Continuous Facility Location Problems

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Introduction

Single-facility Location Problems

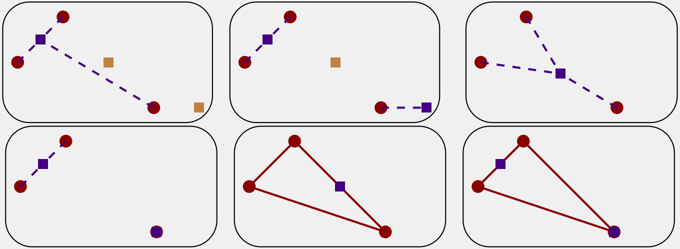
Multi-facility Location Problems

New Approaches

An Extension: Location under Refraction

Location Theory

Given a set of demand points, the goal of classical location problems is to find one or several points for placing new facilities such that they optimize one or several possibly constrained objective functions.



Is LT interesting for the society?

- An expanding market: It will require the addition of more capacity at a certain geographic point, either in an existent facility or in a new one.
- Introduction of new products or services.
- A contracting demand, or changes in the location of the demand: It may require the shut down and/or relocation of operations.
- Obsolescence of a manufacturing facility due to the appearance of new technologies. It means the creation of a new modern plant somewhere else.
- The pressure of the competence. To increase the level of service, it can force the company to increase capacity of certain plants or relocate some of them.
- Change in other resources, like labor conditions or subcontracted components, or change in the political or economic environment in a certain region.
- Mergers and acquisitions. Some facilities may appear as redundants, or bad located with respect to others.

What can/need to be considered in LT?

- Proximity to Customers
- Business Climate
- Total Costs
- Infraestructure
- Quality of Labor
- Suppliers
- Other Facilities
- Political Risks
- Government Barriers
- Trading Blocks
- Environmental Regulation
- Host Community
- Competitive Advantage

Discrete vs. Continuous Location

	DISCRETE	CONTINUOUS
Facilities	To be chosen from an specified finite set	To be chosen from a continuous space.
Costs/Dist	Given	Part of the decision problem.
Forbidden Regions	Filter Potential Facilites (Preprocess)	To be modeled.
Input Data	Matrix of distances	Coordinates of demand points.

Mathematical Programming Framework

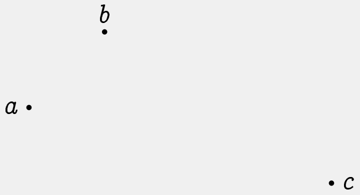
We are given:

- A set of demand points (clients) $\mathcal{A} = \{a_1, \ldots, a_n\}$.
- The number of facilities to be located: p.
- A potential set of facilities $X (= X_1 \times \cdots \times X_p)$.
- A function that measures the cost of locating any set of $(x_1, \ldots, x_p) \in X$: $f_{\mathcal{A}}(x_1, \ldots, x_p)$.

$$\min_{(x_1, \ldots, x_p) \in X} \quad f_{\mathcal{A}}(x_1, \ldots, x_p)$$

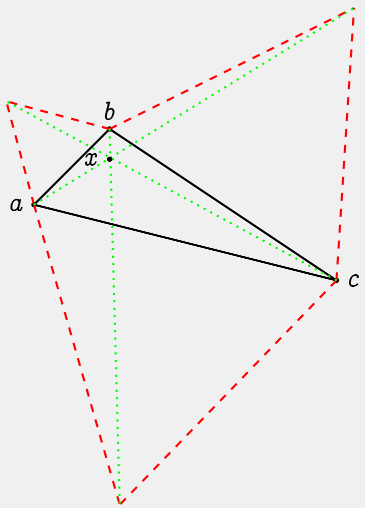
$X \subset \mathbb{R}^d$ (CONTINUOUS)

Weber Problem: The Torricelli Point



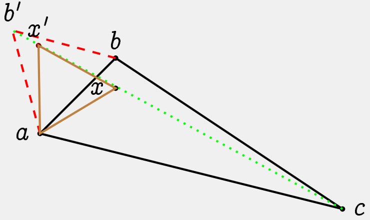
Pierre Fermat (1601-1665): $\min_{x \in \mathbb{R}^2} \|x - a\|_2 + \|x - b\|_2 + \|x - c\|_2$

Weber Problem: The Torricelli Point



Pierre Fermat (1601-1665): $\min_{x \in \mathbb{R}^2} \|x-a\|_2 + \|x-b\|_2 + \|x-c\|_2$

Weber Problem: The Torricelli Point



$$\|x-a\| + \|x-b\| + \|x-c\| = \|x'-x\| + \|x'-b'\| + \|x-c\|$$

x is the unique point which x' is in the *shortest path* (straight line) from b' to c!!! The same for all vertices!

Weber Problem (Simpson, 1705)

Given:

- A set of demand points $\{a_1, \dots, a_n\} \subseteq \mathbb{R}^d$.
- a norm $\|\cdot\|$ in $\mathbb{R}^d$.

$$\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} \|x - a_i\|$$

Weighted Weber problem:

$$\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} w_i \|x - a_i\|$$

for some weights $w_1, \dots, w_n$.

The ℓ_1-norm case

$$\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} w_i \|x - a_i\|_1 = \sum_{i=1}^{n} \sum_{k=1}^{d} w_i \, |x_k - a_{ik}|$$

Linear Programming:

$$\begin{aligned}
\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} w_i \sum_{k=1}^{d} z_{ik} & \\
s.t. z_{ik} &\geq x_k - a_{ik}, \forall i = 1, \ldots, n, k = 1, \ldots, d, \\
z_{ik} &\geq -x_k + a_{ik}, \forall i = 1, \ldots, n, k = 1, \ldots, d, \\
z_{ik} &\geq 0.
\end{aligned}$$

The ℓ_2-norm case: $\mathbb{R}^2$

$$\min_{(x,y)\in\mathbb{R}^2} \quad f(x,y) = \sum_{i=1}^{n} w_i \sqrt{(x-a_i)^2 + (y-b_i)^2}$$

- $\frac{\partial f}{\partial x} = \sum_{i=1}^{n} \frac{w_i(x-a_i)}{\sqrt{(x-a_i)^2+(y-b_i)^2}} = 0.$
- $\frac{\partial f}{\partial y} = \sum_{i=1}^{n} \frac{w_i(y-b_i)}{\sqrt{(x-a_i)^2+(y-b_i)^2}} = 0.$

$$x = \frac{\sum_i \frac{w_i a_i}{\|(x-a_i, y-b_i)\|_2}}{\sum_i \frac{w_i}{\|(x-a_i, y-b_i)\|_2}} \qquad y = \frac{\sum_i \frac{w_i b_i}{\|(x-a_i, y-b_i)\|_2}}{\sum_i \frac{w_i}{\|(x-a_i, y-b_i)\|_2}}$$

Weiszfeld's Algorithm (1937)

Let $x_0 \in \mathbb{R}^d$ an initial feasible solution.

while $x_{k+1} \neq x_k$ **do**

$$x_{k+1} \leftarrow T(x_k) = \begin{cases} \dfrac{\displaystyle\sum_{i=1}^{n} \frac{w_i a_i}{||x_k - a_i||}}{\displaystyle\sum_{i=1}^{n} \frac{w_i}{||x_k - a_i||}} & \text{if } x_k \neq a_i, \forall i, \\ a_i & \text{if } x_k = a_i. \end{cases}$$

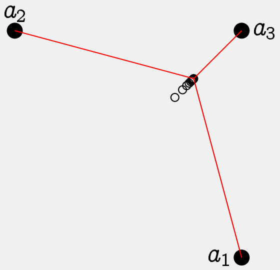
end

Example

$a_1 = (1,0),\ a_2 = (0,1),\ a_3 = (1,1).$
$w_i = 1,\ i = 1,2,3.$

1. $x_0 = (0,0) \rightarrow f(x_1) = 1.984069.$
2. $x_1 = \dfrac{\dfrac{(1,0)}{||(1,0)||} + \dfrac{(0,1)}{||(0,1)||} + \dfrac{(1,1)}{||(1,1)||}}{\dfrac{1}{||(1,0)||} + \dfrac{1}{||(0,1)||} + \dfrac{1}{||(1,1)||}} = (0.6306019374, 0.6306019374) \rightarrow$
   $f(x_2) = 1.945387.$
3. $x_2 = (0.7057929076, 0.7057929076) \rightarrow f(x_2) = 1.936498.$
4. $x_3 = (0.7394342516, 0.7394342516) \rightarrow f(x_3) = 1.933659.$
5. $x_4 = (0.7577268212, 0.7577268212) \rightarrow f(x_4) = 1.932604.$
6. $x_5 = (0.7686160612, 0.7686160612) \rightarrow f(x_5) = 1.932178.$
7. $x_6 = (0.7754326925, 0.7754326925) \rightarrow f(x_6) = 1.931996.$
8. $x_7 = (0.7798314131, 0.7798314131) \rightarrow f(x_7) = 1.931917.$
9. $x_8 = (0.7827248172, 0.7827248172) \rightarrow f(x_8) = 1.931881.$
10. $x_9 = (0.7846518796, 0.7846518796) \rightarrow f(x_9) = 1.931865.$
11. ...

Example



x_k	$f(x_k)$
$(0, 0)$	1.984069
$(0.6306019374, 0.6306019374)$	1.936498
$(0.7057929076, 0.7057929076)$	1.936498
$(0.7394342516, 0.7394342516)$	1.933659
$(0.7577268212, 0.7577268212)$	1.932604
$(0.7686160612, 0.7686160612)$	1.932178
$(0.7754326925, 0.7754326925)$	1.931996
$(0.7798314131, 0.7798314131)$	1.931917
$(0.7827248172, 0.7827248172)$	1.931881
$(0.7846518796, 0.7846518796)$	1.931865
$(0.7859459291, 0.7859459291)$	1.931858
$(0.7868196823, 0.7868196823)$	1.931855
$(0.7874118293, 0.7874118293)$	1.931853
$(0.7878141326, 0.7878141326)$	1.931852
$(0.7880879197, 0.7880879197)$	1.931852

Weiszfeld's algorithm

Weisfeld's converges to the optimal solution (Kuhn 1973; Katz 1974) for Euclidean Distances in $O(n)$ (if the optimum is not one of the demand points).

Accelerations:

- $x(\lambda) = x_k + \lambda(x_{k+1} - x_k)$, $\lambda \in [1, \frac{d}{d-1}]$ (Chen, 1984).
- Update λ's (Drezner, 1992).

For any ℓ_τ-norm:

$$T(x_k)_j = \begin{cases} \dfrac{\displaystyle\sum_{i=1}^{n} \frac{w_i |x_{kj} - a_{ij}|^{\tau-2} a_{ij}}{\|x_k - a_i\|_\tau^{\tau-1}}}{\displaystyle\sum_{i=1}^{n} \frac{w_i |x_{kj} - a_{ij}|^{\tau-2}}{\|x_k - a_i\|_\tau^{\tau-1}}} & \text{if } x_k \neq a_i,\ \forall i, \\ a_{ij} & \text{if } x_{kj} = a_{ij}\ \forall j. \end{cases}$$

The algorithm converges if the generated sequence is *regular*: The nonregular sequences have measure zero in the solution space (Brimberg & Chen; 1998).

Weiszfeld's and beyond

- For ℓ_τ-norms ($\tau \in [1,2]$): (Brimberg & Love; 1992).
- For ℓ_τ-norms ($\tau > 2$): (Rodríguez-Chía & Valero; 2013). aproximate $v \in \mathbb{R}_+$ by $\sqrt{v^2+\varepsilon}$.
- ℓ_τ-norms: Perturbations (Morris & Verdini; 2001): Approximate $\sum_{i=1}^{n} \|x - a_i\|_\tau$ by

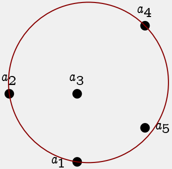
$$\left[\sum_{i=1}^{n} \left((x_k - a_{ik})^2 + \epsilon\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}.$$

- Banach Spaces. (Eckhardt, 1980; Puerto & Rodríguez-Chía, 1999, 2006)
- Sphere. (Zhang, 2003). $\min_{x \in \mathbb{R}^3} \sum_{i=1}^{n} w_i \cos^{-1}(a_i^t x)$.
- Regional Demands.
- Demand sets.
- Radial distances.

The center problem

"It is required to find the least circle which shall contain a given system of points in the plane (Sylvester 1857)"

$$\min_{x \in \mathbb{R}^2} \max_{i=1,\ldots,n} \|x - a_i\|$$

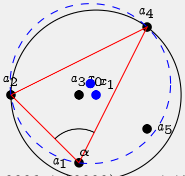


"If a circle is drawn through three points, then two cases arise. If the three points do not lie on the same semicircle, no smaller circle than this one can be drawn that contain the three points. If the points do lie in the same semicircle, it is obvious that a circle described upon the line joining the outer two as a diameter will be smaller than the circle passing through all three and will contain them all. (Sylvester 1860)"

Chrystal–Peirce (Sylvester)'s Algorithm

- Construct a large circle (with center x) which covers all the points and which passes through a_i and a_j.
- Find a_k such that the angle $\alpha = a_i a_k a_j$ is minimum.

if α *is obtuse* **then**
  $x^* = x$ and $f^* = \frac{1}{2}||a_i - a_j||$.
else
  Compute the center of the circle passing through a_i, a_j and a_k: x.
  if *The triangle formed by those points is not obtuse* **then**
    $x^* = x$.
  else
    Drop the point with the obtuse angle and Repeat

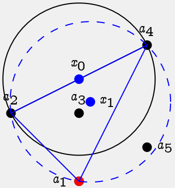


$x^* = (1.16666, 1.16666)$, $r = 1.178511$

Elzinga–Hearn's Algorithm

- Pick any two demand points a_i and a_j.
- Construct a circle based on the segment connecting a_i and a_j.

if *The circle covers all points* **then**
 STOP
else
 Add a point outside the circle a_k.
if *The triangle with the three points at its vertices is obtuse* **then**
 Drop the obtuse vertex and Repeat.
if *The circle passing through the three points covers all points* **then**
 STOP.
if *There is a point outside the circle:* a_ℓ **then**
 Add it as a fourth point. Discard one of then:
 - Keep a_ℓ and it farthest point, a_i.
 - Extend the diameter of the current circle through a_i defining two half planes.
 - Select the point which is not on the same half plane as a_k.



$x^* = (1.16666, 1.16666), r = 1.178511$

Considerations

Euclidean Unweighted Center Problems on the Plane:

- Silvester–Chrystal's needs to construct a initial circle passing through two points and covering the rest. Then, for the iterations, it needs to select the point with minimum angle (law of cosines...). Complexity at most $O(n^3)$ (Chrystal; 1985).
- Elzinga–Hearn's need to compute the circle enclosing an acute triangle (Drezner–Wesolowsky; 1980). Then, finding a point outside the circle. Complexity at least $O(n^2)$ (Drezner & Sheda, 1987).
- (Drezner, 2011) tested for up to 10000 points and determined that "empirically" Erzinga–Hearn's is more efficient for larger problems, although similar for small instances.

Improvements

- (Shamos & Hoey, 1975): $O(n\log(n))$ using Voronoi diagrams.
- (Megiddo, 1983): $O(n)$ linear programming on the plane.
- (Elzinga & Hearn, 1975): Extension to dimension d.
- ℓ_1-norm:

$$
\begin{aligned}
&\min_{x\in\mathbb{R}^d} t\\
&\quad s.t.\, t \geq \sum_{k=1}^{d} z_{ik}, \forall i = 1,\dots,d,\\
&\qquad z_{ik} \geq x_k - a_{ik}, \forall i = 1,\dots,n, k = 1,\dots,d,\\
&\qquad z_{ik} \geq -x_k + a_{ik}, \forall i = 1,\dots,n, k = 1,\dots,d,\\
&\qquad z_{ik} \geq 0.
\end{aligned}
$$

- Weighted Case: Euclidean Case on the plane $O(n)$ (Dyer, 1986; Megiddo, 1983) and $O(3^{d+2^2}n)$ for dimension d.

Ordered Median Objective

$$\min_{x\in\mathbb{R}^d} \sum_{i=1}^{n} \|x - a_i\|$$

The median solutions:

- Concern with the spatial efficiency.
- Remote and low-population density areas are discriminated in terms of accessibility to facilities, as compared with centrally situated and high-population density areas.

$$\min_{x\in\mathbb{R}^d} \max_{i=1,\ldots,n} \|x - a_i\|$$

The center solutions:

- Promote spatial equity.
- But, may cause a large increase in the total distance (losing spatial efficiency).

Ordered Median Objective

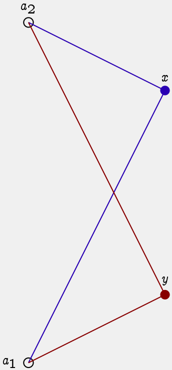
- Denote $z_i(x) = \|x - a_i\|$.
- For a given x, sort z_i: $z_{(1)} \geq z_{(2)} \geq \cdots \geq z_{(n)}$.
- For center problems we would like to minimize $z_{(1)}$.
- For median problems we would like to minimize the sum of $z_{(i)}$ (which equals the sum of z_i).
- What if we wish to minimize the sum of the k-largest z's: $z_{(1)} + z_{(2)} + \cdots + z_{(k)}$.
- What if we wish obtain the solution with minimum range? $z_{(1)} - z_{(n)}$.

These are Ordered median functions!!!

$$OM_\lambda(z_1, \ldots, z_n) = \sum_{i=1}^{n} \lambda_i z_{(i)}$$

Particular choices of λ allow to model many problems!!

Ordered Median Objective



$$OM_\lambda(x) = \lambda_1\|x - a_1\| + \lambda_2\|x - a_2\|.$$
$$OM_\lambda(y) = \lambda_1\|y - a_2\| + \lambda_2\|y - a_1\|$$

Ordered Median Location Problems

$$\min_{x \in \mathbb{R}^d} OM_\lambda(w_1\|x - a_1\|, \ldots, w_n\|x - a_n\|)$$

Special Interesting Cases:

- Weber Problem $\lambda = (1, \ldots, 1)$.
- Center Problem $\lambda = (1, 0, \ldots, 0)$.
- k-centrum Problem $\lambda = (\overbrace{1, \ldots, 1}^{k}, 0, \ldots, 0)$.
  - Introduced by Slater (1978) for discrete facility location.
  - For $k = 1$, Center, for $k = n$, Median.
  - (Rodríguez-Chía, Espejo & Drezner; 2010): Gradient descent method for the planar Euclidean case.
  - (Ogryczak & Tamir, 2003):

$$\min kt + \sum_{i=1}^{n} q_i$$

$$s.t. q_i \geq w_1\|x - a_1\| - t, \forall i = 1, \ldots, n,$$

$$q_i \geq 0, \forall i = 1, \ldots, n,$$

$$x \in \mathbb{R}^d.$$

(Ogryczak & Tamir; 2003)

Let $\Theta_k(z) = \sum_{i=1}^{k} z_{(i)}$ and $h(t) = \sum_{i=1}^{n} (k(t - z_i)_- + (n-k)(z_i - t)_+)$.

- h is piecewise linear and convex, and $z_{(k)}$ is a minimum of h.
- $h(z_{(k)}) = n \sum_{i=1}^{k} z_{(i)} - k \sum_{i=1}^{n} z_{(i)} = n\Theta_k(z) - k \sum_{i=1}^{n} z_i$.
- $\Theta_k(z) = \frac{1}{n} \left(k \sum_{i=1}^{n} z_i + \min_{t\in\mathbb{R}} h(t)\right)$.
- Defining $q_i = (z_i - t)_+$ and $p_i = (z_i - t)_-$:

$$\Theta_k(z) = \min \frac{1}{n} \left(\sum_{i=1}^{n} (kp_i + (n-k)q_i + kz_i) \right)$$
$$s.t.\ \ z_i - t = q_i - p_i, \forall i = 1, \dots, n,$$
$$q_i, p_i \geq 0, \forall i = 1, \dots, n.$$

- Since $p_i = q_i - y_i + t$:

$$\Theta_k(z) = \min_{t\in\mathbb{R}} \left(kt + \sum_{i=1}^{n} q_i \right)$$

Ordered Median Objectives

$$\min kt + \sum_{i=1}^{n} q_i$$

$$s.t. \, q_i \geq w_1 \|x - a_1\| - t, \forall i = 1, \ldots, n,$$

$$q_i \geq 0, \forall i = 1, \ldots, n,$$

$$x \in \mathbb{R}^d.$$

- For *polyhedral norms*: A Linear Program.
- For ℓ_1-norm: $d + 1$ variables and $2^d n$ constraints, for fixed d, solved in $O(n)$ (Megiddo, 1984; Zemel, 1984).

Ordered Median Problems

The result in (Ogryczak & Tamir, 2003) extends to monotone λ: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (assuming $\lambda_{n+1} = 0$):

$$\begin{aligned}
\Lambda_\lambda(z) &= \sum_{i=1}^{n} \lambda_i z_{(i)} \\
&= (\lambda_1 - \lambda_2) z_{(1)} + (\lambda_2 - \lambda_3)(z_{(1)} + z_{(2)}) + (\lambda_3 - \lambda_4)(z_{(1)} + z_{(2)} + z_{(3)}) \\
&+ \cdots + (\lambda_{n-1} - \lambda_n)(z_{(1)} + \cdots + z_{(n-1)}) + \lambda_n (z_{(1)} + \cdots + z_{(n)}) \\
&= \sum_{k=1}^{n} (\lambda_k - \lambda_{k+1}) \Theta_k(z)
\end{aligned}$$

$$\begin{aligned}
\min_{t_1,\ldots,t_n \in \mathbb{R}} & \sum_{i=1}^{n} (\lambda_k - \lambda_{k+1}) \left(k t_k + \sum_{i=1}^{n} q_{ik} \right) \\
s.t\, q_{ik} &\geq z_i - t_k, \forall i, k = 1, \ldots, n, \\
q_{ik} &\geq 0, \forall i, k = 1, \ldots, n.
\end{aligned}$$

General Ordered Median Objectives

- a.k.a. Ordered Weighted Averaging in Multicriteria Analysis (Yager, 1988).
- Introduced in Location Theory (in network location) by (Nickel & Puerto; 1999) extending median, center and cent-dian criteria.
- For the continuous case:
  - (Puerto & Fernández; 1995,2000): Geometric characterization of the solutions.
  - (Puerto, Rodríguez-Chía & Fernández-Palacín; 1997): Semiobnoxious location problems.
  - (Rodríguez-Chía, Nickel, Puerto & Fernández; 2000): Polyhedral gauges on the plane - Polynomially bounded algorithm based on iterating on sorted bisectors and solving LP's.
  - (Drezner & Nickel, 2008): Euclidean planar case (BTST).
  - (Espejo, Rodríguez-Chía & Valero, 2009): Approximated gradient descent method for the convex case for $\tau \in [1, 2]$.
  - (Blanco, ElHaj, Puerto; 2013): A general SDP-relaxation.
  - (Blanco, Puerto, ElHaj; 2014, 2015): SOCP & SDP exact formulations for single/multi-facility problems.

Multifacility Problems

We are given:

- A set of demand points (clients) $\mathcal{A} = \{a_1, \dots, a_n\}$.
- The number of facilities to be located: p.
- A potential set of facilities $X (= X_1 \times \cdots \times X_p)$.
- A function that measures the cost of locating any set of $(x_1, \dots, x_p) \in X$: $f_{\mathcal{A}}(x_1, \dots, x_p)$.

$$\min_{(x_1, \dots, x_p) \in X} \; f_{\mathcal{A}}(x_1, \dots, x_p)$$

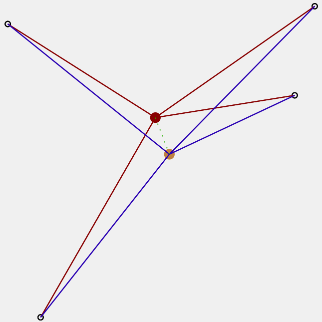
Multiple-allocation Weber Problem

Several facilities (each producing a different product) are to be located in order to minimize the sum of the weighted distances between all facilities and all users as well as between facilities.

Multiple-allocation Weber Problem

w_{ij}: weight between demand point a_i and the facility x_j.
μ_{ij}: weight between facilities x_j and $x_{j'}$.

$$\min_{x_1,\dots,x_p \in \mathbb{R}^d} \sum_{i=1}^{n} \sum_{j=1}^{p} w_{ij} \|x_j - a_i\| + \sum_{j=1}^{p-1} \sum_{j'=j+1}^{p} \mu_{jj'} \|x_j - x_{j'}\|$$



Multiple-allocation Weber Problem

- The multiple-allocation Weber problem is strictly convex if the points are not collinear and $w, \mu \geq 0$.
- For *linear distances* (block norms): LP formulation (Ward & Wendell, 1985).
- For Euclidean distances:
  - Practically efficient: (Calamai & Conn, 1980).
  - Poly-time: (Xue, Rosen & Pardalos, 1996).

Single-allocation Weber Problem

Several facilities (producing the same product) are to be located in order to minimize the sum of the weighted distances between all users as to their closest facility.

Multiple-allocation Weber Problem

w_j: weight for facility x_j.

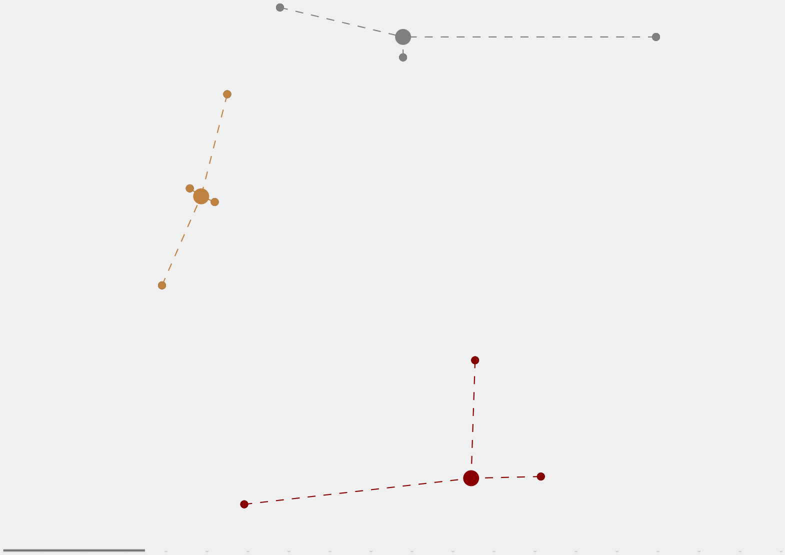
$$\min_{x_1,\ldots,x_p \in \mathbb{R}^d} \sum_{i=1}^{n} \sum_{j=1}^{p} w_j \min_j \{\|x_j - a_i\|\}$$

$$\min_{x_1,\ldots,x_p \in \mathbb{R}^d} \sum_{i=1}^{n} \sum_{j=1}^{p} z_{ij} w_j \|x_j - a_i\|$$

$$s.t. \sum_i z_{ij} = 1, \forall j = 1, \ldots, p,$$

$$z_{ij} \in \{0, 1\}, \forall i = 1, \ldots, n, j = 1, 1, \ldots, p.$$

Multiple-allocation Weber Problem



Multiple-allocation Weber Problem

- The objective function is neither convex or concave (Cooper, 1967).
- Eilon, Watson–Ghandi–Christofides, 1971) found a 50-point data set such that for 5 facilities has 61 local minima (the worst deviated 41% from the best).
- With the Euclidean norm, the optimal locations of new facilities are in the convex hull of existing facilities. (Francis & Cabot, 1972).
- With any norm on the plane, at least one optimal location of each new facility, belongs to the convex hull of existing facilities (Hansen, Perreur & Thisse, 1980).
- When a mixed norm problem on the plane involves ℓ_τ-norms ($\tau \geq 1$) one optimal location of each new facility belongs to the octagonal hull of existing facilities (Hansen, Perreur & Thisse, 1980).
- Optimal locations for all the new facilities can be found in the metric hull (intersection of all metric balls containing them) of existing facilities (Michelot, 1987).
- NP-hard (Megiddo, 1984) – Enumeration of Voronoi partitions of the customer set.

Multiple-allocation Weber Problem

- Some heuristics (for the planar Euclidean case)
  - Iterate on the location–allocation phases until no improvement is made: (Cooper, 1964).
  - Local search (Love & Juel, 1982), (Brimberg, Drezner, Mladenovic & Salhi, 2014).
  - Tabu search (Brimberg & Mladenovic, 1996).
  - p-median based approach (Hansen, Maldenovic & Taillard, 1998).
- Exact methods (on the plane):
  - Euclidean distance: B-&-b partitioning the space (Kuenne & Soland, 1972), descents algorithms ($p = 2$) (Ostresh; 1973, 1975), separating hyperplanes (Drezner; 1984), B-&-b + covering (Rosing, 1992) ...
  - Rectilinear distances: (Love & Morris, 1975)
  - Collinear points (Love, 1976).
  - DC programming for $p = 2$ (Chen, Hansen, GJaumard & Tuy, 1998).
  - BSSS - Column Generation: (Krau, 1997)?

B., Puerto, ElHaj. *Revisiting several problems and algorithms in continuous location with ℓ_τ norms*. COA2013.

Single-Facility Convex Ordered Median Problems

- A set of demand points $\{a_1, \dots, a_n\} \subseteq \mathbb{R}^d$.
- For each demand point a_i a weight ω_i.
- a norm $\|\cdot\|_\tau$ in $\mathbb{R}^d$ ($\tau \geq 1, \tau \in \mathbb{Q}$): $\|x\|_\tau = (\sum_{j=1}^d |x_j|^\tau)^{1/\tau}$.
- A set of weights $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$

$$\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{(i)} \|x - a_{(i)}\|$$

where $\omega_{(1)}\|x - a_{(1)}\|_\tau \geq \ldots \geq \omega_{(n)}\|x - a_{(n)}\|_\tau$.

The formulation

Let us denote $z_i = \omega_i \|x - a_i\|$:

$\min_{x\in\mathbb{R}^d} \displaystyle\sum_{i=1}^{n} \lambda_i \omega_{(i)} \|x - a_{(i)}\| = \min_{x\in\mathbb{R}^d} \sum_{i=1}^{n} \lambda_i z_{(i)}.$

Then, if $\mathcal{P}_n$ is the set of permutations of $\{1, \ldots, n\}$:

$$\sum_{i=1}^{n} \lambda_i z_{(i)} = \max_{\sigma\in\mathcal{P}_n} \sum_{i=1}^{n} \lambda_i z_{\sigma(i)}$$

(Hardy, Littlewood & Pólya; 1934):

Let σ be other permutation of the indices of z, then there exists i, j such that $z_{\sigma(i)} < z_{\sigma(j)}$:

$\lambda_i z_{\sigma(j)} + \lambda_j z_{\sigma(i)} - (\lambda_i z_{\sigma(i)} + \lambda_j z_{\sigma(j)}) = (\lambda_i - \lambda_j)(z_{\sigma(j)} - z_{\sigma(i)}) \geq 0$

Then, we may exchange z_i by z_j... after a finite number of steps...

The formulation

How to represent the feasible set $\mathcal{P}_n$?

$p_{ik} = \begin{cases} 1 & \text{if } z_i \text{ goes in position } k, \\ 0 & \text{otherwise.} \end{cases}$, so:

$$\sum_{i=1}^{n} p_{ik} = 1, \forall k = 1, \ldots, n,$$

$$\sum_{i=1}^{n} p_{ik} = 1, \forall k = 1, \ldots, n,$$

The formulation

$$
\begin{aligned}
\sum_{i=1}^{n} \lambda_i z_{(i)} = \quad & \max \sum_{i=1}^{n} \sum_{k=1}^{n} \lambda_k z_i p_{ik} \\
& s.t \quad \sum_{i=1}^{n} p_{ik} = 1, \; \forall k = 1, \dots, n, \\
& \quad \sum_{k=1}^{n} p_{ik} = 1, \; \forall i = 1, \dots, n, \\
& \quad p_{ik} \in \{0, 1\}.
\end{aligned}
$$

For given $z's$, the problem is TU (is an assignment problem), so equivalent to:

$$
\begin{aligned}
\sum_{i=1}^{n} \lambda_i z_{(i)} = \quad & \max \sum_{i=1}^{n} \sum_{k=1}^{n} \lambda_k z_i p_{ik} \\
& s.t \quad \sum_{i=1}^{n} p_{ik} = 1, \; \forall k = 1, \dots, n, \\
& \quad \sum_{k=1}^{n} p_{ik} = 1, \; \forall i = 1, \dots, n, \\
& \quad p_{ik} \geq 0.
\end{aligned}
$$

So, its solution coincides with the one of its dual:

The formulation

Hence, to compute $\min_{x\in\mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{(i)} \|x - a_{(i)}\|$, we have:

$$\begin{aligned} \min \sum_{k=1}^{n} v_k &+ \sum_{i=1}^{n} w_i \\ s.t \quad v_i + w_k &\geq \lambda_k z_i, \qquad \forall i, k = 1, \ldots, n, \\ z_i &\geq \omega_i \|x - a_i\|_\tau, \quad i = 1, \ldots, n. \end{aligned}$$

The formulation

$$\omega_i \|\bar{x} - a_i\|_\tau \le \bar{z}_i \iff \omega_i \left(\sum_{j=1}^{d} |\bar{x}_j - a_{ij}|^{\frac{r}{s}} \right)^{\frac{s}{r}} \le \bar{z}_i^{\frac{s}{r}} \bar{z}_i^{\frac{1}{\rho}}$$

$$\iff \omega_i \left(\sum_{j=1}^{d} |\bar{x}_j - a_{ij}|^{\frac{r}{s}} \bar{z}_i^{\frac{r}{s}(-\frac{r-s}{r})} \right)^{\frac{s}{r}} \le \bar{z}_i^{\frac{s}{r}}$$

$$\iff \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} |\bar{x}_j - a_{ij}|^{\frac{r}{s}} \bar{z}_i^{-\frac{r-s}{s}} \le \bar{z}_i$$

which holds if and only if $\exists u_i \in \mathbb{R}^d$, $u_{ij} \ge 0$, $\forall j = 1, \ldots, d$ such that

$$|\bar{x}_j - a_{ij}|^{\frac{r}{s}} \bar{z}_i^{-\frac{r-s}{s}} \le u_{ij}, \quad \text{satisfying} \quad \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} \le \bar{z}_i,$$

equivalently $|\bar{x}_j - a_{ij}|^r \le u_{ij}^s \bar{z}_i^{r-s}$, $\quad \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} \le \bar{z}_i$.

The formulation

So, $\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{(i)} \|x - a_{(i)}\|$ is reformulated as:

$$
\begin{aligned}
\min \sum_{k=1}^{n} v_k + \sum_{i=1}^{n} w_i & \\
s.t v_i + w_k \geq \lambda_k z_i, & \qquad \forall i, k = 1, ..., n, \\
y_{ij} - x_j + a_{ij} \geq 0, & \qquad i = 1, \ldots, n, \; j = 1, ..., d, \\
y_{ij} + x_j - a_{ij} \geq 0, & \qquad i = 1, \ldots, n, \; j = 1, ..., d, \\
y_{ij}^{r} \leq u_{ij}^{s} z_i^{r-s}, & \qquad i = 1, \ldots, n, \; j = 1, ..., d, , \\
\omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} \leq z_i, & \qquad i = 1, \ldots, n, \\
u_{ij} \geq 0, & \qquad i = 1, \ldots, n, \; j = 1, \ldots, d.
\end{aligned}
$$

The ℓ_1-norm case

If $\tau = 1$:

$$
\begin{aligned}
\min \sum_{k=1}^{n} v_k + \sum_{i=1}^{n} w_i \\
s.t \quad v_i + w_k \geq \lambda_k z_i, \forall i, k = 1, ..., n, \\
z_i \geq \omega_i \sum_{j=1}^{d} u_{ij}, i = 1, ..., n, \\
x_j - a_{ij} \leq u_{ij}\, i = 1, ..., n,\ j = 1, \dots, d, \\
-x_j + a_{ij} \leq u_{ij}\, i = 1, ..., n,\ j = 1, \dots, d.
\end{aligned}
$$

A General Model

In general for $\tau = \frac{r}{s}$:

$$
\begin{aligned}
& \min \sum_{k=1}^{n} v_k + \sum_{i=1}^{n} w_i \\
& s.t v_i + w_k \geq \lambda_k z_i, && \forall i, k = 1, ..., n, \\
& \quad y_{ij} - x_j + a_{ij} \geq 0, && i = 1, \ldots, n,\ j = 1, ..., d, \\
& \quad y_{ij} + x_j - a_{ij} \geq 0, && i = 1, \ldots, n,\ j = 1, ..., d, \\
& \quad y_{ij}^r \leq u_{ij}^s z_i^{r-s}, && i = 1, \ldots, n,\ j = 1, ..., d, , \\
& \quad \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} \leq z_i, && i = 1, \ldots, n, \\
& \quad u_{ij} \geq 0, && i = 1, \ldots, n,\ j = 1, \ldots, d.
\end{aligned}
$$

How to handle with the constraints $x^r \leq u^s t^{r-s}$??

Handling $x^r \le u^s t^{r-s}$

For $\tau = 2$ $(r = 2, s = 1)$: $x^2 \le u\,t$ (SECOND ORDER CONE CONSTRAINT)
For $\tau = 3$ $(r = 3, s = 1)$: $x^3 \le u\,t^2$ (??)

Let $w = \sqrt{ux}$: $w^2 \le ux$ and $x^4 \le ut^2x = w^2t^2 \Rightarrow x^2 \le wt$.
Actually, if both constraints hold:
$x^4 \le w^2t^2 \le uxt^2 \Rightarrow x^3 \le ut^2$.
so the constraint $x^3 \le u\,t^2$ is equivalent to:

$$w^2 \le ux,$$
$$x^2 \le wt, w \ge 0.$$

SECOND ORDER CONE CONSTRAINTS!!

Handling $x^r \leq u^s t^{r-s}$

Let $\tau = \frac{r}{s} > 1$, $\tau \neq 2$ be such that $r, s \in \mathbb{N} \setminus \{0\}$ and $\gcd(r, s) = 1$.
Let x, u and t be non negative and satisfying

$$x^r \leq u^s t^{r-s}. \tag{1}$$

Let $k = \lfloor \log_2(r) \rfloor$ and $\alpha = bin(s)$, $\beta = bin(r-s)$ and $\gamma = bin(2^k - r) \in \{0, 1\}^k$.
Then, if there exists w such that either:

Handling NonLinear Constraints

1. (x, t, u, w) is a solution of the following system, if $\alpha_i + \beta_i + \gamma_i = 1$, for all $0 < i \leq k-1$.

$$\begin{cases} w_1^2 & \leq u^{\alpha_0} t^{\beta_0} x^{\gamma_0}, \\ w_{i+1}^2 & \leq w_i u^{\alpha_i} t^{\beta_i} x^{\gamma_i}, \ \ i = 1, \ldots, k-2 \\ x^2 & \leq w_{k-1} u^{\alpha_{k-1}} t^{\beta_{k-1}} x^{\gamma_{k-1}}, \end{cases}$$

Handling NonLinear Constraints

2. Let $c = \#\{i : \alpha_i + \beta_i + \gamma_i = 3,\ i = 2, \ldots, k-2\}$, (x, t, u, w) is a solution of the following system, if there exist i_j and $i_{l(j)}$, $j = 1, \ldots, c$ such that:

1. $0 < i_1 < i_2 < \ldots < i_c \leq k-2$,
2. $i_j < i_{l(j)} < i_{j+1}$,
3. $\alpha_{i_j} + \beta_{i_j} + \gamma_{i_j} = 3$, $\alpha_{i_{l(j)}} + \beta_{i_{l(j)}} + \gamma_{i_{l(j)}} = 0$ and $\alpha_h + \beta_h + \gamma_h = 2$ for $h = i_j + 1, \ldots, i_{l(j)-1}$.

$$
\left\{
\begin{array}{lll}
w_1^2 & \leq u^{\alpha_0} t^{\beta_0} x^{\gamma_0}, & \\
w_{i+1}^2 & \leq w_i u^{\alpha_i} t^{\beta_i} x^{\gamma_i}, & i \in \{1, \ldots, i_1 - 1\} \\
\hline
\multicolumn{3}{c}{\text{for each } j = 1, \ldots, c} \\
w_{\theta(j)}^2 & \leq ut, & \\
w_{\theta(j)+1}^2 & \leq w_{\theta(j)-1} x & \\
\left.\begin{array}{ll} w_{\theta(j)+2*s}^2 & \leq w_{\theta(j)+2(s-1)} a_{i_j+s} \\ w_{\theta(j)+2*s+1}^2 & \leq w_{\theta(j)+2s-1} b_{i_j+s} \end{array}\right\}, & & \begin{array}{l} s = 1, \ldots, i_{l(j)} - i_j - 1 \text{ and} \\ a_{i_j+s} + b_{i_j+s} = u^{\alpha_{i_j+s}} t^{\beta_{i_j+s}} x^{\gamma_{i_j+s}}, \end{array} \\
w_{\theta(j)+2(i_{l(j)}-i_j)}^2 & \leq w_{\theta(j)+2(i_{l(j)}-i_j-1)} w_{\theta(j)+2(i_{l(j)}-i_j-1)+1}, & \left\{ \begin{array}{l} \text{if } m - 1 \geq \\ \theta(j) + 2(i_{l(j)} - i_j - 1) \end{array} \right., \\
w_{\theta(j)+2(i_{l(j)}-i_j)+s}^2 & \leq w_{\theta(j)+2(i_{l(j)}-i_j)+s-1} u^{\alpha_{i_{l(j)}+s}} t^{\beta_{i_{l(j)}+s}} x^{\gamma_{i_{l(j)}+s}}, & \left\{ \begin{array}{l} \text{for all } s = 1, \ldots, \\ \quad i_{j+1} - i_{l(j)} - 1 \end{array} \right. \\
\hline
x^2 & \leq w_m d. &
\end{array}
\right.
$$

Conversely, if (x, t, u, w) is a solution of one of those systems then (x, t, u) verifies $x^r \leq u^s t^{r-s}$.

Example

Let us consider $\tau = \frac{100000}{70001}$ which in turns means that $r = 10^5$ and $s = 70001$.

$$x^{100000} \quad \leq \quad u^{70001} t^{29999}$$

In this case $k = \log_2(10^5) = 17$ and

- $bin(70001) = (1,0,0,0,1,1,1,0,1,0,0,0,1,0,0,0,1)$.
- $bin(29999) = (1,1,1,1,0,1,0,0,1,0,1,0,1,1,1,0,0)$.
- $bin(31072) = (0,0,0,0,0,1,1,0,1,0,0,1,1,1,1,0,0)$.

According to the requirement of 2.:

$i_1 = 5$	$i_2 = 8$	$i_3 = 12$
$i_{l(1)} = 7$	$i_{l(2)} = 9$	$i_{l(3)} = 15$
$\theta(1) = 6$	$\theta(2) = 11$	$\theta(3) = 16$

,

the total number of inequalities is $m = 1 + 2 \times 6 + 9 = 22$.

Example

level 1	level 2	level 3	level 4	level 5
$w_1^2 \leq ut$	$w_2^2 \leq w_1 t$	$w_3^2 \leq w_2 t$	$w_4^2 \leq w_3 t$	$w_5^2 \leq w_4 t$

Bloc i_1

level 6	level 7	level 8
$w_6^2 \leq ut$ $w_7^2 \leq w_5 x$	$w_8^2 \leq w_6 u$ $w_9^2 \leq w_7 x$	$w_{10}^2 \leq w_8 w_9$

Bloc i_2

level 8	level 9	level 10
$w_{10}^2 \leq w_8 w_9$	$w_{11}^2 \leq ut$ $w_{12}^2 \leq w_{10} x$	$w_{13}^2 \leq w_{11} w_{12}$

level 11	level 12
$w_{14}^2 \leq w_{13} t$	$w_{15}^2 \leq w_{14} x$

Bloc i_3

level 13	level 14	level 15	level 16
$w_{16}^2 \leq ut$ $w_{17}^2 \leq w_{15} x$	$w_{18}^2 \leq w_{16} t$ $w_{19}^2 \leq w_{17} x$	$w_{20}^2 \leq w_{18} t$ $w_{21}^2 \leq w_{19} x$	$w_{22}^2 \leq w_{20} w_{21}$

level 17
$x^2 \leq w_{22} u$

SDP Representation

$$a^2 \leq bc \quad \Leftrightarrow \begin{pmatrix} b+c & 0 & 2a \\ 0 & b+c & b-c \\ 2a & b-c & b+c \end{pmatrix} \succeq 0,\ b+c \geq 0 \Leftrightarrow \left\| \begin{pmatrix} 2a \\ b-c \end{pmatrix} \right\|_2 \leq b+c,$$

For any set of lambda weights satisfying $\lambda_1 \geq \ldots \geq \lambda_n$ and $\tau = \frac{r}{s}$ such that $r, s \in \mathbb{N} \setminus \{0\}$, $r > s$ and $\gcd(r, s) = 1$, the OM location problem can be represented as a semidefinite programming problem with $n^2 + n(2d+1)$ linear constraints and at most $4nd \log r$ positive semidefinite constraints.

Let $\varepsilon > 0$ be a prespecified accuracy and (X^0, S^0) be a feasible primal-dual pair of initial solutions. An optimal primal-dual pair (X, S) satisfying $X \cdot S \leq \varepsilon$ can be obtained in at most $O(\alpha \log \frac{X^0 \cdot S^0}{\varepsilon})$ iterations and the complexity of each iteration is bounded above by $O(\alpha\beta^3, \alpha^2\beta^2, \alpha^2)$ being $\alpha = 3n + 2nd(1 + \log r)$ and $\beta = p$, the dimension of the dual matrix variable S_p.

Single-facility constrained location problems

Consider the restricted problem:

$$\min_{x \in \mathbf{K} \subset \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \|x - a_{\sigma(i)}\|_\tau . \qquad (2)$$

Assume that any of the following conditions holds:

1. $g_i(x)$ are concave for $i = 1, \ldots, \ell$ and $-\sum_{i=1}^{\ell} \mu_i \nabla^2 g_i(x) \succ 0$ for each dual pair (x, μ) of the problem of minimizing any linear functional $c^t x$ on $\mathbf{K}$ (*Positive Definite Lagrange Hessian* (PDLH)).
2. $g_i(x)$ are sos-concave on $\mathbf{K}$ for $i = 1, \ldots, \ell$ or $g_i(x)$ are concave on $\mathbf{K}$ and strictly concave on the boundary of $\mathbf{K}$ where they vanish, i.e. $\partial \mathbf{K} \cap \partial \{x \in \mathbb{R}^d : g_i(x) = 0\}$, for all $i = 1, \ldots, \ell$.
3. $g_i(x)$ are strictly quasi-concave on $\mathbf{K}$ for $i = 1, \ldots, \ell$.

Then, there exists a constructive finite dimension embedding, which only depends on τ and g_i, $i = 1, \ldots, \ell$, such that (2) is a semidefinite problem.

Experiments: Weber

		d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
τ	n	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
1.5	10	0.20	0.06	10^{-8}	$< 10^{-8}$	0.28	0.06	10^{-8}	$< 10^{-8}$	0.86	0.09	$< 10^{-8}$	$< 10^{-8}$
	100	1.71	0.27	10^{-8}	$< 10^{-8}$	3.16	0.40	10^{-8}	$< 10^{-8}$	10.89	8.77	10^{-8}	$< 10^{-8}$
	500	10.78	15.34	10^{-8}	$< 10^{-8}$	15.84	22.32	10^{-8}	$< 10^{-8}$	51.23	1883.8	$< 10^{-8}$	$< 10^{-8}$
	1000	21.22	128.53	10^{-8}	$< 10^{-8}$	30.67	179.72	10^{-8}	$< 10^{-8}$	103.17	24170.12	10^{-8}	$< 10^{-8}$
	5000	103.50	57013.83	10^{-8}	$< 10^{-8}$	178.50	NaN	10^{-8}	NaN	566.64	NaN	10^{-8}	NaN
	10000	210.22	NaN	10^{-8}	NaN	455.05	NaN	10^{-8}	NaN	1330.36	NaN	10^{-8}	NaN
2	10	0.06	0.04	$< 10^{-8}$	$< 10^{-8}$	0.12	0.04	$< 10^{-8}$	$< 10^{-8}$	0.46	0.04	$< 10^{-8}$	$< 10^{-8}$
	100	0.40	0.05	$< 10^{-8}$	$< 10^{-8}$	0.99	0.07	$< 10^{-8}$	$< 10^{-8}$	4.63	0.04	$< 10^{-8}$	$< 10^{-8}$
	500	1.50	0.12	$< 10^{-8}$	$< 10^{-8}$	5.83	0.14	$< 10^{-8}$	$< 10^{-8}$	21.63	0.14	$< 10^{-8}$	$< 10^{-8}$
	1000	3.27	0.42	$< 10^{-8}$	$< 10^{-8}$	11.08	0.48	$< 10^{-8}$	$< 10^{-8}$	44.04	0.36	$< 10^{-8}$	$< 10^{-8}$
	5000	17.06	19.90	$< 10^{-8}$	$< 10^{-8}$	58.16	22.09	$< 10^{-8}$	$< 10^{-8}$	218.82	15.59	$< 10^{-8}$	$< 10^{-8}$
	10000	33.19	137.81	$< 10^{-8}$	$< 10^{-8}$	118.91	162.98	$< 10^{-8}$	$< 10^{-8}$	455.30	107.06	$< 10^{-8}$	$< 10^{-8}$
3	10	0.19	0.06	10^{-8}	$< 10^{-8}$	0.31	0.14	10^{-8}	10^{-8}	0.99	0.54	10^{-8}	5×10^{-8}
	100	1.88	1.77	10^{-8}	2×10^{-6}	3.60	3.66	10^{-8}	2×10^{-5}	12.71	43.54	10^{-8}	9×10^{-8}
	500	10.82	24.44	10^{-8}	8×10^{-4}	17.87	124.97	10^{-8}	8×10^{-4}	57.91	3362.08	10^{-8}	5×10^{-5}
	1000	21.73	55.03	10^{-8}	7×10^{-4}	33.99	279.74	10^{-8}	2×10^{-3}	118.11	NaN	10^{-8}	NaN
	5000	110.87	NaN	10^{-8}	NaN	181.17	NaN	10^{-8}	NaN	646.46	NaN	10^{-8}	NaN
	10000	245.66	NaN	10^{-8}	NaN	477.38	NaN	10^{-8}	NaN	1616.26	NaN	10^{-8}	NaN
3.5	10	0.33	0.12	10^{-8}	$< 10^{-8}$	0.47	0.16	10^{-8}	2×10^{-8}	1.75	0.43	10^{-8}	$< 10^{-8}$
	100	3.87	3.31	10^{-8}	4×10^{-6}	5.44	5.44	10^{-8}	9×10^{-8}	19.71	124.31	10^{-8}	2×10^{-7}
	500	18.62	49.23	10^{-8}	3×10^{-3}	26.99	242.89	10^{-8}	3×10^{-3}	92.64	16092.68	10^{-8}	10^{-3}
	1000	37.06	190.64	10^{-8}	2×10^{-3}	51.50	799.74	10^{-8}	2×10^{-3}	192.77	21409.00	10^{-8}	7×10^{-4}
	5000	280.27	NaN	10^{-8}	NaN	304.94	NaN	10^{-8}	NaN	1178.17	NaN	10^{-8}	NaN
	10000	964.18	NaN	10^{-8}	NaN	872.29	NaN	10^{-8}	NaN	2431.58	NaN	10^{-8}	NaN

Experiments: Center

		d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
τ	n	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
1.5	10	0.22	0.05	10^{-6}	$< 10^{-8}$	0.39	0.21	$< 10^{-8}$	4×10^{-5}	1.02	0.10	10^{-4}	2×10^{-2}
	100	2.13	0.20	$< 10^{-8}$	$< 10^{-8}$	8.75	0.53	$< 10^{-8}$	4×10^{-5}	43.28	9.03	2×10^{-5}	2×10^{-2}
	500	12.59	8.20	$< 10^{-8}$	$< 10^{-8}$	63.16	38.33	$< 10^{-8}$	9×10^{-5}	237.99	1435.45	8×10^{-6}	8×10^{-3}
	1000	27.11	56.80	$< 10^{-8}$	$< 10^{-8}$	115.35	315.47	$< 10^{-8}$	5×10^{-5}	327.04	NaN	6×10^{-6}	NaN
	5000	150.34	26786.61	10^{-8}	$< 10^{-8}$	357.49	NaN	$< 10^{-8}$	NaN	1231.78	NaN	5×10^{-6}	NaN
	10000	371.39	NaN	$< 10^{-8}$	NaN	1297.23	NaN	$< 10^{-8}$	NaN	2762.51	NaN	7×10^{-6}	NaN
2	10	0.11	0.03	5×10^{-8}	5×10^{-6}	0.17	0.04	10^{-8}	6×10^{-5}	0.44	0.04	2×10^{-5}	10^{-2}
	100	1.34	0.05	10^{-7}	7×10^{-6}	2.09	0.05	10^{-8}	4×10^{-6}	7.68	0.06	2×10^{-4}	8×10^{-3}
	500	8.80	0.09	8×10^{-8}	6×10^{-6}	13.69	0.10	10^{-8}	10^{-5}	50.25	0.14	10^{-5}	5×10^{-3}
	1000	20.85	0.11	5×10^{-7}	5×10^{-6}	30.27	0.16	10^{-8}	10^{-5}	115.64	0.26	10^{-5}	6×10^{-3}
	5000	119.90	0.68	10^{-6}	4×10^{-6}	212.21	0.78	10^{-8}	2×10^{-4}	912.44	1.37	4×10^{-3}	5×10^{-3}
	10000	287.13	1.31	3×10^{-6}	10^{-5}	467.08	1.39	10^{-8}	8×10^{-5}	1510.41	2.48	8×10^{-3}	5×10^{-3}
3	10	0.23	0.07	2×10^{-7}	$< 10^{-8}$	0.37	0.23	10^{-8}	2×10^{-5}	1.08	0.09	10^{-4}	7×10^{-3}
	100	2.32	0.28	$< 10^{-8}$	$< 10^{-8}$	7.48	0.48	$< 10^{-8}$	3×10^{-7}	37.66	9.58	10^{-5}	10^{-2}
	500	14.47	14.78	10^{-8}	$< 10^{-8}$	52.27	34.85	2×10^{-7}	4×10^{-6}	209.46	1446.29	6×10^{-6}	5×10^{-3}
	1000	28.93	110.24	10^{-8}	$< 10^{-8}$	119.96	297.47	6×10^{-7}	4×10^{-5}	293.48	NaN	2×10^{-5}	NaN
	5000	160.96	19057.62	10^{-8}	$< 10^{-8}$	456.11	NaN	10^{-4}	NaN	1223.41	NaN	2×10^{-4}	NaN
	10000	434.79	NaN	$< 10^{-8}$	NaN	6829.70	NaN	3×10^{-5}	NaN	2663.15	NaN	3×10^{-4}	NaN
3.5	10	0.33	0.06	$< 10^{-8}$	10^{-8}	0.56	0.07	6×10^{-6}	5×10^{-5}	1.80	0.17	5×10^{-5}	10^{-2}
	100	4.47	0.38	10^{-7}	$< 10^{-8}$	13.72	0.83	4×10^{-7}	2×10^{-5}	68.91	19.60	2×10^{-5}	6×10^{-3}
	500	21.93	19.68	2×10^{-8}	$< 10^{-8}$	90.65	58.54	2×10^{-7}	9×10^{-6}	373.21	3658.09	2×10^{-6}	6×10^{-3}
	1000	44.82	246.90	2×10^{-8}	$< 10^{-8}$	179.48	963.05	2×10^{-5}	6×10^{-7}	603.41	NaN	10^{-5}	NaN
	5000	244.04	NaN	2×10^{-8}	NaN	551.16	NaN	2×10^{-6}	NaN	2279.93	NaN	2×10^{-4}	NaN
	10000	510.25	NaN	2×10^{-8}	NaN	2618.87	NaN	2×10^{-5}	NaN	4814.26	NaN	6×10^{-5}	NaN

Experiments: 0.5-centrum

		d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
τ	n	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
1.5	10	0.29	0.11	10^{-8}	$< 10^{-8}$	0.42	0.10	$< 10^{-8}$	10^{-8}	1.01	0.13	2×10^{-4}	2×10^{-4}
	100	4.67	0.53	$< 10^{-8}$	$< 10^{-8}$	8.01	1.18	$< 10^{-8}$	$< 10^{-8}$	30.45	17.07	3×10^{-8}	5×10^{-8}
	500	39.92	32.40	10^{-8}	$< 10^{-8}$	49.26	96.39	$< 10^{-8}$	$< 10^{-8}$	205.37	2794.70	2×10^{-8}	10^{-8}
	1000	68.11	214.23	10^{-8}	$< 10^{-8}$	109.52	659.33	$< 10^{-8}$	$< 10^{-8}$	437.95	53531.38	2×10^{-8}	$< 10^{-8}$
	5000	476.95	NaN	$< 10^{-8}$	NaN	668.08	NaN	$< 10^{-8}$	NaN	4738.70	NaN	2×10^{-8}	NaN
	10000	1242.57	NaN	10^{-8}	NaN	2016.01	NaN	$< 10^{-8}$	NaN	15348.57	NaN	2×10^{-8}	NaN
2	10	0.13	0.07	$< 10^{-8}$	$< 10^{-8}$	0.18	0.05	$< 10^{-8}$	$< 10^{-8}$	0.45	0.05	8×10^{-5}	4×10^{-4}
	100	1.36	0.13	$< 10^{-8}$	$< 10^{-8}$	2.03	0.10	$< 10^{-8}$	$< 10^{-8}$	7.56	0.10	10^{-8}	$< 10^{-8}$
	500	8.40	0.68	$< 10^{-8}$	$< 10^{-8}$	15.63	0.48	$< 10^{-8}$	$< 10^{-8}$	53.87	0.47	$< 10^{-8}$	$< 10^{-8}$
	1000	22.38	3.71	$< 10^{-8}$	$< 10^{-8}$	30.90	2.13	$< 10^{-8}$	$< 10^{-8}$	108.22	1.99	$< 10^{-8}$	$< 10^{-8}$
	5000	128.18	397.93	$< 10^{-8}$	$< 10^{-8}$	195.56	293.61	$< 10^{-8}$	$< 10^{-8}$	815.34	228.04	$< 10^{-8}$	$< 10^{-8}$
	10000	337.17	3112.30	$< 10^{-8}$	$< 10^{-8}$	460.98	2129.80	$< 10^{-8}$	$< 10^{-8}$	2373.23	2545.92	$< 10^{-8}$	$< 10^{-8}$
3	10	0.30	0.10	10^{-8}	$< 10^{-8}$	0.42	0.11	$< 10^{-8}$	2×10^{-8}	1.09	0.13	3×10^{-5}	4×10^{-7}
	100	5.65	0.66	10^{-8}	$< 10^{-8}$	10.20	1.46	$< 10^{-8}$	$< 10^{-8}$	40.08	21.77	2×10^{-7}	$< 10^{-8}$
	500	50.36	41.43	10^{-8}	$< 10^{-8}$	72.35	125.78	$< 10^{-8}$	$< 10^{-8}$	225.13	2331.94	4×10^{-8}	$< 10^{-8}$
	1000	100.17	418.97	10^{-8}	$< 10^{-8}$	145.24	774.01	$< 10^{-8}$	$< 10^{-8}$	463.74	NaN	4×10^{-8}	NaN
	5000	582.84	NaN	2×10^{-8}	NaN	894.95	NaN	$< 10^{-8}$	NaN	4067.13	NaN	2×10^{-8}	NaN
	10000	1715.00	NaN	10^{-8}	NaN	2565.21	NaN	2×10^{-8}	NaN	13649.88	NaN	5×10^{-8}	NaN
3.5	10	0.44	0.11	6×10^{-8}	$< 10^{-8}$	0.60	0.12	$< 10^{-8}$	$< 10^{-8}$	1.81	0.26	2×10^{-4}	2×10^{-6}
	100	10.90	1.58	10^{-8}	$< 10^{-8}$	16.97	3.54	$< 10^{-8}$	$< 10^{-8}$	60.45	61.05	4×10^{-8}	10^{-8}
	500	80.28	133.69	2×10^{-8}	$< 10^{-8}$	124.50	347.44	2×10^{-8}	$< 10^{-8}$	379.20	6691.80	4×10^{-8}	$< 10^{-8}$
	1000	171.62	869.75	2×10^{-8}	$< 10^{-8}$	252.79	2367.65	2×10^{-8}	10^{-8}	852.59	211506.04	4×10^{-8}	$< 10^{-8}$
	5000	1033.28	NaN	2×10^{-8}	NaN	1700.71	NaN	2×10^{-8}	NaN	8510.86	NaN	6×10^{-8}	NaN
	10000	2345.25	NaN	2×10^{-8}	NaN	4682.55	NaN	2×10^{-8}	NaN	27723.99	NaN	4×10^{-8}	NaN

Experiments: Random λ's

		d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
τ	n	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
1.5	10	0.24	0.19	$< 10^{-8}$	$< 10^{-8}$	0.40	0.09	$< 10^{-8}$	$< 10^{-8}$	1.19	0.14	3×10^{-6}	10^{-5}
	100	4.03	1.33	10^{-8}	$< 10^{-8}$	6.73	1.88	10^{-8}	$< 10^{-8}$	22.22	18.46	$< 10^{-8}$	$< 10^{-8}$
	500	159.42	75.98	10^{-8}	$< 10^{-8}$	190.77	141.67	10^{-8}	$< 10^{-8}$	380.99	1322.44	$< 10^{-8}$	3×10^{-8}
	1000	1270.76	NaN	10^{-7}	NaN	1730.61	NaN	2×10^{-8}	NaN	2379.03	NaN	$< 10^{-8}$	NaN
2	10	0.14	0.04	3×10^{-8}	10^{-8}	0.18	0.06	10^{-8}	10^{-8}	0.62	0.06	3×10^{-5}	2×10^{-6}
	100	5.11	0.31	3×10^{-6}	10^{-8}	6.94	0.37	2×10^{-8}	10^{-8}	17.07	0.29	3×10^{-8}	$< 10^{-8}$
	500	427.54	12.88	3×10^{-6}	10^{-8}	443.47	15.29	3×10^{-6}	$< 10^{-8}$	1092.70	11.35	5×10^{-7}	$< 10^{-8}$
	1000	2079.97	100.51	3×10^{-6}	10^{-8}	7702.03	127.81	2×10^{-6}	$< 10^{-8}$	9235.59	81.98	7×10^{-7}	$< 10^{-8}$
3	10	0.51	0.10	5×10^{-8}	2×10^{-8}	0.72	0.16	10^{-7}	6×10^{-8}	1.89	0.28	4×10^{-6}	3×10^{-6}
	100	64.14	3.49	5×10^{-6}	10^{-6}	58.10	5.03	10^{-6}	5×10^{-7}	152.45	46.06	10^{-6}	2×10^{-7}
	500	1532.17	272.56	2×10^{-4}	10^{-4}	2269.39	413.24	3×10^{-5}	7×10^{-5}	7950.09	4100.40	9×10^{-6}	2×10^{-5}
	1000	4546.73	2080.30	3×10^{-4}	10^{-4}	5678.17	NaN	9×10^{-5}	NaN	18011.79	29734.01	3×10^{-5}	10^{-5}
3.5	10	0.63	0.16	10^{-6}	10^{-8}	1.48	0.19	10^{-5}	4×10^{-8}	7.05	0.34	2×10^{-6}	2×10^{-5}
	100	33.32	10.01	10^{-6}	4×10^{-6}	302.76	15.03	9×10^{-5}	3×10^{-6}	596.20	163.09	2×10^{-4}	7×10^{-8}
	500	1555.08	438.34	5×10^{-6}	3×10^{-5}	2774.06	817.91	4×10^{-4}	4×10^{-4}	8705.00	19654.02	$\times10^{-4}$	2×10^{-4}
	1000	7625.95	NaN	2×10^{-5}	NaN	7681.10	NaN	6×10^{-4}	NaN	18845.92	NaN	2×10^{-4}	NaN

Range & Non Convex Constraints

		d	
		3	
τ	n	Time(Ave)	Gap(Ave)
2	10	0.46	0.00001623
	100	9.45	0.00457982
	500	80.56	0.00030263
	1000	204.96	0.00094492

$$\mathbb{K} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - 2x_2^2 - 2x_3^2 \geq 0, -2x_1^2 + 5x_2^2 + 4x_3^2 \geq 0\}.$$

Comparisons

		Algorithm		
n	Problem	DN09	ERV09	New
100	Weber	0.47	N/A	0.05
	k-centrum	0.39	1.76	0.13
	random	7.13	0.79	0.31
500	Weber	7.28	N/A	0.12
	k-centrum	3.99	5.63	0.68
	random	85.56	4.69	12.88
1000	Weber	27.69	N/A	0.42
	k-centrum	15.04	25.32	3.71
	random	340.2	17.17	100.51

		$\mathbb{R}^2$				$\mathbb{R}^3$			
		τ				τ			
		2		3		2		3	
n	Problem	BHP12	New	BHP12	New	BHP12	New	BHP12	New
100	Weber	3.55	0.05	5.21	1.77	4.79	0.07	7.32	3.60
	Center	30.83	0.05	34.07	0.28	48.51	0.05	57.85	0.48
	k-centrum	37.58	0.13	34.41	0.66	52.52	0.10	53.87	1.46
500	Weber	17.74	0.12	27.46	10.82	25.32	0.14	37.22	17.87
	Center	305.36	0.09	299.41	14.47	566.29	0.10	600.27	34.85
	k-centrum	285.02	0.68	291.8	41.43	452.85	0.48	449.46	72.35
1000	Weber	39.82	0.42	58.32	21.73	56.86	0.48	84.06	33.99
	Center	736.25	0.11	864.93	28.93	1494.76	0.16	1606.89	119.96
	k-centrum	666.2	3.71	729.3	100.17	1149.9	2.13	1280.1	145.24

Blanco, Puerto, ElHaj. *Continuous multifacility ordered median location problems* . EJOR2015.

Multiple-allocation OM Location Problems

Let us consider a set of demand points $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}^d$. We want to locate p new facilities $X = \{x_1, x_2, \ldots, x_p\}$ which minimize the following expression:

$$f_\lambda^{NI}(x_1, x_2, \ldots, x_p) = \sum_{i=1}^{n} \sum_{j=1}^{p} \lambda_{ij} d_{(i)}(x_j) + \sum_{j=1}^{p-1} \sum_{j'=j+1}^{p} \mu_{jj'} \|x_j - x_{j'}\|_\tau, \quad (3)$$

where for any $x \in \mathbb{R}^d$, $d_i(x) = \omega_i \|a_i - x\|_\tau$ and $d_{(i)}(x)$ is the i-th element in the permutation of $(d_1(x), \ldots, d_n(x))$ such that $d_{(1)}(x) \geq d_{(2)}(x) \geq \ldots \geq d_{(i)}(x) \geq \ldots \geq d_{(n)}(x)$. In this model, it is assumed that:

$$\lambda_{1j} \geq \lambda_{2j} \geq \ldots \geq \lambda_{nj} \geq 0, \ \forall j = 1, \ldots, p. \quad (4)$$

$\mu_{jj'} \geq 0$ for any $j, j' = 1, \ldots, p$ and, as mention above, $d_{(i)}(x_j)$ is the expression, which appears at the i-th position in the ordered version of the list

$$L_j^{NI} := (w_1 \|x_j - a_1\|_\tau, \ldots, w_n \|x_j - a_n\|_\tau) \quad \text{for } j = 1, 2, \ldots, p. \quad (5)$$

Multiple-allocation OM Location Problems

$$\rho_\lambda^{NI} := \min_x \{ f_\lambda^{NI}(x) : x = (x_1, \ldots, x_p),\ x_j \in \mathbb{R}^d,\ \forall j = 1, \ldots, p \}, \qquad (\mathbf{LOCOMF - NI})$$

Theorem

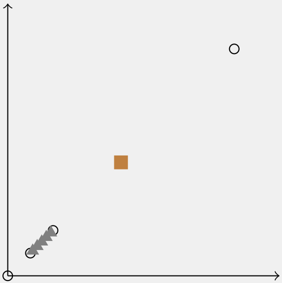
Assume that $\tau \notin \{1, +\infty\}$, the demand points in A are not collinear and for all $i = 1, \ldots, n$ there exists at least one $j \in \{1, \ldots, p\}$ such that $\lambda_{ij} \neq 0$. Then the optimal solution of Problem $(\mathbf{LOCOMF - NI})$ is unique.

Collocation

$A = \{(0,0),(1,1),(2,2),(10,10)\}$, ℓ_2-norm and consider the following weights $w_i = 1$ for all $i = 1, \ldots, 4$, $\mu_{12} = 0$ and

$$\lambda_{11} = \lambda_{21} = \lambda_{31} = \lambda_{41} = 1,$$
$$\lambda_{12} = 1,\ \lambda_{22} = \lambda_{32} = \lambda_{42} = 0.$$

$f^* = 16\sqrt{2}$ which is attained by $x_1^* \in \{(x,x) : x \in [1,2]\}$, $x_2^* = (5,5)$.



Multiple-allocation OM Location Problems

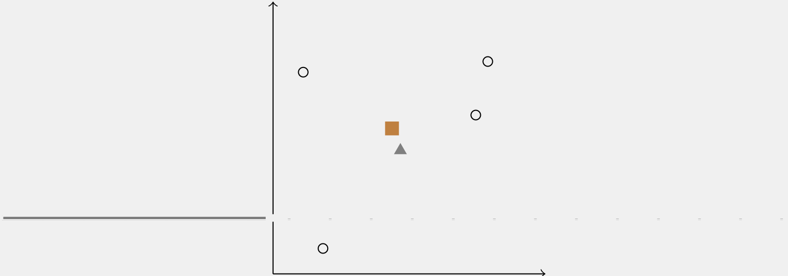
$$\rho_\lambda^{NI} = \min \sum_{i=1}^{n}\sum_{j=1}^{p} v_{ij} + \sum_{\ell=1}^{n}\sum_{j=1}^{p} w_{\ell j} + \sum_{j=1}^{p-1}\sum_{j'=j+1}^{p} t_{jj'} \qquad (\text{NIMFOMP}_\lambda)$$

$$\begin{aligned}
\text{s.t. } & v_{ij} + w_{\ell j} \geq \lambda_{\ell j} u_{ij}, \forall i, \ell = 1, \ldots, n,\ j = 1, \ldots, p && (6)\\
& y_{ijk} - x_{jk} + a_{ik} \geq 0, \forall i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d && (7)\\
& y_{ijk} + x_{jk} - a_{ik} \geq 0, \forall i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d, && (8)\\
& y_{ijk}^{r} \leq \varsigma_{ijk}^{s} u_{ij}^{r-s}, \forall i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d, && (9)\\
& \omega_i^{\frac{r}{s}} \sum_{k=1}^{d} \varsigma_{ijk} \leq u_{ij}, \forall i = 1, \ldots, n,\ j = 1, \ldots, p && (10)\\
& z_{jj'k} - x_{jk} + x_{j'k} \geq 0, \forall j, j' = 1, \ldots, p,\ k = 1, \ldots, d, && (11)\\
& z_{jj'k} + x_{jk} - x_{j'k} \geq 0, \forall j, j' = 1, \ldots, p,\ k = 1, \ldots, d, && (12)\\
& z_{jj'k}^{r} \leq \xi_{jj'k}^{s} t_{jj'}^{r-s}, \forall j, j' = 1, \ldots, p,\ k = 1, \ldots, d, && (13)\\
& \mu_{jj'}^{\frac{r}{s}} \sum_{k=1}^{d} \xi_{jj'k} \leq t_{jj'}, \forall j, j' = 1, \ldots, p, && (14)\\
& \varsigma_{ijk} \geq 0, \forall i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d, && (15)\\
& \xi_{jj'k} \geq 0, \forall j, j' = 1, \ldots, p,\ k = 1, \ldots, d, && (16)\\
& v_{ij} \in \mathbb{R}, w_{\ell j} \in \mathbb{R}, t_{jj'} \geq 0, \forall i = 1, \ldots, n, j, j' = 1, \ldots, p && (17)
\end{aligned}$$

Example

$A = \{(9.46, 9.36), (8.93, 7.00), (2.20, 1.12), (1.33, 8.89)\}$ (A is subset of size 4 of the 50-cities data set from (Eilon, Watson-Gandy & Christofides), and (randomly generated-) lambda weights:
$\lambda_{11} = 147.31$, $\lambda_{21} = 24.44$, $\lambda_{31} = 24.16$, $\lambda_{41} = 10.77$, $\lambda_{12} = 119.08$),
$\lambda_{22} = 0.56$, $\lambda_{32} = 0.00$, $\lambda_{42} = 0.00$, $\mu_{12} = 0.56$ and $w_i = 1$ for all $i = 1, \ldots, n$.

$$\min_{x_1, x_2 \in \mathbb{R}^2} 147.31 d_{(1)}(x_1) + 24.44 d_{(2)}(x_1) + 24.16 d_{(3)}(x_1) + 10.77 d_{(4)}(x_1) + 119.08 d_{(1)}(x_2$$
$$+ 0.56 d_{(2)}(x_2) + 0.00 d_{(3)}(x_2) + 0.00 d_{(4)}(x_2) + 0.56 \|x_1 - x_2\|_2$$



Experiments

Gurobi 5.6 executed in PC with an Intel Core i7 processor at 2x 2.40 GHz and 4 GB of RAM.
(Eilon, Watson-Gandy & Christofides; 1971) Dataset – 50 cities.
Random λ and μ weights.

	τ		
p	1.5	2	3
2	2.5095	2.1157	3.7470
5	12.7794	6.5161	9.8130
10	29.1873	10.5726	19.5455
15	49.4854	19.1129	40.4506
30	148.7449	40.5635	85.5676

Single allocation multifacility location problems

Let us consider a set of demand points $\{a_1, a_2, \dots, a_n\} \subset \mathbb{R}^d$. We want to locate p new facilities $X = \{x_1, x_2, \dots, x_p\}$.
$t_i : \mathbb{R}^{pd} \mapsto \mathbb{R}$, $t_i(x_1, \dots, x_p) := \min_j \|x_j - a_i\|$.
Each client a_i will be allocated to its closest facility:

$$\begin{array}{llll} \tilde{f}_i(x): & \mathbb{R}^{pd} & \mapsto & \mathbb{R} \\ & x = (x_1, \dots, x_p) & \mapsto & t_i := \min\limits_{j=1..p} \{\|x_j - a_i\|\}. \end{array}$$

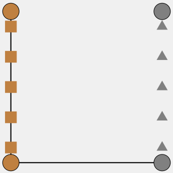
$$\rho_\lambda := \min_x \{\sum_{i=1}^{n} \lambda_i \tilde{f}_{(i)}(x) : x = (x_1, \dots, x_p),\ x_j \in \mathbf{K}, \forall j = 1, \dots, p\}, \tag{LOCOMF}$$

where:

- $\mathbb{K} \subseteq \mathbb{R}^d$ satisfies the Archimedean property. Without loss of generality we shall assume that we know $M > 0$ such that $\|x_j\|_2 \le M$, for all $j = 1, \dots, p$.
- $\tau := \frac{r}{s} \ge 1$, $r, s \in \mathbb{N}$ with $gcd(r, s) = 1$.
- $\lambda_\ell \ge 0$ for all $\ell = 1, \dots, n$

Example

$A = \{\ (0,0),\ (0,1),\ (1,1),\ (1,0)\}$. $\lambda_{ij} = 1\ \forall i, j$.



$x_1^* \in \{0\} \times [0,1]$ and $x_2^* \in \{1\} \times [0,1]$

MISOCO Formulation

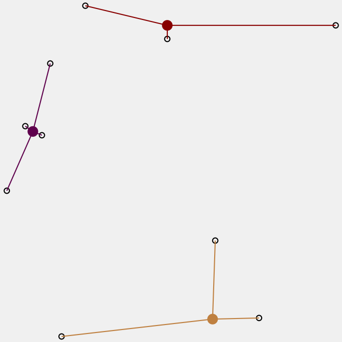
$$
\begin{aligned}
\min \quad & \sum_{\ell=1}^{n} \lambda_\ell \theta_\ell \\
\text{s.t.} \quad & t_i \leq \theta_\ell + UB_i(1 - w_{i\ell}), \forall i = 1, \ldots, n, \; \ell = 1, \ldots, n, \\
& \theta_\ell \geq \theta_{\ell+1}, \forall \ell = 1, \ldots, n-1, \\
& u_{ij} \leq t_i + UB_i(1 - z_{ij}), \forall \; i = 1, \ldots, n, \; j = 1, \ldots, p, \\
& v_{ijk} - x_{jk} + a_{ik} \geq 0, \forall i = 1, \ldots, n, \; j = 1, \ldots, p, \; k = 1, \ldots, d, \\
& v_{ijk} + x_{jk} - a_{ik} \geq 0, \forall i = 1, \ldots, n, \; j = 1, \ldots, p, \; k = 1, .., d, \\
& v_{ijk}^{r} \leq \zeta_{ijk}^{s} u_{ij}^{r-s}, \forall i = 1, \ldots, n, \; j = 1, \ldots, p, \; k = 1, \ldots, d, \\
& \sum_{k=1}^{d} \zeta_{ijk} \leq u_{ij}, \forall i = 1, \ldots, n, \; j = 1, \ldots, p, \\
& \sum_{j=1}^{p} z_{ij} = 1, \forall i = 1, \ldots, n, \\
& \sum_{i=1}^{n} w_{i\ell} = 1, \forall \ell = 1, \ldots, n,
\end{aligned}
$$

MISOCO Formulation

Let x be a feasible solution of the location problem then there exists a solution $(x, z, u, v, \zeta, w, t, \theta)$ of the above problem such that their objective values are equal. Conversely, if $(x, z, u, v, \zeta, w, t, \theta)$ is a feasible solution for the above problem then x is a feasible solution for the location problem. Furthermore, if K satisfies Slater condition then the feasible region of the continuous relaxation of the above problem also satisfies Slater condition and $\rho_\lambda = \hat{\rho}_\lambda$.

Example

$p = 3$, $\tau = \frac{7}{5}$,
$\lambda = (2.25, 1.70, 1.14, 1.11, 1.06, 1.03, 1.01, 1.01, 1.00, 1.00)$.



$x_1^* = (6.19, 1.58)$, $x_2^* = (5.00, 9.36)$, and $x_3^* = (1.44, 6.55)$, with optimal objective value $f^* = 30.1460$

Constrained Case

If any of the following conditions hold:

1. $g_i(x)$ are concave for $i = 1, \ldots, m$ and $-\sum_{i=1}^m \mu_i \nabla^2 g_i(x) \succ 0$ for each dual pair (x, μ) of the problem of minimizing any linear functional $c^t x$ on $\mathbf{K}$ (*Positive Definite Lagrange Hessian* (PDLH)).
2. $g_i(x)$ are sos-concave on $\mathbf{K}$ for $i = 1, \ldots, m$ or $g_i(x)$ are concave on $\mathbf{K}$ and strictly concave on the boundary of $\mathbf{K}$ where they vanish, i.e. $\partial\mathbf{K} \cap \partial\{x \in \mathbb{R}^d : g_i(x) = 0\}$, for all $i = 1, \ldots, m$.
3. $g_i(x)$ are strictly quasi-concave on $\mathbf{K}$ for $i = 1, \ldots, m$.

Then, there exists a constructive finite dimension embedding, which only depends on τ and g_i, $i = 1, \ldots, m$, such that the problem is mixed-integer SDP representable.

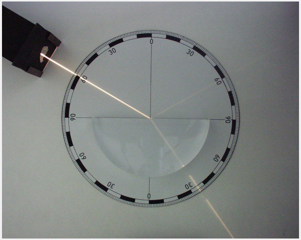
Experiments

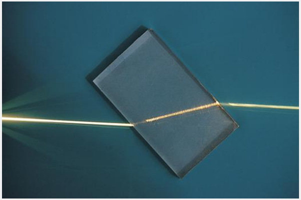
		p-median		p-center		p-25-centrum	
p	τ	CPUTime	f^*	CPUTime	f^*	CPUTime	f^*
2	1.5	22.31	150.955	1.03	4.9452	10.08	100.8474
	2	1.13	135.5222	0.28	4.8209	0.38	95.0892
	3	23.68	130.8560	13.51	4.7880	139.03	89.0238
5	1.5	55.28	78.6074	3.73	2.8831	33.09	53.4995
	2	12.49	72.2369	5.37	2.6610	7.61	49.6932
	3	125.10	68.1791	2.87	2.5094	18.23	46.9844
10	1.5	5.36	45.0525	2.66	1.6929	68.36	30.7137
	2	2.31	41.6851	5.3	1.6113	17.93	28.9017
	3	4.76	39.7222	55.76	1.5950	225.64	27.5376
15	1.5	6.70	30.0543	9.44	1.1139	49.92	22.4165
	2	43.91	27.6282	0.62	1.0717	11.26	20.6536
	3	150.99	26.6047	50.08	1.0530	244.59	20.8544
30	1.5	14.45	9.9488	74.43	1.0080	202.54	9.0806
	2	4.81	8.7963	1.53	0.9192	5.29	8.5216
	3	198.78	8.6995	57.37	0.8508	287.90	8.0016

Blanco, Puerto, Ponce *Continuous location under the effect of refraction*. Submitted. http://arxiv.org/abs/1404.3068

Refraction

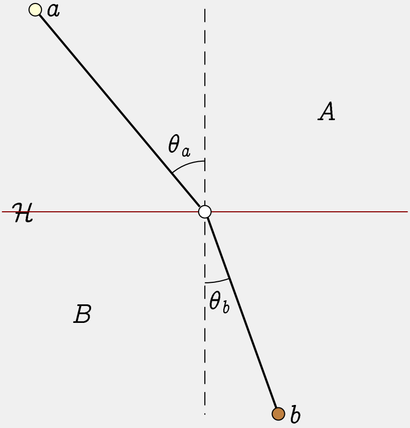
Change in direction of propagation of any wave as a result of its traveling at different speeds at different points along the wave front.





Applications: Transportation Systems connecting urban and rural areas; natural barriers or borders, ...

Euclidean Planar Snell's Law



ω_A, ω_B refraction indices.

$$\min_{x \in \mathcal{H}} \quad \omega_A \|a - x\| + \omega_B \|x - b\|$$

$$\Downarrow$$

$$\boxed{\omega_A \sin \theta_a = \omega_B \sin \theta_b}$$

SP between points separated by a hyperplane

$\mathcal{H} = \{x \in \mathbb{R}^d : \alpha^t x = \beta\}$.

$a \in H_A = \{x \in \mathbb{R}^d : \alpha^t x \leq \beta\}$ ($p_A = \frac{r_A}{s_A}$-norm)

$b \in H_B = \{x \in \mathbb{R}^d : \alpha^t x > \beta\}$ ($p_B = \frac{r_B}{s_B}$-norm)

Lemma

If $1 < p_A, p_B < +\infty$, the length $d_{p_A p_B}(a, b)$ of the shortest weighted path between a and b is

$$d_{p_A p_B}(a, b) = \omega_a \|x^* - a\|_{p_A} + \omega_b \|x^* - b\|_{p_B},$$

where $x^* = (x_1^*, \ldots, x_d^*)^t$, $\alpha^t x^* = \beta$ must satisfy the following conditions:

SP between points separated by a hyperplane

1. For all j such that $\alpha_j = 0$:

$$\omega_a \left[\frac{|x_j^* - a_j|}{||x^* - a||_{p_A}} \right]^{p_A - 1} \mathrm{sg}(x_j^* - a_j) + \omega_b \left[\frac{|x_j^* - b_j|}{||x^* - b||_{p_B}} \right]^{p_B - 1} \mathrm{sg}(x_j^* - b_j) = 0.$$

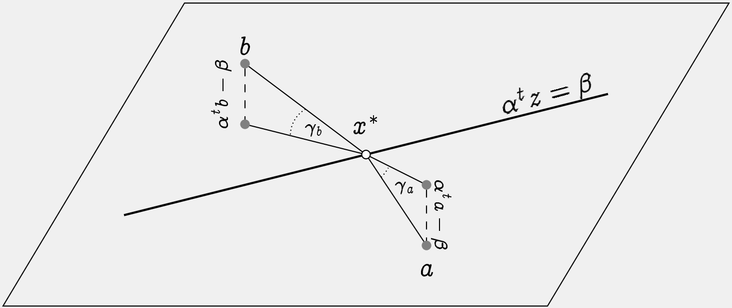
2. For all i, j such that $\alpha_i \alpha_j \neq 0$.

$$\omega_a \left[\frac{|x_i^* - a_i|}{||x^* - a||_{p_A}} \right]^{p_A - 1} \frac{\mathrm{sg}(x_i^* - a_i)}{\alpha_i} + \omega_b \left[\frac{|x_i^* - b_i|}{||x^* - b||_{p_B}} \right]^{p_B - 1} \frac{\mathrm{sg}(x_i^* - b_i)}{\alpha_i} =$$

$$\omega_a \left[\frac{|x_j^* - a_j|}{||x^* - a||_{p_A}} \right]^{p_A - 1} \frac{\mathrm{sg}(x_j^* - a_j)}{\alpha_j} + \omega_b \left[\frac{|x_j^* - b_j|}{||x^* - b||_{p_B}} \right]^{p_B - 1} \frac{\mathrm{sg}(x_j^* - b_j)}{\alpha_j}.$$

Generalized Snell's Law

$$\sin_{p_A} \gamma_{a_j} := \frac{|\alpha_j a_j - \alpha_j x_j^*|}{\|a - x^*\|_{p_A}}, \; j = 1, \ldots, d.$$



Generalized Snell's Law

Corollary (Snell's-like result)

The point x^* in $\mathcal{H}$ must satisfy:

1. *For all j such that $\alpha_j = 0$:*

$$\omega_a \left[\frac{|x_j^* - a_j|}{\|x^* - a\|_{p_A}} \right]^{p_A - 1} \mathrm{sg}(x_j^* - a_j) + \omega_b \left[\frac{|x_j^* - b_j|}{\|x^* - b\|_{p_B}} \right]^{p_B - 1} \mathrm{sg}(x_j^* - b_j) = 0.$$

2. *For all i, j, $\alpha_i \alpha_j \neq 0$.*

$$\omega_a \left[\frac{\sin_{p_A} \gamma_{a_i}}{|\alpha_i|} \right]^{p_A - 1} \frac{\mathrm{sg}(x_i^* - a_i)}{\alpha_i} + \omega_b \left[\frac{\sin_{p_B} \gamma_{b_i}}{|\alpha_i|} \right]^{p_B - 1} \frac{\mathrm{sg}(x_i^* - b_i)}{\alpha_i} =$$

$$\omega_a \left[\frac{\sin_{p_A} \gamma_{a_j}}{|\alpha_j|} \right]^{p_A - 1} \frac{\mathrm{sg}(x_j^* - a_j)}{\alpha_j} + \omega_b \left[\frac{\sin_{p_B} \gamma_{b_j}}{|\alpha_j|} \right]^{p_B - 1} \frac{\mathrm{sg}(x_j^* - b_j)}{\alpha_j},$$

Generalized Snell's Law

Corollary (Snell's Law)

If $d = 2$, $p_A = p_B = 2$ the point x^* satisfies:

$$\omega_A \sin\theta_A = \omega_B \sin\theta_B,$$

where θ_A and θ_B are:

1. *if $\alpha_1 \leq \alpha_2$, the angles between the vectors $a - x^*$ and $(-\alpha_2, \alpha_1)^t$, and $b - x^*$ and $(\alpha_2, -\alpha_1)^t$.*
2. *if $\alpha_1 > \alpha_2$, the angles between the vectors $a - x^*$ and $(\alpha_2, -\alpha_1)^t$, and $b - x^*$ and $(-\alpha_2, \alpha_1)^t$.*

Location under Refraction

Given $\mathcal{H} = \{x \in \mathbb{R}^d : \alpha^t x = \beta\}$, $A \subseteq \mathrm{H}_A$, $B \subseteq \mathrm{H}_B$:

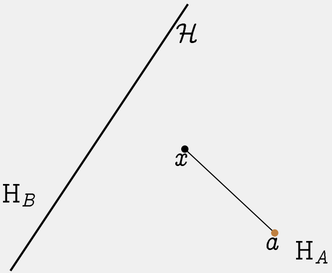
$$f^* := \inf_{x \in \mathbb{R}^d} \sum_{a \in A} \omega_a \; d_{p_A,p_B}(x,a) + \sum_{b \in B} \omega_b \; d_{p_A,p_B}(x,b) \tag{P}$$

where $d_{p_A,p_B}(x,y)$ is the length of the SP between $x, y \in \mathbb{R}^d$.

- Parlar, 1994. (Planar and heuristics).
- Carrizosa & Rodríguez-Chía, 1997. (Rapid transit lines induced by networks on the plane).
- Brimberg, Kakhki & Wesolowsky, 2003, 2005 (Planar and ℓ_1-ℓ_2 norms, bounded regions).
- Zaferanieh, Taghizadeh, Brimberg & Wesolowsky, 2008. (Planar and BSSS based method).
- Fathali & Zaferanieh, 2011. (Planar and block norms).

Our Goal: Exact approach for solving (P) for any d and any ℓ_p-norms.

Shortest paths



Shortest paths

For $x \in \mathbb{R}^d$, then we assume that the shortest path length between x and $a \in \mathrm{H}_A$ or $b \in \mathrm{H}_B$ is:

$$d_{p_A,p_B}(x,a) = \begin{cases} \|x-a\|_{p_A} & \text{if } x \in \mathrm{H}_A, \\ \min\limits_{y \in \mathcal{H}} \|y-a\|_{p_A} + \|x-y\|_{p_B} & \text{if } x \in \mathrm{H}_B, \end{cases}$$

and

$$d_{p_A,p_B}(x,b) = \begin{cases} \|x-b\|_{p_B} & \text{if } x \in \mathrm{H}_B, \\ \min\limits_{y \in \mathcal{H}} \|y-b\|_{p_B} + \|x-y\|_{p_A} & \text{if } x \in \mathrm{H}_A. \end{cases}$$

Theorem

Assume that $\min\{|A|,|B|\} > 2$. *If the points in A or B are not collinear and $p_A < +\infty$, $p_B > 1$ then Problem* (P) *always has a unique optimal solution.*

Formulation

$$
\begin{aligned}
\min \sum_{a\in A} \omega_a Z_a &+ \sum_{b\in B} \omega_b Z_b \\
\text{s.t.}\quad z_a - Z_a &\leq M_a(1-\gamma), && \forall a \in A,\\
\theta_a + u_a - Z_a &\leq M_a\,\gamma, && \forall a \in A,\\
z_b - Z_b &\leq M_b\,\gamma, && \forall b \in B,\\
\theta_b + u_b - Z_b &\leq M_b\,(1-\gamma), && \forall b \in B,\\
z_a &\geq \|x-a\|_{p_A}, && \forall a \in A,\\
\theta_a &\geq \|x-y_a\|_{p_B}, && \forall a \in A,\\
u_a &\geq \|a-y_a\|_{p_A}, && \forall a \in A,\\
z_b &\geq \|x-b\|_{p_B}, && \forall b \in B,
\end{aligned}
$$

$$
\begin{aligned}
\theta_b &\geq \|x-y_b\|_{p_A}, && \forall b \in B,\\
u_b &\geq \|b-y_b\|_{p_B}, && \forall b \in B,\\
\alpha^t x - \beta &\leq M(1-\gamma), &&\\
\alpha^t x - \beta &\geq -M\gamma, &&\\
\alpha^t y_a &= \beta, && \forall a \in A,\\
\alpha^t y_b &= \beta, && \forall b \in B,\\
Z_a, z_a, \theta_a, u_a &\geq 0, && \forall a \in A,\\
Z_b, z_b, \theta_B, u_B &\geq 0, && \forall b \in B,\\
y_a, y_b &\in \mathbb{R}^d, && \forall a \in A, b \in B,\\
\gamma &\in \{0,1\}.
\end{aligned}
$$

Divide et impera

Theorem

Let $x^* \in \mathbb{R}^d$ be the optimal solution of (P)*. Then, x^* is the solution of one of the following two problems:*

$$\min_{x \in \mathrm{H}_A} f * \qquad (\mathrm{P}_A) \qquad\qquad \min_{x \in \mathrm{H}_B} f * \qquad (\mathrm{P}_B)$$

Divide et impera

P_A

$$\min \sum_{a \in A} \omega_a z_a + \sum_{b \in B} \omega_b \theta_b + \sum_{b \in B} \omega_b u_b$$

s.t.

$$z_a \geq \|x - a\|_{p_A}, \forall a \in A,$$
$$\theta_b \geq \|x - y_b\|_{p_A}, \forall b \in B,$$
$$u_b \geq \|b - y_b\|_{p_B}, \forall b \in B,$$
$$\alpha^t y_b = \beta, \forall b \in B,$$
$$\alpha^t x \leq \beta,$$
$$z_a \geq 0, \ \forall a \in A,$$
$$\theta_b, u_b \geq 0, \ \forall b \in B,$$
$$x, y_b \in \mathbb{R}^d.$$

P_B

$$\min \sum_{b \in B} \omega_a z_b + \sum_{a \in A} \omega_a \theta_a + \sum_{a \in A} \omega_a u_a$$

s.t.

$$z_b \geq \|x - b\|_{p_B}, \forall b \in B,$$
$$\theta_a \geq \|x - y_a\|_{p_B}, \forall a \in A,$$
$$u_a \geq \|a - y_a\|_{p_A}, \forall a \in A,$$
$$\alpha^t y_a = \beta, \forall a \in A,$$
$$\alpha^t x \geq \beta,$$
$$z_b \geq 0, \ \forall b \in B,$$
$$\theta_a, u_a \geq 0, \ \forall a \in A,$$
$$x, y_a \in \mathbb{R}^d.$$

NLP Formulation

Lemma

$Z \geq \|X - Y\|_p$, for any $p = \frac{r}{s}$ with $r, s \in \mathbb{N} \setminus \{0\}$, $r > s$ and $\gcd(r, s) = 1$, and X, Y variables in $\mathbb{R}^d$, can be equivalently written as the following set of constraints:

$$
\begin{aligned}
Q_k + X_k - Y_k &\geq 0, \; k = 1, \ldots, d, \\
Q_k - X_k + Y_k &\geq 0, \; k = 1, \ldots, d, \\
Q_k^r &\leq \xi_k^s Z^{r-s}, k = 1, \ldots, d, \\
\sum_{k=1}^{d} \xi_k &\leq Z, \\
\xi_k &\geq 0, k = 1, \ldots, d.
\end{aligned}
$$

NLP Formulation

Theorem

Let $\|\cdot\|_{p_i}$ *be a* ℓ_{p_i}*-norm with* $p_i = \frac{r_i}{s_i} > 1$, $r_i, s_i \in \mathbb{N} \setminus \{0\}$, *and* $\gcd(r_i, s_i) = 1$ *for* $i \in \{A, B\}$. *Then, solving* (P_A) *is equivalent to*

$$
\begin{aligned}
\min \ & \sum_{a \in A} \omega_a z_a + \sum_{b \in B} \omega_j \theta_j + \sum_{b \in B} \omega_b u_b \\
s.t. \ & x \in \mathrm{H}_A, y \in H, \\
& t_{ak} - x_k + a_k \geq 0, \\
& t_{ak} + x_k - a_k \geq 0, \\
& v_{bk} + x_k - y_{bk} \geq 0, \\
& v_{bk} - x_k + y_{bk} \geq 0, \\
& g_{bk} - y_{bk} + b_k \geq 0, \\
& g_{bk} + y_{bk} - b_k \geq 0, \\
& t_{ak}^{r_A} \leq \xi_{ak}^{s_A} z_a^{r_A - s_A}, \\
& v_{bk}^{r_A} \leq \rho_{bk}^{s_A} \theta_b^{r_A - s_A}, \\
& g_{bk}^{r_B} \leq \psi_{bk}^{s_B} u_b^{r_B - s_B},
\end{aligned}
\qquad
\begin{aligned}
& \sum_{k=1}^{d} \xi_{ak} \leq z_a, \\
& \sum_{k=1}^{d} \rho_{bk} \leq \theta_b, \\
& \sum_{k=1}^{d} \psi_{bk} \leq u_b, \\
& \xi_{ak}, t_{ak}, \rho_{bk}, v_{bk}, \psi_{bk}, g_{bk} \geq 0, \\
& z_a, \theta_b, u_b \geq 0, \\
& x, y_b \in \mathbb{R}^d.
\end{aligned}
$$

SOCP Formulation

Corollary

Problem (P_A) *can be represented as a semidefinite programming problem with:*

- $|A|(2d+1) + |B|(4d+3) + 1$ *linear constraints, and*
- *at most* $4d(|A|\log r_A + |B|\log r_A + |B|\log r_B)$ *positive semidefinite constraints.*

Constrained Case

Theorem

Let $\mathbf{K} := \{x \in \mathbb{R}^d : g_j(x) \geq 0,\ j = 1, \ldots, l\}$ *be a basic closed, compact semialgebraic set with nonempty interior, and consider the restricted problem:*

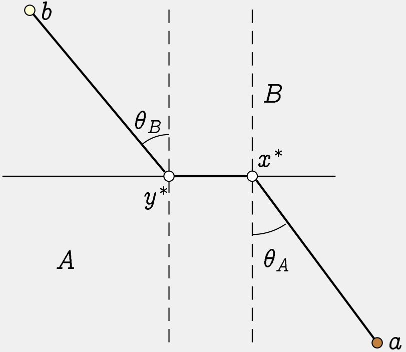
$$\min_{x \in \mathbf{K}} \sum_{a \in A} \omega_a\ d(x, a) + \sum_{b \in B} \omega_b\ d(x, b). \tag{19}$$

Assume that $\mathbf{K}$ *satisfies the Archimedean property and further that any of the following conditions hold:*

1. $g_i(x)$ *are concave for* $i = 1, \ldots, l$ *and* $-\sum_{i=1}^{l} \mu_i \nabla^2 g_i(x) \succ 0$ *for each dual pair* (x, μ) *of the problem of minimizing any linear functional* $c^t x$ *on* $\mathbf{K}$ *(Positive Definite Lagrange Hessian* (PDLH)*).*
2. $g_i(x)$ *are sos-concave on* $\mathbf{K}$ *for* $i = 1, \ldots, l$ *or* $g_i(x)$ *are concave on* $\mathbf{K}$ *and strictly concave on the boundary of* $\mathbf{K}$ *where they vanish, i.e.* $\partial\mathbf{K} \cap \partial\{x \in \mathbb{R}^d : g_i(x) = 0\}$*, for all* $i = 1, \ldots, l$*.*
3. $g_i(x)$ *are strictly quasi-concave on* $\mathbf{K}$ *for* $i = 1, \ldots, l$*.*

Then, there exists a constructive finite dimension embedding, which only depends on p_A*,* p_B *and* g_i*,* $i = 1, \ldots, l$*, such that the solution of* (19) *can be obtained by solving two semidefinite programming problems.*

Hyperplane Endowed with a third norm...



$$d_t(a,b) = \begin{cases} \|a-b\|_{p_i} & \text{if } a, b \in \mathrm{H}_i,\ i \in \{A, B\}, \\ \min\limits_{x,y \in \mathcal{H}} \|x-a\|_{p_A} + \|x-y\|_{p_H} + \|y-b\|_{p_B} & \text{if } a \in \mathrm{H}_A,\ b \in \overline{\mathrm{H}}_B, \end{cases} \tag{DT}$$

Snell's like result

Assume that $\|\cdot\|_{p_A}$, $\|\cdot\|_{p_B}$, $\|\cdot\|_{p_H}$ are ℓ_p-norms with $1 < p < +\infty$. Let $x^*, y^* \in \mathbb{R}^d$, $\alpha^t x^* = \alpha^t y^* = \beta$. Then, x^* and y^* define the shortest weighted path between a and b when traversing the hyperplane is allowed if and only if the following conditions are satisfied:

1. For all j such that $\alpha_j = 0$:

$$\omega_a \left[\frac{|x_j^* - a_j|}{\|x^* - a\|_{p_A}}\right]^{p_A - 1} \mathrm{sg}(x_j^* - a_j) + \omega_H \left[\frac{|x_j^* - y_j^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \mathrm{sg}(x_j^* - y_j^*) = 0,$$

$$\omega_b \left[\frac{|y_j^* - b_j|}{\|y^* - b\|_{p_B}}\right]^{p_B - 1} \mathrm{sg}(y_j^* - b_j) - \omega_H \left[\frac{|x_j^* - y_j^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \mathrm{sg}(x_j^* - y_j^*) = 0.$$

Snell's like result

2. For all i, j, such that $\alpha_i \alpha_j \neq 0$:

$$\omega_a \left[\frac{\sin \gamma_{a_i}}{|\alpha_i|}\right]^{p_A - 1} \frac{\operatorname{sg}(x_i^* - a_i)}{\alpha_i} + \omega_H \left[\frac{|x_i^* - y_i^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \frac{\operatorname{sg}(x_i^* - y_i^*)}{\alpha_i} =$$

$$\omega_a \left[\frac{\sin \gamma_{a_j}}{|\alpha_j|}\right]^{p_A - 1} \frac{\operatorname{sg}(x_j^* - a_j)}{\alpha_j} + \omega_H \left[\frac{|x_j^* - y_j^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \frac{\operatorname{sg}(x_j^* - y_j^*)}{\alpha_j},$$

and

$$\omega_a \left[\frac{\sin \gamma_{b_i}}{|\alpha_i|}\right]^{p_B - 1} \frac{\operatorname{sg}(y_i^* - b_i)}{\alpha_i} - \omega_H \left[\frac{|x_i^* - y_i^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \frac{\operatorname{sg}(x_i^* - y_i^*)}{\alpha_i} =$$

$$\omega_a \left[\frac{\sin \gamma_{b_j}}{|\alpha_j|}\right]^{p_B - 1} \frac{\operatorname{sg}(y_j^* - b_j)}{\alpha_j} - \omega_H \left[\frac{|x_j^* - y_j^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \frac{\operatorname{sg}(x_j^* - y_j^*)}{\alpha_j}.$$

Snell's like result

Corollary

If $d = 2$, $p_A = p_B = p_H = 2$ *and* $\mathcal{H} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$, *the points* x^*, y^* *satisfy one of the following conditions:*

1. $\omega_a \sin\theta_a = \omega_b \sin\theta_b = \omega_H \frac{|y_1^*|}{\|x^* - y^*\|_{p_H}}$ *and* $x^* \neq y^*$, *or*
2. $\omega_a \sin\theta_a = \omega_b \sin\theta_b$ *and* $x^* = y^*$,

where θ_a *is the angle between the vectors* $a - x^*$ *and* $(0, -1)$ *and* θ_b *the angle between* $b - y^*$ *and* $(0, 1)$.

Location if the hyperplane is endowed with third norm

$$\min_{x\in\mathbb{R}^d} \sum_{a\in A} \omega_a\, d_t(x, a) + \sum_{b\in B} \omega_b\, d_t(x, b). \tag{PT}$$

Location if the hyperplane is endowed with third norm

Theorem

Assume that $\min\{|A|, |B|\} > 2$. *If the points in* A *or* B *are not collinear and* $p_B > 1$ *or* $p_A < +\infty$ *then Problem* (PT) *always has a unique optimal solution.*

Proposition

Let $A, B \subseteq \mathbb{R}^d$ *and* $\mathcal{H} = \{x \in \mathbb{R}^d : \alpha^t x = \beta\}$. *Then, if* $p_A \geq p_B \geq p_H$, *Problem* (PT) *reduces to Problem* (P).

Theorem

(PT) *can also be formulated as a SOCP Problem.*

Experiments: Comparisons

SOCP coded in Gurobi 5.6 (PC with an Intel Core i7 processor at 2x 2.40 GHz, and 4GB RAM). Barrier convergence tol. QCP: 10^{-10}.
(*) Parlar '94 (**) Zaferanieh, Taghizadeh, Brimberg & Wesolowsky, '08. $\ell_{p_H} = \frac{1}{4}\ell_\infty$

$\|A \cup B\|$	$\mathcal{H}$	CPUTime	f^*	CPUTime$^{(*),(**)}$	$f^{(*),(**)}$
4 (*)	$y = x$	0.037041	26.951942	49.62	26.951958
18 (*)	$y = 1.5x$	0.057064	112.350633	35.54	112.350702
30 (**)	$y = 0.5x$	0.056049	301.378686	8.25	301.491361
30 (**)	$y = x$	0.076050	265.971645	15.31	265.973315
30 (**)	$y = 1.5x$	0.074053	257.814199	16.94	257.814247
50 (**)	$y = 0.5x$	0.107079	1126.392248	35.00	1127.382313
50 (**)	$y = x$	0.116091	966.377027	30.61	966.377615
50 (**)	$y = 1.5x$	0.095062	939.487369	29.44	939.487629

With a third norm for the hyperplane:

$\|A \cup B\|$	$\mathcal{H}$	CPUTime	f^*	x^*
4 (*)	$y = x$	0.0000	20.5307	$(0.000000, 0.000001)$
18 (*)	$y = 1.5x$	0.0000	108.3362	$(8.811381, 7.119336)$
30 (**)	$y = 0.5x$	0.0156	254.7805	$(6.000000, 3.000000)$
30 (**)	$y = x$	0.0000	230.7513	$(5.234851, 5.234838)$
30 (**)	$y = 1.5x$	0.0156	244.4072	$(5.153294, 5.102873)$
50 (**)	$y = 0.5x$	0.0156	917.1736	$(11.923664, 5.961832)$
50 (**)	$y = x$	0.0156	808.2990	$(10.000020, 9.999995)$
50 (**)	$y = 1.5x$	0.0156	892.4482	$(10.521522, 9.571467)$

Eilon, Watson-Gandy & Christofides data set

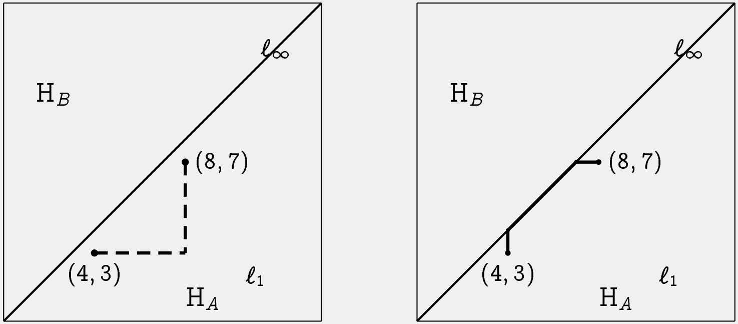
$\lvert A\cup B\rvert = 50$			$\mathcal{H}=\{y=1.5x\}$ ($\lvert A\rvert=15$)		$\mathcal{H}=\{y=x\}$ ($\lvert A\rvert=18$)		$\mathcal{H}=\{y=0.5x\}$ ($\lvert A\rvert=39$)	
p_A	p_B	p_H	CPUTime	f^*	CPUTime	f^*	CPUTime	f^*
1.5	1		0.0000	230.8447	0.0313	212.9341	0.0156	200.6406
2	1		0.0158	227.9991	0.0156	202.6576	0.0000	185.9525
	1.5		0.0313	194.1881	0.0313	189.0401	0.0156	182.1283
3	1		0.0313	223.8203	0.0469	194.1612	0.0156	174.0444
	1.5		0.0156	192.0466	0.0469	180.9279	0.0313	170.3199
	2		0.0156	178.2223	0.0312	174.8964	0.0313	168.5066
∞	1		0.0000	219.8367	0.0000	182.1900	0.0000	161.2033
	1.5		0.0313	188.7783	0.0156	168.9589	0.0000	157.2146
	2		0.0156	175.4420	0.0156	163.6797	0.0000	155.6124
	3		0.0156	164.5924	0.0156	159.3740	0.0156	154.3965
1	1	1.5	0.0156	237.4732	0.0156	224.9178	0.0000	236.1300
		2	0.0000	237.3162	0.0156	218.9480	0.0000	235.4689
		3	0.0156	236.3904	0.0156	213.5591	0.0156	234.9807
		∞	0.0000	233.7967	0.0156	204.3500	0.0000	234.7300
1.5	1	2	0.0156	230.8165	0.0313	206.9512	0.0469	200.5514
		3	0.0625	228.5484	0.0938	201.5863	0.0156	200.3068
		∞	0.0313	225.9387	0.0156	192.4722	0.0156	200.1428
	1.5	2	0.0313	196.5559	0.0469	193.3584	0.0313	196.4864
		3	0.0469	196.5561	0.0469	188.3989	0.0313	196.3008
		∞	0.0156	196.5431	0.0469	179.3396	0.0313	196.1787
2	1	3	0.0156	225.7539	0.0313	197.2805	0.0156	185.9501
		∞	0.0156	223.1421	0.0156	188.1506	0.0156	185.9133
	1.5	3	0.0469	194.1881	0.0469	184.0770	0.0313	182.1271
		∞	0.0156	194.1881	0.0313	175.0117	0.0158	182.0955
	2	3	0.0156	180.1096	0.0156	178.0624	0.0156	180.1097
		∞	0.0156	180.1097	0.0156	169.7842	0.0156	180.0857
3	1	∞	0.0313	221.2011	0.0156	184.9957	0.0313	174.0442
	1.5		0.0313	192.0466	0.0313	171.8455	0.0313	170.3199
	2		0.0156	178.2223	0.0313	166.6027	0.0156	168.5066
	3		0.0312	166.8362	0.0469	162.3214	0.0313	166.8361

Experiments: Larger Instances

			$\lvert A \cup B \rvert = 5000$			$\lvert A \cup B \rvert = 10000$			$\lvert A \cup B \rvert = 50000$		
p_A	p_B	p_H	$d = 2$	$d = 3$	$d = 5$	$d = 2$	$d = 3$	$d = 5$	$d = 2$	$d = 3$	$d = 5$
1.5	1		3.2034	5.4599	10.1520	7.4852	9.2511	19.0804	40.9418	74.9246	115.2941
2	1		1.5939	2.2502	7.6415	5.1255	8.2040	14.0078	21.8708	25.9411	59.7786
	1.5		3.9692	6.0632	4.5474	8.1728	14.0797	23.8067	55.2635	83.8310	154.2883
3	1		3.9222	5.1412	6.9852	6.8132	9.4927	20.6114	42.9964	61.4724	116.4665
	1.5		5.4850	10.0950	13.4449	14.3149	21.0337	34.0574	91.9616	106.6900	206.6997
	2		7.9385	9.8603	10.1802	14.2672	17.7362	38.0629	95.3150	135.0647	180.6230
∞	1		0.3125	0.6940	9.4607	0.8750	1.6096	6.3288	6.0945	25.7856	89.7772
	1.5		1.2346	2.2502	8.6333	5.6724	4.9605	9.1259	18.8410	32.5503	54.0310
	2		0.8908	1.2188	15.9704	1.9534	2.7346	7.9853	18.8615	17.2053	40.5464
	3		3.4691	2.7346	12.0584	9.5637	6.7195	9.5323	71.7654	70.1868	49.5907
1	1	1.5	18.9396	28.7109	15.6735	37.5415	80.9833	401.8414	596.6057	878.6363	3171.6235
		2	13.7043	24.4318	13.2359	29.2056	68.3894	372.3283	354.3334	721.5562	3166.1511
		3	17.5702	25.1258	3.8570	39.3008	93.4990	415.0733	541.8219	1014.1090	3945.8234
		∞	4.9695	11.7517	3.1101	13.7673	26.7468	96.7260	133.7586	632.9736	2492.2830
1.5	1	2	5.2506	8.2509	4.6457	13.7986	16.0956	37.3793	105.4177	103.2694	273.0866
		3	6.2975	11.9545	4.0473	13.2135	24.9720	57.8267	96.9583	128.9880	326.7660
		∞	3.6722	5.5632	4.1409	7.0632	13.1580	31.0345	46.1239	81.3482	118.2435
	1.5	2	12.9546	15.8455	3.7347	23.3466	29.3155	46.6898	138.6629	200.2891	385.1307
		3	13.5232	14.9234	4.5473	22.2837	33.9099	53.9483	171.0538	175.6803	697.5071
		∞	12.0022	11.5482	3.9533	21.8464	22.1743	37.0102	111.1779	144.5975	241.2852
2	1	3	3.5316	7.6883	125.3288	9.8294	11.5794	41.0986	61.4067	62.9410	158.6635
		∞	1.7034	3.3288	145.9833	3.5629	7.7041	15.4610	22.8465	38.9976	98.4269
	1.5	3	5.6255	9.3605	105.3967	13.4234	19.0805	45.4697	71.1114	101.3439	269.3303
		∞	5.1256	5.4850	137.3159	7.6791	16.5075	24.8255	63.0027	85.4602	134.8291
	2	3	6.6725	9.4387	132.3028	12.1731	20.4003	39.2473	79.9453	121.0863	220.7875
		∞	4.6879	5.4607	153.6319	9.4696	14.5639	22.6620	68.1690	63.1358	118.4005
3	1	∞	3.7357	6.5511	17.7052	7.8602	10.1575	34.1457	37.1292	48.5630	140.3546
	1.5		7.7665	10.4455	17.7145	15.2061	26.2626	37.2546	84.7931	119.5438	235.1177
	2		7.6569	10.6885	17.4306	16.5483	23.6745	44.5896	99.2611	227.0411	219.4903
	3		9.8843	10.0948	19.1583	19.2838	21.8153	43.0209	129.5420	153.3979	243.4983

'Easy' Extensions

- Norms for each demand points: Each point provided with two norms $||\cdot||_a^A$ and $||\cdot||_b^B$.
- Critical Reflection angle principle: Shortest paths between points in the same halfspace are allowed to "traverse and reflect".



length $= 8$.

length $= 6$.

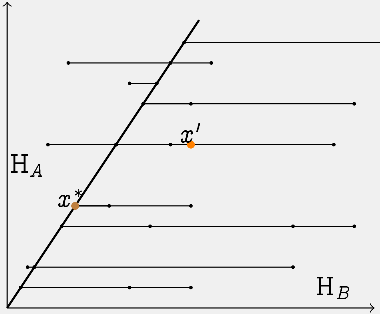
Extensions

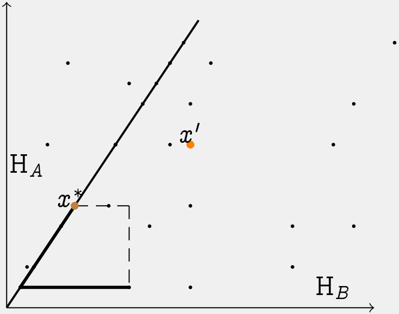
$$
\begin{aligned}
\min \sum_{a\in A} \omega_a z_a + \sum_{b\in B} \omega_b \theta_b + \sum_{b\in B} \omega_b u_b & \\
\text{s.t. } z_a^1 \geq \|x-a\|_{p_A},\ \forall a \in A, & \\
z_a^2 \geq \|x-y_a^1\|_{p_A},\ \forall a \in A, & \\
z_a^3 \geq \|y_a^1-y_a^2\|_{p_B},\ \forall a \in A, & \\
z_a^4 \geq \|y_a^2-a\|_{p_A},\ \forall a \in A, & \\
\theta_b \geq \|x-y_b\|_{p_A},\ \forall b \in B, & \\
u_b \geq \|y_b-b\|_{p_B}\ \forall b \in B, & \quad (\mathrm{P}_A^{\mathrm{EXT}}) \\
z_a \geq z_a^1 + M_a(\delta_a-1), \forall a \in A, & \\
z_a \geq z_a^2 + z_a^3 + z_4^a - M_a\delta_a, \forall a \in A, & \\
\alpha^t x \leq \beta, & \\
\alpha^t y_a^j = \beta,\ \forall j = 1,2, & \\
\alpha^t y_b = \beta,\ \forall a \in A, & \\
\delta_a \in \{0,1\},\ \forall a \in A, & \\
z_a^k \geq 0,\ \forall a \in A, k = 1,2,3,4 & \\
\theta_b, u_b \geq 0,\ \forall b \in B, & \\
x, y_a^1, y_a^2, y_b \in \mathbb{R}^d. &
\end{aligned}
$$

$$
\begin{aligned}
\min \sum_{a\in A} \omega_a z_a + \sum_{b\in B} \omega_b \theta_b + \sum_{b\in B} \omega_b u_b & \\
\text{s.t. } z_b^1 \geq \|x-b\|_{p_B},\ \forall b \in B, & \\
z_b^2 \geq \|x-y_b^1\|_{p_B},\ \forall b \in B, & \\
z_b^3 \geq \|y_b^1-y_b^2\|_{p_A},\ \forall b \in B, & \\
z_b^4 \geq \|y_b^2-b\|_{p_B},\ \forall b \in B, & \\
\theta_a \geq \|x-y_a\|_{p_A},\ \forall a \in A, & \\
u_a \geq \|y_a-a\|_{p_B}\ \forall a \in A, & \quad (\mathrm{P}_B^{\mathrm{EXT}}) \\
z_b \geq z_b^1 + M_b(\delta_b-1), \forall b \in B, & \\
z_b \geq z_b^2 + z_b^3 + z_4^a - M_b\delta_b, \forall b \in B, & \\
\alpha^t x \geq \beta, & \\
\alpha^t y^j = \beta,\ \forall j = 1,2, & \\
\alpha^t y_a = \beta,\ \forall a \in A, & \\
\delta_b \in \{0,1\},\ \forall b \in B, & \\
z_b^k \geq 0,\ \forall a \in A, k = 1,2,3,4 & \\
\theta_b, u_b \geq 0,\ \forall b \in B, & \\
x, y_b^1, y_b^2, y_a \in \mathbb{R}^d. &
\end{aligned}
$$

Extensions

$$\ell_{p_H} = \frac{1}{4}\ell_\infty,\ \ell_{p_A} = \ell_{p_B} = \ell_1$$





$f^* = 128.00$
$f' = 132.9166.$

$d((6,1), x^*) = 6.3333$
$d_1((6,1), x^*) = 6.66666.$

Extensions

$$p_A = p_B = 1 \text{ and } \|\cdot\|_H = \frac{1}{4}\ell_\infty \ , \mathcal{H} = \{(x, y) : y = \alpha_1 \, x\}$$

α_1	N	x'_T	f'_T	CPUTime$_T$	x^*_{Refl}	f^*_{Refl}	CPUTime$_{Refl}$	Improvement
0.5	4	$(5, 2.5)$	16.75	0.0000	$(5, 2.5)$	16.75	0.0156	0.00%
	18	$(9, 4.5)$	97.75	0.0000	$(9, 4.5)$	89.50	0.0313	9.22%
	30	$(6, 3)$	266.50	0.0000	$(6, 3)$	251.00	0.0313	6.18%
	50 (*)	$(12, 6)$	959.75	0.0000	$(11, 5.5)$	911.50	100.1884	5.29%
	50 (**)	$(5.89, 2.945)$	201.55	0.0000	$(5.89, 2.945)$	189.91	100.1563	6.13%
1	4	$(5, 6)$	24.17	0.0000	$(4, 6)$	23.67	0.0156	2.11%
	18	$(9, 8)$	132.92	0.0000	$(3.3333, 5)$	128.00	0.0156	3.84%
	30	$(5, 5)$	299.75	0.0000	$(2.6667, 4)$	269.75	0.0625	11.12%
	50 (*)	$(11, 10)$	1076.58	0.0156	$(5.3333, 8)$	1009.25	100.4547	6.67%
	50 (**)	$(3.7133, 5.570)$	206.37	0.0156	$(3.5, 5.250)$	195.52	100.0241	5.55%
1.5	4	$(0, 0)$	22.50	0.0000	$(5, 5)$	22.50	0.0156	0.00%
	18	$(8, 8)$	123.00	0.0000	$(8, 8)$	105.50	0.0781	16.59%
	30	$(5, 5)$	265.25	0.0000	$(5, 5)$	251.25	1.2971	5.57%
	50 (*)	$(1, 10)$	927.75	0.0000	$(1, 10)$	873.50	100.0066	6.21%
	50 (**)	$(5, 5)$	177.52	0.0000	$(5.57, 5.57)$	170.40	11.6622	4.18%

(*) Zaferanieh, Taghizadeh Kakhki, Brimberg, J. & Wesolowsky, '08.

(**) Eilon, Watson-Gandy & Christofides, '71.

More?

What about locating other objects?

- Segments (Imai, Lee & yang; 1992), (Agarwal, Efrat, Sharir & Toledo, 1993), (Petersen, 1997).
- Rectilinear trajectories (Díaz-Bañez & Mesa, 1996).
- Rectangular facilities: (Carrizosa, Muñoz-Márquez, Puerto; 1998).
- Lines & Hyperplanes: (Schöbel, 1999).
- Point + Segment (Service and Rapid Transit Line): (Espejo & Rodríguez-Chía, 2011).
- Polyhedral Structures.
- Minisum Hyperspheres: (Körner, 2011).
- Competitive/Huff Location models. (Drezner, 1995).
- Any dimension? Any norm? other criteria? other convex bodies? new approaches?