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Geometrical Method for the Determination of the Position and Orientation of a Scanning Laser Doppler Vibrometer

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ABSTRACT

In this paper, a geometrical method is presented to determine the pose (position and orientation) of a Scanning Laser Doppler Vibrometer (SLDV) with respect to a structural coordinate system. Multiple registration points are used simultaneously in a least-squares sense. The structural coordinates and the corresponding scanning angles are known for each of the registration points. In the geometrical method three steps are involved in the determination of the SLDV pose. Its implementation has been tested by simulated data and experimental data. The results have shown that this method and its implementation are correct and effective. Copyright © 1996 Elsevier Science Ltd.

NOMENCLATURE

$\{\mathbf{a}\}$, $\{\mathbf{b}\}$, $\{\mathbf{c}\}$	defined in eqn (20)
a_i, b_i, c_i	difference between direction cosines of vector $\{\mathbf{d}_i\}$ and vector $\{\mathbf{d}_i'\}$
a_X, a_Y, b_X, b_Y	constants in the equation relating the input voltages to scanning angles
$D_{ij,\mathrm{L}}$	distance between points P_i and P_j measured in the laser coordinate system
$D_{ij,\mathrm{S}}$	distance between points P_i and P_j measured in the structural coordinate system
$\{\mathbf{d}_i\}$	vector from average point P_a to registration point P_i

$\{\mathbf{d}'_i\}$	a vector obtained by rotating $\{\mathbf{d}_i\}$ an angle $-\theta$ about axis $\{\mathbf{k}\}$
dl	separation distance of the reflection centres of the two mirrors
$\{\mathbf{k}\}$	equivalent axis for expressing an orientation of a coordinate system
L	third component of the scanning coordinates: ranges from the registration point to the moving laser centre
l, m, n	three direction cosines of a vector
N	number of registration points
O'	moving laser centre
$O_L X_L Y_L Z_L$	laser coordinate system
$O_S X_S Y_S Z_S$	structural coordinate system
P	arbitrary spatial point or registration point
P_a	average point of all registration points
P_i	the ith registration point
Q	sum of the squared differences in distances (defined in eqn (19))
${}^S_L[\mathbf{R}]$	rotation matrix: orientation of the laser coordinate system with respect to the structural coordinate system
SLDV	Scanning Laser Doppler Vibrometer
${}^S_L\{\mathbf{T}\}$	translation vector: position of the laser coordinate system with respect to the structural coordinate system
V_X, V_Y	input voltages to scanner controller for X and Y mirrors, respectively
x_S, y_S, z_S	structural coordinates
x_L, y_L, z_L	laser coordinates
$\varepsilon_{xP_i}, \varepsilon_{yP_i}, \varepsilon_{zP_i}$	errors in the measurements of structural coordinates of point P_i
ϕ_i	angle between the vector $\{\mathbf{d}_i\}$ and axis $\{\mathbf{k}\}$
φ_X, φ_Y	scanning angles
θ	equivalent angle for expressing an orientation of a coordinate system
θ_i	equivalent angle determined from vectors $\{\mathbf{d}_i\}$ and $\{\mathbf{d}'_i\}$
θ'_i	angle between vector $\{\mathbf{d}_i\}$ and vector $\{\mathbf{d}'_i\}$
$\theta_{X_L}, \theta_{Y_L}$	first two components of the scanning coordinates
$\{\ \}, [\ ]$	vector and matrix
$\Vert\ \Vert$	length of a vector

1 INTRODUCTION

A SLDV is a velocity transducer with scanning capability. The SLDV consists of two parts: a Michelson interferometer and a scanner.[1,2] The Michelson interferometer performs the measurement of the velocity. The scanner aims the laser beam at desired measurement points on the structures. The SLDV offers many advantages over the conventional accelerometer in dynamic measurements. It has found many applications in the research laboratory and industry for dynamic testing, modal analysis, noise control and damage identification.[3–5]

Figure 1 shows the definitions of the coordinate systems used in this paper. A Cartesian coordinate system called the structural coordinate system $O_S X_S Y_S Z_S$ (global coordinate system) is established. The points of interest on the structure are defined with respect to this coordinate system, which means the structural coordinates (x_S, y_S, z_S) of the points of interest are known. Although the structural coordinate system is placed on the plate in Fig. 1, it can be put anywhere in space. Another Cartesian coordinate system called the laser coordinate system $O_L X_L Y_L Z_L$ is also established. It is placed inside the laser head. A more detailed description about it will be given in the next section.

In our laser-based six degree-of-freedom mobility measurement system, the scanning laser vibrometer is used for the velocity measurement. The measured velocity is a vector along the line-of-sight of the laser beam. The vector direction of the velocity should be described

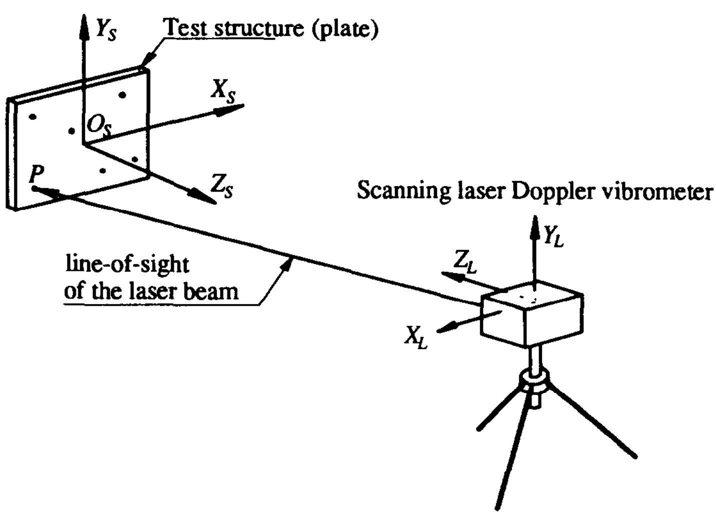

Fig. 1. A Scanning Laser Doppler Vibrometer and coordinate systems.

with respect to the structural coordinate system. The direction of the line-of-sight cannot be directly measured by the SLDV, however, this direction can be obtained once the pose of the laser coordinate system is obtained with respect to the structural coordinate system. Therefore, the objective of this paper is to develop a method to determine the SLDV pose with respect to the structural coordinate system.

Zeng *et al.*[6] developed a geometrical method to determine the pose of the SLDV. In that method, only four registration (or reference) points were required. Montgomery *et al.*[7] developed an iterative method to determine the pose of the SLDV. In their method, multiple points were used. They took the ranges of any three points as independent variables. The pose was determined by adjusting the three ranges to minimize an error function. However, the selection of the three good points becomes very important. This paper presents an improved geometrical method. Four or more registration points are simultaneously used in a least-squares sense to determine the pose. Three steps are involved in the pose determination of the SLDV. First, the ranges are determined. Second, the orientation is obtained. Third, the position is found.

2 SCANNER MODEL AND PROBLEM DEFINITION

2.1 The operation of the scanner: input voltages and scanning angles

The scanner inside the laser head of the SLDV uses two independently controlled scanning mirrors, referred to as the X mirror and Y mirror, to direct the laser beam to different spatial locations on the structure. Two voltages are input to a scanner controller to control the rotation of the X and Y mirrors, respectively. Line HH and line VV in Fig. 2 correspond to zero input voltages for the Y and X mirrors. The angle of the laser beam away from its home position (zero input voltages) is defined as the scanning angle which is proportional to the input voltage. In quadrant I, the laser beam has positive X scanning angle and positive Y scanning angle; in quadrant II, negative X and positive Y; in quadrant III, negative X and negative Y; in quadrant IV, positive X and negative Y. Quite clearly, the laser beam in any position is well defined relative to the home position.

A linear relationship exists between the scanning angle of the laser beam and the input voltage to the scanner controller for both X and Y mirrors[8]

$$\left.\begin{aligned} \varphi_X &= a_X + b_X V_X \\ \varphi_Y &= a_Y + b_Y V_Y \end{aligned}\right\} \tag{1}$$

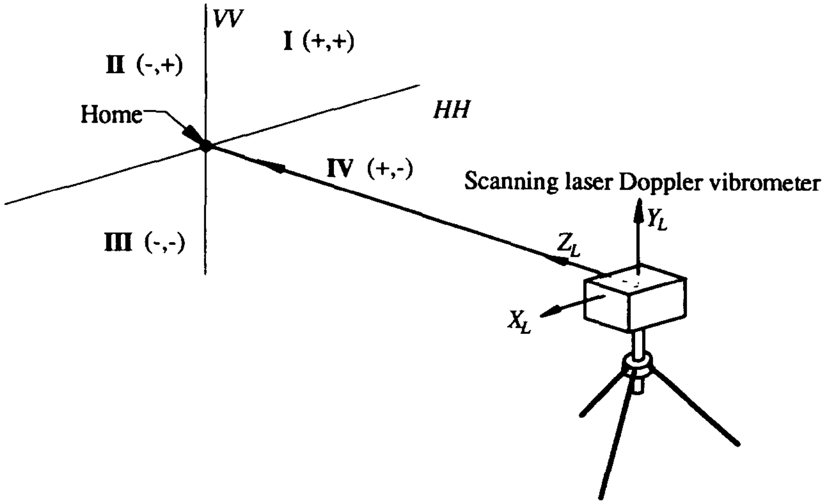

Fig. 2. The SLDV laser head and relative scanning quadrants.

where V_X and V_Y are the input voltages in volts. φ_X and φ_Y are scanning angles in degrees. a_X, b_X, a_Y, and b_Y are constants obtained from calibration.

2.2 Scanner model: scanning coordinates and laser coordinates

Figure 3 shows that the two mirrors are installed such that their rotation axes are perpendicular to each other. The two mirrors are inclined 45° relative to the vertical while in their home position. The distance between the mirror surfaces along the laser beam at the home position is called the separation distance, *dl*, of the reflection centres of the two mirrors, and is a known value.

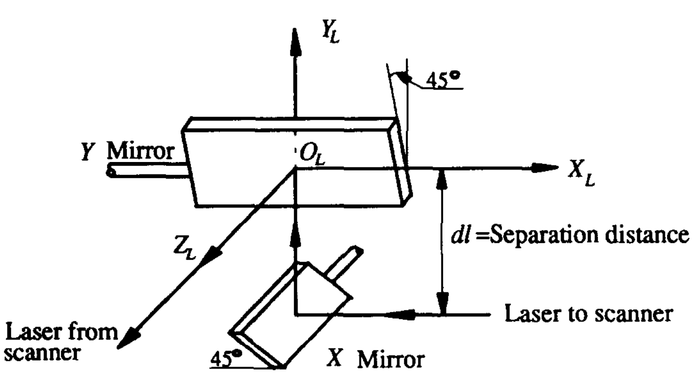

Fig. 3. The laser home position and laser coordinate system.

Since the laser beam is well defined with respect to the home position, we attach the laser coordinate system to the Y scanning mirror when the scanning mirrors are at home position. The coordinate system remains frozen in that position. In this way, the laser beam will be well defined relative to this laser coordinate system. Figure 3 shows that the origin of the laser coordinate system is on the surface of the Y mirror at the reflection point. The X_L axis is the surface and parallel to the rotation axis of the Y mirror. The Z_L axis is along the laser beam at home position. X_L, Y_L, and Z_L form a right-hand coordinate system. The surface-coated scanning mirror is mounted so that the reflective surface is away from the axis of rotation by approximately one-half (1/2) of the glass thickness. For mathematical convenience, the mirror surfaces are assumed to be on the centre of the rotation axis. As a result, the X_L axis is on the centre of the rotation axis of the Y mirror. The error caused by this assumption can be neglected.[8]

A spatial point P seen by the laser beam can be fully determined relative to the laser coordinates system in two ways. The first is through the use of the laser coordinates (x_L, y_L, z_L). The second is through the use of the scanning coordinates $(\theta_{X_L}, \theta_{Y_L}, L)$ defined later. A scanner model is defined as the relationship between the laser coordinates and the scanning coordinates. Since two mirrors are separated, the scanner model cannot be derived without further considerations.

Let us define the scanning coordinates. When the X mirror rotates, the laser beam will move to left or right. It is equivalent to say that laser beam comes from an imaginary point O' (of point O) behind the Y mirror in Fig. 4. The different horizontal positions are achieved by rotating the laser beam an angle θ_{Y_L} about an axis which passes O' and is parallel to the Y_L axis when the Y mirror is at home position. Note that angle θ_{Y_L} does not vary with the rotation of the Y mirror. When

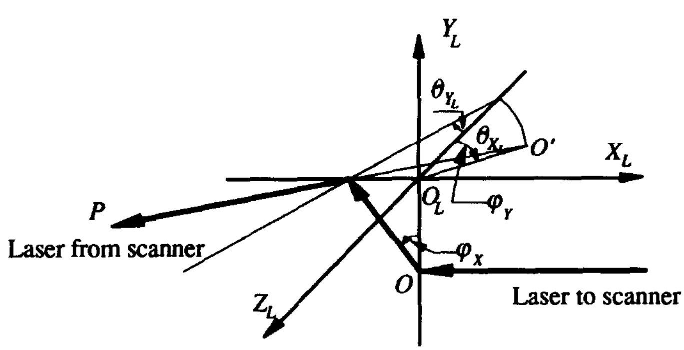

Fig. 4. Relationship between the scanning angles and the scanning coordinates.

the Y mirror rotates, the laser beam goes up and down. This motion of the mirror creates the need to define a moving laser centre, or source of the laser relative to the laser coordinate system. The variable centre is defined by O'. We can say that different vertical laser positions are obtained by rotating the laser beam an angle θ_{X_L} about the X_L axis. The range L is defined as the distance between the spatial point P and the moving laser centre O'. It is clear that the point P is fully determined by its scanning coordinates $(\theta_{X_L}, \theta_{Y_L}, L)$. To make the sign agree between the laser coordinates and the scanning coordinates, we define θ_{X_L} as positive if it moves the laser beam to a positive y_L, and vice versa. The same is true for θ_{Y_L}, which is referenced to the positive and negative x_L. Equation (2) can easily be obtained from Fig. 4. The negative sign in eqn (2) comes from the geometry shown in Fig. 4 and the right-hand rule angular sign convention

$$\left.\begin{aligned} \theta_{X_L} &= \varphi_Y \\ \theta_{Y_L} &= -\varphi_X \end{aligned}\right\} \tag{2}$$

From Fig. 4, we can define the scanner model. Equation (3) defines the transformation of the scanning coordinates to the laser coordinates for a point P

$$\left.\begin{aligned} x_{P,L} &= L_P \sin \theta_{P,Y_L} \\ y_{P,L} &= (L_P \cos \theta_{P,Y_L} - dl) \sin \theta_{P,X_L} \\ z_{P,L} &= (L_P \cos \theta_{P,Y_L} - dl) \cos \theta_{P,X_L} \end{aligned}\right\} \tag{3}$$

and the following equation transfers the laser coordinates to the scanning coordinates

$$\left.\begin{aligned} \theta_{P,X_L} &= \tan^{-1} (y_{P,L}/z_{P,L}) \\ \theta_{P,Y_L} &= \tan^{-1} [x_{P,L}/(\sqrt{y_{P,L}^2 + z_{P,L}^2} + dl)] \\ L_P &= [x_L^2 + (\sqrt{y_{P,L}^2 + z_{P,L}^2} + dl)^2]^{1/2} \end{aligned}\right\} \tag{4}$$

2.3 Coordinate transformation: the pose of the SLDV

A spatial point P can be fully determined by its structural coordinates $(x_{P,S}, y_{P,S}, z_{P,S})$. It can also be fully determined by its laser coordinates $(x_{P,L}, y_{P,L}, z_{P,L})$. The relationship between the structural coordinates and the laser coordinates is the coordinate transformation.

$$\begin{Bmatrix} x_P \\ y_P \\ z_P \end{Bmatrix}_S = {}^S_L\{\mathbf{T}\} + {}^S_L[\mathbf{R}] \begin{Bmatrix} x_P \\ y_P \\ z_P \end{Bmatrix}_L \tag{5}$$

TABLE 1
Various Coordinate Systems

Coordinate systems	*Variables*
Input voltages	V_X and V_Y
Scanning angles	φ_X and φ_Y
Scanning coordinates	θ_{X_L}, θ_{Y_L}, and L
Laser coordinates	x_L, y_L, and z_L
Structural coordinates	x_S, y_S, and z_S

where ${}^{S}_{L}\{\mathbf{T}\}$ is the translation vector. It is the coordinates of the origin O_L measured in the structural coordinate system, i.e. the position of the laser coordinate system. ${}^{S}_{L}[\mathbf{R}]$ is the rotation matrix, i.e. the orientation of the laser coordinate system. Note that the rotation matrix is an orthogonal matrix and its determinant is positive one. Thus, the rotation matrix is defined by three independent variables. The pose of the SLDV is defined as the position and orientation of the laser coordinate system, i.e. ${}^{S}_{L}\{\mathbf{T}\}$ and ${}^{S}_{L}[\mathbf{R}]$.

2.4 Summary of various coordinate systems

We have introduced various coordinate systems when we derived the relationship between the input voltages and the structural coordinate for a spatial point P. The transformations between them are eqns (1)–(5). Table 1 is a summary.

2.5 Problem definition

We have shown how to transform from the input voltages to the structural coordinate, or vice versa, for a spatial point P. After calibration, the scanner constants (parameters) a_X, b_X, a_Y, b_Y, and dl are known. Using eqn (1), we can calculate the scanning angles (φ_X, φ_Y) from the measured input voltages. By eqn (2), we are able to obtain the scanning coordinates (θ_{X_L}, θ_{Y_L}). Thus, the structural coordinates and the scanning coordinates (θ_{X_L}, θ_{Y_L}) are known for each registration point. We will use them to determine the ranges and the pose of the SLDV.

Equation (5) has three equations. It contains seven unknowns: six unknowns in the pose of the SLDV plus an unknown range. It can be seen that three points will give us a determined equation set, i.e. nine equations and nine unknowns. The reason for that conclusion is that each additional point will introduce an unknown range. The solution of the fully-determined equation set is not unique because of the nonlinearity. Therefore, four or more registration points are required to find a unique solution.

On the basis of above discussions, we can define our problem as: given N ($N \geq 4$) registration points whose structural coordinates (x_S, y_S, z_S) are known and whose scanning angles (φ_X, φ_Y) are measured, estimate the position and orientation of the SLDV.

3 GEOMETRICAL METHOD FOR THE POSE DETERMINATION

3.1 Determination of the ranges

The ranges can be determined without the knowledge of the pose of the SLDV since the distance between any two registration points remains constant whether it is measured in the structural coordinate system or it is measured in the laser coordinate system. For any two points P_i and P_j, the distance between them measured in the structural coordinate system is found to be

$$D_{ij,S} = [(x_{P_j,S} - x_{P_i,S})^2 + (y_{P_j,S} - y_{P_i,S})^2 + (z_{P_j,S} - z_{P_i,S})^2]^{1/2} \tag{6}$$

The same thing measured in the laser coordinate system is found to be

$$\begin{aligned} D_{ij,L} = \{ & (L_{P_j} \sin \theta_{P_j,Y_L} - L_{P_i} \sin \theta_{P_i,Y_L})^2 \\ & + [(L_{P_j} \cos \theta_{P_j,Y_L} - dl) \sin \theta_{P_j,X_L} - (L_{P_i} \cos \theta_{P_i,Y_L} - dl) \sin \theta_{P_i,X_L}]^2 \\ & + [(L_{P_j} \cos \theta_{P_j,Y_L} - dl) \cos \theta_{P_j,X_L} - (L_{P_i} \cos \theta_{P_i,Y_L} - dl) \cos \theta_{P_i,X_L}]^2\}^{1/2} \end{aligned} \tag{7}$$

Equation (7) is obtained by replacing the structural coordinates in eqn (6) with the laser coordinates. Since the distance remains the same, one has

$$D_{ij,S} = D_{ij,L} \qquad (\text{for } i = 1, 2, \ldots, N-1, j = i+1, \ldots, N) \tag{8}$$

The left-hand side of eqn (8) is known since the structural coordinates are known. The right-hand side contains two unknown ranges L_{P_i} and L_{P_j}. For N registration points, there will be $N(N-1)/2$ distances. Therefore, there will be $N(N-1)/2$ equations in eqn (8) for N unknown ranges. An overdetermined nonlinear equation set will be obtained if N is equal to or larger than 4. The least-squares method can be applied to eqn (8) to yield a unique solution when constrained by the following

Find L_{P_i} $(i = 1, 2, \ldots, N)$ that satisfy $L_{P_i} > 0$ and

$$\text{minimize } Q = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (D_{ij,\mathrm{S}} - D_{ij,\mathrm{L}})^2 \quad (9)$$

This model may have multiple local minima. An algorithm[9] has been developed to obtain an initial solution of the ranges for all registration points. Using these initial ranges, a global minimum has been obtained for all the tested cases.

3.2 Determination of the rotation matrix

After the range L is found for a registration point, its laser coordinates can be obtained by using eqn (3). Thus, the structural coordinates $(x_\mathrm{S}, y_\mathrm{S}, z_\mathrm{S})$ and the laser coordinates $(x_\mathrm{L}, y_\mathrm{L}, z_\mathrm{L})$ are available for each of the registration points. The relationship between them is given in eqn (5). Considering the measurement error, one can obtain the following equation for point P_i.

$$\begin{Bmatrix} x_{P_i} \\ y_{P_i} \\ z_{P_i} \end{Bmatrix}_\mathrm{S} = {}^\mathrm{S}_\mathrm{L}\{\mathbf{T}\} + {}^\mathrm{S}_\mathrm{L}[\mathbf{R}] \begin{Bmatrix} x_{P_i} \\ y_{P_i} \\ z_{P_i} \end{Bmatrix}_\mathrm{L} = \begin{Bmatrix} \varepsilon_{xP_i} \\ \varepsilon_{yP_i} \\ \varepsilon_{zP_i} \end{Bmatrix}_\mathrm{S} \quad (10)$$

Haralick *et al.*[10] determined the origin coordinates and rotation matrix from eqn (10) by using a weighted least-squares method. However, their method has two problems. First, when all the points are on the same plane, no unique solution can be found. Second, their method cannot insure that the determinant of the computed rotation matrix will be a positive one. The method developed in this paper will not have these problems. A unique solution can always be found and the determinant of the extracted rotation matrix is always positive.

3.2.1 *Elimination of the origin coordinate as unknowns*

The idea of eliminating the origin coordinates as unknowns is equivalent to translationally moving the two coordinate systems to a common point. The common point can be any registration point, or the average registration point. As shown by Haralick *et al.*,[10] moving to the average point will result in a minimum of total error. The average point P_a of the N registration points can be found as

$$\begin{Bmatrix} x_{P_a} \\ y_{P_a} \\ z_{P_a} \end{Bmatrix}_S = \frac{1}{N}\sum_{i=1}^{N} \begin{Bmatrix} x_{P_i} \\ y_{P_i} \\ z_{P_i} \end{Bmatrix}_S, \quad \begin{Bmatrix} x_{P_a} \\ y_{P_a} \\ z_{P_a} \end{Bmatrix}_L = \frac{1}{N}\sum_{i=1}^{N} \begin{Bmatrix} x_{P_i} \\ y_{P_i} \\ z_{P_i} \end{Bmatrix}_L, \tag{11}$$

The coordinate transformation between the two-coordinate system should hold for the average point. Thus

$$\begin{Bmatrix} x_{P_a} \\ y_{P_a} \\ z_{P_a} \end{Bmatrix}_S = {}^S_L\{\mathbf{T}\} + {}^S_L[\mathbf{R}] \begin{Bmatrix} x_{P_a} \\ y_{P_a} \\ z_{P_a} \end{Bmatrix}_L \tag{12}$$

Subtracting both sides of eqn (12) from eqn (10), the following equation can be obtained

$$\begin{Bmatrix} x_{P_i} - x_{P_a} \\ y_{P_i} - y_{P_a} \\ z_{P_i} - z_{P_a} \end{Bmatrix}_S = {}^S_L[\mathbf{R}] \begin{Bmatrix} x_{P_i} - x_{P_a} \\ y_{P_i} - y_{P_a} \\ z_{P_i} - z_{P_a} \end{Bmatrix}_L + \begin{Bmatrix} \varepsilon_{xP_i} \\ \varepsilon_{yP_i} \\ \varepsilon_{zP_i} \end{Bmatrix}_S \tag{13}$$

Defining a new vector $\{\mathbf{d}_i\}$ which is from average point P_a to registration point P_i, eqn (13) can be written as

$$\{\mathbf{d}_i\}_S = {}^S_L[\mathbf{R}]\{\mathbf{d}_i\}_L + (\varepsilon_{xP_i}\varepsilon_{yP_i}\varepsilon_{zP_i})^T_S \tag{14}$$

In fact, $\{\mathbf{d}_i\}_S$ and $\{\mathbf{d}_i\}_L$ present the same vector $\{\mathbf{d}_i\}$ but they are measured in different coordinate systems. $\{\mathbf{d}_i\}_S$ is the measurement of vector $\{\mathbf{d}_i\}$ in the structural coordinate system while $\{\mathbf{d}_i\}_L$ is the measurement of vector $\{\mathbf{d}_i\}$ in the laser coordinate system. The geometrical meaning of eqn (14) is that the two frames now have the same origin at P_a and the relative orientation of the two coordinate systems remains unchanged. In this way, the two coordinate systems are translationally moved to the average point and the origin coordinates are eliminated. Thus, the only unknown is the rotation matrix.

The rotation matrix can be found by directly applying the least-squares method to minimize the total error, as shown by Stephens.[11] However, this method has the same problem as the one developed by Haralick *et al.*[10] A geometrical method is sought here to get over those two problems. The rotation matrix is expressed as a function of the equivalent angle θ and equivalent axis $\{\mathbf{k}\}$ by the following equation[12]

$${}^S_L[\mathbf{R}] = \begin{bmatrix} l_k l_k v\theta + \cos\theta & l_k m_k v\theta - n_k \sin\theta & l_k n_k v\theta + m_k \sin\theta \\ l_k m_k v\theta + n_k \sin\theta & m_k m_k v\theta + \cos\theta & m_k n_k v\theta - l_k \sin\theta \\ l_k n_k v\theta - m_k \sin\theta & m_k n_k v\theta + l_k \sin\theta & n_k n_k v\theta + \cos\theta \end{bmatrix} \tag{15}$$

where $v\theta = 1 - \cos\theta$ and $\{l_{\mathbf{k}}, m_{\mathbf{k}}, n_{\mathbf{k}}\}$ are the direction cosines of axis $\{\mathbf{k}\}$. Our goal is to determine θ and $\{\mathbf{k}\}$ geometrically.

3.2.2 *The rotation matrix from equivalent angle and equivalent axis*

The equivalent angle and equivalent axis can be determined by using some geometrical relationship and the equivalence between the frame rotation and vector rotation.[10] From the last section, we know that the coordinates of vector $\{\mathbf{d}_i\}$ are $\{\mathbf{d}_i\}_S$ in $O_S X_S Y_S Z_S$ and $\{\mathbf{d}_i\}_L$ in $O_L X_L Y_L Z_L$. Vector $\{\mathbf{d}_i'\}$ is obtained by rotating vector $\{\mathbf{d}_i\}$ about axis $\{\mathbf{k}\}$ by an angle $-\theta$. By the equivalent relationship of frame rotation and vector rotation, the coordinates of vector $\{\mathbf{d}_i'\}$ in $O_S X_S Y_S Z_S$ are equal to $\{\mathbf{d}_i\}_L$. Using $\{\mathbf{d}_i\}$ and $\{\mathbf{d}_i'\}$ (for $i = 1, 2, \ldots, N$), the $\{\mathbf{k}\}$ and θ can be found.

During the rotation, the angle between the vector $\{\mathbf{d}_i\}$ and axis $\{\mathbf{k}\}$ does not change. Therefore, the angle between $\{\mathbf{k}\}$ and $\{\mathbf{d}_i\}$ is equal to the angle between $\{\mathbf{k}\}$ and $\{\mathbf{d}_i'\}$, which leads to

$$l_{\mathbf{k}} l_{\mathbf{d}_i} + m_{\mathbf{k}} m_{\mathbf{d}_i} + n_{\mathbf{k}} n_{\mathbf{d}_i} = l_{\mathbf{k}} l_{\mathbf{d}_i'} + m_{\mathbf{k}} m_{\mathbf{d}_i'} + n_{\mathbf{k}} n_{\mathbf{d}_i'} \tag{16}$$

where $(l_{\mathbf{d}_i}, m_{\mathbf{d}_i}, n_{\mathbf{d}_i})$ is the direction cosine of vector $\{\mathbf{d}_i\}$ and $(l_{\mathbf{d}_i'}, m_{\mathbf{d}_i'}, n_{\mathbf{d}_i'})$ is the direction cosine of vector $\{\mathbf{d}_i'\}$. Note that those direction cosines are relative to the structural coordinate system. Equation (16) can be written as

$$a_i l_{\mathbf{k}} + b_i m_{\mathbf{k}} + c_i n_{\mathbf{k}} = 0 \qquad (\text{for } i = 1, 2, \ldots, N) \tag{17}$$

where $a_i = l_{\mathbf{d}_i} - l_{\mathbf{d}_i'}$, $b_i = m_{\mathbf{d}_i} - m_{\mathbf{d}_i'}$, and $c_i = n_{\mathbf{d}_i} - n_{\mathbf{d}_i'}$. There is a constraint for the direction cosine of axis $\{\mathbf{k}\}$, i.e.

$$l_{\mathbf{k}}^2 + m_{\mathbf{k}}^2 + n_{\mathbf{k}}^2 = 1 \tag{18}$$

Equation (17) can be rewritten in matrix form

$$[\{\mathbf{a}\}\{\mathbf{b}\}\{\mathbf{c}\}]\{\mathbf{k}\} = \{0\} \tag{19}$$

where

$$\left.\begin{aligned} \{\mathbf{a}\} &= \{a_1, a_2, \ldots, a_N\}^{\mathrm{T}} \\ \{\mathbf{b}\} &= \{b_1, b_2, \ldots, b_N\}^{\mathrm{T}} \\ \{\mathbf{c}\} &= \{c_1, c_2, \ldots, c_N\}^{\mathrm{T}} \\ \{\mathbf{k}\} &= \{l_{\mathbf{k}}, m_{\mathbf{k}}, n_{\mathbf{k}}\}^{\mathrm{T}} \end{aligned}\right\} \tag{20}$$

Equations (18) and (19) are the desired equations for the determination of axis $\{\mathbf{k}\}$. If N is equal to or larger than 4, eqn (19) will be overdetermined. For a linearly overdetermined equation, the least-squares method can be used to find the solution. Using the least-squares method, two components of $\{\mathbf{k}\}$ can be found as a function of the third component from eqn (19). Then putting the two components into eqn

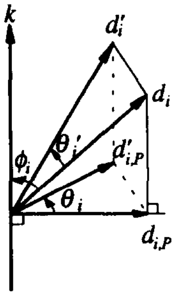

Fig. 5. The determination of rotation angle.

(18), the third component of {**k**} can be solved. The direction of {**k**} found in this way may or may not be correct. However, this uncertainty is not a problem for obtaining the correct rotation matrix. The reason will be given once the rotated angle θ is obtained.

Once the axis {**k**} is obtained, θ can be found. Since θ is measured in a plane perpendicular to axis {**k**}, the angle between {$\mathbf{d}_i$} and {$\mathbf{d}'_i$} is not the required information. The idea is to project {$\mathbf{d}_i$} and {$\mathbf{d}'_i$} onto a plane perpendicular to {**k**} and then find the angle between the projections of {$\mathbf{d}_i$} and {$\mathbf{d}'_i$}. Figure 5 shows the geometry used to determine the rotated angle.

Angle θ'_i between {$\mathbf{d}_i$} and {$\mathbf{d}'_i$} is found to be

$$\cos\theta'_i = l_{\mathbf{d}_i} l_{\mathbf{d}'_i} + m_{\mathbf{d}_i} m_{\mathbf{d}'_i} + n_{\mathbf{d}_i} n_{\mathbf{d}'_i}. \tag{21}$$

Angle ϕ_i between {$\mathbf{d}_i$} and {**k**} is

$$\cos\phi_i = l_{\mathbf{d}_i} l_{\mathbf{k}} + m_{\mathbf{d}_i} m_{\mathbf{k}} + n_{\mathbf{d}_i} n_{\mathbf{k}}. \tag{22}$$

The distance D between the tips of {$\mathbf{d}_i$} and {$\mathbf{d}'_i$} is found by the cosine law.

$$D^2 = \|\{\mathbf{d}_i\}\|^2 + \|\{\mathbf{d}_i\}\|^2 + 2\|\{\mathbf{d}_i\}\|\ \|\{\mathbf{d}_i\}\| \cos\theta'_i \tag{23}$$

The distance D_P between the tips of the $d_{i,P}$ and $d'_{i,P}$ is

$$D_P^2 = \|\{\mathbf{d}_i\}\|^2 \sin^2\phi_i + \|\{\mathbf{d}_i\}\|^2 \sin^2\phi_i - 2\,\|\{\mathbf{d}_i\}\|^2 \sin^2\phi_i \cos\theta_i \tag{24}$$

where $\sin^2\phi_i = 1 - \cos^2\phi_i$. Since the projection does not change the distance, $D = D_P$ gives

$$2\|\{\mathbf{d}_i\}\|^2 (1 - \cos\theta'_i) = 2\,\|\{\mathbf{d}_i\}\|^2 \sin^2\phi_i (1 - \cos\theta_i). \tag{25}$$

From the above equation, one finds the angle θ_i

$$\cos\theta_i = 1 - \frac{1 - \cos\theta'_i}{\sin^2\phi_i} \tag{26}$$

With N registration points, N rotated angles can be found. The estimation of angle θ is to average all the obtained angles

$$\theta = \frac{1}{N} \sum_{i=1}^{N} \theta_i \tag{27}$$

where θ_i is given by eqn (26). The sign of θ_i may be wrong because of the uncertainty of the axis $\{\mathbf{k}\}$'s direction. However, the correct rotation matrix ${}^{\mathrm{S}}_{\mathrm{L}}[\mathbf{R}]$ can be constructed with the derived $\{\mathbf{k}\}$ and θ. Note that the rotation is made according to the right-hand rule. If the derived axis has the correct direction, putting $\{\mathbf{k}\}$ and $-\theta$ into eqn (15) will result in the correct answer. If the derived $\{\mathbf{k}\}$ has the wrong direction, putting $\{\mathbf{k}\}$ and θ into eqn (15) will yield a correct rotation matrix. According to this, what one needs to do to get the correct rotation matrix is to form two rotation matrices. One called ${}^{\mathrm{S}}_{\mathrm{L}}[\mathbf{R}]^{+}$ is obtained by using the derived $\{\mathbf{k}\}$ and positive θ. The other called ${}^{\mathrm{S}}_{\mathrm{L}}[\mathbf{R}]^{-}$ is obtained by using the derived $\{\mathbf{k}\}$ and $-\theta$. One of them must be the correct rotation matrix. To select the correct rotation matrix, it must satisfy eqn (14).

3.3 Determination of the origin coordinate

Once the rotation matrix is obtained, the origin coordinates can be found by using eqn (12)

$${}^{\mathrm{S}}_{\mathrm{L}}[\mathbf{T}\} = \begin{Bmatrix} x_{P_a} \\ y_{P_a} \\ z_{P_a} \end{Bmatrix}_{\mathrm{S}} - {}^{\mathrm{S}}_{\mathrm{L}}[\mathbf{R}] \begin{Bmatrix} x_{P_a} \\ y_{P_a} \\ z_{P_a} \end{Bmatrix}_{\mathrm{L}} \tag{28}$$

As shown by Haralick *et al.*,[10] this is the least-squares solution of eqn (10).

4 TESTING OF THE GEOMETRICAL METHOD AND THE RESULTS

The presented geometrical method has been implemented and incorporated into a software system. A number of simulated and experimental data have been used to test the algorithm and its implementation. Two examples are given below. One is for simulated data. The other is for experimental data. For both examples, 9 registration points were used.

4.1 Simulated data

Table 2 shows the results. The assumed scanning angles (φ_X, φ_Y) are converted to scanning coordinate (θ_{X_L}, θ_{Y_L}) by eqn (2). Then (θ_{X_L}, θ_{Y_L}) plus the assumed range ($L = 2500$ mm for all points) are transferred to the laser coordinates by eqn (3) (assume $dl = 46$ mm). The pose of the SLDV is assumed to be the following.

$$
\begin{aligned}
{}^{S}_{L}\{\mathbf{T}\} &= \begin{Bmatrix} 5{\cdot}000\,000 \\ 8{\cdot}000\,000 \\ 10{\cdot}000\,000 \end{Bmatrix} (\times 10^3\ \text{mm}) \\
{}^{S}_{L}[\mathbf{R}] &= \begin{bmatrix} 0{\cdot}171\,010 & -0{\cdot}613\,092 & -0{\cdot}771\,281 \\ 0{\cdot}296\,198 & -0{\cdot}714\,610 & 0{\cdot}633\,718 \\ -0{\cdot}939\,693 & -0{\cdot}336\,824 & 0{\cdot}059\,391 \end{bmatrix}
\end{aligned} \tag{29}
$$

The assumed pose is used to obtain the (x_S, y_S, z_S) in Table 2. Using the information in Table 2 from column 2 to column 6, the algorithm finds the pose as following

$$
\begin{aligned}
{}^{S}_{L}[\mathbf{T}\} &= \begin{Bmatrix} 5{\cdot}000\,000 \\ 8{\cdot}000\,000 \\ 10{\cdot}000\,000 \end{Bmatrix} (\times 10^3\ \text{mm}) \\
{}^{S}_{L}[\mathbf{R}] &= \begin{bmatrix} 0.171\,010 & -0{\cdot}613\,092 & -0{\cdot}771\,281 \\ 0{\cdot}296\,198 & -0.714\,610 & 0{\cdot}633\,718 \\ -0{\cdot}939\,693 & -0{\cdot}336\,824 & 0{\cdot}059\,391 \end{bmatrix}
\end{aligned} \tag{30}
$$

The last two columns in Table 2 are the residuals of the scanning angles. The residual is defined as the difference between the measured

TABLE 2

Results of Simulated Data

Point	φ_X (deg)	φ_Y (deg)	x_S (mm)*	y_S (mm)*	z_S (mm)*	ε_{φ_X} (deg)	ε_{φ_Y} (deg)
0	−12·5000 0	12·500 0	2·971 523	9·271 496	9·455 807	0·000 000	0·000 000
4	12·500 0	12·500 0	2·786 456	8·950 951	10·472 740	0·000 000	0·000 000
20	−12·500 0	−12·500 0	3·607 074	10·012 285	9·804 970	0·000 000	0·000 000
24	12·500 0	−12·500 0	3·422 008	9·691 740	10·821 903	0·000 000	0·000 000
5	−12·500 0	6·250 0	3·096 657	9·428 539	9·545 102	0·000 000	0·000 000
18	6·250 0	−6·250 0	3·246 178	9·645 684	10·489 197	0·000 000	0·000 000
22	0·000 0	−12·5000 0	3·477 782	9·897 842	10·321 193	0·000 000	0·000 000
9	12·500 0	6·250 0	2·911 590	9·161 994	10·562 035	0·000 000	0·000 000
11	−6·250 00	0·000 0	3·165 281	9·626 344	9·889 110	0·000 000	0·000 000

* $\times 10^3$.

scanning angles and the fitted scanning angles. The fitted scanning angles are obtained from the structural coordinates by using the extracted pose. Equations (2), (4) and (5) can be used to obtain those angles.

4.2 Experimental data

Table 3 shows the results of experimental data. Column 2 to column 6 are measured data. The scanning angles are obtained by using eqn (1)

TABLE 3
Results of Experimental Data

Point	φ_X (deg)	φ_Y (deg)	x_S (mm)	y_S (mm)	z_S (mm)	$\varepsilon_{\varphi X}$ (deg)	$\varphi_{\varphi Y}$ (deg)
1	−0·8670	4·9036	−69·850	298·958	0·000	0·02747	0·01963
2	0·1052	5·0989	0·000	298·958	0·000	0·0279	0·0136
3	1·1008	5·2942	69·850	298·958	0·000	0·0296	0·0029
4	−0·9958	−0·1131	−69·850	0·000	0·000	−0·0124	−0·0148
5	0·0115	0·0334	0·000	0·000	0·000	0·0001	−0·0158
6	1·0305	0·1921	69·850	0·000	0·000	0·0006	−0·0081
7	−1·0895	−5·2762	−69·850	−298·704	0·000	−0·0213	0·0115
8	−0·0822	−5·2030	0·000	−298·704	0·000	−0·0248	−0·0017
9	0·9485	−5·1175	69 850	−298·958	0·000	−0·0297	−0·0000

from the measured input voltages. The separation distance dl is equal to 46 mm. This value is from the manufacturer of the SLDV. Following is the pose from the method described here.

$$
{}^{S}_{L}\{\mathbf{T}\} = \begin{Bmatrix} 1787{\cdot}540 \\ -746{\cdot}884 \\ 2586{\cdot}007 \end{Bmatrix} (\text{mm})
$$

$$
{}^{S}_{L}[\mathbf{R}] = \begin{bmatrix} -0{\cdot}8242 & 0{\cdot}1200 & -0{\cdot}5535 \\ -0{\cdot}0130 & 0{\cdot}9730 & 0{\cdot}2303 \\ 0{\cdot}5662 & 0{\cdot}1970 & -0{\cdot}8004 \end{bmatrix} \tag{31}
$$

4.3 Discussions

For the simulated data, the geometrical method is shown to be able to find the correct pose. The resulting residuals are all zeros up to six digits.

For the experimental data, the true pose is not known. By just studying the extracted pose, one cannot say anything about the method. However, we can analyse the residuals to verify the method. If the residuals are within the measurement errors, we can say that the method is correct and effective. The measurement errors in the scanning angles are included in the following list.

(1) There are errors in converting the measured input voltages into scanning angles. These errors are essentially from the calibration of scanner constants a_X, a_Y, b_X, and b_Y. The standard deviation of the residuals (in the scanning angle) from the calibration is about 0·010 degree.[9]

(2) There are errors in aiming the laser beam at a registration point. These errors are from two sources. One is the limited D–A conversion resolution. The D–A converter used in the experimental data acquisition has about 0·0125° resolution in scanning angles. The maximum error is 0·0063° (half of the resolution). The other source is the finite size of the laser beam spot. After the laser beam is focused and filtered by a very thin blue film, its size is about 1·27 mm in diameter. The centre of the spot is used to check if the laser beam is aimed at a registration point. This can introduce error.

(3) There are errors in measurements of the structural coordinates of the registration points. The resolution was 1·586 mm (1/16″) in the measurement of the structural coordinates. The distance between the laser head and the structure was about 2580 mm. Thus, the maximum resulting error in the scanning angle is about 0·0176°.

Take the maximum error for the input voltage conversion at the 3σ point, i.e. $3\sigma = 0{\cdot}0300°$. Combining this estimate with the other two maximum errors using the sum of variance procedure yields a maximum (3σ) error of about 0·0357°. The maximum absolute residuals (the last two columns in Table 3) is about 0·0297°. It is clear that the residuals are within the measurement error. Therefore, the developed geometrical method and its implementation are correct and effective.

5 CONCLUSIONS

This paper presented a geometrical method for the pose determination of a SLDV. The relationship between the input voltages and the structural coordinates has been derived. The procedure for obtaining the input voltages and obtaining the direction of line-of-sight for a spatial point has been given. The three steps for determining the pose have been derived in detail. The two problems associated with the existing methods for pose determination (after the range is obtained) are overcome by the presented method. After a number of tests with simulated and experimental data, the geometric method is shown to be correct and effective.

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