DD2370 Computational Methods for Electromagnetics

Eigenvalue Problems in CEM

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Eigenvalues Problems

- CEM applications can be either
  1. a driven system, where one seeks the **response to a source**, for instance an antenna, or
  2. an eigenvalue calculation, where one seeks the **natural oscillation frequencies of the system**.
- The field solution to the eigenvalue calculation is a nonzero field that satisfies the homogeneous problem, i.e. the field problem without sources.
  - This type of situation occurs typically for an electromagnetic system with a source that is nonzero for a finite time.
  - Once the source vanishes, the field solution can be expressed as a superposition of eigenmodes, where each eigenmode oscillates at a particular eigenfrequency.

Maxwell's Equations in Frequency Domain

In a linear, dispersion-free medium, Maxwell's equations can be written as the single second-order curl-curl equation for the electric field

$$-\epsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \nabla \times \frac{1}{\mu}\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{J}}{\partial t}$$

$$\omega_m^2 \epsilon \boldsymbol{E}_m = \nabla \times \frac{1}{\mu}\nabla \times \boldsymbol{E}_m$$

In the absence of sources, J=0, and with harmonic time dependence exp(jt), the curl-curl equation gives the eigenvalue problem

Eigenvalues and Eigenfunctions

$$\omega_m^2 \epsilon \boldsymbol{E}_m = \nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_m$$

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- For nontrivial solutions (E_m != 0)
  - w_m = eigenvalue
  - E_m is the corresponding eigenfunction, or eigenmode.
- If the region is a closed cavity with a perfectly conducting boundary (i.e., **n** x **E** = 0 on the boundaries), the operator on the right-hand side, $\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_m$, is **self-adjoin**t, that is

$$\int_\Omega \boldsymbol{E}_1 \cdot L[\boldsymbol{E}_2] dV = \int_\Omega \boldsymbol{E}_2 \cdot L[\boldsymbol{E}_1] dV$$

Eigenvalues in Closed Cavity with PEC

- We can calculate analytically the eigenvalues as

$$\omega^2 = \frac{\int_\Omega \mu^{-1} |\nabla \times \boldsymbol{E}|^2 dV}{\int_\Omega \epsilon |\boldsymbol{E}|^2 dV}$$

- Eigenvalues are **real** and **nonnegative**.
- The eigenfrequencies are real for any lossless region bounded by perfect conductors.
  - Damping can appear if there is dissipation of energy, for example from regions with finite electrical conductivity, or if the region is not completely enclosed by a perfect conductor.

Helmholtz Equation

The self-adjoint curl-curl equation leads us to consider eigenvalue problems of the type

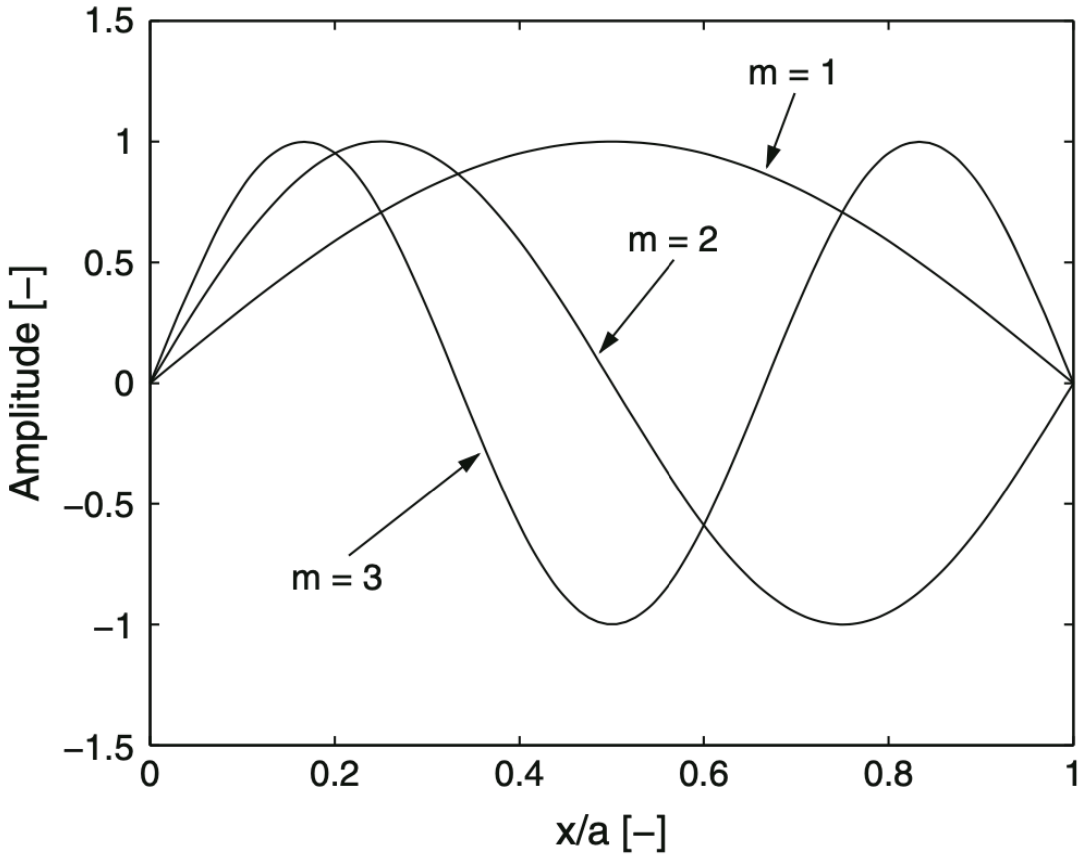
$$L[f] = -\omega^2 f \quad \text{in } \Omega$$

together with a suitable boundary condition on *0* and *a* . We will assume that *L* is a linear self-adjoint operator with nonpositive eigenvalues. As a simple example to illustrate general principles, we will study the 1D **Helmholtz equation**:

$$\frac{d^2 f}{dx^2} = -k^2 f, \quad 0 < x < a, \quad f(0) = f(a) = 0$$

Solutions of Eigenvalue Problem

$$k_m = \frac{m\pi}{a}, \quad m \text{ an integer},$$



Solve Numerically Helmholtz Equation with FD

To discretize the 1D Helmholtz equation by finite differences, we divide the interval *[0,a]* into *N* subintervals of equal length $h = a/N$. The simplest finite difference approximation is

$$\frac{d^2 f}{dx^2} = -k^2 f, \quad 0 < x < a, \quad f(0) = f(a) = 0$$

$$\frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} = -k^2 f_i, \quad i = 1, 2, \ldots, N-1$$

Matrix Formulation of Eigenvalue Problem

The boundary conditions are $f_0 = f_N = 0$, so there is no reason to include f_0 and f_N as unknowns.

$\frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} = -k^2 f_i$ can be written as a linear system with an *(N – 1) x (N – 1)* matrix **A**:

$$\mathbf{Af} = \lambda \mathbf{f}$$

$$\mathbf{A} = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Matlab Solution – Sparse Representation and *eig*

- When n is large, **A** consists mostly of zeros, and this can be exploited by saving the matrix in sparse form.
  - We can use **`spalloc`** function
- The physical eigenvalues $-k^2$ are simply the eigenvalues of the matrix **A**.
  - These eigenvalues can be computed with the MATLAB routine **`eig`**, which computes all eigenvalues and corresponding eigenvectors of an algebraic eigenvalue problem.

Matlab Code

```
% ------------------------------------------------------------
% Compute eigenvalues of 1D Helmholtz equation using FD
% ------------------------------------------------------------
function k = HFD1D(a, N)

% Arguments:
%    a = length of interval
%    N = number of subintervals (equal length)
% Returns:
%    k = eigenvalues

h = a/N;                % Grid size
A = spalloc(N-1, ...    % Allocate sparse matrix
            N-1, ...    % with 3*(N-1) nonzeros
            3*(N-1));
d = -2/h^2;             % Value of diagonal entries
s = 1/h^2;              % Value of upper and lower
                        % diagonal entries

% Initialize the diagonal entries
for i = 1:N-1
  A(i,i) = d;   % Diagonal entries
end

% Initialize the upper and lower diagonal entries
for i = 1:N-2
  A(i,i+1) = s; % Upper diagonal entries
  A(i+1,i) = s; % Lower diagonal entries
end

% Computing the eigenvalues
lambda = eig(A);
k = sqrt(sort(-lambda));
```

Calculate eigenvalues in [0, pi]

```
>> k = HFD1D(pi,10)

k =

    0.9959
    1.9673
    2.8902
    3.7420
    4.5016
    5.1504
    5.6723
    6.0546
    6.2878
```

```
>> k = HFD1D(pi,20)

k =

    0.9990
    1.9918
    2.9723
    3.9345
    4.8725
    5.7804
    6.6527
    7.4839
    8.2690
      …
```

```
>> k = HFD1D(pi,30)

k =

    0.9995
    1.9963
    2.9877
    3.9708
    4.9431
    5.9018
    6.8443
    7.7681
    8.6706
    9.5493
      …
```

We can perform convergence tests and find second order convergence