

Recapitulando:

Eq. laplace. Coord. Esféricas: $\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$

Simetria de rotações em torno do eixo Z: $\rightarrow \frac{\partial V}{\partial \phi} = 0$

Separação de Variáveis: $V(r, \theta) = R(r) \Theta(\theta)$; logo a E.D.P. fica:

$$\underbrace{\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right)}_{l(l+1)} + \underbrace{\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \cdot \frac{d\Theta}{d\theta} \right)}_{-\frac{l(l+1)}{\sin \theta}} = 0$$

$\begin{matrix} \text{const} \\ l(l+1) \end{matrix}$ $\begin{matrix} \text{const} \\ -l(l+1) \end{matrix}$

$\ell = 0, 1, 2, 3, 4, \dots$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Eq. p/ R:

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = l(l+1)R \Rightarrow R(r) = A r^l + \frac{B}{r^{l+1}}$$

↓
C infinitas soluções

Eq. p/ Θ:

$$\frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = -l(l+1) \sin \theta \cdot \Theta; \text{ nova variável } X = \cos \theta$$

$$\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{d}{dx} = -\sin \theta \cdot \frac{d}{dx} \Rightarrow -\sin \theta \frac{d}{dx} \left[\sin \theta \left(-\sin \theta \frac{d\Theta}{dx} \right) \right] = -l(l+1) \sin \theta \cdot \Theta$$

$$\Rightarrow \frac{d}{dx} \left[\left(\sin \theta \right)^2 \frac{d\Theta}{dx} \right] = -l(l+1) \Theta \Rightarrow \frac{d}{dx} \left[(1-x^2) \frac{d\Theta}{dx} \right] = -l(l+1) \Theta \quad \begin{matrix} \text{p/ cada } l \\ x \text{ tem uma nova} \\ \text{equação.} \end{matrix}$$

Solução: $\Theta(x) = P_l(x)$

↳ Polinômio de Legendre

$$\begin{cases} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = \frac{1}{2}(3x^2 - 1) \\ P_3(x) = \frac{1}{2}(5x^3 - 3x) \\ \vdots \end{cases}$$

Fórmula de Rodrigues:

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l$$

Ex: $P_4(x) = ?$; Voltando: $\frac{d}{d\theta} \left(\sin \theta \frac{d\theta}{d\theta} \right) = -l(l+1) \sin \theta \theta$

$\theta(\theta) = P_l(\cos \theta)$ ou seja $\left\{ R(r), \theta(\theta) \right\}$

$1 \quad (l=0)$

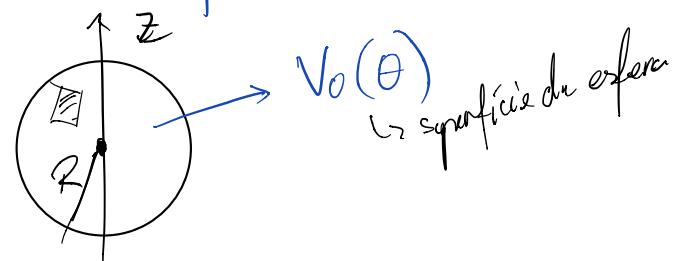
$\cos \theta \quad (l=1)$

$\frac{1}{2} (3[\cos \theta]^2 - 1) \quad (l=2)$

$V(r, \theta) = R(r) \cdot \theta(\theta)$

$V = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) \cdot P_l(\cos \theta)$ solução geral

0 no 0
div no inf
div(0)
0 no inf



Ex: Potencial em uma cavidade esférica

$V(r, \theta)$ dentro da esfera.
o origem está incluída

$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) \cdot P_l(\cos \theta)$ (geral)

diverge na origem \rightarrow é imposto como condição de contorno que $B_l = 0 \forall l$
 \forall que o potencial não diverge na origem.

$\Rightarrow V = \sum_{l=0}^{\infty} A_l r^l \cdot P_l(\cos \theta)$; condição $V(R, \theta) = V_0(\theta) \Rightarrow V_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta)$

$\Rightarrow \sum_{l=0}^{\infty} A_l P_l(\cos \theta) = V_0(\theta) \rightarrow$ Série de Legendre

intervalo $[-1, 1]$

polinômios são ortogonais entre si

$\int_{-1}^1 P_m(x) P_n(x) dx = \delta_{mn} \cdot \frac{2}{2l+1}$ ortogonalidade

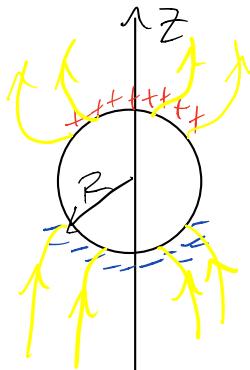
$\int_0^{\pi} P_m(\cos \theta) P_n(\cos \theta) \sin \theta d\theta = \delta_{mn} \cdot \frac{2}{2m+1}$

Ex: mostrar que $A_l = \frac{2l+1}{2} \int_0^{\pi} V_0(\theta) P_l(\cos \theta) \sin \theta d\theta$

Ex: $V_0(\theta) = \cos\theta$ $\alpha_1 = 1$
 $d_m = 0$ p/ m+1

Aula 26/03 :

Esfera condutora em um campo elétrico uniforme:



$$r \gg R$$

$$\vec{E} = E_0 \hat{z}$$

Calcular $V(r, \theta)$

$$V = 0 \text{ na superfície} \quad (r=R)$$

possui simetria em Z

$$V(r, \theta) \neq 0 \quad r > R$$

$$V(r, \theta) \neq 0 \quad r < R$$

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$V_{in}(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \quad r < R \quad B_l = 0, \forall l$$

$$\Rightarrow V_{in}(R, \theta) = \sum_{l=0}^{\infty} \underbrace{A_l R^l}_{A_l} P_l(\cos\theta) = 0$$

$\left. \begin{array}{l} P_l \text{ base; ortogonal completa} \\ A_l = 0, \forall l \Rightarrow A_l R^l = 0 \end{array} \right\} \Rightarrow A_l = 0$

$\boxed{V_{in}(r, \theta) = 0}$

Fórmula esférica: $V = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \quad r > R$

Condições de Contorno:

(a) $V = 0 \quad p/ r = R$

(b) $V \rightarrow -E_0 z = -E_0 r \cos\theta \quad \text{quando } r \gg R$

campo constante

$$V(R, \theta) = 0 = \sum_{l=0}^{\infty} \left(A_l R^l + \frac{B_l}{R^{l+1}} \right) P_l(\cos \theta) = 0 \quad \left\{ \begin{array}{l} \text{Essa soma só pode ser} \\ \text{zero se } A_l = 0, \forall l \end{array} \right.$$

$\underbrace{A_l}_{\alpha_l}$

$$\Rightarrow A_l R^l + \frac{B_l}{R^{l+1}} = 0 \Rightarrow A_l R^l = - \frac{B_l}{R^{l+1}} \Rightarrow \boxed{B_l = - A_l R^{2l+1}}$$

Então temos que: $V(r, \theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos \theta)$

$r \gg R \rightarrow$ se resumi a analisar: $\frac{R^{2l+1}}{r^{l+1}} = \left(\frac{R}{r} \right)^{l+1} \cdot R^l$

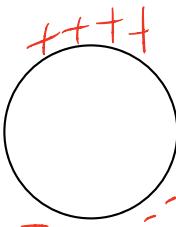
$\xrightarrow{R/r \rightarrow 0} 0 \text{ p/ todo } l$

$$V(r, \theta) \rightarrow \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) = A_0 + A_1 r \cos \theta + \dots = - E_0 r \cos \theta$$

$$\Rightarrow A_0 = 0 ; \quad \underline{A_1 = - E_0} \quad \Rightarrow \boxed{A_{l+1} = 0}$$

Logo: $\left\{ \begin{array}{l} V(r, \theta) = - E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta \\ V(r, \theta) = 0 \quad \text{p/ } r < R \end{array} \right.$

$$= - E_0 z + \frac{E_0 R^3}{r^2} \cos \theta$$

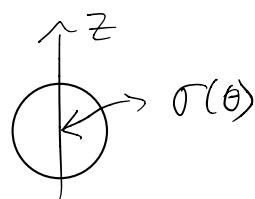


$$\Gamma = -\epsilon_0 \vec{\nabla} V \cdot \hat{n} = -\epsilon_0 \frac{\partial V}{\partial r} \Big|_{r=R}$$

$$\stackrel{\text{ex}}{=} 3E_0 E_0 \cos \theta$$

\hookrightarrow potencial gerado pela distribuição de carga na esfera devido ao campo elétrico.

Exemplo 2:



$$V_{in}(r, \theta) \quad r \leq R$$

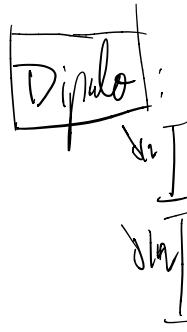
$$V_{out}(r, \theta) \quad r \geq R$$

$$V(r, \theta) = \sum \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

$$\begin{aligned} V_{in} &= \sum (A_\ell r^\ell) P_\ell && \text{Condições de contorno} \\ V_{out} &= \sum \left(\frac{B_\ell}{r^{\ell+1}} \right) P_\ell && V_{in}(R, \theta) = V_{out}(R, \theta) \rightarrow \underline{\text{continuidade do potencial}} \end{aligned}$$



Multipolos:



$$V(\vec{r}) ; \quad V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

$$\vec{r} = \frac{d}{2} \hat{z} + \vec{r}_1 \quad \vec{r}_2 = \frac{d}{2} \hat{z} + \vec{r}$$

$$\begin{cases} \vec{r}_1 = \vec{r} - \frac{d}{2} \hat{z} \\ \vec{r}_2 = \vec{r} + \frac{d}{2} \hat{z} \end{cases} \quad \Rightarrow$$

$$r_1^2 = r^2 - rd\cos\theta + d^2/4 = r^2 \left(1 - \frac{d}{r} \cos\theta + \frac{d^2}{4r^2} \right)$$

$$r_2^2 = r^2 + rd\cos\theta + d^2/4 = r^2 \left(1 + \frac{d}{r} \cos\theta + \frac{d^2}{4r^2} \right)$$

No limite $r \gg d$

$$\begin{aligned} r_1^2 &\approx r^2 \left(1 - \frac{d}{r} \cos\theta \right) \\ r_2^2 &\approx r^2 \left(1 + \frac{d}{r} \cos\theta \right) \end{aligned} \quad \Rightarrow$$

$$\begin{aligned} \frac{1}{r_1} &= \frac{1}{r} \left(1 - \frac{d}{r} \cos\theta \right)^{-1/2} \xrightarrow{\frac{d}{r} \ll r} \frac{1}{r_1} \approx \frac{1}{r} \left(1 + \frac{1}{2} \frac{d}{r} \cos\theta \right) \\ \frac{1}{r_2} &\approx \frac{1}{r} \left(1 - \frac{1}{2} \frac{d}{r} \cos\theta \right) \end{aligned}$$

$$\frac{1}{r_1} - \frac{1}{r_2} \approx \frac{d}{r^2} \cos\theta$$

$$\Rightarrow V(r, \theta) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q \cdot d \cdot \cos\theta}{r^2}$$

Def: $\boxed{\vec{P} = q \cdot d \hat{z}}$

$$\hat{r} = \frac{\vec{r}}{r}$$

$$\Rightarrow V(r, \theta) = \frac{1}{4\pi\epsilon_0} \cdot \frac{\vec{P}_0 \cdot \hat{r}}{r^2}$$

); Vale no limite $d \rightarrow 0$ $\left\{ \begin{array}{l} dq = \text{const} \\ q \rightarrow \infty \end{array} \right.$

monopolo $V \approx \frac{Q}{r}$;

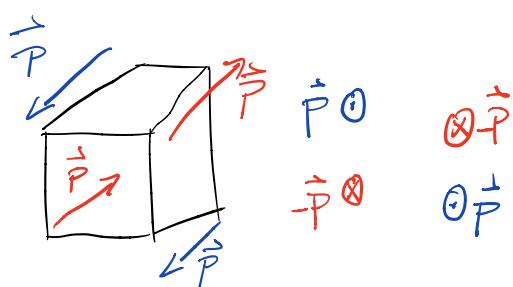
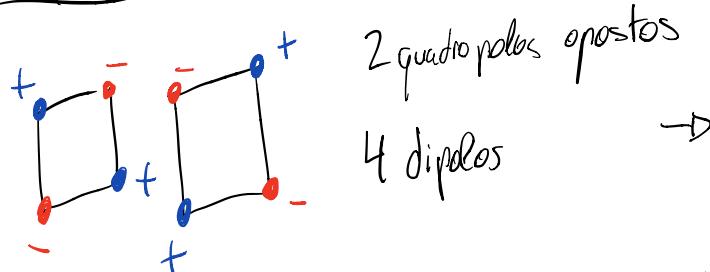
dipolo \vdash : $V \approx \frac{|\vec{P}|}{r^2}$; $Q_{\text{total}} = 0$

Quadrupolo

$\vdash \vec{P} \uparrow \downarrow -\vec{P}; Q_{\text{total}} = 0 \quad \vec{P}_{\text{total}} = 0$ (2 dipolos opostos)

$\begin{matrix} + & & - \\ + & & - \\ - & & + \end{matrix} \quad \begin{matrix} & & \downarrow \\ & & \delta \ll R \end{matrix} \quad V \approx \frac{1}{r^3}$ (4 monopolos)

Octopolo



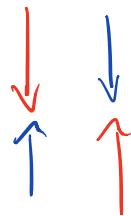
$$V \approx \frac{1}{r^4}$$

→ o número de maneiras de montar os polos dependem das orbitas da mecânica quântica.

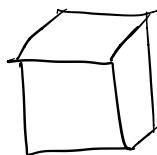
Mais de um tipo de quadrupolo



Mais de um tipo de octopolo:



Hexadecapolo: 2 octopolos opostos



$$V \approx \frac{1}{r^5}$$

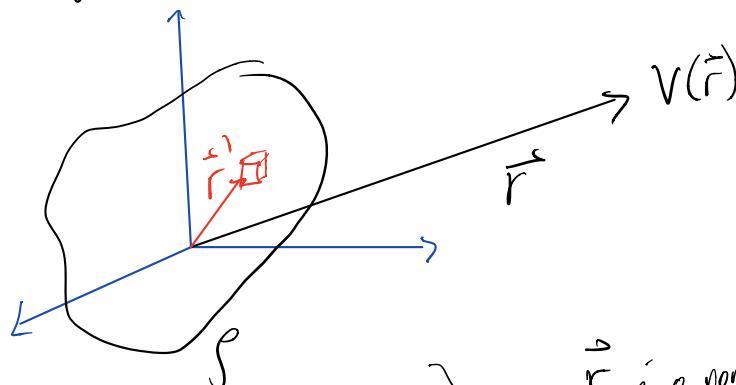
muitas vezes podemos falar em

corpo que em pontos suficientemente distantes o potencial se comporta como o potencial de algum multipolo

O exemplo mais concreto disso é o potencial de algumas moléculas.

4 quadripolos, 6 dipolos, 16 monopolos

Em geral



$$V(\vec{r}) = \alpha_1 \frac{1}{r} + \alpha_2 \frac{1}{r^2} + \alpha_3 \frac{1}{r^3} + \dots$$

\vec{r} é o ponto de interesse para calcular o potencial
 \vec{r}' é um vetor que varre todo o corpo e/ou distribuição de carga.

Expansão em multípolos:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{\|\vec{r} - \vec{r}'\|} d^3x$$

$$\|\vec{r} - \vec{r}'\|^2 = r^2 + (r')^2 - 2rr' \cos\theta = r^2 \left[1 + \underbrace{\left(\frac{r'}{r} \right)^2 - 2 \left(\frac{r'}{r} \right) \cos\theta}_{\epsilon} \right]$$

$\epsilon \rightarrow$ número pequeno

$r \gg r'$

$$\|\vec{r} - \vec{r}'\| = r(1+\epsilon)^{1/2}$$

$$\epsilon = \left(\frac{r'}{r} \right) \left(\frac{r'}{r} - 2 \cos\theta \right)$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} (1 + \epsilon)^{-1/2}$$

\hookrightarrow expansão em série de Taylor

$$f(y + \epsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(y)}{n!} \epsilon^n$$

$$(1 + \epsilon)^{-1/2} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + O(\epsilon^4)$$

$$\epsilon = \underbrace{\left(\frac{r'}{r}\right)}_a \left(\frac{r'}{r} - 2 \underbrace{\cos\theta}_b \right)$$

$$(1 + \epsilon)^{-1/2} = 1 + a \left[\frac{1}{2}(2b - a) + \frac{3}{8}a(a - 2b)^2 + \frac{5}{16}a^2(2b - a)^3 + \dots \right]$$

$$= 1 + ab + a^2 \underbrace{\frac{(3b^2 - 1)}{2}}_{P_1(b)} + a^3 \underbrace{\frac{(5b^3 - 3b)}{2}}_{P_2(b)} + a^4 \underbrace{\dots}_{P_3(b)} \underbrace{1}_{P_4(b)}$$

polinômios de Legendre

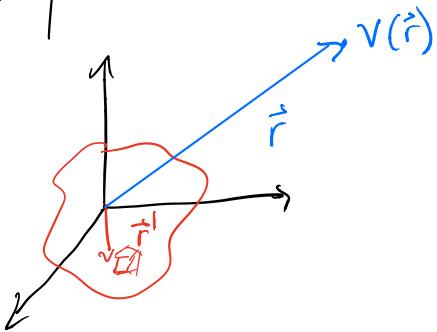
$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta) \rho(r') d^3x$$

$$= \frac{1}{4\pi\epsilon_0} \left[\underbrace{\frac{1}{r} \int \rho d^3x}_{Q_{\text{total}}} + \frac{1}{r^2} \int r' \cos\theta g(r') d^3x + O\left(\frac{1}{r^3}\right) \right]$$

Aula 2 de Abril :

Vimos que:



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta(\vec{r}, \vec{r}')) f(\vec{r}') d\omega'$$

$$= \underbrace{\frac{1}{4\pi\epsilon_0} \frac{1}{r} \int P_0 f(\vec{r}') d\omega'}_{\text{potencial de monopolo}} + \underbrace{\frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int r' P_1(\cos\theta) f(\vec{r}') d\omega'}_{\text{dipolo}} + \underbrace{\frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int r'^2 P_2(\cos\theta) f(\vec{r}') d\omega'}_{\text{quadropolo}} + \dots$$

$$V(\vec{r}) = V_{\text{monop}} + V_{\text{dip}} + V_{\text{quad}} + \dots$$

$$\frac{1}{r} \quad \frac{1}{r^2} \quad \frac{1}{r^3}$$

V_{monop}

$$\int (r')^0 P_0 f(\vec{r}') d\omega' = \int f(\vec{r}') d\omega' = Q \Rightarrow \boxed{V_{\text{monop}} = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r} Q}$$

V_{dip}

$$\int (r')^1 P_1(\cos\theta) f(\vec{r}') d\omega' = \int r' \cos\theta(\vec{r}, \vec{r}') f(\vec{r}') d\omega' \Rightarrow \vec{r}' \cdot \frac{\vec{r}}{r} = r' \cos\theta$$

$$\Rightarrow \int \vec{r}' \cdot \frac{\vec{r}}{r} f(\vec{r}') d\omega' = \underbrace{\frac{\vec{r}}{r} \cdot \int \vec{r}' f(\vec{r}') d\omega'}_{\text{vetor que depende de } f}$$

Definimos então

$$\vec{P} = \int \vec{r}' f(\vec{r}') d\omega' \equiv \text{momento de dipolo}$$

$$\Rightarrow \frac{\vec{r}}{r} \cdot \vec{P} \Rightarrow V_{\text{dip}} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\vec{P} \cdot \vec{r}}{r}$$

$$Q = \int \rho d\omega \quad \vec{P} = \int \vec{r}' \rho d\omega' ; \quad P_x = \int x' \rho d\omega' \\ P_y = \int y' \rho d\omega' \\ P_z = \int z' \rho d\omega'$$

Notação: $x = x_1; x_2 = y; x_3 = z$

$$P_i = \int x'_i \rho(\vec{r}') d\omega' ; \quad i=1,2,3 \rightarrow V_{\text{dip}} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \sum_{i=1}^3 \frac{P_i x_i}{r}$$

Vamos mostrar que

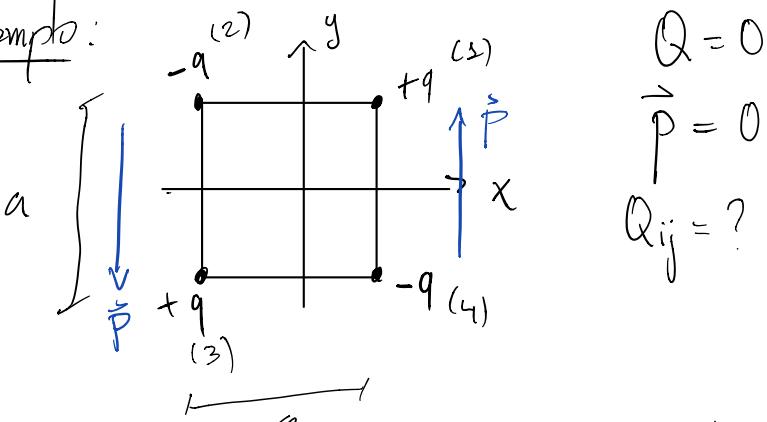
$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r^3} \cdot \frac{1}{2} \sum_i \sum_j \frac{Q_{ij} x^i x^j}{r^2} ; \quad \text{onde} \quad Q_{ij} = \int (3x'_i x'_j - \delta_{ij} r^2) \rho(\vec{r}') d\omega'$$

\nearrow
delta de Kronecker

Octopolo

$$V_8 = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r^4} \left(\frac{1}{2} \sum_{ijk} Q_{ijk} \frac{x_i x_j x_k}{r^3} \right)$$

Exemplo:



$$Q_{ij} = \sum_{\ell=1}^4 \left(3x_i^{(\ell)} x_j^{(\ell)} - r^{(\ell)^2} \delta_{ij} \right) q_\ell \quad \begin{cases} x^1 \cdot x^1 = \left(\frac{a}{2}\right)^2 \\ x^2 \cdot x^2 = \left(-\frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2 \\ x^3 \cdot x^3 = x^4 \cdot x^4 = \left(\frac{a}{2}\right)^2 \\ (r^{(1)})^2 = \left(\frac{\sqrt{2}a}{2}\right)^2 = (r^{(2)})^2 = (r^{(3)})^2 = (r^{(4)})^2 \end{cases}$$

cargas

$$Q_{xx} = \sum_{\ell} \left(3x^\ell x^\ell - (r^{(\ell)})^2 \right) q_\ell$$

$$= 3 \left(\frac{a}{2}\right)^2 - \left(\frac{\sqrt{2}a}{2}\right)^2 \sum_{\ell=1}^4 q_\ell = 0 \quad \text{Analogamente } Q_{yy} = Q_{zz} = Q_{xz} = Q_{yz} = 0,$$

$$Q_{zx} = Q_{zy} \quad \left| \quad Q_{zx} = \sum_{\ell} \left(3z^{(\ell)} x^\ell - (r^{(\ell)})^2 \delta_{ij} \right) q_\ell = 0 \right.$$

$$Q_{xy} = Q_{yx} \neq 0$$

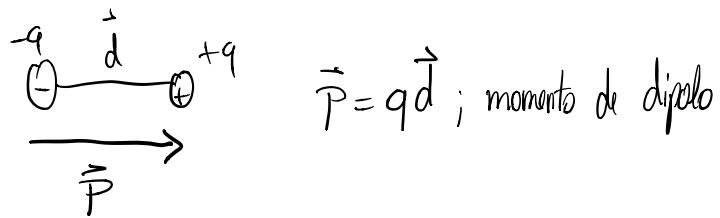
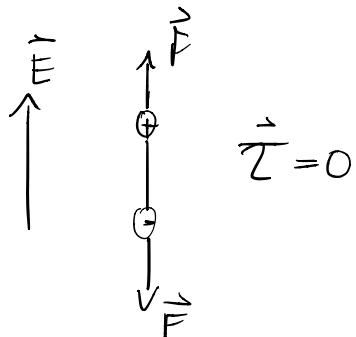
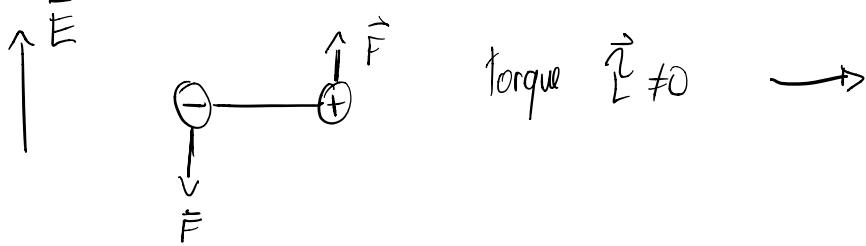
$$Q_{xy} = \sum_{\ell} \left(3x^\ell y^\ell - (r^{(\ell)})^2 \delta_{12} \right) q_\ell = 3 \left[x_1^{(1)} y_1^{(1)} q_1 + x_2^{(2)} y_2^{(2)} q_2 + x_3^{(3)} y_3^{(3)} q_3 + x_4^{(4)} y_4^{(4)} q_4 \right]$$

$$= 3 \left[\left(\frac{1}{2}\right)\left(\frac{a}{2}\right)q + \left(-\frac{a}{2}\right)\left(\frac{a}{2}\right)(-q) + \left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right)q + \left(\frac{a}{2}\right)\left(-\frac{a}{2}\right)(-1) \right] = \underline{3a^2 q = Q_{xy} = Q_{yx}}$$

$$Q_{ij} \rightarrow \begin{pmatrix} 0 & 3a^2q & 0 \\ 3a^2q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

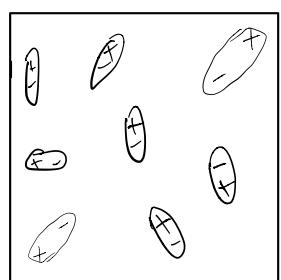
Aula 14 de Abril :

Dielétricos: Dipolo em um campo \vec{E} :



→ O campo elétrico tende a alinhar o momento de dipolo consigo mesmo
(esse alinhamento pode não ser 100% perfeito)

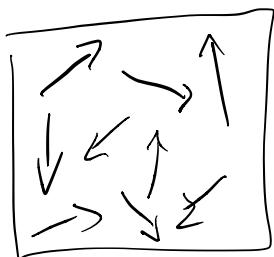
Materiais dielétricos:

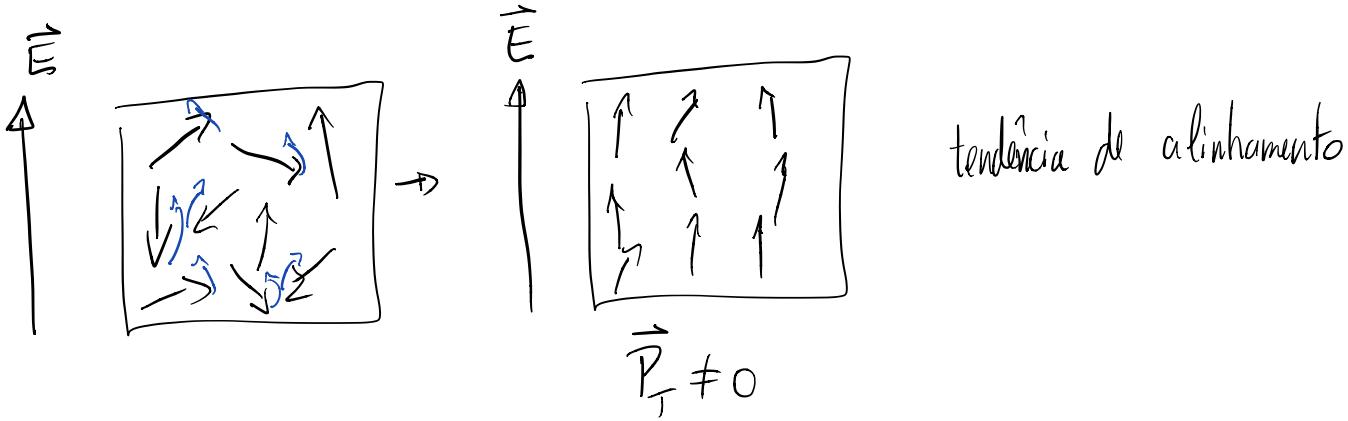


Moleculas polares
aleatoriamente distribuidas

} Coleção de moléculas polares

Diagram showing a rectangular frame containing several molecular dipoles with arrows pointing in various directions. Below the frame is a box containing the equation $\vec{P}_T = 0$.





tendência de alinhamento

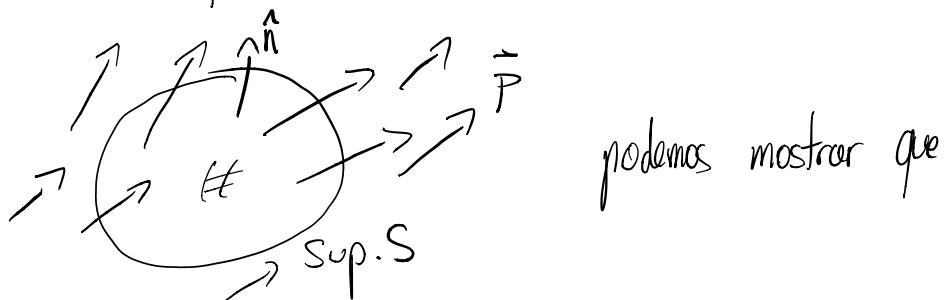
Vetor Polarizacão (Densidade de Polarizacão)

$$\vec{P} := \frac{\vec{P}_f \text{ no volume } \Delta V}{\Delta V}$$

→ Campo vetorial genuino

\vec{P} uniforme → densidade de carga apenas na superfície (σ)

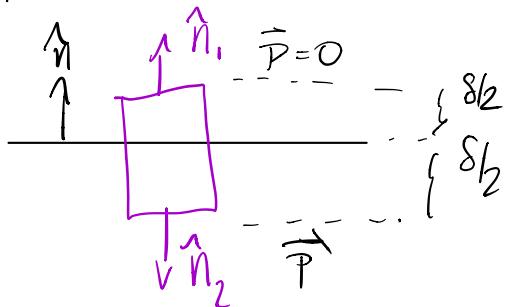
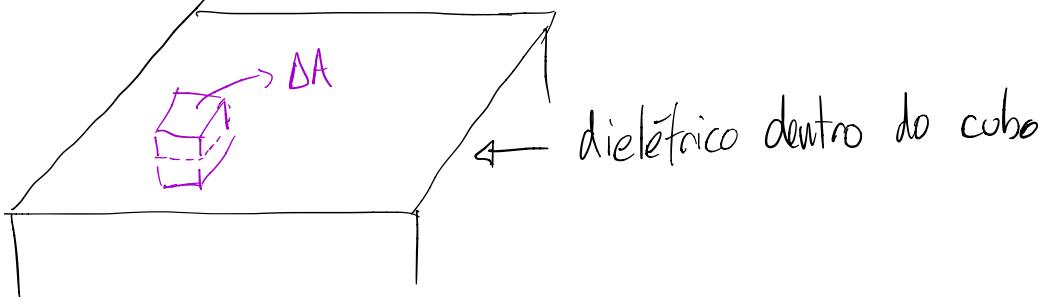
\vec{P} não uniforme → densidade de carga na superfície e no interior (ρ)



$$\oint_S \vec{P} \cdot \hat{n} ds = -Q_p \text{ (dentro de } S\text{)}$$

$Q_p \rightarrow$ Carga de polarização

$$\int \vec{P} \cdot \hat{n} ds = \int_{Vol} \vec{V} \cdot \vec{P} d\sigma = -Q_p = - \int_{Vol} S_p d\sigma \Rightarrow \vec{V} \cdot \vec{P} = -S_p$$



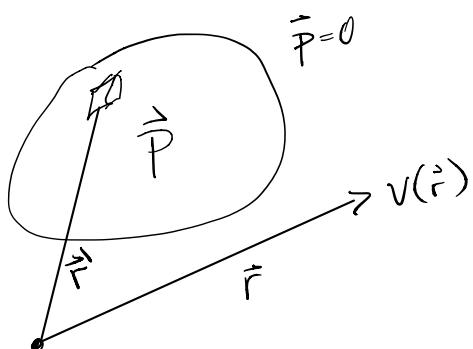
$$\oint \vec{P} \cdot \hat{n} dS = -Q_{\square}$$

$$\lim \delta \rightarrow 0$$

$$\oint \vec{P} \cdot \hat{n} dS = \int_{\text{bottom}} \vec{P} \cdot \hat{n} dS + \int_{\text{top}} \vec{P} \cdot \hat{n} dS + \int_{\text{lateral}} \vec{P} \cdot \hat{n} dS = \cancel{\vec{P} \cdot \hat{n}_1 \Delta A} + \vec{P} \cdot \hat{n}_2 \Delta A = -\vec{P} \cdot \hat{n} \Delta A$$

$\delta \rightarrow 0$

$$Q_{\square} = \sigma_P \cdot \Delta A \quad \rightarrow \quad \boxed{\vec{P} \cdot \hat{n} = \sigma_P}$$



Um volume Δv contribui com $\vec{P} = \vec{P} \Delta v$

$$dV(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{|\vec{r}-\vec{r}'|^2} \frac{(\vec{r}-\vec{r}') \cdot \vec{P} \Delta v}{|\vec{r}-\vec{r}'|}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \cdot \vec{P}(\vec{r}') dV'$$

$$\text{identidades: } \vec{\nabla}^! \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \quad (1)$$

$$\vec{\nabla}^! (f \vec{A}) = \vec{\nabla} f \cdot \vec{A} + f \vec{\nabla} \cdot \vec{A} \quad (2)$$

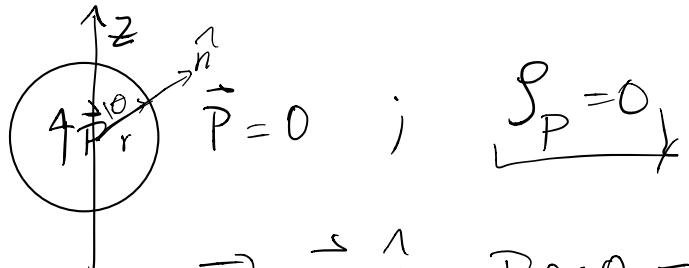
$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \vec{\nabla}^! \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) \cdot \vec{P}(\vec{r}') d\vec{v}'$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\underset{\text{Vol}}{\int} \vec{\nabla}^! \cdot \left(\frac{\vec{P}}{|\vec{r}-\vec{r}'|} \right) d\vec{v}' - \underset{\text{Vol}}{\int} \frac{1}{|\vec{r}-\vec{r}'|} \vec{\nabla}^! \cdot \vec{P} d\vec{v}' \right]$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\underset{\text{Surf}}{\oint} \frac{\vec{P} \cdot \hat{n}}{|\vec{r}-\vec{r}'|} ds + \underset{\text{Vol}}{\int} \frac{1}{|\vec{r}-\vec{r}'|} (-\vec{\nabla}^! \cdot \vec{P}) d\vec{v}' \right]$$

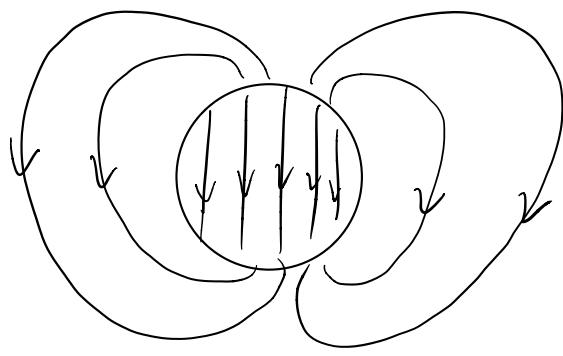
$$\sigma_P = \vec{P} \cdot \hat{n} ; \quad \vec{S}_P = -\vec{\nabla} \cdot \vec{P}$$

Exemplo: Esfera raio R , $\vec{P} = P \hat{z}$, uniforme



$$\rightarrow \vec{P} \cdot \hat{n} = P \cos\theta = \sigma_P$$

$$V(\vec{r}) = \begin{cases} \frac{P}{3\epsilon_0} \vec{r} \cos\theta & r < R \rightarrow \vec{E} = -\frac{P}{3\epsilon_0} \hat{k} \\ \frac{P}{3\epsilon_0} \frac{R^3}{r^2} \cos\theta & r > R \rightarrow \vec{E} \text{ de dipolo} \quad \vec{P} = 4/3 \pi R^3 P \hat{k} \end{cases}$$



Aula 16 de Abril :

$$\oint = \oint_e + \oint_p$$

↓ depende do efeito
 dentro das cargas livres ↓ da polarização

$$\epsilon_0 \cdot \vec{\nabla} \cdot \vec{E} = \oint = \oint_e + \oint_p = \oint_e - \vec{\nabla} \cdot \vec{P} \Rightarrow \epsilon_0 \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P} = \oint_e$$

$$\Rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \oint_e ; \quad \boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$$

vetor de
deslocamento

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{D} = \oint_e}$$

escrever os ds de Maxwell em termo das
cargas livres da qual temos controle

$$\vec{P} = \chi_E \epsilon_0 \vec{E} \rightarrow \text{Dielétricos lineares}$$

↳ Sus. Elétrica ↳ relação linear entre polarização e campo elétrico

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \chi_E \epsilon_0 \vec{E} \rightarrow$$

$\vec{D} = \epsilon_0 (1 + \chi_E) \vec{E}$

χ_E depende
do material

podemos definir ainda $\boxed{E = \epsilon_0 (1 + \chi_E)}$ $\Rightarrow \vec{D} = \epsilon \vec{E}$

$$k = \frac{\epsilon}{\epsilon_0}$$



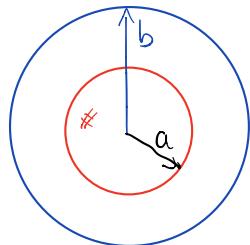
Delimita o que é diferente
o material em questão é
do vácuo

Constante Dieletélica



$$K = 1 + \chi_e$$

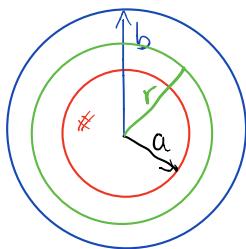
Exemplo:



Carga Q na esfera $\vec{E} = ?$

$$\nabla \cdot \vec{D} = \rho_e \rightarrow$$

$$\oint_S \vec{D} \cdot d\vec{A} = Q_e \text{ (dentro de } S)$$



$$\oint \vec{D} \cdot d\vec{n} ds = D \cdot 4\pi r^2 = Q \Rightarrow \vec{D} = \frac{Q}{4\pi} \cdot \frac{1}{r^2} \hat{r}$$

$$\vec{D} = \epsilon \vec{E} \rightarrow \vec{E} = \frac{Q}{4\pi \epsilon} \cdot \frac{1}{r^2} \hat{r}; a < r < b$$

$$\vec{E} = \frac{Q}{4\pi \epsilon_0} \cdot \frac{1}{r^2} \hat{r} \quad (r > b)$$

↳ não existe polarização do lado
de forma.

$$\left. \begin{array}{l} \vec{P} = \epsilon_0 \chi_e \vec{E} \\ \vec{P} = \frac{\chi_e \epsilon_0 Q}{4\pi} \cdot \frac{1}{r^2} \hat{r} \end{array} \right\}$$

$$\nabla \cdot \vec{P} = -S_p \rightarrow \nabla \cdot \vec{P} = 0; P \sim \frac{1}{r^2} \rightarrow S_p = 0$$

$$\vec{P} \cdot \hat{n} = O_P$$



$$O_P \left\{ \begin{array}{l} \frac{\chi_e \epsilon_0 Q}{4\pi \epsilon} \cdot \frac{1}{b^2} \quad r = b \\ -\frac{\chi_e \epsilon_0 Q}{4\pi \epsilon} \cdot \frac{1}{a^2} \quad r = a \end{array} \right.$$

Eletrostática:

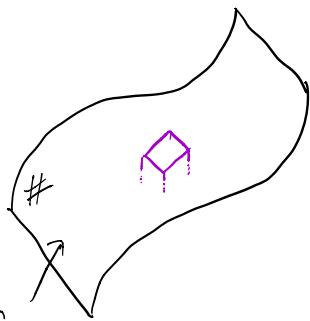
$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho \quad \text{no Vácuo}$$

$$\vec{\nabla} \times \vec{E} = 0$$

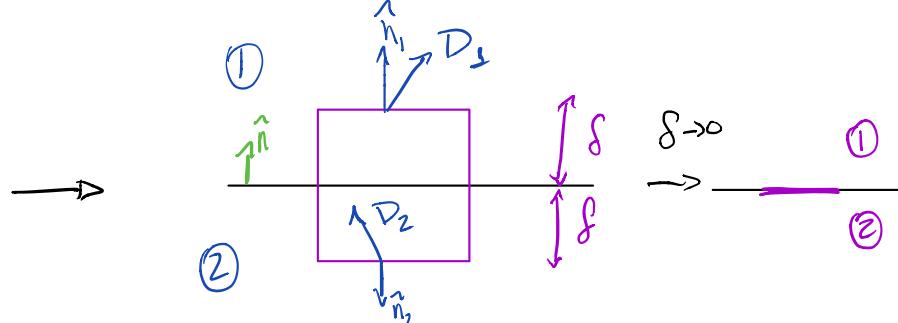
$$\left| \begin{array}{l} \vec{\nabla} \cdot \vec{D} = \rho_e \\ \vec{\nabla} \times \vec{E} = 0 \\ \vec{D} = \epsilon \vec{E} \end{array} \right. \quad \text{dielétrico}$$

↑ inf do material

Condições de Contorno:



interface
que divide
materiais diferentes



$$\int \vec{D} \cdot \vec{n} dA \xrightarrow{\lim \delta \rightarrow 0} \int_{\text{Region 1}} \vec{D}_1 \cdot \vec{n}_1 dA + \int_{\text{Region 2}} \vec{D}_2 \cdot \vec{n}_2 dA + \underbrace{\int_{\text{Lado}} \vec{D} \cdot \vec{n} dA}_{=0}$$

$$= \vec{D}_2 \cdot \vec{n} \Delta A + \vec{D}_2 \cdot (-\vec{n}) \Delta A = \sigma_e \Delta A$$

$$\Rightarrow \boxed{D_2^+ - D_2^- = \sigma_e} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Condições de Contorno}$$

$$\vec{\nabla} \times \vec{E} = 0 \rightarrow \boxed{E_2^+ - E_2^- = 0}$$

$$\vec{\nabla}^2 V = \frac{1}{\epsilon_0} (\rho_e + \rho_p) ; \quad \underline{\text{Supondo que } \rho_e = 0} \quad \text{dentro do material dielétrico}$$

$$\vec{\nabla}^2 V = \frac{1}{\epsilon_0} \rho_p \rightarrow \vec{\nabla}^2 V = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot \vec{P} = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot (\epsilon_0 \chi_E \vec{E})$$

$$\vec{\nabla}^2 V = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot \left(\epsilon_0 \chi_E \vec{E} \cdot \frac{\vec{e}}{E} \right) = -\frac{\epsilon_0 \chi_E}{\epsilon_0 \cdot E} \vec{\nabla} \cdot \vec{D} = -\frac{\chi_E}{E} \vec{\nabla} \cdot \vec{D}$$

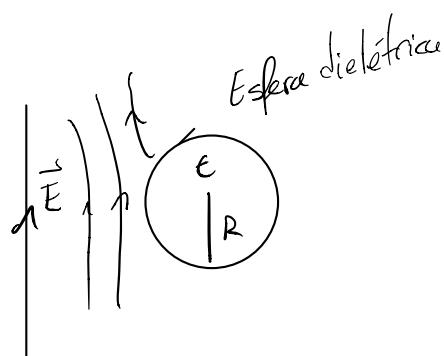
$$\Rightarrow \vec{\nabla}^2 V = -\frac{\chi_E}{E} f_e \rightarrow \boxed{\vec{\nabla}^2 V = 0} \quad \begin{matrix} \rightarrow \text{com novas} \\ \text{condições de contorno} \end{matrix}$$

$\left\{ \begin{array}{l} V_1 = V_2 \\ \epsilon_1 \frac{\partial V}{\partial n} - \epsilon_2 \frac{\partial V}{\partial n} = -\sigma_e \end{array} \right.$
 \rightarrow na interface

Aula 23 de Abril :

Exemplo:

$$\uparrow \uparrow \uparrow \quad \vec{E} = E_0 \hat{z}$$



$$V_{out} = ?$$

$$V_{in} = ?$$

$$\vec{\nabla}^2 V = 0; \quad V_{in}(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{\tilde{B}_l}{r^{l+1}} \right) P_l(\cos\theta)$$

aplicar condições de contorno

$$V_{out}(r, \theta) = \sum_{l=0}^{\infty} \left(\tilde{A}_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

para determinar os coeficientes

$$V_{in} \text{ regular em } r=0 \therefore \underline{\tilde{B}_l = 0}$$

\rightarrow campo uniforme

$$V_{out} \underset{r \gg R}{\approx} \tilde{A}_0 + \tilde{A}_1 r \cos\theta + \tilde{A}_2 r^2 P_2 + \dots \underset{l=1}{\approx} -E_0 z = -E_0 r \cos\theta$$

$$\tilde{A}_0 = 0; \quad \tilde{A}_1 = -E_0; \quad \tilde{A}_n = 0, \forall n \neq 1$$

$$\Rightarrow V_{out} = -E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$V_{in}(R_1\theta) = V_{out}(R_1\theta)$$

$$\sum_{l=0}^{\infty} A_l R^l P_l = -\underbrace{E_0 R \cos\theta}_{-E_0 R P_1} + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+2}} P_l$$

$$\left. \begin{array}{l} A_1 R = -E_0 R + \frac{B_1}{R^2} \quad ; \quad n=1 \\ A_n R^n = \frac{B_n}{R^{n+1}} \quad ; \quad n \neq 1 \end{array} \right\} ; \quad \left. \begin{array}{l} \epsilon \frac{\partial V_{in}}{\partial r} - E_0 \frac{\partial V_{out}}{\partial r} = 0 \\ \Rightarrow \epsilon \frac{\partial V_{in}}{\partial r} = E_0 \frac{\partial V_{out}}{\partial r} \quad ; \quad r=R \end{array} \right\}$$

$$\Rightarrow \boxed{\frac{\epsilon}{\epsilon_0}} \frac{\partial V_{in}}{\partial r} = \frac{\partial V_{out}}{\partial r} ; \quad r=R$$

$\kappa = \text{const. dielectrica}$

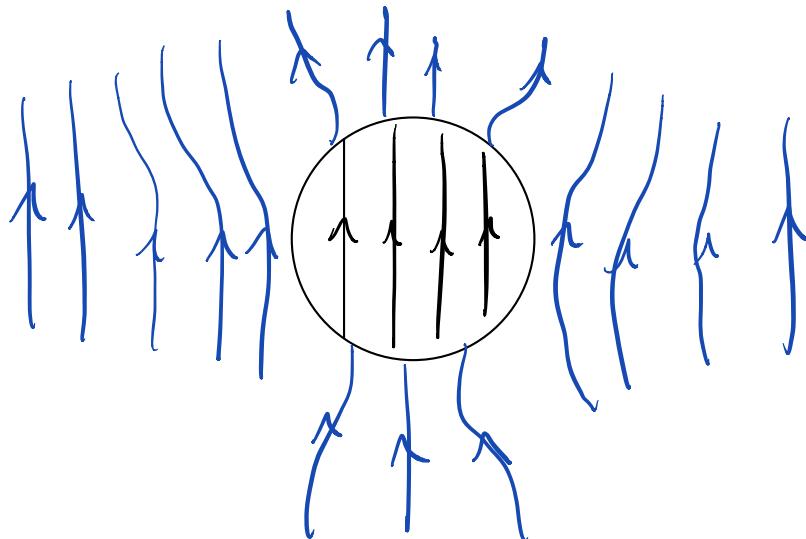
$$\left. \begin{array}{l} K A_1 = -E_0 \sim \frac{2B_1}{R^3} \quad ; \quad l=1 \\ l K A_l R^{l-1} = -\frac{(l+1)}{R^{l+2}} B_l \quad ; \quad l \neq 1 \end{array} \right\} (2) \quad \text{Resolvendo as equações}$$

$$\left. \begin{array}{l} A_n R^n = \frac{B_n}{R^{n+1}} \Rightarrow A_n R^{2n+1} = B_n \\ l K A_l R^{l-1} = -\frac{(l+1) B_l}{R^{l+2}} \Rightarrow n \cancel{K} \underbrace{A_n R^{2n+1}}_{B_n} = -(n+1) B_n \Rightarrow (Kn+n+1) B_n = 0 \end{array} \right\} \Rightarrow \frac{B_n = 0}{n \neq 1} \Rightarrow \underline{A_n = 0} ; \quad n \neq 1$$

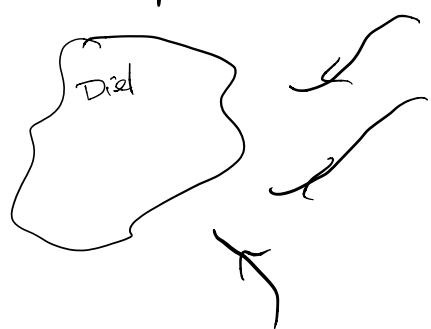
$$\begin{cases} A_1 R - \frac{B_1}{R^2} = -E_0 R \\ KA_1 + 2\frac{B_1}{R^3} = -E_0 \end{cases} \Rightarrow \begin{aligned} A_1 &= -\frac{3}{K+2} E_0 \\ B_1 &= \left(\frac{K-1}{K+2}\right) R^3 E_0 \end{aligned}$$

$$V_{in}(r,\theta) = A_1 r \cos\theta = -\frac{3E_0 z}{K+2}; \quad \vec{E}_{in} = \frac{3E_0}{K+2} \hat{k} \quad \xrightarrow{\text{campo uniforme}}$$

$$V_{out}(r,\theta) = -E_0 z + \frac{B_1}{r^2} \cos\theta = -E_0 z + \left(\frac{K-1}{K+2}\right) R^3 E_0 \cdot \frac{\cos\theta}{r^2} \quad \xrightarrow{\text{campo de dipolo}}$$



Energia do Campo Elétrico:



$$\rho \rightarrow \hat{\rho}_e = \rho_e + \delta\rho_e$$

$$\int W = \int_{\mathbb{R}^3} \delta \hat{\rho}_e V d\sigma$$

$$\vec{\nabla} \cdot \vec{D} = \rho; \quad \tilde{\rho} = \rho + \delta\rho$$

$$\vec{\nabla} \cdot \vec{D} = \hat{\rho} = \underline{\vec{\nabla} \cdot \vec{D} + \delta\rho}$$

$$\tilde{\vec{D}} = \vec{D} + \delta\vec{D} \Rightarrow \tilde{\nabla} \cdot \tilde{\vec{D}} = \vec{\nabla} \cdot \vec{D} + \vec{\nabla} \cdot \delta\vec{D} \Rightarrow \tilde{\vec{P}} = \vec{P} + \vec{\nabla} \cdot \delta\vec{D}$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \delta\vec{D} = \delta\vec{P}}$$

$$\int_W \vec{\nabla} \cdot \delta\vec{D} v \, dv ; \quad \vec{\nabla} \cdot (\delta\vec{D} v) = (\vec{\nabla} \delta\vec{D}) v + \delta\vec{D} \cdot \vec{\nabla} v$$

$$\vec{\nabla} \delta\vec{D} \cdot v = \vec{\nabla} (\delta\vec{D} v) - \delta\vec{D} \cdot \vec{\nabla} v$$

$$\Rightarrow \int_{\mathbb{R}^3} \vec{\nabla} \cdot (\delta\vec{D} v) \, dv + \int_{\mathbb{R}^3} \delta\vec{D} \cdot \vec{E} \, dv$$

$$= \cancel{\oint \delta\vec{D} \cdot n \, d\alpha} + \int \delta\vec{D} \cdot \vec{E} \, dv \Rightarrow \int_W \delta\vec{D} \cdot \vec{E} \, dv$$

Corrente Elétrica :

1D

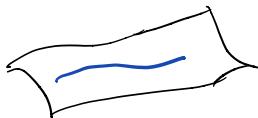
$$\vec{I} = \lambda \vec{v}$$



corrente que passa por um ponto

2D

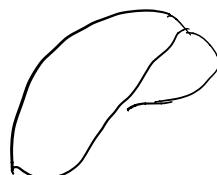
$$\vec{K} = \sigma \vec{v}$$



corrente que passa por uma curva

3D

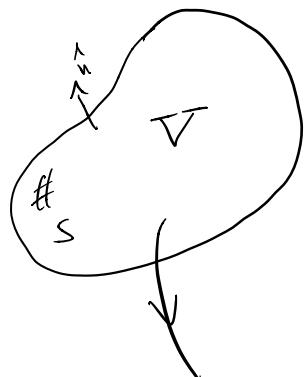
$$\vec{J} = \rho \vec{v}$$



corrente que passa por uma superfície

$$I = \int_{\delta} \vec{K} \cdot d\ell ; \quad I = \iint_S \vec{J} \cdot d\vec{a}$$

Conservação da Carga



$$Q[V] = \text{carga no volume } V$$

A carga nesse corpo só pode variar se entrar carga do lado de fora pr o lado de dentro ou se sair carga do lado de dentro pr lado de fora.
 Se a carga varia sem esse transporte, acarreta na criação/destruição de carga, o que é impossível logo a carga de um corpo só pode variar mediante a uma corrente na superfície

$$\frac{dQ[V]}{dt} = -I[S] \quad \text{as definição} \quad \xrightarrow{\text{(dur nome a algo)}}$$

$$\Rightarrow \frac{d}{dt} \int_S \vec{J} d\sigma = - \oint_S \vec{J} \cdot d\vec{\sigma} = - \int_V \vec{\nabla} \cdot \vec{J} d\tau \Rightarrow \int_V \frac{\partial \vec{J}}{\partial t} d\tau = - \int_V \vec{\nabla} \cdot \vec{J} d\tau$$

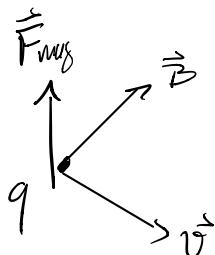
$$\Rightarrow \int_V \left(\frac{\partial \vec{J}}{\partial t} + \vec{\nabla} \cdot \vec{J} \right) d\tau = 0 ; \text{ If volume } V \Rightarrow \frac{\partial P}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \Rightarrow$$

$$\vec{\nabla} \cdot \vec{J} = - \frac{\partial \vec{P}}{\partial t}$$

Equação da Continuidade

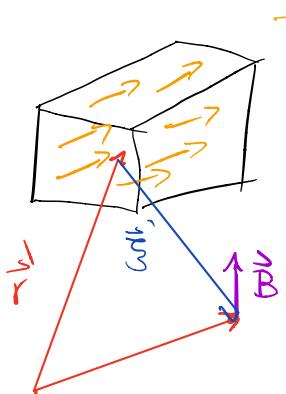
Força Magnética

$$\vec{F}_{\text{mag}} = q \vec{v} \times \vec{B}$$



Fonte do campo magnético: Correntes \vec{J}, \vec{k}, I

Lei de Biot-Savart:



$$\vec{s} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|(\vec{r} - \vec{r}')|^3} d\sigma$$



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K}(\vec{r}') \times (\vec{r} - \vec{r}')}{|(\vec{r} - \vec{r}')|^3} da$$



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\gamma} \frac{\vec{I}(\vec{r}') \times (\vec{r} - \vec{r}')}{|(\vec{r} - \vec{r}')|^3} dl$$

Sabemos que: $\vec{\nabla} \cdot \vec{B} = 0$; $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (magnetostática) \hookrightarrow no ha dependencia temporal

$$\vec{\nabla}_r \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{\nabla}_r \left(\vec{J}(\vec{r}) \times \frac{\vec{\xi}(\vec{r}, \vec{r}')}{\xi^3} \right) d\sigma$$

$$\Rightarrow \vec{\nabla}_r \cdot \left(\vec{J}(\vec{r}) \times \frac{\vec{\xi}(\vec{r}, \vec{r}')}{\xi^3} \right) = \cancel{(\vec{\nabla}_r \times \vec{J}) \frac{\vec{\xi}}{\xi^3}} - \vec{J} \cdot \vec{\nabla}_r \times \left(\frac{\vec{\xi}}{\xi^3} \right)$$

$$\vec{\nabla}_r \times \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \frac{\mu_0}{4\pi} \underbrace{\int \vec{\nabla}_r \times \left(\vec{J}(\vec{r}) \times \frac{\vec{\xi}}{\xi^3} \right) d\sigma}_{=} = \\ &= \vec{J} \vec{\nabla}_r \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) - (\vec{J} \cdot \vec{\nabla}_r) \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \\ &\xrightarrow{=} 0 \end{aligned}$$

$$\vec{\nabla}_r \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \begin{cases} \infty, r = r' & \vec{\xi}(\vec{r}, \vec{r}') \\ 0, r \neq r' & \cancel{\vec{\xi}(\vec{r}, \vec{r}')} \end{cases}$$

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J} \cdot 4\pi \delta(r-r') dv' - \underbrace{(\vec{J} \cdot \vec{\nabla}_r) \cdot \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}}_{=} = 0$$

$$\rightarrow \underline{\mu_0 \vec{J} = \vec{\nabla} \times \vec{B}} /$$

Mostraremos que:

$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \quad \text{para algum } \vec{A}$$

\hookrightarrow Potencial Vetor

$$\vec{B} = \text{dado}; \quad \vec{A}_1; \quad \vec{\nabla} \times \vec{A}_1 = \vec{B} \checkmark$$

$$\vec{A}_2 = \vec{A}_1 + \vec{\nabla} f; \quad f = \text{qualquer}; \quad \vec{\nabla} \times \vec{A}_2 = \vec{B}?$$

\vec{A} não é único

$$\vec{\nabla} \times \vec{A}_2 = \vec{\nabla} \times (\vec{A}_1 + \vec{\nabla} f) = \vec{\nabla} \times \vec{A}_1 + \underbrace{\vec{\nabla} \times \vec{\nabla} f}_{=0} = \vec{\nabla} \times \vec{A}_1 = \vec{B} /$$

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0 \checkmark$$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$$

Suponha que $\vec{\nabla} \cdot \vec{A}_0 \neq 0$; se n^o que $\exists \lambda(x_1, y_1, z)$ tal que $\vec{A} = \vec{A}_0 + \vec{\nabla} \lambda$; $\vec{\nabla} \cdot \vec{A} = 0$?
 $\hookrightarrow \lambda = \lambda(r)$

$$\vec{\nabla} \cdot \vec{A} = 0$$

$$\vec{\nabla} \cdot (\vec{A}_0 + \vec{\nabla} \lambda) = 0 \quad \begin{matrix} \nearrow \text{achar } \lambda \text{ tal que} \\ \text{isso seja verdade} \end{matrix}$$

$$= \vec{\nabla}_0 \vec{A}_0 + \vec{\nabla} \cdot \vec{\nabla} \lambda = 0 \Rightarrow \underbrace{\lambda(\vec{r}) + \vec{\nabla}^2 \lambda = 0}_{\text{dado}} \Rightarrow \boxed{\vec{\nabla}^2 \lambda = -\lambda} \Rightarrow \lambda = \frac{1}{4\pi} \int \frac{\lambda(\vec{r}')}{|\vec{r} - \vec{r}'|} d\omega' \quad \begin{matrix} \uparrow \\ \text{Eq. de Poisson} \end{matrix}$$

$$\Rightarrow \boxed{\vec{A} = \vec{A}_0 + \vec{\nabla} \lambda; \quad \vec{\nabla} \cdot \vec{A} = 0} \quad \begin{matrix} \text{ND} \\ \text{Construção de um novo } \vec{A} \\ \text{que satisfaz } \vec{\nabla} \cdot \vec{A} = 0 \end{matrix}$$

Voltando temos:

$$\underbrace{\vec{\nabla}(\vec{\nabla} \cdot \vec{A})}_{=0} - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$$

$$\vec{A} = \vec{A} + \vec{\nabla} \lambda$$

Sempre existe um \vec{A} tq $\vec{\nabla} \cdot \vec{A} = 0$

$$\Rightarrow \boxed{\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}} \quad \text{na 3eq de Laplace}$$

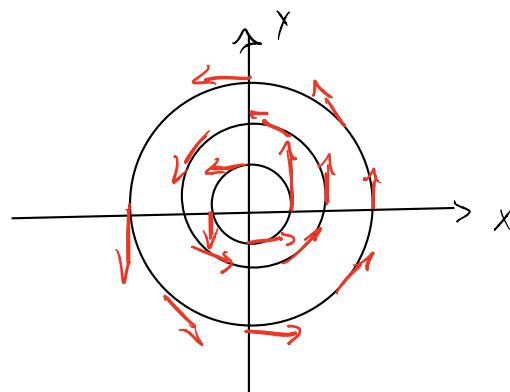
\Rightarrow

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\Omega$$

Se fizermos $\vec{\nabla} \times \vec{A}$ caímos na lei de Biot-Savart

Exemplo: Coord (r, ϕ, z) ; $r = \sqrt{x^2 + y^2}$

$$\vec{J} = \frac{\alpha}{\mu_0} \cdot \frac{1}{r^2} \hat{\phi}$$



Encontre \vec{B}

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\Omega$$

$$\begin{cases} \vec{\nabla} \times \vec{B} = \frac{\alpha}{r^2} \hat{\phi} \\ \vec{\nabla} \cdot \vec{B} = 0 \end{cases} \quad \text{Simétricas}$$

• Translação em z

$$\vec{B} = B_r \hat{r} + B_\phi \hat{\phi} + B_z \hat{z}$$

• Rotação em z

$$\Rightarrow B_r = B_r(r)$$

$$B_\phi = B_\phi(r)$$

$$B_z = B_z(r)$$

\curvearrowright hipótese

$$(\vec{\nabla} \times \vec{B})_r = \frac{1}{r} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} = \mu_0 J_r = 0 \quad (\perp)$$

$$(\vec{\nabla} \times \vec{B})_y = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \mu_0 j_\phi = \frac{\alpha}{r^2} \quad (2)$$

$$(\vec{\nabla} \times \vec{B})_z = \frac{1}{r} \left[\frac{\partial}{\partial r} (r B_\theta) - \frac{\partial B_r}{\partial \phi} \right] = \mu_0 j_z = 0 \quad (3)$$

$$-\frac{\partial B_\theta}{\partial r} = \alpha \frac{1}{r^2} \Rightarrow B_\theta = \frac{\alpha}{r} + C_1 \quad \begin{array}{l} B \rightarrow 0 \text{ quando } r \rightarrow \infty \\ \Rightarrow C_1 = 0 \end{array} \Rightarrow \boxed{B_\theta = \frac{\alpha}{r}}$$

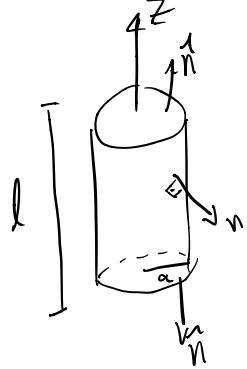
$$\Rightarrow \boxed{B_\phi = \frac{C_2}{r}}$$

$$\begin{aligned} & \vec{\nabla} \cdot \vec{B} = 0 \\ &= \frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{1}{r} \cancel{\frac{\partial B_\theta}{\partial \phi}} + \cancel{\frac{\partial B_\phi}{\partial z}} = 0 \\ & \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r B_r) = 0 \Rightarrow \boxed{B_r = \frac{C_3}{r}} \end{aligned}$$

$$\vec{B} = (B_r, B_\theta, B_\phi)$$

$$= \left(\frac{C_1}{r}, \frac{C_2}{r}, \frac{\alpha}{r} \right); \quad C_2 = ? \quad \text{em } \mathbb{R}^3 - \{0\} \quad \left\{ \begin{array}{l} \text{o que está acontecendo na} \\ \text{origem?} \\ \hookrightarrow \text{analisar forma integral} \end{array} \right.$$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \int_V \vec{\nabla} \cdot \vec{B} = \int_S \vec{B} \cdot \hat{n} d\alpha = 0 \quad \text{VS} \quad \text{S}$$

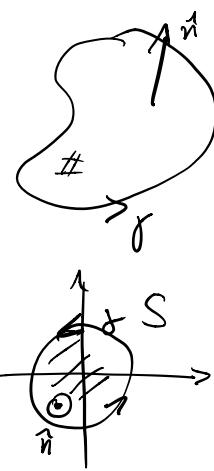


$$\int_S \vec{B} \cdot \hat{n} d\alpha = \int_{\text{lado}} + \int_T + \int_B = \int_{\text{lado}} \frac{C_3}{r} r \cdot \hat{n} + \int_{\text{topo}} \cancel{\frac{\alpha}{r} \hat{z} \cdot \hat{z}} + \int_{\text{base}} \cancel{\frac{\alpha}{r} \hat{z} \cdot (-\hat{z})} = \frac{C_3 \pi r^2}{r} = \frac{C_3 \pi r^2}{r}$$

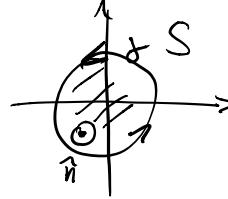
O, N.C cancelam

$$\Rightarrow \int_S \vec{B} \cdot \hat{n} d\alpha = 2\pi C_3 l \Rightarrow \text{isso é nulo apenas se } \boxed{C_3 = 0}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} = \frac{\alpha}{r^2} \hat{r} \quad \rightarrow \quad \int_S (\nabla \times \vec{B}) d\vec{a} = \mu_0 \int_S \vec{j} \cdot \hat{n} dA$$



$$\oint_S \vec{B} \cdot d\vec{l} = \int_S (\nabla \times \vec{B}) d\vec{a} \quad \Rightarrow \quad \oint_S \vec{B} \cdot d\vec{l} = \mu_0 I$$



$$d\vec{l} = \hat{\varphi} dl \quad \oint_S \vec{B} \cdot \hat{\varphi} dl = \int_B B_\varphi dl$$

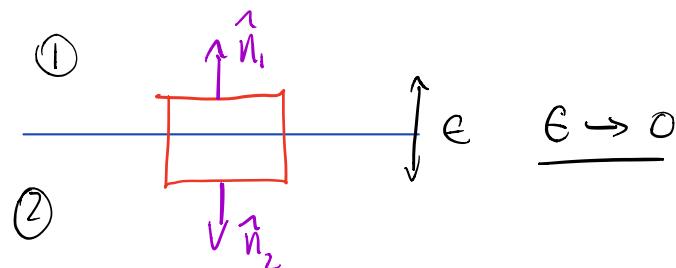
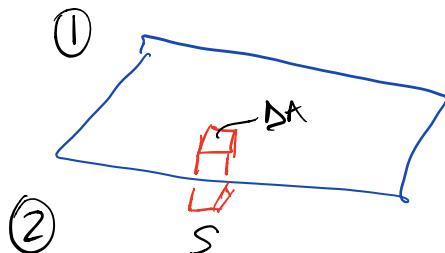
$$= \oint_S \frac{C_2}{a} dl = \frac{C_2}{a} \cdot 2\pi a = 2\pi C_2$$

$$\oint_S \vec{B} \cdot d\vec{l} = 2\pi C_2 \equiv 0 \quad \rightarrow \boxed{C_2 = 0}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\alpha}{r} \hat{z}}$$

Condições de Contorno p/ \vec{B} e \vec{A} :

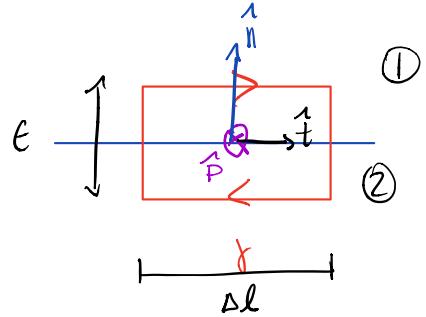
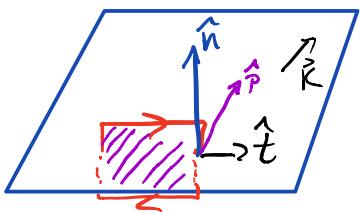
$$\nabla \cdot \vec{B} = 0 ; \quad \oint_S \vec{B} \cdot d\vec{a} = 0$$



$$\oint_S \vec{B} \cdot d\vec{a} = \int_T \vec{B}_1 \cdot \hat{n} dA + \int_B \vec{B}_2 \cdot \hat{n}_2 dA + \int_{\text{lateral}} \vec{B} \cdot \hat{n} dA \xrightarrow{e \rightarrow 0} = (\vec{B}_1 - \vec{B}_2) \cdot \hat{n} \Delta A = 0$$

$$\Rightarrow \boxed{\vec{B}_1^\perp = \vec{B}_2^\perp}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} ; \oint \vec{B} \cdot d\vec{l} = \mu_0 I$$



$$\oint \vec{B} \cdot d\vec{l} = \int_{\gamma} \vec{B}_1 \cdot \vec{t}_1 dl + \int_{\gamma} \vec{B}_2 \cdot \vec{t}_2 dl + \int_{\text{circular}} \vec{B} \cdot d\vec{l} \xrightarrow[c \rightarrow 0]{\text{circular}}$$

$$= (\vec{B}_1 - \vec{B}_2) \cdot \vec{t} \Delta l = \mu_0 \vec{k} \cdot \vec{P} \Delta l$$

$$(\vec{B}_1 - \vec{B}_2) \cdot \vec{t} = \mu_0 \vec{k} \cdot \vec{P}$$

$\vec{t} = \text{queer vetor tangente}$

$$\vec{P} = \hat{n} \times \vec{t} ; \vec{k} \cdot \vec{P} = \vec{k} \cdot (\hat{n} \times \vec{t}) = \vec{t} \cdot (\vec{k} \times \hat{n})$$

$$\Rightarrow \begin{cases} (\vec{B}_1 - \vec{B}_2) \cdot \vec{t} = \mu_0 (\vec{k} \times \hat{n}) \vec{t} \\ (\vec{B}_1 - \vec{B}_2) \cdot \hat{n} = \mu_0 (\vec{k} \times \hat{n}) \hat{n} \end{cases} \xrightarrow{\text{sum}} \boxed{\vec{B}_1 - \vec{B}_2 = \mu_0 \vec{k} \times \hat{n}}$$

Condições de Contorno para \vec{A}

$$\nabla \cdot \vec{A} = 0 \rightarrow \boxed{A_1^\perp = A_2^\perp}$$

P/ os pontos
da interface

$$\nabla \times \vec{A} = \vec{B} \rightarrow \oint \vec{A} \cdot d\vec{l} = \int_S \vec{B} \cdot n ds \rightarrow \boxed{A_1'' = A_2''}$$

$$\Rightarrow \boxed{\vec{A}_1 = \vec{A}_2}$$

Potencial Vetor é contínuo ao longo da superfície

Aproximação de multipolos:



\vec{r}

$$\vec{A}(\vec{r}) = \frac{1}{r} + \frac{1}{r^2} + \dots$$

I

Analizar um loop de corrente de uma distância considerável e analisar qual o potencial vetor gerado.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{e}'}{|\vec{r} - \vec{r}'|} \quad \text{a aproximação entra em } |\vec{r} - \vec{r}'|;$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\theta'}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

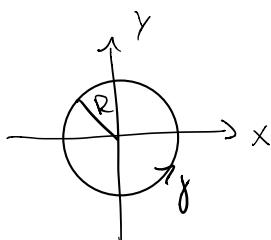
$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint d\vec{e}' + \frac{1}{r^2} \oint r' \cos\theta' d\vec{e}' + \dots \right]$$

$$\vec{d\ell} = (dx, dy, dz) \xrightarrow{\oint} (x_f - x_i; y_f - y_i; z_f - z_i) = \vec{0} \quad (\underline{\text{curva fechada}})$$

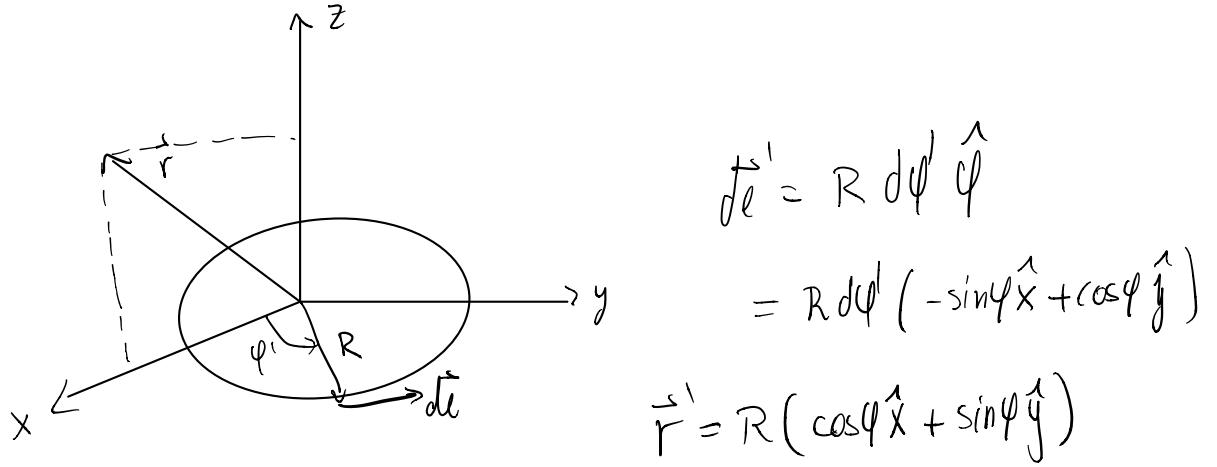
$$\vec{A}_{\text{dip}} = \frac{\mu_0 I}{4\pi} \cdot \frac{1}{r^2} \oint r' \cos\theta' d\vec{e}'$$

$$\oint \left(\vec{r}' \cdot \hat{r} \right) d\vec{e}' = \vec{F}(\hat{r})$$

Exemplo:



Escolher (x_1, y_1, z_1) tal que $\vec{r} = (x_1, 0, z_1)$

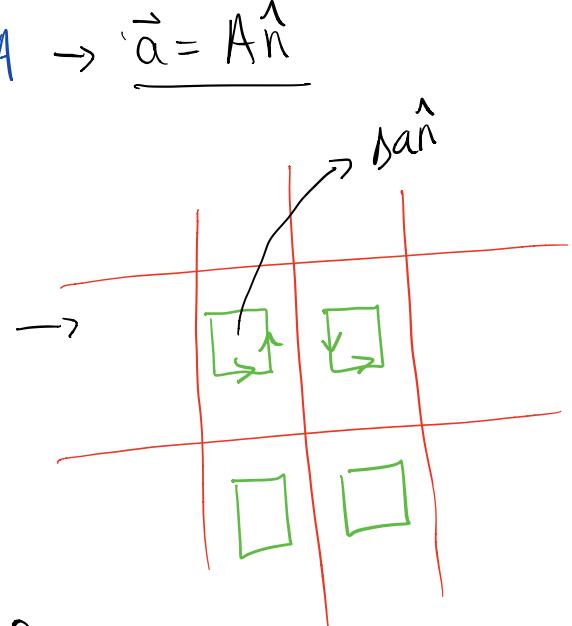
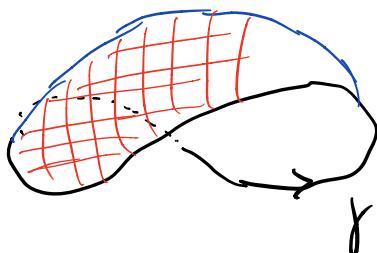
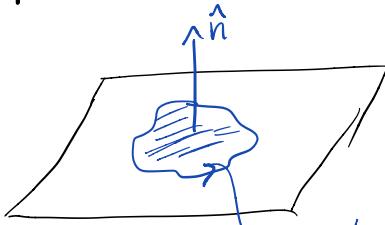


$$\oint (\vec{r} \cdot \vec{r}') \vec{dl} = -\frac{X R^2}{r} \int_0^{2\pi} \underbrace{\sin\varphi \cos\varphi d\varphi}_0 \hat{x} + \frac{X R^2}{r} \int_0^{2\pi} \underbrace{\cos^2\varphi d\varphi}_\pi \hat{y}$$

$$\begin{aligned}&\leq \frac{X}{r} \pi R^2 \hat{y} \\ &= \vec{a} \times \vec{r} = \frac{\vec{a} \times \vec{r}}{r}\end{aligned}$$

$$\begin{aligned}\hat{r} &= \frac{1}{r} (\hat{x} \hat{x} + \hat{z} \hat{z}) \\ \text{inventar} \\ \vec{a} &= \pi R^2 \hat{z} = \text{Area} \cdot \hat{z}\end{aligned}$$

$$\vec{A}_{\text{dip}} = \underbrace{\mu_0 I}_{4\pi} \cdot \frac{\vec{a} \times \hat{r}}{r}$$

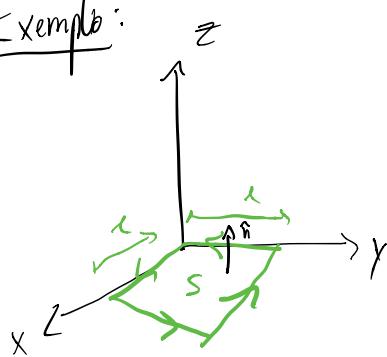


$$\vec{a} = \int_S \hat{n} da ; \quad \oint \hat{n} da = 0$$

→ área vetorial

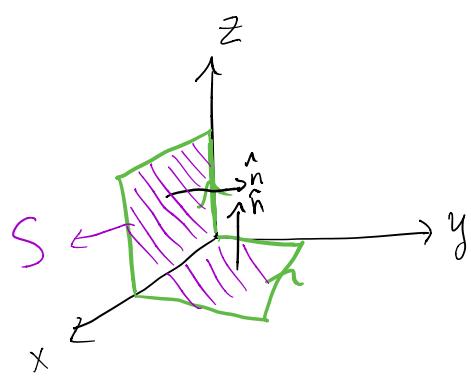
Momento de dipolo: $\vec{m} \equiv I\vec{a}$ $\rightarrow \vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

Exemplo:



$$\vec{A} \approx \frac{\mu_0}{4\pi} \frac{l}{r^2} \vec{m} \times \hat{r}$$

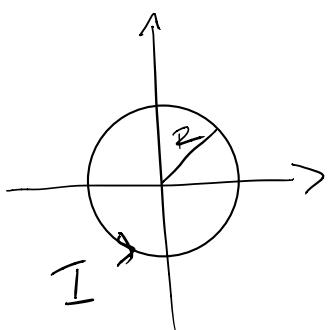
$$\vec{m} = I\vec{a} = I \int_S \vec{n} da = I \hat{z} \int_S da = I l^2 \hat{z} \quad l \ll r$$



$$\vec{a} = \int_S \vec{n} da = \int_{S_1} \vec{n} da + \int_{S_2} \vec{n} da$$

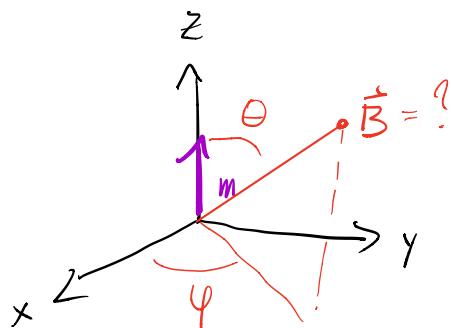
$$\vec{a} = \hat{z} \int_{S_1} da + \hat{y} \int_{S_2} da = l^2 (\hat{z} + \hat{y})$$

$$\vec{m} = I l^2 (\hat{z} + \hat{y})$$



$$\vec{m} = I \pi R^2 \hat{z}$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^3}; \quad \vec{m} = m \hat{z}$$



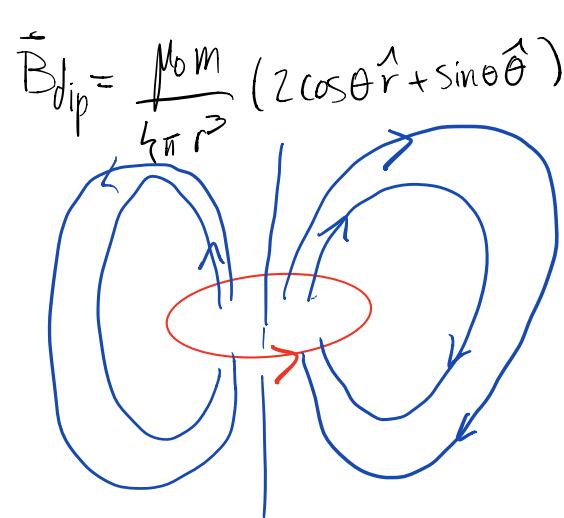
Qual o campo magnético de um dipolo?

$$\vec{B} = \nabla \times \vec{A}$$

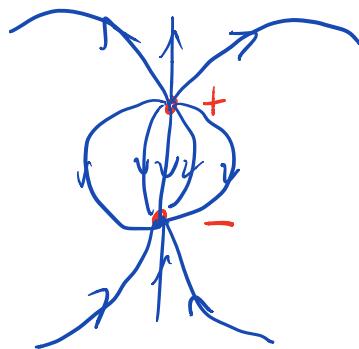
$$\vec{m} \times \vec{r} = (-my, mx, 0) = (-mr\sin\theta\sin\phi, mr\sin\theta\cos\phi, 0)$$

$$= mr\sin\theta(-\sin\phi, \cos\phi, 0) = mr\sin\theta \hat{\vec{q}}$$

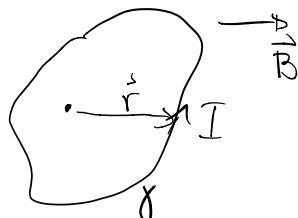
$$\vec{A} = \frac{\mu_0}{4\pi} \frac{m\sin\theta}{r^2} \hat{\vec{\phi}} ; \quad \vec{\nabla} \times \vec{A} = \underbrace{\frac{\mu_0 m}{4\pi r^3} (2\cos\theta \hat{\vec{r}} + \sin\theta \hat{\vec{\theta}})}_{\vec{B}_{\text{dip}}} \quad \text{R analogo ao campo elétrico}$$



$$\vec{E}_{\text{dip}} = \frac{P}{4\pi\epsilon_0} \cdot \frac{1}{r^3} (2\cos\theta \hat{\vec{r}} + \sin\theta \hat{\vec{\theta}})$$



$$\vec{F} = q\vec{v} \times \vec{B}$$



$$d\vec{F} = I d\vec{l} \times \vec{B} = -I \vec{B} \times d\vec{l}$$

$$\vec{F} = -I \oint \vec{B} \times d\vec{l} \quad \text{re } \vec{B} \text{ for uniforme}$$

$$\vec{F} = -IB \times \underbrace{\oint d\vec{l}}_{=0} = 0$$

Torque $d\vec{N} = \vec{I} \vec{r} \times d\vec{F} = \vec{I} \vec{r} \times (\vec{d}l \times \vec{B})$

$$\boxed{\vec{N} = I \oint \vec{r} \times (\vec{d}l \times \vec{B})}$$