Multifacility Location Problem

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Chapter 4
Multifacility Location Problem

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In the previous chapter, we studied the case of a single new facility to be located relative to a number of existing facilities. In this chapter we consider the problem of optimally locating more than one new facilities with respect to locations of a number of existing facilities (demand points), the locations of which are known.

While the problems are natural extension of those of single facility location, there are two important conditions:

1. At least two facilities are to be located
2. Each new facility is linked to at least one other new facility

If the first condition contracted, this problem is considered as a single facility location problem (SFLP) and if the second condition contracted, we can consider the problem as some of independent single facility location problems. Thus the SFLP can be considered as a spatial case of the multifacility location problem (MFLP).

4.1 Applications and Classifications

As it is expected, applications of MFLPs occur in the same contexts as discussed in the chapter "Single Facility Location Problem" by Esmaeel Moradi and Morteza Bidkhori, this volume for SFLP. Ostresh (1977) represented some applications of the MFLP as follows:

1. A system of warehouses is to be established to serve a set of predetermined regions.
2. Industrial and commercial establishments tend to be more concentrated than expected on the basis of minimizing transport costs alone.
3. In large organizations (such as the Federal government) face to face communication must take place between adjacent (usually) levels of the hierarchy.

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A classification of different types of MFLP and their properties is shown in the following statement:

- Area solution: discrete, continual
- The space in which facilities are located: planer location, sphere location
- Objective function: MiniMax, MiniSum
- Type of the distance: rectangular distance, Euclidean distance, squared Euclidean distance, l_p distance
- Parameters: stochastic, deterministic
- How facilities are assumed: point, region

Researchers have worked on a variety of MFLPs. However no research has been conducted on many types of MFLPs resulted from multiplying the above items.

4.2 Models

In this section we introduce MFLP models represented and developed as mathematical models.

4.2.1 MiniSum

The MiniSum multifacility location problem consisting of finding locations of new facilities which will minimize a total cost function consists of a sum of costs directly proportional to the distances between the new facilities, and costs directly proportional to the distances between new and existing facilities.

The general-model for this problem can be stated as follows:

4.2.1.1 Model Assumptions

The assumptions of this model are as follows:

- The area solution is continual
- The space in which facilities are located is planer
- The objective function is MiniSum
- Type of the distance can be either rectangular, Euclidean, squared Euclidean or l_p distance
- Parameters are deterministic
- Facilities are assumed as points

4.2.1.2 Model Inputs

Model inputs of this model are:

n: Number of new facilities
m: Number of existing facilities
w_{ij}: Nonnegative weight between new facility i and existing facility j by a unit distance
v_{ik}: Nonnegative weight between new facilities i and k by a unit distance
$d(X_j, P_i)$: Distance between the location of new facility j and existing facility i
$d(X_j, X_k)$: Distance between the location of new facilities j and k
$P_j : (a_j, b_j)$ The location coordinates of existing facility j

4.2.1.3 Model Output (Decision Variable)

Model output of this model is:

$X_i : (x_i, y_i)$ The location coordinates of new facility i

4.2.1.4 Objective Function

$$\text{Min} \sum_{1 \le j < k \le n} v_{jk}.d(X_j,\ X_k) + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}.d(X_j,\ P_i). \tag{4.1}$$

Thus each of the m new facilities is to be located with respect to the n existing facilities and also with respect to the other new facilities. The location of X_j may depend on the location of some point X_k because of the terms involving v_{jk}. For convenience in the presentation, we assume all the w_{ji} and all the v_{jk} are positive.

Notice that it is the cost proportional to the distance between new facilities which distinguish the MFLP from SFLP. In fact, when terms v_{ji} are zero then (4.1) may be written as

$$\text{Min} \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}.d(X_j,\ P_i). \tag{4.2}$$

And (4.2) just defines a one-facility total cost expression. So that (4.2) is the sum of n different one-facility cost expression and may be written as

$$\text{Min} \sum_{j=1}^{n} \left(\text{Min} \sum_{i=1}^{m} w_{ji}.d(X_j,\ P_i) \right). \tag{4.3}$$

Since the location of one new facility has no effect upon the cost of locating other new facilities for the special case where all v_{ij} are zero, locations of new facilities may be found by solving n SFLP independently.

4.2.1.5 Rectangular Distance MiniSum Location Problem

In this section, we consider MFLP when we have rectangular distance. The objective of this problem is minimization of the weighted sum of the rectangular distance between the locations of new facilities and new and existing facilities. For formulation of this problem if in (4.1) we replace the $d(X_j, P_i)$ and $d(X_j, X_k)$ by (4.4) and (4.5) The Rectangular MFLP problem can be formulated as (4.6) this problem is of added interest since it will be shown that its optimal solution can be used to compute both a lower and an upper bound for the value of the optimal solution to the Euclidean problem.

$$d(X_j\,,\,X_k) \;=\; \left|x_j - x_k\right| + \left|y_j - y_k\right|, \tag{4.4}$$

$$d(X_j, P_i) = \left|x_j - a_i\right| + \left|y_j - b_i\right|, \tag{4.5}$$

$$\begin{aligned} \text{Min} \sum_{1\le j<k\le n} & v_{jk}\left(\left|x_j - x_k\right| + \left|y_j - y_k\right|\right) \\ & + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\left(\left|x_j - a_i\right| + \left|y_j - b_i\right|\right). \end{aligned} \tag{4.6}$$

The objective function is converted into two minimum problems:

$$\text{Min } f = \text{Min } f_1(x) + \text{Min } f_2(x). \tag{4.7}$$

where:

$$f_1(X) = \sum_{1\le j<k\le n} v_{jk}\left|x_j - x_k\right| + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\left|x_j - a_i\right|, \tag{4.8}$$

$$f_2(Y) = \sum_{1\le j<k\le n} v_{jk}\left|y_j - y_k\right| + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\left|y_j - b_i\right|. \tag{4.9}$$

Just like single facility location case in the chapter "Single Facility Location Problem" by Esmaeel Moradi and Morteza Bidkhori, this volume, optimum x coordinate of new facilities can be found independent from optimum y coordinates. As you see the objective function is nonlinear and we should make it linear. The method of linearization is very similar to the method we used in single facility location in the chapter "Single Facility Location Problem" by Esmaeel Moradi and Morteza Bidkhori, this volume.

Linearization

Given numbers a, b, p and q,

If

$$\begin{aligned} & a - b - p + q = 0, \\ & p.q = 0, \\ & p \ge 0,\ q \ge 0. \end{aligned}$$

Then

$$|a-b| = p + q.$$

So minimizing the objective function is equivalent to the following problem:

$$\text{Min} \quad \sum_{1\le j<k\le n} v_{jk}(p_{jk}+q_{jk}) + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}(r_{ji}+s_{ji}), \tag{4.10}$$

Subject to

$$x_j - x_k - p_{jk} + q_{jk} = 0 \qquad 1\le j<k\le n, \tag{4.11}$$

$$x_j - r_{ji} + s_{ji} = a_i \qquad i=1,\ldots,m,\ j=1,\ldots,n, \tag{4.12}$$

$$p_{jk}, q_{jk} \ge 0 \qquad 1\le j<k\le n, \tag{4.13}$$

$$r_{ji}, s_{ji} \ge 0 \qquad i=1,\ldots,m,\ j=1,\ldots,n, \tag{4.14}$$

$$p_{jk}q_{jk} = 0 \qquad 1\le j<k\le n, \tag{4.15}$$

$$r_{ji}.s_{ji} = 0 \qquad i=1,\ldots,m,\ j=1,\ldots,n, \tag{4.16}$$

$$x_j \text{ unrestricted}, \qquad j=1,2,\ldots,n. \tag{4.17}$$

As in the single facility location for any basic feasible solution, if p_{jk} is in the basic feasible solution, q_{jk} will not be and vice versa. Likewise, if r_{ji} is in the basic feasible solution, s_{ji} will not be and vice versa. Since variables not in the basic feasible solution are zero, the multiplicative constraints will be therefore satisfied for every feasible solution. Minimizing f_2 is the same as what was done for f_1.

4.2.1.6 Squared Euclidean Distance MiniSum Location Problem

In this section, we consider MFLP when we have squared Euclidean distance. The objective of this problem is minimization of the weighted sum of the square of the Euclidean distance between the locations of new facilities and new and existing facilities. The problem considered in this section is also referred to as a "quadratic facility location problem" and the "gravity problem".

Although there aren't too many situations where there are physical reasons for using squared Euclidean distance, there are at least two reasons for the gravity problem. First, in some cases the solution to the gravity problem can be used to approximate the solution to the location problem where costs increase linearly with Euclidean distance. Second, there exist location problems where costs increase quadratically with the Euclidean distance between facilities.

For formulation of this problem if in (4.1) we replace the $d(X_j, P_i)$ and $d(X_j, X_k)$ by (4.18) and (4.19) the squared Euclidean MFLP problem can formulate as (4.20)

$$d(X_j, X_k) = (x_j - x_k)^2 + (y_j - y_k)^2, \tag{4.18}$$

$$d(X_j, X_i) = (x_j - a_i)^2 + (y_j - b_i)^2, \tag{4.19}$$

$$\text{Min } f = \sum_{1 \leq j < k \leq n} v_{jk}.\left[(x_j - x_k)^2 + (y_j - y_k)^2\right] + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}.\left[(x_j - a_i)^2 + (y_j - b_i)^2\right]. \tag{4.20}$$

4.2.1.7 Contour Lines for Squared Euclidean MFLP

In MFLP construction of contour lines is possible expect for certain special case when $n = 2$. In this section we want to observe the property of contour lines for squared Euclidean MFLP

Let $f(x, y)$ given in (4.20) be written as $f(x, y) = f(x) + f(y)$ where

$$f(x) = \sum_{1 \leq j < k \leq n} v_{jk}(x_j - x_k)^2 + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}(x_j - a_i)^2. \tag{4.21}$$

And

$$f(y) = \sum_{1 \leq j < k \leq n} v_{jk}(y_j - y_k)^2 + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}(y_j - b_i)^2. \tag{4.22}$$

If $n = 2$, contour lines for $f(x)$ are concentric ellipses centered on (x^*, y^*) and contour lines for $f(y)$ are concentric ellipses centered on (x^*, y^*)

Because of the squared Euclidean MFLP is separable and symmetric in x and y, only $f(x)$ is considered and contour line is defined as the set of all points (x_1, x_2) for which

$$k = v_{12}(x_1 - x_2) + \sum_{i=1}^{m} w_{1i}(x_1 - a_i)^2 + \sum_{i=1}^{m} w_{1i}(x_2 - a_i)^2. \tag{4.23}$$

And k is a constant denoting the value of the contour line. By expanding and collecting terms it is seen that (4.23) can be expressed in the form of a general conic section:

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0, \tag{4.24}$$

where

$$A = v_{12} + \sum_{i=1}^{m} w_{1i},$$
$$B = -2v_{12},$$
$$C = v_{12} + \sum_{i=1}^{m} w_{2i},$$

$$D = -2\sum_{i=1}^{m} w_{1i}a_i,$$

$$E = -2\sum_{i=1}^{m} w_{2i}a_i,$$

$$D = \sum_{i=1}^{m} w_{1i}a_i^2 + \sum_{i=1}^{m} w_{2i}a_i^2 - k.$$

A sufficient condition given by Thomas (1968) for (4.24) to be an ellipse is for the discriminate, $B^2 - 4AC$, to be negative. Direct substitution gives

$$B^2 - 4AC = 4v_{12}^2 - 4v_{12}^2 - 4v_{12}\left(\sum_{i=1}^{m} w_{1i} + \sum_{i=1}^{m} w_{2i}\right) - 4\left(\sum_{i=1}^{m} w_{1i}\right)\left(\sum_{i=1}^{m} w_{2i}\right) < 0. \tag{4.25}$$

Assuming the problem is well formulated (4.24) is the equation for a rotated ellipse as noted by the presence of the x_1x_2 term. Due to symmetry a similar result is found for the y variables. Furthermore, by definition of a contour line since k achieves its smallest value at (x_1^*, x_2^*) the contour line is centered on the optimum location with respect to the nonrotated axes (Eyster and White 1973).

4.2.1.8 Euclidean Distance MiniSum Location Problem

The Euclidean multifacility location problem often assumes that the transportation costs from the new facility to a demand point are proportional to the Euclidean distance between these points, with the factor of proportionality (weight) depending on the demand point.

Here, this problem is mathematically formulated:

$$\begin{aligned} \text{Min} \sum_{1 \le j < k \le n} & v_{jk} \cdot \left[(x_j - x_k)^2 + (y_j - y_k)^2\right]^{1/2} \\ &+ \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji} \cdot \left[(x_j - a_i)^2 + (y_j - b_i)^2\right]^{1/2}. \end{aligned} \tag{4.26}$$

It is interesting to note that when all the existing facility locations are collinear, that is, all lie on a single line, the Euclidean model essentially includes the rectilinear distance model.

4.2.2 *MiniMax*

Minimization of a sum of weighted distances that was introduced in Sect. 4.1.1 may not be a proper goal when the facilities to be located must provide services of an urgent nature. Therefore in this section we introduce some multifacility location problems with MiniMax objective function and by different type of distances.

4.2.2.1 Model Assumptions

Model assumptions of this model are as follows:

- The area solution is continual
- The space in which facilities are located is planer
- The objective function is MiniMax
- Type of the distance can be either rectangular, Euclidean, squared Euclidean or l_p distance
- Parameters are deterministic
- Facilities are assumed as points

4.2.2.2 Model Inputs

We have all the inputs in previous model.

4.2.2.3 Model Outputs (Decision Variables)

The outputs of this model are similar to the previous model.

4.2.2.4 Objective Function

$$\text{Min } f = \text{Max}\{ v_{jk}.d(X_j,\ X_k), w_{ji}.d(X_j,\ P_i)\}. \tag{4.27}$$

4.2.2.5 Rectangular Distance MiniMax Location Problem

The problem considered is that of locating n new facilities among m existing facilities with the objective of minimizing the maximum weighed Rectangular distance among all facilities.

$$\begin{aligned} \text{Min } f = \text{Max}\, \big\{ & w_{ji} \left| x_j - a_i \right| + \left| y_j - b_i \right| \; j = 1, \ldots, n, \;\; i = 1, \ldots, m; \\ & v_{jk} \left| x_j - x_k \right| + \left| y_j - y_k \right| 1 \le j < k < n \big\}. \end{aligned} \tag{4.28}$$

4.2.2.6 Euclidean Distance MiniMax Location Problem

The problem considered is that of locating n new facilities among m existing facilities with the objective of minimizing the maximum weighed Euclidean distance among all facilities.

Elzinga et al. formulated this problem in 1976:

$$\begin{aligned}\text{Min } f = \text{Max}\,\{&w_{ji}[(x_j - a_i)^2 + (y_j - b_i)^2]^{1/2}\ j = 1, \ldots, n, i = 1, \ldots, m;\\ &v_{jk}[(x_j - x_k)^2 + (y_j - y_k)^2]^{1/2} 1 \leq j < k < n\}\,.\end{aligned} \tag{4.29}$$

4.2.3 *Other Models*

4.2.3.1 Rectangular Multi Product Multifacility Location Problem

This model considers multi products in MFLP. Sherali and Shetty (1978) formulated the rectangular multi product MFLP.

4.2.3.2 Model Assumptions

Model assumptions of this model are as follows:

- The area solution is continual
- The space in which facilities are located is planer
- The objective function is MiniSum
- The distances are rectangular
- Parameters are deterministic
- Facilities are assumed as points

4.2.3.3 Model Inputs and Outputs (Decision Variables)

Model inputs and outputs of this model are as follows:

c_{ijk}: The cost per unit of product k to be transported from a new facility i to a new or existing facility j
u_{ijk}: The known amount of product k to be transported from a new facility i to a new or existing facility j at a cost of c_{ijk} per unit of the product, per unit distance.

(x_j, y_i): decision variables for $i = 1, 2, \ldots, n$ and are fixed and known for each facility $n + i$, $i = 1, 2, \ldots, m$'.

4.2.3.4 Objective Function

The objective function of this model and its related constraints are as follows:

$$\text{Min} \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{j=1}^{n+m'} c_{ijk} u_{ijk} \left\{ \left| x_i - x_j \right| + \left| y_i - y_j \right| \right\}. \tag{4.30}$$

The objective function may be written as:

$$\text{Min} \sum_{k=1}^{p} \sum_{i=1}^{n} w_{ij} \left\{ \left| x_i - x_j \right| + \left| y_i - y_j \right| \right\}, \tag{4.31}$$

$$w_{ij} = \sum_{k=1}^{p} (c_{ijk} u_{ijk} + c_{jik} u_{jik}) \qquad j = n+1, \ldots, n+m'. \tag{4.32}$$

4.2.3.5 Multifacility Location Problem on Sphere

As it is known, one of the assumptions when locating facilities is concerned with the size of the area covering the destinations (or the demand points). If the area covering the demand points is sufficiently small, then this part of the earth's surface can be approximated by a plane. When the destination points are widely separated, the area covering these points can no longer be approximated by a plane and the formulations we discussed so far is not suitable.

Problems concerning location of international headquarters, distribution/ marketing centers, detection station placement, and placement of radio transmitters for long range communication may fall into this category.

The location problem on a sphere is more complicated than its counterpart on the plane, because unlike on plane the solution space is not convex on sphere.

Dhar and Rao (1982) formulated this problem. Any point on the sphere can be defined by its latitude $-\pi/2 \leq \Phi \leq \pi/2$ and longitude $-\pi \leq \theta \leq \pi$ and d_{ij} is the shortest distance between points i and j.

4.2.3.6 Model Assumptions

Model assumptions of this model are as follows:

- The area solution is continual
- The space in which facilities are located is on sphere
- The objective function is MiniSum
- Parameters are deterministic
- Facilities are assumed as points
- The distance is defined as

$$d_{ij} = \text{Arc}\cos[\cos \Phi_i \cos \Phi_j \cos(\theta_i - \theta_j) + \sin \Phi_i \sin \Phi_j]. \tag{4.33}$$

4.2.3.7 Model Inputs

We have all the inputs in MiniSum models.

4.2.3.8 Model Outputs (Decision Variables)

The outputs of this model are similar to the MiniSum models.

4.2.3.9 Objective Function

The objective function of this model and its constraints are as follows:

$$\text{Min}_{\Phi_j \theta_j} \sum_{1 \le j < k \le n} v_{jk}(d_{jk}) + \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji}(d_{ji}). \tag{4.34}$$

Subject to

$$-\pi/2 \le \Phi_j \le \pi/2, \tag{4.35}$$

$$-\pi \le \theta \le \pi. \tag{4.36}$$

4.2.3.10 Multifacility Location Problem with Rectangular Regions

So far we assumed each existing and new facility as a point. Wesolowsky and Love applied the concept of Rectangular regions to MFLP in 1971a, b. They stated that in many different contexts it is proper to treat the destination to be served as a region, rather than a point such as in the location of a library or emergency services or other public service facility designed to serve either a neighborhood or a densely populated urban area.

Even when the users of the new facility are discretely distributed, the number of users may become so large that it may be infeasible in terms of data collection and computational efficiency to represent each customer by a point. In this situation it is necessary to assume regional destinations in order to solve the problem.

4.2.3.11 Model Assumptions

Model assumptions of this model are as follows:

- The area solution is continual
- The space in which facilities are located is planer
- The objective function is MiniSum
- The distances are rectangular

- Parameters are deterministic
- Facilities are assumed as regions

4.2.3.12 Model Inputs

We have all the inputs in MiniSum models in addition to the following inputs:
$R_i = \left[a_{i_1}, a_{i_2}\right]$ Rectangular region i, where $a_{i_1} < a_{i_2}$ and $b_{i_1} < b_{i_2}$
A_i: The area of R_i

4.2.3.13 Model Outputs (Decision Variables)

The outputs of this model are similar to MiniSum models.

4.2.3.14 Objective Function

The objective function of this model and its constraints are as follows:

$$\text{Min} \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{w_{ij}}{A_i} \int_{R_i}\int \left(\left|x_j - a_i\right| + \left|y_j - b_i\right|\right) da_i \, db_i + \sum_{i \leq j < k \leq n} v_{jk} \left(\left|x_j - x_k\right| + \left|y_j - y_k\right|\right). \qquad (4.37)$$

4.2.3.15 Stochastic Multifacility Location Problem

In many cases the parameters of the model are not deterministic. This model is represented and solved by Seppalla (1975). In order to define a stochastic decision model, we should first determine the stochastic parameters, how they are distributed, whether they are correlated, and which decision criterion will be used in industrial applications. The set of stochastic elements of planning models are often restricted to consist only of demands for products, while other parameters or variables, such as unit costs, capacities and locations, are considered to be fixed.

4.2.3.16 Model Assumptions

Model assumptions of this model are as follows:

- The area solution is continual
- The space in which facilities are located is planer
- The objective function is MiniSum
- The distances are Euclidean

- Parameters are stochastic
- Facilities are assumed as points
- The weights v_{jk}, w_{ji} for all i, j and k are normally distributed random variables

4.2.3.17 Model Inputs

The inputs of this model are similar to the MiniSum model in addition to the following:

α: is a predetermined probability
δ: is an assisting
$P\{.\}$: is a probability operator

4.2.3.18 Model Outputs (Decision Variables)

The output of this model is similar to the MiniSum model.

4.2.3.19 Objective Function

The objective function of this model and its constraints are as follows:

$$\text{Min } \delta. \tag{4.38}$$

Subject to

$$P\left\{ \sum_{1 \le j < k \le n} v_{jk}.\left[(x_j - x_k)^2 + (y_j - y_k)^2\right]^{1/2} + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}.\left[(x_j - a_i)^2 + (y_j - b_i)^2\right]^{1/2} \le \delta \right\} \ge \alpha. \tag{4.39}$$

4.3 Solution Techniques

4.3.1 MiniSum

4.3.1.1 Rectangular Distance MiniSum Location Problem

The rectangular distance multifacility location problem always has a minimum cost solution where the x coordinate of each new facility is equal to the x coordinate of some existing facility and the same is true for y coordinate (Francis et al. 1992).

If we call p_1 the linear programming problem obtained in (4.10) a straight forward linear programming of p_1 can be time consuming when a large number of new and existing facilities are involved.

Depending on the characteristics of a particular multifacility Rectangular location problem, the optimum solution can be sometimes obtained in an iterative mode by solving some single facility location problems. Also, a dual formulation of the linear programming problem (p_1) can provide more efficient solution to the rectangular MFLP (Francis et al. 1992).

- Francis (1964) solved a special case of the MFLP, with rectilinear distance when the weights are equal.
- Cabot et al. (1970) decomposed the location problem into two independent subproblems, each of which is equivalent to a linear programming problem which is essentially the dual of a minimal cost network flow problem. The dual variables in each of the optimum tableaus to the two flow problems give the x and y coordinates respectively of the optimum locations of the new facilities.
- Pritsker and Ghare (1970) suggested a gradient technique for this problem. The basic contribution of them was a derivation of necessary conditions for an optimal solution and an algorithm for obtaining optimal solutions to the decomposed problems.
- Rao (1973) considered a direct search approach to the RMFLP in detail, however he demonstrated that the gradient technique was basically a primal simplex-based linear programming approach, and in the presence of degeneracy, the optimality conditions were not sufficient. A necessary condition for optimality to be sufficient in special cases and the main difficulties associated with the direct search approach were discussed.
- Wesolowsky and Love (1971a, b) and Morris (1975) showed that the problem with linear locational constraints could be solved by linear programming.
- A thorough set of necessary and sufficient optimality conditions were developed by Juel and Love (1976).
- Idrissi et al. (1989) developed a dual problem for the constrained multifacility minisum location problems involving mixed norms. General optimality conditions were obtained providing new algorithms which are decomposition methods based on the concept of partial inverse of a multifunction.
- A nonlinear approximation method was developed by Wesolowsky and Love (1972), where any number of linear and (or) nonlinear constraints defining a convex feasible region can be included.
- The hyperboloid approximation procedure for solving the perturbed rectilinear distance MFLP was also proposed by Eyster et al. (1973).
- Morris (1975) used the dual problem of Rectangular MFLP and then reduced it to a problem with substantially fewer variables and constraints. He stated that linear and pairwise constraints limiting distances between new points and between new and existing points can be imposed to restrict the location of new points.
- Picard and Ratliff (1978) solved the problem via at most $(m-1)$ minimum cut problems on derived networks containing at most $(n-2)$ vertices. They showed

that the optimum location of new facilities is dependent on the relative orderings of old facilities but not on the distances between them.

- Subsequently, Kolen (1981) exhibited the equivalence of the method of Sherali and Shetty and Picard and Ratliff. The main difference between these two procedures was principally in the computational implementation. Moreover, this type of approach is known to be the most effective way of solving the rectilinear distance MFLP.
- A modified version of the method of Picard and Ratliff (1978) was proposed by Cheung (1980).
- Dax (1986) presented a new method that, as he stated, handles efficiently the rectilinear distance Problem MFLP having large clusters, i.e. where several new facilities are located together at one point. This paper states and proves a new necessary and sufficient optimality condition. This condition transforms the problem of computing a descent direction into a constrained linear least-squares problem. The latter problem is solved by a relaxation method that takes advantage of its special structure. The new technique is incorporated into the direct search method.
- As an alternative to linear programming, a simple approach which sometimes finds optimal locations was presented by (Francis et al. 1992) as coordinate descent. By deleting the term that shows the relationship between new facilities in objective function, the problem is converted to some single facility location problems to which we can apply median conditions. The first coordinate we choose is the first variable and the second coordinate is the second variable, and so on. It is continued until we obtain the same vector by coordinate descent that we have obtained previously by coordinate descent, at which point we stop.
- Allen (1995) developed a dual-based lower bound to the multifacility ℓp distance location problem and he stated that the bound is as good or better than other bounds.

4.3.1.2 Squared Euclidean Distance MiniSum Location Problem

In this problem, it is obvious that the function that is to be minimized is strictly convex, and unlike the Euclidean distance case, it has continuous first partial derivatives with respect to x and y.

Consequently the optimal solution is unique and the approach to finding optimal solution is the same for the SFLP; partial derivation of (4.20) with respect to each variable are computed and set to equal to zero.

The result of the partial derivation computation is two sets of line equation on involving the x coordinate (and y coordinate) of the new facilities.

To compute the partial derivations, it is convenient to define a new quantity $\hat{v}_{ik}$ where

$$\hat{v}_{ik} = \begin{cases} v_{ij} & k > j \\ v_{kj} & k \le j \end{cases} . \tag{4.40}$$

Computing the partial derivation of (4.21) with respect to x_i and (4.22) with respect to y_i gives, for $j = 1, \ldots, n$,

$$\frac{\partial f}{\partial x_j} = 2\sum_{k=1}^{n} \hat{v}_{ik}(x_j - x_k) + 2\sum_{i=1}^{m} w_{ji}(x_j - a_i). \tag{4.41}$$

And

$$\frac{\partial f}{\partial y_j} = 2\sum_{k=1}^{n} \hat{v}_{ik}(y_j - y_k) + 2\sum_{i=1}^{m} w_{ji}(y_j - b_i). \tag{4.42}$$

If we set the (4.41) and (4.42) to zero and solve it, the optimum values of the x and y coordinates for the new facilities are related by the following expressions:

$$x_j = \frac{\sum_{k=1}^{n} \hat{v}_{ik} x_k + \sum_{i=1}^{m} w_{ji} a_i}{\sum_{k=1}^{n} \hat{v}_{ik} + \sum_{i=1}^{m} w_{ji}}. \tag{4.43}$$

And

$$y_j = \frac{\sum_{k=1}^{n} \hat{v}_{ik} y_k + \sum_{i=1}^{m} w_{ji} b_i}{\sum_{k=1}^{n} \hat{v}_{ik} + \sum_{i=1}^{m} w_{ji}}. \tag{4.44}$$

Furthermore, White (1971) established that the optimum solution to the multifacility problem is given by

$$x^* = A^{-1} W a. \tag{4.45}$$

and

$$y^* = A^{-1} W b, \tag{4.46}$$

where x^* and y^* are $n \times 1$ column vectors giving the optimum coordinate locations for the new facilities, a and b are $m \times 1$ column vectors giving the x and y coordinate locations, respectively, for the existing facilities W is an $n \times m$ matrix containing the weights w_{ij}, and A is an $n \times n$ nonsingular matrix given as follows:

$$A = \begin{bmatrix} \sum_{k=1}^{n} \hat{v}_{1k} + \sum_{i} w_{1i} & -\hat{v}_{12} & \ldots. & -\hat{v}_{1n} \\ -\hat{v}_{21} & \sum_{k=1}^{n} \hat{v}_{2k} + \sum_{i} w_{2i} & \ldots & -\hat{v}_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -\hat{v}_{n1} & -\hat{v}_{n2} & \ldots. & \sum_{k=1}^{n} \hat{v}_{nk} + \sum_{i} w_{ni} \end{bmatrix}. \tag{4.47}$$

The matrix A is strictly diagonally dominant, allowing (4.45) and (4.46) to be solved using the method of simultaneous displacements, an iterative solution procedure based on (4.43) and (4.44) (Eyster and White 1973).

It is interesting to note that the solution of the squared Euclidean distance problem has been used to obtain a good starting solution for the corresponding Euclidean MFLP (Francis et al. 1992).

4.3.1.3 Euclidean Distance MiniSum Location Problem

The objective function of Euclidean MFLP is convex, since it is the sum of norms that are convex functions and Francis and Cabot (1972) have proven that a necessary and sufficient condition for the objective function to be strictly convex is that for each new facility i, the set $S_i = \{j : w_{ij} > 0\}$ is nonempty and that the location of the points in S_i are non-collinear. As it is known, the optimal solution of Euclidean MFLP problem exists and lies in the convex hull of the existing facilities, and therefore, this optimal solution can be expressed as the convex combination of the existing facilities (Francis et al. 1983).

In the previous treatment of the rectilinear distance problem and the squared Euclidean distance problem; we found that multifacility problems were not substantially more difficult to solve than the corresponding single facility version. Such is not the case for Euclidean distance problem. The main difficulty with the Euclidean SFLP arises because its objective function is not differentiable at the points $a_1, \ldots, a_n$, For Euclidean MFLP the function f is nondifferentiable not only at a set of isolated points, but also on linear subspaces $X_i = X_k$.

Eyster et al. (1973) used an extension of the Weiszfeld algorithm. In this procedure, in order to avoid difficulties of the partial derivatives of the distance function not existing at points P_i and at points where other new facilities have been located

$$d(X_j, X_k) = [(x_j - x_k)^2 + (y_j - y_k)^2 + \varepsilon]^{1/2}, \tag{4.48}$$

$$d(X_j, X_i) = [(x_j - a_i)^2 + (y_j - b_i)^2 + \varepsilon]^{1/2}. \tag{4.49}$$

This procedure is labeled the hyperboloid approximation procedure (HAP) and is probably the most common procedure for solving the multifacility location problem, using Euclidean distances or even rectilinear distances. Rosen and Xue (1993) proved the global convergence of HAP when applied to the problems MFLP. Hap comes from computing partial derivatives of the function f, setting them to zero, and solving for new facility locations.

To simplify, we define terms $\alpha_t(X_1, \ldots, X_n), \beta_t(X_1, \ldots, X_n), \Gamma_t(X_1, \ldots, X_n)$, and for $t = 1, \ldots, n$ as follows:

$$\alpha_t(X_1, \ldots, X_n) = \sum_{j=1}^{n} v_{tj} \frac{x_j}{[(x_t - x_j)^2 + (y_t - y_j)^2 + \varepsilon]^{1/2}}$$
$$+ \sum_{i=1}^{m} w_{ti} \frac{a_i}{[(x_t - a_i)^2 + (y_t - b_i)^2 + \varepsilon]^{1/2}} \quad t = 1, \ldots, n, \tag{4.50}$$

$$\beta_t(X_1,\ldots,X_n) = \sum_{j=1}^{n} v_{tj}\frac{y_j}{[(x_t-x_j)^2+(y_t-y_j)^2+\varepsilon]^{1/2}}$$

$$+\sum_{i=1}^{m} w_{ti}\frac{b_i}{[(x_t-a_i)^2+(y_t-b_i)^2+\varepsilon]^{1/2}} \qquad t=1,\ldots,n, \tag{4.51}$$

$$\Gamma_t(X_1,\ldots,X_n) = \sum_{j=1}^{n} \frac{v_{ij}}{[(x_t-x_j)^2+(y_t-y_j)^2+\varepsilon]^{1/2}}$$

$$+\sum_{i=1}^{m} \frac{w_{ti}}{[(x_t-a_i)^2+(y_t-b_i)^2+\varepsilon]^{1/2}} \qquad t=1,\ldots,n. \tag{4.52}$$

Then it may be shown that the partial derivatives of f with respect to x_t and y_t is as follows:

$$\frac{\partial f}{\partial x_t} = \Gamma_t(X_1,\ldots,X_n)x_t - \alpha_t(X_1,\ldots,X_n) \qquad t=1,\ldots,n, \tag{4.53}$$

and

$$\frac{\partial f}{\partial y_t} = \Gamma_t(X_1,\ldots,X_n)y_t - \beta_t(X_1,\ldots,X_n) \qquad t=1,\ldots,n. \tag{4.54}$$

By setting the partial derivatives to zero for x_1 and y_1 we have:

$$x_t = \frac{1}{\Gamma_t(X_1,\ldots,X_n)}\alpha_t(X_1,\ldots,X_n) \qquad t=1,\ldots,n. \tag{4.55}$$

And

$$y_t = \frac{1}{\Gamma_t(X_1,\ldots,X_n)}\beta_t(X_1,\ldots,X_n) \qquad t=1,\ldots,n. \tag{4.56}$$

The iteration procedure determines x^{k+1} and y^{k+1}in terms of x^k and y^k using the following:

$$x_t^{k+1} = \frac{1}{\Gamma_t(X_1^k,\ldots,X_n^k)}\alpha_t(X_1^k,\ldots,X_n^k) \qquad t=1,\ldots,n, \tag{4.57}$$

$$y_t^{k+1} = \frac{1}{\Gamma_t(X_1^k,\ldots,X_n^k)}\beta_t(X_1^k,\ldots,X_n^k) \qquad t=1,\ldots,n. \tag{4.58}$$

By making an initial choice of new facility locations, say $\alpha_t(X_1^{(0)},\ldots,X_n^{(0)})$, use the algorithm to compute new answers for new facility locations. In using HAP to solve multifacility location problems, it has been observed that the larger the value of ε the faster the convergences to optimum value of approximating function. However, the accuracy of approximation decrease with increasing values of ε consequently, in solving location problem using HAP a large value of ε is used initially; the solution obtained used as starting solution, using a smaller value of ε; and the process is continued by successively reducing the value of ε until no signification decrease in the value of either (x_i, x_j).interestingly, HAP can be used to solve the rectangular location problem and can also be used to handle situation involving a mixture of recliner and Euclidean distance (Eyster et al. (1973).

4.3.1.4 Some other Kinds of Algorithms in Euclidean MFLP

- Eyster et al. (1973) used an extension of the Weiszfeld algorithm.
- Calamai and Conn (1980) have proposed a pseudo-gradient technique that classifies the new facilities into distinct categories based on their coincidence with other facilities in order to derive a descent method for solving Euclidean MFLP.
- Chatelon (1978) have also approached Euclidean MFLP by using a general e-subgradient method in which search directions are generated based on the subdifferential of the objective function over a neighborhood of the current iterate.
- Sequential unconstrained minimization techniques used by Love (1969) and the Weiszfeld fixed-point iterative method as utilized by Rado (1988), are also among other efforts to solve Euclidean MFLP.
- Several second-order methods have also been designed to solve the Euclidean MFLP. Calamai and Conn (1980) were the first to propose a projected gradient-based algorithm.
- Various quadratic convergence approaches have also been developed by Calamai and Conn (1982, 1987), in which specialized line-searches are used in conjunction with projected second-order techniques.
- Rosen and Xue (1992) developed an algorithm which, from any initial point, generates a sequence of points that converges to the closed convex set of optimal solutions to the Problem Euclidean MFLP.
- For the multifacility location problem with no constraints on the location of the new facilities, Juel and Love (1980) derived some sufficient conditions for the coincidence of facilities that are valid in a general symmetric metric.
- The results of Juel and Love (1980) were later extended by Lefebvre et al. (1991) to be applicable to some location problems having certain locational constraints.
- Mazzerella and Pesamosca (1996) have used the optimality conditions of Euclidean MFLP as a tool for obtaining both stopping rules for some computational algorithms such as the projected Newton procedure of Calamai and Conn (1987), and the analytical solution of many simple problems.
- Love (1969) applied convex programming to the problem in three dimensions.
- Carrizosa et al. (1993) derived the geometrical characterizations for the set of efficient, weakly efficient and properly efficient solutions of the Euclidean MFLP when it includes certain convex locational constraints.
- Love and Yoeng (1981), Elzinga and Hearn (1983), Juel (1984), and Love and Dowling (1989) explored the bounding method that continuously updates a lower

bound on the optimal objective function value during each iteration. This method is based on the idea that the convex hull and the current value of the gradient determine an upper bound on the objective function's improvement.

- Wendell and Petersen (1984) have derived a lower bound from the dual to Euclidean MFLP.
- Love (1974) developed the dual problem corresponding to a hyperbolic approximation of the constrained multifacility location problem with l_p distances.
- White (1976) gave a Varignon frame interpretation of the dual problem.
- Sinha (1966) have used duality results involving general quadratic forms.
- Francis (1972) derived a differentiable, convex quadratically constrained dual optimization problem, and achieved several useful relationships between the dual and Euclidean MFLP.
- Xue et al. (1996) have suggested the use of polynomial-time interior point algorithm to solve this dual problem based on this idea, they presented a procedure in which an approximate optimum to Euclidean MFLP can be recovered by solving a sequence of linear equations, each associated with an iterate of the interior point algorithm used to solve the dual problem.
- Love and Kraemer (1973) gave a dual decomposition method for solving the constrained Euclidean MFLP.
- Love (1974) developed the dual problem corresponding to a hyperbolic approximation of the objective function for the constrained MFLP with l_p distance.

4.3.2 MiniMax

4.3.2.1 Rectangular Distance MiniMax Location Problem

Some procedures for solving rectangular MiniMax MFLP are shown here:

- Wesolowsky (1972) converted the rectangular MiniMax MFLP into a parametric linear programming problem with $5mn + 5/2n(n-1)$ constraints and $2mn + n(n-1) + 2n$ variables in addition to the parameters.
- Elzinga and Hearn (1973) recognized some simplifications in linear program presented by Wesolowsky.
- Dearing and Francis (1974) showed that the problem can be decomposed into two sub problems that have identical structures and that may be solved independently each of which had $2mn + n(n-1)$ constraints and $n + 1$ variables. Each problem is solved efficiently by converting it into an equivalent network flow problem.
- Morris (1973) has introduced this problem with linear constraints which (a) limit the new facilities location and (b) enforce upper bounds on the distances between new and existing facilities and between new facilities. He uses dual variables that provide information about the complete range of new facility locations which satisfies the MiniMax criterion.

- Drezner and Wesolowsky (1978) presented a method involved the numerical integration of ordinary differential equations and was computationally superior to methods using nonlinear programming.

4.3.2.2 Euclidean Distance MiniMax Location Problem

The application of nonlinear duality theory shows Euclidean minimax MFLP can always be solved by maximizing a continuously differentiable concave objective subject to a small number of linear constraints. This leads to a solution procedure which produces very good numerical results. Love et al. (1973) presented a nonlinear programming method for computing the solution to MFLPs using Euclidean distances when the MiniMax criterion is to be satisfied.

4.3.3 Solution Techniques for other Models

In this section we introduce some solution techniques for the models shown in Sect. 1.3

- A specialized simplex based-algorithm was derived by Sherali and Shetty (1978) for solving rectangular multiproduct MFLP.
- Dhar and Rao (1982) presented an iterative solution for MFLP on sphere. The procedure involved the approximation of the domain of objective function which in the limit approaches to that of the original objective function. Aykin and Babu (1987) considered Euclidean, squared Euclidean and the great circle distances. They formulated an algorithm and investigated its convergence properties.
- Wesolowsky and Love (1971a, b) considered the problem of MFLP with rectangular regions for the cases $n = 1$ or 2. They used a simple gradient reduction technique to solve the single facility problem. As they stated, the procedure becomes very complex when $n > 2$. Aly and Marucheck (1982) stated that if there is interfacility interaction among the new facilities, a gradient-free nonlinear search algorithm is utilized. Computational experience suggests that this algorithm is expedient even in the solution of large problems.
- Seppalla (1975) stated that in order to be able to solve the stochastic MFLP, we must transform it into the deterministic equivalent forms.

4.3.4 Some Heuristic and Metaheuristic Methods

There are a few heuristics methods for solving MFLP:

- Vergin and Rogers (1967) introduced a simple heuristic for solving MFLP with Euclidean distance. This procedure locates each of new facilities in a temporary location at each step and locates the next new facility according to the facilities located so far. After all n new facilities are located in this manner the process is repeated and the readjustment process is continued until no further movements occur during a complete round of adjustment evaluations.

- Davoud Pour and Nosraty (2006) solved the MFLP with ant-colony optimization metaheuristic when the distances are rectangular and Euclidian. This algorithm produces optimal solutions for problem instances of up to 20 new facilities.

4.4 Case Study

Smallwood (1965) introduced a model for the placement of n detection stations so as to maximize the probability that at least one of them will detect any enemy event occurring within the area. This research was done within boundaries of USA and USSR. The model is based on five assumptions that one of them is the assumption of plane area that was made in order to allow the use of the relatively convenient Cartesian coordinate system. Each of these assumptions can be relaxed to reduce the error of the model.

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