Subsets

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1 Definitions

1.1 Preorder

Definition 1.1 (Reflexivity). $A \subseteq A$

Definition 1.2 (Transitivity). $A \subseteq B$ and $B \subseteq C \implies A \subseteq C$

Definition 1.3 (Isomorphic). $A = B \iff A \subseteq B$ and $B \subseteq A$

Lemma 1.1. $A = B \iff (A \subseteq B \impliedby TRUE \implies B \subseteq A)$

1.2 Yoneda

Definition 1.4 (Yoneda $\subseteq$). $A \subseteq B \iff \forall X(X \subseteq A \implies X \subseteq B)$

Lemma 1.2 (Yoneda 1 =). $A = B \iff \forall X(X \subseteq A \iff X \subseteq B)$

Lemma 1.3 (Yoneda 2 =). $A = B \iff \forall X(A \subseteq X \iff B \subseteq X)$

1.3 Initial and Terminal Object ($\emptyset$ and U)

Definition 1.5 (Initial). $\emptyset \subseteq A$

Definition 1.6 (Terminal). $A \subseteq U$

1.4 Meets ($A \cap B$)

Definition 1.7 (Meet). $A \subseteq B \cap C \iff A \subseteq B$ and $A \subseteq C$

Lemma 1.4 (Zero Element). $A \cap U = A$

Proof.

$$
\begin{aligned}
& X \subseteq A \cap U \\
\iff & \quad \{ \text{ Meet } \} \\
& X \subseteq A \text{ and } X \subseteq U \\
\iff & \quad \{ \text{ Terminal } \}
\end{aligned}
$$

$$
\begin{aligned}
& X \subseteq A \text{ and } TRUE \\
\iff & \quad \{ \text{ A and TRUE = A } \} \\
& X \subseteq A
\end{aligned}
$$

□

Lemma 1.5 (Absolute Element). $A \cap \emptyset = \emptyset$

Proof.

$$
\begin{aligned}
& \emptyset \subseteq A \cap \emptyset \\
\iff & \quad \{ \text{ Initial } \} \\
& TRUE \\
\iff & \quad \{ \text{ Reflexivity } \} \\
& A \cap \emptyset \subseteq A \cap \emptyset \\
\iff & \quad \{ \text{ Meet } \} \\
& A \cap \emptyset \subseteq A \text{ and } A \cap \emptyset \subseteq \emptyset \\
\implies & \quad \{ \text{ A and B } \implies \text{ B } \} \\
& A \cap \emptyset \subseteq \emptyset
\end{aligned}
$$

□

Lemma 1.6 (Associativity). $A \cap (B \cap C) = (A \cap B) \cap C$

Proof.

$$
\begin{aligned}
& X \subseteq A \cap (B \cap C) \\
= & \quad \{ \text{ Meet } \} \\
& X \subseteq A \text{ and } X \subseteq B \cap C \\
= & \quad \{ \text{ Meet } \} \\
& X \subseteq A \text{ and } (X \subseteq B \text{ and } X \subseteq C) \\
= & \quad \{ \text{ Associativity of And } \} \\
& (X \subseteq A \text{ and } X \subseteq B) \text{ and } X \subseteq C \\
= & \quad \{ \text{ Meet } \} \\
& X \subseteq A \cap B \text{ and } X \subseteq C \\
= & \quad \{ \text{ Meet } \} \\
& X \subseteq (A \cap B) \cap C
\end{aligned}
$$

□

Lemma 1.7 (Commutativity). $A \cap B = B \cap A$

Proof.

$$\begin{aligned}
& X \subseteq A \cap B \\
= \quad & \{ \text{ Meet } \} \\
& X \subseteq A \text{ and } X \subseteq B \\
= \quad & \{ \text{ Commutativity of And } \} \\
& X \subseteq B \text{ and } X \subseteq A \\
= \quad & \{ \text{ Meet } \} \\
& X \subseteq B \cap A
\end{aligned}$$

□

1.5 Joins ($A \cup B$)

Definition 1.8 (Join)**.** $A \cup B \subseteq C \iff A \subseteq C$ and $B \subseteq C$

Lemma 1.8 (Zero Element)**.** $A \cup \emptyset = A$

Proof.

$$\begin{aligned}
& A \cup \emptyset \subseteq X \\
\iff \quad & \{ \text{ Join } \} \\
& A \subseteq X \text{ and } \emptyset \subseteq X \\
\iff \quad & \{ \text{ Initial } \} \\
& A \subseteq X \text{ and } TRUE \\
\iff \quad & \{ \text{ A and TRUE = A } \} \\
& A \subseteq X
\end{aligned}$$

□

Lemma 1.9 (Absolute Element)**.** $U \cup A = U$

Proof.

$$\begin{aligned}
& U \cup A \subseteq U \\
\iff \quad & \{ \text{ Terminal } \} \\
& TRUE \\
\iff \quad & \{ \text{ Reflexivity } \} \\
& U \cup A \subseteq U \cup A \\
\iff \quad & \{ \text{ Join } \} \\
& U \subseteq U \cup A \text{ and } A \subseteq U \cup A \\
\implies \quad & \{ \text{ A and B } \implies \text{ A } \} \\
& U \subseteq U \cup A
\end{aligned}$$

□

Lemma 1.10 (Associativity). $A \cup (B \cup C) = (A \cup B) \cup C$

Proof.

$$
\begin{array}{ll}
& A \cup (B \cup C) \subseteq X \\
\Longleftrightarrow & \quad \{ \text{ Join } \} \\
& A \subseteq X \text{ and } B \cup C \subseteq X \\
\Longleftrightarrow & \quad \{ \text{ Join } \} \\
& A \subseteq X \text{ and } (B \subseteq X \text{ and } C \subseteq X) \\
\Longleftrightarrow & \quad \{ \text{ Associativity of And } \} \\
& (A \subseteq X \text{ and } B \subseteq X) \text{ and } C \subseteq X \\
\Longleftrightarrow & \quad \{ \text{ Join } \} \\
& A \cup B \subseteq X \text{ and } C \subseteq X \\
\Longleftrightarrow & \quad \{ \text{ Join } \} \\
& (A \cup B) \cup C \subseteq X
\end{array}
$$

□

Lemma 1.11 (Commutativity). $A \cup B = B \cup A$

Proof.

$$
\begin{array}{ll}
& A \cup B \subseteq X \\
= & \quad \{ \text{ Join } \} \\
& A \subseteq X \text{ and } B \subseteq X \\
= & \quad \{ \text{ Commutativity of And } \} \\
& B \subseteq X \text{ and } A \subseteq X \\
= & \quad \{ \text{ Join } \} \\
& B \cup A \subseteq X
\end{array}
$$

□

Lemma 1.12 (Golden Rule). $A \subseteq A \cap B \iff B \cup A \subseteq B$

Proof.

$$
\begin{array}{ll}
& A \subseteq A \cap B \\
\Longleftrightarrow & \quad \{ \text{ Meet } \} \\
& A \subseteq A \text{ and } A \subseteq B \\
\Longleftrightarrow & \quad \{ \text{ Reflexivity } \} \\
& TRUE \text{ and } A \subseteq B \\
\Longleftrightarrow & \quad \{ \text{ Reflexivity } \} \\
& B \subseteq B \text{ and } A \subseteq B \\
\Longleftrightarrow & \quad \{ \text{ Join } \}
\end{array}
$$

$$B \cup A \subseteq B$$

□

1.6 Adjoints (+ and /)

Definition 1.9 (Adjoint). $A + B \subseteq C \iff A \subseteq C/B$

Definition 1.10 (Associativity of +). $A + (B + C) = (A + B) + C$

Definition 1.11 (Commutativity of +). $A + B = B + A$

Lemma 1.13 (+ distributes over Joins). $(A \cup B) + C = (A + C) \cup (B + C)$

Proof.

$$
\begin{array}{ll}
& (A \cup B) + C \subseteq X \\
\iff & \quad \{ \text{ Adjoint } \} \\
& A \cup B \subseteq X/C \\
\iff & \quad \{ \text{ Join } \} \\
& A \subseteq X/C \text{ and } B \subseteq X/C \\
\iff & \quad \{ \text{ Adjoint } \} \\
& A + C \subseteq X \text{ and } B + C \subseteq X \\
\iff & \quad \{ \text{ Join } \} \\
& (A + C) \cup (B + C) \subseteq X
\end{array}
$$

□

Lemma 1.14 (/ distributes over Meets). $(A \cap B)/C = (A/C) \cap (B/C)$

Proof.

$$
\begin{array}{ll}
& X \subseteq (A \cap B)/C \\
\iff & \quad \{ \text{ Adjoint } \} \\
& X + C \subseteq A \cap B \\
\iff & \quad \{ \text{ Meet } \} \\
& X + C \subseteq A \text{ and } X + C \subseteq B \\
\iff & \quad \{ \text{ Adjoint } \} \\
& X \subseteq A/C \text{ and } X \subseteq B/C \\
\iff & \quad \{ \text{ Meet } \} \\
& X \subseteq (A/C) \cap (B/C)
\end{array}
$$

□

Lemma 1.15 (Preservation of infima). $\emptyset + A = \emptyset$

Proof.

$$
\begin{aligned}
& \emptyset + A \subseteq X \\
\iff & \quad \{ \text{ Adjoint } \} \\
& \emptyset \subseteq X/A \\
\iff & \quad \{ \text{ Initial } \} \\
& TRUE \\
\iff & \quad \{ \text{ Initial } \} \\
& \emptyset \subseteq X
\end{aligned}
$$

□

Lemma 1.16 (Preservation of suprema). $U/A = U$

Proof.

$$
\begin{aligned}
& X \subseteq U/A \\
\iff & \quad \{ \text{ Adjoint } \} \\
& X + A \subseteq U \\
\iff & \quad \{ \text{ Terminal } \} \\
& TRUE \\
\iff & \quad \{ \text{ Terminal } \} \\
& X \subseteq U
\end{aligned}
$$

□

Lemma 1.17 (Left cancellation law). $(A/B) + B \subseteq A$

Proof.

$$
\begin{aligned}
& (A/B) + B \subseteq A \\
\iff & \quad \{ \text{ Adjoint } \} \\
& A/B \subseteq A/B \\
\iff & \quad \{ \text{ Reflexivity } \} \\
& TRUE
\end{aligned}
$$

□

Lemma 1.18 (Right Cancelation law). $A \subseteq (A + B)/B$

Proof.

$$
\begin{aligned}
& A \subseteq (A + B)/B \\
\iff & \quad \{ \text{ Adjoint } \} \\
& A + B \subseteq A + B \\
\iff & \quad \{ \text{ Reflexivity } \}
\end{aligned}
$$

$$TRUE$$

$\square$

Lemma 1.19 (Monotonicity of +). $A \subseteq B \implies A + C \subseteq B + C$

Proof.

$$
\begin{aligned}
& A \subseteq B \\
\iff & \quad \{ \text{ A and TRUE = A } \} \\
& A \subseteq B \text{ and } TRUE \\
\iff & \quad \{ \text{ Right Cancellation Law } \} \\
& A \subseteq B \text{ and } B \subseteq (B + C)/C \\
\implies & \quad \{ \text{ Transitivity of } \subseteq \} \\
& A \subseteq (B + C)/C \\
\iff & \quad \{ \text{ Adjoint } \} \\
& A + C \subseteq B + C
\end{aligned}
$$

$\square$

Lemma 1.20 (Monotonicity of /). $A \subseteq B \implies A/C \subseteq B/C$

Proof.

$$
\begin{aligned}
& A \subseteq B \\
\iff & \quad \{ \text{ TRUE and A = A } \} \\
& TRUE \text{ and } A \subseteq B \\
\iff & \quad \{ \text{ Left Cancellation Law } \} \\
& (A/C) + C \subseteq A \text{ and } A \subseteq B \\
\implies & \quad \{ \text{ Transitivity of } \subseteq \} \\
& (A/C) + C \subseteq B \\
\iff & \quad \{ \text{ Adjoint } \} \\
& A/C \subseteq B/C
\end{aligned}
$$

$\square$

Lemma 1.21 (Weak-inverse +). $A + B = ((A + B)/B) + B$

Proof.

$$
\begin{aligned}
& ((A + B)/B) + B \subseteq A + B \\
\iff & \quad \{ \text{ Left Cancellation Law } \} \\
& TRUE \\
\iff & \quad \{ \text{ Right Cancellation Law } \}
\end{aligned}
$$

$$
\begin{aligned}
& A \subseteq (A+B)/B \\
\Longrightarrow & \quad \{ \text{ Monotonicity of } + \} \\
& A+B \subseteq ((A+B)/B)+B
\end{aligned}
$$

□

Lemma 1.22 (Weak-inverse /)**.** $A/B = ((A/B)+B)/B$

Proof.

$$
\begin{aligned}
& A/B \subseteq ((A/B)+B)/B \\
\Longleftrightarrow & \quad \{ \text{ Right Cancellation Law } \} \\
& TRUE \\
\Longleftrightarrow & \quad \{ \text{ Left Cancellation Law } \} \\
& (A/B)+B \subseteq A \\
\Longrightarrow & \quad \{ \text{ Monotonicity of } / \} \\
& ((A/B)+B)/B \subseteq A/B
\end{aligned}
$$

□

Lemma 1.23 (/ distributes over +)**.** $A/(B+C) = (A/B)/C$

Proof.

$$
\begin{aligned}
& X \subseteq (A/B)/C \\
\Longleftrightarrow & \quad \{ \text{ Adjoint } \} \\
& X+C \subseteq A/B \\
\Longleftrightarrow & \quad \{ \text{ Adjoint } \} \\
& (X+C)+B \subseteq A \\
\Longleftrightarrow & \quad \{ \text{ Associativity of } + \} \\
& X+(C+B) \subseteq A \\
\Longleftrightarrow & \quad \{ \text{ Adjoint } \} \\
& X \subseteq A/(C+B)
\end{aligned}
$$

□

Lemma 1.24 (Duality)**.** $A \cap B \subseteq (A+B)/(B \cup A)$

Proof.

$$
\begin{aligned}
& X \subseteq A \cap B \\
\Longleftrightarrow & \quad \{ \text{ Meet } \} \\
& X \subseteq A \text{ and } X \subseteq B \\
\Longrightarrow & \quad \{ \text{ Monotonicity of } + \}
\end{aligned}
$$

$$
\begin{array}{ll}
 & X + B \subseteq A + B \text{ and } X + A \subseteq B + A \\
\iff & \quad \{ \text{ Commutativity of } + \ \} \\
 & B + X \subseteq A + B \text{ and } A + X \subseteq A + B \\
\iff & \quad \{ \text{ Adjoint } \} \\
 & B \subseteq (A + B)/X \text{ and } A \subseteq (A + B)/X \\
\iff & \quad \{ \text{ Join } \} \\
 & B \cup A \subseteq (A + B)/X \\
\iff & \quad \{ \text{ Adjoint } \} \\
 & (B \cup A) + X \subseteq A + B \\
\iff & \quad \{ \text{ Commutativity of } + \ \} \\
 & X + (B \cup A) \subseteq A + B \\
\iff & \quad \{ \text{ Adjoint } \} \\
 & X \subseteq (A + B)/(B \cup A)
\end{array}
$$

□

2 Exercises

1. (Weakening) $A \subseteq A \cup B$
2. (Projection) $A \cap B \subseteq A$
3. (Idempotency) $A \cup A = A$
4. (Meet $\subseteq$ Join) $A \cap B \subseteq A \cup B$
5. (Monotonicity of $\cup$) $A \subseteq B$ and $C \subseteq D \implies A \cup C \subseteq B \cup D$
6. $A + A/A = A$
7. (Self-Distributivity) $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$
8. (Absorption) $A \cap (A \cup B) = A$