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LETTERS

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SOLUTION OF STELLARATOR BOUNDARY VALUE PROBLEMS WITH EXTERNAL CURRENTS

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ABSTRACT. Neumann boundary value problems are solved for stellarator fields generated by external currents. The method can be applied to the construction of external coil configurations of stellarators and to studies of the properties of vacuum fields. A computer code, NESCOIL, has been developed and applied to the Helias and ATF stellarator configurations.

1. INTRODUCTION

Three-dimensional MHD equilibrium codes [1–3] have been extensively applied to studies of magnetohydrodynamically stable stellarators [4, 5]. The equilibrium problem is solved with these codes in the plasma region for a prescribed boundary.

The experimental realization of such a solution involves the problem of finding a distribution of external currents which produces a magnetic field surrounding the plasma equilibrium and satisfying the boundary conditions at the plasma boundary $\vec{B}_V \cdot \vec{n} = 0$ and $\vec{B}_V^2 = \vec{B}_p^2 + 2p$ ($\vec{n}$ is the exterior normal to the boundary).

Continuation of the magnetic field into the vacuum region leads to a Cauchy-type initial value problem, which is not well posed for an elliptic partial differential equation. Furthermore, it may happen in the event of exact continuation that the singularities, which correspond to the currents, appear too close to the plasma surface.

These difficulties can be avoided if one does not insist on a rigorous solution of the boundary value problem. The external vacuum field can be represented by superposing harmonic functions such as Dommaschk potentials [6] so that a solution of the boundary value problem is approximated.

A different method which was similarly applied to the tokamak equilibrium problem [7] is proposed in the present letter. The external magnetic field is produced by a current distribution on a closed surface surrounding the plasma region (Fig. 1). The surface current is then determined such that the normal component of the field $\vec{B}$ produced is minimized at the plasma boundary $\vec{B} \cdot \vec{n} = 0$. As the results show, this can be achieved, with convincing accuracy, for an appropriately chosen outer surface. This approximate solution of a boundary value problem yields a vacuum field which is regular in the whole domain bounded by the outer surface and approximates the vacuum field of the plasma equilibrium configuration. This vacuum field is uniquely determined by the shape of the plasma boundary. Equilibrium solutions with finite plasma pressure can then be obtained by applying the NEMEC (= VMEC + NESTOR) free boundary code [8, 9].

The vacuum field solution can be used to find the position of coils generating the magnetic field. The outer current carrying surface has to be shaped in such a way that a surface current distribution whose closed current lines are not too complicated can be suitably modelled to discretize it into a finite number of coils.

The method of solution proposed here is described in Section 2, and applications are presented in Section 3.

2. METHOD OF SOLUTION

The magnetic field $\vec{B}$ (curl $\vec{B} = \vec{j}$, div $\vec{B} = 0$) produced by a surface current $\vec{j}$ on the outer toroidal surface ∂D can be computed by the Biot-Savart formula:

$$\vec{B} = \frac{1}{4\pi} \int_{\partial D} df' \frac{\vec{j}' \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \tag{1}$$

The plasma boundary ∂R and the outer boundary ∂D are assumed to consist of n_p toroidal periods. If angle-like variables u and v are introduced, one period of the boundaries ∂R and ∂D is given by mapping the unit square $0 \leqslant u < 1$, $0 \leqslant v < 1$ onto the surfaces, which reads for ∂D:

$$r = \sum_{m=0,n=-n_b}^{m_b,n_b} \hat{r}_{mn} \cos 2\pi(mu + nv)$$

$$z = \sum_{m=0,n=-n_b}^{m_b,n_b} \hat{z}_{mn} \sin 2\pi(mu + nv) \tag{2}$$

$$\varphi = \frac{2\pi}{n_p} v$$

where (r, φ, z) are cylindrical co-ordinates and the usual stellarator symmetry, $r(u, v) = r(-u, -v)$, $z(u, v) = -z(-u, -v)$, is assumed.

The surface current density with the same periodicity can be expressed by a potential $\Phi(u, v)$ [10] defined on the surface,

$$\vec{j} = \vec{n} \times \text{Grad}\, \Phi(u,v) \tag{3}$$

where Grad is the gradient operator on the surface ($\vec{x}_u \cdot \text{Grad}\, \Phi = \Phi_u$, $\vec{x}_v \cdot \text{Grad}\, \Phi = \Phi_v$), $\vec{n} = -(\vec{x}_u \times \vec{x}_v)/|\vec{x}_u \times \vec{x}_v|$ is the exterior normal to ∂D, and $\vec{x}_u$, $\vec{x}_v$ are the derivatives with respect to u and v, respectively.

The general ansatz for $\Phi(u, v)$ can be written as

$$\Phi(u,v) = \sum_{m=0,n=-N}^{M,N} \hat{\Phi}_{mn} \sin 2\pi(mu + nv) - \frac{I_p}{n_p} v - I_t u \tag{4}$$

where I_p and I_t are the net poloidal and toroidal surface currents, respectively. The potential is antisymmetric with respect to u and v, because of the assumed stellarator symmetry.

With prescribed values of I_t and I_p, the solution of the boundary value problem is approximated by determining the Fourier coefficients $\hat{\Phi}_{mn}$ such that the component of $\vec{B}$ normal to the plasma surface ∂R is minimized:

$$\int_{\partial R} df (\vec{B} \cdot \vec{n})^2 = \min \tag{5}$$

The periodic part of the potential is written as a truncated Fourier series. The values of M and N used depend on the desired accuracy of the solution. High mode numbers contribute strongly oscillating surface currents with little effect on the magnetic field at the plasma boundary. Fourier harmonics with wavelengths shorter than the plasma-to-outer-surface distance can usually be neglected.

Inserting Eqs (2), (3), and (4) into Eq. (5) and integrating by parts the term with the periodic part of the potential, we obtain the expression

$$\int_0^1 \int_0^1 \frac{du\,dv}{|\vec{N}|} \Big(\sum_{m=0,n=-N}^{M,N} \hat{\Phi}_{mn} g_{mn}(u,v) + h(u,v) \Big)^2 = \min \tag{6}$$

with

$$g_{mn}(u,v) = \int_0^1 \int_0^1 du'dv'\, g(u',v',u,v) \sin 2\pi(mu' + nv')$$

$$g(u',v',u,v) = -\frac{1}{4\pi} \sum_{\ell=0}^{n_p-1} \Big(\frac{\vec{N} \cdot \vec{N}'^{(\ell)}}{|\vec{x} - \vec{x}'^{(\ell)}|^3} - \frac{3\vec{N} \cdot (\vec{x} - \vec{x}'^{(\ell)})\, \vec{N}'^{(\ell)} \cdot (\vec{x} - \vec{x}'^{(\ell)})}{|\vec{x} - \vec{x}'^{(\ell)}|^5} \Big) \tag{7}$$

and the inhomogeneous term

$$h(u,v) = \frac{1}{4\pi} \int_0^1 \int_0^1 du'dv' \sum_{\ell=0}^{n_p-1} \Big[\Big(I_t \vec{x}_{v'}'^{(\ell)} - \frac{I_p}{n_p} \vec{x}_{u'}'^{(\ell)} \Big) \times \frac{(\vec{x} - \vec{x}'^{(\ell)})}{|\vec{x} - \vec{x}'^{(\ell)}|^3} \Big] \vec{N} \tag{8}$$

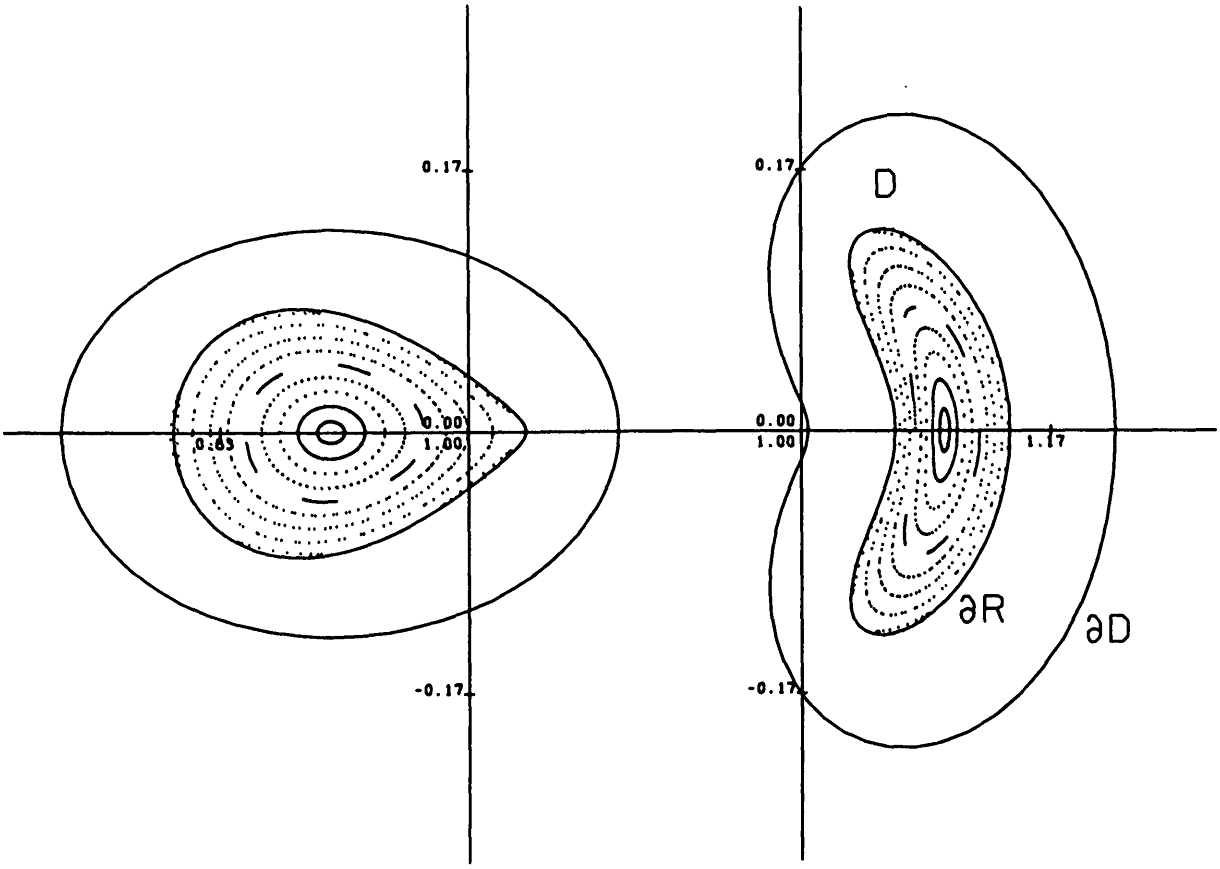

FIG. 1. Poincaré plot of Helias vacuum field $\vec{B}$, where $\vec{B}$ is generated by surface current on ∂D so that $\vec{B}\cdot\vec{n}$ on ∂R is minimized. Number of Fourier harmonics $\hat{\Phi}_{mn}$ taken into account: $m \leqslant 8$, $|n| \leqslant 8$. Number of field periods $n_p = 5$.

In terms of Cartesian co-ordinates the vectors $\vec{x}'^{(l)}$ are defined by

$$\vec{x}'^{(\ell)} = \begin{pmatrix} r' \cos 2\pi \ (\ell + v')/n_p \\ r' \sin 2\pi \ (\ell + v')/n_p \\ z' \end{pmatrix}, \ell = 0, \ n_p - 1 \tag{9}$$

and $\vec{N} = -\vec{x}_u \times \vec{x}_v$, $\vec{N}'^{(\ell)} = -\vec{x}'^{(\ell)}_{u'} \times \vec{x}'^{(\ell)}_{v'}$ are vectors normal to the surfaces. Primed quantities refer to the outer surface ∂D, and unprimed to the first toroidal period of the plasma boundary ∂R.

Differentiation of expression (6) with respect to $\hat{\Phi}_{mn}$ leads to a set of linear equations for the $\hat{\Phi}_{mn}$ determining the minimum.

Concerning the manifold of solutions, the values of I_p and I_t can be prescribed independently, but the magnetic field in D is independent of the net toroidal current I_t; it depends only on the net poloidal current I_p. This is because a solution of the exterior Neumann problem with respect to ∂D can be added to each solution, which does not change the field in D. Such a

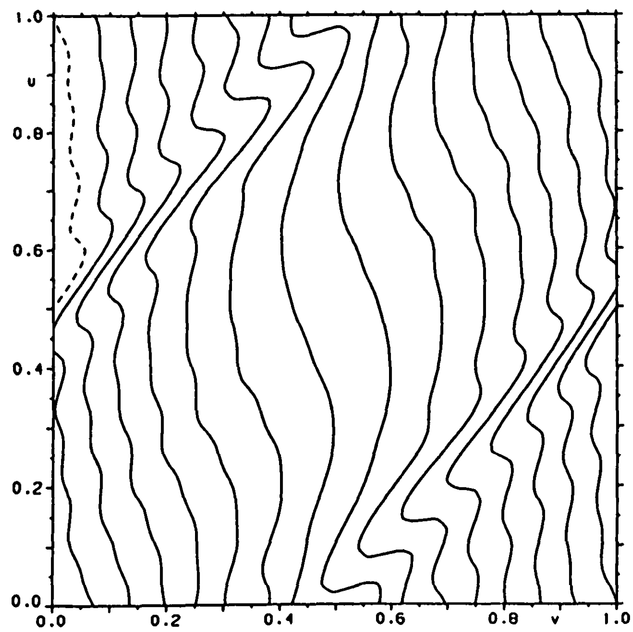

FIG. 2. Surface current lines of Helias vacuum field (Fig. 1) in the (u,v) plane. Net toroidal current $I_t = 0$. Number of Fourier harmonics $\hat{\Phi}_{mn}$ taken into account: $m \leqslant 8$, $|n| \leqslant 8$.

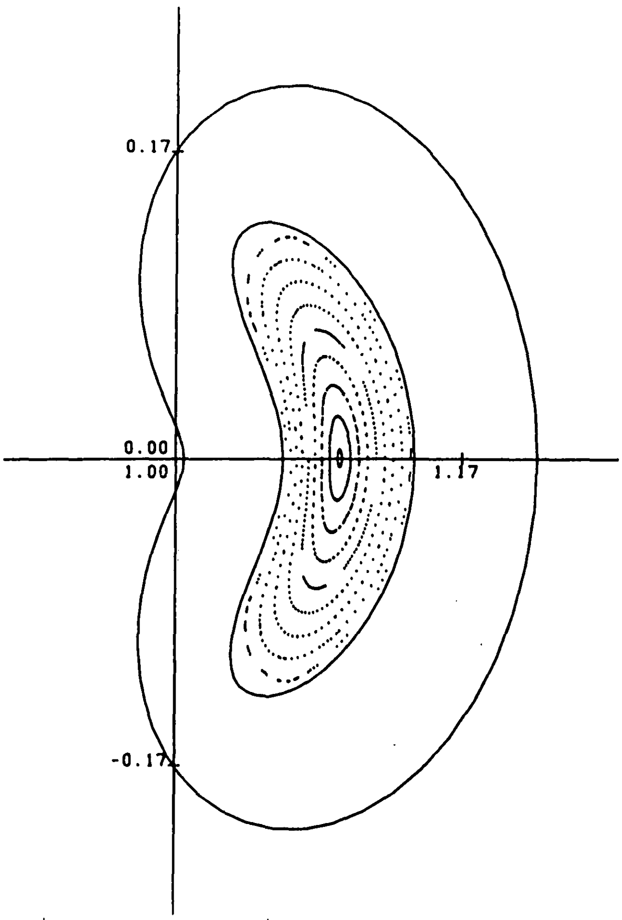

FIG. 3. Poincaré plot of Helias vacuum field. Number of Fourier harmonics $\hat{\Phi}_{mn}$: $m \leqslant 4$, $|n| \leqslant 4$.

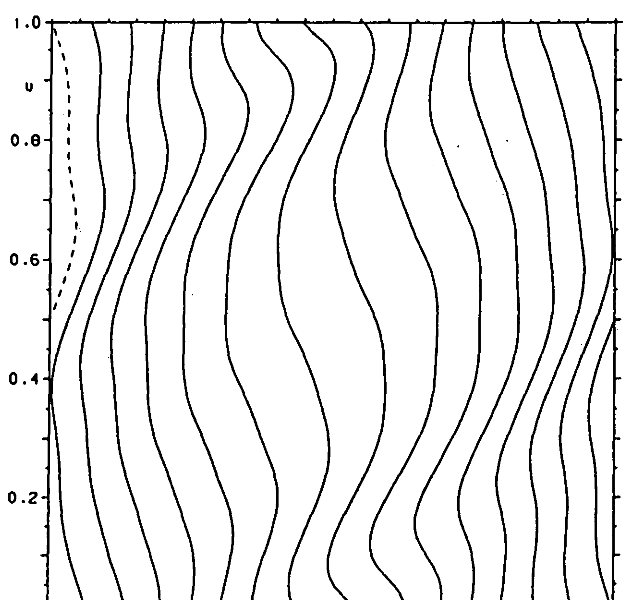

FIG. 4. Surface current lines of Helias vacuum field in the (u, v) plane. Number of Fourier harmonics $\hat{\Phi}_{mn}$: $m \leqslant 4$, $|n| \leqslant 4$.

Neumann field produced by a toroidal current distribution is tangential on ∂D and vanishes in D. This means that a one-parameter set of current distributions exists for each vacuum field solution.

3. RESULTS

The NESCOIL (NEumann Solver for fields produced by external COILs) computer code is developed for solving the boundary value problem. The boundaries given by Fourier series are discretized by means of an equally spaced mesh in the (u, v) and (u′, v′) planes. In all cases presented, a fine mesh with a number of mesh points $N_u = N_v = N'_{u'} = N'_{v'} = 64$ is used to keep the real space discretization error small. The numerical integrations in Eqs (7) and (8) and the solution of the set of linear equations for $\hat{\Phi}_{mn}$ are performed by standard methods.

Two applications are presented. In the first of these, a possible coil configuration for the Helias stellarator device is computed. An MHD stable Helias-type stellarator configuration with $n_p = 5$ periods, an aspect ratio of A = 11.5 and a β value of ⟨β⟩ = 0.09 has been proposed [4, 5]. Figure 1 shows the shape ∂R of the Helias equilibrium at v = 0, π and an outer surface ∂D with an aspect ratio of A = 6.5. The current distribution on ∂D is computed by approximating the Helias vacuum field. Figure 2 shows the surface current lines of the solution with no net toroidal current $I_t = 0$. The current lines poloidally closed can easily be discretized and represented by a finite number of modular coils. Figure 1 shows the Poincaré plot of the field produced by 15 coils per period.

The Helias configuration is characterized by a strongly reduced parallel current density and a magnetic well. The solution shown in Fig. 1 approximates the Helias

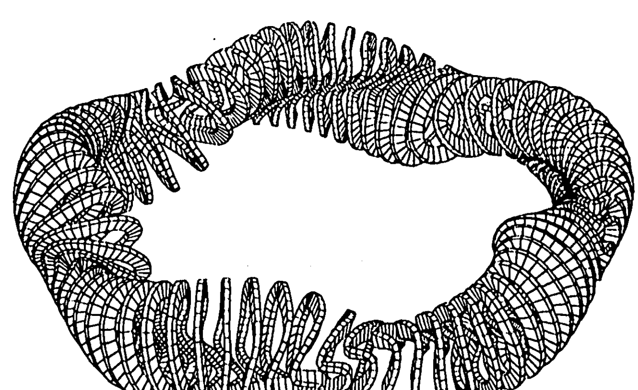

FIG. 5. Modular coils for Helias stellarator. Number of coils per period: $N_c = 15$. Aspect ratio of coil configuration: $A = 6.5$.

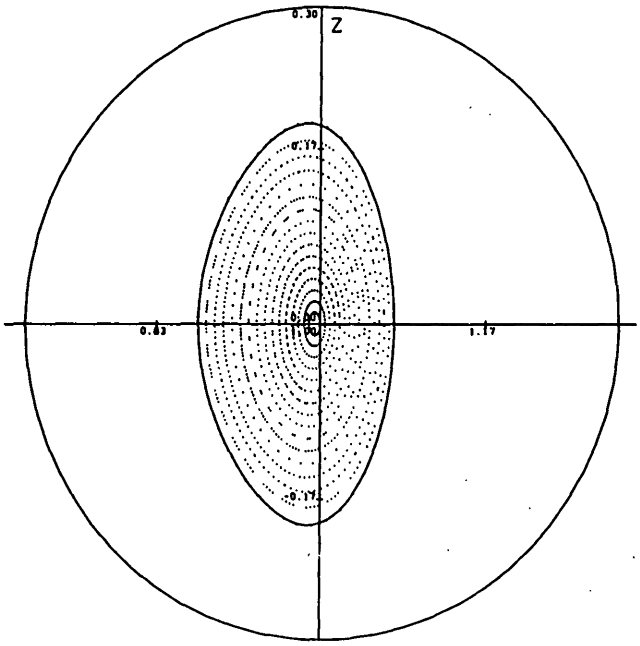

FIG. 6. Poincaré plot of ATF vacuum field produced by surface current on torus with circular cross-section and aspect ratio of $A = 3.3$. Number of Fourier harmonics $\hat{\Phi}_{mn}$ taken into account: $m \leqslant 8$, $|n| \leqslant 8$.

vacuum field with good accuracy. The magnetic well of the field is about 1%. The rotational transform approaches a value of $\iota = \frac{1}{5}$ per period at the edge, where the corresponding islands are observed.

For the solution shown in Figs 1 and 2, Fourier harmonics of $\hat{\Phi}_{mn}$ up to $m \leqslant 8$, $|n| \leqslant 8$ are taken into account. A smoother shape of the coils can be obtained by reducing the number of Fourier harmonics $\hat{\Phi}_{mn}$. In Fig. 3 the solution for the case with $\hat{\Phi}_{mn}$, $m \leqslant 4$, $|n| \leqslant 4$ is shown. Again, the field is generated by 15 coils per period. The boundary value problem is solved with less accuracy, but a magnetic well of still 1% is obtained. The current density shown in Fig. 4 is considerably smoother and is more readily achieved with external coils. A modular coil configuration with 15 finite size coils per period which is suitable for producing a field of 3 T is plotted in Fig. 5.

In the second case the magnetic field of the ATF configuration is considered [11]. The original ATF field is generated by a pair of torsatron coils and four toroidal coils shaping mainly the vertical field. Figure 6 shows the Poincaré plot of the ATF field produced by a surface current on a torus with circular cross-section and an aspect ratio of $A = 3.3$. Fourier harmonics of $\hat{\Phi}_{mn}$ up to $m \leqslant 8$, $|n| \leqslant 8$ are taken into account. The radial positions $r = 0.772$ and $r = 1.228$ of the ATF torsatron coils are inside the torus domain D. The solution in Fig. 6 shows an accurate approximation of the field in the plasma region by external currents at a larger distance from the plasma.

4. CONCLUSIONS

A method of approximate solution of Neumann boundary value problems for stellarator fields produced by external currents is described. The NESCOIL computer code for implementing this method has been developed and applied successfully. The cases presented show that the method appears to be an effective tool to study stellarator coil configurations and to investigate properties of stellarator vacuum fields.

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