Journal of Physics C: Solid State Physics

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To cite this article: R J Baxter 1973 *J. Phys. C: Solid State Phys.* **6** L445

J. Phys. C: Solid State Phys., Vol. 6, 1973. Printed in Great Britain. © 1973

LETTER TO THE EDITOR

Potts model at the critical temperature

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Received 9 October 1973

Abstract. It is shown that the two-dimensional q-component Potts model is equivalent to a staggered ice-type model. It is deduced that the model has a first-order phase transition for $q > 4$, and a higher-order transition for $q \leqslant 4$. The free energy and latent heat at the transition are calculated.

The Potts (1952) model is a generalization of the Ising model such that each spin σ can have q values, and nearest-neighbour spins have interaction energy $-\epsilon$ if they are alike, zero if they are different. Thus the partition function is

$$Z = \sum \exp\left(\beta\epsilon \sum_{\mathrm{nn}} \delta_{\sigma,\sigma'}\right) \tag{1}$$

where the summation inside the exponential is over all nearest-neighbour pairs on the lattice, and the other is over all spin configurations. Here we consider only the two-dimensional model on a square lattice of N sites (N large).

This model has not been solved exactly, and the nature of the transition is not completely known (Golner 1973 and references therein). Here we show that the model is equivalent to a staggered ice-type model. Not all the relevant properties of this latter model have been established with full mathematical rigour, but they are sufficiently well understood to imply very strongly that the Potts model has a first-order transition for $q > 4$, and a higher-order transition for $q \leqslant 4$. We obtain exact expressions for the free energy at the critical temperature T_c, and the latent heat. Unfortunately we are not able to obtain the critical exponent α for $q = 3$ or 4.

If $v = \exp(\beta\epsilon) - 1$, then (1) can be rewritten

$$Z = \sum \prod_{\mathrm{nn}} (1 + v\delta_{\sigma,\sigma'}). \tag{2}$$

It is known (Kasteleyn and Fortuin 1969) that the Potts model is related to the connectivity and percolation problems in graph theory. One way to see this is to expand the product in (2), and draw a line on the corresponding edge of the lattice if one takes the $v\delta_{\sigma\sigma'}$ term, no line if one takes unity. The summations over spins can then be performed for each term in the expansion, giving

$$Z = \sum q^C v^l \tag{3}$$

where the summation is over all graphs G (ie ways of drawing lines on the edges of the lattice), C is the number of connected pieces (including isolated sites) in G, and l is the number of lines.

In the language of graph theory, (3) is a Whitney (1932) polynomial. Temperley and Lieb (1971) have shown that for a simply connected lattice such polynomials can be expressed as staggered ice-type models. In these one draws arrows on the edges of the square lattice so that at each vertex there are two arrows in and two out. There are then six possible configurations of arrows at each vertex, as shown in figure 1. In general, one divides the lattice into A and B sublattices, such that an A (B) vertex has only B (A) vertices as neighbours. One then associates weights $\omega_1, \ldots, \omega_6$ with the six arrow configurations on A sites, and weights $\omega_1', \ldots, \omega_6'$ with those on B sites.

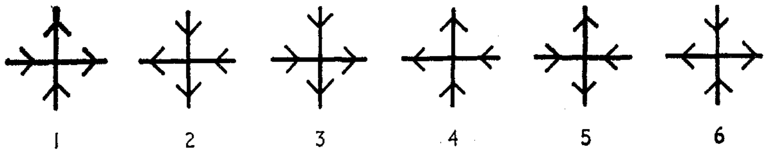

Figure 1

Using Temperley and Lieb's results, we find that

$$Z = v^N Z^* \tag{4}$$

where Z^* is the partition function of a staggered ice-type problem on a lattice of $2N$ sites, with weights

$$\begin{aligned} \omega_1, \ldots, \omega_6 &= 1, 1, x, x, 1 + x \exp\theta, 1 + x \exp(-\theta) \\ \omega_1', \ldots, \omega_6' &= 1, 1, x^{-1}, x^{-1}, 1 + x^{-1} \exp\theta, 1 + x^{-1} \exp(-\theta) \end{aligned} \tag{5}$$

where

$$2 \cosh\theta = q^{1/2}, \qquad x = q^{-1/2} v. \tag{6}$$

Clearly Z^* is invariant with respect to inverting x, ie to replacing v by q/v. This duality relation has been known for some time (eg Mittag and Stephen 1971). It implies that if the critical temperature T_c is unique, then it must occur when

$$x = 1, \qquad v = q^{1/2} \tag{7}$$

(we take $\epsilon > 0$).

Stephen and Mittag (1972) have shown that at this temperature one can construct classes of commuting transfer matrices. From the example of the eight-vertex model (Baxter 1972), this suggests that perhaps the Potts model is exactly soluble at T_c. Indeed it is, for when $x = 1$ we see from (5) that the weights in the A and B sublattices are the same. Using the fact that vertex configurations 5 and 6 occur in pairs, it follows that Z^* is the partition function of the F model, which has been obtained by Lieb (1967). Thus at T_c the free energy f per site of the Potts model is given by

$$-\beta f = \lim_{N\to\infty} N^{-1} \ln Z$$

$$= \tfrac{1}{2} \ln q + \theta + 2 \sum_{n=1}^{\infty} n^{-1} \exp(-n\theta) \tanh n\theta \tag{8a}$$

$$= \ln 2 + 4 \ln [\Gamma(\tfrac{1}{4})/2\Gamma(\tfrac{3}{4})] \tag{8b}$$

$$= \tfrac{1}{2} \ln q + \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x} \tanh \mu x \, \frac{\sinh(\pi-\mu)x}{\sinh \pi x} \tag{8c}$$

where θ is defined by equation (6): the form (*a*) applies for $q > 4$, $\theta > 0$; (*b*) for $q = 4$ $\theta = 0$; and (*c*) for $q < 4$, $\theta = \mathrm{i}\mu$, $0 < \mu < \frac{1}{2}\pi$.

To obtain the behaviour near T_c we need to consider the model (5) when x is close to, but not equal to, one. To do this, multiply the A, B, 5, 6 weights by $x^{-1/2}$, $x^{1/2}$, $\exp(-\frac{1}{2}\theta)$, $\exp(\frac{1}{2}\theta)$, respectively. This leaves Z^* unchanged, and the weights become

$$\begin{aligned}\omega_1, \ldots, \omega_6 &= \exp(-u), \exp(-u), \exp u, \exp u, c \exp s, c \exp(-s) \\ \omega_1', \ldots, \omega_6' &= \exp u, \exp u, \exp(-u), \exp(-u), c \exp(-s), c \exp s \end{aligned} \tag{9}$$

where

$$c^2 = 2(\cosh u + \cosh \theta) \tag{10}$$

$$\exp(2u) = x, \qquad \exp(2s) = (1 + x \exp \theta)/(x + \exp \theta). \tag{11}$$

Thus Z^* is a special case of a more general model, where θ, u, s are arbitrary and the weights are defined by (9) and (10). Let the free energy per vertex of this model be $F(\theta, u, s)$. We are interested in the behaviour for u, s small, since this corresponds to the transition region of the Potts model.

Some properties are known. When $u = 0$ the general model becomes the staggered F model. This is rigorously known (Brascamp *et al* 1973) to have a spontaneous polarization P for sufficiently large θ—ie the derivative wrt s of $\beta F(\theta,0,s)$ has a jump discontinuity of magnitude $2P$ at $s = 0$. This polarization has been evaluated (Baxter 1973) and found to be nonzero for all real θ. It vanishes at $\theta = 0$.

When θ is pure imaginary, the derivatives wrt s is presumably continuous, but non-analytic, at $s = 0$. Certainly this is so for $\theta = \mathrm{i}\pi/2$ (Baxter 1970).

From θ-large expansions, $F(\theta,u,s)$ appears to be an analytic function of u at $u = s = 0$ for sufficiently large θ. Since this is the ordered limit of the system, we conjecture that this analyticity is true for all θ, real or pure imaginary.

From (9), $F(\theta,u,s)$ is an even function of both u and s. Putting these observations together, near $u = s = 0$ we expect that

$$\beta[F(\theta,u,s) - F(\theta,0,0)] = -P(\theta)|s| + \mathrm{O}(u^2), \quad \theta \text{ real} \tag{12a}$$

$$= \mathrm{o}(|s|,u), \quad \theta \text{ imaginary} \tag{12b}$$

where

$$P(\theta) = \prod_{n=1}^{\infty} (\tanh n\theta)^2 \tag{13}$$

is the spontaneous staggered polarization of the F model.

Now consider the Potts model, in which u and s are defined by (11). Considered as functions of the temperature T, they have zeros (in general simple) at T_c. From (12*a*) the derivative of the free energy therefore has a jump discontinuity at T_c if θ is real and nonzero. Thus if $q > 4$ the transition is first-order, with latent heat per site

$$\begin{aligned} L &= 4P(\theta)(\mathrm{d}s/\mathrm{d}\beta)_{T=T_c} \\ &= 2\epsilon(1 + q^{-1/2}) P(\theta) \tanh \tfrac{1}{2}\theta. \end{aligned} \tag{14}$$

This vanishes for $q = 4$. For $q \leqslant 4$ it follows from (12*b*) that the transiti on should be of order higher than one.

The above reasoning can be generalized to a Potts model with different horizontal and vertical interactions, leading to the same conclusions regarding the order of the transition.

The author thanks Professors H N V Temperley and F Y Wu for helpful correspondence and discussions regarding the relation between the Whitney polynomial and the staggered ice-type model.

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