THE THIRD POSSIBILITY

NOT only in mathematics, but also in the ordinary business of life as well as in the sciences, we use logic of what is often a rigorous kind. An example is the use of what logicians call the " law of the excluded middle ". According to this, with regard to any assertion or proposition, such as " two plus three equals four " or " my hair is black ", only two mutually exclusive possibilities exist: the proposition asserted is the case or the negative of the proposition is the case. Thus, for the example cited, one must either have " two plus three equals four " or " two plus three does not equal four ", and " my hair is black " or " my hair is not black ". Most people, including most scientists and philosophers, but excluding at least one important school of philosophers of mathematics, would support the view that in *all* cases there are only two mutually exclusive possibilities, that a third possibility does not exist.

I wish to maintain that such a view is myopic and wrong. To take firstly an old example, if one asserts that so-and-so is bald, there will clearly be cases where both the proposition and its negative apply, because depending on his standards in these things, one person may legitimately describe Mr. Smith as bald, while another equally legitimately state the opposite; there will even be cases where one could legitimately maintain that neither of the two possibilities is the case: one would say " it is not certain whether Mr. Smith is bald or not bald ". The explanation of this situation usually given is that it does not exhibit the breakdown of a logical law, but merely the vagueness and unsuitability of the concept involved, that of " baldness ". It would be maintained that if we explicated the concept of baldness, by say, defining it as " having less than N hairs on the crown of the head " (N being some suitably chosen number), then there would only be two possibilities. Even if this were the case—and the matter is not quite as simple as it seems—it would merely mean that a concept has now been fabricated to which the law of the excluded middle applies; but the latter does nevertheless not apply to the concept of baldness we started with and which is the commonly used one. Similar considerations hold for concepts of the special sciences, such as " being alive " when applied to viruses in biology. Defenders of the all-or-nothing view may maintain that the law in question applies only to assertions employing concepts of a sufficient degree of precision. But what is a " sufficient degree of precision "? I doubt whether it could be defined otherwise than by a requirement that the law of the excluded middle be obeyed. If this were so, one would be left with the not very helpful knowledge that the law applies only to assertions involving concepts to which the law applies.

What about its application in mathematics, which should have an adequate degree of precision? If we look upon mathematics as a system of strict deduction, in which all theorems are ultimately derived from axioms by means of explicit rules of inference, then the natural meaning to be ascribed to the law of the excluded middle is that, for any proposition A, the only possibilities are either that A should be deducible or that not-A should be deducible. But this is clearly not the case in general. For example, generations of mathematicians failed to deduce Euclid's "parallel postulate" from his other axioms and postulates: this postulate is equivalent to the proposition that for a given straight line and a point not on it, one and only one straight line can be drawn through the point parallel to the straight line; nor did any of them deduce the negation of this proposition. The reason, we now know, is that perfectly good non-Euclidean geometries (like those of Bolyai and Lobachevsky in the nineteenth century) can be constructed with all Euclid's other axioms and postulates as well as one that contradicts the "parallel postulate". Another, even more striking and more recent, example is from elementary arithmetic, *i.e.* the arithmetic which uses only the positive and zero integers as well as their addition and multiplication. In 1931, Gödel constructed a certain proposition about the integers in such an arithmetic, for which he *proved* that neither it nor its negative was deducible by the rules of inference, if the arithmetic was consistent.

Unless, then, one ascribes some other meaning to the law of the excluded middle in mathematics—and it is difficult to see what other meaning is clear and feasible—it cannot hold in general. Some thinkers have accepted totally the implications of this view. Bearing in mind that "deducible", as ordinarily understood, means just what it says, namely, "*able* to be arrived at by a *finite* number of applications of the rules of inference", they have come to accept a rather revolutionary view of what is meant by mathematical truth. Take the example often used by the Intuitionist school of mathematical philosophy, led by Brouwer:—The number $\pi = 3{\cdot}14159\ldots$ is known to be one which when expressed as a decimal never terminates. There are perfectly explicit methods by which successive digits of the decimal expansion can be calculated as far as one cares to go. Now consider the proposition: "Somewhere in the decimal expansion of π occur the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 in succession in this order." Nobody has ever yet found this occurrence, but maybe an electronic digital computer will one day. Most mathematicians and scientists would, however, say that this proposition is either true or not true. But, nobody has yet indicated any *finite* procedure by which its truth or falsity could definitely be established. Therefore, these thinkers (and I) would say that neither of the two possibilities, "true" and "not true" applies to this case, but a third possibility which I shall non-committally refer to as "U" applies. *A tentative description of U is: no finite procedure is known by which the proposition's validity can be definitely decided.* The usual objection to this view is that the "truth value" of propositions thereby becomes subject to change with time: if tomorrow a computer comes out with the (checked) result that the ten digits following the hundred-millionth one in the decimal expansion are the ones referred to in the proposition, then the proposition will cease to be U and become true. Quite so—and the only objection to this state of affairs arises from the (platonic) presupposition that all truths must be absolute, independent of time and even of the existence of thinking beings. But there are no good grounds for this presupposition: in fact, it would be difficult to point to any truth which approximates to this recipe.

Leaving the field of mathematics now, I wish to show by a striking and interesting example how the recognition of this "third possibility" enables one to clear up simply an otherwise most paradoxical situation. A captain has decided to order his men to do a route-march on the following Wednesday, but—it being now Sunday—he tells them merely that on one of the following six week-days he will give them a *surprise* route-march; he defines a surprise route-march quite precisely, namely as one which occurs on a day such that on the morning of that day they will not be able to deduce that it is going to occur on that day. It immediately strikes all the men that it could not occur on the following Saturday, because if they reached Saturday morning without its having occurred they would know it was going to occur on that, the last possible, day and so it would not be a surprise march. But one of the men asserts that it cannot occur on *any* day of the week and proves this by the correct use of the traditional logic we normally use in the ordinary affairs of life as well as in the sciences: Firstly, it cannot occur on Friday, because by Friday morning—if it has not occurred yet—the men would know that it could not occur on Saturday, therefore that it must occur on Friday, and therefore that it would not be a surprise march. Secondly, it cannot occur on Thursday, because by a similar argument concerned with the state of affairs on Thursday morning it would then also not be a surprise march. In this way he argues back day by day and shows that Wednesday, Tuesday and Monday are also ruled out as possible days for a surprise march.

Nevertheless, we know that when the march is ordered on Wednesday it will be a surprise in the strict sense defined above; namely, the men will not know that morning that it is going to occur. The logic is therefore at fault: it has led to a wrong conclusion.

But, if the third possibility is allowed, the situation can be cleared up as follows. One may reason that if by Saturday morning no march had occurred, then either it is deducible that it will occur on that day or it is deducible that it will not occur on that day or—and this is the third possibility—there is no finite procedure by which its occurrence or non-occurrence on that day could be definitely deduced. The third possibility is the correct one, because, if the captain did not lie, the first implies that the march will not be a surprise one and the second contradicts the assertion that there would be a march during the week; if the captain did allow himself to lie then the third possibility most obviously holds also. This would be the logical situation on the morning of Saturday. Therefore, the men's view that the march could not occur on Saturday is not logically tenable, and all the argument about its possible occurrences on all the other days back to Monday collapses.

In view, however, of the novelty of this kind of logic, it remains pertinent to ask what the logical situation would be if the captain did indeed order the route-march on Saturday. In this case there is no difficulty: for in view of the third possibility having been the correct one, such a march would indeed have been a surprise one. Suppose, however, he ordered no march at all, not even on Saturday. Then the only conclusion is that the captain had lied on Sunday. It cannot be argued that such a contingency was implicitly excluded in the formulation and the "third-possibility" analysis of the problem, for if one looks carefully at the presentation and discussion, it will be seen that this is not the case.

However, an opponent of the logic here advocated may therefore propose a somewhat different situation: Instead of a captain, one has a suitable machine fitted with an output loudspeaker; it is programmed so that it will definitely select by a random process *one* of the six possible days and announce the route-march on that day. But this would be an entirely different situation: for, while the march could now be assumed to take place on some day, it can now not be assumed to be a surprise one—for it might take place on Saturday.

The reader may find it entertaining to attempt to solve a more famous paradox such as that of the Cretan liar by a similar method.

I am very grateful to Dr. D. Michie and Mr. R. S. McGowan for interesting discussions of the route-march problem.

P.S.—The route-march problem, sometimes referred to as the "prediction paradox", has been discussed from other points of view in many articles since the end of the war. Two of the most interesting are those of Quine (MIND, lxii (1953), 65), who also allows the possibility of the march occurring on the last day, and that of O'Beirne (NEW SCIENTIST, x (25 May 1961), 464), who emphasises that from the point of view of classical logic the actual truth of the captain's statements *prohibits* the men from assuming they are true.

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