Towards BCRT for FPTS
Best-case response time lower bound

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1 BCRT lower bound for FPTS

In this section, we propose a lower bound for the *best-case response time* of a task scheduled under FPTS. In addition, we show that the lower bound that is proposed is tighter than other trivial lower bounds for FPTS.

1.1 Hypothetical best-case interval for execution of a task τ_i

Definition 1. *The hypothetical best-case interval $HI_i(y,\alpha)$ is defined as the length of the shortest interval $[t_s, t_e)$ in which an amount of time $y \in \mathbb{R}^+$ is available for the execution of τ_i assuming that all jobs of delaying tasks activated at or after $t_e - \alpha$ are postponed after t_e, where $\alpha \in \mathbb{R}^+ \cup \{0\}$. Furthermore, all tasks are scheduled under FPPS.*

Note that $HI_i(y,\alpha)$ assumes that all tasks are scheduled under FPPS. Therefore, $HI_i(y,\alpha)$ can be seen as a generalization of the *best-case interval* $BI_i(y)$ defined in [1], with the addition that jobs of *delaying* tasks activated after or at $t_e - \alpha$ do not influence the amount of time available for task τ_i in the interval. Clearly, $HI_i(y,\alpha) = BI_i(y)$ when there are no *delaying* tasks or when $\alpha = 0$. Based on this, we formulate the following theorem.

Theorem 1. *The hypothetical best-case interval $HI_i(y,\alpha)$ in which an amount of time $y \in \mathbb{R}^+$ is available for task τ_i is given by the largest $x \in \mathbb{R}^+$ satisfying*

$$x = y + \sum_{h:\pi_h > \theta_i} \left(\left\lceil \frac{x}{T_h} \right\rceil - 1 \right)^+ BC_h + \sum_{d:\theta_i \geq \pi_d > \pi_i} \left(\left\lceil \frac{x-\alpha}{T_d} \right\rceil - 1 \right)^+ BC_d, \tag{1}$$

where the notation w^+ stands for $\max(w,0)$.

Proof. Since the only difference between the notion of *hypothetical best-case interval* $HI_i(y,\alpha)$ and *best-case interval* $BI_i(y)$ relies on *delaying* tasks, the first part of Theorem 1 remains the same for *preemptive* tasks as the analysis of $BI_i(y)$. Hence, we only have to prove that the last term $\sum_{d:\theta_i \geq \pi_d > \pi_i} \left(\left\lceil \frac{x-\alpha}{T_d} \right\rceil - 1 \right)^+ BC_d$ addresses the influence of delaying tasks in the interval $[t_s, t_e)$.

By Definition 1, jobs of *delaying* tasks activated after time $t_e - \alpha$ do not influence the length $HI_i(y,\alpha)$ of the interval $[t_s, t_e)$. Therefore, we only have to calculate the minimum interference provoked by *delaying* tasks in the interval $[t_s, t_e - \alpha)$. Since this interval has a length of $HI_i(y,\alpha) - \alpha$, the minimum interference is simply given by $\sum_{d:\theta_i \geq \pi_d > \pi_i} \left(\left\lceil \frac{HI_i(y,\alpha)-\alpha}{T_d} \right\rceil - 1 \right)^+ BC_d$. Therefore, concluding the proof. □

To calculate $HI_i(y,\alpha)$, we can use an iterative procedure starting with an upper-bound. An appropriate upper-bound to start the iterative procedure is the following.

Lemma 1. *An appropriate initial value for the iterative procedure to determine $HI_i(y,\alpha)$ is*

$$HI_i^{(0)}(y,\alpha) = \frac{y}{1 - BU_{i-1}}. \tag{2}$$

Proof. Similar to the proof of Lemma 3 in [1]. □

1.2 Hypothetical shortest response time

In this section, we first introduce the notion of *hypothetical shortest response time* and subsequently formulate a number of lemmas related with such a notion.

Definition 2. *The hypothetical shortest response time $\Psi_i(\alpha)$ with $\alpha \in \mathbb{R}^+ \cup \{0\}$ is defined as the shortest response time of a job k of τ_i assuming that delaying tasks activated after or at time $f_{i,k} - \alpha$ are postponed after $f_{i,k}$. Furthermore, all tasks are scheduled under FPPS.*

Note that Definition 2 assumes that all tasks are scheduled under FPPS. Hence, $\Psi_i(\alpha)$ can be seen as a generalization of the *best-case response time* analysis for FPPS and arbitrary deadlines proposed in [1], with the addition that jobs of *delaying* tasks activated after or at $f_{i,k} - \alpha$ do not influence the response time of $\iota_{i,k}$. Clearly, when there are no *delaying* tasks or when $\alpha = 0$, $\Psi_i(\alpha)$ is the same as the *best-case response time* under FPPS. Recall that the best-case analysis for FPPS investigates whether previous jobs induce blocking in the job experiencing the optimal instant in order to determine its response time. This is done using the notion of *best-case interval* $BI_i(y)$. Based on these observations, we can simply extent the best-case analysis for FPPS with the notion of *hypothetical best-case interval* $HI_i(y,\alpha)$ to determine $\Psi_i(\alpha)$ as follows.

Theorem 2. *Let all tasks of a task-set $\mathcal{T}$ be strictly periodic and let wl_i exist. The hypothetical shortest response time $\Psi_i(\alpha)$ of a task $\tau_i \in \mathcal{T}$ is given by*

$$\Psi_i(\alpha) = \max_{1 \le k \le wl_i} (HI_i(k \cdot BC_i, \alpha) - (k-1)T_i). \tag{3}$$

As an example of the *hypothetical shortest response time* $\Psi_i(\alpha)$ of a task τ_i, consider the the task-set with characteristics as describe in Table 1. Note that τ_1 is a *preemptive* task of τ_3, and τ_2 is a *delaying* task. Figure 1 shows a timeline where the *hypothetical shortest response time* $\Psi_3(\alpha)$ for task τ_3 is found when choosing $\alpha = 2$. As can be seen, the start of the job of task τ_2 activated at time $t = -2$ is postponed after the completion of the job of τ_3. Furthermore, observe that the job of τ_2 activated at time $t = -6$ can preempt τ_3 because the Definition 2 assumes that tasks are scheduled under FPPS.

Table 1: Task set $\mathcal{T}_1$.

	T_i	C_i	π_i	θ_i	wl_i	WR_i
τ_1	3	1	3	3	1	1
τ_2	4	1	2	2	3	9
τ_3	12	5	1	2	1	12

The *least common multiple* of the periods is 12 and $U^{\mathcal{T}_1} = 1$.

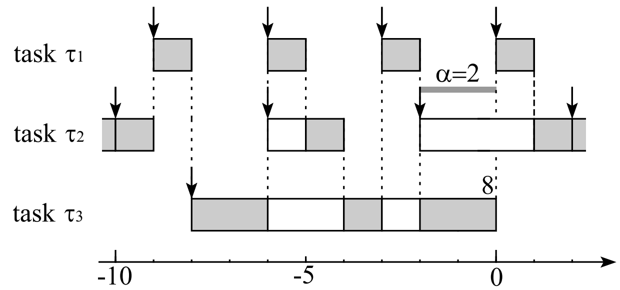

Figure 1: A timeline for $\mathcal{T}_1$ showing the *hypothetical shortest response time* $\Psi_3(2) = 8$ for task τ_3.

We now present some corollaries and lemmas related with the notion of *hypothetical shortest response time*. From Definition 2, we can conclude that α restricts the number of preemptions that *delaying* tasks can provoke in the *hypothetical shortest response time* of a task. In fact, α can be seen as a forbidden region where *delaying* tasks cannot execute. Therefore, when increasing the value of α in $\Psi_i(\alpha)$, we are increasing the forbidden region of *delaying* tasks, and consequently reducing its influence on the shortest response time of τ_i. From this observation, we formulate the following corollary.

Corollary 1. *The function $\Psi_i(\alpha)$ describing the hypothetical shortest response time of a task τ_i is a monotonically decreasing function.*

Since $\Psi_i(\alpha)$ is a monotonically decreasing function, its smallest value is found when α tends to infinity. Clearly, this means that we are simply forbidding the execution of *delaying* tasks completely. Therefore, for this case, the shortest response time of task τ_i is only influenced by its *preemptive* tasks.

Corollary 2. *A lower bound for the hypothetical shortest response time $\Psi_i(\alpha)$ is the shortest hold time H_i^{lb} of τ_i when considering only preemptive tasks, i.e $\Psi_i(\alpha) \geq H_i^{lb}$ for all $\alpha \in \mathbb{R}^+ \cup \{0\}$.*

We now introduce a lemma showing a lower bound for the best-case response time of a task scheduled under FPTS.

Lemma 2. *In FPTS, the best case response time BR_i of a task τ_i is bounded from below by the hypothetical shortest response time $\Psi_i(BR_i)$, i.e. $BR_i \geq \Psi_i(BR_i)$.*

Proof. Let k^{bcrt} be a job of τ_i with $R_{i,k^{bcrt}} = BR_i$. Note that the response time of job k^{bcrt} is not affected by delaying tasks activated after $f_{i,k^{bcrt}} - HT_{i,k^{bcrt}}$ because such tasks cannot preempt τ_i. Now assume that all *delaying* tasks activated after or at time $f_{i,k^{bcrt}} - BR_i$ are postponed after $f_{i,k^{bcrt}}$. Clearly, under this assumption, the new response time $R'_{i,k^{bcrt}}$ of job k^{bcrt} can at most be equal to BR_i because we are reducing the influence of *delaying* tasks in the response time of k^{bcrt}. Therefore, $BR_i \geq R'_{i,k^{bcrt}}$.

Recall that $\Psi_i(BR_i)$ is, by definition, the shortest response time of a job k of τ_i when assuming that delaying tasks activated after or at time $f_{i,k} - BR_i$ are postponed after $f_{i,k}$. Therefore, we can conclude that $R'_{i,k^{bcrt}} \geq \Psi_i(BR_i)$ and consequently $BR_i \geq \Psi_i(BR_i)$ □

1.3 Best-case response time lower bound for FPTS

Theorem 3. *Let all tasks of a task-set $\mathcal{T}$ be strictly periodic and let wl_i exist. A lower bound BR_i^{lb} for the best-case response time of a task τ_i scheduled under FPTS is given by*

$$BR_i^{lb} = \min_{H_i^{lb} \leq \alpha \leq \Psi_i(H_i^{lb})} (\max\{\alpha, \Psi_i(\alpha)\}), \tag{4}$$

where H_i^{lb} is the shortest hold time of τ_i when considering only preemptive tasks.

Proof. We first show that $\min\limits_{H_i^{lb} \leq \alpha \leq \infty}(\max\{\alpha, \Psi_i(\alpha)\})$ is a lower bound for the *best-case response time* of τ_i. Since we are looking for the minimum result of $\max\{\alpha, \Psi_i(\alpha)\}$ considering the interval $H_i^{lb} \leq \alpha \leq \infty$, it is sufficient to show that at least for one value of α within that interval the result of $\max\{\alpha, \Psi_i(\alpha)\}$ is a lower bound for the *best-case response time*. Let $\alpha = BR_i$, clearly this value is in the interval $H_i^{lb} \leq \alpha \leq \infty$. Therefore, we only have to prove that $BR_i \geq \max\{BR_i, \Psi_i(BR_i)\}$. Using Lemma 2, it holds that $\max\{BR_i, \Psi_i(BR_i)\} = BR_i$; hence, concluding the first part of the proof.

We now want to show that the minimum of $\max\{\alpha, \Psi_i(\alpha)\}$ can never be found for $\alpha > \Psi_i(H_i^{lb})$. In order to prove this, first note that by choosing $\alpha = H_i^{lb}$ and using Corollary 2, it holds that $\max\{H_i^{lb}, \Psi_i(H_i^{lb})\} = \Psi_i(H_i^{lb})$. Since for any value of $\alpha > \Psi_i(H_i^{lb})$ the result of $\max\{\alpha, \Psi_i(\alpha)\}$ is always greater than $\Psi_i(H_i^{lb})$, we conclude that the minimum can never be found for such values; therefore, a proper upper bound for α is $\Psi_i(H_i^{lb})$. □

1.4 Tightness of BR_i^{lb}

A trivial lower bound for the *best-case response time* of a task τ_i scheduled under FPTS is the shortest hold-time H_i^{lb} of such a task when ignoring *delaying* tasks, i.e. when considering only preemptive tasks. We want to show that the lower bound BR_i^{lb} described in Theorem 3 is tighter than H_i^{lb} for many cases.

We first show that BR_i^{lb} is always larger than H_i^{lb}. From Lemma 2, we know that $\Psi_i(\alpha) \geq H_i^{lb}$, we therefore derive that $\max\{\alpha, \Psi_i(\alpha)\} \geq H_i^{lb}$. Hence, $BR_i^{lb} \geq H_i^{lb}$ in general.

We now show the cases where BR_i^{lb} is tighter than H_i^{lb}, i.e. when $BR_i^{lb} > H_i^{lb}$. Firstly, note that for $\alpha > H_i^{lb}$, it holds that $\max\{\alpha, \Psi_i(\alpha)\}) > H_i^{lb}$. In addition, when $\Psi(H_i^{lb}) > H_i^{lb}$, it also holds that $\max\{\alpha, \Psi_i(\alpha)\} > H_i^{lb}$ for $\alpha = H_i^{lb}$. Therefore, we conclude that for the cases where $\Psi(H_i^{lb}) > H_i^{lb}$, the lower bound BR_i^{lb} presented in Theorem 3 is tighter than H_i^{lb}.

1.5 An algorithm for computing the BCRT lower bound

Theorem 2 gives a lower bound for the *best-case response time* of a task τ_i. However, since $\alpha \in \mathbb{R}^+ \cup \{0\}$ is a continuous variable, the set of values where $H_i^{lb} \leq \alpha \leq \Psi_i(H_i^{lb})$ holds may be infinite. In order to solve this, Algorithm 1 shows a procedure to derive the lower bound of the *best-case response time* presented in Theorem 2 for a task τ_i. Nevertheless, instead of trying all possible values of α in the interval $H_i^{lb} \leq \alpha \leq \Psi_i(H_i^{lb})$, Algorithm 1 starts with $\alpha = H_i^{lb}$ and only increases α in discrete steps that can decrease the result of $\max\{\alpha, \Psi_i(\alpha)\}$. If the result of $\max\{\alpha, \Psi_i(\alpha)\}$ starts increasing instead of decreasing, the algorithm terminates and the minimum is found. Note that this is a good point to stop the algorithm because $\Psi_i(\alpha)$ is a monotonically decreasing function. Therefore, if the result of the expression $\max\{\alpha, \Psi_i(\alpha)\}$ starts increasing when increasing α, it means that α becomes the dominant term in $\max\{\alpha, \Psi_i(\alpha)\}$, and the result of $\Psi_i(\alpha)$ will continue increasing.

The procedure in Algorithm 1 first assigns to α the hold time of τ_i when considering only preemptive tasks. In Line 3, $\max\{\alpha, \Psi_i(\alpha)\}$ with the initial value of α is chosen as an initial candidate for the best-case lower bound value BR_i^{lb}. Furthermore, in case that BR_i^{lb} is higher than α, it means that it is possible to increase α hopefully leading to a reduction in the best-case lower bound. Hence, Lines 5-7 increase the value of α. More precisely, the job k^{tight} of task τ_i that induces some blocking on the start time of the job experiencing the *hypothetical shortest response time* is determined in Line 5. Moreover, Lines 6 and 7 determine the time the forbidden region α of delaying tasks should be increased in order to prevent job k^{tight} from inducing some blocking. At this point, the result of $\max\{\alpha, \Psi_i(\alpha)\}$ with the new value of α is assigned to BR_i^{lb} only if it is smaller than its current value. This is expressed in Line 8. Finally, if the new value of α is still smaller than the new value of BR_i^{lb}, the procedure is repeated. Otherwise, the algorithm terminates.

Table 2 shows an overview of the variables used in Algorithm 1.

Table 2: Terminology.

Name	Descriptions
BR_i^{lb}	Lower bound of the best-case response time of τ_i.
α	Length of the forbidden region of *delaying* tasks. It is assumed that *delaying* jobs activated in this region are postponed.
k^{tight}	Job of τ_i inducing some blocking on the start time of the job experiencing the *hypothetical shortest response time*.
DI_i	Length of the time interval between the activation of the job k^{tight} of τ_i and the start of the forbidden region of *delaying* tasks with length α.

Algorithm 1 Algorithm to derive a lower bound for the *best-case response time* of task τ_i.

```
procedure bcrtLowerBound(𝒯,i)
    α ← shortest hold time of τ_i when considering only preemptive tasks;
    BR_i^lb ← max{α, Ψ_i(α)};
    while α < BR_i^lb do
        k^tight ← the smallest k with 1 ≤ k ≤ wl_i that leads to BR_i^lb;
        DI_i ← BR_i^lb + (k^tight − 1)T_i − α;
        α ← min_{d:θ_i ≥ π_d > π_i} (DI_i mod T_d) + α;
        BR_i^lb ← min{BR_i^lb, max{α, Ψ_i(α)}};
    end while
    return BR_i^lb;
```

1.6 An example

Consider the set of tasks $\mathcal{T}_3$ with characteristics as described in Table 3. We will apply Algorithm 1 to derive a lower bound BR_4^{lb} for the *best-case response time* of task τ_4. Figure 2 shows an example of how each variable of the algorithm would be represented in a timeline for $\mathcal{T}_3$ after the first iteration. Furthermore, note that tasks τ_1 and τ_2 are *preemptive* tasks of τ_4, whereas τ_3 is a *delaying* task. As can be seen, $\alpha^{(0)} = 22$ represents the initial value of α, and it is initially equal to the hold time of the last job of τ_4. Furthermore, the job of τ_4 activated at time $t = 560$ denoted as k^{tight} is the one that leads to the *hypothetical response time* of $\Psi_4(\alpha^{(0)}) = 36$, and subsequently to $BR_4^{lb(0)} = 36$.

Note that, in Figure 2, it is possible to increase the forbidden region of *delaying* task τ_3 in order to allow more space for the execution of τ_4. Therefore, the algorithm increments α till an activation of τ_3 coincides with the activation of job k^{tight} of τ_4. This time increment can be calculated as (DI_4 mod T_3), and it leads to $\alpha^{(1)} = 26$. Note that, at this point, $BR_4^{lb(0)}$ is still larger than $\alpha^{(1)}$; hence, the algorithm will recalculate BR_4^{lb}.

Table 3: Task set $\mathcal{T}_3$.

	T_i	$WC_i = BC_i$	π_i	θ_i	wl_i	WR_i	BR_i
τ_1	35	5	4	4	1	5	5
τ_2	35	5	3	3	1	10	5
τ_3	50	20	2	2	2	62	20
τ_4	70	22	1	2	5	66	27

The *least common multiple* of the periods is 350 and $U^{\mathcal{T}_3} = 1$.

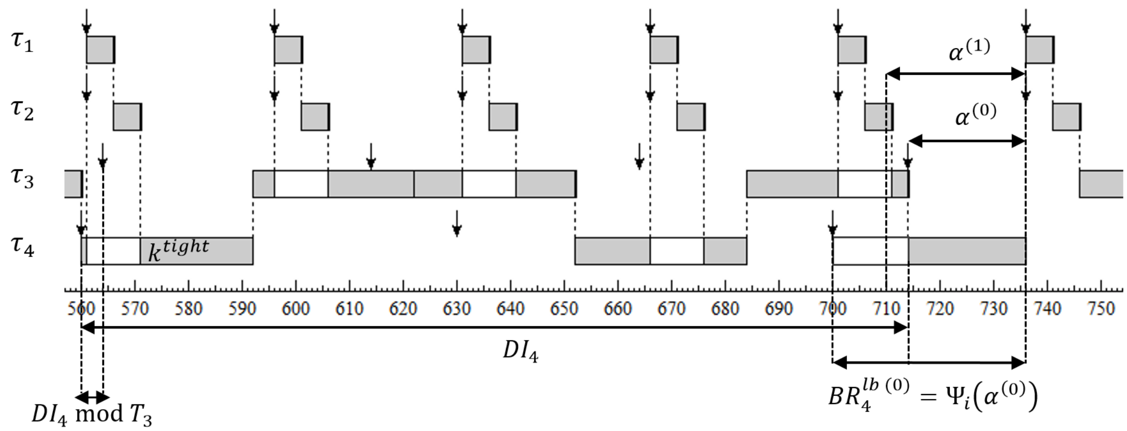

Figure 2: Example after the first iteration of $bcrtLowerBound(\mathcal{T}_3, 4)$.

Figure 3 shows the timeline for the second iterations of $bcrtLowerBound(\mathcal{T}_3, 4)$. As can be seen, the activations of the *delaying* task τ_3 was pushed to an earlier moment in time, allowing to decrease the *hypothetical response time* $\Psi_4(\alpha^{(1)}) = 22$ of the last job. At this point $BR_4^{lb(1)} = \alpha^{(1)} = 26$; hence, the algorithm terminates because there is no previous job restricting the response time of the last job of τ_4. Therefore, $BR_4^{lb} = 26$ is a proper lower bound for the *best-case response time* of τ_4.

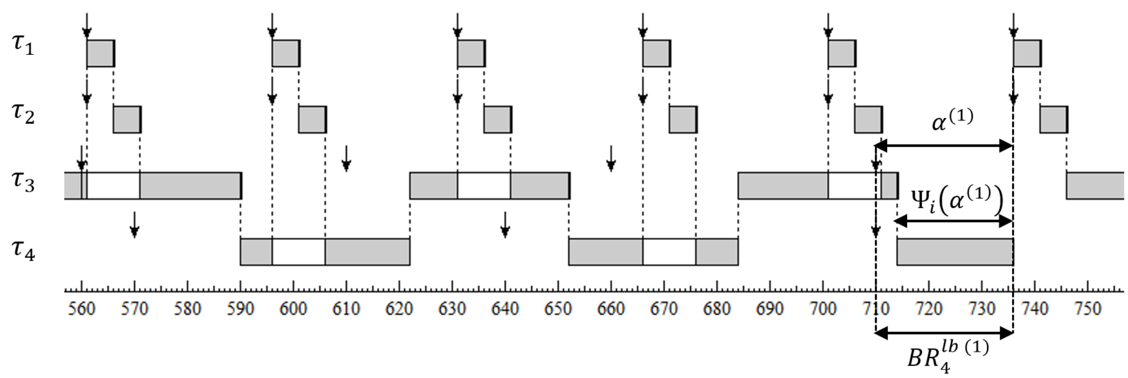

Figure 3: Example after the second iteration of $bcrtLowerBound(\mathcal{T}_3, 4)$.

References

[1] R.J. Bril, J.J. Lukkien, and R.H. Mak. Best-case response times and jitter analysis of real-time tasks with arbitrary deadlines. In Proc. 21st International Conference on Real-Time Networks and Systems (RTNS), ACM, pp. 193-202, October 2013.