IE5403 - Chapter 3

Facility location

Definitions

Facility = Something (plant, office, warehouse, etc) built or established to serve a purpose

Facility location = The determination of which of several possible locations should be operated in order to min. or max. some objectives, such as profit, cost, distance, etc.)

Location Examples

- Locate a new warehouse relative to production facilities and customers
- Locate an emergency service (police station, fire station, blood bank, etc.)
- Locate a branch office for banks
- Locate supply centers for construction projects

Transportation Method- Example

* The system has three plants in different cities (P1, P2, P3) and two warehouses (W1, W2)
* A new warehouse will be located, candidate locations are (W3, W4)
* Transportation costs, damands and supply capacities are as follows:

	W1	W2	W3	W4	Supply
P1	5	9	16	17	800
P2	9	4	8	9	800
P3	15	8	3	5	600
Demand	700	700	700	700	

Example - solution

	W1	W2	W3	Dummy	Supply
P1	5	9	16		800
P2	9	4	8		800
P3	15	8	3		600
Demand	700	700	700	100	

Σ Demand = 2100 Σ Supply = 2200

Example - solution

	W1	W2	W3	Dummy	Supply
P1	$\boxed{5}$ 700	$\boxed{9}$	$\boxed{16}$	100	0
P2	$\boxed{9}$	$\boxed{4}$ 700	$\boxed{8}$ 100		0
P3	$\boxed{15}$	$\boxed{8}$	$\boxed{3}$ 600		0
Demand	0	0	0	0	

Σ Cost = 3500 + 2800 + 800 + 1800 = 8900 YTL

Example - solution

	W1	W2	W4	Dummy	Supply
P1	5	9	17		800
P2	9	4	9		800
P3	15	8	5		600
	700	700	700	100	

Example - solution

	W1	W2	W4	Dummy	Supply
P1	5 700	9	17	100	0
P2	9	4 700	9 100		0
P3	15	8	5 600		0
	0	0	0	0	

Σ Cost = 3500 + 2800 + 900 + 3000 = 10,200 YTL

Least cost alternative is selected

Σ Cost for W3 = 8900 YTL

Σ Cost for W4 = 10,200 YTL

W3 will be the new warehouse location

Multi-facility location problem

Assumptions:

- m facilities to be located
- n available sites
- $m \leq n$
- Each facility will require only one site, and it is not possible to locate more than one facility at any given site
- No interaction between the pairs of new facilities

Multi-facility location problem

Objective:

Locate m facilities such that, the sum of the setting up and operating costs of the system are minimized

Variables and parameters

$$Y_{ij} = \begin{cases} 1 & \text{, if } i^{th} \text{ facility is located at the } j^{th} \text{ site} \\ 0 & \text{, otherwise} \end{cases}$$

c_{ij} = fixed cost, operating costs and transportation costs resulting from locating i^{th} facility at the j^{th} site

Model formulations

$$min\ Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\ Y_{ij}$$

s.t.

$$\sum_{j=1}^{n} Y_{ij} = 1 \qquad , \forall i$$

Each facility should be located at one site

$$\sum_{i=1}^{m} Y_{ij} \le 1 \qquad , \forall j$$

Each site can be used to locate at most one facility

$$Y_{ij} = 0,1 \qquad , \forall i,j$$

Assignment Problem

If $m = n$, then constraint set (2) will become as follows:

$$\sum_{i=1}^{m} Y_{ij} = 1 \quad , \forall j \quad \left\{ \begin{array}{l} \text{Each site can be used} \\ \text{to locate only one facility} \end{array} \right.$$

If $m < n$, then create $(n - m)$ dummy facilities with $c_{ij} = 0$ to solve the assignment problem

Example 5.2

Suppose that we have to locate four new facilities at four alternative sites. Given fixed and operating costs below, solve the facility location problem using the Hungarian algorithm to minimize the total assignment cost

Fixed and operating costs:

	Sites			
Facilities	**1**	**2**	**3**	**4**
1	80	60	50	70
2	60	50	30	40
3	70	80	40	60
4	60	70	50	60

The Hungarian algorithm

Sites

Facilities

				Row Min
80	60	50	70	$p_1 = 50$
60	50	30	40	$p_2 = 30$
70	80	40	60	$p_3 = 40$
60	70	50	60	$p_4 = 50$

Step 1: From each entry of a row, we subtract the minimum value in that row and get the following reduced cost matrix:

Sites

Facilities

30	10	0	20
30	20	0	10
30	40	0	20
10	20	0	10

Column Minimum $q_1=10$ $q_2=10$ $q_3=0$ $q_4=10$

Step 2: From each entry of a column, we subtract the minimum value in that column and get the following reduced cost matrix:

Sites

Facilities

20	0	0	10
20	10	0	0
20	30	0	10
0	10	0	0

Step 3: Now we test whether an assignment can be made as follows. If such an assignment is possible, it is the optimal assignment.

(a)Examine the first row. If there is only one zero in that row, surround it by a square($\boxed{0}$) and cross ($\cancel{0}$) all the other zeros in the **column** passing through the surrounded zero.

Next examine the other rows and repeat the above procedure for each row having only one zero.

If a row has more than one zero, do nothing to that row and pass on to the next row.

Step 3(a) gives the following table.

Sites

Facilities

20	$\boxed{0}$	~~0~~	10
20	10	~~0~~	$\boxed{0}$
20	30	$\boxed{0}$	10
$\boxed{0}$	10	~~0~~	~~0~~

Two possible optimal solutions PASS for now

Step 3(b): Now repeat the above procedure for columns. (if necessary)

Example 5.2 (solution)

If there is now a surrounded zero in each row and each column, the optimal assignment is obtained.

Facility 1 is located at *Site 2*

Facility 2 is located at *Site 4*

Facility 3 is located at *Site 3*

Facility 4 is located at *Site 1*

Total cost = *200*

Facility location-allocation problems

Determine:

(i) number of new facilities to be located

(ii) locations of the new facilities

(iii) allocation of item movement between the facilities and customers

Model 1: (Simplest case)

Assumptions:

→ single product

→ uncapacitated

→ One stage (warehouse $\rightarrow$ customer) distribution system

Variables and parameters

$$Y_i = \begin{cases} 1 & \text{, if a warehouse is located at site } i \\ 0 & \text{, otherwise} \end{cases}$$

f_i = the fixed cost of openning a warehouse at site i

X_{ij} = the quantity of product that flows from warehouse at site i to customer j

c_{ij} = the cost of supplying one unit of product from warehouse at site i to customer j

d_j = the demand of customer j

Model formulations

$$\min Z = \sum_{i=1}^{m} f_i Y_i + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} X_{ij}$$

s.t.

$$\sum_{i=1}^{m} X_{ij} = d_j \quad , \forall j$$

Demands of all customers should be satisfied

$$\sum_{j=1}^{n} X_{ij} \le Y_i \sum_{j=1}^{n} d_j \quad , \forall i$$

if no warehouse opened at site ***i***, no flow can take between ***i*** → ***j***

$$X_{ij} \geq 0 \quad , \forall i,j$$

$$Y_i = 0,1 \quad , \forall i$$

Continuous Space Models

- A facility can be located anywhere in the two dimensional space ($\equiv$ infinitely many candidate locations)
- Total cost is proportional to distance

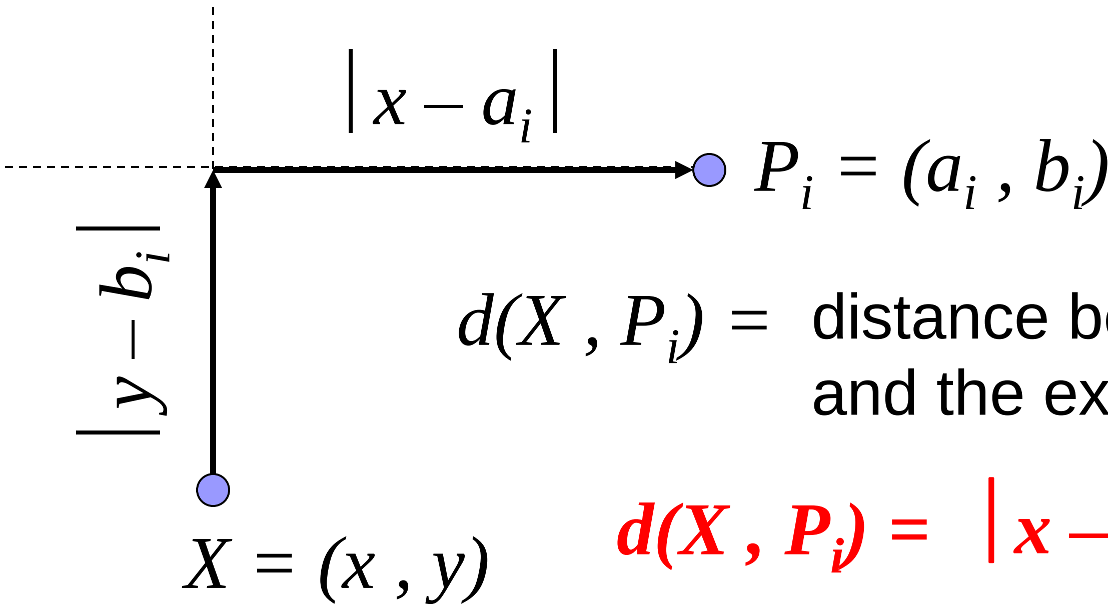
Distance measures

- Rectilinear distance: distances between the new facility and existing facilities are measured along the paths that are parallel to a set of perpendicular axes.
- Euclidean distance: distances between the new facility and existing facilities are measured along a straight line path connecting two points.

Rectilinear distance

$X = (x\ ,\ y)$: coordinates of the new facility

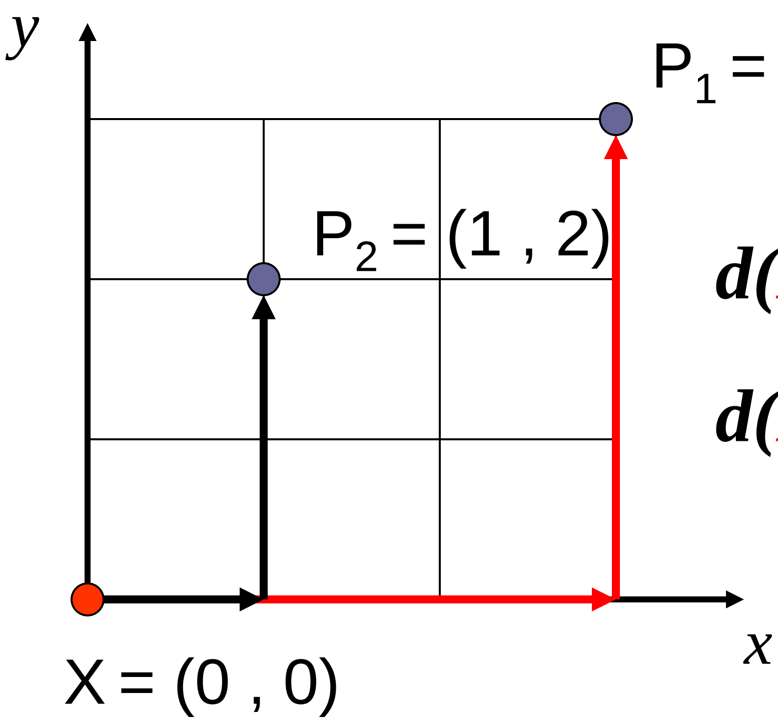
$P_i = (a_i\ ,\ b_i)$: coordinates of the existing facility ***i***



$d(X\ ,\ P_i) =$ distance between the new facility and the existing facility ***i***

$$\boldsymbol{d(X\ ,\ P_i) = |x - a_i| + |y - b_i|}$$

Rectilinear distance – Example



$$d(X , P_1) = |0 - 3| + |0 - 3| = 6$$

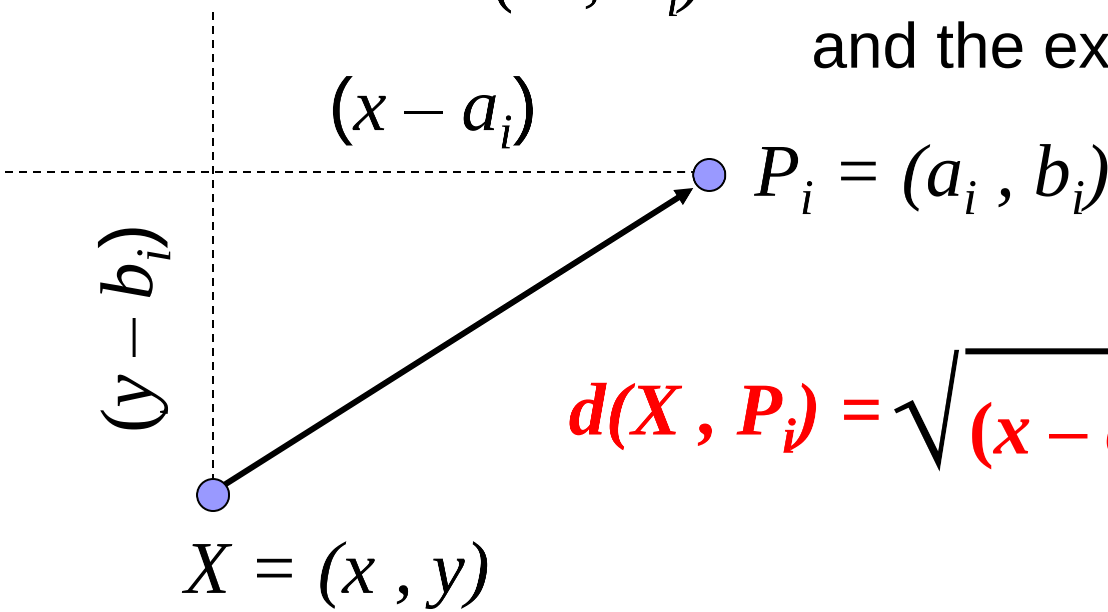
$$d(X , P_2) = |0 - 1| + |0 - 2| = 3$$

Euclidean distance

$X = (x\ ,\ y)$: coordinates of the new facility

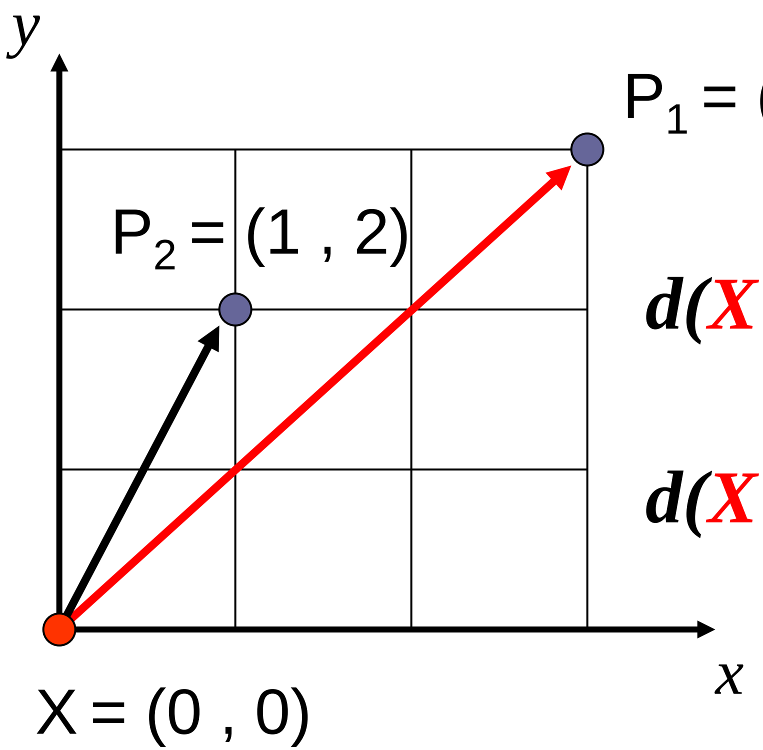
$P_i = (a_i\ ,\ b_i)$: coordinates of the existing facility ***i***

$d(X\ ,\ P_i)$ = distance between the new facility and the existing facility ***i***



$$d(X\ ,\ P_i) = \sqrt{(x - a_i)^2 + (x - a_i)^2}$$

Euclidean distance – Example



$$d(X, P_1) = \sqrt{(0-3)^2 + (0-3)^2} = 4.24$$

$$d(X, P_2) = \sqrt{(0-1)^2 + (0-2)^2} = 2.24$$

Single facility location with rectilinear distance

Objective:

Total weighted rectilinear distance is minimized

$$\min f(x,y) = \sum_{i=1}^{m} w_i \left(|x - a_i| + |y - b_i| \right)$$

$$= \underbrace{\min \sum_{i=1}^{m} w_i \left(|x - a_i| \right)}_{f(x)} + \underbrace{\min \sum_{i=1}^{m} w_i \left(|y - b_i| \right)}_{f(y)}$$

Rules of optimality

Rule 1:

x-coordinate of the new facility will be the same as the x-coordinate of some existing facility.

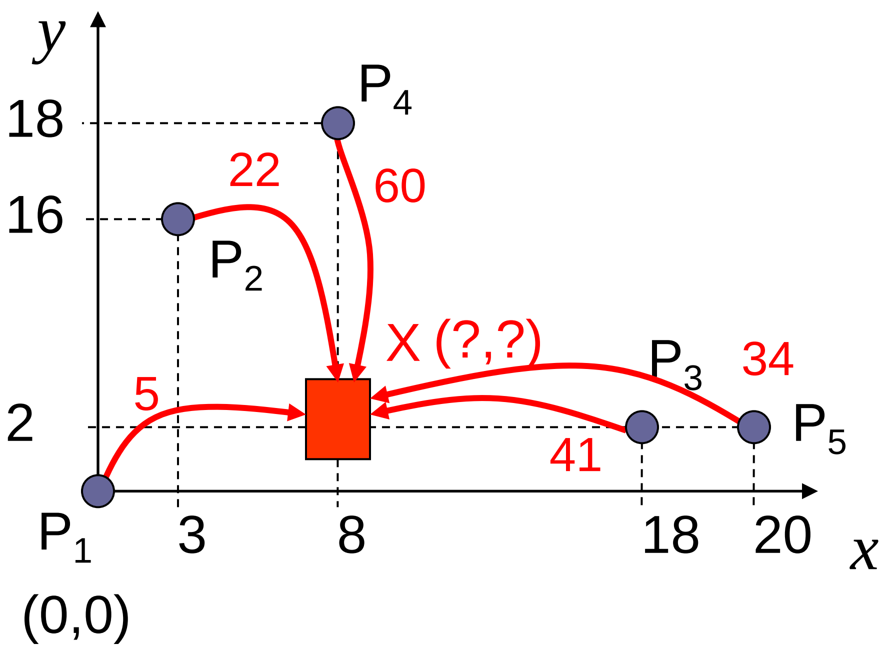
y-coordinate of the new facility will be the same as the y-coordinate of some existing facility.

Rule 2: (median location rule)

The optimum x coordinate will be such that no more than one half of the total weight will be to the left of x coordinate, and no more than one half of the total weight will be to the right of x coordinate. Same is true for y coordinate

Example 5.3

A rent-a-car company has 5 offices located in a large city. Locations of the offices are as follows:



The company wishes to locate a maintenance facility in the city to give service to the cars.

The number of cars transported par day between the maintenance facility and the offices are expected to be 5, 22, 41, 60 and 34 respectively.

Example 5.3 (solution to x coordinate)

Step 1: Order the facilities by their increasing x-coordinate values

Existing facility	x-coordinate value
1	0
2	3
4	8
3	18
5	20

Example 5.3 (solution to x coordinate)

Step 2: Cumulative weights are computed in two directions (top → down and bottom → up)

Existing facility	x-coordinate value	Weight	Cumulative Weight (top-down)	Cumulative Weight (bottom-up)
1	0	5	2	162
2	3	22	27	157
4	8	60	87	135
3	18	41	128	75
5	20	34	162	34

Example 5.3 (solution to x coordinate)

Step 3:
Calculate the half of the total weight = 162 / 2 = 81

Existing facility	x-coordinate value	Weight	Cumulative Weight (top-down)	Cumulative Weight (bottom-up)
1	0	5	2	162
2	3	22	27	157
4	8	60	87	135
3	18	41	128	75
5	20	34	162	34

Example 5.3 (solution to *x* coordinate)

Step 4: Find the median of the x-values that shows the optimum coordinate

Existing facility	*x*-coordinate value	Weight	Cumulative Weight (top-down)	Cumulative Weight (bottom-up)
1	0	5	2	162
2	3	22	27 < 81	157
4	$\boxed{8}$	60	87	135
3	18	41	128	75 < 81
5	20	34	162	34

Optimum x^*-coordinate = 8

Example 5.3 (solution to *y* coordinate)

Step 1: Order the facilities by their increasing *y*-coordinate values

Existing facility	*y*-coordinate value
1	0
3	2
5	2
2	16
4	18

Example 5.3 (solution to *y* coordinate)

Step 2: Cumulative weights are computed in two directions (top → down and bottom → up)

Existing facility	*y*-coordinate value	Weight	Cumulative Weight (top-down)	Cumulative Weight (bottom-up)
1	0	5	5	162
3	2	41	46	157
5	2	34	80	116
2	16	22	102	82
4	18	60	162	60

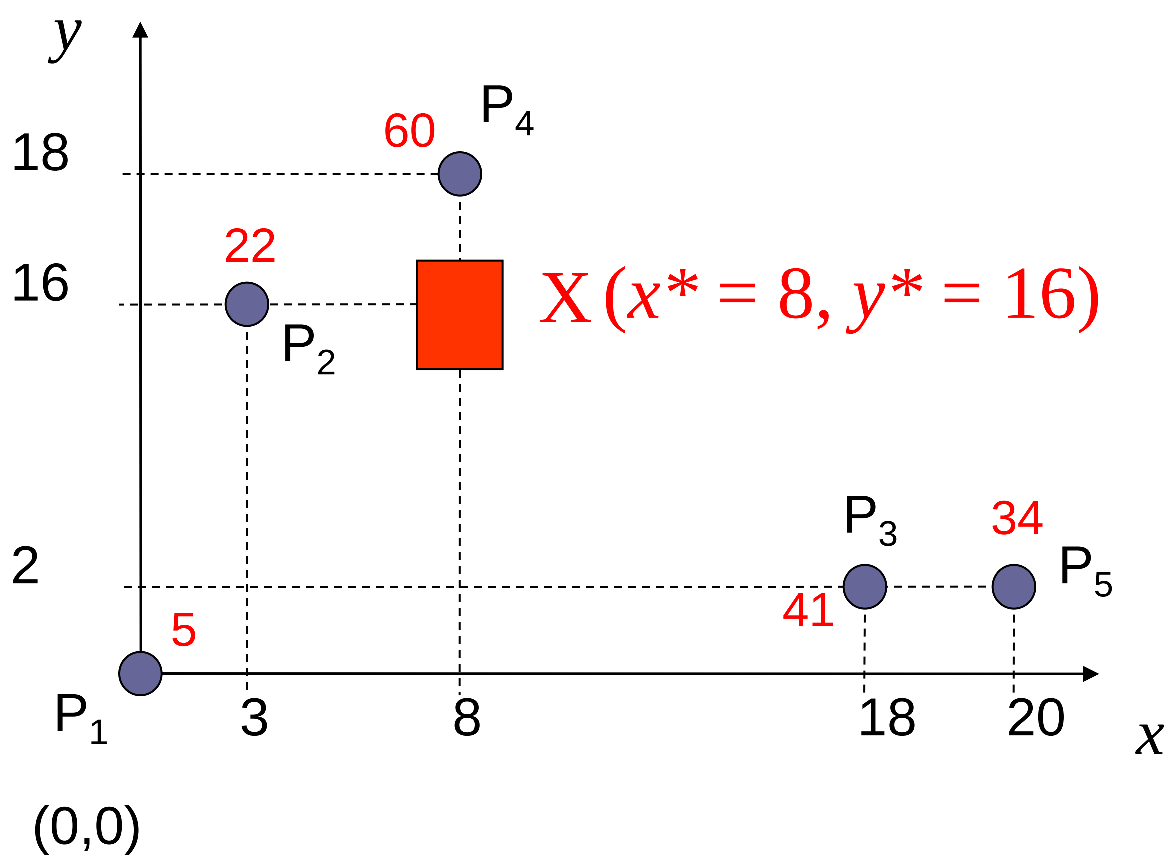
Example 5.3 (solution to *x* coordinate)

Step 4: Find the median of the y-values that shows the optimum coordinate

Existing facility	*x*-coordinate value	Weight	Cumulative Weight (top-down)	Cumulative Weight (bottom-up)
1	0	5	5	162
3	2	41	46	157
5	2	34	80 < 81	116
2	$\boxed{16}$	22	102	82
4	18	60	162	60 < 81

Optimum y^*-coordinate = 16

Optimum location that minimizes weighted rectilinear distances



Single facility location with squared Euclidean distance

Objective:

Total weighted squared Euclidean distance is minimized

$$\min f(x,y) = \sum_{i=1}^{m} w_i \, [\, (\mathbf{x} - \mathbf{a_i})^2 + (\mathbf{y} - \mathbf{b_i})^2 \,]$$

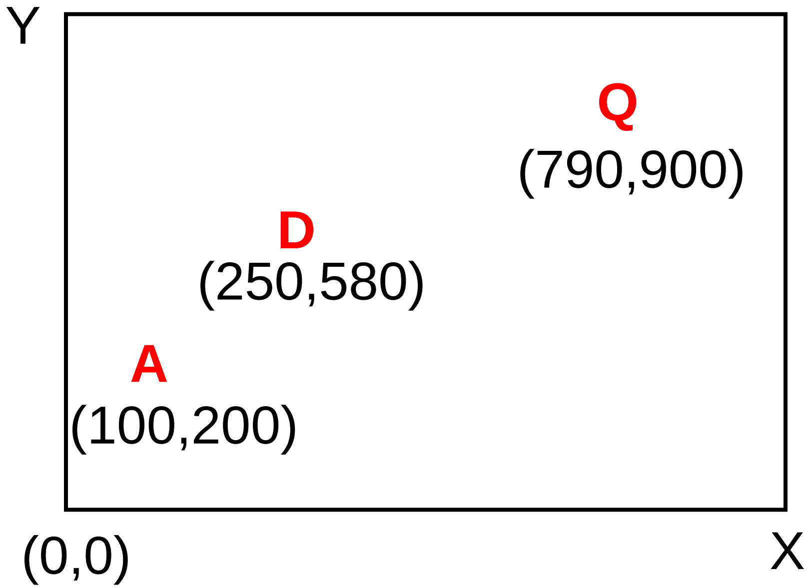
Optimal x^*, y^* coordinates (Center of gravity)

$$\frac{\partial f(x,y)}{\partial x} = 2\sum_{i=1}^{m} w_i\,(\mathbf{x} - \boldsymbol{a_i}) = 0 \;\rightarrow\; \boxed{\mathbf{x^*} = \frac{\sum_{i=1}^{m} w_i\,\boldsymbol{a_i}}{\sum_{i=1}^{m} w_i}}$$

$$\frac{\partial f(x,y)}{\partial y} = 2\sum_{i=1}^{m} w_i\,(\mathbf{y} - \boldsymbol{b_i}) = 0 \;\rightarrow\; \boxed{\mathbf{y^*} = \frac{\sum_{i=1}^{m} w_i\,\boldsymbol{b_i}}{\sum_{i=1}^{m} w_i}}$$

Example 5.4 (center of gravity method)

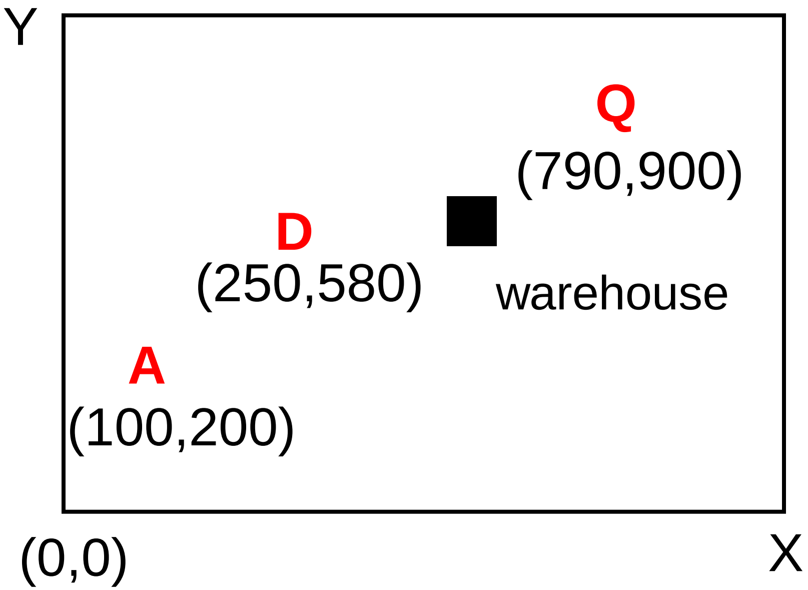
Several automobile showrooms are located according to the following grid which represents coordinate locations for each Showroom as follows:



Showroom	No of Z-Mobiles sold per month
A	1250
D	1900
Q	2300

What is the best location for a new Z-Mobile warehouse that will serve showrooms? Use center of gravity method.

Example 5.4 (solution)



Showroom	No of Z-Mobiles sold per month
A	1250
D	1900
Q	2300

$$x^* = \frac{1250\,(100) + 1900\,(250) + 2300\,(790)}{1250 + 1900 + 2300} = 443.49$$

$$y^* = \frac{1250\,(200) + 1900\,(580) + 2300\,(900)}{1250 + 1900 + 2300} = 627.89$$