DD2370 Computational Methods for Electromagnetics

Time-Domain Eigenvalue Calculation

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Stability Analysis

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Eigenvalues Calculation with Time-Domain FD

One common way of determining eigenfrequencies in CEM is

1. Time-step a solution, using for example a finite difference program
2. Record the field at some location
3. Fourier transform this signal to locate its main frequency components.

Time-domain

$$L[f] = -\omega^2 f \quad \Rightarrow \quad \frac{\partial^2 f}{\partial t^2} = L[f] \quad \Rightarrow \quad \frac{f^{(n+1)} - 2f^{(n)} + f^{(n-1)}}{(\Delta t)^2} = L[f^{(n)}]$$

An important advantage of this formulation is that the time-stepping is ***explicit*****, that is, no matrix inversion is needed to compute $f^{(n+1)}$**

$$\Downarrow$$

$$\boxed{f^{(n+1)} = 2f^{(n)} - f^{(n-1)} + (\Delta t)^2 L[f^{(n)}]}$$

Time-Stepping - How to Choose the Time-Step

$$f^{(n+1)} = 2f^{(n)} - f^{(n-1)} + (\Delta t)^2 L[f^{(n)}]$$

- These **time-stepping schemes**, often referred to as ***leap-frog***, are very efficient, and allow determination of the complete eigenvalue spectrum.
- An important issue for explicit **time-stepping schemes** is how to choose the time-step Delta t.
  - This is mainly determined by **stability.**

Stability Analysis – Amplifications Factors

- The analysis is based on the fact that any *discrete* time equation, which has no explicit time dependence, has solutions of the form
    - $\mathbf{f^{(n)} = f_{omega}\ rho^{n}}$, that is geometrical sequences in discrete time.
- Here, *rho* is called the **amplification factor** of the eigenmode f_{omega}
- Stability requires ***I rho I <= 1*** for *all* eigenmodes.

Stability Analysis II – Calculate rho for FD

- Substituting $f^{(n)} = f_\omega \rho^n$
- into $f^{(n+1)} = 2f^{(n)} - f^{(n-1)} + (\Delta t)^2 L[f^{(n)}]$
- and using $L[f_\omega] = -\omega^2 f_\omega$, we obtain a quadratic equation for the amplification factor (divide by f_{omega} and rho^{n-1})

$$\rho^2 - [2 - (\omega \Delta t)^2]\rho + 1 = 0$$

- with the solutions:

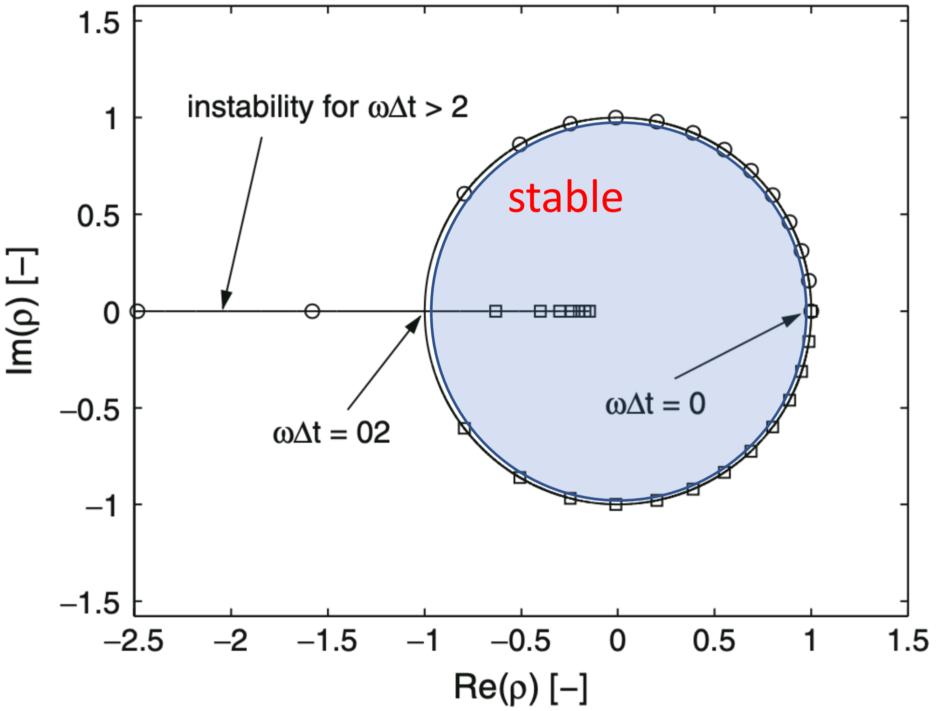
$$\boxed{\rho = 1 - \frac{1}{2}(\omega \Delta t)^2 \pm j\omega \Delta t \sqrt{1 - \frac{1}{4}(\omega \Delta t)^2}}$$

Conditional Stability – I rho I <= 1

- The roots stay on the unit circle Irhol < 1 as long as I w Delta tI < = 2, but when **I w Delta tI > 2**, one root has modulus larger than unity → the solution will grow exponentially in time, and the scheme for time-stepping is **unstable**.
- Since **I w Delta tI <= 2** has to hold for *all* the eigenmodes the condition on the time-step for the explicit scheme is

$$\boxed{\Delta t \leq \frac{2}{|\omega_{\max}|}}$$

$\rho = 1 - \frac{1}{2}(\omega \Delta t)^2 \pm j\omega \Delta t \sqrt{1 - \frac{1}{4}(\omega \Delta t)^2}$ <= 1



Apply Stability to Space Discretization: Courant Condition

- If we apply this stability limit to the operator $L = d^2/dx^2$ iscretized on a uniform grid with cell size *h*, the largest numerical eigenvalue is w_{max} = 2/h , and stability requires ***Delta t < = 2/ w_{max} = h***
  - Therefore, **time-step for our simple explicit scheme for the wave equation should not be larger than the space step, for stability reasons**.
- In the FEM chapter, we will also study ***implicit*** **time-stepping schemes**, which make it possible to remove the limit on the time-step.