Reference Frame Transformations (GPS)

This section presents methods for transforming points and vectors between rectangular coordinate systems. The axes of each coordinate system are assumed to be right-handed and orthogonal. Three dimensions are used throughout the discussion; however, the discussion is equally valid for $\mathbb{R}^n$.

The Direction Cosine Matrix

Let ϕ_1 represent a right-handed orthogonal coordinate system. Let $\mathbf{v}_1$ be a vector from the origin O_1 of the ϕ_1 frame to the point P. The representation of the vector $\mathbf{v}_1$ with respect to frame ϕ_1 is

$$\mathbf{v}_1 \qquad x_1\mathbf{I}_1 + y_1\mathbf{J}_1 + z_1\mathbf{K}_1 \qquad (2.13)$$

axes and where

$$\mathbf{I}_1, \mathbf{J}_1, \mathbf{K}_1$$

are unit vectors along the ϕ_1 axes and

$$\begin{aligned} x_1 &= (P - O_1) \cdot \mathbf{I}_1 \\ y_1 &= (P - O_1) \cdot \mathbf{J}_1 \\ z_1 &= (P - O_1) \cdot \mathbf{K}_1 . \end{aligned}$$

The vector $[\mathbf{v}_1]^1 = [x_1, y_1, z_1]^\top$ contains the coordinates of the point P with respect to the axes of ϕ_1 and is the representation of the vector $\mathbf{v}_1$ with respect to ϕ_1. The physical interpretations of the coordinates is that they are the projections of the vector $\mathbf{v}_1$ onto the ϕ_1 axes. For the two-dimensional x — y plane, the discussion of this paragraph is depicted in Figure 2.9.

A vector v can be defined without reference to a specific reference frame. When convenient, as discussed above, the representation of $\mathbf{v} \in \mathbb{R}^3$ with respect to the axes of frame ϕ_1 is

$$[\mathbf{v}]^1 = \begin{bmatrix} \mathbf{I}_1 \cdot \mathbf{v} \\ \mathbf{J}_1 \cdot \mathbf{v} \\ \mathbf{K}_1 \cdot \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_1^\top \\ \mathbf{J}_1^\top \\ \mathbf{K}_1^\top \end{bmatrix} \mathbf{v}. \qquad (2.14)$$

Eqn. (2.14) is used in derivations later in this subsection.

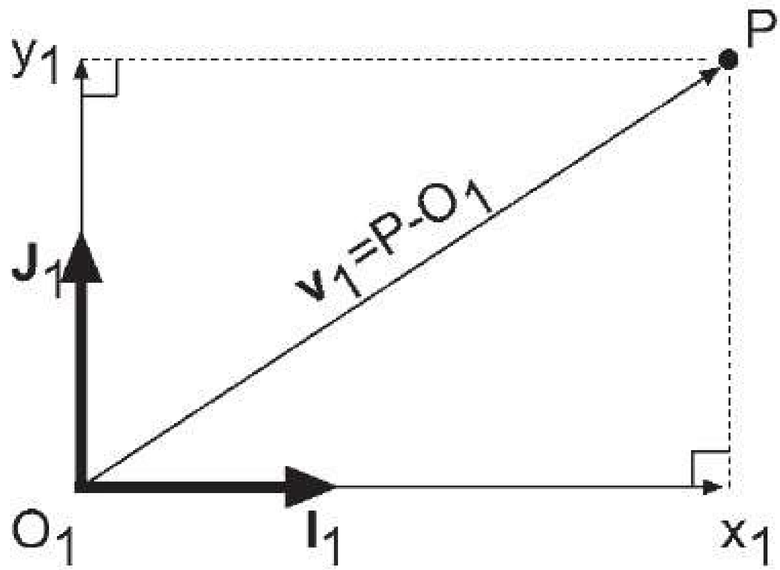

Figure 2.9: Two dimensional representation of the determination of the coordinates of a point P relative to the origin O_1 of reference frame.

With two distinct reference frames ϕ_1 and ϕ_2, the same point can be represented by a different sets of coordinates in each reference frame.The remainder of this section discusses the important question of how to use the coordinates of a point in one frame-of-reference to compute the coordinates of the same point with respect to a different frame-of-reference. The transformation of point coordinates from one frame-of-reference to another will require two operations: translation and rotation.

From the above discussion, $\mathbf{v}_1 = P - O_1$ is the vector from O_1 to P and

$\mathbf{v}_2 = P - O_2$ is the vector from O_2 to P. Define $\mathbf{O}_{12} = O_2 - O_1$ as the vector

from O_1 to O_2. Therefore, we have that $\mathbf{v}_1 = \mathbf{O}_{12} + \mathbf{v}_2$.

This equation must hold whether the vectors are represented in the coordinates of the ϕ_1 frame or the ϕ_2 frame.

Denote the components of vector $\mathbf{v}_1$ relative to the ϕ_1 frame as $[\mathbf{v}_1]^1 = [x_1, y_1, z_1]^1$, the components of $\mathbf{v}_2$ relative to the ϕ_2 frame as $[\mathbf{v}_2]^2 = [x_2, y_2, z_2]^2$ and the components of $\mathbf{O}_{12}$ relative to the ϕ_1 frame as $[\mathbf{O}_{12}]^1 = [x_O, y_O, z_O]^1$. Assume that $[\mathbf{v}_2]^2$, $[\mathbf{O}_{12}]^1$ and the relative orientation of the two reference frames are known. Then, the position of P with respect to the ϕ_1 frame can be computed as

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}^1 = \begin{bmatrix} x_O \\ y_O \\ z_O \end{bmatrix}^1 + [\mathbf{v}_2]^1. \qquad (2.15)$$

Because it is the only unknown term in the right hand side, the present question of interest is how to calculate $[\mathbf{v}_2]^1$ based on the available information.

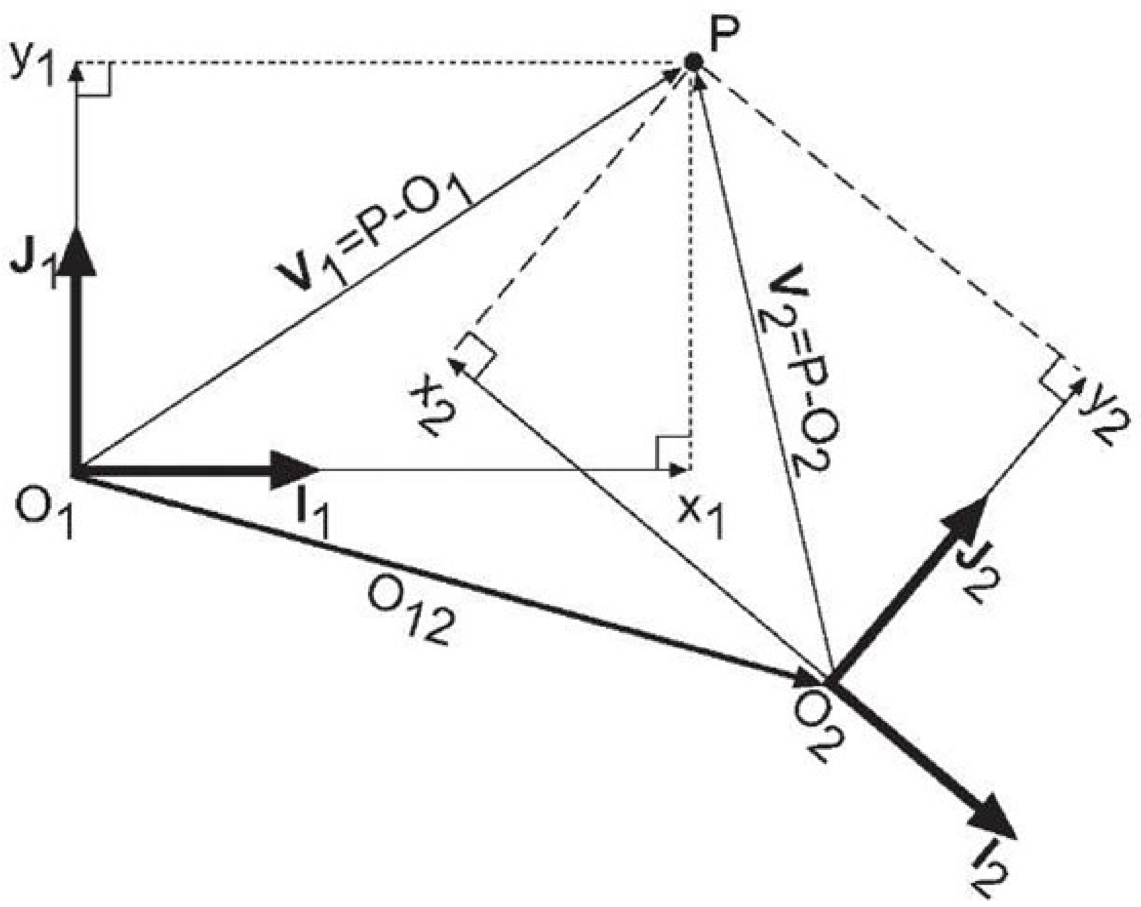

Figure 2.10: Definition of the coordinates of a point P with respect to two frames-of-reference ϕ_1 and ϕ_2.

Let $\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2$ represent the unit vectors along the axes. As discussed relative to eqn. (2.14), vectors $\mathbf{v}_I, \mathbf{v}_J$, and $\mathbf{v}_K$ defined as

$$\mathbf{v}_I = \begin{bmatrix} \mathbf{I}_1 \cdot \mathbf{I}_2 \\ \mathbf{J}_1 \cdot \mathbf{I}_2 \\ \mathbf{K}_1 \cdot \mathbf{I}_2 \end{bmatrix}, \mathbf{v}_J = \begin{bmatrix} \mathbf{I}_1 \cdot \mathbf{J}_2 \\ \mathbf{J}_1 \cdot \mathbf{J}_2 \\ \mathbf{K}_1 \cdot \mathbf{J}_2 \end{bmatrix}, \text{ and } \mathbf{v}_K = \begin{bmatrix} \mathbf{I}_1 \cdot \mathbf{K}_2 \\ \mathbf{J}_1 \cdot \mathbf{K}_2 \\ \mathbf{K}_1 \cdot \mathbf{K}_2 \end{bmatrix}$$

represent the unit vectors in the direction of the ϕ_2 coordinate axes that are resolved in the ϕ_1 reference frame. Since $\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2$ are orthonormal, so are $\mathbf{v}_I, \mathbf{v}_J, \mathbf{v}_K$. Therefore, the matrix

$$\mathbf{R}_2^1 = [\mathbf{v}_I, \mathbf{v}_J, \mathbf{v}_K]$$

is an orthonormal matrix

$$(\text{i.e., } (\mathbf{R}_2^1)^\top \mathbf{R}_2^1 = \mathbf{R}_2^1 (\mathbf{R}_2^1)^\top = \mathbf{I}).$$

Each element of $\mathbf{R}_2^1$ is the cosine of the angle between one of $\mathbf{I}_1, \mathbf{J}_1, \mathbf{K}_1$ and one of $\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2$. To see this, consider the element in the third row second column:

$$\begin{aligned}[\mathbf{R}_2^1]_{3,2} &= \mathbf{K}_1 \cdot \mathbf{J}_2 \\ &= \|\mathbf{K}_1\|_2 \|\mathbf{J}_2\|_2 \cos(\beta_3) \\ &= \cos(\beta_3)\end{aligned}$$

where β_3 is the angle between $\mathbf{K}_1$ and $\mathbf{J}_2$ and we have used the fact that $\|\mathbf{K}_1\|_2 = \|\mathbf{J}_2\|_2 = 1.$

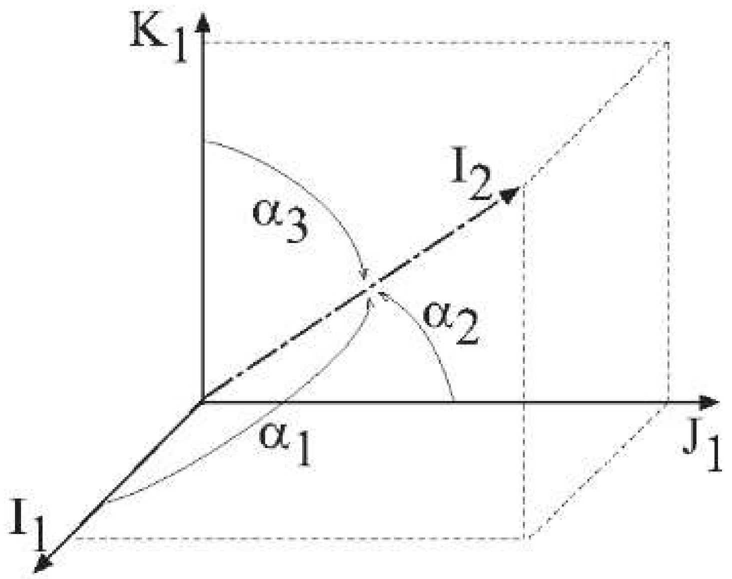

Figure 2.11: Definition of α_i for $i = 1, 2, 3$ **in eqn. (2.16).**

Because each element of $\mathbf{R}_2^1$ is the cosine of the angle between one of the coordinate axes of $\phi 1$ and one of the coordinate axes of $\phi 2$, the matrix $\mathbf{R}_2^1$ is referred to as a direction cosine matrix:

$$\mathbf{R}_2^1 = \begin{bmatrix} \cos(\alpha_1) & \cos(\beta_1) & \cos(\gamma_1) \\ \cos(\alpha_2) & \cos(\beta_2) & \cos(\gamma_2) \\ \cos(\alpha_3) & \cos(\beta_3) & \cos(\gamma_3) \end{bmatrix}. \qquad (2.16)$$

Figure 2.11 depicts the angles . α_i for $i = 1, 2, 3$ that define the first column of $\mathbf{R}_2^1$. The β_i and γ_i angles are defined similarly. When the relative orientation of two reference frames is known, the direction cosine matrix $\mathbf{R}_2^1$ is unique and known.

Although the direction cosine matrix has nine elements, due to the three orthogonality constraints and the three normality constraints, there are only three degrees of freedom.

$$\mathbf{v}_2 = \mathbf{I}_2 x_2 + \mathbf{J}_2 y_2 + \mathbf{K}_2 z_2$$

where $[\mathbf{v}_2]^2 = [x_2, y_2, z_2]^\top$ and by eqn. (2.14)

$$[\mathbf{v}_2]^1 = \begin{bmatrix} \mathbf{I}_1^\top \\ \mathbf{J}_1^\top \\ \mathbf{K}_1^\top \end{bmatrix} \mathbf{v}_2.$$

Therefore,

$$\begin{aligned} [\mathbf{v}_2]^1 &= \begin{bmatrix} \mathbf{I}_1^\top \\ \mathbf{J}_1^\top \\ \mathbf{K}_1^\top \end{bmatrix} (\mathbf{I}_2 x_2 + \mathbf{J}_2 y_2 + \mathbf{K}_2 z_2) \\ &= \begin{bmatrix} \mathbf{I}_1 \cdot \mathbf{I}_2 \\ \mathbf{J}_1 \cdot \mathbf{I}_2 \\ \mathbf{K}_1 \cdot \mathbf{I}_2 \end{bmatrix} x_2 + \begin{bmatrix} \mathbf{I}_1 \cdot \mathbf{J}_2 \\ \mathbf{J}_1 \cdot \mathbf{J}_2 \\ \mathbf{K}_1 \cdot \mathbf{J}_2 \end{bmatrix} y_2 + \begin{bmatrix} \mathbf{I}_1 \cdot \mathbf{K}_2 \\ \mathbf{J}_1 \cdot \mathbf{K}_2 \\ \mathbf{K}_1 \cdot \mathbf{K}_2 \end{bmatrix} z_2 \\ &= \mathbf{R}_2^1 [\mathbf{v}_2]^2. \end{aligned} \tag{2.17}$$

Point Transformation

When eqn. (2.17) is substituted into eqn. (2.15) it yields the desired equation for the transformation of the coordinates of P with respect to frame 2, as represented by $[\mathbf{v}_2]^2$, to the coordinates of P with respect to frame 1, as represented by $[\mathbf{v}_1]^1$:

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}^1 &= \begin{bmatrix} x_O \\ y_O \\ z_O \end{bmatrix}^1 + \mathbf{R}_2^1 [\mathbf{v}_2]^2 \\ [\mathbf{v}_1]^1 &= [\mathbf{O}_{12}]^1 + \mathbf{R}_2^1 [\mathbf{v}_2]^2. \end{aligned} \tag{2.18}$$

The reverse transformation is easily shown from eqn. (2.18) to be

$$
\begin{aligned}
\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}^2 &= \mathbf{R}_1^2 \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}^1 - \begin{bmatrix} x_O \\ y_O \\ z_O \end{bmatrix}^1 \right) \\
[\mathbf{v}_2]^2 &= \mathbf{R}_1^2 \left([\mathbf{v}_1]^1 - [\mathbf{O}_{12}]^1 \right) \qquad (2.19)
\end{aligned}
$$

where we have used the fact that $\mathbf{R}_1^2 = \left(\mathbf{R}_2^1\right)^{-1} = \left(\mathbf{R}_2^1\right)^{T}$, where the last equality is true due to the orthonormality of RJ. Note that the point transformation between reference systems involves two operations: translation to account for separation of the origins, and rotation to account for non-alignment of the axis.

Vector Transformation

Consider two points P_1 and P_2. Let the vector v denote the directed line segment from P_1 to P_2. Relative to ϕ_1, v can be described as

$$
\begin{aligned}
\mathbf{v} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}^1 &= \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}^1 - \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}^1 \\
&= \left(\begin{bmatrix} x_O \\ y_O \\ z_O \end{bmatrix}^1 + \mathbf{R}_2^1 \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}^2 \right) \\
&\quad - \left(\begin{bmatrix} x_O \\ y_O \\ z_O \end{bmatrix}^1 + \mathbf{R}_2^1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{R}_2^1 \left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}^2 - \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}^2 \right) \\
\mathbf{v}^1 &= \mathbf{R}_2^1 \mathbf{v}^2. \qquad (2.20)
\end{aligned}
$$

Eqn. (2.20) is the vector transformation between coordinate systems. This relation is valid for any vector quantity. As discussed in detail in [95], it is important to realized that vectors, vector operations, and relations between vectors are invariant relative to any two particular coordinate representations as long as the coordinate systems are related through eqn. (2.20). This is important, as it corresponds to the intuitive notion that the physical properties of a system are invariant no matter how we orient the coordinate system in which our analysis is performed.

In the discussion of this section, the two frames have been considered to have no relative motion. Issues related to relative motion will be critically important in navigation systems and are discussed in subsequent sections.

Throughout the text, the notation $\mathbf{R}_a^b$ will denote the rotation matrix transforming vectors from frame a to frame b. Therefore,

$$\mathbf{v}^b = \mathbf{R}_a^b \mathbf{v}^a \qquad (2.21)$$

$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b \qquad (2.22)$$

where $\mathbf{R}_b^a = \left(\mathbf{R}_a^b\right)$.

Matrix Transformation

In some instances, a matrix will be defined with respect to a specific frame of reference. Eqns. (2.21-2.22) can be used to derive the transformation of such matrices between frames-of-reference.

Let $\mathbf{\Omega}^a \in \mathbb{R}^{3\times 3}$ be a matrix defined with respect to frame a and $\mathbf{v}_1^a, \mathbf{v}_2^a \in \mathbb{R}^3$ be two vectors defined in frame a. Let b be a second frame of reference. If $\mathbf{v}_1^a$ and $\mathbf{v}_2^a$ are related by

$\mathbf{v}_1^a = \mathbf{\Omega}^a \mathbf{v}_2^a$, then eqns. (2.21-2.22) show that

$$\mathbf{R}_b^a \mathbf{v}_1^b = \mathbf{\Omega}^a \mathbf{R}_b^a \mathbf{v}_2^b$$

or

$$\mathbf{v}_1^b = \mathbf{\Omega}^b \mathbf{v}_2^b$$

where

$$\mathbf{\Omega}^b = \mathbf{R}_a^b \mathbf{\Omega}^a \mathbf{R}_b^a \qquad (2.23)$$

is the representation of the matrix $\mathbf{\Omega}$ with respect to frame b.

Next post: Specific Vector Transformations (GPS) Part 1

Previous post: ECEF Coordinate Systems (GPS)

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