Algorithms for Computational Logic

Overview of Satisfiability Modulo Theories

based on slides of João Marques-Silva and Mikoláš Janota

IST, ULisboa

Context

- Propositional satisfiability (SAT)
  - Modern SAT algorithms – CDCL
- Propositional encodings
  - Cardinality constraints, PB constraints, etc.

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- Duality
- Satisfiability Modulo Theories (SMT)
- Other topics:
  - Answer Set Programming (ASP)

Outline

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What is SMT?

Application Examples

- Encoding a scheduling problem
- Encoding symbolic execution

A Glimpse of SMT

Eager Encodings

Lazy Solving

FOL and Terminology

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- term: $f(c)$, atom: $f(c) > d$, literal: $\neg(f(c) > d)$, proposition: $(c < d) \vee (d > z)$
- congruence: $(t_1 = s_1 \wedge t_2 = s_2) \rightarrow f(t_1, t_2) = f(s_1, s_2)$

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- Solve:

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\begin{aligned}
&((x_4 - x_2 \leq 3) \vee (x_4 - x_3 \geq 5)) \wedge (x_4 - x_3 \leq 6) \wedge \\
&(x_1 - x_2 \leq -1) \wedge (x_1 - x_3 \leq -2) \wedge (x_1 - x_4 \leq -1) \wedge \\
&(x_2 - x_1 \leq 2) \wedge (x_3 - x_2 \leq -1) \wedge \\
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- Integer difference logic (with Boolean structure)
- Unsatisfiable (**Why?**)
- How to solve formulas like the above?

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A scheduling example

- Standard job-shop scheduling formulation: [Moura&Bjorner'11]
  - n jobs, each composed of m tasks to be performed chronologically on m machines
    - $d_{i,j}$: duration of task j for job i

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    - All jobs must terminate by a time limit *max*
  - An example (task has number of time units and assigned machine):

$d_{i,j}$	Machine 1	Machine 2
Job 1	2	1
Job 2	3	1
Job 3	2	3

with $max = 8$

A scheduling example (Cont.)

- An SMT model for job-shop scheduling:
  - $t_{i,j}$: start time for task j of job i
  - Example:

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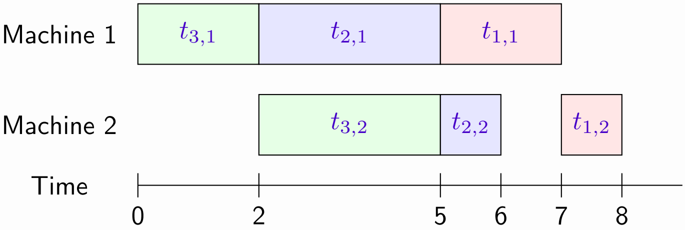

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  - Formulation:

$$
\begin{array}{l}
(t_{1,1} \geq 0) \land (t_{1,2} \geq t_{1,1} + 2) \land (t_{1,2} + 1 \leq 8) \land \\
(t_{2,1} \geq 0) \land (t_{2,2} \geq t_{2,1} + 3) \land (t_{2,2} + 1 \leq 8) \land \\
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  - Integer difference logic with Boolean structure
    - Model: $t_{1,1} = 5; t_{1,2} = 7; t_{2,1} = 2; t_{2,2} = 6; t_{3,1} = 0; t_{3,2} = 3;$

Software testing with symbolic execution

- Example C program: [E.g. Moura&Bjorner'11]

```
int GCD(int x, int y) {
  while (true) {
    int m = x % y;
    if (m == 0) return y;
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- Can the while loop test be executed exactly twice?
  - If so, which inputs allow this to happen?

Software testing with symbolic execution (Cont.)

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First build SSA Program
by loop unrolling
(now variables do not change)

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Path Formula in SMT

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\neg(m_0 = 0) & \wedge \\
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(y_1 = m_0) & \wedge \\
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- **Note**: SSA denotes static single assignment form

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Solution

- Model:
  $x_0 = 2; y_0 = 4; m_0 = 2;$
  $x_1 = 4; y_1 = 2; m_1 = 0;$

- Function call: GCD(2,4)

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- **Recall:** This testing approach is known as **dynamic symbolic execution**
  - Example tools: CUTE, Klee, DART, SAGE, Pex, Yogi

Remark on SW Verification and Testing

- SW verification & testing one of the main driving forces behind SMT
- Testing: Increasing test coverage
  (which input reaches a given path?)
- Verification: proving correctness, absence of crashes, security, etc.
- Implications for security: prove that your program/protocol is secure

Outline

Example Theories – EUF

- Equality with Uninterpreted Functions (EUF)
- Is this formula satisfiable?

$$[a \times (f(b) + f(c)) = d] \wedge [b \times (f(a) + f(c)) \neq d] \wedge [a = b]$$

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- **Useful**: Abstract non-supported operations (functions)
  – E.g. multiplication; ALUs in circuits; etc.

Example Theories – Arithmetic

- Wide range of applications
- Variables are either integers or reals

- Fragments can be solved with more efficient methods
  - Bounds
    - $x \bowtie k$, $\bowtie \in \{<, >, \leq, \geq, =\}$
  - Difference Logic
    - $x - y \bowtie k$, $\bowtie \in \{<, >, \leq, \geq, =\}$
  - UTVPI (Unit Two-Variable Per Inequality)
    - $\pm x \pm y \bowtie k$, $\bowtie \in \{<, >, \leq, \geq, =\}$
  - Linear Arithmetic
    - $\sum a_i \, x_i \bowtie k$, $\bowtie \in \{<, >, \leq, \geq, =\}$
  - Non-Linear Arithmetic
    - E.g. $3\,x\,y - 4\,x^2\,z - 4\,y \leq 10$

Other Theories

- Bit vectors

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- Quantified theories, e.g. quantified linear integer arithmetic is decidable

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- Lazy approaches
  - Embed SAT solver with theory solver(s) (many SAT calls); theory solvers are responsible for deciding conjunctions of literals

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\begin{gathered}
a = b \rightarrow A = B \\
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\end{gathered}
$$

- Formula with only equality and n constants is satisfiable iff it has a model of size at most n. (small model property). [E.g. Pnueli et al.'02]. Constants encoded as bounded integers $1..n$.

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- equality: $(a_k, \ldots, a_0) = (b_k, \ldots, b_0) \rightsquigarrow \bigwedge_{i \in 0..k} (a_i \Leftrightarrow b_i)$
- inequality: $(a_k, \ldots, a_0) < (b_k, \ldots, b_0) \rightsquigarrow$
  $(\neg a_k \wedge b_k) \vee ((a_k \Leftrightarrow b_k) \wedge (a_{k-1}, \ldots, a_0) < (b_{k-1}, \ldots, b_0))$

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- **Note:** log-encoding smaller but unary tends to give better behavior in SAT solvers.

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- **Why?** For any larger model $\mathcal{A}'$ construct $\mathcal{A}$ by considering only elements used to interpret the variables in the formula.
- $(x_1 = x_2) \vee (x_1 = x_3) \wedge (x_3 \neq x_2)$ is decided by looking for a model up to size 3.

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  - "Compress" any solution while preserving the satisfiability of the same literals.

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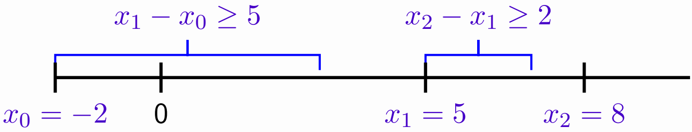

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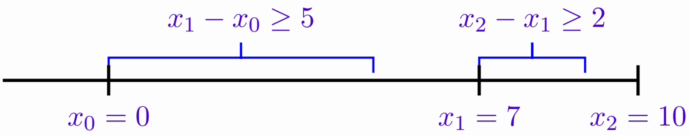

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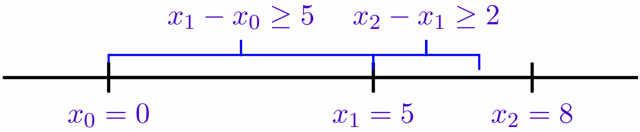

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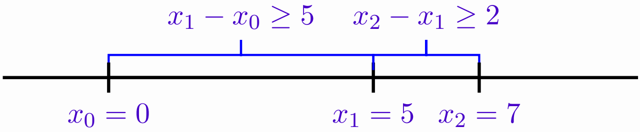

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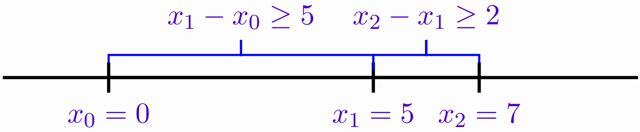

- Max number: $0 + 5 + 2$
- Note that: $5 = 4 + 1$, that is why we add 1 for each variable since original constraints may be negated

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Careful: Need to consider also sub-terms.

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Outline

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  - SAT(α): SAT solver for formula α determines if SAT or UNSAT. If SAT, provides model as a set of true literals.

Lazy approaches – example

- Example SMT formula:

$$g(a) = c \wedge (f(g(a)) \neq f(c) \vee g(a) = d)) \wedge c \neq d$$

- Represent Boolean structure as CNF formula:

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Lazy solving

input : formula ϕ in theory $\mathcal{T}$
output: truth value

$\alpha \leftarrow \mathcal{T}2\mathcal{B}(\phi)$ // abstract input formula
while true **do**
  $(\mathsf{res}, \tau) \leftarrow \mathsf{SAT}(\alpha)$ // Boolean model
  if res = false **then return** false
  $\mathcal{L} \leftarrow \bigcup_{l \in \tau} \mathcal{B}2\mathcal{T}(l)$ // convert to theory model
  $(\mathsf{res}, \mathcal{L}') \leftarrow \mathcal{T}\text{-}\mathsf{SAT}(\mathcal{L})$ // theory check
  if res = true **then return** true
  $\alpha \leftarrow \alpha \wedge \bigvee_{l \in \mathcal{L}'} \neg \mathcal{T}2\mathcal{B}(l)$ // block explanation
end

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  - Repeat until no congruence applies.

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- Build congruence closure as a partitioning of S.
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  - For each $(s = t) \in E$, merge partitions of s and t.
  - For $s_1, \ldots, s_k$ and $t_1, \ldots, t_k$ s.t. s_i is in the same partition as t_i, merge partitions of $f(s_1, \ldots, s_k)$ and $f(t_1, \ldots, t_k)$.
  - Repeat until no congruence applies.
- If there is a $s \neq t \in D$, s.t. s and t are in the same partition, return unsatisfiable otherwise satisfiable.

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  - Add additional source vertex x_0
  - Add edge from x_0 to x_i, for each other vertex x_i
  - Use Bellman-Ford algorithm to check for negative cycles
    - **Negative cycle:** Elimination of variables in (some) inequalities yields $0 \leq -k$, $k > 0$

Integer Difference Logic (Cont.)

$$(x_4 - x_2 \leq 3) \wedge (x_4 - x_3 \leq 6) \wedge (x_1 - x_2 \leq -1) \wedge$$
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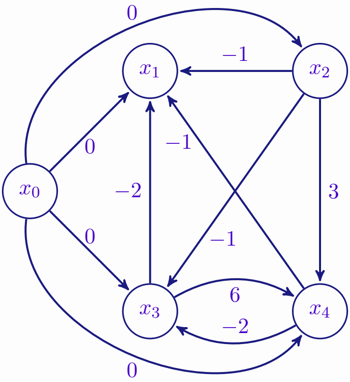

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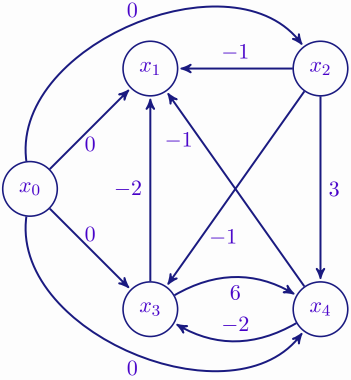

Satisfiable:

$$x_1 = -4$$
$$x_2 = 0$$
$$x_3 = -2$$
$$x_4 = 0$$

$$x_i = \mathsf{dist}(x_0, x_i)$$

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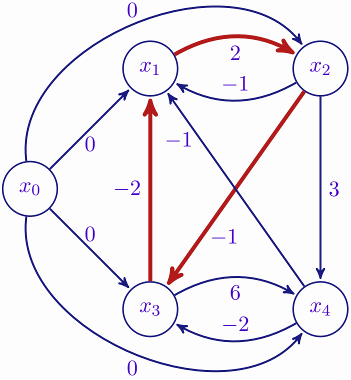

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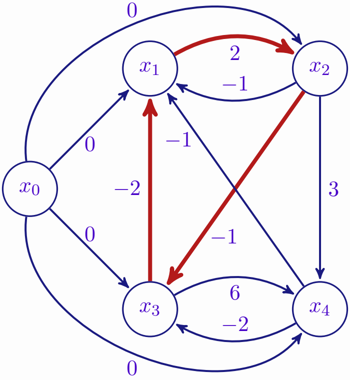

Unsatisfiable

$x_2 - x_1 \leq 2$

$x_3 - x_2 \leq -1$

$x_1 - x_3 \leq -2$

Sum: $0 \leq -1$

Note: *Cycle serves as explanation, UNSAT as long as present.*

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- **Note:** Also possible on reals/rationals but care is needed because we cannot directly convert strict inequalities to non-strict ones.

Recap example for IDL

- Example SMT formula:
$((x_4 - x_2 \leq 3) \vee (x_4 - x_3 \geq 5)) \wedge (x_4 - x_3 \leq 6) \wedge$
$(x_1 - x_2 \leq -1) \wedge (x_1 - x_3 \leq -2) \wedge (x_1 - x_4 \leq -1) \wedge (x_2 - x_1 \leq 2) \wedge$
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- Represent Boolean structure as CNF formula:

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- Interaction between SAT solver & theory solver (IDL):

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UNSAT	$(e) \wedge (g) \wedge (h) \wedge (\neg e \vee \neg g \vee \neg h)$		

Lazy approaches – remarks

- Why are they called lazy?
  - Theory solver called only as needed, to check T-consistency
- Key properties:
  - Avoiding large encodings
  - Modular implementation
    - Easy to add theory solvers
  - Currently, the most efficient algorithms
  - Clear separation between Boolean and theory domains
- Widely used by modern SMT solvers
  - Z3, Yices, OpenSMT, MathSAT, CVC, Barcelogic, etc.

Lazy approaches – Key Techniques

- Key techniques in all efficient SMT solvers:
  - Check T-consistency of partial assignments
  - Given T-inconsistent assignment M, compute $M' \subseteq M$ and add $\neg M'$ as a clause
  - Given T-inconsistent assignment, backtrack to where assignment is T-consistent

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  - **Cannot use**: pure literals, blocked literals, etc.
- T-Solver:
  - Checks T-consistency of conjunctions of literals
  - Performs theory propagation
  - Computes explanations of inconsistency
  - **Note:** T-propagation should be incremental and backtrackable