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Lie Derivatives and Killing Vectors

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Abstract: Followed along George Matsas Lectures Notes and Appendix C of Wald's book

Contents

1 Smooth functions and Lie Derivatives

Let N and M both be manifolds, and lets consider a diffeomorphism between them $\phi : M \to N$. And consider also an smooth function $f : N \to \mathbb{R}$. Let's discuss some key concepts;

- *Pull back* can be thought as a way of making the function f acting on a element of M , this can be achieved by composition with the diffeomorphism in the following way:
$$f \circ \phi : M \to \mathbb{R} \tag{1.1}$$
- *Push Forward*; in the same way as f, the diffeomorphism maps one point $p \in M$ to another point $\phi(p) \in N$ so naturally it also carries p's tangent vectors. By this we can define a map $\phi^* : V_p \to V_{\phi(p)}$ in which this new tangent vector acts like:
$$(\phi^* v)(f) \equiv v(f \circ \phi) \tag{1.2}$$

Also extending these concepts we might want to pull back a dual vector from $\phi(p) \in N$ to its corespondent dual vector at $p \in M$. We define the map $\phi_* : V^*_{\phi(p)} \to V^*_p$ by requiring that $\forall v^a \in V_p$:

$$(\phi_* \mu)_a v^a = \mu_a (\phi^* v)^a \tag{1.3}$$

And thus one can generalize this process to a tensors of type (k, l) both at $\phi(p)$ and p by noticing since ϕ^* is a diffeomorphism it has and inverse $(\phi^{-1})^*$ which takes vectors from $V_{\phi(p)}$ to V_p and thus by a tensor $T^{b_1 \ldots b_k}{}_{a_1 \ldots a_l}$ we define $(\phi^* T)^{b_1 \ldots b_k}{}_{a_1 \ldots a_l}$ by:

$$T^{b_1 \ldots b_k}{}_{a_1 \ldots a_l} (\phi_* \mu_1)_{b_1} \ldots ([\phi^{-1}]^* t_l)^a_l = (\phi^* T)^{b_1 \ldots b_k}{}_{a_1 \ldots a_l} (\mu_1)_{b_1} \ldots (t_l)^a_l \tag{1.4}$$

If we now have some diffeomorphism $\phi : M \to M$ this means that on our manifold we will have the same geometric properties, and statements affirmed in one frame can be translated fully to the other frame, with that we can compare tensors fields such as T and $\phi^* T$, where T is a tensor field on our manifold M. One important scenario is when ϕ is a symmetry transformation for the tensor field T i.e. $\phi^* T = T$. A very special case is with the metric tensor g_{ab}, if exist some symmetry transformation for ϕ for g_{ab} such that $(\phi^* g_{ab}) = g_{ab}$, ϕ is called a *isometry*. Base on this we can also define some Gauge Freedom on general relativity.

Suppose we have two space-times described by manifold and metric (N, g) and other by (M, g'). If exist an diffeomorphism $\phi : N \to M$ and $g' = \phi^* g$ then the two space-times are said to be indistinguishable, they describe the same physics and any law or statement made in one space-time can be translated to the other, we say this is a *Gauge Freedom* in general relativity.

more details

If we have a parameter in this diffeomorphism this would define tangent vectors v^a, since we can now compare different tensor fields by the action of ϕ, it's possible to analyse the difference between the tensor field and its diffeomorphism counterpart as the parameter goes to zero, this will be exactly the definition of the Lie derivative. The notion of a derivative of a tensor field with respect to some tangent vector v^a.

Definition 1 (Lie Derivative)**.** Let $\phi_t : M \to M$ be a diffeomorphism at one parameter and $v^a = (\frac{\mathrm{d}}{\mathrm{d}t})^a$ its tangent vector. We define the Lie derivative of a tensor field $T \in \mathcal{T}(k,l)$ by:

$$\pounds_v T = \lim_{t \to 0} \frac{\phi^*_{-t} T_{\phi(t)} - T}{t} \tag{1.5}$$

The Lie derivative have some interesting properties:

proofs

- $\pounds_v T \in \mathcal{T}(k,l) \to \mathcal{T}(k,l)$
- $\pounds_v (T \otimes S) = \pounds_v T \otimes S + T \otimes \pounds_v S$
- $\pounds_v (\lambda_1 S + \lambda_2 T) = \lambda_1 \pounds_v S + \lambda_2 \pounds_v T$
- $\pounds_v \Big[C_{ij}(T) \Big] = C_{ij}(\pounds_v T)$
- $\pounds_v f = v(f)$
- $\pounds_v w = [v, w]$

Note also if the Lie derivative is zero everywhere, i.e. $\pounds_v T = 0$ if and only if ϕ_t is a symmetry transformation of T $\forall t$. Analysing the Lie derivative and also remembering the commutator using an derivative operator ∇_b we find that:

$$\pounds_v \mu_a = v^b \nabla_b \mu_a + \mu_b \nabla_a v^b \tag{1.6}$$

And for an arbitrary tensor field ${T^{a_1 \ldots a_k}}_{b_1 \ldots b_l}$:

$$\pounds_v {T^{a_1 \ldots a_k}}_{b_1 \ldots b_l} = v^c \nabla_c {T^{a_1 \ldots a_k}}_{b_1 \ldots b_l} - \sum_{i=1}^{k} {T^{a_1 \ldots c \ldots a_k}}_{b_1 \ldots b_l} \nabla_c v^{a_i} + \sum_{j=1}^{l} {T^{a_1 \ldots a_k}}_{b_1 \ldots c \ldots b_l} \nabla_{b_j} v^c \tag{1.7}$$

2 Killing Vector Fields

In the special case where our diffeomorphism one parameter group is a isometries, i.e. $\phi_t^* g_{ab} = g_{ab}$, the vector field which generate ϕ_t , ξ^a is called a *Killing Vector Field*. Note that if ϕ_t is a group of isometries than the Lie derivative of the metric is zero, so a sufficient condition is $\pounds_\xi g_{ab} = 0$ which lead us to Killing's equation:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{2.1}$$

Where ∇_a is the derivative operator associated with the metric.

Proposition. Let ξ^a be a Killing vector field and let γ be a geodesic with tangent u^a. Then $\xi_a u^a$ is constant along γ

proof

References

[1] Matsas, George, *Notas de Aula Relatividade Geral*

[2] Wald, Robert *General Relativity*,