# DD2370 Computational Methods for Electromagnetics

## *Staggered Grids & Yee Lattice*

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# 3D Maxwell's Equations – First Order Formulation

The wave equation is a second-order differential equation for the electric field only. It can also be stated as **a system of coupled first-order differential equations** for both **E** and **H**.

In three dimensions, Maxwell's equations in a source-free region give six scalar equations, 3 for Ampere's law and 3 for Faraday's law.

$$\epsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \qquad \mu \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y},$$

$$\epsilon \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \qquad \mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z},$$

$$\epsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}, \qquad \mu \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}.$$

# *1D Plane Wave*

Consider a **plane wave propagating in the *z*-direction** through a medium such that all quantities are constant in planes perpendicular to the *z*-axis. We assume that the **electric field is oriented in the x-direction, and the magnetic field in the y-direction**. Then, 3D equation reduce to

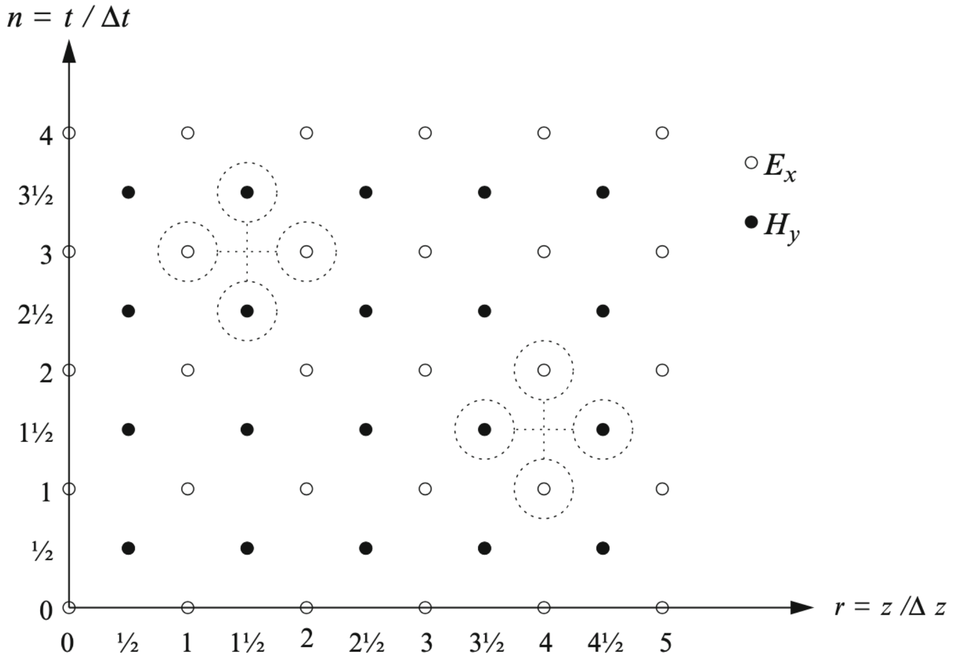
$$\epsilon \frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z},$$

$$\mu \frac{\partial H_y}{\partial t} = -\frac{\partial E_x}{\partial z}.$$

# Staggered Space-Time Grid

- The “trick” used to get a good algorithm is to put the different **E-** and **H-** components at different positions on the grid.
- *First-order* derivatives are much more accurately evaluated on staggered grids:
  - A variable is located on the integer grid, its first derivative is best evaluated on the half-grid, and vice versa.
  - This holds with respect to both space and time.
- Therefore, if we choose to place ***Ex*** **on the integer points both in space and in time**, ***Hy*** **should be on the half-grids in both variables**

Space-Time Plot



# Discretization in Space and Time

- $\epsilon \frac{\partial E_x}{\partial t}$ is applied at integer space points (indexed by r) and integer time points (indexed by n, n+1) using centered and local finite differences in both *z* and t.

$$\frac{E_x|_r^{n+1} - E_x|_r^n}{\Delta t} = -\frac{1}{\epsilon}\frac{H_y|_{r+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{r-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z}$$

- $\mu \frac{\partial H_y}{\partial t}$ is applied at half-integer space points (indexed by r + 1/2) and half-integer time points (indexed by n+1/2) points, also using centered and local finite differences in both *z* and t.

$$\frac{H_y|_{r+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{r+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t} = -\frac{1}{\mu}\frac{E_x|_{r+1}^{n} - E_x|_r^n}{\Delta z}$$

# Three-Dimensions – Yee Scheme

- The **Yee scheme** extends the staggering to three dimensions with a special arrangement of all the components of *E* and *H*.
- The electric field components are computed at "integer" time-steps and the magnetic field at "half-integer" time-steps.

## Numerical Solution of Initial Boundary Value Problems Involving Maxwell's Equations in Isotropic Media

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*t*—Maxwell's equations are replaced by a set of finite ›equations. It is shown that if one chooses the field points ly, the set of finite difference equations is applicable for ⁊ condition involving perfectly conducting surfaces. An given of the scattering of an electromagnetic pulse by a ›nducting cylinder.

### Introduction

ƷTIONS to the time-dependent Maxwell's equa-ıs in general form are unknown except for w special cases. The difficulty is due mainly to ›sition of the boundary conditions. We shall this paper how to obtain the solution numer-.en the boundary condition is that appropriate ˈect conductor. In theory, this numerical attack mployed for the most general case. However, ɔf the limited memory capacity of present day rs, numerical solutions to a scattering problem h the ratio of the characteristic linear dimen-ːhe obstacle to the wavelength is large still be impractical. We shall show by an example he case of two dimensions, numerical solutions ːical even when the characteristic length of the obstacle is moderately large compared to that of an in-coming wave.

A set of finite difference equations for the system of partial differential equations will be introduced in the early part of this paper. We shall then show that with an appropriate choice of the points at which the various field components are to be evaluated, the set of finite difference equations can be solved and the solution will satisfy the boundary condition. The latter part of this paper will specialize in two-dimensional problems, and an example illustrating scattering of an incoming pulse by a perfectly conducting square will be presented.

### Maxwell's Equation and the Equivalent Set of Finite Difference Equations

Maxwell's equations in an isotropic medium [1] are:[1]

$$\frac{\partial B}{\partial t} + \nabla \times E = 0, \tag{1a}$$

$$\frac{\partial D}{\partial t} - \nabla \times H = J, \tag{1b}$$

$$B = \mu H, \tag{1c}$$

$$D = \epsilon E, \tag{1d}$$

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thor is with the Lawrence Radiation Lab., University of Livermore, Calif.

[1] In MKS system of units.

# The 3D Maxwell's Equations – Yee Scheme

$$\mu\frac{H_x|^{n+\frac{1}{2}}_{p,q+\frac{1}{2},r+\frac{1}{2}} - H_x|^{n-\frac{1}{2}}_{p,q+\frac{1}{2},r+\frac{1}{2}}}{\Delta t} = \frac{E_y|^{n}_{p,q+\frac{1}{2},r+1} - E_y|^{n}_{p,q+\frac{1}{2},r}}{\Delta z} - \frac{E_z|^{n}_{p,q+1,r+\frac{1}{2}} - E_z|^{n}_{p,q,r+\frac{1}{2}}}{\Delta y},$$

$$\mu\frac{H_y|^{n+\frac{1}{2}}_{p+\frac{1}{2},q,r+\frac{1}{2}} - H_y|^{n-\frac{1}{2}}_{p+\frac{1}{2},q,r+\frac{1}{2}}}{\Delta t} = \frac{E_z|^{n}_{p+1,q,r+\frac{1}{2}} - E_z|^{n}_{p,q,r+\frac{1}{2}}}{\Delta x} - \frac{E_x|^{n}_{p+\frac{1}{2},q,r+1} - E_x|^{n}_{p+\frac{1}{2},q,r}}{\Delta z},$$

$$\mu\frac{H_z|^{n+\frac{1}{2}}_{p+\frac{1}{2},q+\frac{1}{2},r} - H_z|^{n-\frac{1}{2}}_{p+\frac{1}{2},q+\frac{1}{2},r}}{\Delta t} = \frac{E_x|^{n}_{p+\frac{1}{2},q+1,r} - E_x|^{n}_{p+\frac{1}{2},q,r}}{\Delta y} - \frac{E_y|^{n}_{p+1,q+\frac{1}{2},r} - E_y|^{n}_{p,q+\frac{1}{2},r}}{\Delta x}.$$

$$\epsilon\frac{E_x|^{n+1}_{p+\frac{1}{2},q,r} - E_x|^{n}_{p+\frac{1}{2},q,r}}{\Delta t} = \frac{H_z|^{n+\frac{1}{2}}_{p+\frac{1}{2},q+\frac{1}{2},r} - H_z|^{n+\frac{1}{2}}_{p+\frac{1}{2},q-\frac{1}{2},r}}{\Delta y} - \frac{H_y|^{n+\frac{1}{2}}_{p+\frac{1}{2},q,r+\frac{1}{2}} - H_y|^{n+\frac{1}{2}}_{p+\frac{1}{2},q,r-\frac{1}{2}}}{\Delta z},$$

$$\epsilon\frac{E_y|^{n+1}_{p,q+\frac{1}{2},r} - E_y|^{n}_{p,q+\frac{1}{2},r}}{\Delta t} = \frac{H_x|^{n+\frac{1}{2}}_{p,q+\frac{1}{2},r+\frac{1}{2}} - H_x|^{n+\frac{1}{2}}_{p,q+\frac{1}{2},r-\frac{1}{2}}}{\Delta z} - \frac{H_z|^{n+\frac{1}{2}}_{p+\frac{1}{2},q+\frac{1}{2},r} - H_z|^{n+\frac{1}{2}}_{p-\frac{1}{2},q+\frac{1}{2},r}}{\Delta x},$$

$$\epsilon\frac{E_z|^{n+1}_{p,q,r+\frac{1}{2}} - E_z|^{n}_{p,q,r+\frac{1}{2}}}{\Delta t} = \frac{H_y|^{n+\frac{1}{2}}_{p+\frac{1}{2},q,r+\frac{1}{2}} - H_y|^{n+\frac{1}{2}}_{p-\frac{1}{2},q,r+\frac{1}{2}}}{\Delta x} - \frac{H_x|^{n+\frac{1}{2}}_{p,q+\frac{1}{2},r+\frac{1}{2}} - H_x|^{n+\frac{1}{2}}_{p,q-\frac{1}{2},r+\frac{1}{2}}}{\Delta y},$$

# Yee Scheme

- The Yee scheme, or FDTD scheme, has proven very successful for microwave problems.
  - All derivatives are centered and as **compact as possible**, that is, they are taken across a single cell.

# Bonus Exercise – 3D Cubical Cavity

- Use 3D Yee Scheme and find the eigenfrequencies in a 3D Cubical Cavity with PEC BC.