NETWORK SCIENCE OF ONLINE INTERACTIONS

Critical Phenomena on Networks

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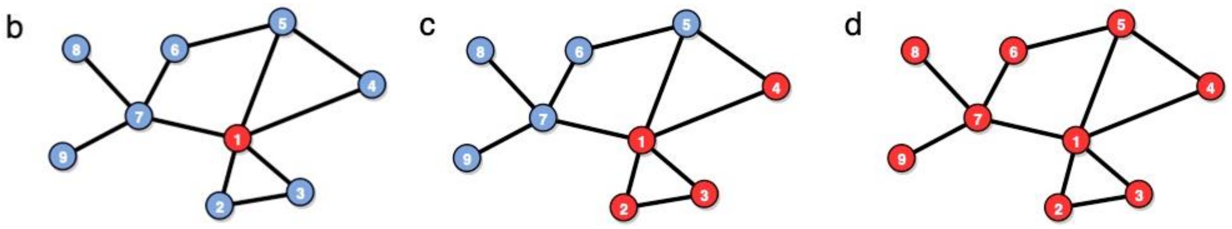
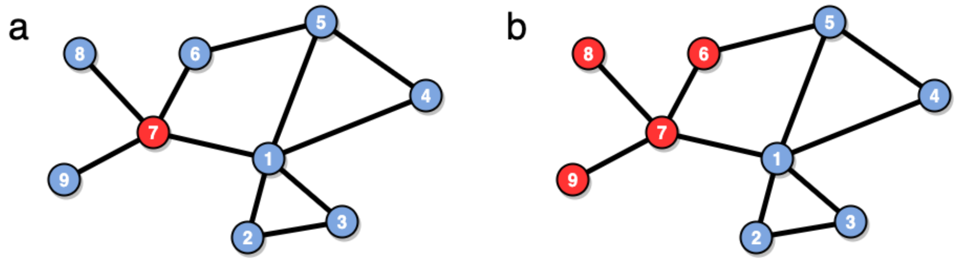
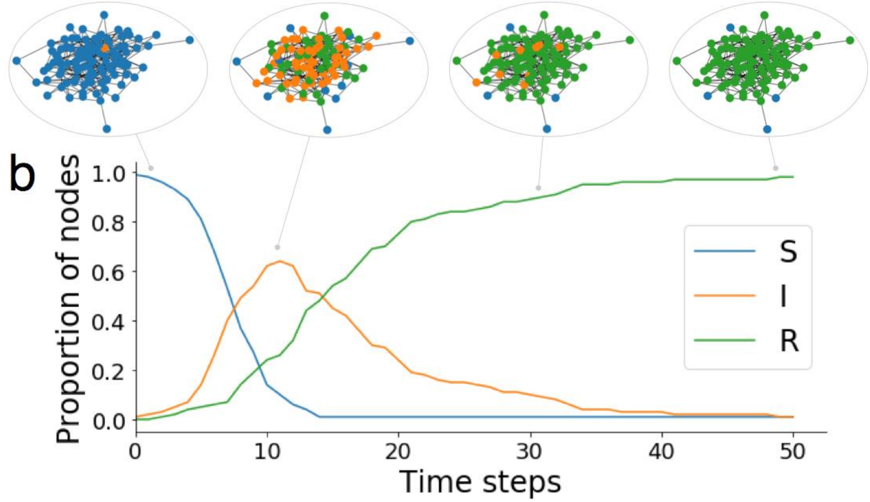
16/Jun/2023

SUMMARY

- Many models with different applicability
- Threshold models
  - Deterministic compounded influence of neighbours
  - Depends on network and starting point
- Cascade models
  - Probabilistic single-node influence
- Epidemic spreading
  - Various components for different types of diseases
- Opinion dynamics
  - Different spreading rules: majority, probabilistic, bounded effect







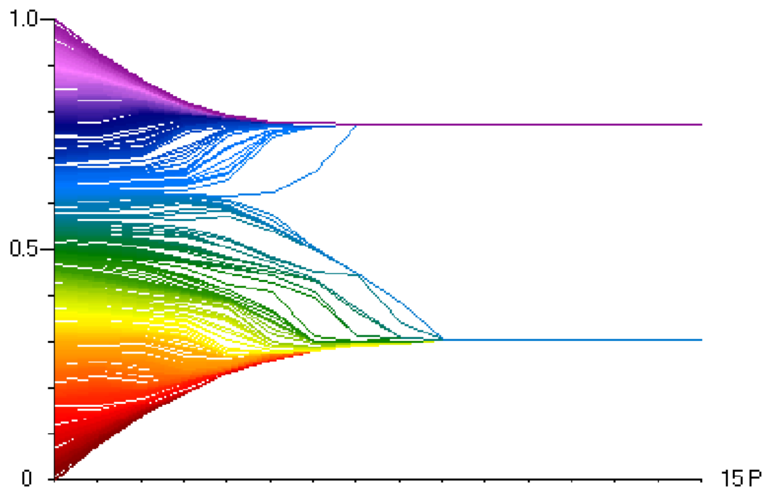
DYNAMICS ON NETWORKS

- Dynamics
  - Models on networks
    - Threshold models
    - Cascade models
    - Epidemic spreading
  - Opinion dynamics

- Critical Phenomena on Networks
  - Adaptive networks
  - Phase Transitions
  - Branching and autoregressive processes
  - Properties of networks at criticality
  - Self-organized criticality

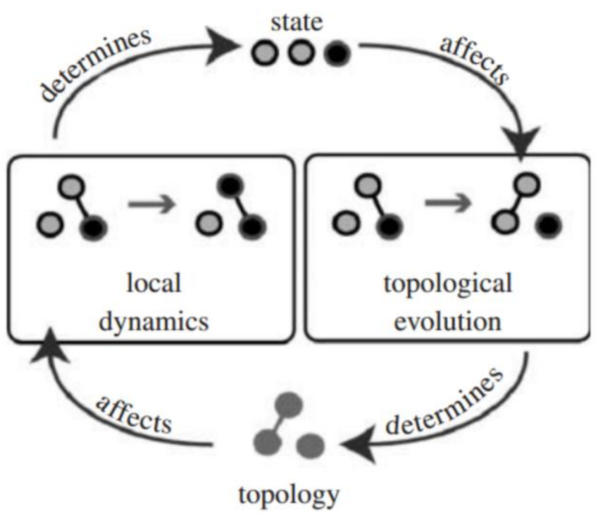
ADAPTIVE NETWORKS

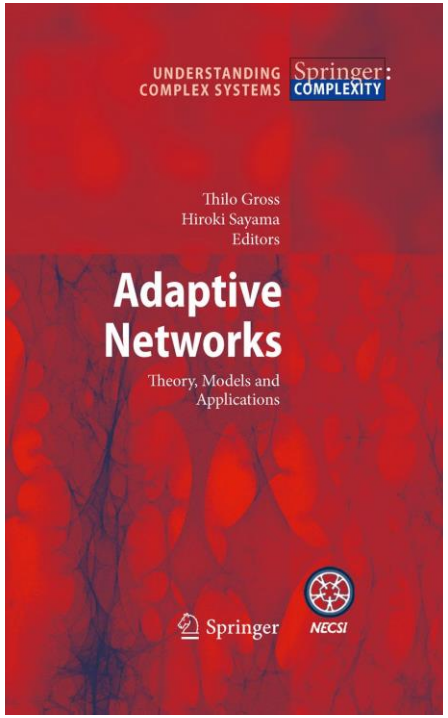
- Networks can evolve (temporal networks)
- Networks can have dynamical models on them
- Previously: bounded confidence model
  - Essentially a temporal network with random $K = 1$ rewiring at each timestep
  - Models instantaneous, independent interactions
  - Fairly robust dynamics (convergence speed doesn't matter)
- What if interactions are not independent (e.g. complex contagion) ?
- What if the network structure changes more slowly than the dynamics on it?



ADAPTIVE NETWORKS

- **Adaptive networks**: co-evolution between topology and dynamics
  - Can lead to much more sophisticated dynamics
  - Including conditions not possible otherwise
  - Many examples
    - Brain networks
    - Technological networks
    - Social networks
      - **Disease spreading**
      - **Polarization dynamics**
    - Reference: book by Gross & Sayama





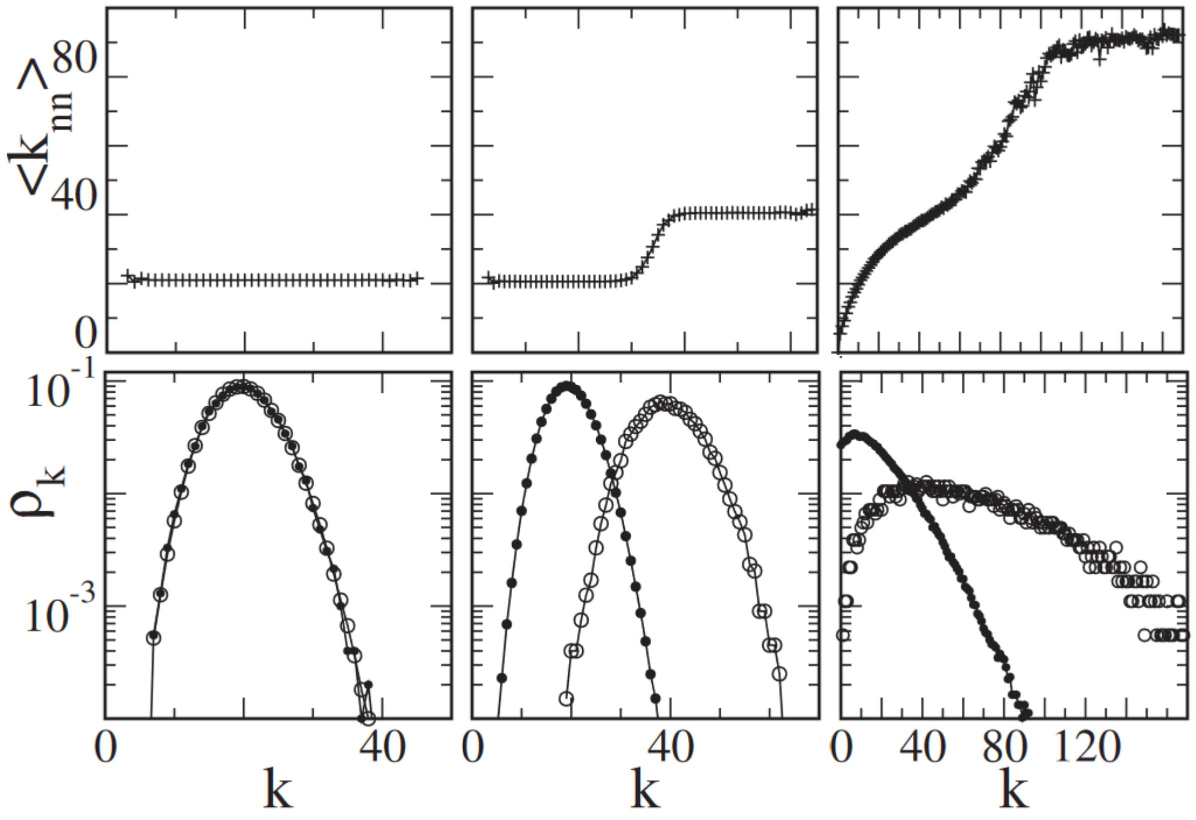
ADAPTIVE NETWORKS

- SIS network
  - Random network with average degree K
  - Nodes spread a disease with probability p
  - Nodes recover with probability r
  - $R_0 = pK/r$
  - Critical transmission rate: $p^\star = r/K$
  - Reminder: no $p^\star$ in scale-free networks
- SIS adaptive network [1]
  - An S node can rewire a link from an I node to another S node with probability w
  - Rewiring stops transmission, but also can create hyperconnected susceptible hubs
  - Critical transmission rate: $p^\star_{AN} = \frac{w}{K(1-\exp(-w/r))}$
- Term expansion: $p^\star_{AN} \approx r/K + w/2K + O(w^2)$
  - Recovers non-adaptive case for $w = 0$
  - If $w \gg r, p^\star_{AN} \approx w/K$
- Increasing rewiring (e.g. isolation) stops the spreading
- Decreasing K also does it
- Can also model content spreading in social networks
  - The topic is the disease, stays in the mind of users for a variable amount of time
  - No threshold because social media is scale-free

[1] Gross et al. *Physical Review Letters* 96: 208701 (2006)

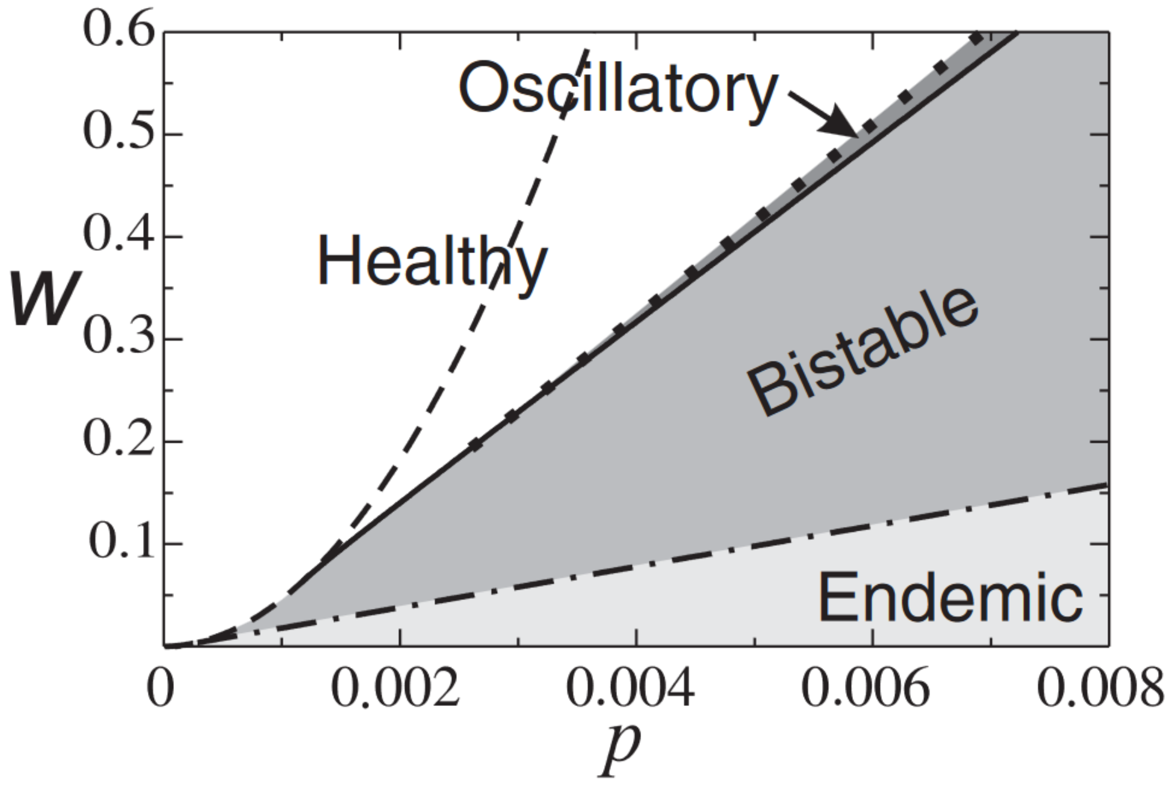
ADAPTIVE NETWORKS

- What happens to the network structure?
  - Degree distribution of susceptible (circles) and infected (dots)
  - Completely random rewiring: random network
  - No disease spreading: two unconnected clusters
  - Adaptive model: **broad tail**
    - Broader with susceptible
    - Creates hubs
    - Structure is dynamic: if a hub gets infected it stops being a hub
  - Degree correlation: $\langle k_{nn} \rangle (k)$
    - Adaptive model is assortative



ADAPTIVE NETWORKS

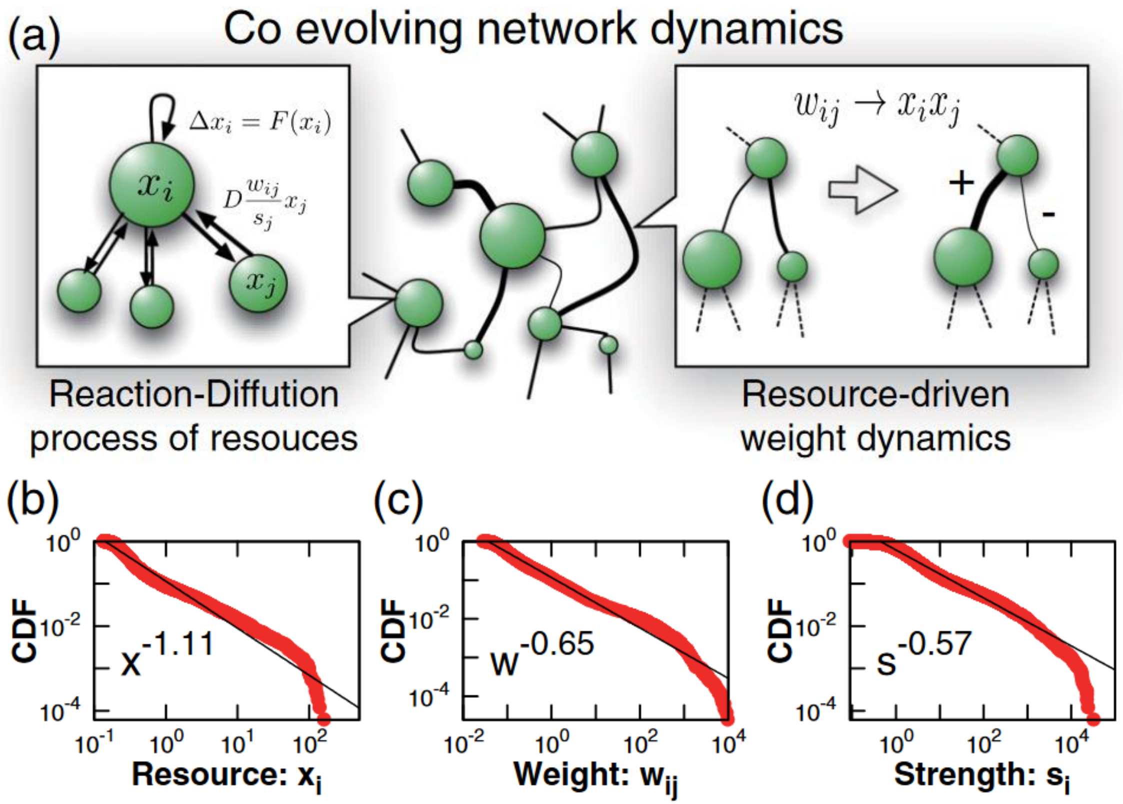
- What happens to the dynamics?
  - Base SIS:
    - Endemic if $\mathrm{p} > r/K$
    - Healthy if $\mathrm{p} < r/K$
  - Adaptive model
    - Complex phase space
    - Can generate **bistable dynamics**
    - Can generate **oscillatory dynamics**



- Dynamics gets much richer by only by adding a simple rewiring rule

ADAPTIVE NETWORKS

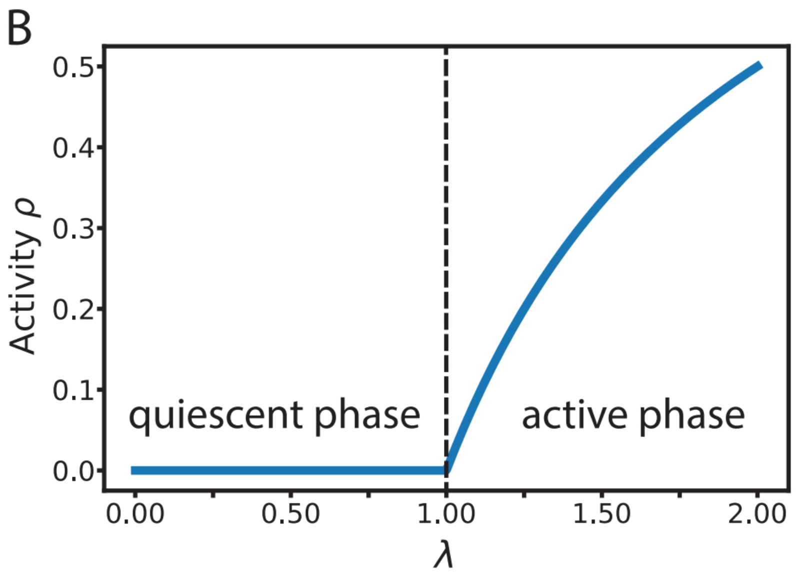
- Models can get even richer with more sophisticated dynamics
- Example: resource diffusion model [1]
  - Nodes produce a resource (e.g. value) x
  - Resource is produced/dissipated to $x \to 1$
  - Resource diffuses between nodes depending on connection strength
  - Adaptation: connection gets stronger between richer nodes
  - Results:
    - Power-laws in resource, weight and strength distributions
    - Creates inequality even if basal resource generation is equal

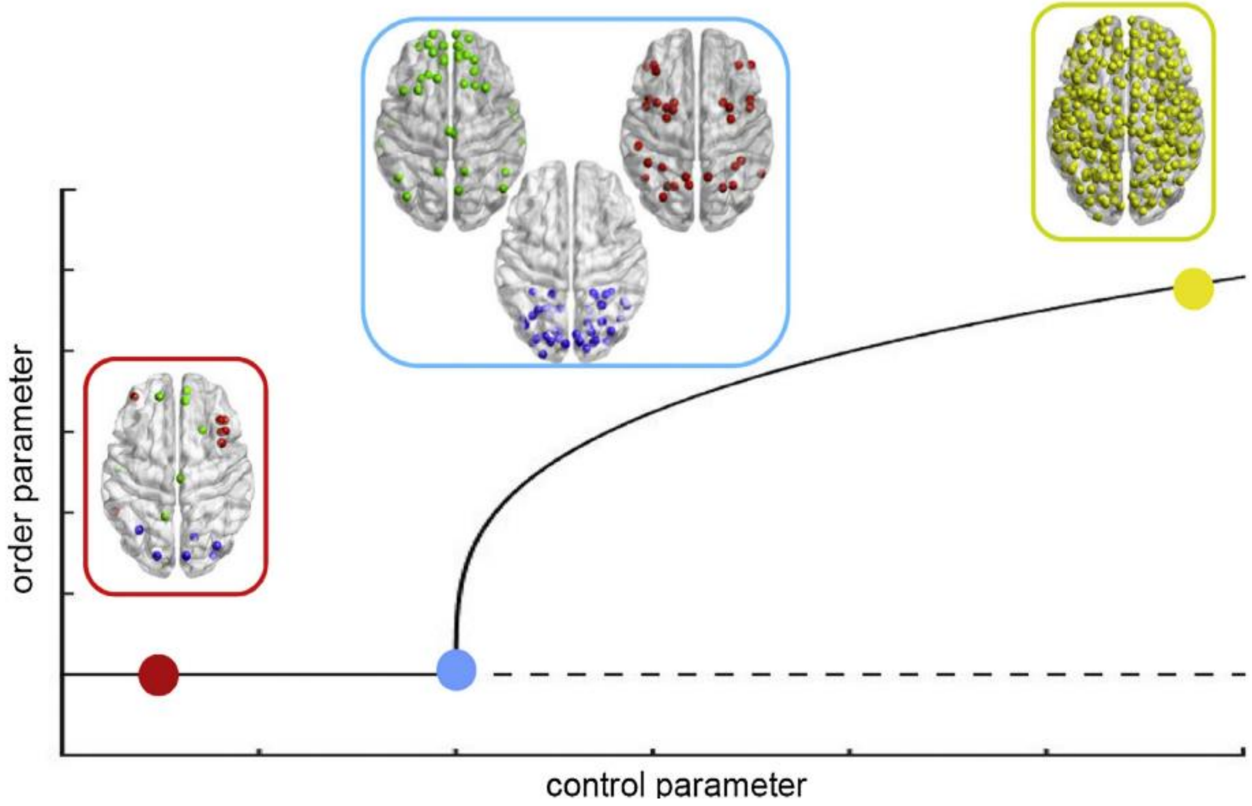


[1] Aoki & Aoyagi. *Physical Review Letters* 109, no. 20 (2012): 208702.

PHASE TRANSITIONS

- Brief definition
  - Phenomena with distinct *phases*, with a **critical point** separating them
  - The critical point has many interesting properties

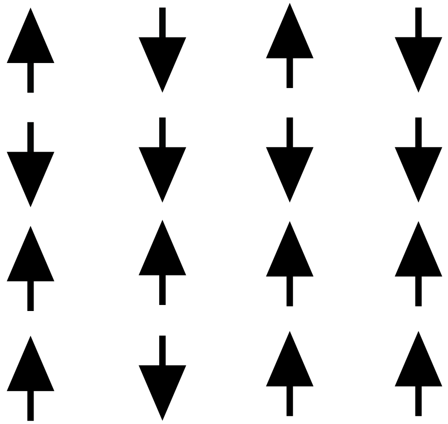


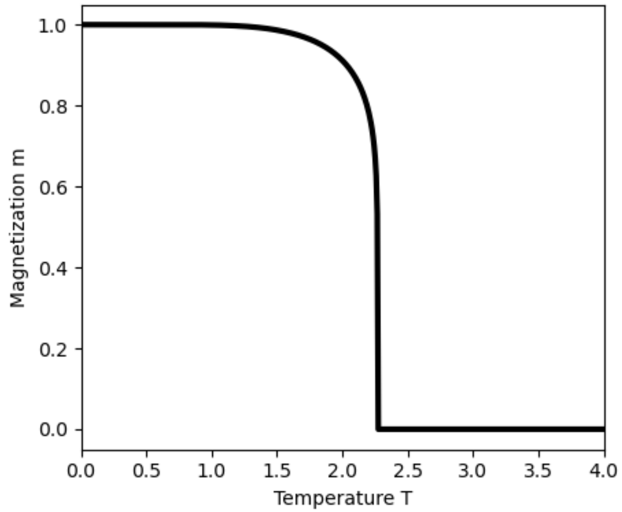


(Cocchi et al, 2017)

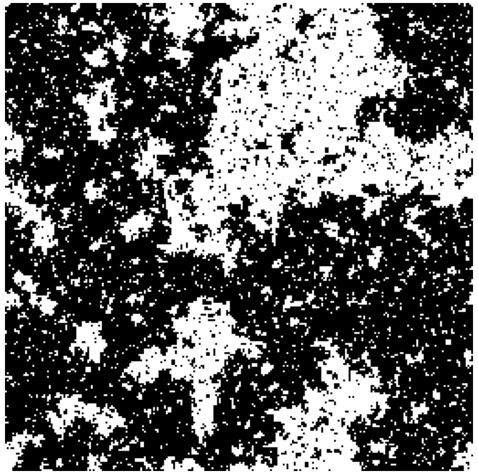
PHASE TRANSITIONS

- Example: 2D Ising model
  - N magnetic spins with orientation $s_i = \pm 1$
  - Hamiltonian $H(\vec{s}) = \sum_{\langle ij \rangle} s_i s_j$, $\langle ij \rangle$ neighbors
  - Probability of observing a state $\vec{s}$ is $P(\vec{s}) \sim e^{-H(\vec{s})/T}$
    - Temperature T
    - Magnetization $m = \frac{1}{N} \sum_i s_i$

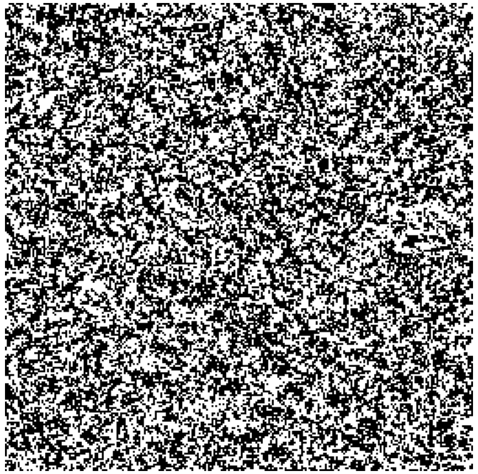




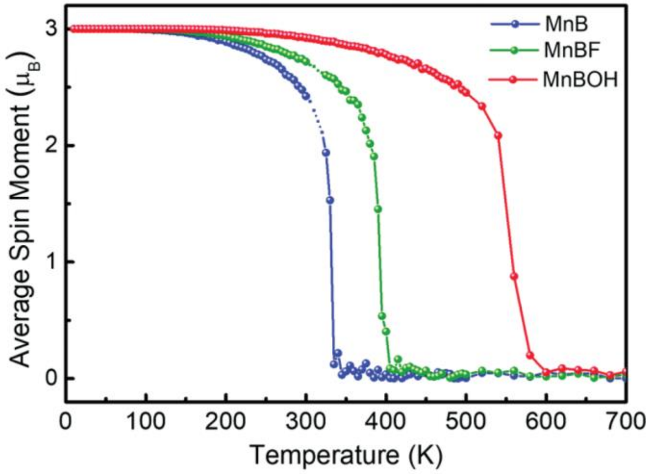
$T = 1$



$T = T_c \approx 2.27$



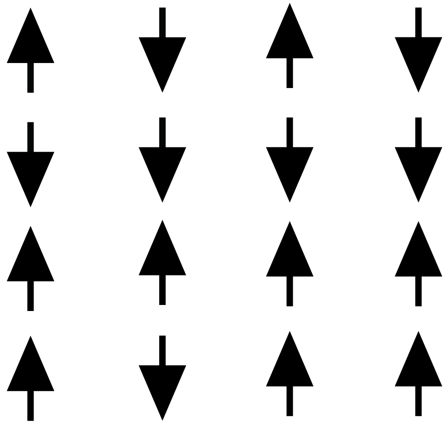
$T = 4$

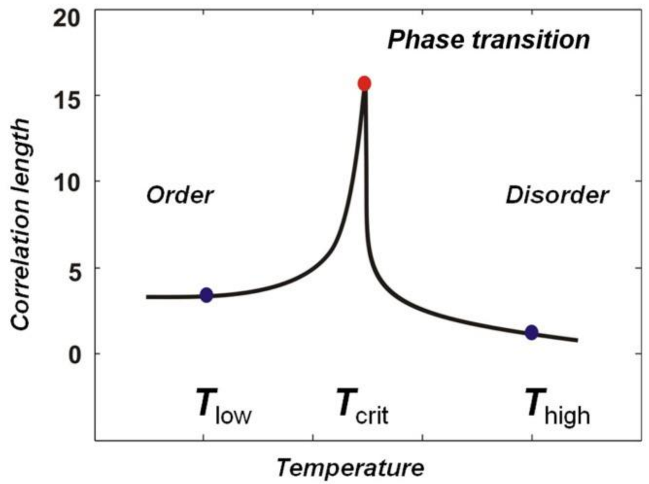


(Jiang et al, 2018)

PHASE TRANSITIONS

- Example: 2D Ising model
  - Correlation function:
    - $\Gamma(i-j) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$
  - Correlation length $\xi(T)$:
    - $\Gamma(x) \sim e^{-x/\xi(T)}$
  - Scaling: $\xi(T) \sim |T - T_c|^{-1}$
- Correlation length diverges at criticality
- Many computationally-useful properties are maximized at criticality
  - Computational power [1-6]
  - Dynamical repertoire [7-9]
  - Dynamic range [10]





(Beggs & Timme, 2012)

[1] Chris G. Langton. Phys. D Nonlinear Phenom. 42.1-3 (1990)
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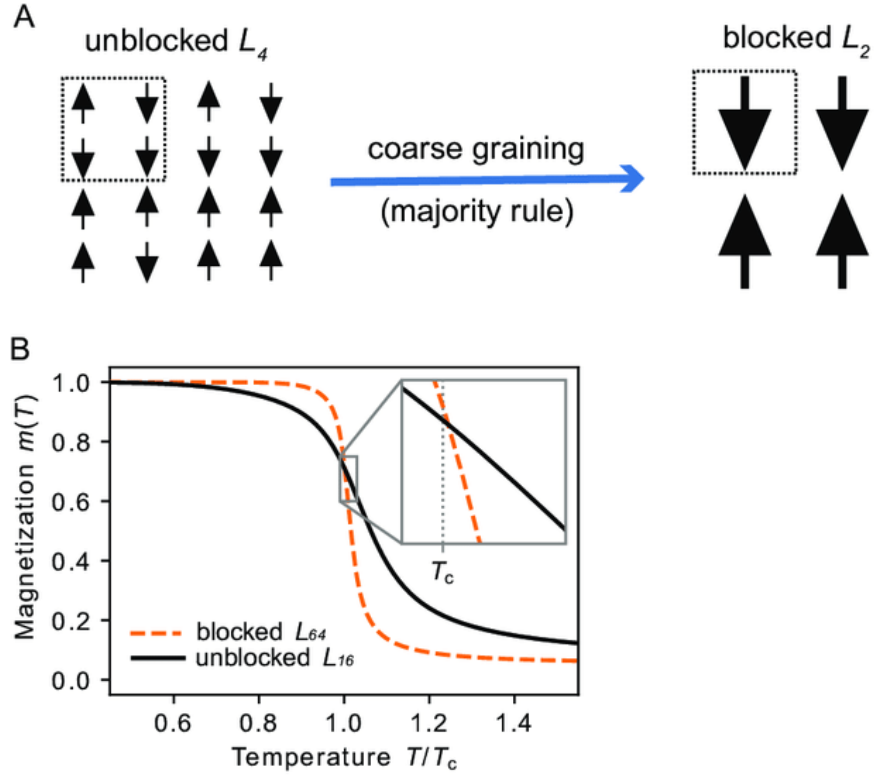
[10] Osame Kinouchi and Mauro Copelli. Nature Physics, 2(5), 348–351 (2006)

COARSE GRAINING/SCALE-FREE DYNAMICS

- Systems at a critical point can also be **coarse-grained**
- The idea: at the critical point, the *dynamics* is scale-free
- Thus we can rescale the system and keep its properties
- But it is only valid at the critical point

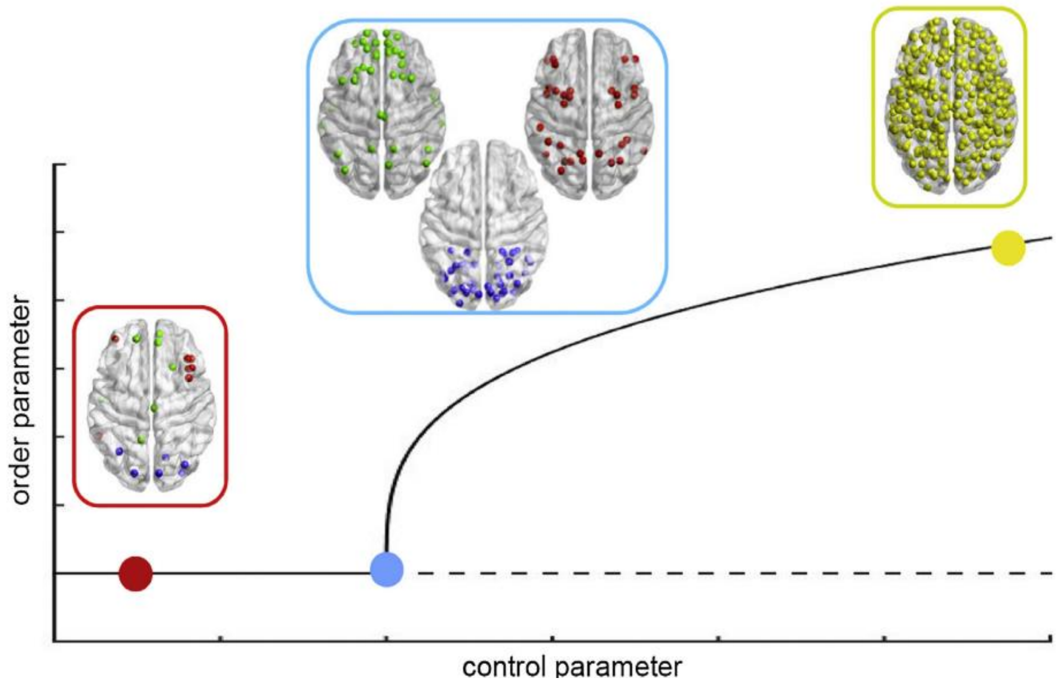


PHASE TRANSITIONS

- For a phase transition you need
  - A control parameter
  - An order parameter
  - A *continuous* transition from zero to non-zero values of the order parameter
- Properties of critical systems
  - Around the critical point x_c, some observables χ scale as power-laws

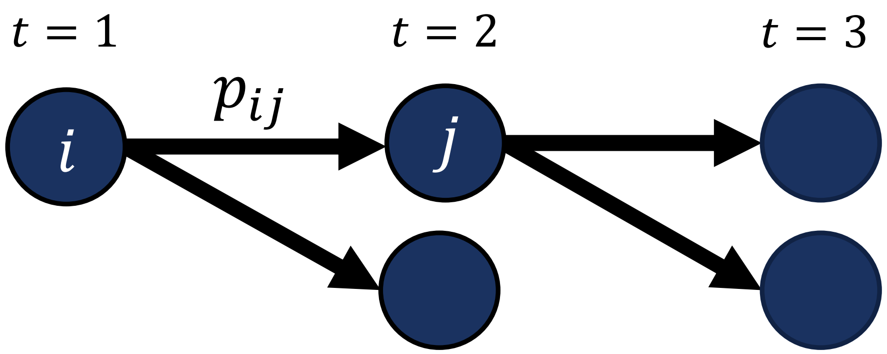
$$\chi \sim |x - x_c|^{-\gamma}$$

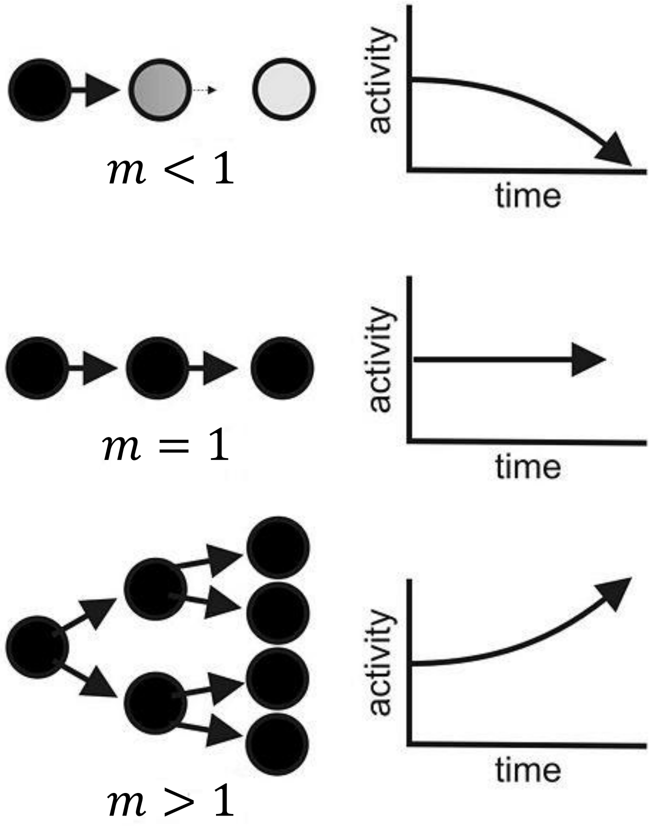
  - Universality: as $x \to x_c$, model details vanish
  - The **critical exponents** γ determines the universality class of the dynamics
  - These exponents are subject to **scaling laws**



BRANCHING AND AUTOREGRESSIVE PROCESSES

- The branching process:
  - Stochastic process where each node at time t produces a random number of offspring at time $t+1$

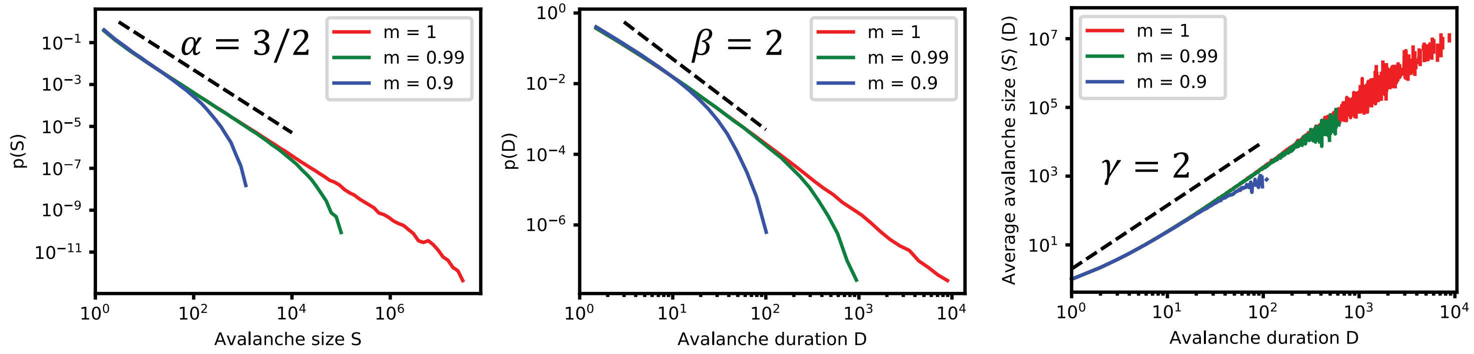


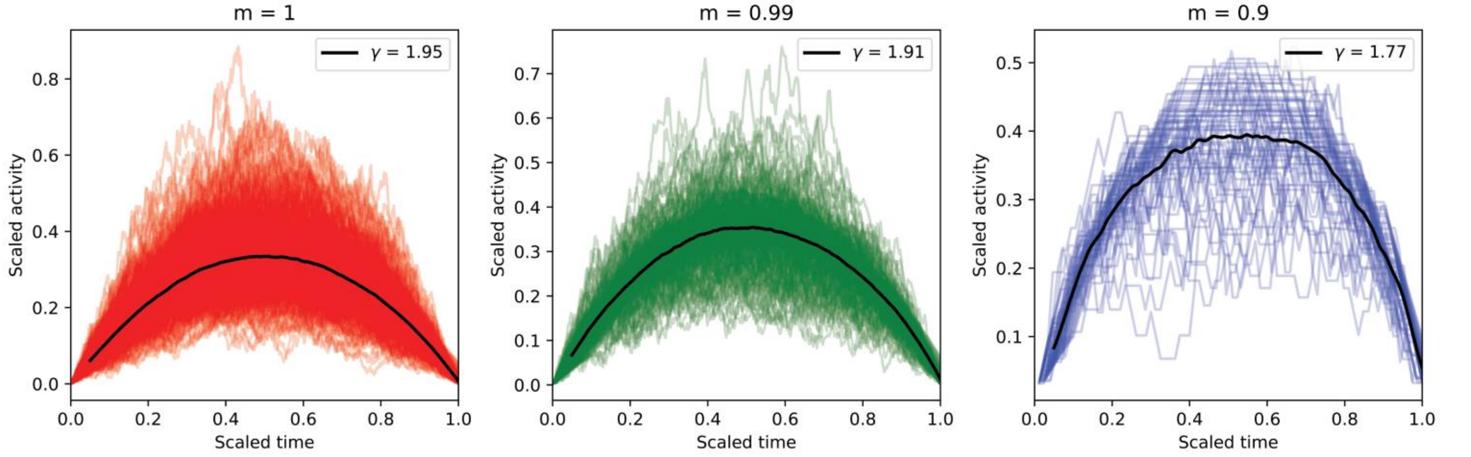


  - **Branching parameter** $m = \langle \sum_j p_{ij} \rangle$
  - Criticality: $m = 1$
  - Spreading timescale: $\tau \sim 1/\ln m$
    - As $m \to 1$, the dynamics spreads *exponentially* longer

BRANCHING AND AUTOREGRESSIVE PROCESSES

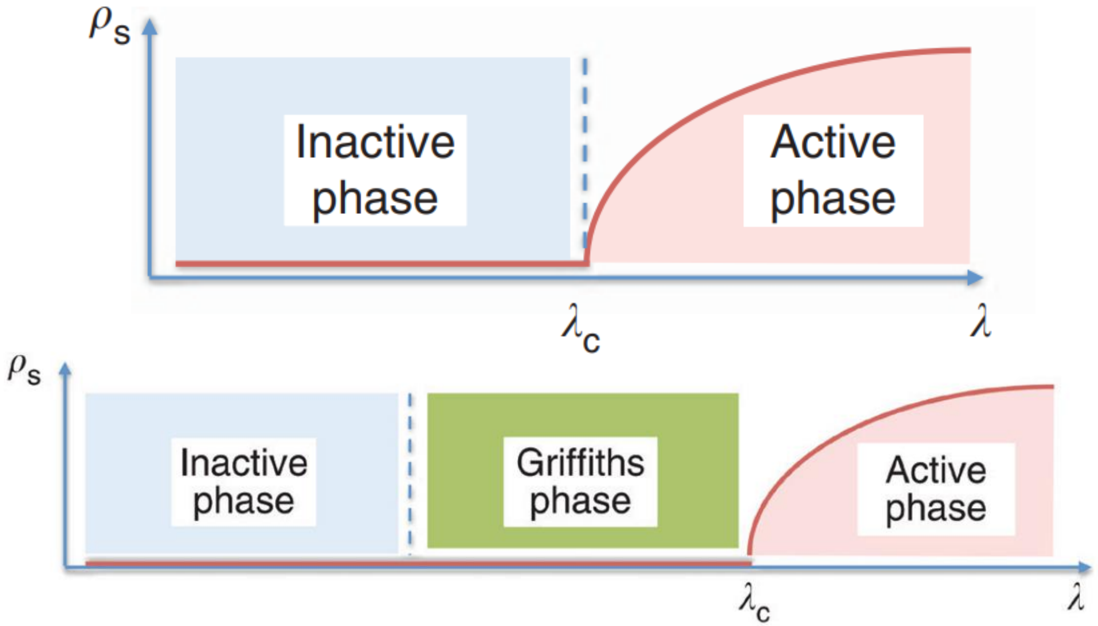
- What can we measure out of a critical branching process?
- Full set of observables
  - $p(S) \sim S^{-\alpha}$ with $\alpha = 3/2$
  - $p(D) \sim D^{-\beta}$ with $\beta = 2$
  - Average size
    - $\langle S \rangle \sim D^{\gamma}$ with $\gamma = 2$
  - Avalanche profile
    - $s(t, D) \sim D^{\gamma-1} F(t/D)$
    - Plotting t/D vs $s(t, D)/D^{\gamma-1}$ results in a data collapse
  - Scaling law
    - $(\beta - 1)/(\alpha - 1) = \gamma$





SELF-ORGANIZED CRITICALITY AND GRIFFITHS PHASES

- Critical dynamics has interesting and often-observed properties
- But it requires parameter fine-tuning (e.g. $m = 1$)
- Two ways of getting the critical dynamics properties
  - Griffiths Phase
  - Self-Organized Criticality
- **Griffiths Phase**
  - The idea: get critical system properties (power-laws, etc) by introducing *heterogeneity* in the structure
  - Example: dynamics on hierarchical modular networks



nature COMMUNICATIONS

ARTICLE

Received 15 Apr 2013 | Accepted 28 Aug 2013 | Published 3 Oct 2013

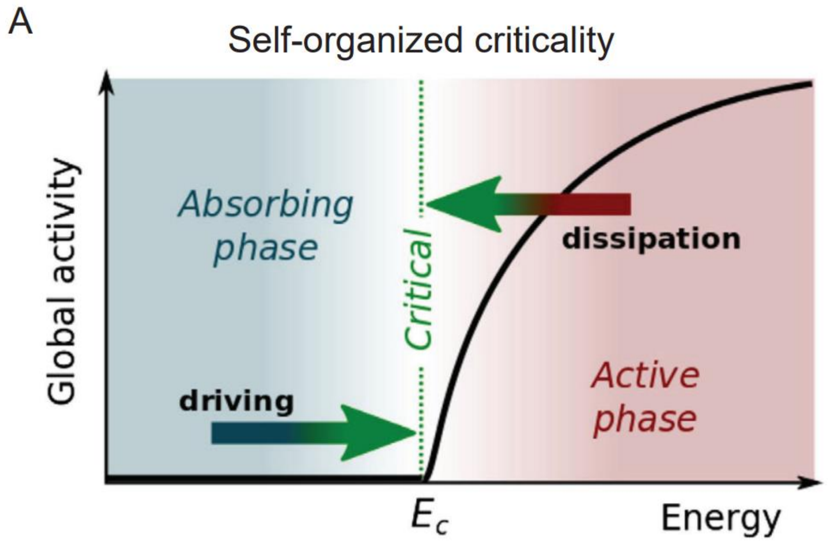
DOI: 10.1038/ncomms3521

Griffiths phases and the stretching of criticality in brain networks

Paolo Moretti[1] & Miguel A. Muñoz[1]

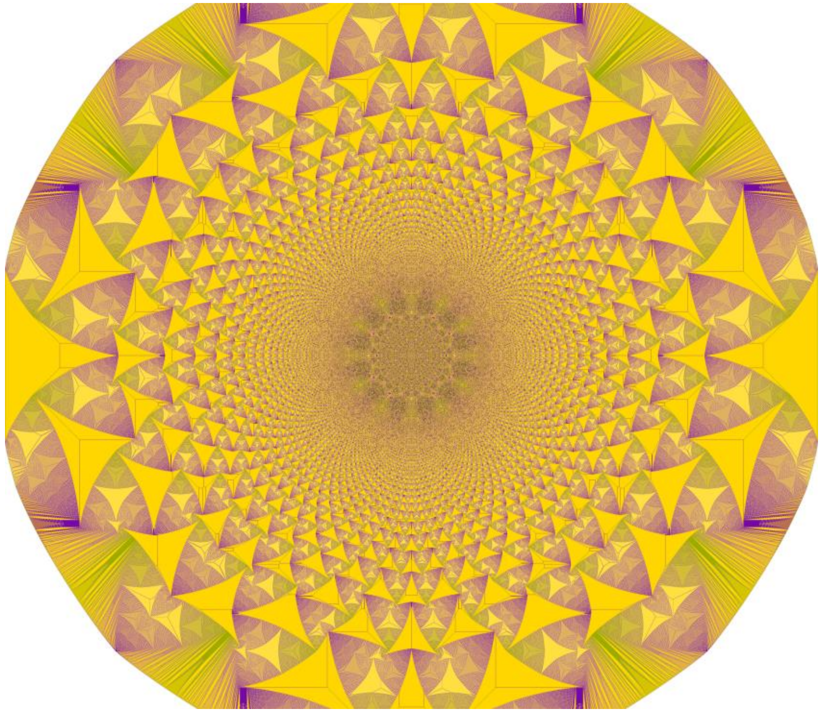
SELF-ORGANIZED CRITICALITY

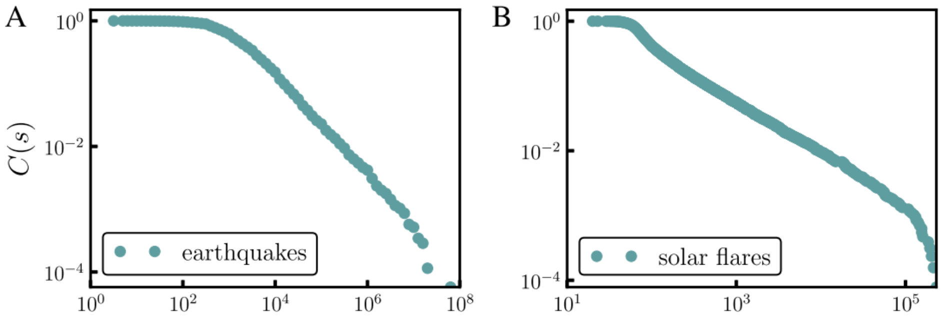
- **Self-Organized Criticality** (SOC)
  - Widely studied in complexity science in the 1990s
  - Some rule (e.g. adaptive network) drives the system towards the phase transition
  - Relies on the concept of drive and dissipation
    - A slow adaptation drives the system towards the critical point
    - A fast dissipation drives it away
    - Most famous example: the **Abelian sandpile** model



SELF-ORGANIZED CRITICALITY

- Abelian sandpile model
  - Nodes in a 2D grid, with a resource (sand grain) Z
  - If $Z > 4$, the node topples and sends one grain to each neighbour
  - One variation: random initial condition, keep adding grains in the middle and dissipating at the borders
  - Results:
    - Fractal structures
    - Power-law avalanche size distribution
    - Power-law event sizes with similar exponents are seen in many phenomena





SUMMARY

- Adaptive networks can have much richer dynamics
  - They can reach states and topologies not possible in the static networks
- Branching processes can easily create many of the power-laws we've seen in the course
- Mechanisms like self-organized criticality can also explain those power-laws without fine-tuning