THE PRINCIPLES OF MATHEMATICS

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CHAPTER LIV.

MOTION.

442. Much has been written concerning the laws of motion, the possibility of dispensing with Causality in Dynamics, the relativity of motion, and other kindred questions. But there are several preliminary questions, of great difficulty and importance, concerning which little has been said. Yet these questions, speaking logically, must be settled before the more complex problems usually discussed can be attacked with any hope of success. Most of the relevant modern philosophical literature will illustrate the truth of these remarks: the theories suggested usually repose on a common dogmatic basis, and can be easily seen to be unsatisfactory. So long as an author confines himself to demolishing his opponents, he is irrefutable; when he constructs his own theory, he exposes himself, as a rule, to a similar demolition by the next author. Under these circumstances, we must seek some different path, whose by-ways remain unexplained. "Back to Newton" is the watchword of reform in this matter. Newton's scholium to the definitions contains arguments which are unrefuted, and so far as I know, irrefutable: they have been before the world two hundred years, and it is time they were refuted or accepted. Being unequal to the former, I have adopted the latter alternative.

The concept of motion is logically subsequent to that of occupying a place at a time, and also to that of change. Motion is the occupation, by one entity, of a continuous series of places at a continuous series of times. Change is the difference, in respect of truth or falsehood, between a proposition concerning an entity and a time T and a proposition concerning the same entity and another time T', provided that the two propositions differ only by the fact that T occurs in the one where T' occurs in the other. Change is continuous when the propositions of the above kind form a continuous series correlated with a continuous series of moments. Change thus always involves (1) a fixed entity, (2) a three-cornered relation between this entity, another entity, and some but not all, of the moments of time. This is its bare minimum. Mere existence at some but not all moments constitutes change on this definition. Consider pleasure, for example. This, we know, exists at some moments, and we may suppose that there are moments when it does not exist. Thus there is a relation between pleasure, existence, and some moments, which does not subsist between pleasure, existence, and other moments. According to the definition, therefore, pleasure changes in passing from existence to non-existence or *vice versâ*. This shows that the definition requires emendation, if it is to accord with usage. Usage does not permit us to speak of change except where what changes is an existent throughout, or is at least a class-concept one of whose particulars always exists. Thus we should say, in the case of pleasure, that my mind is what changes when the pleasure ceases to exist. On the other hand, if my pleasure is of different magnitudes at different times, we should say the pleasure changes its amount, though we agreed in Part III that not pleasure, but only particular amounts of pleasure, are capable of existence. Similarly we should say that colour changes, meaning that there are different colours at different times in some connection; though not colour, but only particular shades of colour, can exist. And generally, where both the class-concept and the particulars are simple, usage would allow us to say, if a series of particulars exists at a continuous series of times, that the class-concept changes. Indeed it seems better to regard this as the only kind of change, and to regard as unchanging a term which itself exists throughout a given period of time. But if we are to do this, we must say that wholes consisting of existent parts do not exist, or else that a whole cannot preserve its identity if any of its parts be changed. The latter is the correct alternative, but some subtlety is required to maintain it. Thus people say they change their minds: they say that the mind changes when pleasure ceases to exist in it. If this expression is to be correct, the mind must not be the sum of its constituents. For if it were the sum of *all* its constituents throughout time, it would be evidently unchanging; if it were the sum of its constituents at one time, it would lose its identity as soon as a former constituent ceased to exist or a new one began to exist. Thus if the mind is anything, and if it can change, it must be something persistent and constant, to which all constituents of a psychical state have one and the same relation. Personal identity could be constituted by the persistence of this term, to which all a person's states (and nothing else) would have a fixed relation. The change of mind would then consist merely in the fact that these states are not the same at all times.

Thus we may say that a term changes, when it has a fixed relation to a collection of other terms, each of which exists at some part of time, while all do not exist at exactly the same series of moments. Can we say, with this definition, that the universe changes? The universe is a somewhat ambiguous term: it may mean all the things that exist at a single moment, or all the things that ever have existed or will exist, or the common quality of whatever exists. In the two former senses it cannot change: in the last, if it be other than existence, it can change. Existence itself would not be held to change, though different terms exist at different times; for existence is involved in the notion of change as commonly employed, which applies only in virtue of the difference between the things that exist at different times. On the whole, then, we shall keep nearest to usage if we say that the fixed relation, mentioned at the beginning of this paragraph, must be that of a simple class-concept to simple particulars contained under it.

443. The notion of change has been much obscured by the doctrine of substance, by the distinction between a thing's nature and its external relations, and by the pre-eminence of subject-predicate propositions. It has been supposed that a thing could, in some way, be different and yet the same: that though predicates define a thing, yet it may have different predicates at different times. Hence the distinction of the essential and the accidental, and a number of other useless distinctions, which were (I hope) employed precisely and consciously by the scholastics, but are used vaguely and unconsciously by the moderns. Change, in this metaphysical sense, I do not at all admit. The so-called predicates of a term are mostly derived from relations to other terms; change is due, ultimately, to the fact that many terms have relations to some parts of time which they do not have to others. But every term is eternal, timeless, and immutable; the relations it may have to parts of time are equally immutable. It is merely the fact that different terms are related to different times that makes the difference between what exists at one time and what exists at another. And though a term may cease to exist, it cannot cease to be; it is still an entity, which can be counted as *one*, and concerning which some propositions are true and others false.

444. Thus the important point is the relation of terms to the times they occupy, and to existence. Can a term occupy a time without existing? At first sight, one is tempted to say that it can. It is hard to deny that Waverley's adventures occupied the time of the '45, or that the stories in the 1,001 Nights occupy the period of Harun al Raschid. I should not say, with Mr Bradley, that these times are not parts of real time; on the contrary, I should give them a definite position in the Christian Era. But I should say that the *events* are not real, in the sense that they never existed. Nevertheless, when a term exists at a time, there is an ultimate triangular relation, not reducible to a combination of separate relations to existence and the time respectively. This may be shown as follows. If "A exists now" can be analyzed into "A is now" and "A exists," where *exists* is used without any tense, we shall have to hold that "A is then" is logically possible even if A did not exist then; for if occupation of a time be separable from existence, a term may occupy a time at which it does not exist, even if there are other times when it does exist. But, on the theory in question, "A is then" and "A exists" constitute the very meaning of "A existed then," and therefore, when these two propositions are true, A must have existed then. This can only be avoided by denying the possibility of analyzing "A exists now" into a combination of two-term relations; and hence non-existential occupation of a time, if possible at all, is radically different from the existential kind of occupation.

It should be observed, however, that the above discussion has a merely philosophical interest, and is strictly irrelevant to our theme. For existence, being a constant term, need not be mentioned, from a mathematical point of view, in defining the moments occupied by a term. From the mathematical point of view, change arises from the fact that there are propositional functions which are true of some but not all moments of time, and if these involve existence, that is a further point with which mathematics as such need not concern itself.

445. Before applying these remarks to motion, we must examine the difficult idea of occupying a place at a time. Here again we seem to have an irreducible triangular relation. If there is to be motion, we must not analyze the relation into occupation of a place and occupation of a time. For a moving particle occupies many places, and the essence of motion lies in the fact that they are occupied at different times. If "A is here now" were analyzable into "A is here" and "A is now," it would follow that "A is there then" is analyzable into "A is there" and "A is then." If all these propositions were independent, we could combine them differently: we could, from "A is now" and "A is there," infer "A is there now," which we know to be false, if A is a material point. The suggested analysis is therefore inadmissible. If we are determined to avoid a relation of three terms, we may reduce "A is here now" to "A's occupation of this place is now." Thus we have a relation between *this time* and a complex concept, A's occupation of this place. But this seems merely to substitute another equivalent proposition for the one which it professes to explain. But mathematically, the whole requisite conclusion is that, in relation to a given term which occupies a place, there is a correlation between a place and a time.

446. We can now consider the nature of motion, which need not, I think, cause any great difficulty. A simple unit of matter, we agreed, can only occupy one place at one time. Thus if A be a material point, "A is here now" excludes "A is there now," but not "A is here then." Thus any given moment has a unique relation, not direct, but *via* A, to a single place, whose occupation by A is at the given moment; but there need not be a unique relation of a given place to a given time, since the occupation of the place may fill several times. A moment such that an interval containing the given moment otherwise than as an end-point can be assigned, at any moment within which interval A is in the same place, is a moment when A is at rest. A moment when this cannot be done is a moment when A is in motion, provided A occupies *some* place at neighbouring moments on either side. A moment when there are such intervals, but all have the said moment as an end-term, is one of transition from rest to motion or *vice versâ*. Motion consists in the fact that, by the occupation of a place at a time, a correlation is established between places and times; when different times, throughout any period however short, are correlated with different places, there is motion; when different times, throughout some period however short, are all correlated with the same place, there is rest.

We may now proceed to state our doctrine of motion in abstract logical terms, remembering that material particles are replaced by many-one relations of all times to some places, or of all terms of a continuous one-dimensional series t to some terms of a continuous three-dimensional series s. Motion consists broadly in the correlation of different terms of t with different terms of s. A relation R which has a single term of s for its converse domain corresponds to a material particle which is at rest throughout all time. A relation R which correlates all the terms of t in a certain interval with a single term of s corresponds to a material particle which is at rest throughout the interval, with the possible exclusion of its end-terms (if any), which may be terms of transition between rest and motion. A time of momentary rest is given by any term for which the differential coefficient of the motion is zero. The motion is continuous if the correlating relation R defines a continuous function. It is to be taken as part of the definition of motion that it is continuous, and that further it has first and second differential coefficients. This is an entirely new assumption, having no kind of necessity, but serving merely the purpose of giving a subject akin to rational Dynamics.

447. It is to be observed that, in consequence of the denial of the infinitesimal, and in consequence of the allied purely technical view of the derivative of a function, we must entirely reject the notion of a *state* of motion. Motion consists *merely* in the occupation of different places at different times, subject to continuity as explained in Part V. There is no transition from place to place, no consecutive moment or consecutive position, no such thing as velocity except in the sense of a real number which is the limit of a certain set of quotients. The rejection of velocity and acceleration as physical facts (*i.e.* as properties belonging *at each instant* to a moving point, and not merely real numbers expressing limits of certain ratios) involves, as we shall see, some difficulties in the statement of the laws of motion; but the reform introduced by Weierstrass in the infinitesimal calculus has rendered this rejection imperative.

CHAPTER LV.

CAUSALITY.

448. A GREAT controversy has existed in recent times, among those who are interested in the principles of Dynamics, on the question whether the notion of causality occurs in the subject or not. Kirchoff* and Mach, and, in our own country, Karl Pearson, have upheld the view that Dynamics is purely descriptive, while those who adhere to the more traditional opinion maintain that it not merely registers sequences, but discovers causal connections. This controversy is discussed in a very interesting manner in Professor James Ward's *Naturalism and Agnosticism*, in which the descriptive theory is used to prove that Dynamics cannot give metaphysical truths about the real world. But I do not find, either in Professor Ward's book or elsewhere, a very clear statement of the issue between the two schools. The practical mathematical form of the question arises as regards *force*, and in this form, there can be no doubt that the descriptive school are in the right: the notion of force is one which ought not to be introduced into the principles of Dynamics. The reasons for this assertion are quite conclusive. Force is the supposed cause of acceleration: many forces are supposed to concur in producing a resultant acceleration. Now an acceleration, as was pointed out at the end of the preceding chapter, is a mere mathematical fiction, a number, not a physical fact; and a component acceleration is doubly a fiction, for, like the component of any other vector sum, it is not part of the resultant, which alone could be supposed to exist. Hence a force, if it be a cause, is the cause of an effect which never takes place. But this conclusion does not suffice to show that causality never occurs in Dynamics. If the descriptive theory were strictly correct, inferences from what occurs at some times to what occurs at others would be impossible. Such inferences must involve a relation of implication between events at different times, and any such relation is in a general sense causal. What does appear to be the case is, that the only causality occurring in Dynamics requires the whole configuration of the material world as a datum, and does not yield relations of particulars to par-

* *Vorlesungen über mathematische Physik*, Leipzig, 1883, Vorrede.

ticulars, such as are usually called causal. In this respect, there is a difficulty in interpreting such seeming causation of particulars by particulars as appears, for example, in the law of gravitation. On account of this difficulty, it will be necessary to treat causation at some length, examining first the meaning to be assigned to the causation of particulars by particulars as commonly understood, then the meaning of causality which is essential to rational Dynamics, and finally the difficulty as regards component acceleration.

449. The first subject of the present chapter is the logical nature of causal propositions. In this subject there is a considerable difficulty, due to the fact that temporal succession is not a relation between events directly, but only between moments*. If two events could be successive, we could regard causation as a relation of succession holding between two events without regard to the time at which they occur. If "A precedes B" (where A and B are actual or possible temporal existents) be a true proposition, involving no reference to any actual part of time, but only to temporal succession, then we say *A causes B*. The law of causality would then consist in asserting that, among the things which actually precede a given particular existent B *now*, there is always one series of events at successive moments which would necessarily have preceded B *then*, just as well as B *now*; the temporal relations of B to the terms of this series may then be abstracted from all particular times, and asserted *per se*.

Such would have been the account of causality, if we had admitted that events can be successive. But as we have denied this, we require a different and more complicated theory. As a preliminary, let us examine some characteristics of the causal relation.

A causal relation between two events, whatever its nature may be, certainly involves no reference to constant particular parts of time. It is impossible that we should have such a proposition as "A causes B now, but not then." Such a proposition would merely mean that A *exists* now but not then, and therefore B will exist at a slightly subsequent moment, though it did not exist at a time slightly subsequent to the former time. But the causal relation itself is eternal: if A *had* existed at any other time, B would have existed at the subsequent moment. Thus "A causes B" has no reference to constant particular parts of time.

Again, neither A nor B need ever exist, though if A should exist at any moment, B must exist at a subsequent moment, and *vice versâ*. In all Dynamics (as I shall prove later) we work with causal connections; yet, except when applied to concrete cases, our terms are not existents. Their non-existence is, in fact, the mark of what is called rational Dynamics. To take another example: All deliberation and choice, all decisions as to policies, demand the validity of causal series whose terms

* See my article in *Mind*, N.S., No. 39, "Is position in time and space absolute or relative?"

do not and will not exist. For the rational choice depends upon the construction of two causal series, only one of which can be made to exist. Unless both were valid, the choice could have no foundation. The rejected series consists of equally valid causal connections, but the events connected are not to be found among existents. Thus all statesmanship, and all rational conduct of life, is based upon the method of the frivolous historical game, in which we discuss what the world would be if Cleopatra's nose had been half an inch longer.

A causal relation, we have seen, has no essential reference to existence, as to particular parts of time. But it has, none the less, some kind of connection with both. If one of its terms is among existents, so is the other: if one is non-existent, the other is also non-existent. If one of the terms is at one moment, the other is at a later or earlier moment. Thus if A causes B, we have also "A's existence implies B's" and "A's being at this moment implies B's being at a subsequent moment." These two propositions are implied by "A causes B"; the second, at least, also implies "A causes B," so that we have here a mutual implication. Whether the first also implies "A causes B," is a difficult question. Some people would hold that two moments of time, or two points of space, imply each other's existence; yet the relation between these cannot be said to be causal.

It would seem that whatever exists at any part of time has causal relations. This is not a distinguishing characteristic of what exists, since we have seen that two non-existent terms may be cause and effect. But the absence of this characteristic distinguishes terms which cannot exist from terms which might exist. Excluding space and time, we may define as a *possible* existent any term which has a causal relation to some other term. This definition excludes numbers, and all so-called abstract ideas. But it admits the entities of rational Dynamics, which might exist, though we have no reason to suppose that they do.

If we admit (what seems undeniable) that whatever occupies any given time is both a cause and an effect, we obtain a reason for either the infinity or the circularity of time, and a proof that, if there are events at any part of time, there always have been and always will be events. If, moreover, we admit that a single existent A can be isolated as the cause of another single existent B, which in turn causes C, then the world consists of as many independent causal series as there are existents at any one time. This leads to an absolute Leibnizian monadism—a view which has always been held to be paradoxical, and to indicate an error in the theory from which it springs. Let us, then, return to the meaning of causality, and endeavour to avoid the paradox of independent causal series.

450. The proposition "A causes B" is, as it stands, incomplete. The only meaning of which it seems capable is "A's existence at any time implies B's existence at some future time." It has always been customary to suppose that cause and effect must occupy consecutive moments; but as time is assumed to be a compact series, there cannot be any consecutive moments, and the interval between any two moments will always be finite. Thus in order to obtain a more complete causal proposition, we must specify the interval between A and B. A causal connection then asserts that the existence of A at any one time implies the existence of B after an interval which is independent of the particular time at which A existed. In other words, we assert: "There is an interval t such that A's existence at any time t_1 implies B's existence at a time $t_1 + t$." This requires the measurement of time, and consequently involves either temporal distance, or magnitude of divisibility, which last we agreed to regard as not a motion of pure mathematics. Thus if our measure is effected by means of distance, our proposition is capable of the generalization which is required for a purely logical statement.

451. A very difficult question remains—the question which, when the problem is precisely stated, discriminates most clearly between monism and monadism. Can the causal relation hold between particular events, or does it hold only between the whole present state of the universe and the whole subsequent state? Or can we take a middle position, and regard one group of events now as causally connected with one group at another time, but not with any other events at that other time?

I will illustrate this difficulty by the case of gravitating particles. Let there be three particles A, B, C. We say that B and C both cause accelerations in A, and we compound these two accelerations by the parallelogram law. But this composition is not truly addition, for the components are not *parts* of the resultant. The resultant is a new term, as simple as its components, and not by any means their sum. Thus the effects attributed to B and C are never produced, but a third term different from either is produced. This, we may say, is produced by B and C together, taken as one whole. But the effect which they produce as a whole can only be discovered by supposing each to produce a separate effect: if this were not supposed, it would be impossible to obtain the two accelerations whose resultant is the actual acceleration. Thus we seem to reach an antinomy: the whole has no effect except what results from the effects of the parts, but the effects of the parts are non-existent.

The examination of this difficulty will rudely shake our cherished prejudices concerning causation. The laws of motion, we shall find, actually contradict the received view, and demand a quite different and far more complicated view. In Dynamics, we shall find (1) that the causal relation holds between events at three times, not at two; (2) that the whole state of the material universe at two of the three times is necessary to the statement of a causal relation. In order to provide for this conclusion, let us re-examine causality in a less conventional spirit.

452. Causality, generally, is the principle in virtue of which, from a sufficient number of events at a sufficient number of moments, one or more events at one or more new moments can be inferred. Let us suppose, for example, that, by means of the principle, if we are given e_1 events at a time t_1, e_2 at a time $t_2, \ldots e_n$ at a time t_n, then we can infer e_{n+1} events at a time t_{n+1}. If, then, $e_{r+1} \geqq e_r$, and if the times t_r are arbitrary, except that t_{r+1} is after t_r, it follows that, from the original data, we can infer certain events at all future times. For we may choose e_1 of the events $e_2, \ldots e_n$ of the events e_{n+1}, and infer e_{n+1} events at a new time t_{n+2}. Hence by means of our supposed law, inference to future times is assured. And if, for any value of r, $e_{r+1} > e_r$, then more than e_{n+1} events can be inferred at the time t_{n+2}, since there are several ways of choosing e_r events out of e_{r+1} events. But if for any value of r, $e_{r+1} > e_r$, then inference to the past becomes in general impossible. In order that an *unambiguous* inference to the past may be possible, it is necessary that the implication should be reciprocal, *i.e.* that e_1 events at time t_1 should be implied by e_2 at $t_2 \ldots e_{n+1}$ at t_{n+1}. But some inference to the past is possible without this condition, namely, that at time t_1 there were e_1 events implying, with the others up to t_n, the e_{n+1} events at time t_{n+1}. But even this inference soon fails if, for any value of r, $e_{r+1} > e_r$, since, after inferring e_1 events at time t_1, e_r for the next inference takes the place of e_{r+1}, but is too small to allow the inference. Thus if unambiguous inference to any part of time is to be possible, it is necessary and sufficient (1) that any one of the $n+1$ groups of events should be implied by the other n groups; (2) that $e_r = e_{r+1}$ for all values of r. Since causality demands the possibility of such inference, we may take these two conditions as satisfied.

Another somewhat complicated point is the following. If $e_1\, e_2 \ldots e_n$ cause e_{n+1}, and $e_2 \ldots e_{n+1}$, cause e_{n+2} and so on, we have an independent causal series, and a return to monadism, though the monad is now complex, being at each moment a group of events. But this result is not necessary. It may happen that only certain groups $e_1\, e_2 \ldots e_n$ allow inference to e_{n+1}, and that $e_2\, e_3 \ldots e_n$, e_{n+1} is not such a group. Thus suppose $e'_1\, e'_2 \ldots e'_4$ simultaneous with $e_1 \ldots e_n$, and causing e'_{n+1}. It may be that $e_2\, e_3 \ldots e_n\, e'_{n+1}$ and $e'_2\, e'_3 \ldots e'_n\, e_{n+1}$ form the next causal groups, causing e_{n+2} and e'_{n+2} respectively. In this way no independent causal series will arise, in spite of particular causal sequences. This however remains a mere possibility, of which, so far as I know, no instance occurs.

Do the general remarks on the logical nature of causal propositions still hold good? Must we suppose the causal relation to hold directly between the *events* $e_1\, e_2 \ldots e_{n+1}$, and merely to imply their temporal succession? There are difficulties in this view. For, having recognized that consecutive times are impossible, it has become necessary to assume finite intervals of time between e_1 and e_2, e_2 and e_3 etc. Hence the length of these intervals must be specified, and thus a mere reference to events, without regard to temporal position, becomes impossible. All we can say is, that only relative position is relevant. Given a causal relation in which the times are t_r, this relation will still be valid for times $T + t_r$. Thus the ultimate statement seems to be: given m events at any moment, m other events at a moment whose distance from the first is specified, and so on till we have n groups of events, then m new events can be inferred at any new moment whose distance from the first is specified, provided m and n have suitable values, and the groups of events be suitably chosen—where, however, the values to be assigned to m and n may depend upon the nature of the events in question. For example, in a material system consisting of N particles, we shall have $m = N$, $n = 2$. Here m depends upon the nature of the material system in question. What circumstances obtain in Psychology, it is as yet impossible to say, since psychologists have failed to establish any strict causal laws.

Thus rational Dynamics assume that, in an independent material system, the configurations at any two moments imply the configuration at any other moment. This statement is capable of translation into the language of pure mathematics, as we shall see in the next chapter. But it remains a question what we are to say concerning such causation of particulars by particulars as *appears* to be involved in such principles as the law of gravitation. But this discussion must be postponed until we have examined the so-called laws of motion.