DD2370 Computational Methods for Electromagnetics

Review of Maxwell Equations

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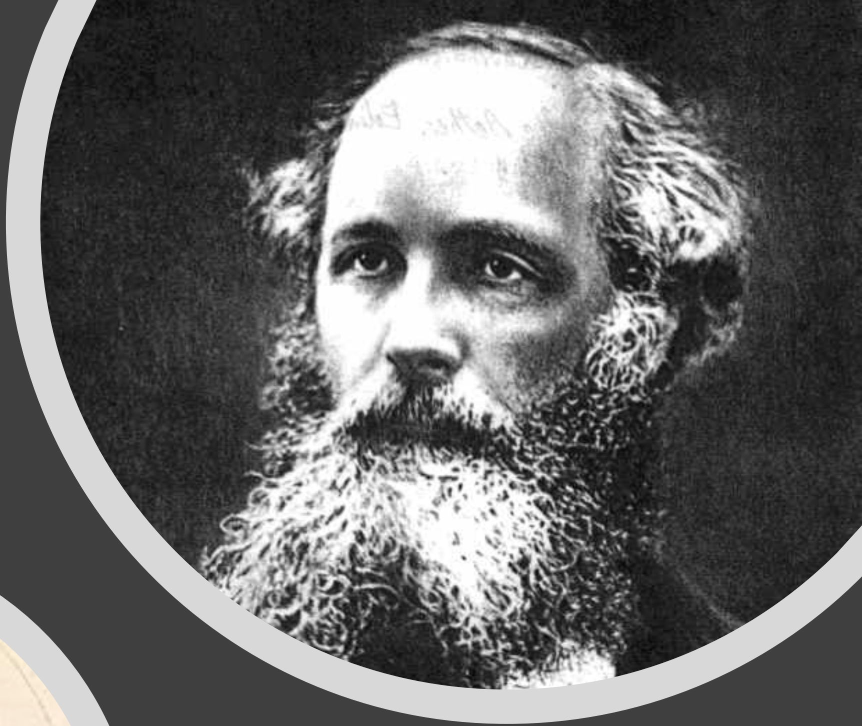
Maxwell's Equations

- Set of 4 mathematical equations that relate precisely the electric and magnetic fields to their sources that are electric charges and currents
- They were established by James Clerk **Maxwell** (1831-1879) based on experimental discoveries by
  - Andre Marie **Ampere** (1775 – 1836)
  - Michael **Faraday** (1791 – 1867)
  - Carl Friederich **Gauss** (1777 – 1855)
- Reformulated in vector form by **Heinrich Hertz** (1857 – 1894) and Oliver **Heaviside** (1850 – 1925)



Two Formulations of Maxwell Equations

- **Differential form** (using differential operators like *divergence* and *curl*) → we will use them for deriving **FD** and **FEM**
- **Integral form** (using volume and surface integrals) → We will use them for deriving **MoM** and **boundary conditions**

Maxwell's Equations Differential form	Maxwell's Equations Integral form
$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$	$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$
$\nabla \times \vec{E} = \boxed{-\frac{\partial \vec{B}}{\partial t}}$	$\oint \vec{E} \cdot d\vec{l} = \boxed{-\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}}$
$\nabla \cdot \vec{B} = 0$	$\oint \vec{B} \cdot d\vec{a} = 0$
$\nabla \times \vec{B} = \mu_0 \vec{J} + \boxed{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$	$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \boxed{\mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t}}$

Differential Formulation of Maxwell's Equations

$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t}$	Ampere's Law	
$\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$	Faraday's Law	
$\nabla \cdot \boldsymbol{D} = \rho$	Poisson's Law	
$\nabla \cdot \boldsymbol{B} = 0$	Condition of solenoidal magnetic flux density	

H = magnetic field
J = current density
D = electric displacement
E = electric field
B = magnetic flux density
ϱ = the electric charge density
t = time

Constitutive Relations

Medium has significant impact on electromagnetic fields; constitutive relations account for the effect of a medium on electromagnetics fields.

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}, \quad \boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

M is the magnetization and **P** is the polarization. In this course we only use **linear**, **isotropic** and **non-dispersive** materials for which the constitutive relations are simply

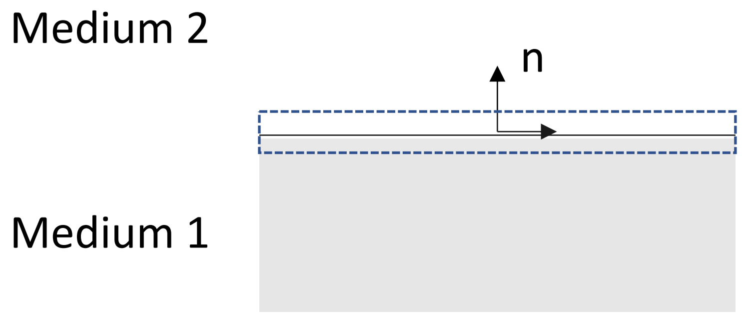
$$\boldsymbol{B} = \mu \boldsymbol{H}, \quad \boldsymbol{D} = \epsilon \boldsymbol{E}$$

Boundary Conditions for B and E

- Derived from integral form of Maxwell's Equations.
- Used in numerical schemes

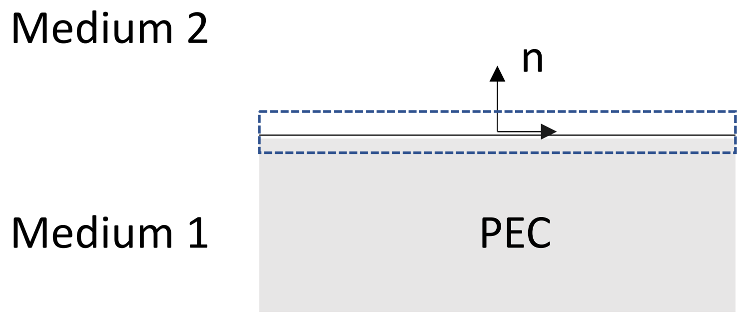
$$\int_V \nabla \cdot \boldsymbol{B}\, dV = \oint_{\partial V} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}\, dS \rightarrow \quad \hat{\boldsymbol{n}} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0$$

$$\int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{S} = \oint_{\partial S} \boldsymbol{E} \cdot d\boldsymbol{l} \rightarrow \quad \hat{\boldsymbol{n}} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}$$



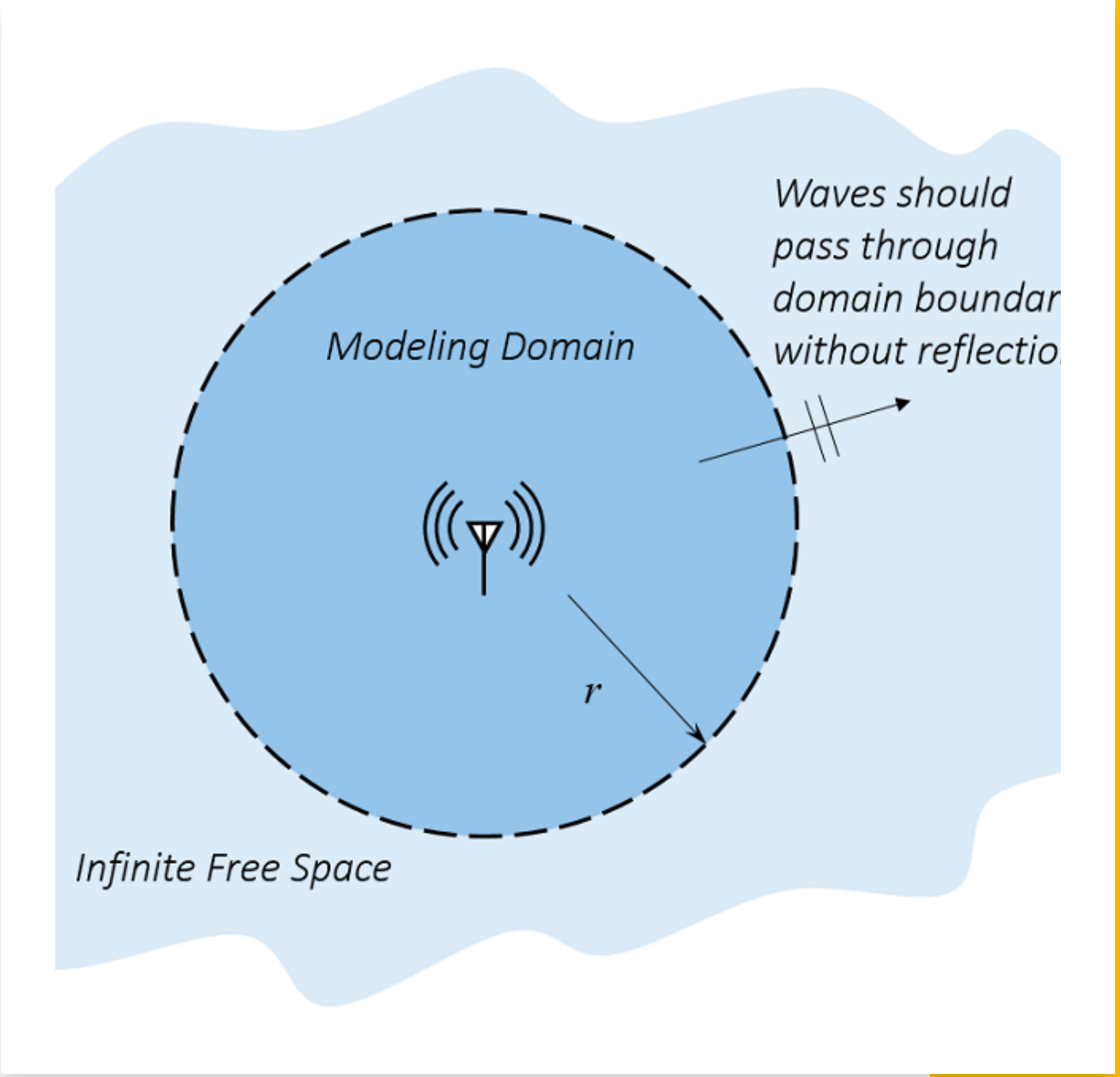
Perfect Electric Conductor (PEC) Boundary Conditions

- Widely used BC in CEM.
- The electric field inside a conductor is 0 and also the electric displacement.
  - We get $\hat{\boldsymbol{n}} \times \boldsymbol{E}_2 = \boldsymbol{0}$ $\quad \hat{\boldsymbol{n}} \cdot \boldsymbol{D}_2 = \rho_s$
- Faraday's law yields that the magnetic flux density is zero inside a PEC
  - We get $\hat{\boldsymbol{n}} \cdot \boldsymbol{B}_2 = 0$ $\quad \hat{\boldsymbol{n}} \times \boldsymbol{H}_2 = \boldsymbol{J}_s$



Absorbing Boundary Conditions (ABC) / Open

- These are used to truncate the computational domain in case of open region problems and can be implemented using a variety of techniques. The most popular ABC is the *perfectly matched layer* (PML)
- Also called Open Boundary Conditions



Energy Relations Associated to E and B

- Important concepts for numerical schemes
  - **Conservation of total energy** should be preserved also in numerical schemes
  - **Unstable methods** can be thought as methods that make growth total energy of the systems

$$w_{\mathrm{e}} = \epsilon |\boldsymbol{E}|^2/2 \qquad w_{\mathrm{m}} = |\boldsymbol{B}|^2/(2\bar{\mu})$$

For a time-varying electromagnetic field, we have the energy density $w_e + w_m$ and this quantity is often used to form energy conservation expressions

Time Evolution of E and B

- Maxwell's Equations **are an overdetermined system** with dependent equations
  - The divergence equations can be thought as initial conditions, if they are preserved at the beginning they will be preserved during the simulation
- Time evolution of E and B is specified by

$$\epsilon \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \boldsymbol{J},$$

$$\mu \frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E}.$$

Curl-Curl Equation or Wave Equation

- The system of two first-order equations can be combined to a single second-order equation for **E**

- The initial conditions that need to be specified are the electric field and its time derivative.

$$\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{J}}{\partial t}$$

Curl-Curl in Frequency Domain (Vector Helmholtz Equation)

- **FEM** is generally used to solve the frequency domain form of the curl-curl equation (vector Helmholtz equation)
  - exp(jωt) time dependence is assumed
  - Time derivative d/dt is replaced by jω, where *j* is the **imaginary unit** and ω is the **angular frequency**

$$\omega_m^2 \epsilon \boldsymbol{E}_m = \nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_m$$

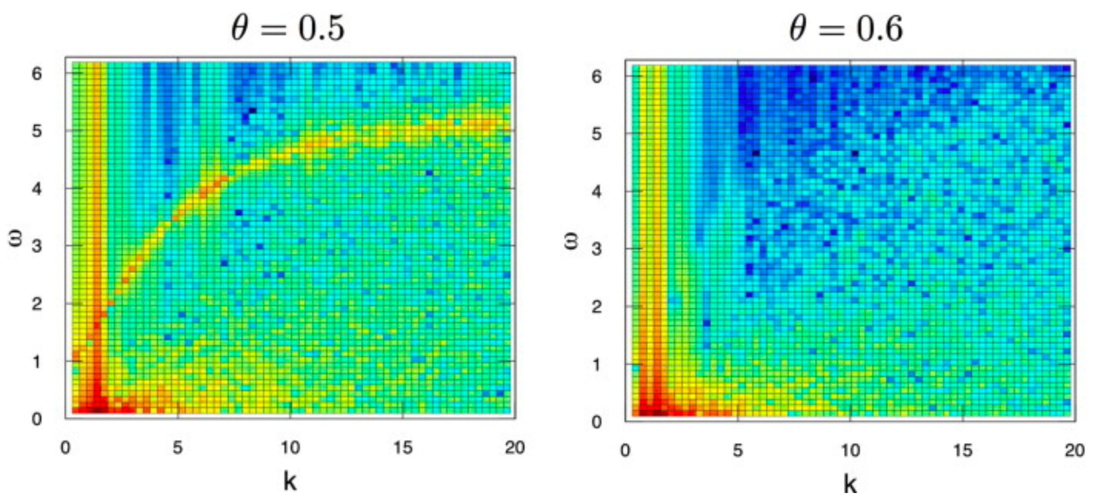
In absence of sources **J**

Dispersion Relation

- The propagation of electromagnetic waves is characterized in terms of the *dispersion relation*, which relates spatial and temporal variation of a monochromatic solution by means of its wavevector **k** and frequency ω.
  - **nondispersive situations** where the frequency is directly proportional to the wavenumber k.
  - When the frequency is not proportional to the wavenumber, we have dispersion
    - wave propagation in some media and waveguides.

Numerical Dispersion Relation

- The **discretization process** may also cause dispersion, called **numerical dispersion**.
- Dispersion implies that a wave packet containing several different spatial frequencies will change shape as it propagates.
    - In our experiments, it is important that the **numerical dispersion is small in comparison to the physical dispersion of interest**
    - It is possible to calculate analytically dispersion relation of numerical methods



Dispersion Relation in 1D Wave Equation

$$\frac{\partial^2}{\partial t^2} E(z,t) = c^2 \frac{\partial^2}{\partial z^2} E(z,t) \quad \rightarrow \quad E(z,t) = E^+(z - ct) + E^-(z + ct)$$

To obtain the dispersion relation for the 1D wave equation, we substitute $E = \exp(j\omega t - jkz)$ and then divide both sides by $\exp(j\omega t - jkz)$

$$\omega^2 = c^2 k^2 \quad \rightarrow \quad \omega = ck$$

The angular frequency ω is a *linear* function of the wavenumber k → all frequency components of a transient wave propagate with the same velocity

Integral Formulation

A simple special case is electrostatics, where there is no time-dependence. For static conditions

$$\nabla \cdot (\epsilon \nabla \phi) = -\rho \qquad \text{where} \qquad \boldsymbol{E} = -\nabla \phi$$

In three dimensions, the “solution” to Poisson’s equation in free space is

$$\phi(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}')dV'}{4\pi\epsilon_0|\boldsymbol{r} - \boldsymbol{r}'|}$$

This formulation is used in the **MoM**. Similar reformulations exist also for the time-dependent Maxwell system.