Unital Commutative Quantales

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1 Definitions

1.1 Preorder

Definition 1.1 (Reflexivity). $A \leq A$

Definition 1.2 (Transitivity). $A \leq B$ and $B \leq C \implies A \leq C$

Definition 1.3 (Isomorphic). $A \simeq B \iff A \leq B$ and $B \leq A$

Lemma 1.1. $A \simeq B \iff (A \leq B \impliedby TRUE \implies B \leq A)$

1.2 Yoneda

Definition 1.4 (Yoneda $\leq$). $A \leq B \iff \forall X(X \leq A \implies X \leq B)$

Lemma 1.2 (Yoneda 1 $\simeq$). $A \simeq B \iff \forall X(X \leq A \iff X \leq B)$

Lemma 1.3 (Yoneda 2 $\simeq$). $A \simeq B \iff \forall X(A \leq X \iff B \leq X)$

1.3 Initial and Terminal Object ($\perp$ and $\top$)

Definition 1.5 (Initial). $\perp \leq A$

Definition 1.6 (Terminal). $A \leq \top$

1.4 Meets ($A \wedge B$)

Definition 1.7 (Meet). $A \leq B \wedge C \iff A \leq B$ and $A \leq C$

Lemma 1.4 (Zero Element). $A \wedge \top \simeq A$

Proof.

$$
\begin{aligned}
& X \leq A \wedge \top \\
\iff & \quad \{ \text{ Meet } \} \\
& X \leq A \text{ and } X \leq \top \\
\iff & \quad \{ \text{ Terminal } \}
\end{aligned}
$$

$$
\begin{aligned}
& X \leq A \text{ and } TRUE \\
\Longleftrightarrow \quad & \quad \{ \text{ A and TRUE = A } \} \\
& X \leq A
\end{aligned}
$$

□

Lemma 1.5 (Absolute Element)**.** $A\wedge \perp \simeq \perp$

Proof.

$$
\begin{aligned}
& \perp \leq A \wedge \perp \\
\Longleftrightarrow \quad & \quad \{ \text{ Initial } \} \\
& TRUE \\
\Longleftrightarrow \quad & \quad \{ \text{ Reflexivity } \} \\
& A \wedge \perp \leq A \wedge \perp \\
\Longleftrightarrow \quad & \quad \{ \text{ Meet } \} \\
& A \wedge \perp \leq A \text{ and } A \wedge \perp \leq \perp \\
\Longrightarrow \quad & \quad \{ \text{ A and B } \Longrightarrow \text{ B } \} \\
& A \wedge \perp \leq \perp
\end{aligned}
$$

□

Lemma 1.6 (Associativity)**.** $A \wedge (B \wedge C) \simeq (A \wedge B) \wedge C$

Proof.

$$
\begin{aligned}
& X \leq A \wedge (B \wedge C) \\
= \quad & \quad \{ \text{ Meet } \} \\
& X \leq A \text{ and } X \leq B \wedge C \\
= \quad & \quad \{ \text{ Meet } \} \\
& X \leq A \text{ and } (X \leq B \text{ and } X \leq C) \\
= \quad & \quad \{ \text{ Associativity of And } \} \\
& (X \leq A \text{ and } X \leq B) \text{ and } X \leq C \\
= \quad & \quad \{ \text{ Meet } \} \\
& X \leq A \wedge B \text{ and } X \leq C \\
= \quad & \quad \{ \text{ Meet } \} \\
& X \leq (A \wedge B) \wedge C
\end{aligned}
$$

□

Lemma 1.7 (Commutativity)**.** $A \wedge B \simeq B \wedge A$

Proof.

$$
\begin{aligned}
& X \leq A \wedge B \\
= & \quad \{ \text{ Meet } \} \\
& X \leq A \text{ and } X \leq B \\
= & \quad \{ \text{ Commutativity of And } \} \\
& X \leq B \text{ and } X \leq A \\
= & \quad \{ \text{ Meet } \} \\
& X \leq B \wedge A
\end{aligned}
$$

□

1.5 Joins ($A \vee B$)

Definition 1.8 (Join)**.** $A \vee B \leq C \iff A \leq C$ and $B \leq C$

Lemma 1.8 (Zero Element)**.** $A \vee \perp \simeq A$

Proof.

$$
\begin{aligned}
& A \vee \perp \leq X \\
\iff & \quad \{ \text{ Join } \} \\
& A \leq X \text{ and } \perp \leq X \\
\iff & \quad \{ \text{ Initial } \} \\
& A \leq X \text{ and } TRUE \\
\iff & \quad \{ \text{ A and TRUE = A } \} \\
& A \leq X
\end{aligned}
$$

□

Lemma 1.9 (Absolute Element)**.** $\top \vee A \simeq \top$

Proof.

$$
\begin{aligned}
& \top \vee A \leq \top \\
\iff & \quad \{ \text{ Terminal } \} \\
& TRUE \\
\iff & \quad \{ \text{ Reflexivity } \} \\
& \top \vee A \leq \top \vee A \\
\iff & \quad \{ \text{ Join } \} \\
& \top \leq \top \vee A \text{ and } A \leq \top \vee A \\
\implies & \quad \{ \text{ A and B } \implies \text{ A } \} \\
& \top \leq \top \vee A
\end{aligned}
$$

□

Lemma 1.10 (Associativity). $A \vee (B \vee C) \simeq (A \vee B) \vee C$

Proof.

$$
\begin{aligned}
& A \vee (B \vee C) \leq X \\
\iff & \quad \{ \text{ Join } \} \\
& A \leq X \text{ and } B \vee C \leq X \\
\iff & \quad \{ \text{ Join } \} \\
& A \leq X \text{ and } (B \leq X \text{ and } C \leq X) \\
\iff & \quad \{ \text{ Associativity of And } \} \\
& (A \leq X \text{ and } B \leq X) \text{ and } C \leq X \\
\iff & \quad \{ \text{ Join } \} \\
& A \vee B \leq X \text{ and } C \leq X \\
\iff & \quad \{ \text{ Join } \} \\
& (A \vee B) \vee C \leq X
\end{aligned}
$$

□

Lemma 1.11 (Commutativity). $A \vee B \simeq B \vee A$

Proof.

$$
\begin{aligned}
& A \vee B \leq X \\
= & \quad \{ \text{ Join } \} \\
& A \leq X \text{ and } B \leq X \\
= & \quad \{ \text{ Commutativity of And } \} \\
& B \leq X \text{ and } A \leq X \\
= & \quad \{ \text{ Join } \} \\
& B \vee A \leq X
\end{aligned}
$$

□

Lemma 1.12 (Golden Rule). $A \leq A \wedge B \iff B \vee A \leq B$

Proof.

$$
\begin{aligned}
& A \leq A \wedge B \\
\iff & \quad \{ \text{ Meet } \} \\
& A \leq A \text{ and } A \leq B \\
\iff & \quad \{ \text{ Reflexivity } \} \\
& TRUE \text{ and } A \leq B \\
\iff & \quad \{ \text{ Reflexivity } \} \\
& B \leq B \text{ and } A \leq B \\
\iff & \quad \{ \text{ Join } \}
\end{aligned}
$$

$$B \vee A \leq B$$

□

1.6 Adjoints (+ and −)

Definition 1.9 (Adjoint). $A + B \leq C \iff A \leq C - B$

Definition 1.10 (Associativity of +). $A + (B + C) \simeq (A + B) + C$

Definition 1.11 (Commutativity of +). $A + B \simeq B + A$

Lemma 1.13 (+ distributes over Joins). $(A \vee B) + C \simeq (A + C) \vee (B + C)$

Proof.

$$\begin{array}{ll} & (A \vee B) + C \leq X \\ \iff & \quad \{ \text{ Adjoint } \} \\ & A \vee B \leq X - C \\ \iff & \quad \{ \text{ Join } \} \\ & A \leq X - C \text{ and } B \leq X - C \\ \iff & \quad \{ \text{ Adjoint } \} \\ & A + C \leq X \text{ and } B + C \leq X \\ \iff & \quad \{ \text{ Join } \} \\ & (A + C) \vee (B + C) \leq X \end{array}$$

□

Lemma 1.14 (− distributes over Meets). $(A \wedge B) - C \simeq (A - C) \wedge (B - C)$

Proof.

$$\begin{array}{ll} & X \leq (A \wedge B) - C \\ \iff & \quad \{ \text{ Adjoint } \} \\ & X + C \leq A \wedge B \\ \iff & \quad \{ \text{ Meet } \} \\ & X + C \leq A \text{ and } X + C \leq B \\ \iff & \quad \{ \text{ Adjoint } \} \\ & X \leq A - C \text{ and } X \leq B - C \\ \iff & \quad \{ \text{ Meet } \} \\ & X \leq (A - C) \wedge (B - C) \end{array}$$

□

Lemma 1.15 (Preservation of infima). $\bot + A \simeq \bot$

Proof.

$$
\begin{aligned}
& \bot + A \leq X \\
\iff & \quad \{ \text{ Adjoint } \} \\
& \bot \leq X - A \\
\iff & \quad \{ \text{ Initial } \} \\
& TRUE \\
\iff & \quad \{ \text{ Initial } \} \\
& \bot \leq X
\end{aligned}
$$

□

Lemma 1.16 (Preservation of suprema)**.** $\top - A \simeq \top$

Proof.

$$
\begin{aligned}
& X \leq \top - A \\
\iff & \quad \{ \text{ Adjoint } \} \\
& X + A \leq \top \\
\iff & \quad \{ \text{ Terminal } \} \\
& TRUE \\
\iff & \quad \{ \text{ Terminal } \} \\
& X \leq \top
\end{aligned}
$$

□

Lemma 1.17 (Left cancellation law)**.** $(A - B) + B \leq A$

Proof.

$$
\begin{aligned}
& (A - B) + B \leq A \\
\iff & \quad \{ \text{ Adjoint } \} \\
& A - B \leq A - B \\
\iff & \quad \{ \text{ Reflexivity } \} \\
& TRUE
\end{aligned}
$$

□

Lemma 1.18 (Right Cancelation law)**.** $A \leq (A + B) - B$

Proof.

$$
\begin{aligned}
& A \leq (A + B) - B \\
\iff & \quad \{ \text{ Adjoint } \} \\
& A + B \leq A + B \\
\iff & \quad \{ \text{ Reflexivity } \}
\end{aligned}
$$

$$TRUE$$

$\square$

Lemma 1.19 (Monotonicity of +)**.** $A \leq B \implies A + C \leq B + C$

Proof.

$$\begin{array}{ll} & A \leq B \\ \iff & \quad \{ \text{ A and TRUE = A } \} \\ & A \leq B \text{ and } TRUE \\ \iff & \quad \{ \text{ Right Cancellation Law } \} \\ & A \leq B \text{ and } B \leq (B + C) - C \\ \implies & \quad \{ \text{ Transitivity of } \leq \} \\ & A \leq (B + C) - C \\ \iff & \quad \{ \text{ Adjoint } \} \\ & A + C \leq B + C \end{array}$$

$\square$

Lemma 1.20 (Monotonicity of −)**.** $A \leq B \implies A - C \leq B - C$

Proof.

$$\begin{array}{ll} & A \leq B \\ \iff & \quad \{ \text{ TRUE and A = A } \} \\ & TRUE \text{ and } A \leq B \\ \iff & \quad \{ \text{ Left Cancellation Law } \} \\ & (A - C) + C \leq A \text{ and } A \leq B \\ \implies & \quad \{ \text{ Transitivity of } \leq \} \\ & (A - C) + C \leq B \\ \iff & \quad \{ \text{ Adjoint } \} \\ & A - C \leq B - C \end{array}$$

$\square$

Lemma 1.21 (Weak-inverse +)**.** $A + B \simeq ((A + B) - B) + B$

Proof.

$$\begin{array}{ll} & ((A + B) - B) + B \leq A + B \\ \iff & \quad \{ \text{ Left Cancellation Law } \} \\ & TRUE \\ \iff & \quad \{ \text{ Right Cancellation Law } \} \end{array}$$

$$
\begin{aligned}
& A \leq (A+B) - B \\
\Longrightarrow \quad & \quad \{ \text{ Monotonicity of } + \ \} \\
& A + B \leq ((A+B) - B) + B
\end{aligned}
$$

□

Lemma 1.22 (Weak-inverse $-$)**.** $A - B \simeq ((A-B)+B) - B$

Proof.

$$
\begin{aligned}
& A - B \leq ((A-B)+B) - B \\
\Longleftrightarrow \quad & \quad \{ \text{ Right Cancellation Law } \} \\
& TRUE \\
\Longleftrightarrow \quad & \quad \{ \text{ Left Cancellation Law } \} \\
& (A-B)+B \leq A \\
\Longrightarrow \quad & \quad \{ \text{ Monotonicity of } - \ \} \\
& ((A-B)+B) - B \leq A - B
\end{aligned}
$$

□

Lemma 1.23 ($-$ distributes over $+$)**.** $A - (B+C) \simeq (A-B) - C$

Proof.

$$
\begin{aligned}
& X \leq (A-B) - C \\
\Longleftrightarrow \quad & \quad \{ \text{ Adjoint } \} \\
& X + C \leq A - B \\
\Longleftrightarrow \quad & \quad \{ \text{ Adjoint } \} \\
& (X+C) + B \leq A \\
\Longleftrightarrow \quad & \quad \{ \text{ Associativity of } + \ \} \\
& X + (C+B) \leq A \\
\Longleftrightarrow \quad & \quad \{ \text{ Adjoint } \} \\
& X \leq A - (C+B)
\end{aligned}
$$

□

Lemma 1.24 (Duality)**.** $A \wedge B \leq (A+B) - (B \vee A)$

Proof.

$$
\begin{aligned}
& X \leq A \wedge B \\
\Longleftrightarrow \quad & \quad \{ \text{ Meet } \} \\
& X \leq A \text{ and } X \leq B \\
\Longrightarrow \quad & \quad \{ \text{ Monotonicity of } + \ \}
\end{aligned}
$$

$$
\begin{array}{ll}
& X + B \leq A + B \text{ and } X + A \leq B + A \\
\iff & \quad \{ \text{ Commutativity of } + \ \} \\
& B + X \leq A + B \text{ and } A + X \leq A + B \\
\iff & \quad \{ \text{ Adjoint } \} \\
& B \leq (A + B) - X \text{ and } A \leq (A + B) - X \\
\iff & \quad \{ \text{ Join } \} \\
& B \vee A \leq (A + B) - X \\
\iff & \quad \{ \text{ Adjoint } \} \\
& (B \vee A) + X \leq A + B \\
\iff & \quad \{ \text{ Commutativity of } + \ \} \\
& X + (B \vee A) \leq A + B \\
\iff & \quad \{ \text{ Adjoint } \} \\
& X \leq (A + B) - (B \vee A)
\end{array}
$$

□

2 Exercises

1. (Weakening) $A \leq A \vee B$
2. (Projection) $A \wedge B \leq A$
3. (Idempotency) $A \vee A \simeq A$
4. (Meet $\leq$ Join) $A \wedge B \leq A \vee B$
5. (Monotonicity of $\vee$) $A \leq B$ and $C \leq D \implies A \vee C \leq B \vee D$
6. $A + A - A \simeq A$
7. (Self-Distributivity) $A \wedge (B \wedge C) \simeq (A \wedge B) \wedge (A \wedge C)$
8. (Absorption) $A \wedge (A \vee B) \simeq A$