ESCOLA SUPERIOR DE TECNOLOGIA IPCA	**Math Foundations for Artificial Intelligence** **2023/2024**

Analytic Geometry

1. Let be the vectors $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$. Compute the length x, and the angle and the distance between x and y .

2. Let be the following vectors of $\mathbb{R}^3$:

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \ v = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

Calculate:

2.1. $w = v - \left(\frac{\langle u,v \rangle}{\langle u,u \rangle} \right) u;$

2.2. $\langle u,w \rangle$. What can you conclude?

3. Determine which of the following pairs of vectors are orthogonal.

3.1. $u = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, \ v = \begin{bmatrix} -2 \\ -3 \end{bmatrix};$

3.2. $u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \ v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix};$

3.3. $u = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \ v = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix};$

3.4. $u = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \ v = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}.$

4. Consider the points $A = (1,-1)$, $B = (0,0)$ and $C = (x,6)$. Knowing that triangle ABC is a right triangle, determine x.

5. Consider the vectors $u=\begin{bmatrix}1\\-1\end{bmatrix}$ and $v=\begin{bmatrix}a\\b\end{bmatrix}$, belonging to $\mathbb{R}^2$, and the vector subspace $S=ger\{u\}$. Determine the relationship between a and b so that it is orthogonal to all elements of S.

6. Show that the following sets are orthogonal and identify the sets that are orthogonal:

   6.1. $\left\{\begin{bmatrix}2\\5\end{bmatrix},\begin{bmatrix}15\\-6\end{bmatrix}\right\}$;

   6.2. $\left\{\begin{bmatrix}\frac{\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\end{bmatrix},\begin{bmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{bmatrix}\right\}$;

   6.3. $\left\{\begin{bmatrix}2\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}3\\0\\6\end{bmatrix}\right\}$;

   6.4. $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}\frac{2\sqrt{5}}{5}\\0\\\frac{-\sqrt{5}}{5}\end{bmatrix},\begin{bmatrix}\frac{\sqrt{5}}{5}\\0\\\frac{2\sqrt{5}}{5}\end{bmatrix}\right\}$.

7. Let be the following vectors of $\mathbb{R}^4$ $v_1=\begin{bmatrix}2\\1\\0\\3\end{bmatrix}, v_2=\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}, v_3=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, u=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$ and $S=span[v_1,v_2,v_3]$.

   7.1. Verify if $B=\{v_1,v_2,v_3\}$ is an orthogonal basis of S.

   7.2. Verify if $u\in S^{\perp}$.

8. Consider the vector subspace of $\mathbb{R}^3$, $S=\left\{\begin{bmatrix}x\\y-x\\x\end{bmatrix}: y,x\in\mathbb{R}\right\}$. Check whether the following vectors belong to $S^{\perp}$:

   8.1. $u=\begin{bmatrix}-2\\0\\2\end{bmatrix}$;

   8.2. $u=\begin{bmatrix}1\\1\\0\end{bmatrix}$.

9. Consider the vector subspace $S = span[\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}]$ and the vector $v = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. Determine.

   9.1. The orthogonal projection of v onto S;

   9.2. The distance between v and S.

10. Consider the vector subspace $S = span[\begin{bmatrix} -1 \\ 1 \end{bmatrix}]$. Determine a vector v in $\mathbb{R}^2$, such that:

$\pi_S(v) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$;

$dist(v, S) = 2$.

11. Consider the Euclidean vector space $\mathbb{R}^4$ with the dot product. A subspace $U \subseteq \mathbb{R}^4$ and $x \in \mathbb{R}^4$ are given by

$$U = span[\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}] \text{ and x= } \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

   11.1. Determine the orthogonal projection $\pi_U(x)$ of x onto U.

   11.2. Determine the distance $d(x, U)$.

12. Consider in the Euclidean vector space $\mathbb{R}^4$ with the dot product the subspace

$$U = \left\{ \begin{bmatrix} y+z \\ y \\ z \\ w \end{bmatrix} : y, z, w \in \mathbb{R} \right\} \text{ and the vector } x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } x_0 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

   12.1. Find an orthonormal basis of U.

   12.2. Determine the orthogonal projection $\pi_U(x)$ of x onto U.

   12.3. Consider the plane in $\mathbb{R}^4$ which is defined by $P := U + x_0$. Determine the distance between P and x.