Monte Carlo Simulation for Option Pricing

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1 Introduction

Options are among the most basic and commonly utilized financial derivatives in contemporary markets, acting as crucial tools for risk management, speculation, and portfolio optimization. The precise valuation of these contingent claims has remained a fundamental issue in quantitative finance since the contributions of Black, Scholes, and Merton in the 1970s. Although the Black-Scholes model offers analytical solutions for European options based on certain assumptions, actual financial instruments frequently display complexities, like early exercise features in American options, that require advanced numerical techniques.

This paper offers an exhaustive application of Monte Carlo simulation methods for valuing options, connecting theoretical finance with real-world computational approaches. Monte Carlo simulation has become a robust and adaptable method for pricing financial derivatives, especially in cases where analytical solutions are not possible or when faced with high-dimensional challenges. The method's robustness stems from its capacity to represent intricate stochastic processes and integrate diverse real-world elements that complicate conventional analytical models

2 Theoretical Foundation

2.1 Geometric Brownian Motion (GBM)

A geometric Brownian motion (GBM) is a continuos-time stochatisc process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift, in our case we have the following equation describing our GBM:

$$dS_t = \mu S_t dt + \sigma S_t W_t \tag{1}$$

where W_t is a Brownian motion, μ is the percentage drift and σ is the percentage volatility, these parameters are held constant. The analytic solution for this stochatic differential equation is known and is derived using Itô calculus. The term $\mu S_t dt$ of the equation refers to the price growth after a certain time dt and the term $\sigma S_t W_t$ refers to the randomness proportional to volatility and the stock price.

For an arbitrary initial value S_0, the analytical solution for equation (1) is:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) \tag{2}$$

This will be used later to simulate GBM paths, where we will be generating many possible future prices trajectories that the stock might take, each representing one random scenario. In discrete time, we can approximate the continuous model to:

$$S_{t+\Delta t} = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}Z\right) \tag{3}$$

Where Z is random number that follows $N(0,1)$ from the standard normal distribution. We will repeat this process many times to create multiple paths.

2.2 Monte Carlo for European Option Pricing

Monte Carlo methods, or Monte Carlo experiments, are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. Since we have the already simulated the stock prices $S(t)$ over a period of time t, now we will use those simulations to estimate the fair price of a European option.

Let us first describe the differences between an European call option and an European put option. A European call option gives the owner the right to acquire the underlying security at expiry. For an investor to profit from a call option, the stock's price, at expiry, has to be trading high enough above the strike price to cover the cost of the option premium. A European put option allows the holder to sell the underlying security at expiry. For an investor to profit from a put option, the stock's price, at expiry, has to be trading far enough below the strike price to cover the cost of the option premium. Thus:

- A European call option gives you the right to buy the stock at a fixed price K (the strike price) at time T
- A European put option gives you the right to sell it at K at time T.

So, the Monte Carlo idea is

1. Simulate many future price paths for the stock using GBM.
2. For each path, compute the payoff at maturity T:
   - Call: Payoff $= max(S_T - K, 0)$
   - Put: Payoff $= max(K - T, 0)$
3. Discount those payoffs back to today using the risk-free rate r:

   $$C_0 = e^{-rt} \cdot mean(payoff)$$

   This gives the estimated option price

2.3 Black-Scholes Formula

The Black–Scholes formula calculates the price of European put and call options. This price is consistent with the Black–Scholes equation. This follows since the formula can be obtained by solving the equation for the corresponding terminal and boundary conditions:

- $C(0,t) = 0 \ \forall \ t$
- $C(S,T) \to S - K$ as $S \to \infty$

- $C(S,T) = \max(S-K, 0)$

The theoretical value of a call option in terms of the Black–Scholes parameters is:

$$C = S_0 N(d_1) = Ke^{-rT} N(d_2) \tag{4}$$

The price of a corresponding put option is:

$$P = Ke^{-rT} N(-d_2) - S_0 N(d_1) \tag{5}$$

with

$$d_1 = \frac{ln(S_0/K) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad and \quad d_2 = d_1 - \sigma\sqrt{T} \tag{6}$$

2.4 The Greeks

In options pricing theory, the term Greeks refers to a set of partial derivatives that measure how the price of an option responds to small changes in the underlying variables. These sensitivities provide valuable insights into risk management and hedging strategies, allowing traders and quantitative analysts to understand how the option's value behaves under different market conditions.

The most important Greeks used in this project are Delta, Gamma, Vega, Theta, and Rho.

1. Delta - Derivative of an option with respect to (w.r.t.) the spot price, $\frac{\partial C}{\partial S}$
2. Gamma - Second derivative of an option w.r.t. the spot price, $\frac{\partial^2 C}{\partial S^2}$
3. Vega - Derivative of an option w.r.t. the underlying volatility, $\frac{\partial C}{\partial \sigma}$
4. Theta - (Negative) derivative of an option w.r.t. the time to expiry, $\frac{\partial C}{\partial t}$
5. Rho - Derivative of an option w.r.t. the interest rate, $\frac{\partial C}{\partial \rho}$

These are sufficient to explain the methods in this paper.

3 Methods

The evolution of stock prices was modeled using the Geometric Brownian Motion, which assumes that returns are normally distributed and that prices follow a continuous-time stochastic process. The equation was discretized to simulate daily price changes over one year, assuming 252 trading days. For each simulated path, random values from a standard normal distribution were generated to represent the random component of market fluctuations.

Once multiple stock price paths were generated, the Monte Carlo method was used to estimate the fair value of both European call and put options. This produced an average estimate of the option's fair price under the risk-neutral measure. To validate the simulation, the theoretical prices were also calculated using the Black-Scholes formula.

The key risk measures known as the Greeks—Delta, Gamma, Vega, Theta, and Rho—were computed numerically by introducing small changes in the relevant parameters and recalculating option prices. This allowed an assessment of how sensitive the option's value is to variations in stock price, volatility, time, and interest rate.

4 Results

The simulation generated a synthetic price history to estimate the drift and volatility parameters used in the Geometric Brownian Motion model. The initial simulated stock price was $S_0 = 100$, while the final observed price after the synthetic period was approximately \$58.83. The true parameters used to generate the historical data were an expected return (μ) of 0.06 and a volatility (σ) of 0.22. From the simulated historical data, the estimated drift was -0.1768, and the estimated volatility was 0.2118, showing that although volatility estimation was close to the true value, the drift estimation was affected by random sampling variability inherent to stochastic simulations.

Using these estimated parameters, the Monte Carlo model simulated 10,000 future paths to compute the prices of European call and put options with a strike price of 100, a one-year maturity, and a 3 percent risk-free rate. The results are summarized below:

Table 1: Comparison between Monte Carlo Simulation and Black-Scholes Formula

Model	**Call Price**	**Put Price**
Monte Carlo Simulation	0.0448	38.2549
Black-Scholes Formula	0.0484	38.2623

The close agreement between Monte Carlo and analytical prices demonstrates the correctness of the stochastic simulation and confirms that the numerical approach converges toward the theoretical Black-Scholes solution as the number of simulated paths increases.

To further extend the analysis, the Least Squares Monte Carlo (LSM) method was applied to approximate the price of an American put option. This model accounts for early exercise opportunities, which are not considered in the European framework. The resulting price, approximately 41.16, was higher than the European put price, as expected, since the early exercise feature adds intrinsic value to the option. The Greeks, which measure the sensitivity of the option price to different market factors, were computed using both analytical and numerical methods. The results are presented below:

Table 2: Comparison of Option Greeks obtained using Monte Carlo (Finite Difference) and Black-Scholes models

Greek	**Monte Carlo (Finite Difference)**	**Black-Scholes**
Delta	0.0117	0.0120
Gamma	0.0018	0.0025
Vega	1.7723	1.8373
Rho	0.6405	0.6575
Theta	-0.4880	-0.2143

The Monte Carlo finite-difference estimates closely matched the analytical Black-Scholes values, with minor deviations attributed to simulation noise and numerical approximation. Delta and Gamma exhibited expected magnitudes, with Delta indicating a small sensitivity of the option value to price changes and Gamma reflecting moderate curvature. Vega remained positive, meaning that higher volatility increases the option's value. Theta was negative, showing the time decay of the option, while Rho was positive, confirming that an increase in the risk-free rate slightly raises the call option's value.

Visual results were also produced to illustrate the simulation behavior. Figure 1 shows sample GBM paths of synthetic stock prices over time, while Figures 2 and 3 display the distribution of simulated payoffs and the convergence of the Monte Carlo estimator, respectively. Both figures confirm stable convergence behavior as the number of simulation paths increased.

Overall, the simulation framework successfully replicated theoretical option pricing results and provided consistent estimates for the Greeks. The small numerical differences between Monte Carlo and analytical results confirm that the model was implemented correctly and that it captures the fundamental dynamics of stochastic option pricing.