DD2370 Computational Methods for Electromagnetics

FEM in 1D

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Problem to be Solved

- As the first model problem we choose a second-order ordinary differential equation, namely the 1D Helmholtz equation:

$$-\frac{d}{dx}\left(\alpha\frac{df}{dx}\right)+\beta f = s, \quad a < x < b,$$

$$f(a) = f_a,$$

$$f(b) = f_b.$$

- Here *f = f (x)* is the sought solution, and the material properties and $\alpha = \alpha(x)$ and the source $s = s(x)$ are prescribed functions of x.

Physical systems

- There are many physical systems that are modeled by

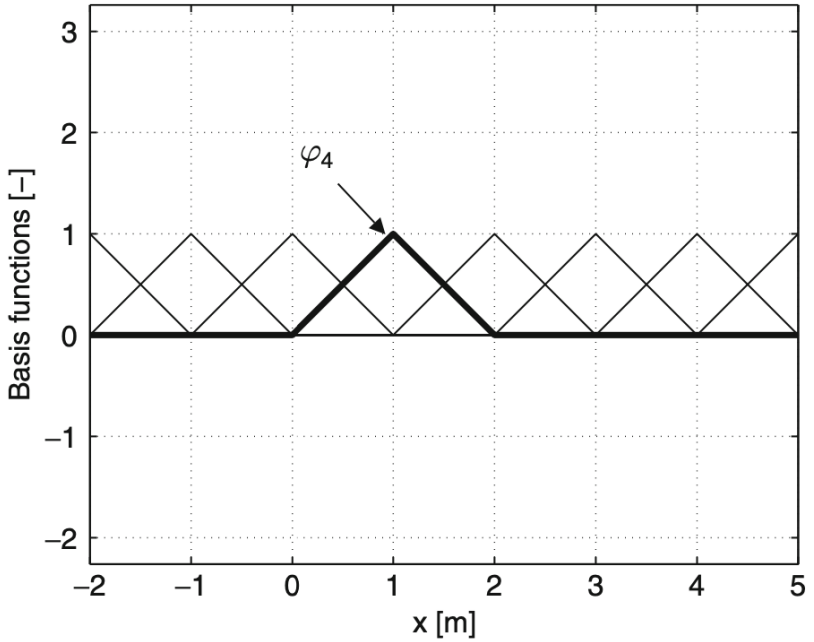
$$-\frac{d}{dx}\left(\alpha\frac{df}{dx}\right)+\beta f = s, \quad a < x < b,$$
$$f(a) = f_a,$$
$$f(b) = f_b.$$

- For example, a **transversal wave in a 1D medium**, such as a light wave propagating and being reflected in **dielectric layers**.

Discretization and Nodal Basis Function – Tent Functions

- We seek the function *f(x)* on the interval a < x < b.
- We first divide this interval into subintervals (elements).
- Let us assume, for example, *a = -2* and *b = 5* and divide the x-axis into 7 equally large elements.
- We call the **endpoints of each element *nodes***, and they have the coordinates $x_i = i - 3$ where *i=1,2,…,8.*
- We introduce the *nodal basis functions* ϕ_i : **linear on each interval, one at node i and zero at all other nodes**
- These basis functions are often called ***tent functions*.**

1D linear elements. The basis function ϕ_4 is emphasized by a thick line.



Approximate Solution Using Nodal Basis Functions

- We seek approximate solutions that are expanded in the basis functions (in the following, *f* will denote this approximate solution):

$$f(x) = \sum_{j=1}^{8} f_j \varphi_j(x)$$

- Note that $f(x_i) = f_i$, so that the **expansion coefficients are the values of f at the nodes**.
- Since f(a) = f_a and f(b) = f_b are known, **we set $f_1 = f_a$ and $f_8 = f_b$.**

Galerkin Approximation

- In the next step, we follow Galerkin's method and choose the test functions wi = ϕ_i where i = 2, 3,…, 7
  - the boundary points are excluded because the corresponding function values are known
- We multiply the residual by the test function w_i and integrate from $x = a$ to $x = b$.
- To move one of the derivatives from f to the test function w_i, we **use integration by parts.** This gives the *weak form* of the original problem, which is the weighted average of the residual:

$$\int_a^b \left(\alpha w_i' f' + \beta w_i f - w_i s\right)\, dx = 0.$$

- In this case, the boundary term $[w_i \alpha f']_a^b$ vanishes since $w_i(a) = w_i(b) = 0$

Getting the Equations from Weak Form

- By substituting

$$f(x) = \sum_{j=1}^{8} f_j \varphi_j(x) \quad \Longrightarrow \quad \int_a^b \left(\alpha w_i' f' + \beta w_i f - w_i s\right)\, dx = 0.$$

- and choosing $w_2 = \varphi_2$
- we generate an equation involving six unknowns: the coefficients f_j for the interior nodes x_j, where $j = 2,3,\ldots,7$.
- Next, we pick $w_3 = \varphi_3$ to generate a second equation, and so on.

Matrix Form of our System

- In the end, we have six equations and six unknowns, and this is formulated as a system of linear equations **Az = b** with

$$A_{ij} = \int_a^b \left(\alpha\varphi_i'\varphi_j' + \beta\varphi_i\varphi_j\right)\, dx$$

$$z_j = f_j,$$

$$b_i = \int_a^b \varphi_i s\, dx.$$

- i = 2,3,…,7 (**for the equations**) and j = 1,2,…,8 (**for the coefficients**)
  - **A** has 8 columns and 6 rows
  - **z** has 8 rows, and **b** has 6 rows.
  - The coefficients f_1 and f_8 are known from the boundary conditions and can be moved to the right-hand side:

Final System – Stiffness Matrix A

Stiffness matrix

$$\begin{pmatrix} A_{22} & A_{23} & \dots & A_{27} \\ A_{32} & A_{33} & \dots & A_{37} \\ \vdots & \vdots & \ddots & \vdots \\ A_{72} & A_{73} & \dots & A_{77} \end{pmatrix} \begin{bmatrix} f_2 \\ f_3 \\ \vdots \\ f_7 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \\ \vdots \\ b_7 \end{bmatrix} - \begin{bmatrix} A_{21} f_1 + A_{28} f_8 \\ A_{31} f_1 + A_{38} f_8 \\ \vdots \\ A_{71} f_1 + A_{78} f_8 \end{bmatrix}$$

- The part of the system matrix **A** that remains on the left-hand side is square; that is, we have as many unknowns as equations.
- The matrix **A** is called ***stiffness matrix*** and is **sparse because the basis functions give only nearest neighbor coupling of the unknowns**.
- Also note that **A** is **symmetric**, $A_{ij} = A_{ji}$. This is related to the fact that the Helmholtz operator is self-adjoint and we used Galerkin's method.