DD2370 Computational Methods for Electromagnetics

Discretizing First-Order Derivative and Analyzing Their Numerical Properties

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From Poisson (2^{nd} Order) to Maxwell (1^{st} Order)

- For Laplace’s equation (second order derivative), e.g. Poisson equation, a straightforward application of finite differences works well.
- However, when **derivatives of odd** order **(in particular first order derivative like in Maxwell’s Equations)** are involved, a different technique is required to get good results.

Motivation - Ampere's and Faraday's laws in **Frequency Domain**

- Consider a **one-dimensional wave (only z direction)** on the form **E**= Ex (z) **i** and **H** = Hy (z) **j** which propagates in vacuum.
- Only the x-component is nonzero for Ampere's law and only the y- component is nonzero for Faraday's law:

$$\frac{dE_x(z)}{dz} = -j\omega\mu_0 H_y(z)$$

$$-\frac{dH_y(z)}{dz} = j\omega\epsilon_0 E_x(z)$$

- These equations involve the **first-order derivative**

Two Solutions for Approximating the 1st Order Derivative

- We want a truncation error that is proportional to h^2 → **Centered**
- Two different alternatives for the representation of the first-order derivative:

1. Finite-difference approximation that extends **over two cells**
2. Finite-difference approximation that extends **over only one cell**

1.

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$$

2.

$$\frac{f(x+h) - f(x)}{h} = f'\left(x + \frac{h}{2}\right) + O(h^2)$$

First Solution – Two Cells Solution

- Introduce a computational grid with the grid points ***z_i = ih***, where the electric field is represented by **Ex_i (*z*)** and the magnetic field is represented by **Hy_i(z). Ex and By are collocated.**
- Finite-difference approximations of the first-order derivative that extends over two cells:

$$\frac{E_x(z_{i+1}) - E_x(z_{i-1})}{2h} = -j\omega\mu_0 H_y(z_i)$$

$$-\frac{H_y(z_{i+1}) - H_y(z_{i-1})}{2h} = j\omega\epsilon_0 E_x(z_i)$$

- **Second order accurate**

Second Solution – **Staggering E and H**

- The alternative solution is to use the finite-difference approximation that extends **over only one cell**.

$$\frac{E_x(z_{i+1}) - E_x(z_i)}{h} = -j\omega\mu_0 H_y\left(z_{i+\frac{1}{2}}\right)$$

$$-\frac{H_y\left(z_{i+\frac{1}{2}}\right) - H_y\left(z_{i-\frac{1}{2}}\right)}{h} = j\omega\epsilon_0 E_x(z_i)$$

- Thus, the unknowns for the magnetic field are **shifted half a cell** with respect to the unknowns for the electric field and this is referred to as **staggered grids**.

Dispersion Relation – Choosing the Right Solution

- We can understand by means the dispersion relation which states that **w = c k**.
- The frequency w represents the time variation of the electromagnetic field → As the frequency *w* increases, the time variation becomes more and more rapid.
- The wavenumber k is proportional to the frequency → **an increase in frequency implies an increase in the wavenumber.**

Wavenumber – Discrete Spatial Variation

- The wavenumber *k = 2pi/lambda* represents the spatial variation of the electromagnetic field, where *lambda* is the wavelength → the **spatial variation** becomes more and more rapid as the wavenumber increases
- The spatial variation is not explicitly represented by the wavenumber in our finite-difference scheme → **the finite-difference approximation of the first-order derivatives that *evaluates* the spatial variation**

Complex Exponential for Analysis. Why?

- All functions can be decomposed as sums over complex exponentials (the Fourier transform).
- The complex exponentials ***exp(jkx),*** where ***j*** is the imaginary unit and **k** is the wavenumber are eigenfunctions of the derivative operator

$$(\partial/\partial x)\exp(jkx) = jk\exp(jkx)$$

Methodology

- We consider a uniform 1D grid with grid points

$$x_i = ih, \quad i = \ldots, -2, -1, 0, 1, 2, \ldots,$$

- We examine the difference approximations by evaluating them for **complex exponentials**, ***f = exp(jkx).***
- The wavenumbers can be restricted so that *I kh I <= pi*. This is because, when any harmonic function is represented on a grid of points with spacing *h*, one can always shift *kh* by any integer multiple of *2pi* without changing the value of f at any grid point.
- Derivative operators can be defined as

$$D_x = f'/f, \quad D_{xx} = f''/f,$$

and for *f = exp(jkx),* the **exact analytical results are**

$$D_x = jk \ , \ D_{xx} = D_x^2 = -k^2$$

Solution 1 Two Cells – Effective Numerical Wavenumber

- Equation derivative across two cells, f′ on the "integer grid":

$$D_x = \frac{f'(x_i)}{f(x_i)} = \frac{f(x_i+h) - f(x_i-h)}{2h\ f(x_i)} = \frac{e^{jkh} - e^{-jkh}}{2h} = \frac{j}{h}\sin(kh)$$

- This gives an effective numerical wavenumber

$$k_{\text{num}}^{\text{two-cell}} = \frac{\sin kh}{h} = k\left(1 - \frac{k^2h^2}{6} + \cdots\right)$$

- The leading term in the expansion is correct; and the relative error is **$k^2h^2/6$**

Solution 2 – Effective Numerical Wavenumber

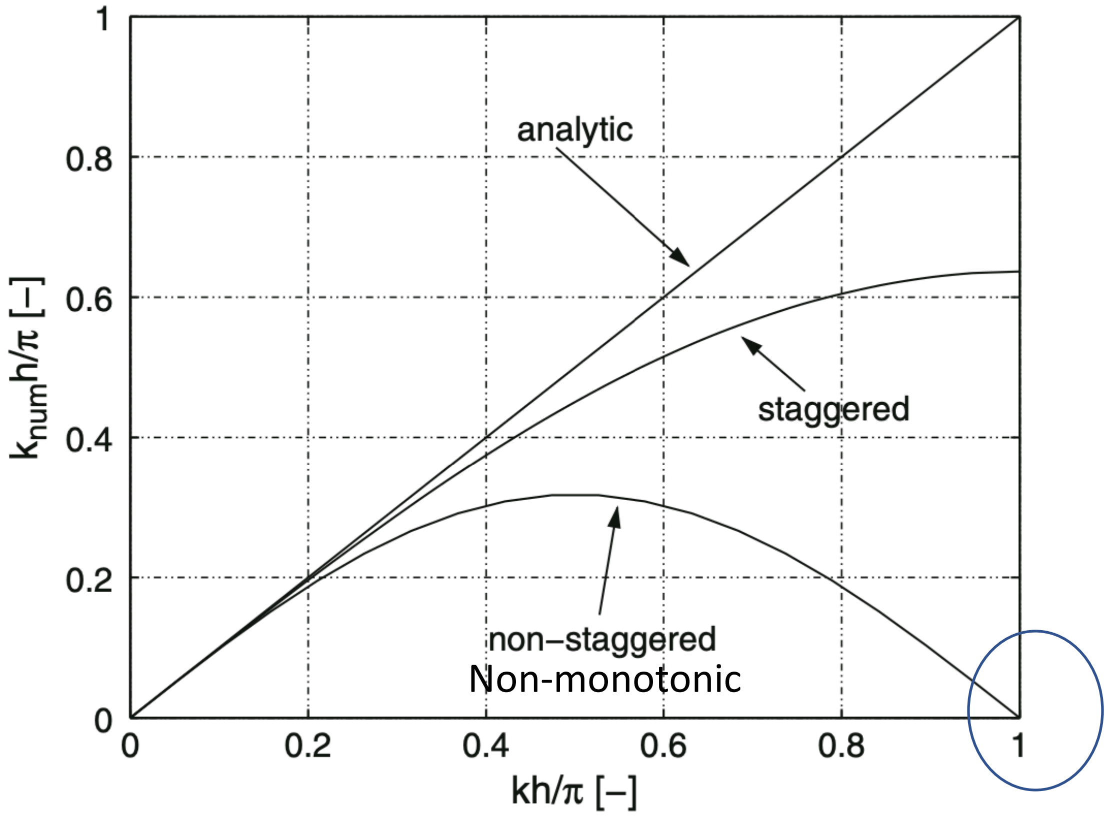
- Equation derivative across one cell, f′ on the half-grid

$$D_x = \frac{f'(x_{i+\frac{1}{2}})}{f(x_{i+\frac{1}{2}})} = \frac{f(x_i + h) - f(x_i)}{hf(x_i + h/2)} = \frac{e^{jkh/2} - e^{-jkh/2}}{h} = \frac{2j}{h} \sin\left(\frac{kh}{2}\right)$$

- This gives an **approximation with a smaller error**

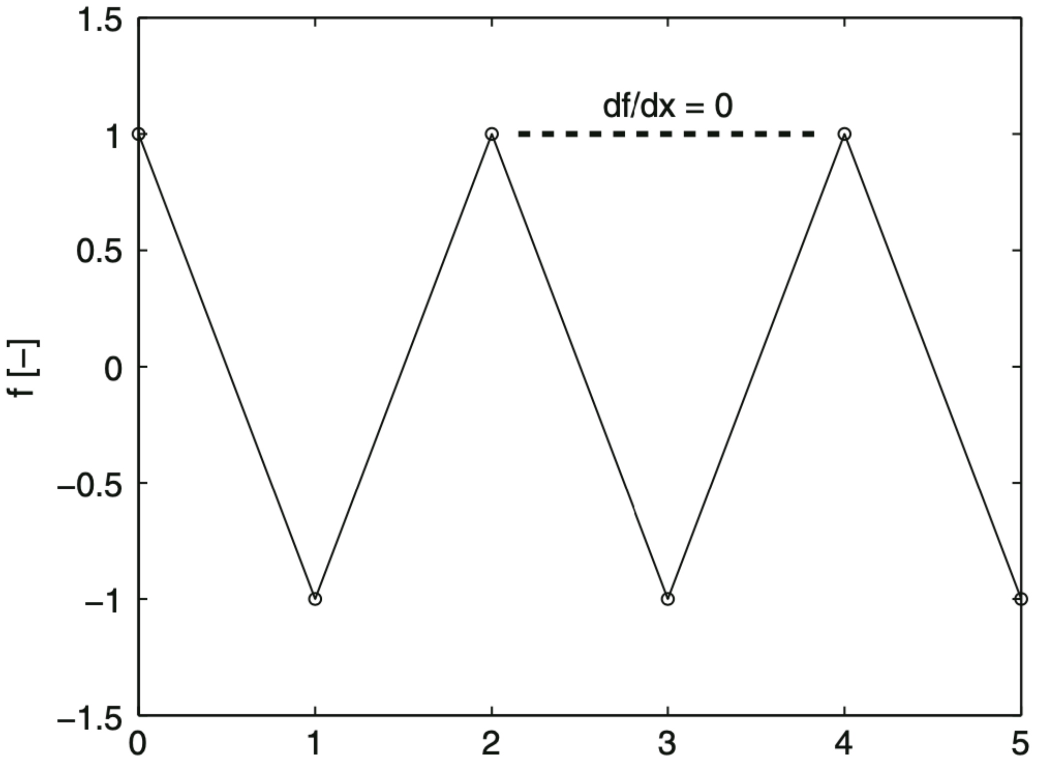
$$k_{\text{num}}^{\text{staggered}} = \frac{2}{h} \sin\frac{kh}{2} = k\left(1 - \frac{k^2h^2}{24} + \cdots\right)$$

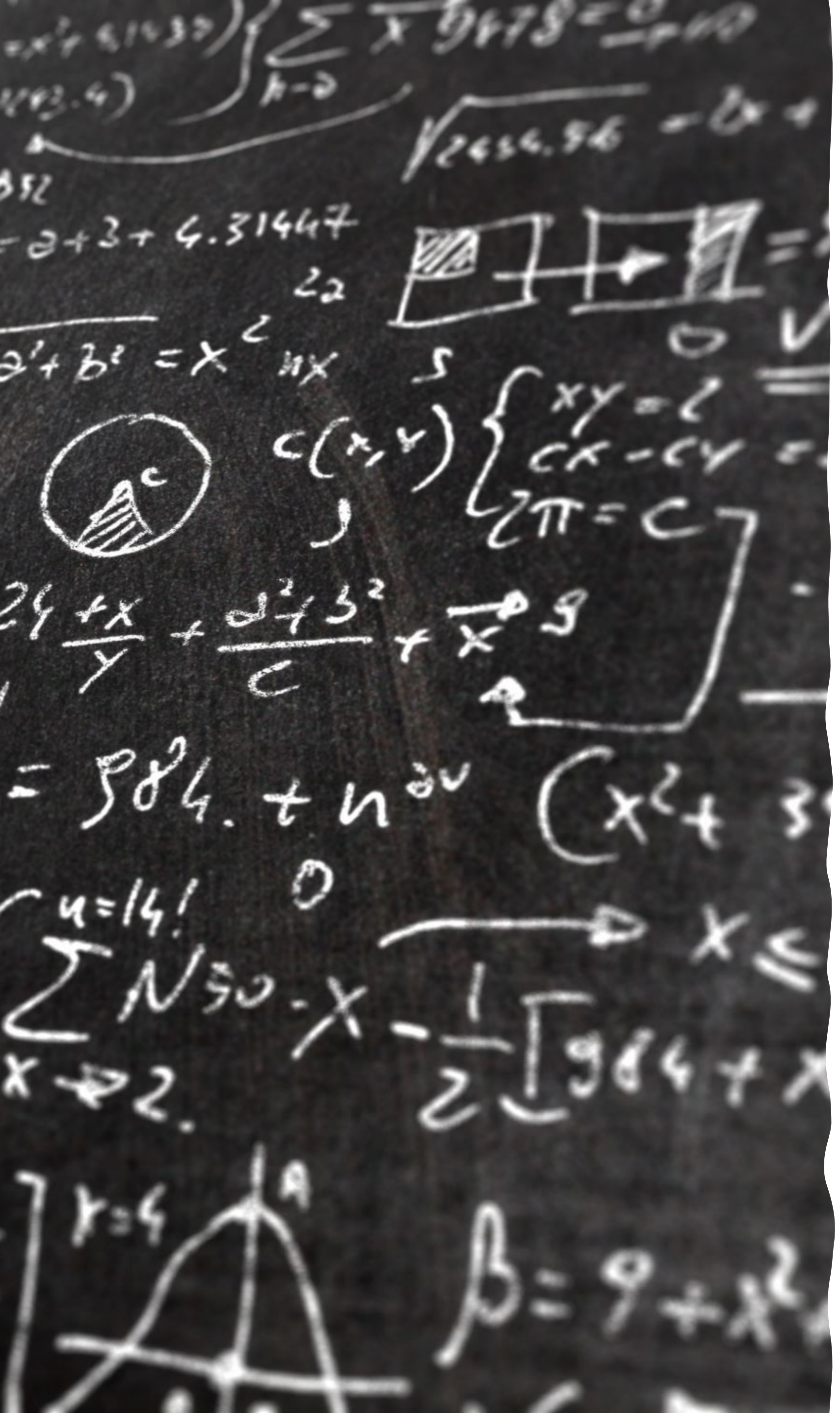
Finite Difference Approximation of Wavenumber



The Problem with a Fast-Oscillating Function (Solution 1)

- The difference formula across two cells gives very poor results when $kh > pi/2$. For $kh = pi \rightarrow f' = 0$ and $k_{two\text{-}cell} = 0$.
- Why: when $kh = pi$, $f(x_i)$ jumps between plus and minus the same value between neighboring points.
- Points at the distance of $2h$ have the same value of $f \rightarrow f' = 0$ at every point on the integer grid. Thus, the most rapidly oscillating function has the derivative equal to zero everywhere on the integer grid.





Spurious Solutions

- The inability of the difference formula across two cells to see rapid oscillations can cause difficulties known as ***spurious modes***.
- By spurious modes we mean solutions of a discretized equation that do not correspond to an analytic (or *physical*) solution.

Example of Spurious Modes

Discretization

$$f' = j\lambda f, \quad x > 0, \quad f(0) = 1 \quad \longrightarrow \quad \frac{f(x_{i+1}) - f(x_{i-1})}{2h} = j\lambda f(x_i).$$

- This will have solutions of the form *exp(jkx),* and the wavenumber can be determined from $k_{\text{two-cell}}$ = lambda.
- This gives two solutions, because $k_{\text{two-cell}}$ (kh) is **nonmonotonic** as shown.
  - One is an acceptable num approximation k_1 *h = asin(lambda h)*
  - *Spurious mode*, having $k_{spurious}$ *h = pi –asin(lambda h) = pi –* $k_1 h$. If *lambda h* is small, this branch for *kh* approaches *pi*, so that the solution resembles the most rapidly oscillating function (see figure in previous slide), even though the correct solution varies slowly on the scale of the grid.