# DD2370 Computational Methods for Electromagnetics
## *Derivation of Higher-Order Finite-Difference Stencils*

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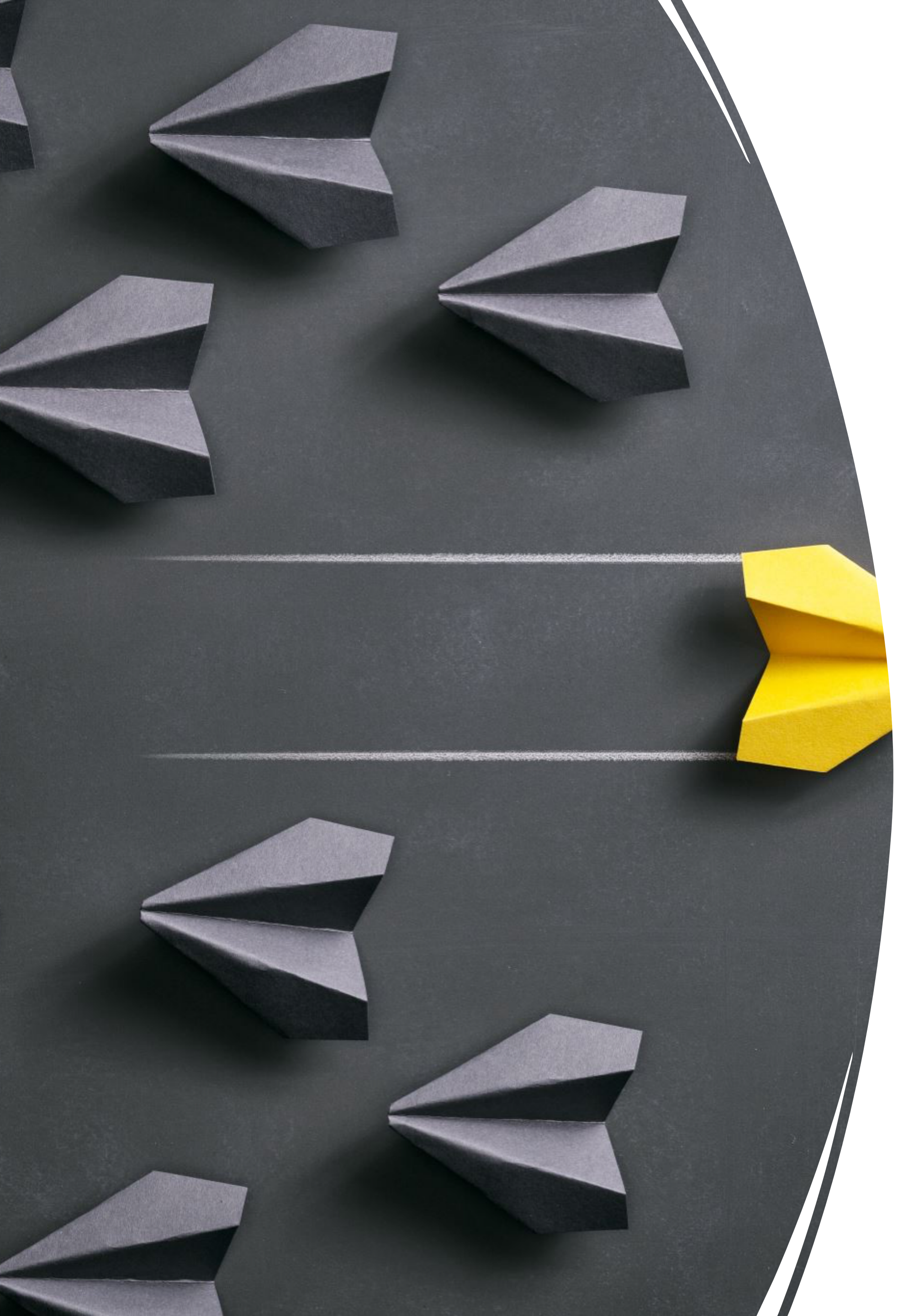
# Higher-Order Finite Differences

- Here, we introduce a procedure to derive finite-difference stencils for the approximation of a given derivative operator.
- In addition, the procedure allows for a rather arbitrary number of grid points to be involved in the finite-difference stencil.
- We expect that a larger number of grid points yields a leading error term that decays faster as the grid is refined, i.e. we achieve a higher order of convergence.
- Clearly, a larger number of grid points also implies that a larger number of floating-point operations must be executed to evaluate the finite-difference approximation.

# Taylor Expansion: Basic Building-Block of FD

$$f(x+\delta) = f(x) + \delta f'(x) + \frac{\delta^2}{2} f''(x) + \frac{\delta^3}{6} f'''(x) + \cdots$$



# 4-Points Stencils

- First, we intend to derive a finite-difference stencil that resembles the one-cell finite-difference approximation
  - We want to include one additional grid point on each side of the lowest-order stencil.
- Consequently, we consider an approximation to the first-order derivative that involves the four grid points *-3h/2*, *-h/2*, *h/2* and *3h/2*.

# Taylor Expansion of f Values on 4 point on the grid

$$f_{i-\frac{3}{2}} = f\left(x - \frac{3h}{2}\right) = f(x) - \frac{3h}{2} f'(x) + \frac{9h^2}{8} f''(x) - \frac{9h^3}{16} f'''(x) + \cdots$$

$$f_{i-\frac{1}{2}} = f\left(x - \frac{h}{2}\right) = f(x) - \frac{h}{2} f'(x) + \frac{h^2}{8} f''(x) - \frac{h^3}{48} f'''(x) + \cdots$$

$$f_{i+\frac{1}{2}} = f\left(x + \frac{h}{2}\right) = f(x) + \frac{h}{2} f'(x) + \frac{h^2}{8} f''(x) + \frac{h^3}{48} f'''(x) + \cdots$$

$$f_{i+\frac{3}{2}} = f\left(x + \frac{3h}{2}\right) = f(x) + \frac{3h}{2} f'(x) + \frac{9h^2}{8} f''(x) + \frac{9h^3}{16} f'''(x) + \cdots$$

# $1^{st}$ Step in Derivation – Introduce Unknowns

We now introduce the unknown constants $a_{i-3/2}$ , $a_{i-1/2}$ , $a_{i+1/2}$ and $a_{i+3/2}$, which are used as coefficients in the finite-difference approximation. Thus, we have the following finite-difference approximation of the first order derivative

$$
\begin{aligned}
& a_{i-\frac{3}{2}} f_{i-\frac{3}{2}} + a_{i-\frac{1}{2}} f_{i-\frac{1}{2}} + a_{i+\frac{1}{2}} f_{i+\frac{1}{2}} + a_{i+\frac{3}{2}} f_{i+\frac{3}{2}} = \\
& \quad = \left( a_{i-\frac{3}{2}} + a_{i-\frac{1}{2}} + a_{i+\frac{1}{2}} + a_{i+\frac{3}{2}} \right) f(x) \\
& \quad + \left( -\frac{3h}{2} a_{i-\frac{3}{2}} - \frac{h}{2} a_{i-\frac{1}{2}} + \frac{h}{2} a_{i+\frac{1}{2}} + \frac{3h}{2} a_{i+\frac{3}{2}} \right) f'(x) \\
& \quad + \left( \frac{9h^2}{8} a_{i-\frac{3}{2}} + \frac{h^2}{8} a_{i-\frac{1}{2}} + \frac{h^2}{8} a_{i+\frac{1}{2}} + \frac{9h^2}{8} a_{i+\frac{3}{2}} \right) f''(x) \\
& \quad + \left( \frac{-9h^3}{16} a_{i-\frac{3}{2}} - \frac{h^3}{48} a_{i-\frac{1}{2}} + \frac{h^3}{48} a_{i+\frac{1}{2}} + \frac{9h^3}{16} a_{i+\frac{3}{2}} \right) f'''(x) + \cdots
\end{aligned}
$$

# 2nd Step – Zero Coefficients for f'',f''' and constant Terms

$$0 = a_{i-\frac{3}{2}} + a_{i-\frac{1}{2}} + a_{i+\frac{1}{2}} + a_{i+\frac{3}{2}}$$

$$1 = -\frac{3h}{2}a_{i-\frac{3}{2}} - \frac{h}{2}a_{i-\frac{1}{2}} + \frac{h}{2}a_{i+\frac{1}{2}} + \frac{3h}{2}a_{i+\frac{3}{2}}$$

$$0 = \frac{9h^2}{8}a_{i-\frac{3}{2}} + \frac{h^2}{8}a_{i-\frac{1}{2}} + \frac{h^2}{8}a_{i+\frac{1}{2}} + \frac{9h^2}{8}a_{i+\frac{3}{2}}$$

$$0 = \frac{-9h^3}{16}a_{i-\frac{3}{2}} - \frac{h^3}{48}a_{i-\frac{1}{2}} + \frac{h^3}{48}a_{i+\frac{1}{2}} + \frac{9h^3}{16}a_{i+\frac{3}{2}}$$

$$f'(x_i) \approx \frac{f_{i-\frac{3}{2}} - 27f_{i-\frac{1}{2}} + 27f_{i+\frac{1}{2}} - f_{i+\frac{3}{2}}}{24h}$$

$$f'_{i+\frac{1}{2}} \approx \frac{f_{i-1} - 27f_i + 27f_{i+1} - f_{i+2}}{24h}$$

$$a_{i-\frac{3}{2}} = \frac{1}{24h} \qquad a_{i-\frac{1}{2}} = -\frac{9}{8h} \qquad a_{i+\frac{1}{2}} = \frac{9}{8h} \qquad a_{i+\frac{3}{2}} = -\frac{1}{24h}$$

# Complex Exponential

- Given this higher-order approximation of the first-order derivative, we perform the analysis with complex exponentials, which yields

$$D_x = \frac{e^{-j3kh/2} - 27e^{-jkh/2} + 27e^{jkh/2} - e^{j3kh/2}}{24h}$$
$$= \frac{j}{3h}\left(7 - \cos^2\left(\frac{kh}{2}\right)\right)\sin\left(\frac{kh}{2}\right).$$

- A Taylor expansion around *kh = 0* gives the numerical wavenumber

$$k_{\text{num}}^{\text{4th order}} = k\left(1 - \frac{3k^4h^4}{640} + \cdots\right)$$

- which has a leading error term that is **proportional to $h^4$** → reduction of the cell size h by a factor of two reduces the error in the wavenumber by a factor 16!

# Difficulties for Higher-Order Approximations

- **Boundary Conditions:**
  - For the lowest-order approximation stencils that are located closest to the boundary, the outermost points of the stencil do not normally extend beyond the boundary but, instead, they would be located on the boundary which makes it possible to incorporate information from the boundary conditions in a natural manner.
  - For a higher-order finite-difference stencil, however, such a situation would imply that some points of the stencil are located outside of the computational domain.
    - A common solution to this problem is to use the lowest-order stencil in the very vicinity of the boundary in combination with the higher-order stencil applied to internal grid-points

# Mixing Different Orders Numerical Schemes for BC

- BCs might require numerical scheme that mixes different types of approximations and → the error analysis becomes more difficult.
- Order of convergence is reduced since it is determined by the poorest approximation present the numerical scheme.
- A better solution is to use **higher-order finite element methods.** → they can exploit the **higher-order approximation also in the very vicinity of the boundary** and, in addition, the boundary can be curved and does not have to coincide with the Cartesian coordinate axes.