Simulating Charged Particle Trajectories in Earth's Magnetosphere with Python to study Van Allen belts

Project 6 - DD2730, 2024/2025

João Pedro Ferreira Biu

Objectives

- Study the performance of 4^{th} order Runge-Kutta and Boris method to simulate particles in the Van Allen Belt
- Study conservation of quantities for various pitch angles
- Conclude which model and method to use for protons and electrons

Particle Dynamics in EM field: Lorentz Force

Lorentz Force Equation:

$$\boxed{\frac{d(\gamma m\mathbf{v})}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

No electric field :
Only gyration motion around magnetic field lines

Cyclotron Frequency: $\omega_C = \frac{|q|B}{m}$

Larmor Radius $r_L = \frac{mv_\perp}{|q|B}$

The circular motion is done around the **guiding center**

In this case, we compute the trajectory of the particle (in blue).

Particle Dynamics in EM field: Guiding Center

$$\frac{dR}{dt} = \frac{\gamma m v^2}{2qB^2}\left(1 + \frac{v_\parallel^2}{v^2}\right)\hat{\mathbf{b}} \times \nabla B + v_\parallel \hat{\mathbf{b}}$$

$$\frac{dv_\parallel}{dt} = -\frac{\mu}{\gamma^2 m}\hat{\mathbf{b}} \cdot \nabla B$$

$$\mu = \frac{m v_\perp^2}{2B}$$

- Valid approximation when $r_L \ll \frac{B}{|\nabla B|}$ – the field does not change significantly within a cyclotron radius
- Compute the trajectory of the Guiding Center
- Removes the need to resolve the cyclotron motion – the smallest length scale of the system

In this case, we compute the trajectory of the guiding center (in red).

Dipole Model Of Earth's Magnetic Field

Near Earth, where the effects of solar winds are not significant, we can approximate Earth's magnetic field to a dipole, this is $\mathrm{r}\sim 2R_E - 3R_E$.

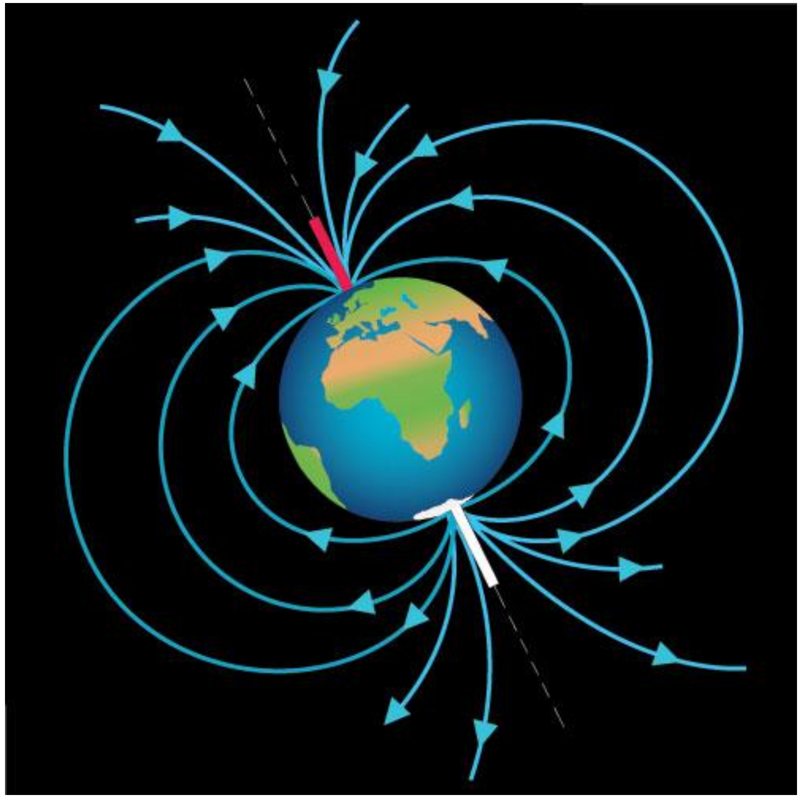
$$B_x = -\frac{3M\mu_0}{4\pi r^5}xz = -\frac{3B_0R_E^3}{r^5}xz$$

$$B_y = -\frac{3M\mu_0}{4\pi r^5}yz = -\frac{3B_0R_E^3}{r^5}yz$$

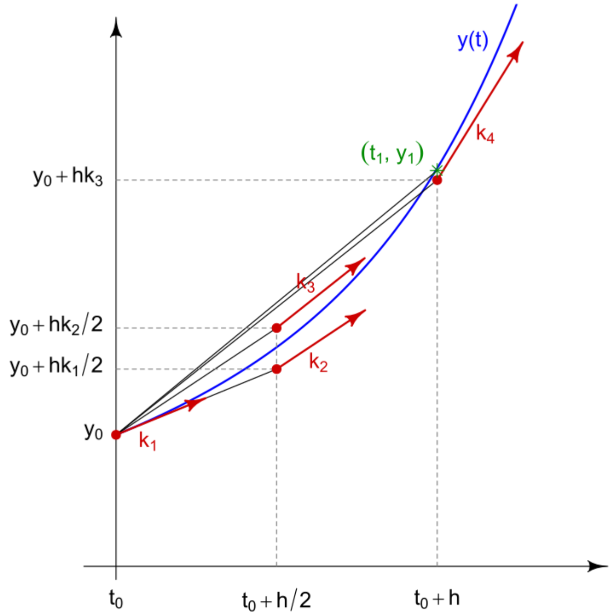
$$B_z = -\frac{M\mu_0}{4\pi r^5}(2z^2 - x^2 - y^2) = -\frac{B_0R_E^3}{r^5}(2z^2 - x^2 - y^2)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Where B_0 is the intensity of the magnetic field at the surface of Earth, and R_E is Earth's radius.



Methods to solve the ODE's



[1]

Runge-Kutta of 4th order

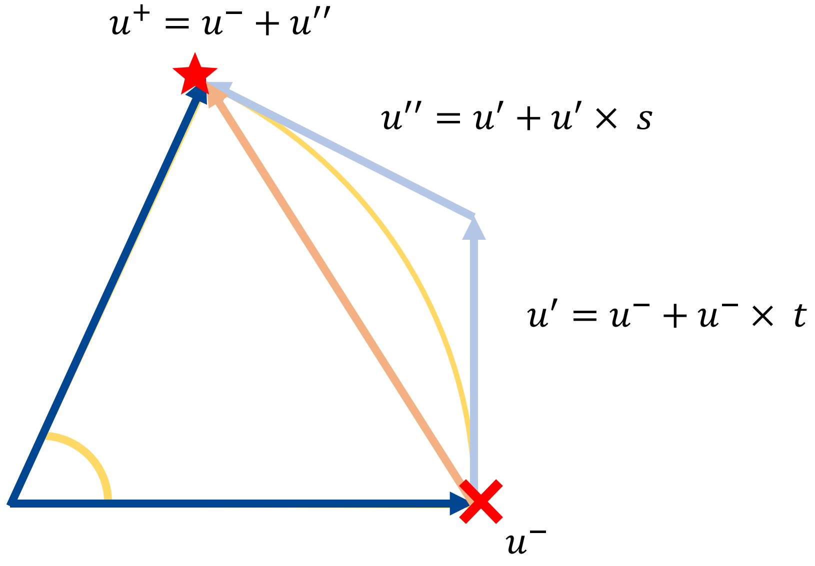
- General Purpose ODE solver
- Easy to implement and generalizable
- Error of order $O(h^4)$
- Non-symplectic
- Needs to resolve gyromotion, so $\Delta t \ll \frac{2\pi}{\omega_C}$

Start at state-i: (x_i, v_i) → Compute state k_1, k_2, k_3, k_4 at intermediate steps → Compute new position and velocity, (x_{i+1}, v_{i+1}) using the weighted average of slopes → Update state $(x_i, v_i) \leftarrow (x_{i+1}, v_{i+1})$

[1] https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods

Methods to solve the ODE's

[2]



Boris Pusher

- Specialized method to solve Lorentz Equation
- Used in particle tracer and Particle-in-Cell codes to compute the motion of particles in EM fields
- Preserves phase-space volume
- Long term accuracy
- Velocity rotation step allows to use larger step size

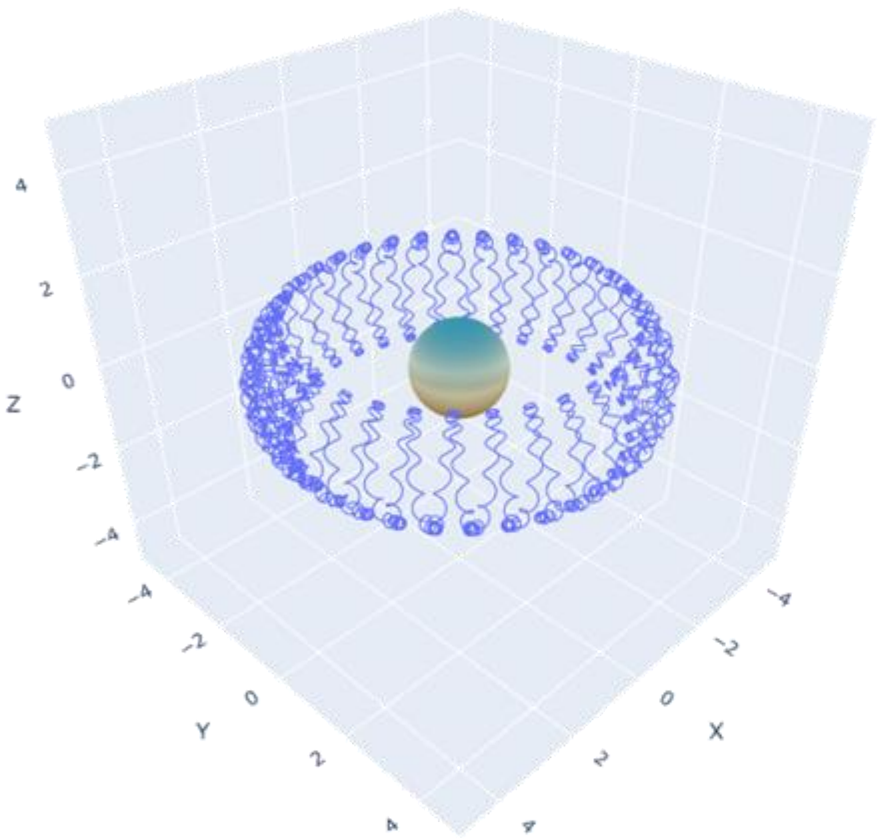
Start at state-i: (x_i, v_i) → Update velocity due to **E** field → Rotate velocity due to **B** field → Update state $(x_i, v_i) \leftarrow (x_{i+1}, v_{i+1})$

What is Evaluated?

- Kinetic energy conservation, $\mathrm{K} = mc^2(\gamma - 1)$
- First adiabatic invariant, $\mu = \frac{\gamma m v_\perp^2}{2B}$
- Second adiabatic invariant, $\int \frac{v_\parallel^2}{v} dt$
- Third adiabatic invariant, $\phi = \int 2\pi r B_0 \frac{R_E^3}{r^3} dr = \frac{2\pi B_0 R_E^3}{R}$

Simulating Protons

- Smallest time scale: $\Delta t \ll \frac{2\pi}{\omega_c} \approx 1.16 \times 10^{-6}\ [s]$
- RK cannot be used for long times in feasible running time
- $\Delta t = 1 \times 10^{-4}$ for Boris and $\Delta t = 1 \times 10^{-7}$ forRK
- Pitch angles $\alpha = [15º, 30º, 45º, 60º, 90º]$

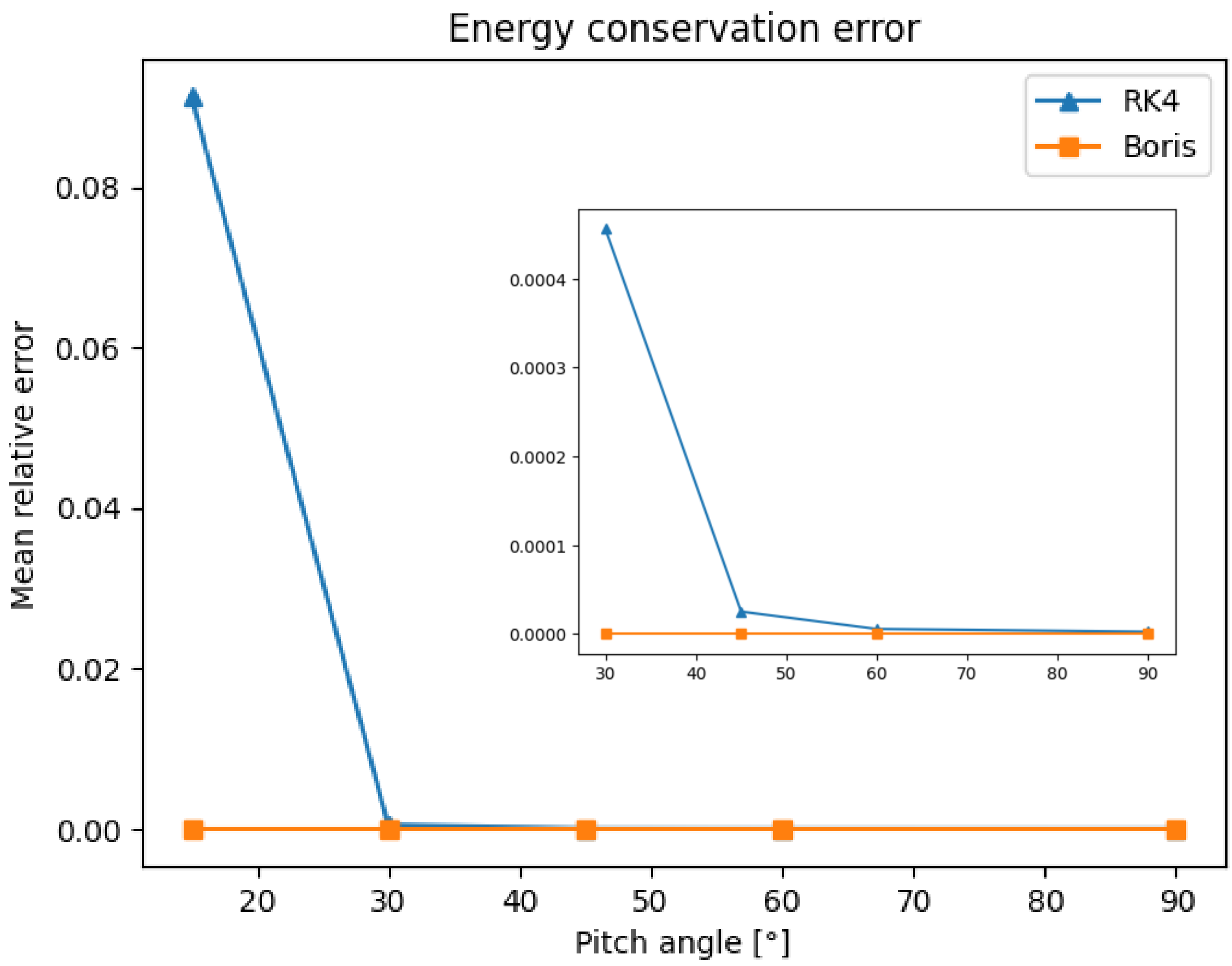


Lorentz Force: Protons

- Initial Kinetic Energy $K_0 = 10\ MeV$
- Mean relative error

$$\overline{\delta A} = \left\langle \left| \frac{A - A_0}{A_0} \right| \right\rangle \cdot 100$$

- Boris: conserves energy
- Runge-Kutta: K-conservation gets worse as the pitch angle gets lower



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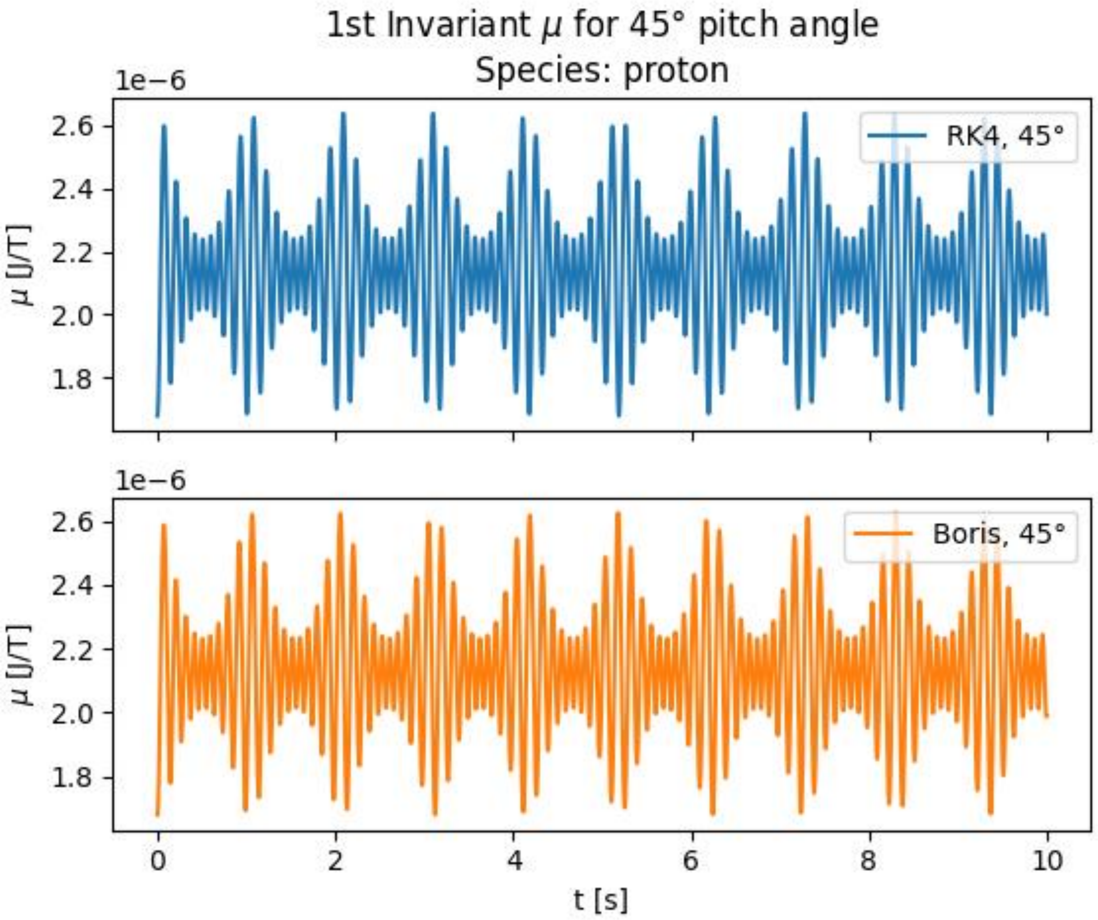

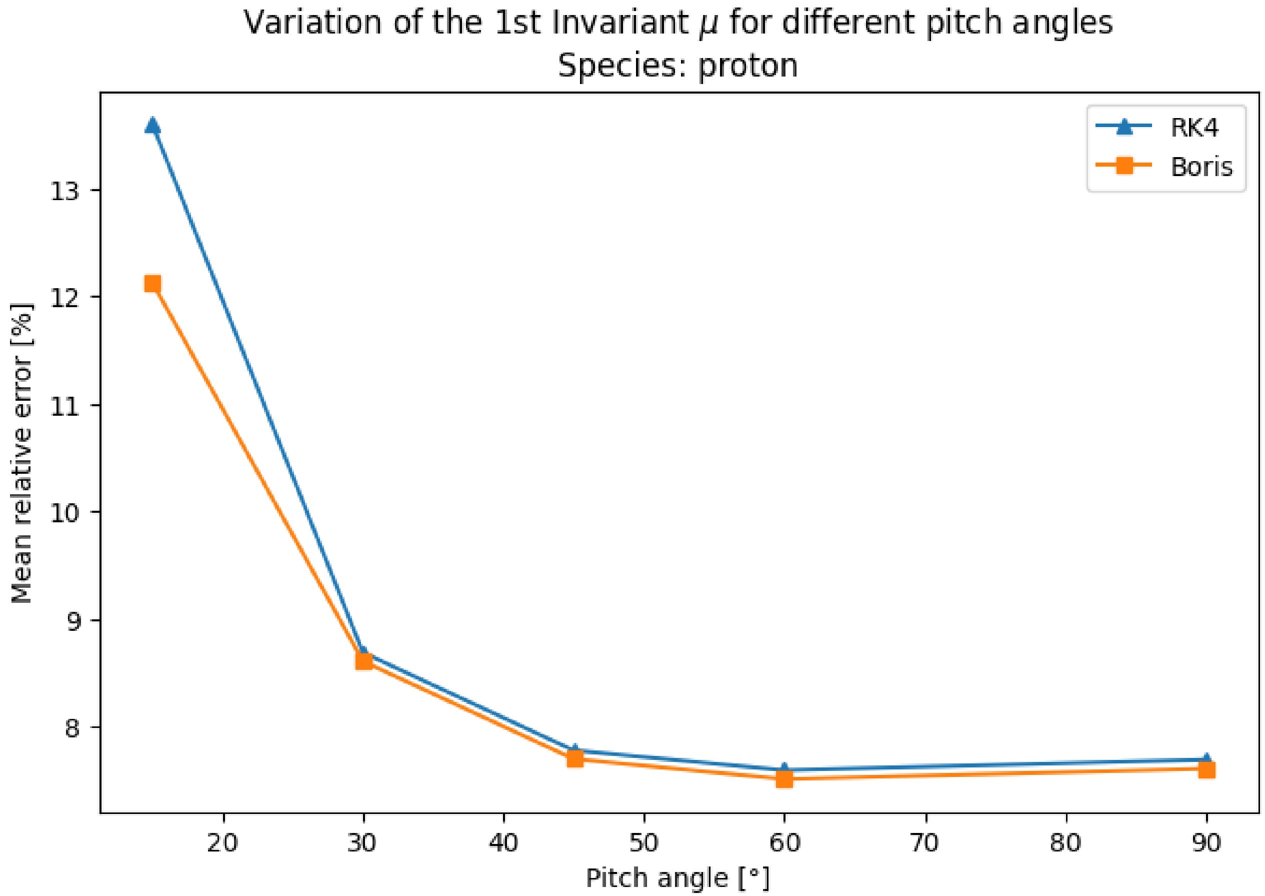

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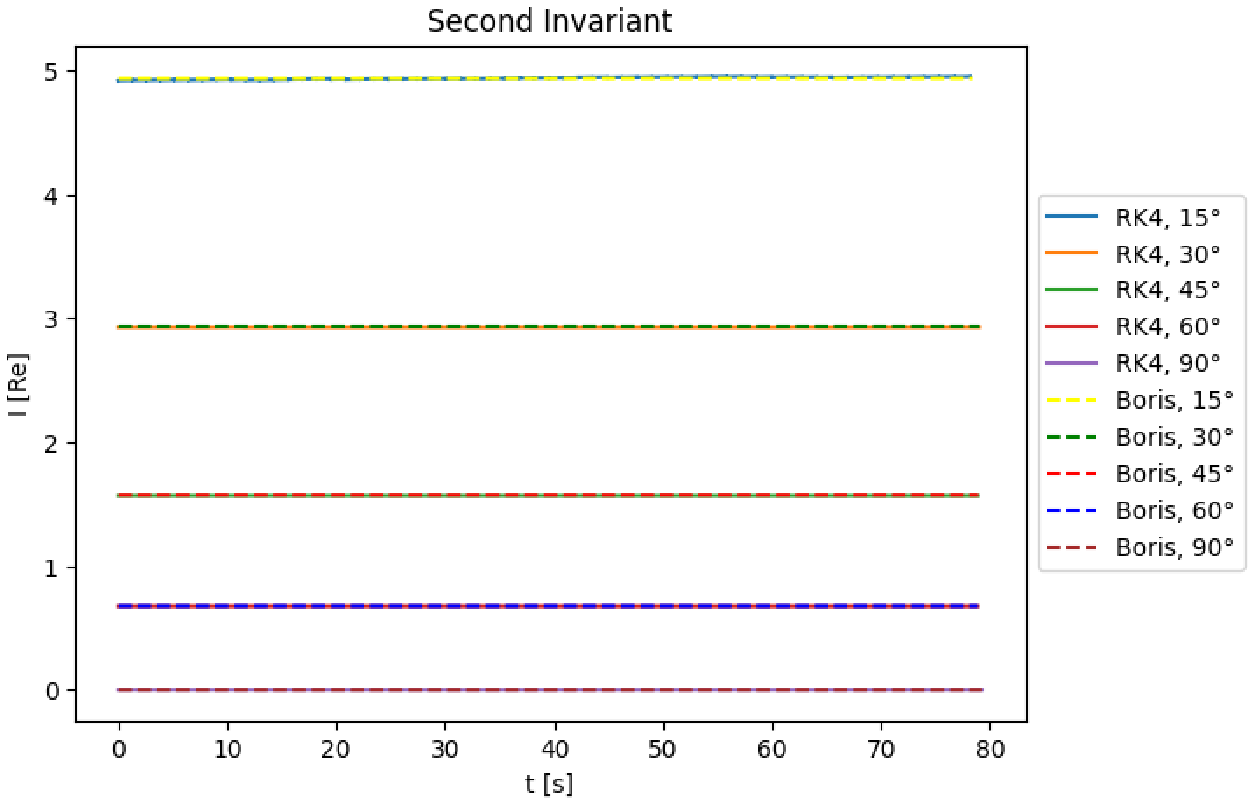

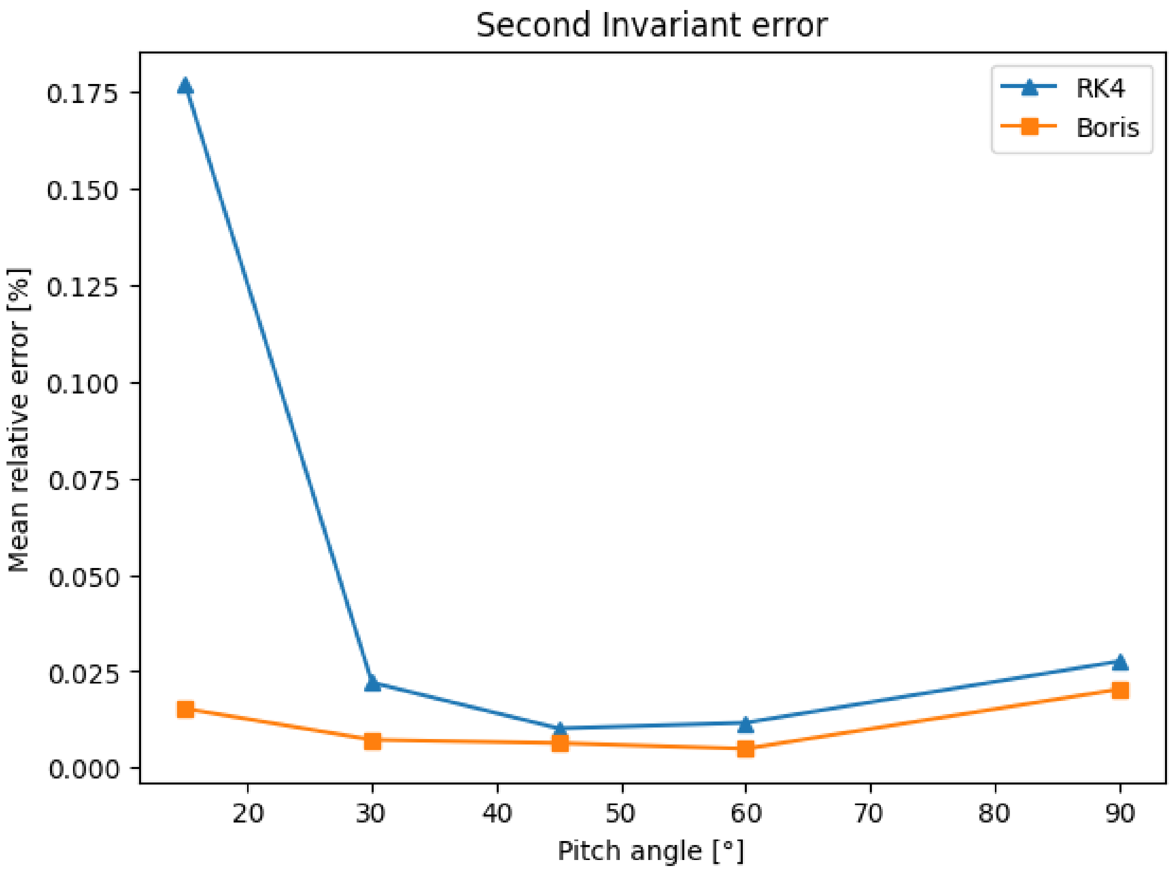

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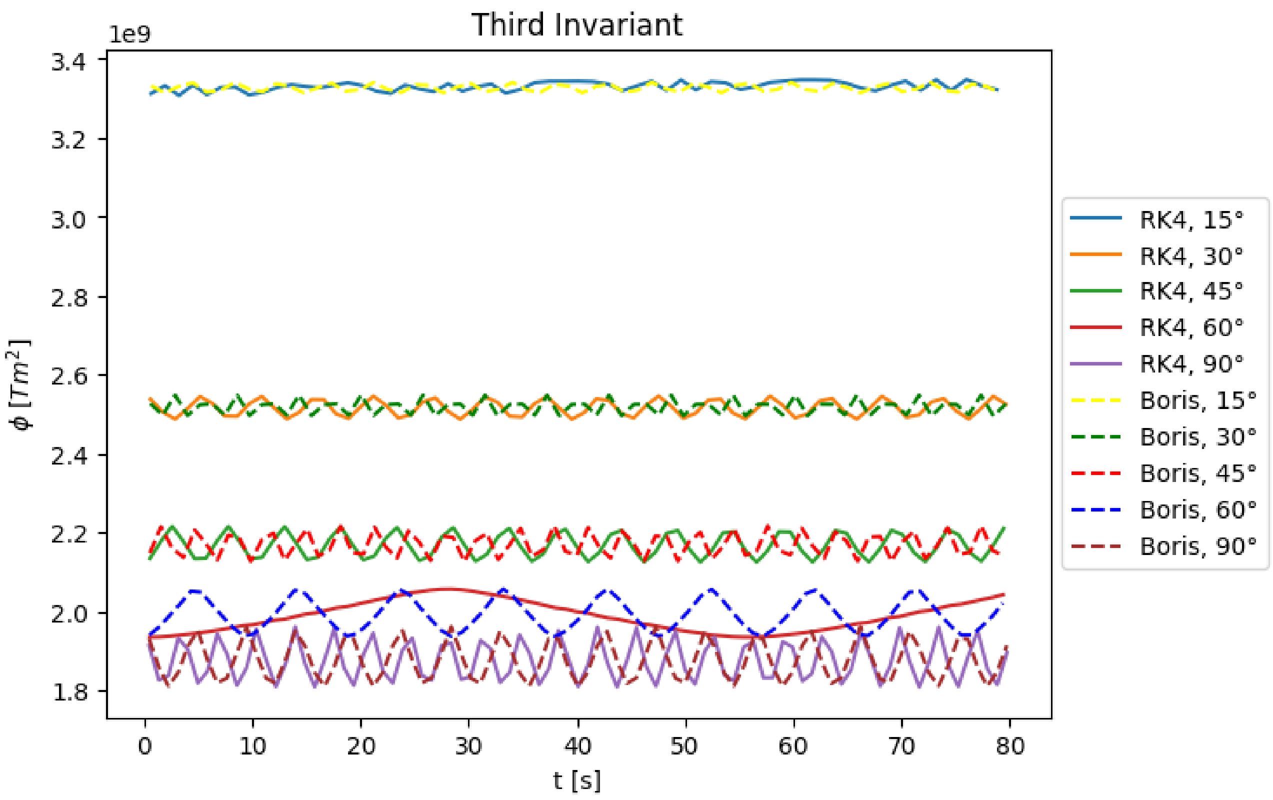
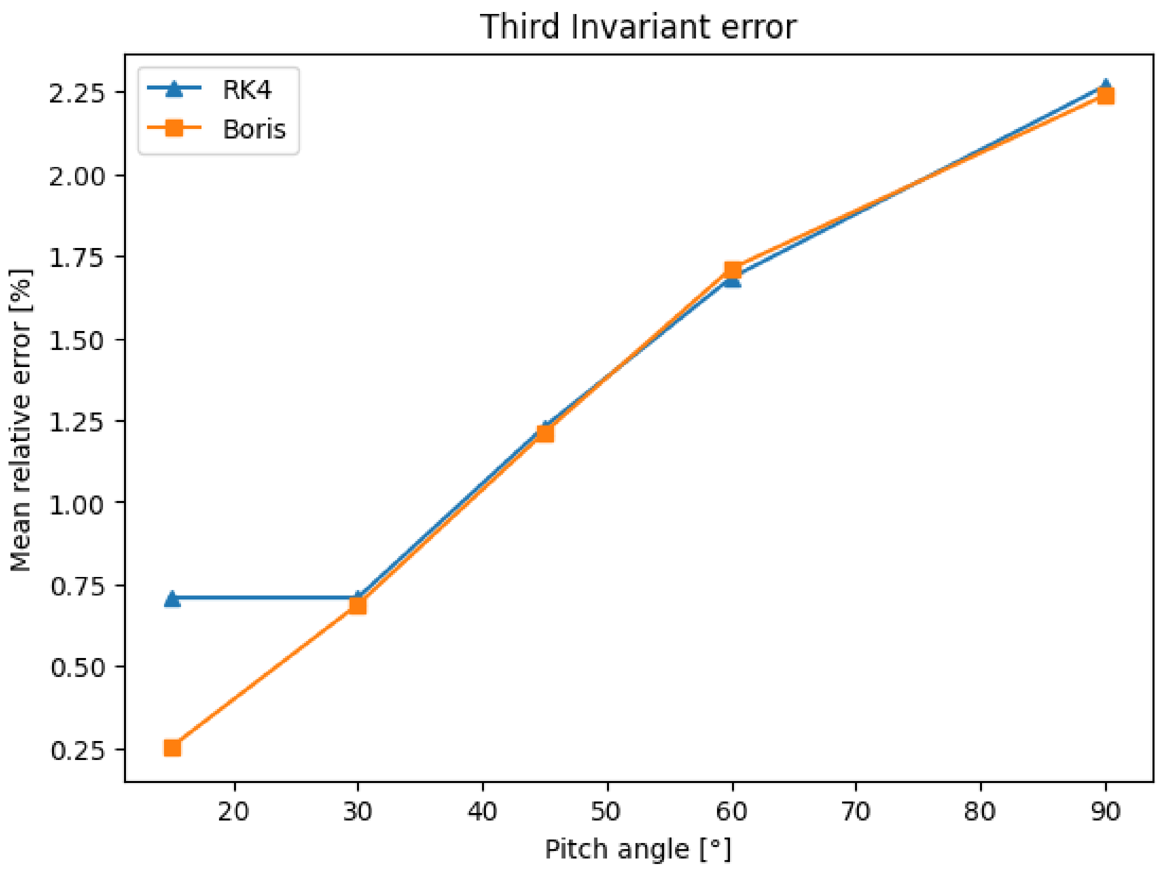
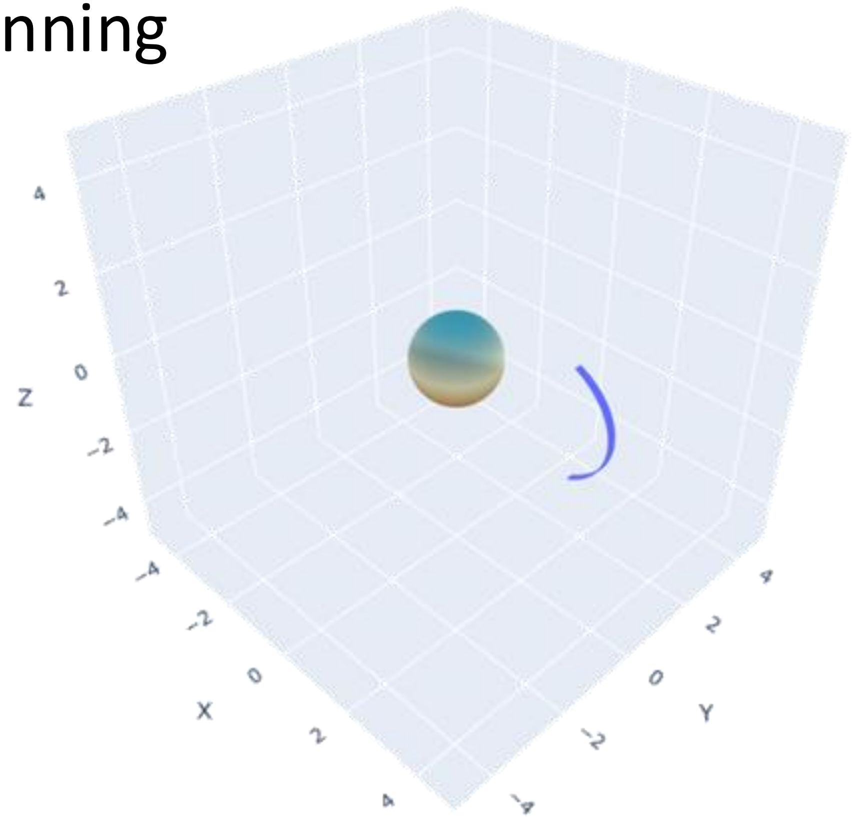




Simulating Electrons

- Smallest time scale: $\Delta t \ll \frac{2\pi}{\omega_c} \approx 1.16 \times 10^{-6}\ [s]$
- RK can be used for long times in feasible running time
- $\Delta t = 0.001$ for both methods
- Pitch angles $\alpha = [15º, 30º, 45º, 60º, 90º]$



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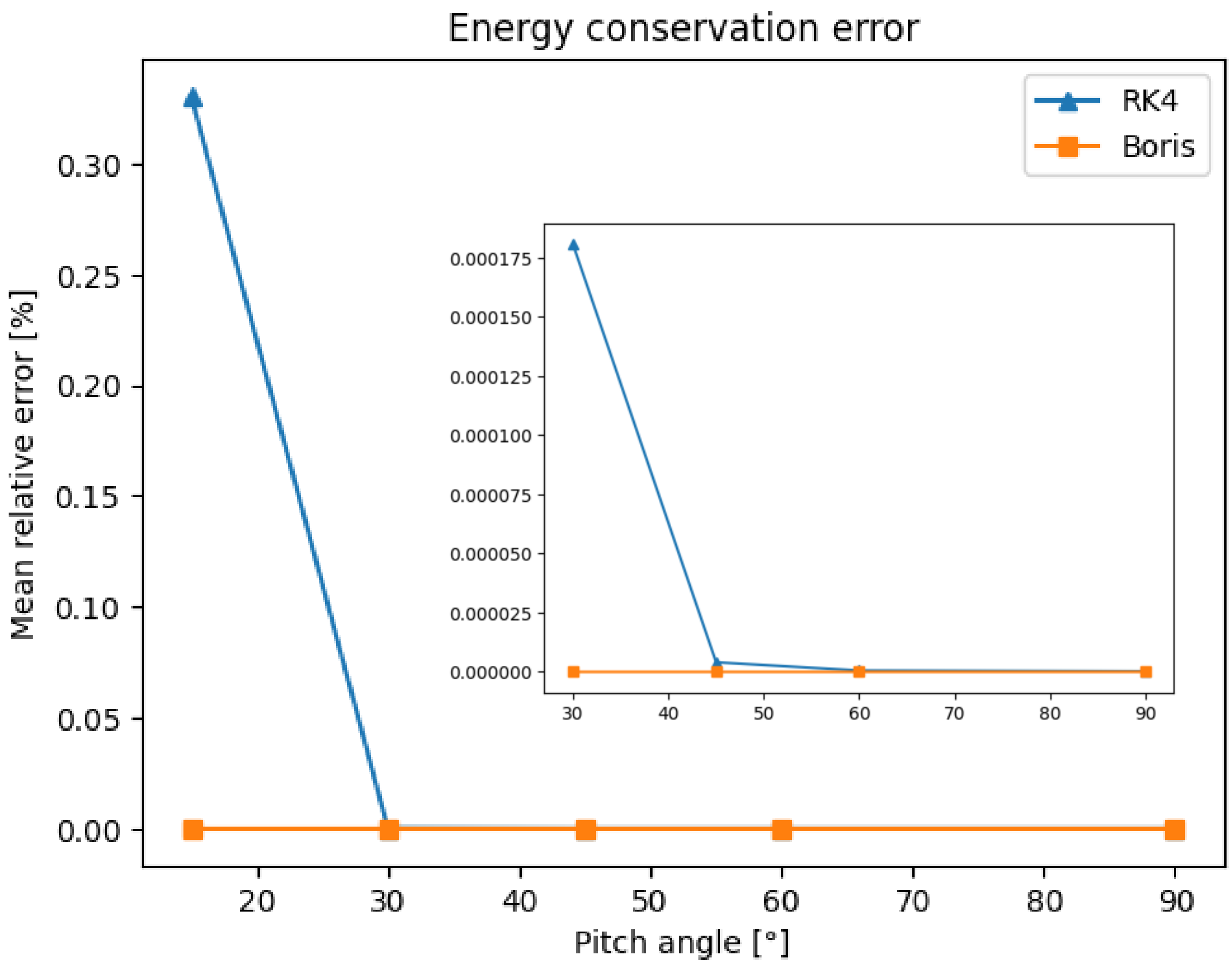

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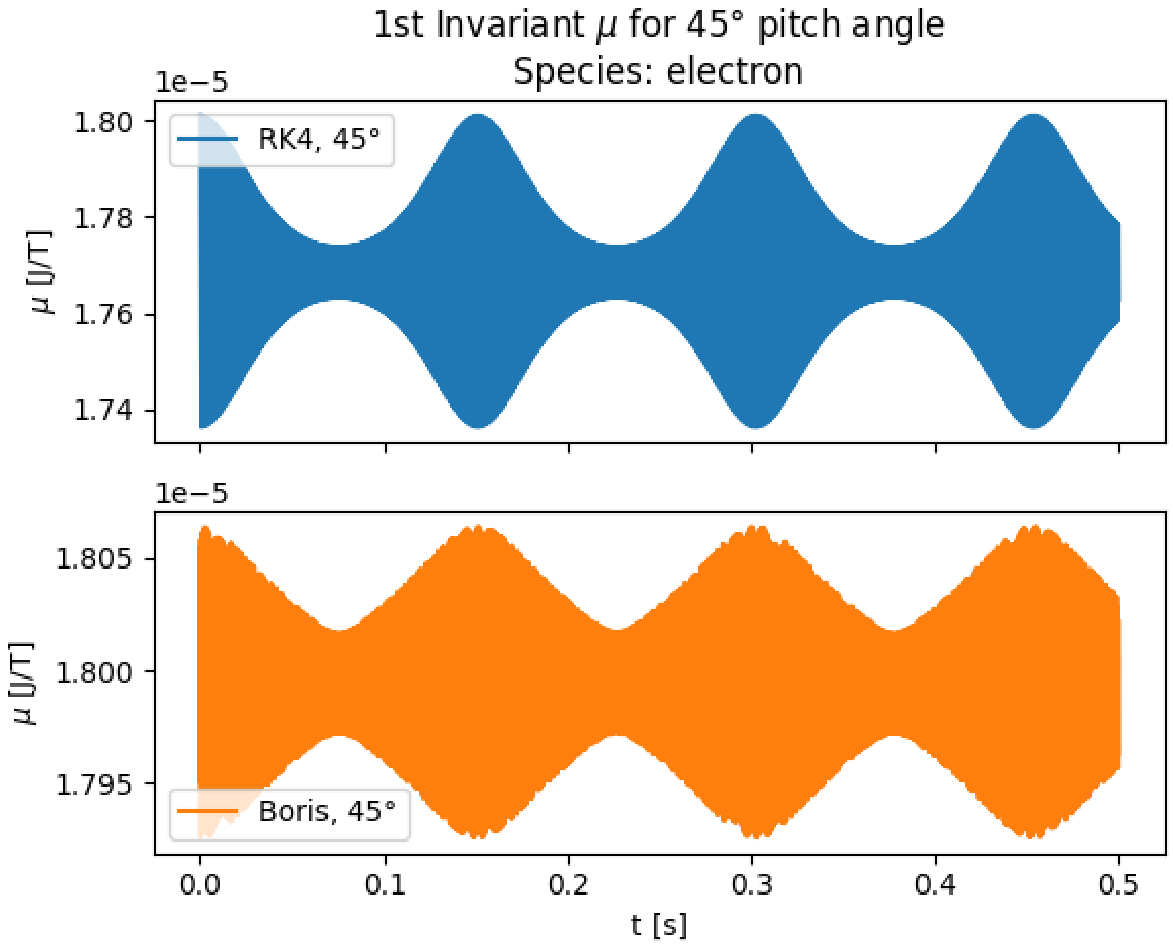

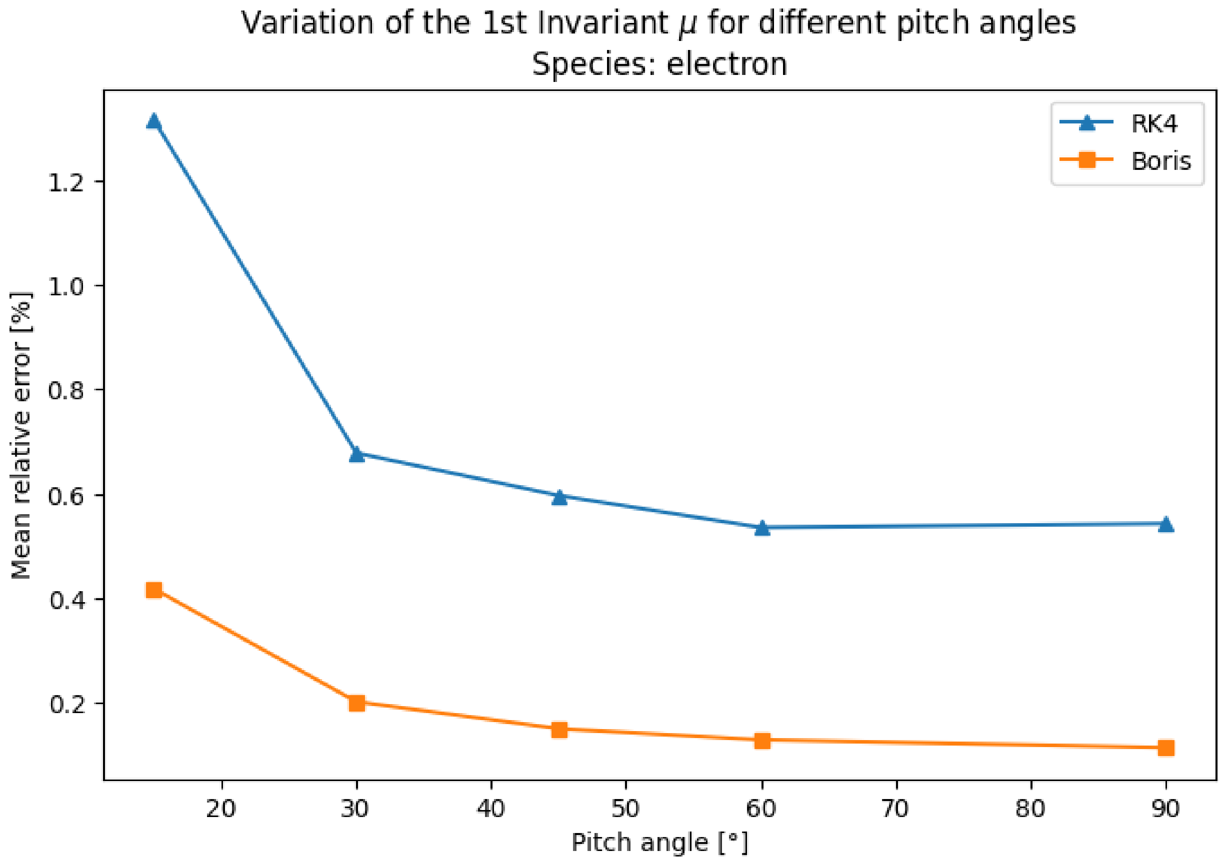

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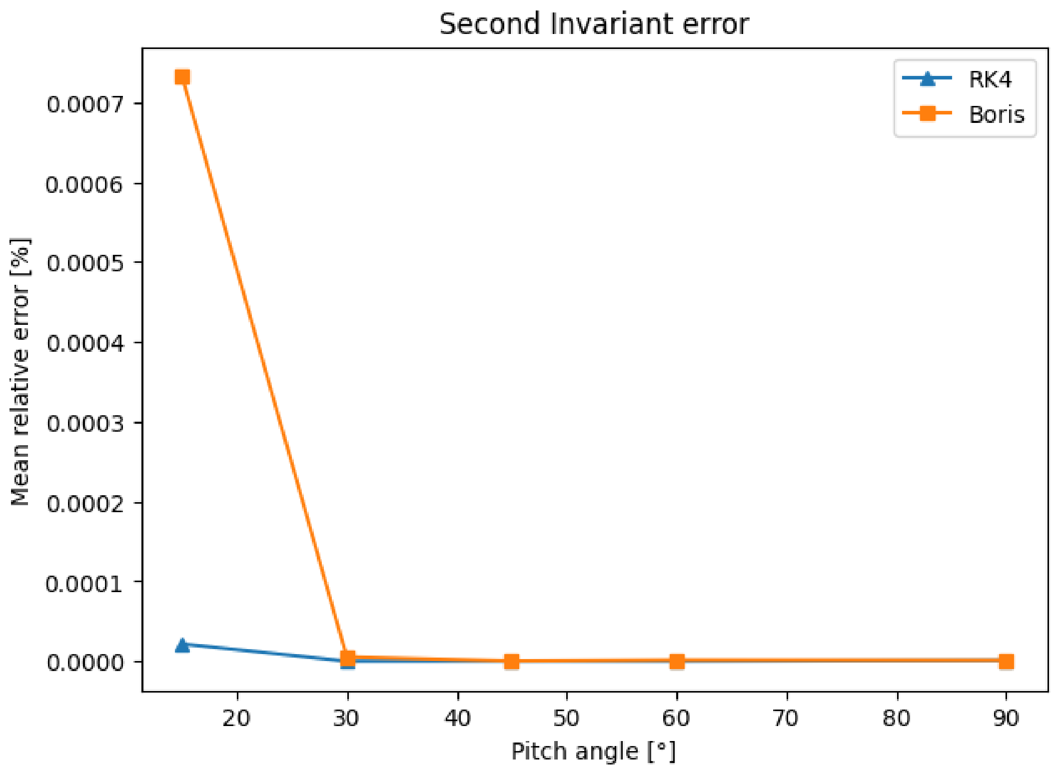
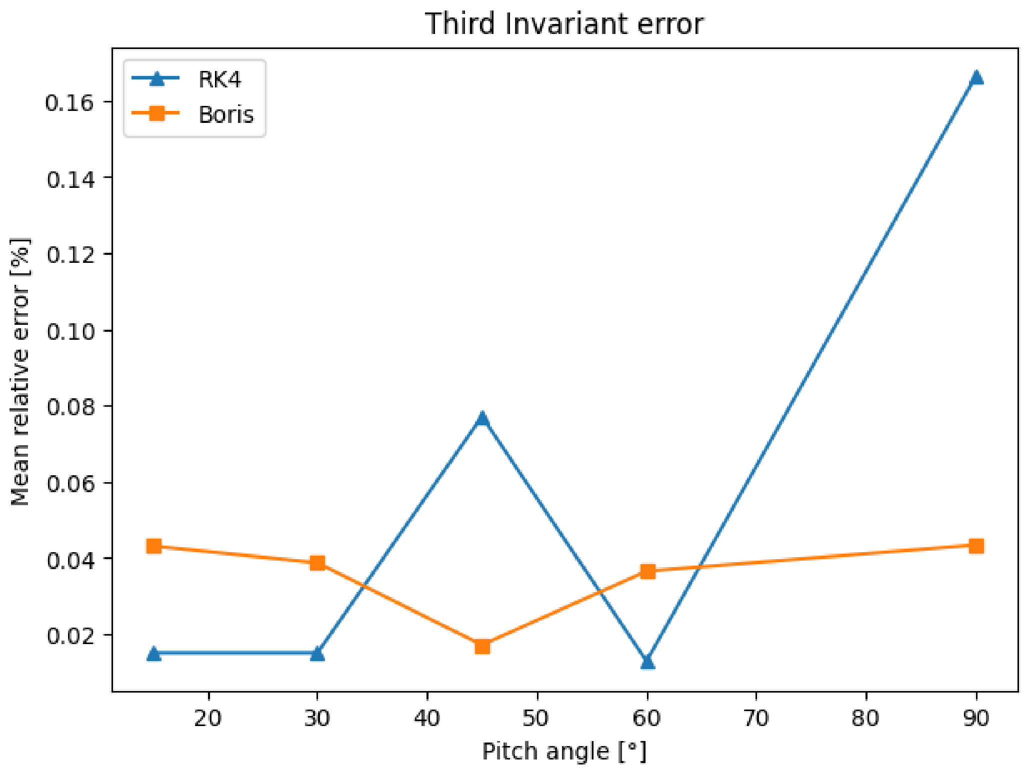




Method Comparison for Electrons

Runge-Kutta 4:

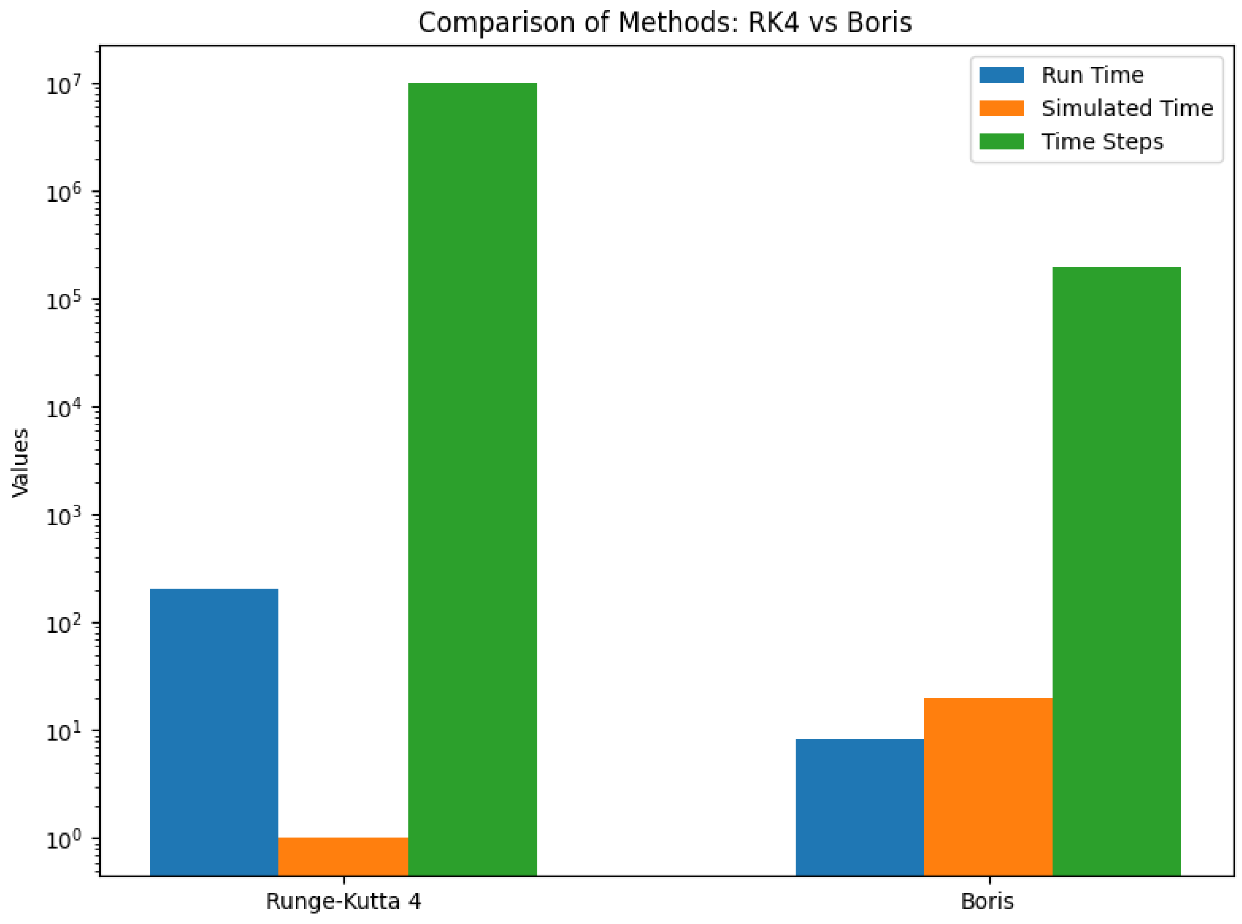
- $\Delta t = 1 \times 10^{-7}$
- $t_{max} = 1$
- $\#steps = 1 \times 10^{7}$

- Diagnostics:
  - $\overline{\delta K} = 8.1 \times 10^{-6}\ \%$
  - $\overline{\delta \mu} = 5.7 \times 10^{-1}\ \%$
  - $\overline{\delta I} = 9.2 \times 10^{-8}\ \%$
  - $\overline{\delta K} = 7.4 \times 10^{-2}\ \%$

Boris Method:

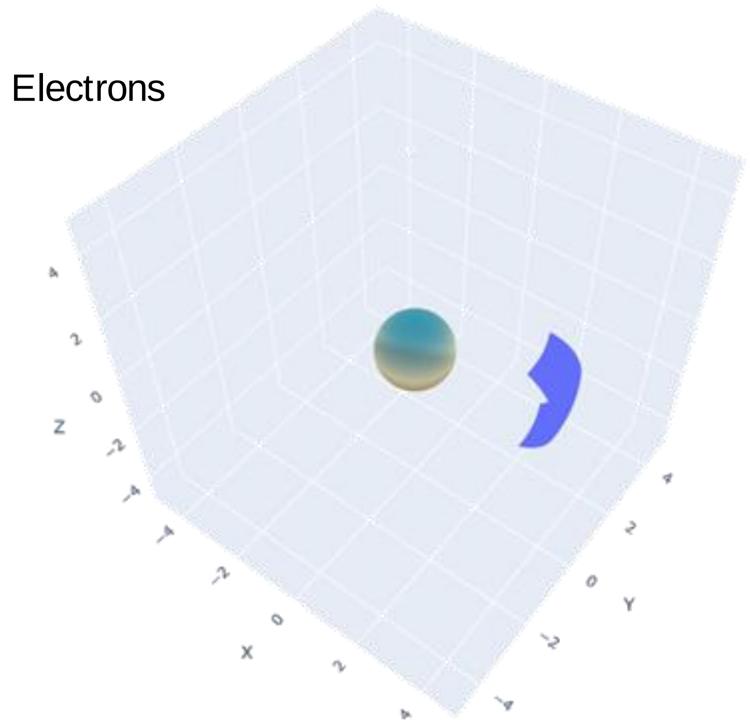
- $\Delta t = 1 \times 10^{-4}$
- $t_{max} = 20$
- $\#steps = 2 \times 10^{-5}$

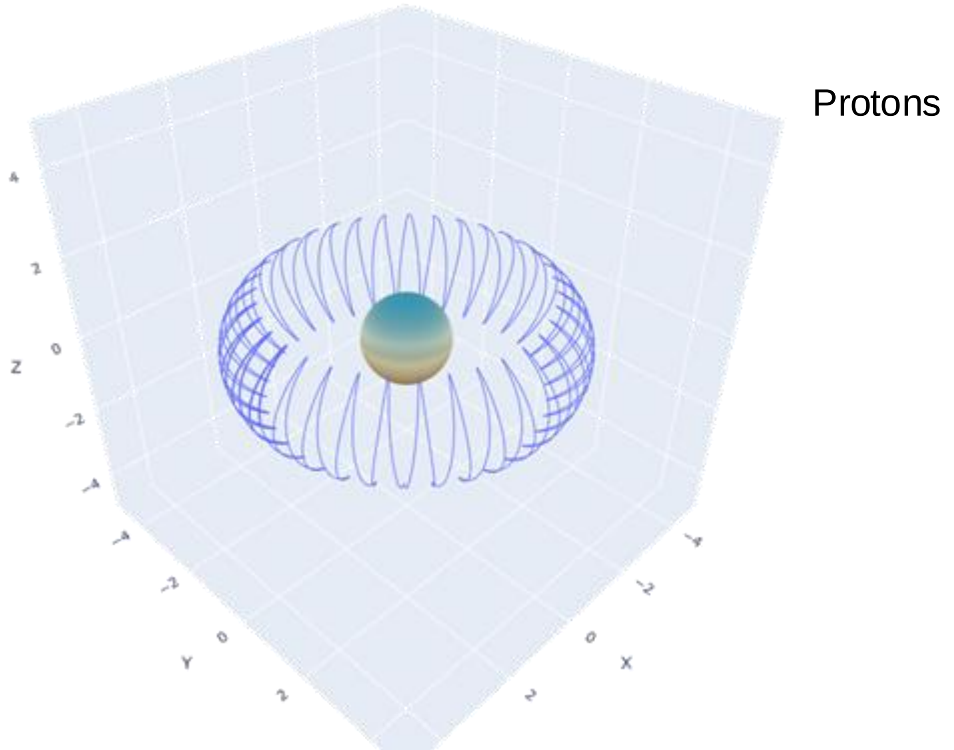
- Diagnostics:
  - $\overline{\delta K} = 1.1 \times 10^{-9}\ \%$
  - $\overline{\delta \mu} = 1.5 \times 10^{-1}\ \%$
  - $\overline{\delta I} = 5.1 \times 10^{-7}\ \%$
  - $\overline{\delta K} = 3.6 \times 10^{-2}\ \%$



Guiding Center

- The gyromotion is averaged – good approximation when the Larmor radius is small, e.g. for the electrons
- No gyromotion = no need to resolve gyrofrequency (we can use RK4)
- Allow us to capture the essential dynamics and run simulations for a long time
- 1st adiabatic invariant is guaranteed to be conserved!





Conclusion and Take-Aways

- The smaller the pitch angle the larger the instabilities and errors in conservation – contributing to less confinement.

- The Boris method allows the use of larger time steps, has better energy and invariant conservation.

- The Guiding Center approximation is a better choice to simulate electrons if we are not interested in small-scale phenomena
  - To use the Lorentz Equation we can use optimized algorithms or develop the simulations using compiled languages, e.g. Fortran, C, C++.