Inequalities

João Paixão and Lucas Rufino

October 24, 2020

1 Definitions

1.1 Preorder

Definition 1.1 (Reflexivity). $A \leq A$

Definition 1.2 (Transitivity). $A \leq B$ and $B \leq C \implies A \leq C$

Definition 1.3 (Isomorphic). $A = B \iff A \leq B$ and $B \leq A$

Lemma 1.1. $A = B \iff (A \leq B \impliedby TRUE \implies B \leq A)$

1.2 Yoneda

Definition 1.4 (Yoneda $\leq$). $A \leq B \iff \forall X(X \leq A \implies X \leq B)$

Lemma 1.2 (Yoneda 1 $=$). $A = B \iff \forall X(X \leq A \iff X \leq B)$

Lemma 1.3 (Yoneda 2 $=$). $A = B \iff \forall X(A \leq X \iff B \leq X)$

1.3 Initial and Terminal Object (0 and ∞)

Definition 1.5 (Initial). $0 \leq A$

Definition 1.6 (Terminal). $A \leq \infty$

1.4 Meets ($A \min B$)

Definition 1.7 (Meet). $A \leq B \min C \iff A \leq B$ and $A \leq C$

Lemma 1.4 (Zero Element). $A \min \infty = A$

Proof.

$$
\begin{aligned}
& X \leq A \min \infty \\
\iff & \quad \{ \text{ Meet } \} \\
& X \leq A \text{ and } X \leq \infty \\
\iff & \quad \{ \text{ Terminal } \}
\end{aligned}
$$

$$
\begin{array}{ll}
& X \leq A \text{ and } TRUE \\
\Longleftrightarrow & \quad \{ \text{ A and TRUE = A } \} \\
& X \leq A
\end{array}
$$

□

Lemma 1.5 (Absolute Element). $A \min 0 = 0$

Proof.

$$
\begin{array}{ll}
& 0 \leq A \min 0 \\
\Longleftrightarrow & \quad \{ \text{ Initial } \} \\
& TRUE \\
\Longleftrightarrow & \quad \{ \text{ Reflexivity } \} \\
& A \min 0 \leq A \min 0 \\
\Longleftrightarrow & \quad \{ \text{ Meet } \} \\
& A \min 0 \leq A \text{ and } A \min 0 \leq 0 \\
\Longrightarrow & \quad \{ \text{ A and B } \Longrightarrow \text{ B } \} \\
& A \min 0 \leq 0
\end{array}
$$

□

Lemma 1.6 (Associativity). $A \min(B \min C) = (A \min B) \min C$

Proof.

$$
\begin{array}{ll}
& X \leq A \min(B \min C) \\
= & \quad \{ \text{ Meet } \} \\
& X \leq A \text{ and } X \leq B \min C \\
= & \quad \{ \text{ Meet } \} \\
& X \leq A \text{ and } (X \leq B \text{ and } X \leq C) \\
= & \quad \{ \text{ Associativity of And } \} \\
& (X \leq A \text{ and } X \leq B) \text{ and } X \leq C \\
= & \quad \{ \text{ Meet } \} \\
& X \leq A \min B \text{ and } X \leq C \\
= & \quad \{ \text{ Meet } \} \\
& X \leq (A \min B) \min C
\end{array}
$$

□

Lemma 1.7 (Commutativity). $A \min B = B \min A$

Proof.

$$\begin{aligned}
& X \le A \min B \\
= & \quad \{ \text{ Meet } \} \\
& X \le A \text{ and } X \le B \\
= & \quad \{ \text{ Commutativity of And } \} \\
& X \le B \text{ and } X \le A \\
= & \quad \{ \text{ Meet } \} \\
& X \le B \min A
\end{aligned}$$

□

1.5 Joins ($A \max B$)

Definition 1.8 (Join)**.** $A \max B \le C \iff A \le C$ and $B \le C$

Lemma 1.8 (Zero Element)**.** $A \max 0 = A$

Proof.

$$\begin{aligned}
& A \max 0 \le X \\
\iff & \quad \{ \text{ Join } \} \\
& A \le X \text{ and } 0 \le X \\
\iff & \quad \{ \text{ Initial } \} \\
& A \le X \text{ and } TRUE \\
\iff & \quad \{ \text{ A and TRUE = A } \} \\
& A \le X
\end{aligned}$$

□

Lemma 1.9 (Absolute Element)**.** $\infty \max A = \infty$

Proof.

$$\begin{aligned}
& \infty \max A \le \infty \\
\iff & \quad \{ \text{ Terminal } \} \\
& TRUE \\
\iff & \quad \{ \text{ Reflexivity } \} \\
& \infty \max A \le \infty \max A \\
\iff & \quad \{ \text{ Join } \} \\
& \infty \le \infty \max A \text{ and } A \le \infty \max A \\
\implies & \quad \{ \text{ A and B } \implies \text{ A } \} \\
& \infty \le \infty \max A
\end{aligned}$$

□

Lemma 1.10 (Associativity)**.** $A \max (B \max C) = (A \max B) \max C$

Proof.

$$
\begin{aligned}
& A \max (B \max C) \leq X \\
\iff & \quad \{ \text{ Join } \} \\
& A \leq X \text{ and } B \max C \leq X \\
\iff & \quad \{ \text{ Join } \} \\
& A \leq X \text{ and } (B \leq X \text{ and } C \leq X) \\
\iff & \quad \{ \text{ Associativity of And } \} \\
& (A \leq X \text{ and } B \leq X) \text{ and } C \leq X \\
\iff & \quad \{ \text{ Join } \} \\
& A \max B \leq X \text{ and } C \leq X \\
\iff & \quad \{ \text{ Join } \} \\
& (A \max B) \max C \leq X
\end{aligned}
$$

□

Lemma 1.11 (Commutativity)**.** $A \max B = B \max A$

Proof.

$$
\begin{aligned}
& A \max B \leq X \\
= & \quad \{ \text{ Join } \} \\
& A \leq X \text{ and } B \leq X \\
= & \quad \{ \text{ Commutativity of And } \} \\
& B \leq X \text{ and } A \leq X \\
= & \quad \{ \text{ Join } \} \\
& B \max A \leq X
\end{aligned}
$$

□

Lemma 1.12 (Golden Rule)**.** $A \leq A \min B \iff B \max A \leq B$

Proof.

$$
\begin{aligned}
& A \leq A \min B \\
\iff & \quad \{ \text{ Meet } \} \\
& A \leq A \text{ and } A \leq B \\
\iff & \quad \{ \text{ Reflexivity } \} \\
& TRUE \text{ and } A \leq B \\
\iff & \quad \{ \text{ Reflexivity } \} \\
& B \leq B \text{ and } A \leq B \\
\iff & \quad \{ \text{ Join } \}
\end{aligned}
$$

$$B \max A \leq B$$

□

1.6 Adjoints (+ and −)

Definition 1.9 (Adjoint)**.** $A + B \leq C \iff A \leq C - B$

Definition 1.10 (Associativity of +)**.** $A + (B + C) = (A + B) + C$

Definition 1.11 (Commutativity of +)**.** $A + B = B + A$

Lemma 1.13 (+ distributes over Joins)**.** $(A \max B) + C = (A + C) \max (B + C)$

Proof.

$$\begin{array}{ll} & (A \max B) + C \leq X \\ \iff & \quad \{ \text{ Adjoint } \} \\ & A \max B \leq X - C \\ \iff & \quad \{ \text{ Join } \} \\ & A \leq X - C \text{ and } B \leq X - C \\ \iff & \quad \{ \text{ Adjoint } \} \\ & A + C \leq X \text{ and } B + C \leq X \\ \iff & \quad \{ \text{ Join } \} \\ & (A + C) \max (B + C) \leq X \end{array}$$

□

Lemma 1.14 (− distributes over Meets)**.** $(A \min B) - C = (A - C) \min (B - C)$

Proof.

$$\begin{array}{ll} & X \leq (A \min B) - C \\ \iff & \quad \{ \text{ Adjoint } \} \\ & X + C \leq A \min B \\ \iff & \quad \{ \text{ Meet } \} \\ & X + C \leq A \text{ and } X + C \leq B \\ \iff & \quad \{ \text{ Adjoint } \} \\ & X \leq A - C \text{ and } X \leq B - C \\ \iff & \quad \{ \text{ Meet } \} \\ & X \leq (A - C) \min (B - C) \end{array}$$

□

Lemma 1.15 (Preservation of infima)**.** $0 + A = 0$

Proof.

$$\begin{array}{ll} & 0 + A \leq X \\ \iff & \quad \{ \text{ Adjoint } \} \\ & 0 \leq X - A \\ \iff & \quad \{ \text{ Initial } \} \\ & TRUE \\ \iff & \quad \{ \text{ Initial } \} \\ & 0 \leq X \end{array}$$

□

Lemma 1.16 (Preservation of suprema)**.** $\infty - A = \infty$

Proof.

$$\begin{array}{ll} & X \leq \infty - A \\ \iff & \quad \{ \text{ Adjoint } \} \\ & X + A \leq \infty \\ \iff & \quad \{ \text{ Terminal } \} \\ & TRUE \\ \iff & \quad \{ \text{ Terminal } \} \\ & X \leq \infty \end{array}$$

□

Lemma 1.17 (Left cancellation law)**.** $(A - B) + B \leq A$

Proof.

$$\begin{array}{ll} & (A - B) + B \leq A \\ \iff & \quad \{ \text{ Adjoint } \} \\ & A - B \leq A - B \\ \iff & \quad \{ \text{ Reflexivity } \} \\ & TRUE \end{array}$$

□

Lemma 1.18 (Right Cancelation law)**.** $A \leq (A + B) - B$

Proof.

$$\begin{array}{ll} & A \leq (A + B) - B \\ \iff & \quad \{ \text{ Adjoint } \} \\ & A + B \leq A + B \\ \iff & \quad \{ \text{ Reflexivity } \} \end{array}$$

$$TRUE$$

$\square$

Lemma 1.19 (Monotonicity of +)**.** $A \leq B \implies A + C \leq B + C$

Proof.

$$
\begin{array}{ll}
 & A \leq B \\
\Longleftrightarrow & \quad \{ \text{ A and TRUE = A } \} \\
 & A \leq B \text{ and } TRUE \\
\Longleftrightarrow & \quad \{ \text{ Right Cancellation Law } \} \\
 & A \leq B \text{ and } B \leq (B + C) - C \\
\Longrightarrow & \quad \{ \text{ Transitivity of } \leq \ \} \\
 & A \leq (B + C) - C \\
\Longleftrightarrow & \quad \{ \text{ Adjoint } \} \\
 & A + C \leq B + C
\end{array}
$$

$\square$

Lemma 1.20 (Monotonicity of −)**.** $A \leq B \implies A - C \leq B - C$

Proof.

$$
\begin{array}{ll}
 & A \leq B \\
\Longleftrightarrow & \quad \{ \text{ TRUE and A = A } \} \\
 & TRUE \text{ and } A \leq B \\
\Longleftrightarrow & \quad \{ \text{ Left Cancellation Law } \} \\
 & (A - C) + C \leq A \text{ and } A \leq B \\
\Longrightarrow & \quad \{ \text{ Transitivity of } \leq \ \} \\
 & (A - C) + C \leq B \\
\Longleftrightarrow & \quad \{ \text{ Adjoint } \} \\
 & A - C \leq B - C
\end{array}
$$

$\square$

Lemma 1.21 (Weak-inverse +)**.** $A + B = ((A + B) - B) + B$

Proof.

$$
\begin{array}{ll}
 & ((A + B) - B) + B \leq A + B \\
\Longleftrightarrow & \quad \{ \text{ Left Cancellation Law } \} \\
 & TRUE \\
\Longleftrightarrow & \quad \{ \text{ Right Cancellation Law } \}
\end{array}
$$

$$
\begin{aligned}
& A \leq (A+B)-B \\
\Longrightarrow \quad & \quad \{ \text{ Monotonicity of } + \ \} \\
& A+B \leq ((A+B)-B)+B
\end{aligned}
$$

□

Lemma 1.22 (Weak-inverse $-$)**.** $A-B = ((A-B)+B)-B$

Proof.

$$
\begin{aligned}
& A-B \leq ((A-B)+B)-B \\
\Longleftrightarrow \quad & \quad \{ \text{ Right Cancellation Law } \} \\
& TRUE \\
\Longleftrightarrow \quad & \quad \{ \text{ Left Cancellation Law } \} \\
& (A-B)+B \leq A \\
\Longrightarrow \quad & \quad \{ \text{ Monotonicity of } - \ \} \\
& ((A-B)+B)-B \leq A-B
\end{aligned}
$$

□

Lemma 1.23 ($-$ distributes over $+$)**.** $A-(B+C) = (A-B)-C$

Proof.

$$
\begin{aligned}
& X \leq (A-B)-C \\
\Longleftrightarrow \quad & \quad \{ \text{ Adjoint } \} \\
& X+C \leq A-B \\
\Longleftrightarrow \quad & \quad \{ \text{ Adjoint } \} \\
& (X+C)+B \leq A \\
\Longleftrightarrow \quad & \quad \{ \text{ Associativity of } + \ \} \\
& X+(C+B) \leq A \\
\Longleftrightarrow \quad & \quad \{ \text{ Adjoint } \} \\
& X \leq A-(C+B)
\end{aligned}
$$

□

Lemma 1.24 (Duality)**.** $A \min B \leq (A+B)-(B \max A)$

Proof.

$$
\begin{aligned}
& X \leq A \min B \\
\Longleftrightarrow \quad & \quad \{ \text{ Meet } \} \\
& X \leq A \text{ and } X \leq B \\
\Longrightarrow \quad & \quad \{ \text{ Monotonicity of } + \ \}
\end{aligned}
$$

$$
\begin{array}{ll}
& X + B \leq A + B \text{ and } X + A \leq B + A \\
\iff & \quad \{ \text{ Commutativity of } + \ \} \\
& B + X \leq A + B \text{ and } A + X \leq A + B \\
\iff & \quad \{ \text{ Adjoint } \} \\
& B \leq (A + B) - X \text{ and } A \leq (A + B) - X \\
\iff & \quad \{ \text{ Join } \} \\
& B \max A \leq (A + B) - X \\
\iff & \quad \{ \text{ Adjoint } \} \\
& (B \max A) + X \leq A + B \\
\iff & \quad \{ \text{ Commutativity of } + \ \} \\
& X + (B \max A) \leq A + B \\
\iff & \quad \{ \text{ Adjoint } \} \\
& X \leq (A + B) - (B \max A)
\end{array}
$$

$\square$

2 Exercises

1. (Weakening) $A \leq A \max B$
2. (Projection) $A \min B \leq A$
3. (Idempotency) $A \max A = A$
4. (Meet $\leq$ Join) $A \min B \leq A \max B$
5. (Monotonicity of max) $A \leq B$ and $C \leq D \implies A \max C \leq B \max D$
6. $A + A - A = A$
7. (Self-Distributivity) $A \min (B \min C) = (A \min B) \min (A \min C)$
8. (Absorption) $A \min (A \max B) = A$