KTH Royal Institute of Technology

Simulating Charged Particle Trajectories in Earth's Magnetosphere with Python to study Van Allen belts

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Abstract

The Van Allen Belts are regions of Earth's magnetosphere where charged particles get trapped. The magnetic field in these belts acts as a magnetic bottle, shielding the surface of our planet from solar winds and cosmic rays. However, this radiation concentration poses a problem to space exploration and satellites. In this project, we implemented a particle tracer that models the behavior of electrons and positrons on Van Allen Belts. Two methods were used to solve the equations of motion, the Runge-Kutta of 4^{th} order, and the Boris method. Our study concluded that particle confinement weakens as the pitch angle decreases and that the Boris method is generally better suited for simulating these systems. The guiding center equations were also simulated, to demonstrate the usefulness of this approximation.

1 Introduction

Van Allen Belts [1] are regions of Earth's magnetosphere where charged particles are trapped by our planet's magnetic field. The magnetosphere acts as a protecting shield, protecting the surface of Earth from energetic particles from solar winds and cosmic rays. However, while the magnetic field prevents high-energy particles from reaching us, it also traps them, increasing the radiation level in the Van Allen Belts, which poses risks to satellites, spacecraft, and astronauts that cross the magnetosphere. The radiation can interact with the electronics in these devices, affect the shielding, and increase health problems due to radiation exposure [2, 3]. The risks caused by the radiation in Van Allen Belts make them of the utmost interest to study, in order to better predict the radiation effects and equip our space exploration devices accordingly. Understanding the physics in these regions also proves to be beneficial for better space weather predictions, which is crucial for space exploration. Many studies of the dynamics of particles in Van Allen Belts are made, using satellites in orbit, theoretical models, and computational simulations [4, 5, 6, 7, 8].

Earth's magnetic field acts as a magnetic bottle trapping mainly electrons and protons in the Van Allen Belts. To study this trapping, computational simulations of the magnetic field and particle tracing techniques are a powerful tool, yet simple to implement. In this work, a computational tool for tracing particles in magnetic fields was developed, and applied to a dipole model of Earth's magnetic field, allowing the study of the particle trapping for various pitch angles and allowing to compare the effectiveness of two different ordinary differential equations (ODE) solving methods applied on this scenario. These two methods were the Runge-Kutta of 4^{th} order and the Boris pusher method. A small remark on the guiding center approximation is also made. All the models and methods used in this project were implemented in Python.

This work is organized as follows. In Section 2 the mathematical models to describe particle trajectories are formulated and explained. The dipole model of Earth's magnetic field is also presented. In Section 3 the two methods for solving ODE are detailed. In Section 4 the results of applying the computational methods described in solving the particles' trajectories are presented and compared, allowing us to grasp which methods are more efficient to use in this scenario. Finally, the conclusion follows in Section 5.

This work was done for the Computational Method for Electromagnetics (DD2370) at KTH, in the second period of the year 2024/2025, for an expected grade of A.

2 Theoretical Background

2.1 Lorentz force equation

The Lorentz equation is simply a formulation of Newton's Second Law of motion, for the force applied by an electromagnetic field in a charged particle. The equation is formulated in its relativistic form as,

$$\frac{d(m_0\gamma\mathbf{v})}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1}$$

where m is the rest mass of the particle, γ is the Lorentz factor, $\mathbf{v}$ is the velocity, q is the particle's charge and $\mathbf{E}$ and $\mathbf{B}$ are the electrical and magnetic field, respectively. This formulation of the particle's equation of motion can be solved by knowing the electric and magnetic fields. In the study case of this project, since we are studying the effects of the magnetic field on the particles' trajectory, we consider the electric field to be zero.

An important detail that one can infer from Eq. (1), is that the magnetic field imposes on charged particles a circular motion around a guiding center. This is called gyromotion, and is characterized by its gyrofrequency,

$$\omega_c = \frac{|q|B}{\gamma m}, \tag{2}$$

and its radius, called the Larmor radius,

$$r_L = \frac{\gamma m v_\perp}{|q|B}, \tag{3}$$

where $v_\perp$ is the value of the velocity perpendicular to the magnetic field. These are also called cyclotron frequency and cyclotron radius, respectively. These two quantities will define the smallest time and length scales of our problem, imposing constraints on the resolution used in the ODE solver methods.

2.2 Guiding center approximation

The guiding center is the center of the gyromotion. It follows the magnetic field lines when the magnetic field is uniform, but for non-uniform magnetic fields, it is susceptible to a drift [9]. The guiding center approximation consists of approximating the particles' trajectories to the trajectory of their guiding center [10]. This is valid in the scenario where the field does not change significantly within a cyclotron radius - this is, when the scale of the system is much larger than the cyclotron radius, making this motion irrelevant for the system.

This is done by considering the particles' position to be $\mathbf{r} = \mathbf{R} + \rho$, where $\mathbf{R}$ is the position of the guiding center and ρ is the position of the particle relative to the guiding center. Assuming the Larmor radius is much smaller than the field's length scale, we can expand the magnetic field $\mathbf{B(r)}$ in a Taylor series around $\mathbf{R}$, $\mathbf{B}(\mathbf{r}) \approx \mathbf{B}(\mathbf{R}) + (\rho \cdot \nabla)\mathbf{B}$. Substituting this in Eq. (1) averaging the equation over a gyro-period, and separating the velocity in the components parallel and perpendicular to the magnetic field, we obtain the acceleration of guiding center approximation,

$$\frac{d\mathbf{R}}{dt} = \frac{\gamma m v^2}{2qB^2}\left(1 + \frac{v_\parallel^2}{v^2}\right)\hat{\mathbf{b}} \times \nabla B + v_\parallel \hat{\mathbf{b}}, \tag{4}$$

$$\frac{dv_\parallel}{dt} = -\frac{\gamma m v_\perp^2}{2B}\hat{\mathbf{b}} \cdot \nabla B, \tag{5}$$

with $\hat{\mathbf{b}}$ being the direction of the magnetic field.

Although this formulation loses information about the small-scale dynamics of the system, such as the cyclotron motion, it has various advantages when studying large-scale systems. Since the smallest time scale is no longer defined by ω_c^{-1}, we do not need to resolve this time scale, allowing us to use larger time steps in our numerical methods. This means we can simulate longer times, and take less time to run large simulations.

2.3 Dipole model of Earth's magnetic field

Earth's magnetic field is very complex due to the geometry of our planet and its rotation motion, the interaction with charged particles, and even the effects of external magnetic fields, due to the long-range nature of the electromagnetic force. On the innermost regions of the magnetic field, where the field is less influenced by external fields, a good first-order approximation is to consider it a magnetic dipole. Using this approximation, we describe the components of the magnetic field as,

$$\mathbf{B}(x, y, z) = -\frac{3B_0 R_e}{r^5}(xz,\ yz,\ 2z^2 - x^2 - y^2) \quad [\mathrm{T}], \tag{6}$$

where $B_0 = 3.07 \times 10^{-5}$ T is the mean value of the magnetic field at the magnetic equator on the Earth's surface and $R_e = 6378$ km is the mean radius of Earth.

3 Implementation

3.1 Runge-Kutta of 4^{th} order

The Runge-Kutta methods are iterative methods to solve ordinary differential equations. The most known method of this family is the Runge-Kutta of 4^{th} order (RK4), a general-purpose method to solve ODE, knowing the system and the initial conditions. This method is widely used due to its accuracy and efficiency, and also for being simple to implement. The total accumulated error of RK4 is of the order $O(h^4)$ where h is the step-size used. This method works by computing the slope of the function at four points within a single step and calculating the next step by doing a weighted average of the slopes at the midpoints. Considering an initial value problem,

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \tag{7}$$

where t_0 and y_0 are the initial conditions, this method works by:

1. At the time step i, it computes the slope at four intermediate steps:

$$\begin{aligned} k_1 &= f(t_i, y_i), \\ k_2 &= f\left(t_i + \frac{h}{2}, y_n + h\frac{k_1}{2}\right), \\ k_3 &= f\left(t_i + \frac{h}{2}, y_n + h\frac{k_2}{2}\right), \\ k_4 &= f\left(t_i + h, y_n + hk_3\right). \end{aligned} \tag{8}$$

2. Computing the solution at the next step, $y_{i+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

3. Updating the time, $t_{i+1} = t_i + h$.

4. Repeating steps 1-3 until it reaches the final time.

This method is not symplectic since it does not conserve the geometric structure of the Hamiltonian of the system, this is, its phase-space volume. Another important detail is that while the RK4 method conserves energy for a short period of time, it does not conserve the system's energy over long periods due to accumulated error.

3.2 Boris Method

The Boris pusher algorithm is a widely used method in plasma physics, especially in particle-in-cell codes [11], to solve the equations of motions of particles in electromagnetic fields. This method is an explicit method that conserves the phase-space volume of the system, making it have excellent long-term accuracy and does not need information about the quantities on the next step to push the particles forward. This method decouples the magnetic and electric field and takes advantage of the rotation motion of the particle to compute half-steps. This is the standard method for computing the equation of motion of particles. The Boris method uses a staggered grid - considering positions to be computed in the integer grid and velocities in the half-integer grid, this method works as follows:

1. Start at the time step t_i, on the position $\mathbf{x}_i$ and velocity $\mathbf{v}_{i-1/2}$

2. Compute the effects of the electric field on the velocity, resulting in

$$\mathbf{u}^- = \mathbf{v}_{i-1/2} + \frac{q\Delta t}{2m}\mathbf{E}_i.$$

3. Compute the magnetic field effects by first computing an auxiliary vector

$$\mathbf{u}' = \mathbf{u}^- + \mathbf{u}^- \times \frac{qB_i\Delta t}{2m},$$

and correct the rotation by computing

$$\mathbf{u}^+ = \mathbf{u}^- + \mathbf{u}' \times \frac{qB_i\Delta t}{m}\frac{1}{1 + |\frac{qB_i\Delta t}{2m}|^2}.$$

4. Apply the second half of the electric field effect, updating the velocity to the next step,

$$\mathbf{v}_{i+1/2} = \mathbf{u}^+ + \frac{q\Delta t}{2m}\mathbf{E}_i.$$

5. Update the position to $\mathbf{x}_{i+1} = \mathbf{x}_i + \mathbf{v}_{i+1/2}\Delta t$.

6. Repeat steps 2 to 5.

In our case, since we do not have an electric field, steps 2 and 4 are not done. A schematic of the rotation applied with this method can be found in Fig. 1.

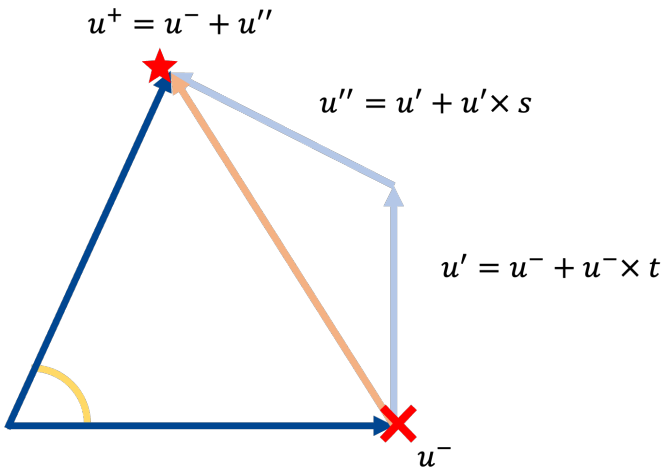


Figure 1: Schematic of the magnetic field effect on the velocity update. In this scheme, $t = \frac{qB_i \Delta t}{2m}$ and $s = \frac{qB_i \Delta t}{m(1+|qB_i \Delta t/2m|^2)}$, the red cross indicates the initial velocity, and the red star the final velocity after the update.

4 Results

In this work, we simulated two species of particles, protons and electrons. We used both the RK4 method and the Boris algorithm to solve the equations of motion and compute the trajectories of the particles. With the results from solving the equations of motion, we evaluated the three adiabatic invariants. The adiabatic invariants are quantities that are conserved due to the periodic motion of the particles in the magnetic field. The first adiabatic invariant is the magnetic moment, defined as

$$\mu = \frac{\gamma^2 m v_\perp^2}{2B}. \tag{9}$$

This quantity expresses the conservation of the magnetic moment of a charged particle moving in a periodic circular path. The second adiabatic invariant, I, is defined as

$$I = \int_0^{T_{bounce}/2} \frac{v_\parallel^2}{v} dt, \tag{10}$$

and it is related to the longitudinal motion of the particles and it is computed as an integral over half the period of bounce between two mirror points (the points where the particle inverts its longitudinal motion). Finally, the third adiabatic invariant is related to the particle's drift motion and expresses that the magnetic flux enclosed by a particle's drift orbit should be constant. Is defined as

$$\phi = \int_{R_0}^{\infty} B_0 \frac{R_E^3}{r^3} 2\pi r dr = 2\pi B_0 \frac{R_E^3}{R}. \tag{11}$$

Finally, we also analyzed the conservation of kinetic energy, defined as

$$K = mc^2(\gamma - 1). \tag{12}$$

To compute if the quantities were conserved, we used the mean relative error, defined for a quantity A as

$$\overline{\delta A} = \left\langle \left| \frac{A - A_0}{A} \right| \right\rangle \times 100 \quad [\%], \tag{13}$$

where A_0 is the initial value, for the case of the energy, and the mean value, for the adiabatic invariants.

4.1 Lorentz equations for protons

Simulating protons can be done easily by employing both methods. For this species, we have that the shortest time scale that needs to be resolved is $2\pi/\omega_c \approx 0.0021$ [s]. This means that for the RK4 method, we need to use a time step $\Delta t \ll 0.0021$ [s], which is large enough for this method to be fast to run long periods.

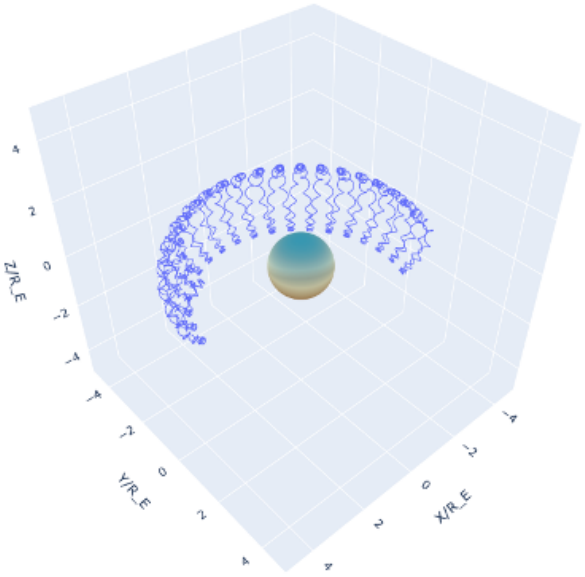


Figure 2: Simulation of a proton's trajectory on Earth's magnetic field, with $K_0 = 10$ MeV and $\alpha = 45^{\text{o}}$, using the Boris method with $\Delta t = 0.001$ s.

We used a time step $\Delta t = 0.001$ [s] for both methods and simulated particles with initial kinetic energy $K_0 = 10$ [MeV], for the pitch angles $\alpha = [15^{\text{o}}, 30^{\text{o}}, 45^{\text{o}}, 60^{\text{o}}, 90^{\text{o}}]$. A three-dimensional plot of the proton's trajectory can be found in Fig. 2 (at the end of this report), where the gyromotion can be clearly seen.

For the case of the protons, we obtained that the Boris method achieved better conservation results overall. The mean relative errors were smaller for the energy values and for all the adiabatic invariants. The plots of the mean relative error of the energy and the three adiabatic invariants for the pitch angles simulated can be found in Fig. 5, for both the Runge-Kutta method and the Boris pusher.

As we can conclude, the Boris method proves to be more efficient in conserving energy, and the first and second adiabatic invariants. Meanwhile, for the third adiabatic invariant the difference of the mean error value for the two algorithms is very small, allowing us to conclude that when studying this quantity, the choice of algorithm should be based on other parameters such as running time. Overall there is a tendency for the mean relative error to be larger as the pitch angle gets smaller. When the pitch angle is smaller, the particle trapping is more difficult and particles travel more along the field line, therefore the fluctuations in the quantities evaluated should be larger, as they are.

4.2 Lorentz equations for electrons

Electrons are much more difficult to simulate than protons. This is due to the fact that the mass of the electron is much smaller than the mass of the proton, which makes the velocity of the electrons much higher, for the same value of energy. It also makes the Larmor radius and the cyclotron frequency much smaller, which poses a problem for simulating these particles with RK4. Since RK4 needs a time step that resolves the smallest time scale of the system, for the electrons we have that $\Delta t \ll 1.16 \times 10^{-6}$ s. This makes RK4 not feasible to simulate the system for long periods of time. For this species, we used a time step $\Delta t = 1 \times 10^{-4}$ s for the Boris method, and $\Delta t = 1 \times 10^{-7}$ s for the RK4, and the same pitch angles as for the proton.

Once again, for the electrons, we obtained that the Boris method was a better method to solve Lorentz equations and obtain the trajectory of the electrons, as it can be used with larger time steps, allowing to simulation of the system for longer periods of time without accumulating considerable errors, while also resolving the cyclotron motion. As we can access from Fig. 6 (at the end of this report), the Boris algorithm achieves smaller mean relative errors in all quantities except for the third invariant. This could be due to simulating the system for a short amount of time, not being enough for it to stabilize.

To compare the performance of the two methods, for the electrons we evaluated the time it took to simulate the system. In Fig. 3 it is possible to verify that simulating one second with a time step of $\Delta t = 1 \times 10^{-7}$ s using RK4 takes more than 10 times more time than simulating 20 seconds with a time step $\Delta t = 1 \times 10^{-4}$ s, using the Boris method. Comparing the values obtained for the mean relative errors of the quantities analyzed, found in Table 1, we

also conclude that the Boris method is a better choice to simulate electrons in Earth's magnetic field.

Quantity	RK4	Boris
$\overline{\delta K}$	8.1×10^{-6} %	1.1×10^{-11}%
$\overline{\delta \mu}$	5.7×10^{-1} %	1.5×10^{-1}%
$\overline{\delta I}$	9.2×10^{-8} %	5.1×10^{-7}%
$\overline{\delta \phi}$	7.4×10^{-2} %	3.6×10^{-2}%

Table 1: Comparison of the mean relative error of the quantities studied for the RK4 and the Boris method.

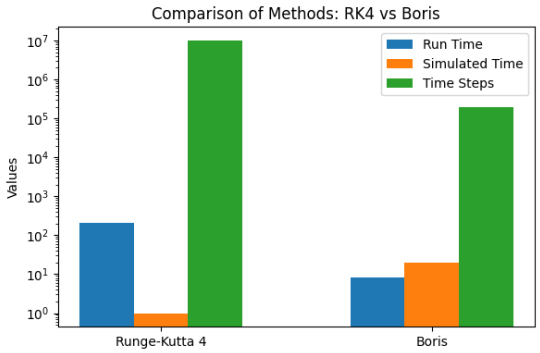
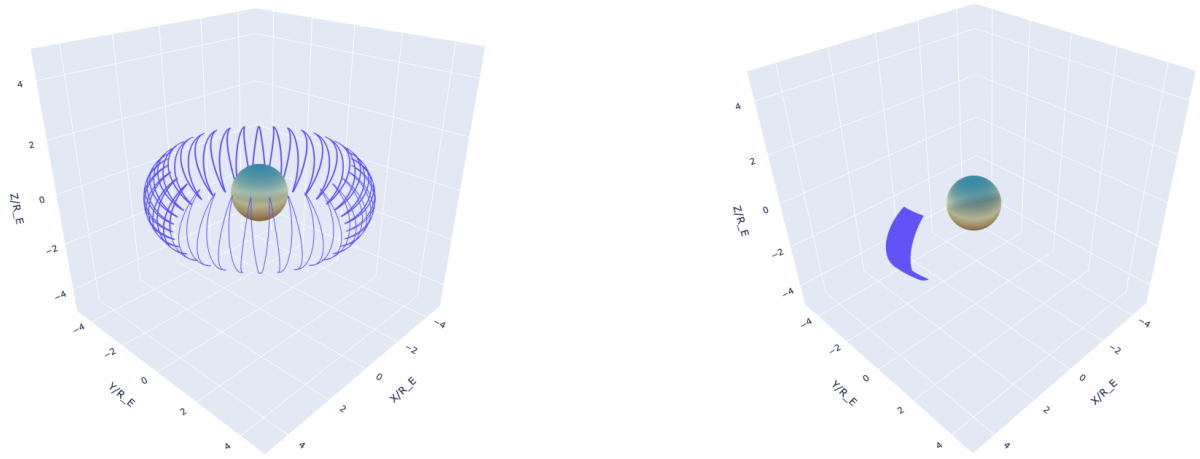


Figure 3: Barplot of the run time, the simulated time, and the number of time steps of electron simulations using RK4 and Boris method.

4.3 Guiding center approximation

The guiding center approximation is a powerful model of a particle's trajectories in electromagnetic fields. In our case, since the dimension of Earth's magnetic field is way larger than the Larmor radius, of both protons and electrons, we can safely use this approximation. Since there is no cyclotron motion, we do not need to resolve this movement, allowing us to use larger time steps. This means that we can capture the essential dynamics of the system and run the simulations for a longer time period. Finally, another advantage of this approximation is that according to its formulation, the first adiabatic invariant is always conserved - this means that the conservation of μ depends only on the numerical method used to solve the system. In Fig. 4 it is possible to find the trajectories of an electron and a proton using the guiding center approximation. As we can see, the cyclotron motion is not present.



(a) Trajectory of a proton using the guiding center approximation.

(b) Trajectory of an electron using the guiding center approximation.

Figure 4: Trajectories of particles using the guiding center approximation, using RK4 with $\Delta t = 0.001$ [s], a $t_{max} = 120$ [s].

5 Conclusion

In this project we were able to implement a particle tracer code fully in Python, using two different methods to solve the equations of motion of the system. These were the Runge-Kutta of 4^{th} order and the Boris algorithm. By solving the Lorentz equations for protons and electrons on Earth's magnetic field, and computing the conservation of the kinetic energy and the three adiabatic invariants, we were able to conclude that the Boris method is better suited for this task than the RK4. We accessed that while RK4 was able to solve the system for a proton and simulate the system for a long time period in feasible running time, for an electric this was not the case. For electrons, due to their mass, the RK4 method cannot be used to simulate the system for long periods, since it needs a very short time step to resolve the cyclotron motion. Since the Boris algorithm resolves the cyclotron motion explicitly in its formulation, we can use larger time steps and accuracy is maintained throughout the simulation. Finally, we used the guiding center approximation to show that, in scenarios where the cyclotron motion is not relevant, and the Larmor radius is small compared with the system's dimension, this approximation is a powerful method to capture the essential dynamics of the system. This method can be used for long simulations and can be used with larger time steps.

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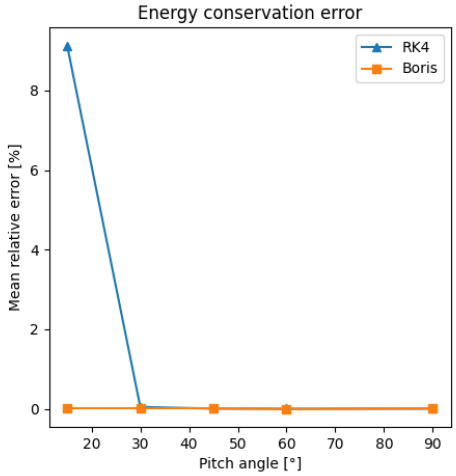
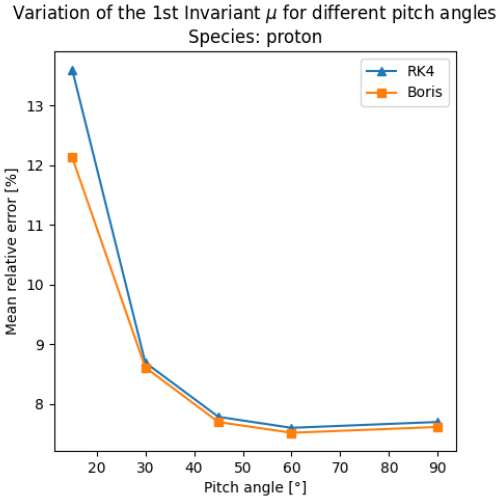
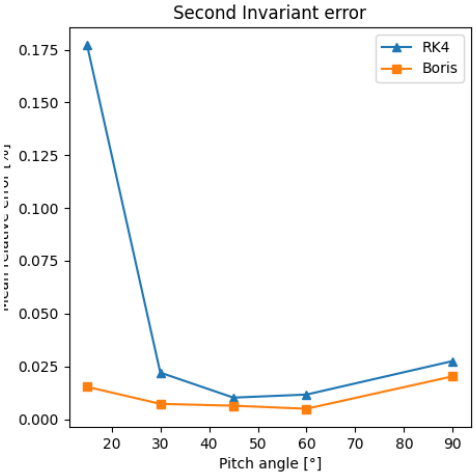


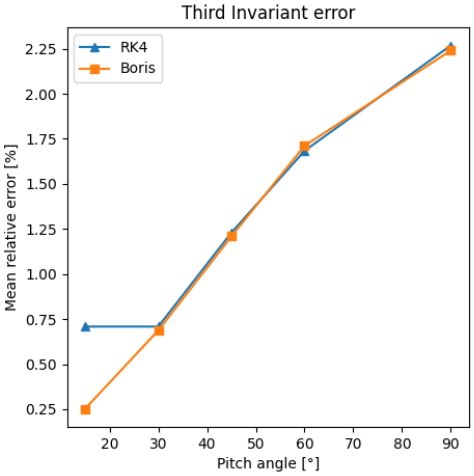
(a) Mean relative error for the kinetic energy.



(b) Mean relative error for μ.

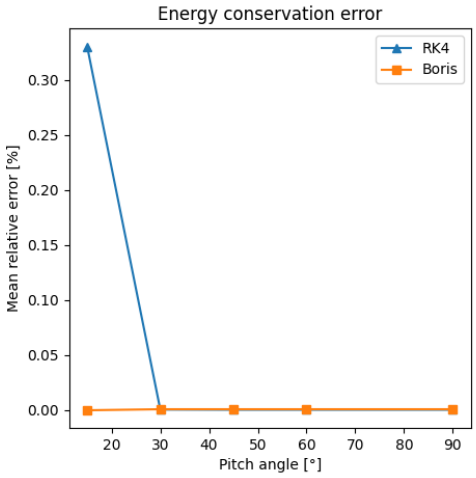


(c) Mean relative error for I.

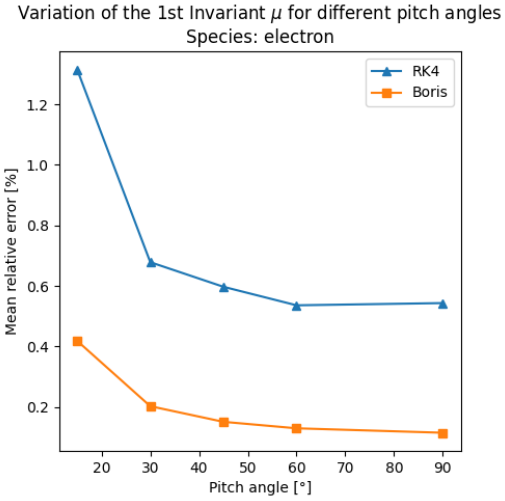


(d) Mean relative error for ϕ.

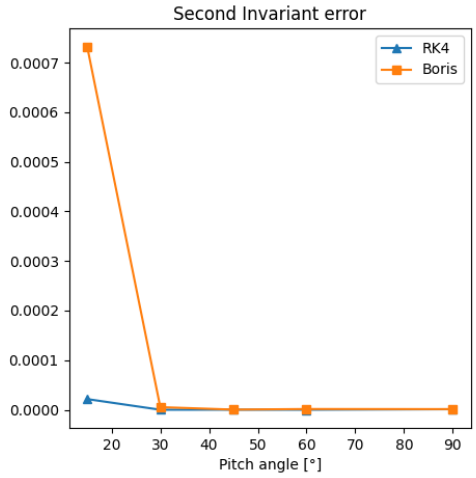
Figure 5: Mean relative error of the kinetic energy and the three adiabatic invariants varying with the pitch angle, for protons.



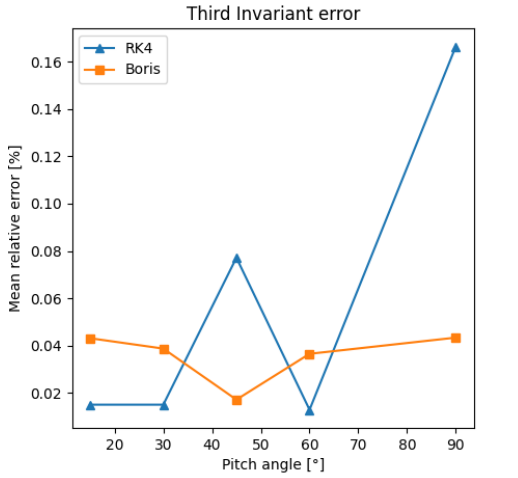
(a) Mean relative error for the kinetic energy.



(b) Mean relative error for μ.



(c) Mean relative error for I.



(d) Mean relative error for ϕ.

Figure 6: Mean relative error of the kinetic energy and the three adiabatic invariants varying with the pitch angle, for electrons.