Simulation of Charged Particles Motion with the Guiding Center Approximation

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The Guiding Center (GC) Approximation

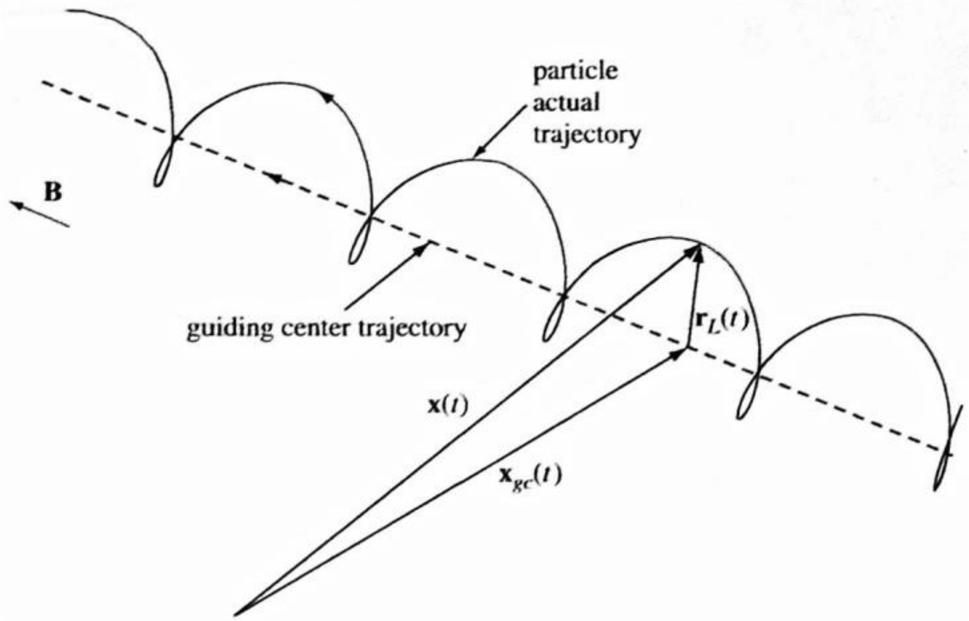
- The guiding center is the geometric center of cyclotron motion.
- We will calculate the trajectory of the guiding center.
- The particle position r is substituted with:

$$\mathbf{r} = \mathbf{R} + \rho$$

- Assuming that the cyclotron radius is much smaller than the length scale of the Field, we can expand B around R to the first order in the Taylor series:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{R}) + (\rho \cdot \nabla)\mathbf{B}$$

- This expression is substituted into the Newton-Lorenz equation, and the equation is **averaged over a gyro-period**, eliminating rapidly oscillating terms containing ρ and its derivatives.



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GC Approximation

1

$$\frac{d\mathbf{R}}{dt} = \frac{mv^2}{2qB^2}\left(1 + \frac{v_{//}^2}{v^2}\right)\hat{\mathbf{b}} \times \nabla B + v_{//}\hat{\mathbf{b}}$$

2

$$\frac{dv_{//}}{dt} = -\frac{\mu}{m}\hat{\mathbf{b}} \cdot \nabla B$$

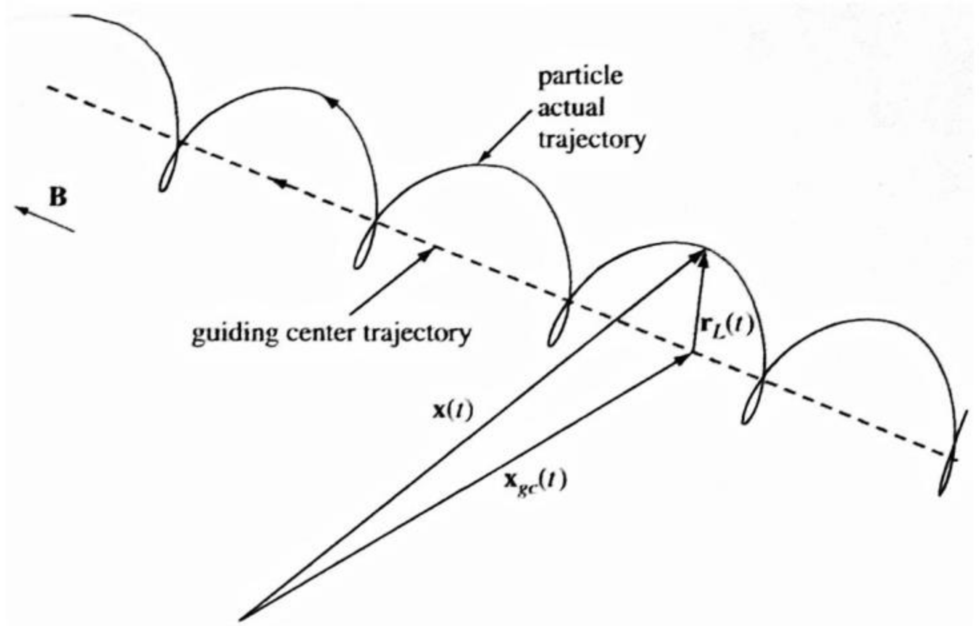
$$\mu = \frac{mv_\perp^2}{2B}$$

Magnetic Moment

The unknowns are the three coordinates of the guiding center and the parallel (to **B**) velocity

Advantages of GC approach

- For numerical stability reasons, the time step in the computer simulations needs to be a fraction of the fastest time scale in the system (in this case, the gyro-period).
  - T = 2*pi*m/(abs(e)*B);
- When we simulate electrons, the gyro-period can become very small, imposing a tiny simulation time step (simulation will take a long time).
- Because the GC approximation removes the gyration motion from the model, we take time steps larger than the gyro period, and the simulation will take less time.



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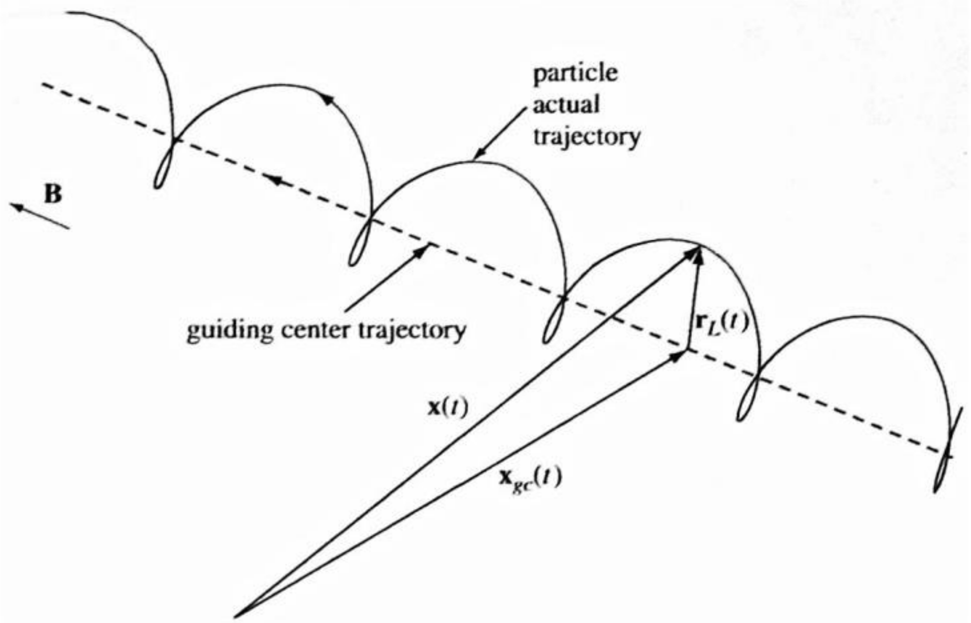
Validity of GC Approximation

- We derived the GC equation assuming that the <u>Larmor radius is much smaller than the length scale of the field.</u>
  - This approximation is often used in the simulation of fusion devices where you have a very strong magnetic field.
- GC approximation is not valid in the presence of magnetic null or weak magnetic field.
- When you have highly energetic particles, the Larmor radius might become comparable or larger than the length scale of the magnetic field (in the case of Earth's magnetic dipole).
  - In this case, the GC approximation is not valid.



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```python
def newton_lorentz_gc(t, x_vect):

x, y, z, vpar = x_vect
vsq = v_mod**2
fac1 = -B0 * Re**3 / (x**2 + y**2 + z**2)**2.5
Bx = 3 * x * z * fac1
By = 3 * y * z * fac1
Bz = (2 * z**2 - x**2 - y**2) * fac1
B_mod = np.sqrt(Bx**2 + By**2 + Bz**2)
# Magnetic moment (adiabatic invariant)
mu = m * (vsq - vpar**2) / (2 * B_mod)

# Gradient of B_mod (numerical)
d = 1e-3 * Re # Small perturbation for finite difference
gradB_x = (getBmod(x + d, y, z) - getBmod(x - d, y, z)) / (2 * d)
gradB_y = (getBmod(x, y + d, z) - getBmod(x, y - d, z)) / (2 * d)
gradB_z = (getBmod(x, y, z + d) - getBmod(x, y, z - d)) / (2 * d)
# Unit vector along B
b_unit_x = Bx / B_mod
b_unit_y = By / B_mod
b_unit_z = Bz / B_mod

# Cross product b_unit x grad(B)
bxgB_x = b_unit_y * gradB_z - b_unit_z * gradB_y
bxgB_y = b_unit_z * gradB_x - b_unit_x * gradB_z
bxgB_z = b_unit_x * gradB_y - b_unit_y * gradB_x
# Dot product b_unit · grad(B)
dotpr = b_unit_x * gradB_x + b_unit_y * gradB_y + b_unit_z * gradB_z

# Guiding center velocity components
fac = m / (2 * q * B_mod**2) * (vsq + vpar**2)
dxdt = fac * bxgB_x + vpar * b_unit_x
dydt = fac * bxgB_y + vpar * b_unit_y
dzdt = fac * bxgB_z + vpar * b_unit_z
dvpar_dt = -mu / m * dotpr

return [dxdt, dydt, dzdt, dvpar_dt]
```

ODE to Solve

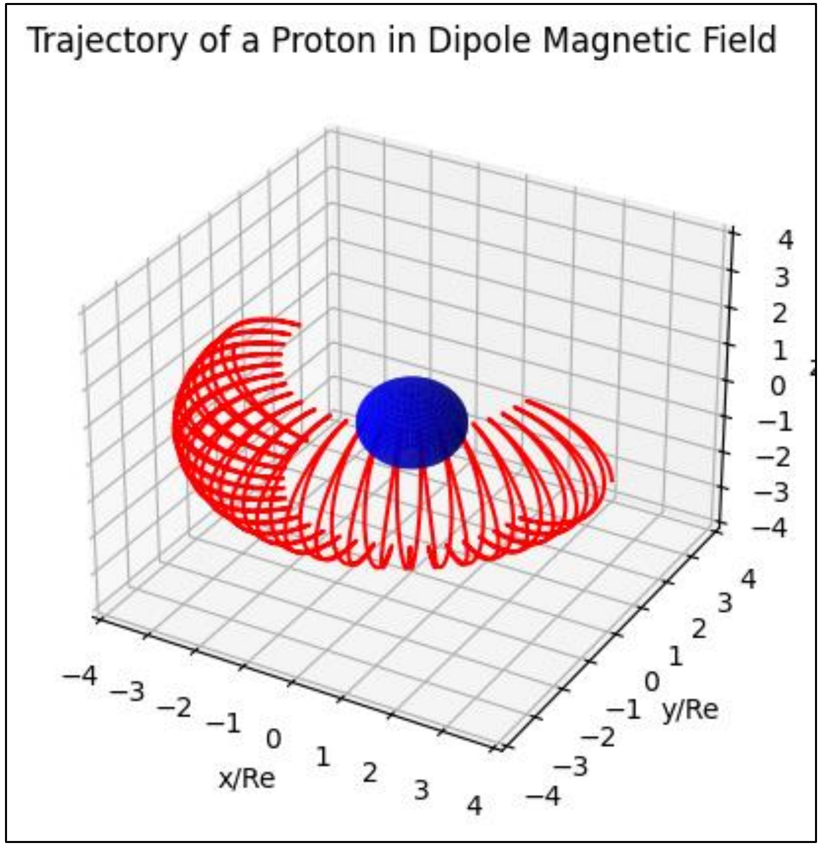
```python
# Magnetic field computation
def getBmod(x, y, z):
fac1 = -B0 * Re**3 / (x**2 + y**2 + z**2)**2.5
Bx = 3 * x * z * fac1
By = 3 * y * z * fac1
Bz = (2 * z**2 - x**2 - y**2) * fac1
return np.sqrt(Bx**2 + By**2 + Bz**2)
```

$$\frac{d\mathbf{R}}{dt} = \frac{mv^2}{2qB^2}\left(1 + \frac{v_{//}^2}{v^2}\right)\hat{\mathbf{b}} \times \nabla B + v_{//}\hat{\mathbf{b}}$$

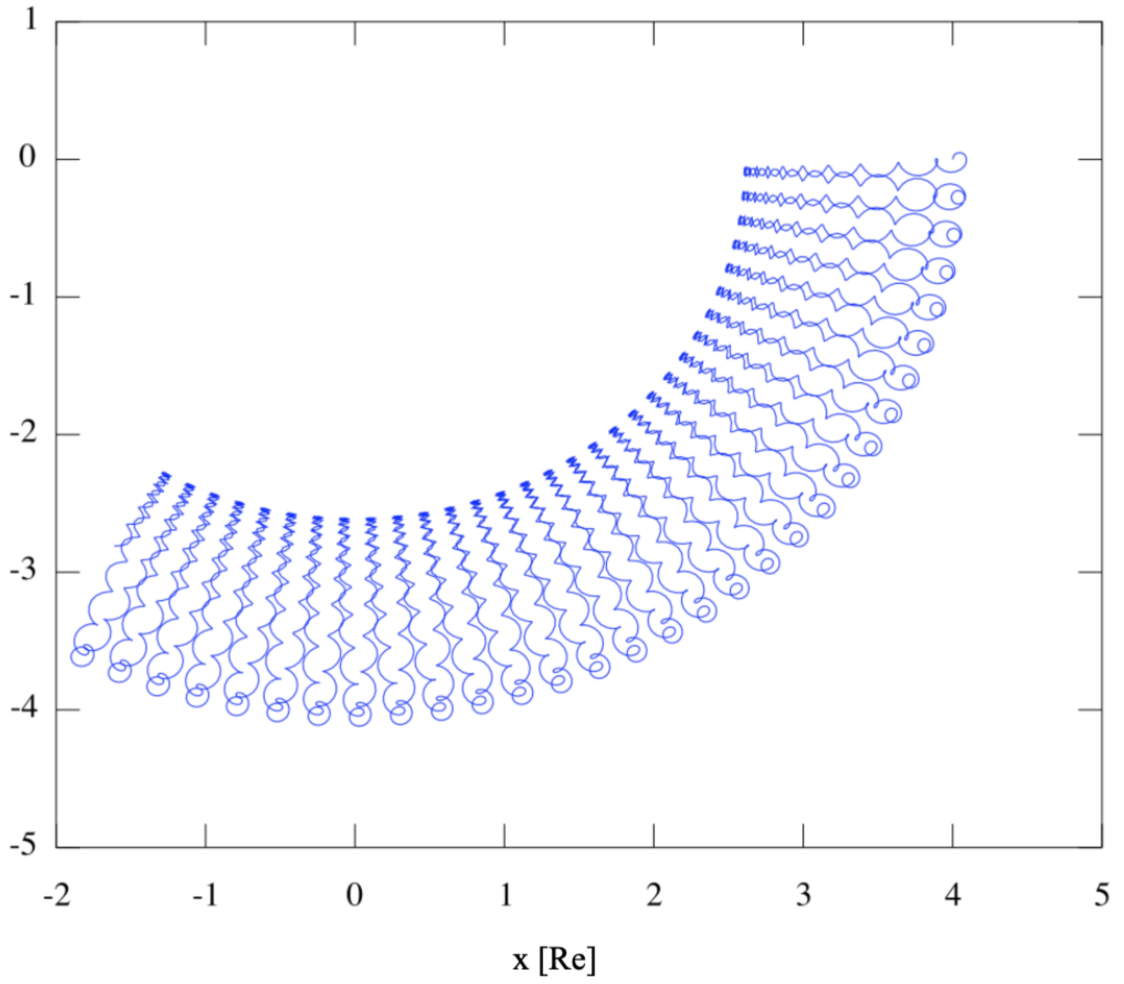
$$\frac{dv_{//}}{dt} = -\frac{\mu}{m}\hat{\mathbf{b}} \cdot \nabla B$$

Main Code

```python
# Initial conditions
x0 = 4 * Re
y0 = 0
z0 = 0
initial_conditions = [x0, y0, z0, v_par0]

# Time span
tfin = 80.0 # seconds
time = np.arange(0, tfin, 0.01)

# Solve the ODE
solution = solve_ivp(newton_lorentz_gc, [0, tfin], initial_conditions,
t_eval=time, method='RK45')

# Extract solution
x_sol = solution.y.T # Transpose for easier handling

# Plot trajectory
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot(x_sol[:, 0] / Re, x_sol[:, 1] / Re, x_sol[:, 2] / Re, 'r')
ax.set_xlabel('x / Re')
ax.set_ylabel('y / Re')
ax.set_zlabel('z / Re')
ax.set_title('Guiding Center Trajectory')
plt.show()
```



No Approximation



GC Approximation