear programming relaxation polytope associated with $P_I(A)$.

**Claim 5.7** *$G$ has an independent dominating set $S$ of size at least $k$ if and only if there exists $\beta \in \mathcal{M}_3^A(J_0)$ such that $\beta\hat{x} - 3 \leq \eta$.*

**Proof:** Note that if $J_0 = \phi$ and $A = A^G$, then $\alpha x \geq 3$ defined in step 2 of procedure *GenIneq* is $\sum_{j \in V} 2x_j \geq 3$. By *Lemma 5.5*, $\alpha x \geq 3$ is a $J_3$-minimal refined valid inequality whose generator hypergraph is isomorphic to $G$. Let $f$ be the weight function as defined in *Lemma 5.1*. Then by *Lemma 5.1*, there exists $\beta \in \mathcal{M}_3^A(J_0)$ such that $\beta\hat{x} - 3 \leq \eta$ if and only if the residual graph of $\alpha x \geq 3$ i.e $G$, has an independent dominating set of weight at least $\sigma - 3 - \eta$, where $\sigma = \sum_{j \in V} 2\hat{x}_j = \frac{2n}{n_1}$. Since $\eta = \frac{2n}{n_1} - 3 - \frac{k}{n_1}$, we get that there exists $\beta \in \mathcal{M}_3^A(J_0)$ such that $\beta\hat{x} - 3 \leq \eta$ if and only if $G$ has an independent dominating set of weight at least $\frac{k}{n_1}$. Since $\hat{x}_j = \frac{1}{n_1}$ $\forall j \in V$, $G$ has an independent dominating set of weight at least $\frac{k}{n_1}$ if and only if it has an independent dominating set of size at least $k$. ■

Hence if we can solve *SepProb* in polynomial time, then we can solve in polynomial time the decision problem of whether a connected graph $G$ with $\gamma_G \geq 2$ has an independent dominating set of size at least $k$. Hence *SepProb* is $NP-Complete$. ■

Next we discuss some results which can be used to choose $J_0$ is step 1 of the procedure *GenIneq* to cut off a given fractional solution to the linear programming relaxation of the set-covering problem.

**Lemma 5.8** *Let $\hat{x}$ be a fractional solution to $Ax \geq \mathbf{1}$, $\mathbf{0} \leq x \leq \mathbf{1}$, with $I = \{j \in N | \hat{x}_j = 1\}$. If $\alpha \in \mathcal{M}_3^A$ such that $\alpha\hat{x} < 3$, then*

1. $I \cap J_3^\alpha = \phi$
2. $I \cap J_2^\alpha = \phi$
3. $|I \cap J_1^\alpha| \leq 1$
4. *If $\nexists \gamma \in \mathcal{M}_2^A$ such that $\gamma\hat{x} < 2$, then $I \cap J_1^\alpha = \phi$*

**Proof:** It is easy to see that statement 1 holds true, since otherwise $\alpha\hat{x} \geq 3$.

**Proof of statement 2**: Suppose $\exists j \in I \cap J_2^\alpha$. Since $j \notin J_3^\alpha$, there exists $i \in M(J_0^\alpha)$ such that $A_{ij} = 0$. To see this, note that if $\forall i \in M(J_0^\alpha), A_{ij} = 1$, then columns in $\{j\} \cup J_0^\alpha$ cover all the rows of $A$, and the corresponding $0 \setminus 1$ incident vector, say $\tilde{x} \in P_I(A)$, satisfies $\alpha\tilde{x} \leq 2$, contradicting the fact that $\alpha x \geq 3$ is a valid inequality. Let $a^i x \geq 1$, be the $i^{th}$ row of $A$. Since $\hat{x}$ is a solution to $Ax \geq \mathbf{1}$ $\mathbf{0} \leq x \leq \mathbf{1}$, we have that $a^i\hat{x} \geq 1$. Also, $\alpha \in \mathcal{M}_3^A(J_0^\alpha)$ implies that $\alpha\hat{x} \geq a^i\hat{x} + 2\hat{x}_j \geq 3$, contradicting the fact that at $\alpha\hat{x} < 3$. Hence $I \cap J_2^\alpha = \phi$.

**Proof of statement 3**: Suppose $\exists j_1, j_2 \in I \cap J_1^\alpha$. Since $\alpha \in \mathcal{M}_3^A$, it follows that columns $A_{j_1}, A_{j_2}$ do not cover all the rows of in $M(J_0^\alpha)$, which implies that $\exists i \in M(J_0^\alpha)$ such that $A_{ij_1} = A_{ij_2} = 0$. Since $\hat{x}$ is a solution to $Ax \geq \mathbf{1}$ $\mathbf{0} \leq x \leq \mathbf{1}$, we have that $a^i\hat{x} \geq 1$. Also, $\alpha \in \mathcal{M}_3^A(J_0^\alpha)$ implies that $\alpha\hat{x} \geq a^i\hat{x} + \hat{x}_{j_1} + \hat{x}_{j_2} \geq 3$, contradicting the fact that at $\alpha\hat{x} < 3$. Hence $|I \cap J_1^\alpha| \leq 1$.

**Proof of statement 4**: Suppose $\nexists \gamma \in \mathcal{M}_2^A$ such that $\gamma\hat{x} < 2$ and $\exists j \in I \cap J_1^\alpha$. Let $N_0 = J_0 \cup \{j\}$ and let $S = M(N_0)$. Note that $S \neq \phi$. Let $N_2 = \{k \in N | A_{ik} = 1, \forall i \in S\}$ and $N_1 = N \setminus (N_0 \cup N_2)$. Let $\beta x \geq 2$ be the following inequality $\sum_{j \in N_2} 2x_j + \sum_{j \in N_1} x_j \geq 2$. It is easy to see that $\beta \in \mathcal{I}_2^A$.

**Claim 5.9** $J_1^\alpha \subseteq N_0 \cup N_1$.

**Proof:** (by contradiction) Suppose $\exists k \in J_1^\alpha \cap N_2$, then clearly $k \neq j$. Furthermore, we have by definition of $N_2$, $A_{ik} = 1, \forall i \in S = M(N_0)$. Hence columns in $\{j,k\} \cup J_0^\alpha$ cover all rows of $A$, and hence the $0 \setminus 1$ incident vector, say $\tilde{x}$, of $\{j,k\} \cup J_0^\alpha$ is in $P_I(A)$. But $\alpha\tilde{x} = 2 < 3$, contradicting the fact that $\alpha \in \mathcal{I}_3^A$. ∎

Hence $\alpha_k \geq \beta_k, \forall k \in N \setminus \{j\}$, $\alpha_j = 1$ and $\beta_j = 0$. Hence $\beta\hat{x} \leq \alpha\hat{x} - \hat{x}_j < 3 - 1 = 2$. Hence $\beta\hat{x} < 2$. Let $\gamma \in \mathcal{M}_2^A$ such that $\gamma \leq \beta$ (such a vector always exists by Theorem 2.1 on Page 60 of [3]). Thus we have $\gamma \in \mathcal{M}_2^A$ and $\gamma\hat{x} \leq \beta\hat{x} < 2$, contradicting the hypothesis that $\nexists\gamma \in \mathcal{M}_2^A$ such that $\gamma\hat{x} < 2$. ∎

**Remark 5.10** The above lemma suggests the following approach to apply results in this section in cutting off fractional solutions. Let $\hat{x}$ be a fractional solution to $Ax \geq \mathbf{1}$, $\mathbf{0} \leq x \leq \mathbf{1}$ and let $I = \{j \in N | \hat{x}_j = 1\}$. First, try to determine if there exists $\alpha \in \mathcal{M}_2^A$, such that $\alpha\hat{x} < 2$. If there exists such an $\alpha$, then add the corresponding inequality to cut off the solution. Seperation heuristics for inequalities $\alpha x \geq 2$ such that $\alpha \in \mathcal{M}_2^A$ can be found in work of Balas and Ng [3, 4]. If the heuristic fails to find an $\alpha \in \mathcal{M}_2^A$ which cuts off the fractional solution, then apply procedure *GenIneq* by choosing maximal set $J_0$ in step 1 such that $I \subseteq J_0$, to generate the generator hypergraph, $\mathcal{G}^\alpha$, in step 3 of the procedure. Inequalities cutting off the fractional solution can then be generated from the (constrained) independent dominating sets of the generator hypergraph. *Lemma 5.1* shows that the most violated inequality in this family of inequalities can be generated by finding the maximum weight (constrained) independent dominating set in the generator hypergraph using the weighting scheme given in that lemma. Furthermore, any suitable version of procedure *GenIneq*, such as *GenIneq*$^\star$ or *GenIneq*$^\#$ can be employed to generate violated residual minimal valid inequalities (if any).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