# Matrix reviews

- Rows and columns
- Square matrix
- Null matrix
- Diagonal matrix
- Identity Matrix
- Echelon form
- Row reduce echelon form
- Addition of Matrices
- Scalar multiplication of Matrices
- Multiplication matrices
- Inverse and transpose

## Systems of linear equations

- Elementary transformations
- Gauss-Jordan Elimination method

## Vector space

**Definition 2.7** (Group). Consider a set $\mathcal{G}$ and an operation $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ defined on $\mathcal{G}$. Then $G := (\mathcal{G}, \otimes)$ is called a *group* if the following hold:

1. *Closure* of $\mathcal{G}$ under $\otimes$: $\forall x, y \in \mathcal{G} : x \otimes y \in \mathcal{G}$
2. *Associativity:* $\forall x, y, z \in \mathcal{G} : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3. *Neutral element:* $\exists e \in \mathcal{G}\, \forall x \in \mathcal{G} : x \otimes e = x$ and $e \otimes x = x$
4. *Inverse element:* $\forall x \in \mathcal{G}\, \exists y \in \mathcal{G} : x \otimes y = e$ and $y \otimes x = e$. We often write $x^{-1}$ to denote the inverse element of $x$.

**Definition 2.9** (Vector Space). A real-valued *vector space* $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

vector space

$$+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \tag{2.62}$$
$$\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V} \tag{2.63}$$

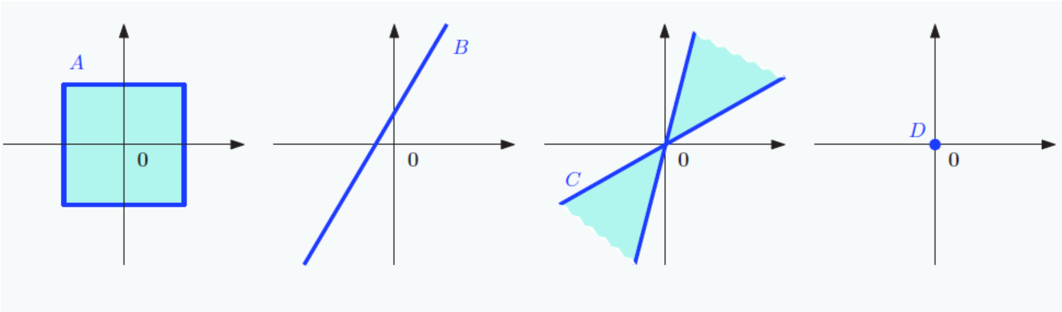
where

1. $(\mathcal{V}, +)$ is an Abelian group
2. Distributivity:
   1. $\forall \lambda \in \mathbb{R}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V} : \lambda \cdot (\boldsymbol{x} + \boldsymbol{y}) = \lambda \cdot \boldsymbol{x} + \lambda \cdot \boldsymbol{y}$
   2. $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V} : (\lambda + \psi) \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x} + \psi \cdot \boldsymbol{x}$
3. Associativity (outer operation): $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V} : \lambda \cdot (\psi \cdot \boldsymbol{x}) = (\lambda\psi) \cdot \boldsymbol{x}$
4. Neutral element with respect to the outer operation: $\forall \boldsymbol{x} \in \mathcal{V} : 1 \cdot \boldsymbol{x} = \boldsymbol{x}$

- Vectors
- Zero vector
- Vector addition
- Scalars
- Multiplication by scalar

**Definition 2.10** (Vector Subspace). Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{U} \subseteq \mathcal{V}, \mathcal{U} \neq \emptyset$. Then $U = (\mathcal{U}, +, \cdot)$ is called *vector subspace* of $V$ (or *linear subspace*) if $U$ is a vector space with the vector space operations $+$ and $\cdot$ restricted to $\mathcal{U} \times \mathcal{U}$ and $\mathbb{R} \times \mathcal{U}$. We write $U \subseteq V$ to denote a subspace $U$ of $V$.

1. $\mathcal{U} \neq \emptyset$, in particular: $\mathbf{0} \in \mathcal{U}$
2. Closure of $U$:
    a. With respect to the outer operation: $\forall \lambda \in \mathbb{R} \, \forall \boldsymbol{x} \in \mathcal{U} : \lambda \boldsymbol{x} \in \mathcal{U}$.
    b. With respect to the inner operation: $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{U} : \boldsymbol{x} + \boldsymbol{y} \in \mathcal{U}$.




## Linear Independence

**Definition 2.11** (Linear Combination)**.** Consider a vector space $V$ and a finite number of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in V$. Then, every $\boldsymbol{v} \in V$ of the form

$$\boldsymbol{v} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_k \boldsymbol{x}_k = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \in V \tag{2.65}$$

with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ is a *linear combination* of the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$.

**Definition 2.12** (Linear (In)dependence)**.** Let us consider a vector space $V$ with $k \in \mathbb{N}$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in V$. If there is a non-trivial linear combination, such that $\mathbf{0} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$ with at least one $\lambda_i \neq 0$, the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are *linearly dependent*. If only the trivial solution exists, i.e., $\lambda_1 = \ldots = \lambda_k = 0$ the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are *linearly independent*.

- $k$ vectors are either linearly dependent or linearly independent. There is no third option.
- If at least one of the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ is $\mathbf{0}$ then they are linearly dependent. The same holds if two vectors are identical.
- The vectors $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k : \boldsymbol{x}_i \neq \mathbf{0}, i = 1, \ldots, k\}$, $k \geqslant 2$, are linearly dependent if and only if (at least) one of them is a linear combination of the others. In particular, if one vector is a multiple of another vector, i.e., $\boldsymbol{x}_i = \lambda \boldsymbol{x}_j$, $\lambda \in \mathbb{R}$ then the set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k : \boldsymbol{x}_i \neq \mathbf{0}, i = 1, \ldots, k\}$ is linearly dependent.

## Basis and Rank

**Definition 2.13** (Generating Set and Span)**.** Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and set of vectors $\mathcal{A} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq \mathcal{V}$. If every vector $\boldsymbol{v} \in \mathcal{V}$ can be expressed as a linear combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$, $\mathcal{A}$ is called a *generating set* of $V$. The set of all linear combinations of vectors in $\mathcal{A}$ is called the *span* of $\mathcal{A}$. If $\mathcal{A}$ spans the vector space $V$, we write $V = \mathrm{span}[\mathcal{A}]$ or $V = \mathrm{span}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k]$.

**Definition 2.14** (Basis)**.** Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and $\mathcal{A} \subseteq \mathcal{V}$. A generating set $\mathcal{A}$ of $V$ is called *minimal* if there exists no smaller set $\tilde{\mathcal{A}} \subsetneq \mathcal{A} \subseteq \mathcal{V}$ that spans $V$. Every linearly independent generating set of $V$ is minimal and is called a *basis* of $V$.

- $\mathcal{B}$ is a basis of $V$.
- $\mathcal{B}$ is a minimal generating set.
- $\mathcal{B}$ is a maximal linearly independent set of vectors in $V$, i.e., adding any other vector to this set will make it linearly dependent.
- Every vector $\boldsymbol{x} \in V$ is a linear combination of vectors from $\mathcal{B}$, and every linear combination is unique, i.e., with

$$\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{b}_i = \sum_{i=1}^{k} \psi_i \boldsymbol{b}_i \tag{2.77}$$

  and $\lambda_i, \psi_i \in \mathbb{R}$, $\boldsymbol{b}_i \in \mathcal{B}$ it follows that $\lambda_i = \psi_i,\ i = 1, \ldots, k$.

- A vector space can be many bases, but all bases possess the same number of elements.
- Dimension of a vector space
- Dimension subspaces

If $U \subseteq V$ is a subspace of $V$, them $\dim(U) \leq dim(V)$.

- $\dim(V) = 0$ if and only if $V = \{0_V\}$.
- $\dim(V) = \dim(U)$ if and only if $V = U$.

*Remark.* A basis of a subspace $U = \mathrm{span}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m] \subseteq \mathbb{R}^n$ can be found by executing the following steps:

1. Write the spanning vectors as columns of a matrix $\boldsymbol{A}$
2. Determine the row-echelon form of $\boldsymbol{A}$.
3. The spanning vectors associated with the pivot columns are a basis of $U$.

## Rank

The number of linearly independent columns of a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ equals the number of linearly independent rows and is called the *rank* of $\boldsymbol{A}$ and is denoted by $\mathrm{rk}(\boldsymbol{A})$.

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- $\mathrm{rk}(\boldsymbol{A}) = \mathrm{rk}(\boldsymbol{A}^\top)$, i.e., the column rank equals the row rank.
- The columns of $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ span a subspace $U \subseteq \mathbb{R}^m$ with $\dim(U) = \mathrm{rk}(\boldsymbol{A})$. Later we will call this subspace the *image* or *range*. A basis of $U$ can be found by applying Gaussian elimination to $\boldsymbol{A}$ to identify the pivot columns.
- The rows of $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ span a subspace $W \subseteq \mathbb{R}^n$ with $\dim(W) = \mathrm{rk}(\boldsymbol{A})$. A basis of $W$ can be found by applying Gaussian elimination to $\boldsymbol{A}^\top$.
- For all $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ it holds that $\boldsymbol{A}$ is regular (invertible) if and only if $\mathrm{rk}(\boldsymbol{A}) = n$.
- For all $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and all $\boldsymbol{b} \in \mathbb{R}^m$ it holds that the linear equation system $\boldsymbol{Ax} = \boldsymbol{b}$ can be solved if and only if $\mathrm{rk}(\boldsymbol{A}) = \mathrm{rk}(\boldsymbol{A}|\boldsymbol{b})$, where $\boldsymbol{A}|\boldsymbol{b}$ denotes the augmented system.
- For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ the subspace of solutions for $\boldsymbol{Ax} = \boldsymbol{0}$ possesses dimension $n - \mathrm{rk}(\boldsymbol{A})$. Later, we will call this subspace the *kernel* or the *null space*.
- A matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ has *full rank* if its rank equals the largest possible rank for a matrix of the same dimensions. This means that the rank of a full-rank matrix is the lesser of the number of rows and columns, i.e., $\mathrm{rk}(\boldsymbol{A}) = \min(m, n)$. A matrix is said to be *rank deficient* if it does not have full rank.

## Coordinates

- $B = \{b_1, b_2, \dots, b_n\}$ is unordered basis
- $B = (b_1, b_2, \dots, b_n)$ is ordered basis

**Definition 2.18** (Coordinates)**.** Consider a vector space $V$ and an ordered basis $B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$ of $V$. For any $\boldsymbol{x} \in V$ we obtain a unique representation (linear combination)

$$\boldsymbol{x} = \alpha_1 \boldsymbol{b}_1 + \dots + \alpha_n \boldsymbol{b}_n \tag{2.90}$$

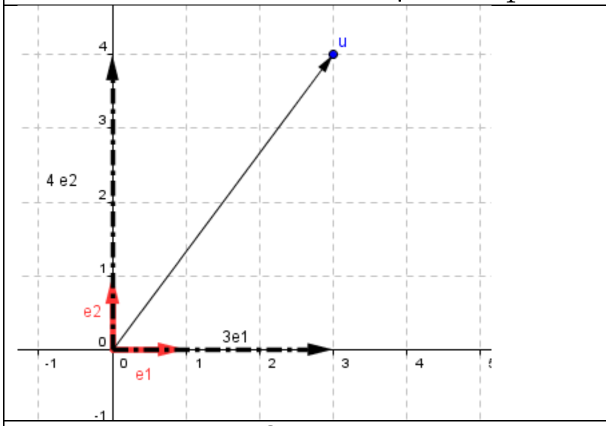
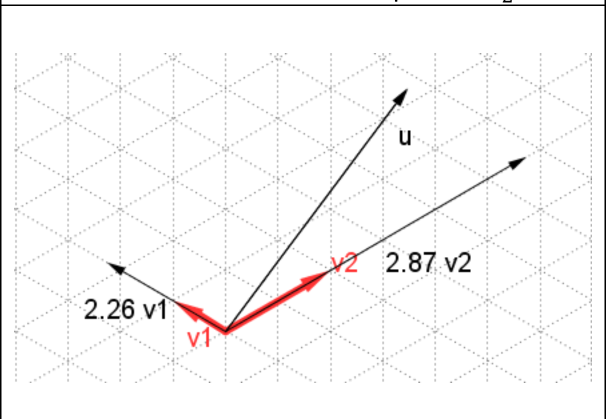
of $\boldsymbol{x}$ with respect to $B$. Then $\alpha_1, \dots, \alpha_n$ are the *coordinates* of $\boldsymbol{x}$ with respect to $B$, and the vector

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n \tag{2.91}$$

is the *coordinate vector/coordinate representation* of $\boldsymbol{x}$ with respect to the ordered basis $B$.

**Example**

In $\mathbb{R}^2$ we will be considered two basis $B_1 = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ and $B_2 = \left\{\begin{bmatrix}-0.87\\0.5\end{bmatrix}, \begin{bmatrix}1.73\\1\end{bmatrix}\right\}$ and the vector $u = \begin{bmatrix}3\\4\end{bmatrix}$.

<table>
<tr><th>Coordinates of $u$ with respect to $B_1$</th><th>Coordinates of $u$ with respect to $B_2$</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$u = \begin{bmatrix}3\\4\end{bmatrix} = (3,4)_{B_1}$</td><td>$u = \begin{bmatrix}3\\4\end{bmatrix} = (2.26, 2.87)_{B_2}$</td></tr>
</table>

- Coordinates of a subspace

## Linear Mappings

$$\Phi: V \rightarrow W$$

Is a linear mapping if preserve the structure of the vector spaces. This mean:

$$\Phi(\boldsymbol{x}+\boldsymbol{y}) = \Phi(\boldsymbol{x}) + \Phi(\boldsymbol{y}) \tag{2.85}$$

$$\Phi(\lambda\boldsymbol{x}) = \lambda\Phi(\boldsymbol{x}) \tag{2.86}$$

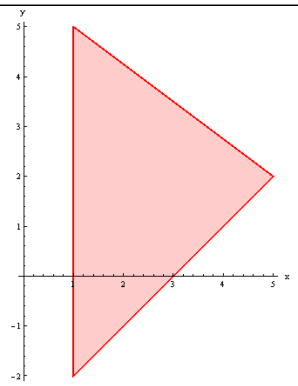
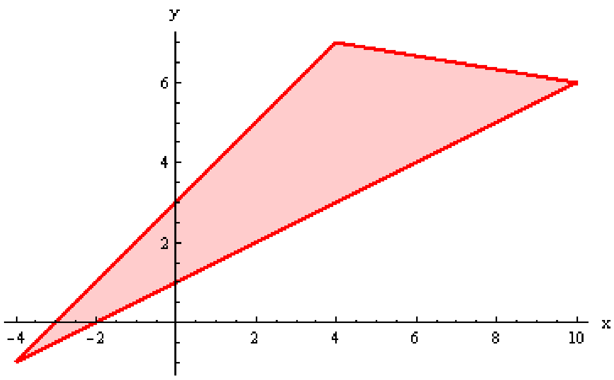
for all $\boldsymbol{x}, \boldsymbol{y} \in V$ and $\lambda \in \mathbb{R}$. We can summarize this in the following definition:

**Definition 2.15** (Linear Mapping)**.** For vector spaces $V, W$, a mapping $\Phi : V \rightarrow W$ is called a *linear mapping* (or *vector space homomorphism/ linear transformation*) if

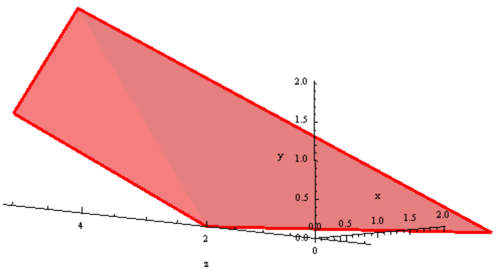
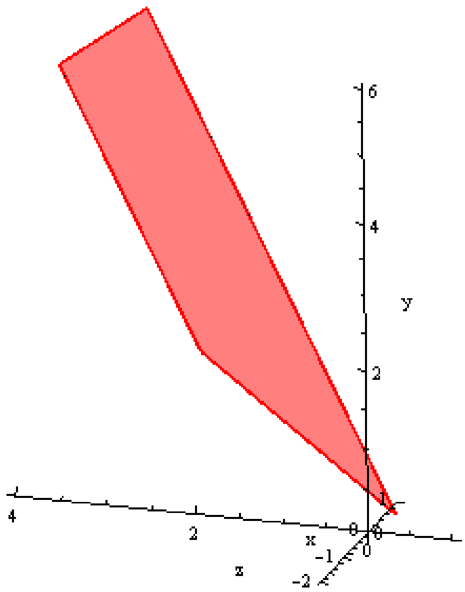
$$\forall \boldsymbol{x}, \boldsymbol{y} \in V\ \forall \lambda, \psi \in \mathbb{R} : \Phi(\lambda\boldsymbol{x} + \psi\boldsymbol{y}) = \lambda\Phi(\boldsymbol{x}) + \psi\Phi(\boldsymbol{y})\,. \tag{2.87}$$

**Same examples of linear mapping**

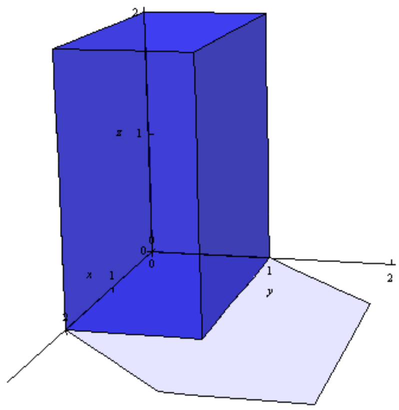
1) $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\Phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2y \\ x+y \end{bmatrix}$.

| $A = \begin{bmatrix} 1 & 1 & 5 \\ 5 & -2 & 2 \end{bmatrix}$ | $\Phi(A) = \begin{bmatrix} 10 & -4 & 4 \\ 6 & -1 & 7 \end{bmatrix}$ |
|---|---|
|  |   |

2) $\Phi: \mathbb{R}^3 \to \mathbb{R}^3$ with $\Phi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y - x \\ x + z \\ -y + z \end{bmatrix}$.

| | |
|---|---|
| $B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 5 & 5 & 2 & -1 \end{bmatrix}$ | $\Phi(B) = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 6 & 5 & 2 & 1 \\ 3 & 4 & 2 & -1 \end{bmatrix}$ |
|  |  |

3) $\Phi: \mathbb{R}^3 \to \mathbb{R}^2$ with $\Phi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + \frac{1}{2}z \\ y + \frac{1}{2}z \end{bmatrix}$.

| | |
|---|---|
| $B = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{bmatrix}$ | $\Phi(B) = \begin{bmatrix} 0 & 2 & 2 & 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \end{bmatrix}$ |
|  | |

**Remark**

Let be $\Phi: V \to W$, a linear mapping , then:

1. $\Phi(-x) = -\Phi(x)$;
2. $\Phi(x-y) = \Phi(x) - \Phi(y)$;
3. $\Phi(0_V) = 0_W$ ;
4. If $V' \subseteq V$ is subspace of $V$, then $\Phi(V')$ is subspace of $W$ ;
5. If $W' \subseteq W$ is subspace of W, then $\Phi^{-1}(W')$ is subspace of $V$, with $\Phi^{-1}(W') = \{x \in V : \Phi(x) \in W'\}$.

**Definition 2.16** (Injective, Surjective, Bijective). Consider a mapping $\Phi : \mathcal{V} \to \mathcal{W}$, where $\mathcal{V}, \mathcal{W}$ can be arbitrary sets. Then $\Phi$ is called

- *Injective* if $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V} : \Phi(\boldsymbol{x}) = \Phi(\boldsymbol{y}) \implies \boldsymbol{x} = \boldsymbol{y}$.
- *Surjective* if $\Phi(\mathcal{V}) = \mathcal{W}$.
- *Bijective* if it is injective and surjective.

- *Isomorphism:* $\Phi : V \to W$ linear and bijective
- *Endomorphism:* $\Phi : V \to V$ linear
- *Automorphism:* $\Phi : V \to V$ linear and bijective
- We define $\mathrm{id}_V : V \to V$, $\boldsymbol{x} \mapsto \boldsymbol{x}$ as the *identity mapping* or *identity automorphism* in $V$.

**Theorem 2.17** (Theorem 3.59 in Axler (2015)). *Finite-dimensional vector spaces $V$ and $W$ are isomorphic if and only if* $\dim(V) = \dim(W)$.

## Matrix Representation of Linear Mappings

**Definition 2.19** (Transformation Matrix). Consider vector spaces $V, W$ with corresponding (ordered) bases $B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$ and $C = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_m)$. Moreover, we consider a linear mapping $\Phi : V \to W$. For $j \in \{1, \dots, n\}$,

$$\Phi(\boldsymbol{b}_j) = \alpha_{1j}\boldsymbol{c}_1 + \cdots + \alpha_{mj}\boldsymbol{c}_m = \sum_{i=1}^{m} \alpha_{ij}\boldsymbol{c}_i \tag{2.92}$$

is the unique representation of $\Phi(\boldsymbol{b}_j)$ with respect to $C$. Then, we call the $m \times n$-matrix $\boldsymbol{A}_\Phi$, whose elements are given by

$$A_\Phi(i, j) = \alpha_{ij}\,, \tag{2.93}$$

the *transformation matrix* of $\Phi$ (with respect to the ordered bases $B$ of $V$ and $C$ of $W$).

**Example**

Consider a homomorphism $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$ such that $\Phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ 2y \\ 2x - 4y \end{bmatrix}$ and the ordered bases $B = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ of $\mathbb{R}^2$ and $C = \left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$ of $\mathbb{R}^3$. Determine the transformation matrix $A_\Phi$ with respect to $B$ and $C$.

If $\hat{x}$ x is the coordinate vector of $x \in V$ with respect to $B$ and $\hat{y}$ the coordinate vector of $y = \Phi(x)$ $\in W$ with respect to C, then

$$\hat{\boldsymbol{y}} = \boldsymbol{A}_\Phi \hat{\boldsymbol{x}}\,. \tag{2.94}$$

**Example**

Consider the ordered bases $B = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ of $\mathbb{R}^2$ and $C = \left\{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$ of $\mathbb{R}^3$, and the homomorphism $\Phi: \mathbb{R}^3 \to \mathbb{R}^2$ whose t transformation matrix $A_\Phi$ with respect to $C$ and $B$, is

$$A_{\phi} = \begin{bmatrix} 5 & 7 & 1 \\ 6 & 8 & 2 \end{bmatrix}$$

Determine $\Phi\left(\begin{bmatrix} 12 \\ 4 \\ 10 \end{bmatrix}\right)$.

**If we considered the standard basis of $\mathbb{R}^{\boldsymbol{n}}$**

**Example**

Consider a homomorphism $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that $\Phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ 2y \\ 2x - 4y \end{bmatrix}$. Determine the transformation matrix $A_{\Phi}$ with respect to **standard basis** of $\mathbb{R}^2$ and $\mathbb{R}^3$.

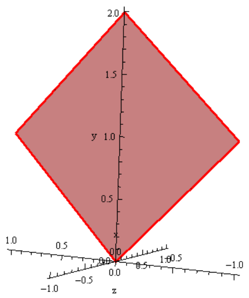
**Example**

Consider homomorphism $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^2$ whose t transformation matrix $A_{\Phi}$ with respect to **standard basis** of $\mathbb{R}^3$ and $\mathbb{R}^2$ is:

$$A_{\phi} = \begin{bmatrix} 5 & 7 & 1 \\ 6 & 8 & 2 \end{bmatrix}$$

a) Determine $\Phi\left(\begin{bmatrix} 12 \\ 4 \\ 10 \end{bmatrix}\right)$.

b) The following picture




can be represented by the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$. Find the picture transformed by $\Phi$.

## Image and Kernel

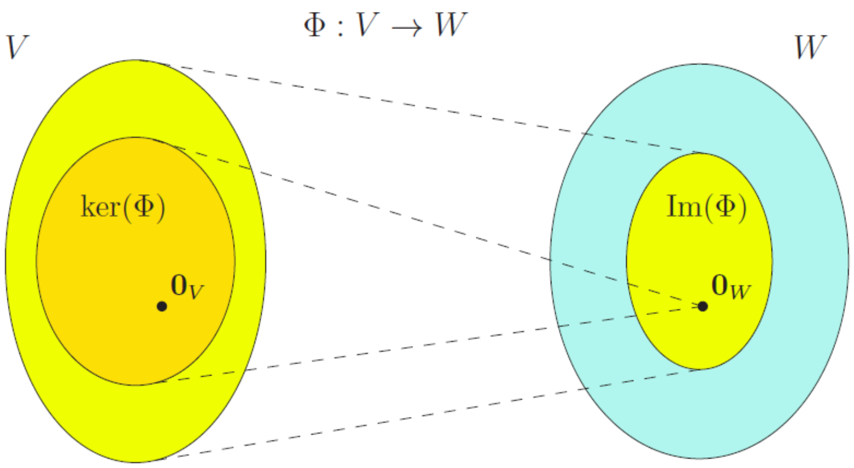
**Definition 2.23** (Image and Kernel).
For $\Phi : V \to W$, we define the *kernel/null space*

$$\ker(\Phi) := \Phi^{-1}(\mathbf{0}_W) = \{\boldsymbol{v} \in V : \Phi(\boldsymbol{v}) = \mathbf{0}_W\} \tag{2.122}$$

and the *image/range*

$$\mathrm{Im}(\Phi) := \Phi(V) = \{\boldsymbol{w} \in W | \exists \boldsymbol{v} \in V : \Phi(\boldsymbol{v}) = \boldsymbol{w}\}\,. \tag{2.123}$$

We also call $V$ and $W$ also the *domain* and *codomain* of $\Phi$, respectively.




- It always holds that $\Phi(\mathbf{0}_V) = \mathbf{0}_W$ and, therefore, $\mathbf{0}_V \in \ker(\Phi)$. In particular, the null space is never empty.
- $\mathrm{Im}(\Phi) \subseteq W$ is a subspace of $W$, and $\ker(\Phi) \subseteq V$ is a subspace of $V$.

**Example**

Let be the homomorphism $\Phi: \mathbb{R}^3 \to \mathbb{R}^4$ such that $\Phi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ -x + y - 3z \\ x + 2y \\ 3x + 4y \end{bmatrix}$.

a) Determine the kernel of $\Phi$.
b) Determine the image of $\Phi$.
c) Determine the transformation matrix $A_\Phi$ with respect to **standard basis** of $\mathbb{R}^3$ and $\mathbb{R}^4$.

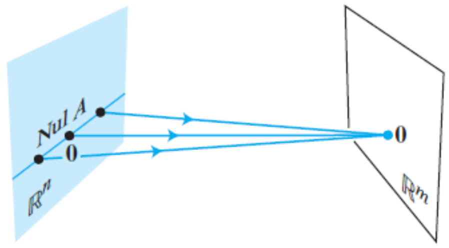
**Example**

Let be the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & -3 \\ 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$, and the homomorfism $\Phi: \mathbb{R}^3 \to \mathbb{R}^4$ such that $\Phi(x) = Ax$

a) Determine the kernel of $\Phi$.
b) Determine the image of $\Phi$.

**Definition:** Let us consider the matrix $A$ with $m$ rows and $n$ columns. The null space of matrix $A$, written as $Null\ A$, is the set of all solutions of the homogeneous equation $Ax = 0$. In set notation,

$$Nul(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$




- The kernel/null space $\ker(\Phi)$ is the general solution to the homogeneous system of linear equations $\boldsymbol{Ax} = \boldsymbol{0}$ and captures all possible linear combinations of the elements in $\mathbb{R}^n$ that produce $\boldsymbol{0} \in \mathbb{R}^m$.
- The kernel is a subspace of $\mathbb{R}^n$, where $n$ is the "width" of the matrix.
- The kernel focuses on the relationship among the columns, and we can use it to determine whether/how we can express a column as a linear combination of other columns.

**Definition:** Let us consider the matrix $A$ with $m$ rows and $n$ columns. The column space of matrix $A$, written as Col(A) $A$, is the set of all of all linear combinations of the columns of A. If $A = [c_1\ c_2 \dots c_n]$ , then

$$col(A) = span[c_1, c_2, \dots, c_n\ ].$$

.

- For $\boldsymbol{A} = [\boldsymbol{a}_1, \dots, \boldsymbol{a}_n]$, where $\boldsymbol{a}_i$ are the columns of $\boldsymbol{A}$, we obtain

$$\mathrm{Im}(\Phi) = \{\boldsymbol{Ax} : \boldsymbol{x} \in \mathbb{R}^n\} = \left\{\sum_{i=1}^{n} x_i \boldsymbol{a}_i : x_1, \dots, x_n \in \mathbb{R}\right\} \quad (2.124a)$$

$$= \mathrm{span}[\boldsymbol{a}_1, \dots, \boldsymbol{a}_n] \subseteq \mathbb{R}^m\,, \quad (2.124b)$$

  i.e., the image is the span of the columns of $\boldsymbol{A}$, also called the *column space*. Therefore, the column space (image) is a subspace of $\mathbb{R}^m$, where $m$ is the "height" of the matrix.
- $\mathrm{rk}(\boldsymbol{A}) = \dim(\mathrm{Im}(\Phi))$.

**Theorem** Let be the homomorphism $\Phi: V \to W$

- $\Phi$ is injective if and only if $Ker(\Phi) = 0_V$.
- $\Phi$ is surjective if and only if $\dim(Im(\Phi)) = \dim(W)$.

**Theorem 2.24** (Rank-Nullity Theorem). *For vector spaces $V, W$ and a linear mapping $\Phi : V \to W$ it holds that*

$$\dim(\ker(\Phi)) + \dim(\mathrm{Im}(\Phi)) = \dim(V) . \qquad (2.129)$$

**Remark** Let us consider the matrix $A$ with $m$ rows and $n$ columns, then

$$n = \dim\big(Nul(A)\big) + \dim\,(Col(A)).$$

- If $\dim(\mathrm{Im}(\Phi)) < \dim(V)$, then $\ker(\Phi)$ is non-trivial, i.e., the kernel contains more than $\mathbf{0}_V$ and $\dim(\ker(\Phi)) \geqslant 1$.
- If $\boldsymbol{A}_\Phi$ is the transformation matrix of $\Phi$ with respect to an ordered basis and $\dim(\mathrm{Im}(\Phi)) < \dim(V)$, then the system of linear equations $\boldsymbol{A}_\Phi \boldsymbol{x} = \mathbf{0}$ has infinitely many solutions.
- If $\dim(V) = \dim(W)$, then the following three-way equivalence holds:
  - $\Phi$ is injective
  - $\Phi$ is surjective
  - $\Phi$ is bijective

  since $\mathrm{Im}(\Phi) \subseteq W$.

## Compose and Inverse of a homomorphism

**Theorem:**

- For vector spaces $V, W$ e $U$ and linear mappings $\Phi : V \to W$ e $\Psi : W \to U$, then the mapping $\Psi \circ \Phi : V \to U$ is also linear.

- Let be $\Phi : V \to W$ e $\Psi : W \to U$, linear mappings, $A_\Phi$ be the transformation matrix of $\Phi$ with respect to $S_1$, an ordered basis of $V$ and $S$, an ordered basis of f $W$, and $B_\Psi$ be the transformation matrix of $\Psi$ with respect to $S$, an ordered basis of $W$ and $S_2$, an ordered basis of $U$. Then the transformation matrix of $\Psi \circ \Phi$, $M_{\Psi \circ \Phi}$ with respect to $S_1$ and $S_2$ is given by

$$M_{\Psi \circ \Phi} = B_\Psi A_\Phi$$

**Example:**

Consider homomorphisms $\Phi: \mathbb{R}^3 \to \mathbb{R}^2$ and $\Psi: \mathbb{R}^2 \to \mathbb{R}^3$ ,

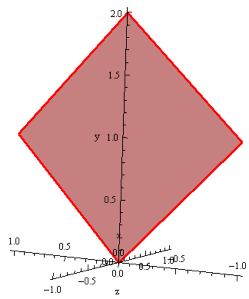
- $\Phi$ is represented by the transformation matrix $A_\Phi$ with respect to **standard basis** of $\mathbb{R}^3$ and $\mathbb{R}^2$:

$$A_\phi = \begin{bmatrix} 5 & 7 & 1 \\ 6 & 8 & 2 \end{bmatrix}$$

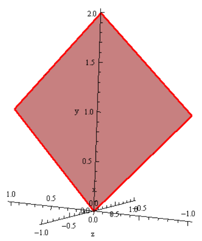
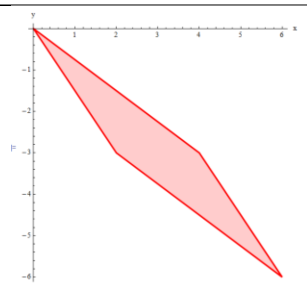
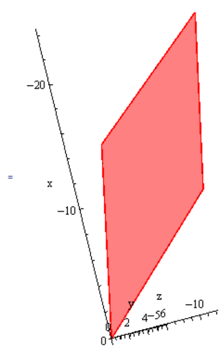
- $\Psi$ is represented by the transformation matrix $B_\Psi$ with respect to **standard basis** of $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$B_\Psi = \begin{bmatrix} -2 & 7 \\ 1 & 0 \\ 6 & 8 \end{bmatrix}$$

The following picture



can be represented by the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$. Find the picture transformed by $\Psi \circ \Phi$.

| Original | $\Phi(A)$ | $\Psi(\Phi(A))$ |
|---|---|---|
| $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ | $\boldsymbol{\Phi(A)} = \begin{bmatrix} 0 & 4 & 6 & 2 & 0 \\ 0 & -3 & -6 & -3 & 0 \end{bmatrix}$ | $\Psi(\Phi(A)) = \begin{bmatrix} 0 & -14 & -24 & -10 & 0 \\ 0 & 4 & 6 & 2 & 0 \\ 0 & 0 & -12 & -12 & 0 \end{bmatrix}$ |
|  |  |  |

**Definition**

For vector spaces $V, W$ and linear mappings $\Phi : V \to W$ e $\Psi: W \to V$, such that $\Psi \circ \Phi = \mathrm{id}_V$ and $\Phi \circ \Psi = \mathrm{id}_W$, then $\Psi$ is the inverse of $\Phi$. And we say that is an invertible linear mapping.

**Remark**

- $(\Phi^{-1})^{-1} = \Phi;$
- $(\Psi \ o\ \Phi)^{-1} = \Phi^{-1}\, o \Psi^{-1}$ .
- If $\Phi$: $V \to W$ is an invertible linear mapping, then $dim\ (V) = dim$ (W).

**Theorem** Let be $\Phi : V \to W$ a linear mapping, $A_\Phi$ be the transformation matrix of $\Phi$ with respect to $S$, an ordered basis of $V$ and $B$, an ordered basis of $W$. Then the transformation matrix of $\Phi^{-1}$, $A_{\Phi^{-1}}$ with respect to$B$ and $S$is given by

$$A_{\Phi^{-1}} = (A_\Phi)^{-1}$$

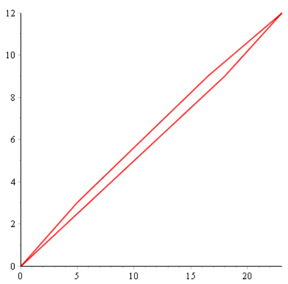
**Example:**

Consider homomorphisms $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ whose the transformation matrix $A_\Phi$ with respect to **standard basis** of $\mathbb{R}^2$ is

$$A_\Phi = \begin{bmatrix} \mathbf{6} & \mathbf{5} \\ \mathbf{3} & \mathbf{3} \end{bmatrix}$$

One picture was transformed by $\Phi$ , and the final picture is given by the following matrix:

$\boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \mathbf{18} & \mathbf{23} & \mathbf{16.5} & \mathbf{5} \\ \mathbf{0} & \mathbf{9} & \mathbf{12} & \mathbf{9} & \mathbf{3} \end{bmatrix}$. Discover the original picture.

| **Transformed picture** | **Original ?** |
| --- | --- |
|  | |

## Basis changes

**Theorem 2.20** (Basis Change). *For a linear mapping $\Phi : V \to W$, ordered bases*

$$B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n), \quad \tilde{B} = (\tilde{\boldsymbol{b}}_1, \dots, \tilde{\boldsymbol{b}}_n) \tag{2.103}$$

*of $V$ and*

$$C = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_m), \quad \tilde{C} = (\tilde{\boldsymbol{c}}_1, \dots, \tilde{\boldsymbol{c}}_m) \tag{2.104}$$

*of $W$, and a transformation matrix $\boldsymbol{A}_\Phi$ of $\Phi$ with respect to $B$ and $C$, the corresponding transformation matrix $\tilde{\boldsymbol{A}}_\Phi$ with respect to the bases $\tilde{B}$ and $\tilde{C}$ is given as*

$$\tilde{\boldsymbol{A}}_\Phi = \boldsymbol{T}^{-1}\boldsymbol{A}_\Phi\boldsymbol{S}\,. \tag{2.105}$$

*Here, $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ is the transformation matrix of $\mathrm{id}_V$ that maps coordinates with respect to $\tilde{B}$ onto coordinates with respect to $B$, and $\boldsymbol{T} \in \mathbb{R}^{m\times m}$ is the transformation matrix of $\mathrm{id}_W$ that maps coordinates with respect to $\tilde{C}$ onto coordinates with respect to $C$.*

$$\begin{tikzcd}
V \arrow[r, "\Phi"] & W \\
B \arrow[r, "\Phi_{CB}", "\boldsymbol{A}_\Phi"'] & C \\
\tilde{B} \arrow[u, "\Psi_{B\tilde{B}}", "\boldsymbol{S}"'] \arrow[r, "\tilde{\boldsymbol{A}}_\Phi", "\Phi_{\tilde{C}\tilde{B}}"'] & \tilde{C} \arrow[u, "\boldsymbol{T}", "\Xi_{C\tilde{C}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\Phi"] & W \\
B \arrow[r, "\Phi_{CB}", "\boldsymbol{A}_\Phi"'] & C \arrow[d, "\boldsymbol{T}^{-1}"', "\Xi_{\tilde{C}C} = \Xi_{C\tilde{C}}^{-1}"] \\
\tilde{B} \arrow[u, "\Psi_{B\tilde{B}}", "\boldsymbol{S}"'] \arrow[r, "\tilde{\boldsymbol{A}}_\Phi", "\Phi_{\tilde{C}\tilde{B}}"'] & \tilde{C}
\end{tikzcd}$$

**Example**

Consider homomorphisms $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ whose the transformation matrix $A_\Phi$ with respect to **standard basis** of $\mathbb{R}^2$ is

$$A_\Phi = \begin{bmatrix} \mathbf{6} & \mathbf{5} \\ \mathbf{3} & \mathbf{3} \end{bmatrix}$$

Find the transformation matrix $\widehat{A_\Phi}$ with respect to the ordered basis $B = \left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$.

## Affine subspaces

**Definition 2.25** (Affine Subspace)**.** Let $V$ be a vector space, $\boldsymbol{x}_0 \in V$ and $U \subseteq V$ a subspace. Then the subset

$$L = \boldsymbol{x}_0 + U := \{\boldsymbol{x}_0 + \boldsymbol{u} : \boldsymbol{u} \in U\} \tag{2.130a}$$

$$= \{\boldsymbol{v} \in V | \exists \boldsymbol{u} \in U : \boldsymbol{v} = \boldsymbol{x}_0 + \boldsymbol{u}\} \subseteq V \tag{2.130b}$$

is called *affine subspace* or *linear manifold* of $V$. $U$ is called *direction* or *direction space*, and $\boldsymbol{x}_0$ is called *support point*. In Chapter 12, we refer to such a subspace as a *hyperplane*.

*Remark.* Consider two affine subspaces $L = \boldsymbol{x}_0 + U$ and $\tilde{L} = \tilde{\boldsymbol{x}}_0 + \tilde{U}$ of a vector space $V$. Then, $L \subseteq \tilde{L}$ if and only if $U \subseteq \tilde{U}$ and $\boldsymbol{x}_0 - \tilde{\boldsymbol{x}}_0 \in \tilde{U}$.

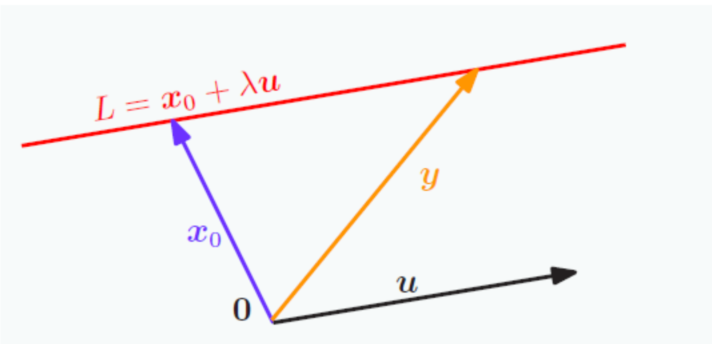
Affine subspaces are often described by *parameters*: Consider a $k$-dimensional affine space $L = \boldsymbol{x}_0 + U$ of $V$. If $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k)$ is an ordered basis of $U$, then every element $\boldsymbol{x} \in L$ can be uniquely described as

$$\boldsymbol{x} = \boldsymbol{x}_0 + \lambda_1 \boldsymbol{b}_1 + \ldots + \lambda_k \boldsymbol{b}_k \,, \tag{2.131}$$

where $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. This representation is called *parametric equation* of $L$ with directional vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ and *parameters* $\lambda_1, \ldots, \lambda_k$. ◇

**Example 2.26 (Affine Subspaces)**

- One-dimensional affine subspaces are called *lines* and can be written as $\boldsymbol{y} = \boldsymbol{x}_0 + \lambda \boldsymbol{x}_1$, where $\lambda \in \mathbb{R}$, where $U = \text{span}[\boldsymbol{x}_1] \subseteq \mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$. This means that a line is defined by a support point $\boldsymbol{x}_0$ and a vector $\boldsymbol{x}_1$ that defines the direction. See Figure 2.13 for an illustration.



- Two-dimensional affine subspaces of $\mathbb{R}^n$ are called *planes*. The parametric equation for planes is $\boldsymbol{y} = \boldsymbol{x}_0 + \lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $U = \operatorname{span}[\boldsymbol{x}_1, \boldsymbol{x}_2] \subseteq \mathbb{R}^n$. This means that a plane is defined by a support point $\boldsymbol{x}_0$ and two linearly independent vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ that span the direction space.
- In $\mathbb{R}^n$, the $(n-1)$-dimensional affine subspaces are called *hyperplanes*, and the corresponding parametric equation is $\boldsymbol{y} = \boldsymbol{x}_0 + \sum_{i=1}^{n-1} \lambda_i \boldsymbol{x}_i$, where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-1}$ form a basis of an $(n-1)$-dimensional subspace $U$ of $\mathbb{R}^n$. This means that a hyperplane is defined by a support point $\boldsymbol{x}_0$ and $(n-1)$ linearly independent vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-1}$ that span the direction space. In $\mathbb{R}^2$, a line is also a hyperplane. In $\mathbb{R}^3$, a plane is also a hyperplane.

*Remark* (Inhomogeneous systems of linear equations and affine subspaces). For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, the solution of the linear equation system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is either the empty set or an affine subspace of $\mathbb{R}^n$ of dimension $n - \operatorname{rk}(\boldsymbol{A})$. In particular, the solution of the linear equation $\lambda_1 \boldsymbol{x}_1 + \ldots + \lambda_n \boldsymbol{x}_n = \boldsymbol{b}$, where $(\lambda_1, \ldots, \lambda_n) \neq (0, \ldots, 0)$, is a hyperplane in $\mathbb{R}^n$.

**Definition 2.26** (Affine Mapping)**.** For two vector spaces $V, W$, a linear mapping $\Phi : V \to W$, and $\boldsymbol{a} \in W$, the mapping

$$\phi : V \to W \tag{2.132}$$

$$\boldsymbol{x} \mapsto \boldsymbol{a} + \Phi(\boldsymbol{x}) \tag{2.133}$$

is an *affine mapping* from $V$ to $W$. The vector $\boldsymbol{a}$ is called the *translation vector* of $\phi$.

- Every affine mapping $\phi : V \to W$ is also the composition of a linear mapping $\Phi : V \to W$ and a translation $\tau : W \to W$ in $W$, such that $\phi = \tau \circ \Phi$. The mappings $\Phi$ and $\tau$ are uniquely determined.
- The composition $\phi' \circ \phi$ of affine mappings $\phi : V \to W$, $\phi' : W \to X$ is affine.
- Affine mappings keep the geometric structure invariant. They also preserve the dimension and parallelism.