3 Problem 3

Consider a 1D regular lattice in which all sites are occupied by one individual. Individuals can be in one of two possible states: *healthy* ('○') or *infected* ('●'). The disease is transmitted through contacts between nearest neighbors: infected individuals pass the disease to its healthy neighbors at rate λ/q where q is the lattice coordination number and $\lambda > 0$ is a transmission rate. Infected sites recover at unit rate, and are immediately susceptible to reinfection.

- What are the transition rates for this one-dimensional disease-spreading process?

$$\boxed{\begin{array}{l} ●—○—○ \xrightarrow{\lambda/2} ●—●—○ \\ ○—○—● \longrightarrow ○—●—● \end{array}} \; ; \quad \boxed{●—○—● \xrightarrow[\lambda]{} ●—●—●}$$

$$\boxed{\begin{array}{l} ○—●—○ \longrightarrow ○—○—○ \\ ●—●—○ \longrightarrow ●—○—○ \\ ○—●—● \longrightarrow ○—○—● \\ ●—●—● \underbrace{\longrightarrow}_{1} ●—○—● \end{array}}$$

●—○—●

↳ $P(○ ● ●) = P(\eta_0, \eta_1, \eta_2)$

$$\boxed{○ \to \eta = 0 \,, \quad ● \to \eta = 1}$$

- Derive the master equations for $\rho = P(●, t)$ and $\sigma = P(○●, t)$, which give the density of 'infected' lattice sites and the density of 'healthy/infected' interfaces, respectively.

* Infection by a single neighbor:

2 Possibilities $\left\{ \begin{array}{l} ●—○—○ \\ ○—○—● \end{array} \right.$,

1 infected neighbor: $2 P(○ ● ○) \cdot \frac{\lambda}{2} = \lambda \underline{P(○ ● ○)}_{P(0,1,0)}$

* Infection by two neighbors:

1 possibility: ●—○—●

2 infected neighbors: $1 \cdot P(○ ● ●) \cdot \frac{2\lambda}{2} = \lambda \underline{P(○ ● ●)}_{P(0,1,1)}$

The transition rate from healthy to infected is then:

$$T_{0\to 1} = \lambda P(\circ\bullet\circ) + \lambda P(\circ\bullet\bullet) = \lambda P(0,1,0) + \lambda P(0,1,1).$$

And the transition back to healthy only depends on the focal site being infected:

$$T_{1\to 0} = P(\bullet) = P(1)$$

$$P(\eta_0,\eta_1,\eta_2) = P(\eta_0)P(\eta_1)P(\eta_2),$$

$$P(1) = \rho \Rightarrow P(0) = 1-\rho$$

$$\therefore \boxed{\frac{d\rho}{dt} = \lambda\left[P(0,1,0) + P(0,1,1)\right] - P(1)} \Rightarrow$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} = \lambda\left[\rho(1-\rho)^2 + \rho^2(1-\rho)\right] - \rho}$$

Now, about the density of healthy/infected ($\circ\bullet$) interfaces:

* Infection w/ 1 infected neighbor:

2 possibilities $\left\{ \begin{array}{l} \bullet - \circ - \circ \;\to\; \bullet - \bullet - \circ \\ \circ - \circ - \bullet \;\to\; \circ - \bullet - \bullet \end{array} \right\}$

↳ <u>no variation</u> in the interface number

* Infection w/ 2 infected neighbors:

1 possibility: $\bullet - \circ - \bullet \;\to\; \bullet - \bullet - \bullet$

variation: -2

transition rate: $\frac{2\lambda}{2} = \lambda$

$$\therefore -2\cdot 1\cdot P(\circ\bullet\bullet)\cdot\lambda = \underline{-2\lambda P(\circ\bullet\bullet)}$$

* Recovery beside 1 infected neighbor:

2 possibilities ● — ● — ○ → ● — ○ — ○

○ — ● — ● → ○ — ○ — ●

Variation: 0.

* Recovery between 2 infected neighbors:

1 possibility: ● — ● — ● → ● — ○ — ●

Variation: 2

T.R.: 1

$\therefore\ 2 \cdot 1 \cdot P(\bullet \bullet \bullet) \cdot 1 = 2P(\bullet \bullet \bullet)$

* Recovery between two healthy:

1 possibility: ○ — ● — ○ → ○ — ○ — ○

Variation: -2

T.R.: 1

$\therefore\ -2 \cdot 1 \cdot P(\bullet \circ \circ) \cdot 1 = -2P(\bullet \circ \circ)$

After considering all possible cases, we can write the M.E.:

$$\boxed{\begin{aligned} \frac{d\sigma}{dt} &= -2\lambda P(\circ \bullet \bullet) + 2P(\bullet \bullet \bullet) - 2P(\bullet \circ \circ) = \\ &= -2\lambda P(0,1,1) + 2P(1,1,1) - 2P(1,0,0) \end{aligned}}$$

$$* \; P(\eta_0, \eta_1, \eta_2) = \frac{P(\eta_0, \eta_1) P(\eta_0, \eta_2)}{P(\eta_0)}$$

$$P(1,1) = P(1) - P(1,0) = \rho - \sigma$$

$$\cdot\, P(0,1,1) = \frac{P(0,1)^2}{P(0)} = \frac{\sigma^2}{1-\rho} \; ; \; \cdot\, P(1,1,1) = \frac{P(1,1)^2}{P(1)} = \frac{(\rho-\sigma)^2}{\rho}$$

$$\cdot\ P(1,0,0) = \frac{P(1,0)^2}{P(1)} = \frac{\sigma^2}{\rho}$$

$$\therefore\ \frac{d\sigma}{dt} = -\frac{2\lambda\sigma^2}{1-\rho} + \frac{2(\rho-\sigma)^2}{\rho} - \frac{2\sigma^2}{\rho} = -\frac{2\lambda\sigma^2}{1-\rho} + 2\,\frac{\rho^2-2\rho\sigma}{\rho} \Rightarrow$$

$$\Rightarrow \boxed{\frac{d\sigma}{dt} = 2\rho - 4\sigma - 2\,\frac{\lambda\sigma^2}{1-\rho}}$$

- Apply a simple mean field approximation and obtain the steady-state density of infected sites as a function of the control parameter λ.

$$\frac{d\rho}{dt} = \lambda\left[P(0,1,0) + P(0,1,1)\right] - P(1) = \lambda\left[\rho(1-\rho)^2 + \rho^2(1-\rho)\right] - \rho = F(\rho)$$

$$\frac{d\rho}{dt} = 0 \Rightarrow \lambda\left[\rho(1+\rho^2-2\rho) + \rho^2 - \rho^3\right] - \rho = \lambda\left[\rho + \cancel{\rho^3} - 2\rho^2 + \rho^2 - \cancel{\rho^3}\right] - \rho =$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} = (\lambda-1)\rho - \lambda\rho^2.}$$

We know the solution for this equation:

$$\boxed{\rho(t) = \frac{(\lambda-1)\rho(0)\,e^{(\lambda-1)t}}{\lambda-1-\lambda\rho(0)\left(1-e^{(\lambda-1)t}\right)}}\ ,\quad \text{With steady states:}$$

$$\boxed{\cdot \lim_{t\to\infty} \rho(t) = \begin{cases} 0, & \lambda < 1 \\ \dfrac{\lambda-1}{\lambda}, & \lambda > 1 \end{cases}}\ .$$

$$\text{If } \lambda = 1: \frac{d\rho}{dt} = -\lambda\rho^2 \Rightarrow \lambda t = \int \frac{d\rho}{-\rho^2} = \frac{1}{\rho} + C \Rightarrow \rho(t) = \frac{1}{\lambda t - C} \xrightarrow[t\to\infty]{} \boxed{0}$$

$$C = \frac{-1}{\rho(0)}$$

- Apply a pair-correlation approximation and obtain, again, the steady-state density of infected sites and healthy-infected interfaces.

$$\frac{d\rho}{dt} = \lambda\left[P(0,1,0) + P(0,1,1)\right] - P(1)$$

$$\cdot P(1) = \rho; \quad \cdot P(0,1,0) = \frac{P(0,1)P(0,0)}{P(0)} = \frac{\sigma\left(P(0)-P(0,1)\right)}{1-\rho} = \frac{\sigma}{1-\rho}\left(1-\rho-\sigma\right) =$$

$$= \sigma\left(1 - \frac{\sigma}{1-\rho}\right)$$

$$\cdot P(0,1,1) = \frac{P(0,1)^2}{P(0)} = \frac{\sigma^2}{1-\rho}$$

$$\therefore \frac{d\rho}{dt} = \lambda\left[\sigma\left(1 - \cancel{\frac{\sigma}{1-\rho}}\right) + \cancel{\frac{\sigma^2}{1-\rho}}\right] - \rho$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} = \lambda\sigma - \rho} \; ; \; \boxed{\frac{d\sigma}{dt} = 2\rho - 4\sigma - 2\frac{\lambda\sigma^2}{1-\rho}} .$$

$$\frac{d\rho}{dt} = 0 \Rightarrow \rho = \lambda\sigma; \qquad \sigma = \frac{\rho}{\lambda}$$

$$\frac{d\sigma}{dt} = 0 \Rightarrow 2\rho - 4\sigma - 2\frac{\lambda\sigma^2}{1-\rho} = 0 \Rightarrow \rho(1-\rho) - 2\sigma(1-\rho) - \lambda\sigma^2 = 0 \Rightarrow$$

$$\Rightarrow \rho(1-\rho) - \frac{2}{\lambda}\rho(1-\rho) - \lambda\frac{\rho^2}{\lambda^2} = 0 \Rightarrow \rho - \rho^2 - \frac{2}{\lambda}(\rho-\rho^2) - \frac{\rho^2}{\lambda} = 0 \Rightarrow$$

$$\Rightarrow \rho\left(1-\frac{2}{\lambda}\right) - \rho^2\left(1-\frac{1}{\lambda}\right) = 0 \Rightarrow \boxed{\rho = 0 \Rightarrow \sigma = 0}, \text{ or}$$

$$\text{or } \boxed{\rho = \frac{\lambda-2}{\lambda-1} \Rightarrow \sigma = \frac{\lambda-2}{\lambda^2-\lambda}} .$$

* $\rho = \frac{\lambda-2}{\lambda-1} > 0 \iff \lambda > 2$ or $\lambda < 1$.

* $\rho = \frac{\lambda-2}{\lambda-1} \leqslant 1 \iff$ if $\lambda > 1$: $\lambda-2 \leqslant \lambda-1 \rightarrow \rho \leqslant 1 \;\forall \lambda > 1$.

if $\lambda < 1$: $\lambda-2 \geq \lambda-1 \rightarrow \rho$ is not $\leqslant 1$.

$\therefore 0 < \rho = \frac{\lambda-2}{\lambda-1} \leqslant 1 \iff \lambda > 2$.

* If $\lambda = 1$, it follows directly that our only fixed point is $\rho = \sigma = 0$.

Let's check the stability of the fixed points, starting by the trivial one.

$$\frac{d\rho}{dt} = f(\rho, \sigma), \quad \frac{d\sigma}{dt} = g(\rho, \sigma)$$

$$\Rightarrow \frac{\partial F}{\partial \rho} = -1; \quad \frac{\partial F}{\partial \sigma} = \lambda;$$

$$\frac{\partial g}{\partial \rho} = 2 - \frac{2\lambda\sigma^2}{(1-\rho)^2}; \quad \frac{\partial g}{\partial \sigma} = -4 - \frac{4\lambda\sigma}{1-\rho}$$

$$\therefore M = \begin{pmatrix} -1 & \lambda \\ 2-\frac{2\lambda\sigma^2}{(1-\rho)^2} & -4-\frac{4\lambda\sigma}{1-\rho} \end{pmatrix}$$

* For $\rho = \sigma = 0$: $M = \begin{pmatrix} -1 & \lambda \\ 2 & -4 \end{pmatrix} \Rightarrow \det M = 4 - 2\lambda$,

$\mathrm{Tr} M = -5$

$\therefore$ $\underline{\rho = \sigma = 0 \text{ is stable if } \det M = 4 - 2\lambda > 0 \Rightarrow \lambda < 2.}$

$*$ For $\sigma = \rho/\lambda$ and $\rho = \dfrac{\lambda - 2}{\lambda - 1}$:

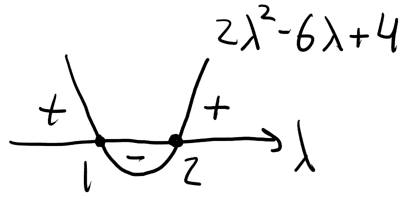
$$\frac{\sigma}{1-\rho} = \frac{1}{\lambda}\,\frac{\rho}{1-\rho} = \frac{1}{\lambda}\cdot\frac{\frac{\lambda-2}{\lambda-1}}{1-\frac{\lambda-2}{\lambda-1}} = \frac{1}{\lambda}\,\frac{\lambda-2}{\lambda-1-(\lambda-2)} = \frac{1}{\lambda}\,\frac{\lambda-2}{1} = \frac{\lambda-2}{\lambda}$$

$$\therefore M = \begin{pmatrix} -1 & \lambda \\ 2 - 2\frac{(\lambda-2)^2}{\lambda} & -4-4(\lambda-2) \end{pmatrix} \Rightarrow \mathrm{Tr}\,M = -1-4-4(\lambda-2) = -5-4\lambda+8 =$$
$$= 3 - 4\lambda < 0 \Leftrightarrow \underline{\lambda > 3/4}$$

$\cdot\ \det M = 4 + 4(\lambda-2) - 2\lambda + 2(\lambda-2)^2 = 4 + 4\lambda - \cancel{8} - 2\lambda + 2\lambda^2 - 8\lambda + \cancel{8} =$

$= 2\lambda^2 - 6\lambda + 4 > 0$?

$2\lambda^2 - 6\lambda + 4 = 0 \Rightarrow \lambda = 1$ or $\lambda = 2$ $\Rightarrow$



$\therefore$ $\det M > 0$ if $\lambda < 1$ or $\lambda > 2$

$$\Rightarrow \boxed{\underbrace{\rho = \frac{\lambda-2}{\lambda-1} \text{ and } \sigma = \rho/\lambda}_{\det M > 0 \text{ and } \mathrm{Tr}\,M < 0} \text{ is stable if } \lambda > 2.}$$