Recursivity and the Estimation of Dynamic Games with Continuous Controls

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Introduction

We revisit the estimation of **dynamic games with continuous controls** (DGCCs)

- Investments in R&D, quality, capacity, advertising, political campaign spending . . .

Related literature.

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First, we propose **estimators based on firms' optimal policies.**

- "Recursive Estimators" (REs).
- We evaluate their performance relative to Bajari et al. (2007) (BBL)
  - BBL: VF inequalities from randomly drawn policies.
- We establish
  - REs are computationally tractable.
  - They substantially outperform BBL.

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- We establish
  - REs are computationally tractable.
  - They substantially outperform BBL.

Second, we construct a **semi-parametric estimator for DGCCs.**

- We establish
  - Misspecification of the cost shock distribution can bias estimation a great deal.
  - The SP estimator works reasonably well. More work needed, in progress.

Related literature.

Roadmap

- A **model** of entry, exit, and quality upgrading.
- Introduce estimators based on recursive equilibrium conditions (REs).
- Monte Carlo simulations of REs and BBL.
- Semi-parametric estimation of DGCCs.

Model: Market Structure and Profits

Pakes and McGuire (1994)

$\bar{N}$ single-product firms in the market

- N active firms, $N \leq \bar{N}$; N endogenous.
- $\bar{N} - N$ inactive firms.

Firms have quality levels $\xi_j \in \Xi \subset \mathbb{R} \cup \{-\infty\}$

- Being inactive indicated by $\xi_j = -\infty$.

The industry state is $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{\bar{N}})$.

Firms earn flow profits that depend on their qualities: $\pi_j(\boldsymbol{\xi})$.

- $\pi_j(\boldsymbol{\xi})$ are *symmetric* – Doraszelski and Satterthwaite (2010):
  - $\pi_j(\xi_j, \xi_2, \ldots, \xi_{j-1}, \xi_1, \xi_{j+1}, \ldots, \xi_{\bar{N}}) = \pi_1(\boldsymbol{\xi}) = \pi(\boldsymbol{\xi})$ for all $j = 2, \ldots, \bar{N}$
  - $\pi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_{-1}) = \pi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_{p(-1)})$ for any permutation $p(\cdot)$.

Model: Quality Transitions

Transitions:

- Firms can invest to affect the evolution of quality.
- Transitions are conditionally independent and depend only on own quality and investment:

$$F(\boldsymbol{\xi}'|\boldsymbol{\xi}, \boldsymbol{x}) = \prod_{j=1}^{\bar{N}} F(\xi_j'|\xi_j, x_j) \ .$$

Investment is costly: $c(x, \nu)$

- $\nu \sim F_\nu$ is an investment cost shock.
- $\partial_x^2 c(x, \nu) \geq 0$ for all $x \in \mathbb{R}_+, \nu \in \mathrm{supp}(F_\nu)$.
- $\partial_{\nu x}^2 c(x, \nu) > 0$ for all $x \in \mathbb{R}_+, \nu \in \mathrm{supp}(F_\nu)$.

Model: Timing and Equilibrium

1	2	3	4	5	6
Incumbents earn $\pi_j(\boldsymbol{\xi})$	$\rho \sim F_\rho$ $\phi \sim F_\phi$	Entry/exit decisions	$\nu \sim F_\nu$ (active firms)	Investment decisions	Transitions $\xi' \sim F(\xi' \mid \xi, x)$

Definition (Symmetric MPE)

A **Symmetric Markov Perfect Equilibrium** (SMPE) is a tuple

$$(\alpha^E(\boldsymbol{\xi}, \phi), \alpha^I(\boldsymbol{\xi}, \rho), \sigma^x(\boldsymbol{\xi}, \nu))$$

such that, given that all firms behave according to this tuple,

1. $\alpha^E(\boldsymbol{\xi}, \phi) \in \{0, 1\}$ maximizes the potential entrant's ENPV of profits;
2. $\alpha^I(\boldsymbol{\xi}, \rho) \in \{0, 1\}$ and $\sigma^x(\boldsymbol{\xi}, \nu)$ maximize the incumbent's ENPV of profits.

Roadmap

- A model of entry, exit, and quality upgrading.
- Introduce estimators based on **recursive equilibrium conditions** (REs).
- Monte Carlo simulations of REs and BBL.
- Semi-parametric estimation of DGCCs.

Recursive Estimator

The firm's investment problem can be written as:

$$\max_{x \in \mathbb{R}_+} \left\{ -c(x, \nu; \boldsymbol{\theta}_x) + \beta \int_{\xi_1'} \hat{W}(\xi_1' \mid \boldsymbol{\xi}, \hat{F}^{\sigma}; \boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho) \, \mathrm{d}\hat{F}(\xi_1' \mid \xi_1, x) \right\}$$

▸ W, F^{σ}, concavity

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‣ W, F^{σ}, concavity

$\Rightarrow$ Predicted levels of investment $T_{\{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho\}}(\boldsymbol{\xi}, \nu; \hat{\Phi})$, where $\hat{\Phi} = \left(\widehat{\bar{V}_I}((\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho)), \hat{F}, \hat{F}^{\sigma} \right)$

- The rough **idea** is to compare these values with observed ones.
  - Could do so in different ways.
  - We obtain good results with an Indirect Inference estimator. ‣ Details

‣ $\bar{V}_I$: Closed Form Solution

Roadmap

- A model of entry, exit, and quality upgrading.
- Introduce estimators based on recursive equilibrium conditions (REs).
- **Monte Carlo** simulations of REs and BBL.
- Semi-parametric estimation of DGCCs.

Monte Carlo Setup

We specialize the framework above following Hashmi and van Biesebroeck (2016) (HvB):

- We **add entry** to their framework.
- $\Xi = \{-\infty, \xi_{\text{min}}, \xi_{\text{min}} + \delta, \ldots, \xi_{\text{max}} - \delta, \xi_{\text{max}}\}$.
- $\pi(\boldsymbol{\xi})$ is derived from
  - Single-product firms facing nested logit demand. [Mixed logit in HvB]
  - Constant marginal cost $mc(\xi)$.
  - Nash-Bertrand competition.
  - Flow Profit Derivation
- Transitions
  - $\xi' \in \{\xi - \delta, \xi, \xi + \delta\}$.
  - $P(\xi + \delta \mid \xi, x)$: $\uparrow x, \downarrow \xi$.
  - Our specification of P differs from HvB to ensure concavity of the investment problem.
  - Computational payoff of local transitions + absorbing exit
- $c(x, \nu) = \theta_{x1} x + \theta_{x2} x^2 + \theta_{x3} \nu x$.

Setup details. First stage details.

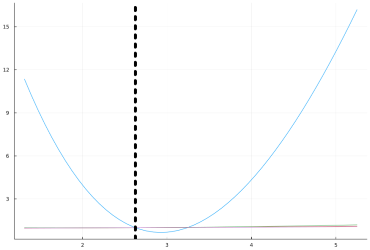
Monte Carlo Results

	Value	Indirect Inference	BBL		
			Asymptotic	Multiplicative	Additive
θ_{x1}	2.625	2.872	6.888	1.944	0.08
		0.734	2.635	2.21	0.442
θ_{x2}	1.624	1.581	10.0	5.481	0.055
		0.148	0.004	0.813	0.335
θ_{x3}	0.5096	0.466	5.714	9.611	9.067
		0.097	2.694	1.142	2.003
μ_ρ	-1.0	-1.127	-1.329	-1.103	-2.0
		0.39	0.101	0.081	0.001
σ_ρ	0.75	0.746	0.69	0.589	1.038
		0.13	0.036	0.031	0.167
μ_ϕ	0.625	0.864	1.809	1.851	1.898
		0.616	0.074	0.063	0.268
σ_ϕ	0.5	0.735	2.998	2.998	3.0
		0.468	0.015	0.019	0.0

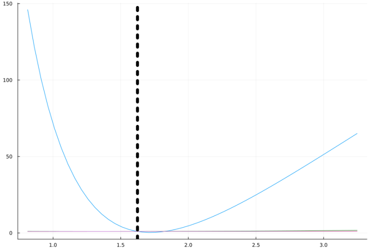
- Qualitatively similar results with sample size as in Ryan (2012).

Policy and Deviation Details
Estimate distributions.

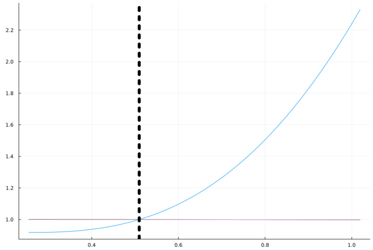
Objective Function: Investment Cost Parameters



θ_{x1}



θ_{x2}



θ_{x3}

$$c(x,\nu) = \theta_{x1}x + \theta_{x2}x^2 + \theta_{x3}x\nu$$

θ^{II}_{x3} is minimized away from zero in this sample.

Objective Function: F_ρ and F_ϕ

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- Monte Carlo simulations of REs and BBL.
- **Semi-parametric** estimation of DGCCs.

Semi-parametric Estimation

This is preliminary.

Model + data + $\boldsymbol{\theta}$ induce an operator $T_{\boldsymbol{\theta}} : \mathcal{F}_\nu \to \mathcal{F}_\nu$ as follows:

$$F_\nu \underbrace{\mapsto}_{(i)} \sigma^x(\boldsymbol{\xi}, \nu) \underbrace{\mapsto}_{(ii)} \bar{V}_I \underbrace{\mapsto}_{(iii)} W(\xi' \mid \boldsymbol{\xi}, F^\sigma) \mapsto F_\nu$$

(i) $F_X(\sigma(\boldsymbol{\xi}, \nu) \mid \boldsymbol{\xi}) = 1 - F_\nu(\nu)$. (ii) ▸ $\bar{V}_I$ (iii) ▸ W (iv) FOC

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If $T_{\boldsymbol{\theta}}$ has a unique fixed point $F_\nu(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$,

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If $T_{\boldsymbol{\theta}}$ has a unique fixed point $F_\nu(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$,

Then, we can attempt estimation of $\boldsymbol{\theta}$ by solving

$$\min_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, F_\nu(\boldsymbol{\theta}))$$

- $Q(\boldsymbol{\theta}, F_\nu)$ might be the II objective above.
- In the example below, $\sum_\xi d\left(\hat{F}_X(\cdot \mid \xi), F_X(\cdot \mid \xi, \boldsymbol{\theta})\right)$, where $d(F, G) = \|F^{-1} - G^{-1}\|_\infty$

Semi-parametric Estimation: Example

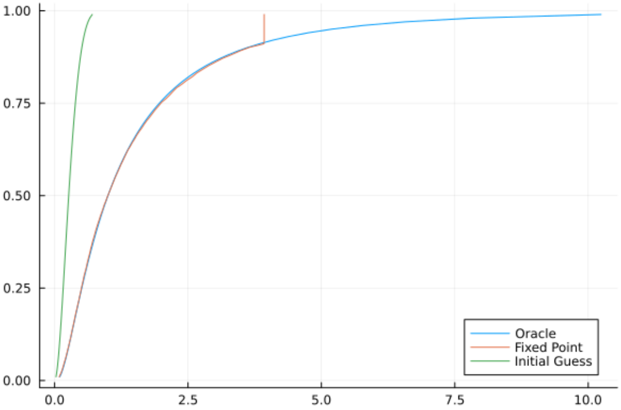
We test the feasibility of the idea above in a very simple setting.

- A monopolist solving a dynamic decision problem.
- $D(p; \xi) = \xi - p \Rightarrow \pi(\xi)$. Zero cost.
- $\xi \in \{\xi_L, \xi_H\}$.
- Monopolist controls dynamics of ξ via $P(\xi_L \mid \xi, x)$.
- $c(x, \nu)$ as above.

This is a rather difficult DGP for the semi-parametric estimator.

- Only two states.
- Stationary distribution has $P(\xi_H) = 0.9$.
- And still ...

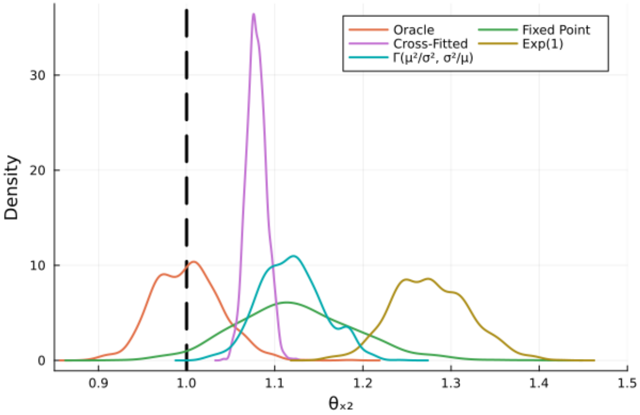
Semi-parametric Estimation: $T_{\boldsymbol{\theta}}$



T_θ seems rather stable in this example.

- Convergence looks linear with a rather low error ratio.
- We have found convergence for all $\boldsymbol{\theta}$ and F_ν^0.

Semi-parametric Estimation: Monte Carlo with Misspecification



Objective

Conclusion

Many of the firm decisions we wish to model are

(i) **Dynamic**, and...
(ii) Naturally treated as **continuous**.

Yet, applications of DGCCs are not as plentiful as one might expect. **Why?**

- Many barriers to estimating a DGCC.
- An important one, we hypothesize, is the performance of available estimators.

We **show** that estimators of DGCCs based on firms' optimality conditions

- Are sufficiently **cheap** to compute in an empirically-relevant setting.
- Deliver substantial **performance gains** relative to alternatives.

We have also given a preliminary outline of a **semi-parametric** estimator for DGCCs.

- They perform reasonably well in a simple example.
- We show that misspecification of F_ν can impart significant bias on $\hat{\boldsymbol{\theta}}$.
- To-do list:
  - Proof of convergence.
  - Application based on Hashmi and van Biesebroeck (2016).

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Related Literature and Contributions

Related Literature

- Bajari et al. (2007) and applications, see references above.
- Srisuma (2013)
  - Similar intuition.
  - Known F_ν.
  - Concrete proposal has its challenges.
    - $\rightarrow$ Non-smooth objective, requires a lot of data and computation.
    - $\Rightarrow$ Simple, static, Monte Carlo experiments.
- Schrimpf (2011)
  - Identification and estimation in the fully non-parametric case.
  - Theory only.

Contributions.

- Estimators based on recursive equilibrium conditions are **computationally tractable**.
- Recursive estimators can **outperform** the BBL inequality estimator in a relevant model.
- **Semi-parametric** estimation of DGCCs (very much in progress).

Back to intro.

A Modicum of Characterization – 1/3

Let

1. $V_I(\boldsymbol{\xi}, \rho)$ be an incumbent's ENPV of profits given $(\boldsymbol{\xi}, \rho)$.
2. $\bar{V}_I(\boldsymbol{\xi}) := \int_\rho V_I(\boldsymbol{\xi}, \rho)\, \mathrm{d}F_\rho$.

It is possible to show that

$$\mathbb{E}_\sigma[V_I(\boldsymbol{\xi}', \rho)|\boldsymbol{\xi}, x] = \int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}, F^\sigma)\, \mathrm{d}F(\xi_1' \mid \xi_1, x)\ ,$$

where

- $F^\sigma(\xi_j' \mid \boldsymbol{\xi}_j) := \int_{\varepsilon_j} F(\xi_j' \mid \xi_j, \sigma_j(\boldsymbol{\xi}_j, \varepsilon_j))\, \mathrm{d}G_{\varepsilon_j}$;
- $W(\xi_1' \mid \boldsymbol{\xi}, F^\sigma) := \int_{\xi_2'} \cdots \int_{\xi_{\bar{N}}'} \bar{V}_I(\xi_1', \boldsymbol{\xi}_{-1}')\, \mathrm{d}F^\sigma(\xi_{\bar{N}}' \mid \boldsymbol{\xi}_{\bar{N}}) \ldots \mathrm{d}F^\sigma(\xi_2' \mid \boldsymbol{\xi}_2)$;

Back to recursive estimators. Back to timeline & equilibrium.

A Modicum of Characterization – 2/3

The investment problem can then be written as

$$\max_{x \in \mathbb{R}_+} \left\{ - c(x, \nu) + \beta \int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}, F^{\sigma}) \, \mathrm{d}F(\xi_1' \mid \xi_1, x) \right\}$$

Assumption (A1)

Let Ξ be a compact subset of the real line, and denote its minimum and maximum by ξ_m and ξ_M. Let Ξ° be the interior of Ξ. The family of distributions $F(\cdot \mid \xi, x)$ is such that

(a) *$F(\xi' \mid \xi, x)$ is strictly decreasing and strictly convex in x, for all $\xi \in \Xi$ and $\xi' \in \Xi^{\circ}$;*

(b) *$F(\xi_m \mid \xi, x)$ is decreasing and convex in x, , for all $\xi \in \Xi$.*

A Modicum of Characterization – 3/3

Proposition

Assume

(a) $c(x, \nu)$ *is convex in* x*;*

(b) $F(\xi' \mid \xi, x)$ *is twice continuously differentiable in* x *for all* ξ', ξ*;*

(c) $W(\xi' \mid \boldsymbol{\xi}, F^\sigma)$ *is increasing in* ξ' *for all* $\boldsymbol{\xi}$*.*

(d) *A1 holds.*

Then the incumbent's investment problem is strictly concave in x*.*

- Useful for computing equilibria and for our approach to estimation.
- N.B.: F^σ is the one endogenous parameter to the investment problem.
  - Existence: suffices to establish the existence of a fixed point for F^σ.

Back to solution concept.

Additional characterization Details

MPE Computation

Estimating Continuation Values

We show that

$$[I - \beta \boldsymbol{M}(\boldsymbol{P})]\bar{\boldsymbol{V}}_I = \boldsymbol{\pi} - \boldsymbol{K}(\boldsymbol{\theta}_x) + \boldsymbol{\Sigma}(F_\rho)$$

Estimating Continuation Values

We show that

$$[I - \beta \boldsymbol{M}(\boldsymbol{P})]\bar{\boldsymbol{V}}_I = \boldsymbol{\pi} - \boldsymbol{K}(\boldsymbol{\theta}_x) + \boldsymbol{\Sigma}(F_\rho)$$

where

$$\boldsymbol{K}(\boldsymbol{\theta}_x) = \left[\mathbb{P}(\alpha^I(\boldsymbol{\xi}, \rho) = 1 \mid \boldsymbol{\xi}) \int c(\sigma^x(\boldsymbol{\xi}, \nu), \nu; \boldsymbol{\theta}_x)\, \mathrm{d}F_\nu\right]_{\{\boldsymbol{\xi} \in \Xi_I\}}$$

$$\boldsymbol{\Sigma}(F_\rho) = \left[[1 - \mathbb{P}(\alpha^I(\boldsymbol{\xi}, \rho) = 1 \mid \boldsymbol{\xi})]\mathbb{E}[\rho \mid F_\rho(\rho) > \mathbb{P}(\alpha^I(\boldsymbol{\xi}, \rho) = 1 \mid \boldsymbol{\xi})]\right]_{\{\boldsymbol{\xi} \in \Xi_I\}}$$

$$\boldsymbol{M}(\boldsymbol{P}) = \left[\prod_{j=1}^{\bar{N}} F^\sigma(\xi_j' \mid \boldsymbol{\xi}) : \boldsymbol{\xi}', \boldsymbol{\xi} \in \Xi_I\right]$$

- $\bar{V}_I$ estimable up to $\boldsymbol{\theta}$ given $\hat{\mathbb{P}}(\alpha^I(\boldsymbol{\xi}, \rho) = 1 \mid \boldsymbol{\xi})$, $\hat{F}^\sigma(\xi_j' \mid \boldsymbol{\xi})$, and $\hat{\sigma}^x(\boldsymbol{\xi}, \nu)$.
- $\sigma^x(\boldsymbol{\xi}, \nu) = F_X(1 - F_\nu(\nu) \mid \boldsymbol{\xi})$.
- Similar calculations in Jofre-Bonet and Pesendorfer (2003) and Pakes et al. (2007).

Back to recursive estimator.

Recursive Estimator: Indirect Inference

Investment

We run

$$T_{\{\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho\}}(\boldsymbol{\xi}_i,\nu_i;\hat{\Phi}) = \sum_{k=1}^{B} \lambda_k^x \Psi_k^x(\boldsymbol{\xi}_i) + \zeta_i^x \quad \Rightarrow \quad \hat{\boldsymbol{\lambda}}_{\text{Investment}}(\{\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho\};\hat{\Phi})$$

for draws $\nu_i \sim F_\nu$ and the analogous regression on observed investment.

Entry/Exit

Solving the max in the Bellman Equation yields $\widehat{\bar{V}}_I^A(\boldsymbol{\xi};\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho) \Rightarrow \hat{\mathbb{P}}(\text{ Exit} \mid \boldsymbol{\xi};\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho)$.

- Run regressions of these probabilities on functions of $\boldsymbol{\xi}$.
- Run analogous regressions on observed exit decisions.
- Similarly for entry decisions.

The Indirect Inference estimator solves

$$\min_{\boldsymbol{\theta}} \left[\hat{\boldsymbol{\lambda}}(\boldsymbol{\theta};\hat{\Phi}) - \hat{\boldsymbol{\gamma}}\right]^\top \boldsymbol{\Omega} \left[\hat{\boldsymbol{\lambda}}(\boldsymbol{\theta};\hat{\Phi}) - \hat{\boldsymbol{\gamma}}\right]$$

Indirect Inference Details
Other Recursive Estimators

Flow Profit Specification

Demand

- Nested Logit:

$$u_{ij} = \begin{cases} \epsilon_i^{\text{out}} + (1-\varsigma)\varepsilon_{ij} & \text{if } j = 0 \\ \alpha p_j + \xi_j + \epsilon_i^{\text{in}} + (1-\varsigma)\varepsilon_{ij} & \text{if } j = 1, \ldots, N \end{cases},$$

- Inside and Outside good nests; $\epsilon_i^{\text{g}} + (1-\varsigma)\varepsilon_{ij} \sim T1EV, \text{g} = \{\text{in, out}\}$.

Pricing

- Marginal cost: $\mu(\xi_j) = \exp(\theta_{c1} + \theta_{c2}\xi_j)$.
- Firms compete à la Bertrand
  - Caplin and Nalebuff (1991) $\Rightarrow \exists!$ Eqm $\Rightarrow \pi_j(\boldsymbol{\xi})$ well-defined.

▸ Back to Monte Carlo setup.

Monte Carlo Setup

Parameter	Value
Data Structure	
Number of Markets	100
Number of Periods	40
Maximum Number of Firms	5
Own State Space	[-∞, -1.4, -1.2, ... , 1.2, 1.4]
Model Parameters	
Discount Factor	0.925
Investment Cost Parameters	[2.625, 1.624, 0.5096]
Scrap Value Distribution	$\log N(-1.0, 0.75)$
Entry Cost Distribution	$\log N(0.625, 0.5)$
II Estimator	
Number of Investment Repetitions	5
Weight Matrix	Bootstrap
Number of Bootstrap Samples	1000
BBL Estimators	
Number of Inequalities	5000
Number of Simulated Paths	500
Simulation Horizon	80

Two-parameter F_ρ, F_ϕ, as opposed to common exponential.

Back to Monte Carlo setup.

First Stage: II and BBL

- $\pi(\boldsymbol{\xi})$
  - We treat as known.
  - In practice, estimated offline.

- $P(\xi_j' \mid \boldsymbol{\xi})$
  - Flexible MNL on functions of $\boldsymbol{\xi}$.
- $P(\xi' \mid \xi, x)$
  - Flexible MNL on functions of ξ and x.
- $\sigma^x(\boldsymbol{\xi}, \nu)$
  - $\sigma^x(\boldsymbol{\xi}, \nu) = F_X^{-1}(1 - F_\nu(\nu) \mid \boldsymbol{\xi})$
  - We estimate evenly spaced **quantile regressions** of investment on functions of $\boldsymbol{\xi}$.
  - Commonly used $\mathbb{E}[x \mid \boldsymbol{\xi}]$ ignores dependence on ν.
- Probabilities of entry and exit.
  - Logit on functions of $\boldsymbol{\xi}$.

Back to Monte Carlo setup.

Bajari et al. (2007)

If σ is a SMPE then $\forall\ \boldsymbol{\xi}, \sigma'$

$$\bar{V}_I(\boldsymbol{\xi}; \sigma, \sigma, \boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho) \geq \bar{V}_I(\boldsymbol{\xi}; \sigma', \sigma, \boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho);$$

- Analogous condition holds for $\bar{V}_E$.

Define, for entrants and incumbents,

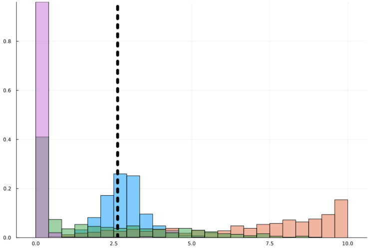
$$g(\boldsymbol{\xi}, \sigma'; \sigma, \boldsymbol{\theta}) := \bar{V}(\boldsymbol{\xi}; \sigma, \sigma, \boldsymbol{\theta}) - \bar{V}(\boldsymbol{\xi}; \sigma', \sigma, \boldsymbol{\theta})$$

The **BBL estimator** is

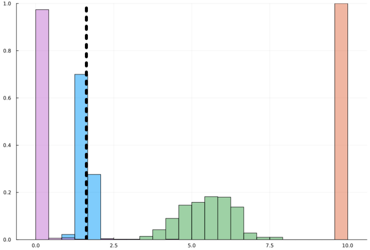
$$\hat{\boldsymbol{\theta}}_{BBL} = \arg\min_{\boldsymbol{\theta}} \hat{Q}(\boldsymbol{\theta}, \hat{\sigma}) = \frac{1}{n_I} \sum_{i=1}^{n_I} \left(\min \left\{ \hat{g}(\boldsymbol{\xi}_i, \sigma_i'; \hat{\sigma}, \boldsymbol{\theta}), 0 \right\} \right)^2$$

- For $n_I < \infty$, $\hat{\boldsymbol{\theta}}_{BBL}$ may be a set even if the model is point-identified.
- Estimates are a function of the chosen **deviations** σ'.
  - How to choose deviations σ'?
  - Choice does matter: Srisuma (2013) and simulations below.

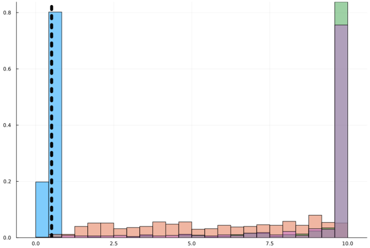
Investment Cost Parameters Estimates



θ_{x1}



θ_{x2}



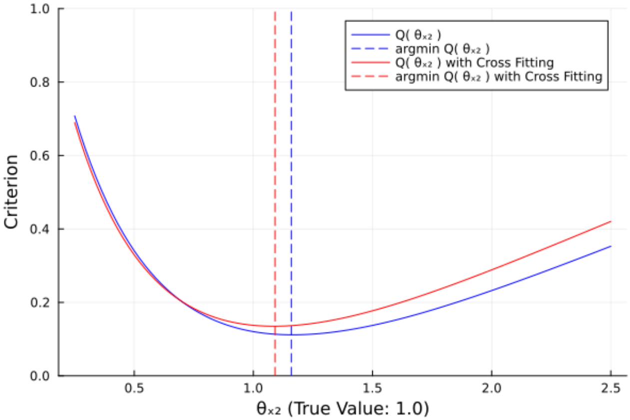
θ_{x3}

$$c(x, \nu) = \theta_{x1}x + \theta_{x2}x^2 + \theta_{x3}x\nu$$

Indirect Inference | BBL - Asymptotic | BBL - Multiplicative | BBL - Additive

▸ Indirect Inference Only ▸ F_ρ and F_ϕ Estimates ▸ Back to Monte Carlo results.

Semi-parametric Estimation: Example



Back to distributions.

Value Functions: The Incumbent's Investment Problem

The expected net present value (ENPV) to the incumbent entering the period is

$$\bar{V}_I(\boldsymbol{\xi}) = \int_\rho V_I(\boldsymbol{\xi}, \rho) dF_\rho = \int_\rho \max \left\{ \pi(\boldsymbol{\xi}) + \rho, \int V_I^A(\boldsymbol{\xi}, \nu) dF_\nu \right\} dF_\rho$$

where $V_I^A(\boldsymbol{\xi}, \nu)$ is the ENPV of having chosen to be active next period, i.e.

$$V_I^A(\boldsymbol{\xi}, \nu) = \max_{x \in \mathbb{R}_+} \left\{ \pi(\boldsymbol{\xi}) - c(x, \nu) + \beta \mathbb{E}[V_I(\boldsymbol{\xi}', \rho) \mid \boldsymbol{\xi}, x, \sigma] \right\} \qquad \text{(Investment)}$$

and the continuation value is

$$\mathbb{E}[V_I(\boldsymbol{\xi}', \rho) \mid \boldsymbol{\xi}, x, \sigma] = \int_{\boldsymbol{\varepsilon}_{-1}} \int_{\boldsymbol{\xi}'} \bar{V}_I(\boldsymbol{\xi}) \mathrm{d}F(\boldsymbol{\xi}' \mid \boldsymbol{\xi}, x, \sigma_{-1}(\boldsymbol{\xi}, \boldsymbol{\varepsilon}_{-1})) \mathrm{d}G_{\boldsymbol{\varepsilon}_{-1}}$$

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and the continuation value is , by indep. of ε_j

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$$\mathbb{E}[V_I(\boldsymbol{\xi}', \rho) \mid \boldsymbol{\xi}, x, \sigma] = \int_{\xi_1'} \int_{\xi_{\bar{N}}'} \cdots \int_{\xi_2'} \bar{V}_I(\boldsymbol{\xi}) \prod_{j=2}^{\bar{N}} \int_{\varepsilon_j} \mathrm{d}F(\xi_j' \mid \xi_j, \sigma_j(\boldsymbol{\xi}_j, \varepsilon_j)) \mathrm{d}G_{\varepsilon_j} \mathrm{d}F(\xi_1' \mid \xi_1, x)$$

Value Functions: The Incumbent's Investment Problem

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Value Functions: The Incumbent's Investment Problem

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and the continuation value is , by indep. of ε_j and by conditional indep. of $\boldsymbol{\xi}' \mid \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{x}$

$$\mathbb{E}[V_I(\boldsymbol{\xi}', \rho) \mid \boldsymbol{\xi}, x, \sigma] = \int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}; F^\sigma) \mathrm{d}F(\xi_1' \mid \xi_1, x) \qquad \text{(Continuation)}$$

$$W(\xi_1' \mid \boldsymbol{\xi}; F^\sigma) = \int_{\xi_{\bar{N}}'} \cdots \int_{\xi_2'} \bar{V}_I(\boldsymbol{\xi}) \mathrm{d}F^\sigma(\xi_2' \mid \boldsymbol{\xi}_2) \ldots \mathrm{d}F^\sigma(\xi_{\bar{N}}' \mid \boldsymbol{\xi}_{\bar{N}})$$

Value Functions: The Incumbent's Investment Problem

Value Functions: Exit

Incumbents must commit to exiting before knowing ν. Their ENPV given ρ is

$$V_I(\boldsymbol{\xi}, \rho) = \max\left\{\pi(\boldsymbol{\xi}) + \rho, \int V_I^A(\boldsymbol{\xi}, \nu) dF_\nu\right\} \tag{Exit}$$

Hence the conditional probability of exiting is

$$\mathbb{P}(\alpha^I(\boldsymbol{\xi}, \rho) = 1 | \boldsymbol{\xi}) = 1 - F_\rho\left(\int V_I^A(\boldsymbol{\xi}, \nu) dF_\nu - \pi(\boldsymbol{\xi})\right)$$

Value Functions: Entry

Entrants are short-lived: they either enter or perish. Their ENPV given ϕ is

$$V_E(\boldsymbol{\xi}_{-1}, \phi) = \max\left\{0, \bar{V}_E^A(\boldsymbol{\xi}_{-1}) - \phi\right\} \tag{Entry}$$

where

$$\bar{V}_E^A(\boldsymbol{\xi}_{-1}) = \int_\nu \max_{x \in \mathbb{R}_+} \left\{ -c(x,\nu) + \beta \int_{\xi_1'} W(\xi_1' | (-\infty, \boldsymbol{\xi}_{-1}), F^\sigma) \mathrm{d}F(\xi_1' | \xi_E, x) \right\} \mathrm{d}F_\nu$$

where $\xi_E \in \Xi$ is an exogenously specified quality for the entrant.
Hence the conditional probability of entering is

$$\mathbb{P}(\alpha^E(\boldsymbol{\xi}_{-1}, \phi) = 1 | \boldsymbol{\xi}_{-1}) = F_\phi(\bar{V}_E^A(\boldsymbol{\xi}_{-1}))$$

Back

Investment: First Order Condition

Consider firm 1's problem (wlog given symmetry and anonymity). Then

$$-\partial_x c(x,\nu) + \beta\partial_x\left(\int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}, F^\sigma) dF(\xi_1' \mid \xi_1, x)\right) \le 0 \tag{1}$$

where recall

$$W(\xi_1' \mid \boldsymbol{\xi}; F^\sigma) := \int_{\xi_2'} \cdots \int_{\xi_{\bar{N}}'} \bar{V}_I(\xi_1', \boldsymbol{\xi}_{-1}') dF(\xi_{\bar{N}}' \mid \xi_{\bar{N}}, x) \dots dF(\xi_2' \mid \xi_2, x)$$

and

$$\bar{V}_I(\boldsymbol{\xi}) = \int_\rho V_I(\boldsymbol{\xi}, \rho) dF_\rho = \int_\rho \max\left\{\pi(\boldsymbol{\xi}) + \rho, \int V_I^A(\boldsymbol{\xi}, \nu) dF_\nu\right\} dF_\rho$$

and

$$V_I^A(\boldsymbol{\xi}, \nu) = \max_{x \in \mathbb{R}_+}\left\{\pi(\boldsymbol{\xi}) - c(x,\nu) + \beta\mathbb{E}[V_I(\boldsymbol{\xi}', \rho) \mid \boldsymbol{\xi}, x, \sigma]\right\}$$

Back to model.

Investment: Assumptions

Assumption (Technical)

$F(\xi' \mid \xi, x)$ is twice continuously differentiable: $\partial_x^2 F(\xi' \mid \xi, x)$ exists and is continuous.

Assumption (Diminishing Returns)

Let $\Xi^o = \Xi \setminus \{\xi_{min}, \xi_{max}\}$.
For all $\xi \in \Xi, \xi' \in \Xi^o$: $\partial_x F(\xi' \mid \xi, x) < 0, \partial_x^2 F(\xi' \mid \xi, x) > 0$.
For all $\xi \in \Xi$: $\partial_x F(\xi_{min} \mid \xi, x) \leq 0, \partial_x^2 F(\xi_{min} \mid \xi, x) \geq 0$.

$\partial_x(1 - F(\xi' \mid \xi, x)) > 0 \Rightarrow FOSD$

Assumption (Convex Costs)

$c(x, \nu)$ is convex in x: $\partial_x c(x, \nu) > 0$, $\partial_x^2 c(x, \nu) > 0$.

Back to model.

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Assumption (Convex Costs)

$c(x, \nu)$ is convex in x: $\partial_x c(x, \nu) > 0$, $\partial_x^2 c(x, \nu) > 0$.

Assumption (Higher Quality is 'Desirable')

$W(\xi' | \boldsymbol{\xi}, F^\sigma)$ is increasing in ξ' for all $\boldsymbol{\xi}$.

Back to model.

Investment: Uniqueness

Under the assumptions made in the previous slide,

$$\pi(\boldsymbol{\xi}) - c(x,\nu) + \beta \int_{\xi_j'} W(\xi_j' \mid \boldsymbol{\xi}, F^{\sigma}) \mathrm{d}F(\xi_j' \mid \xi_j, x)$$

is strictly concave.

<u>Proof</u>

$$\begin{aligned}\int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, F^{\sigma})\, \mathrm{d}F(\xi_i' \mid \xi_i, x) = &- \int_{\xi_j'} F(\xi' \mid \xi, x)\, \mathrm{d}W(\xi' \mid \boldsymbol{\xi}, F^{\sigma}) \\ &+ W(\xi_{max} \mid \boldsymbol{\xi}, F^{\sigma}) \underbrace{F(\xi_{max} \mid \xi, x)}_{=1} - W(\xi_{min} \mid \boldsymbol{\xi}, F^{\sigma}) F(\xi_{min} \mid \xi, x)\end{aligned}$$

Therefore,

$$\frac{\partial^2}{\partial x^2}\left(\int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, \sigma)\, \mathrm{d}F(\xi_i' \mid \xi_i, x)\right) = -\int_{\xi_j'} \partial_x^2 F(\xi' \mid \xi, x)\, \mathrm{d}W(\xi' \mid \boldsymbol{\xi}, \sigma) - W(\xi_{min} \mid \boldsymbol{\xi}, \sigma) \partial_x^2 F(\xi_{min} \mid \xi, x) < 0$$

▸ Back to model.

Computing Markov Perfect Equilibria

1. Guess $\bar{V}_I(\boldsymbol{\xi})$ and $F^\sigma(\xi' \mid \boldsymbol{\xi})$.
2. Solve incumbents' investment problem at all $\boldsymbol{\xi}$ and judiciously chosen ν.
   - Gives $\bar{V}_I^A := \int V_I^A(\boldsymbol{\xi}, \nu)\, \mathrm{d}F_\nu$.
3. Compute incumbents' optimal exit decisions.
4. Use incumbents' optimal choices to update $\bar{V}_I(\boldsymbol{\xi})$.
5. Solve potential entrants' investment problems at all $\boldsymbol{\xi}_{-1}$ and judiciously chosen ν to obtain $\bar{V}_E^A := \int V_E^A(\boldsymbol{\xi}_{-1}, \nu)\, \mathrm{d}F_\nu$.
6. Compute potential entrants' optimal entry decisions.
7. Iterate to convergence in $\bar{V}_I$ and F^σ.

N.B.:

- We need only compute $\bar{V}_I$.
  - Potential entrants' decisions depend on $\bar{V}_I$.
  - There is no continuation value conditional on not entering: short-lived PEs assumption.
  - Symmetry reduces the computational burden significantly: Pakes and McGuire (1994).

Back to model.

Indirect Inference I: Investment

Predicted investment $T_{\{\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho\}}(\boldsymbol{\xi},\nu;\hat{\Phi})$ can be used in multiple ways.

Our preferred estimator is based on Indirect Inference:

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1. Compute $\{T_{\{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho\}}(\boldsymbol{\xi}_i, \nu_i; \hat{\Phi})\}_{i=1}^{MK}$ for K randomly drawn ν per observation.
2. Estimate

$$T_{\{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho\}}(\boldsymbol{\xi}_i, \nu_i; \hat{\Phi}) = \sum_{k=1}^{B} \lambda_k^x \Psi_k^x(\boldsymbol{\xi}_i) + \zeta_i^x \quad \Rightarrow \quad \hat{\boldsymbol{\lambda}}_{\text{Investment}}(\{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho\}; \hat{\Phi})$$

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3. Estimate

$$x_m = \sum_{k=1}^{B} \gamma_k^x \Psi_k^x(\boldsymbol{\xi}_m) + \eta_m^x \quad \Rightarrow \quad \hat{\boldsymbol{\gamma}}_{\text{Investment}} := \left[\hat{\gamma}_1^x, \ldots, \hat{\gamma}_B^x, \hat{S}_\eta^x\right]$$

Indirect Inference IIa: Exit

1. Compute

$$\widehat{\mathbb{P}}(\mathsf{Exit} \mid \boldsymbol{\xi}; \{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho\}) = 1 - F_\rho\left(\int_\nu \widehat{V}_I^A(\boldsymbol{\xi}; \{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho\}) \mathrm{d}F_\nu - \pi(\boldsymbol{\xi}); \boldsymbol{\theta}_\rho \right)$$

2. Estimate, as in Collard-Wexler (2013),

$$\widehat{\mathbb{P}}(\mathsf{Exit} \mid \boldsymbol{\xi}; \{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho\}) = \sum_{k=1}^{B} \lambda_k^E \Psi_k^E(\boldsymbol{\xi}_i) + \zeta_i^E \quad \Rightarrow \quad \hat{\boldsymbol{\lambda}}_{\mathsf{Exit}}(\{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho\}; \hat{\Phi})$$

3. Estimate

$$\mathbb{1}\{\xi_m' = -\infty\} = \sum_{k=1}^{B} \gamma_k^E \Psi_k^E(\boldsymbol{\xi}_m) + \eta_m^E \quad \Rightarrow \quad \hat{\boldsymbol{\gamma}}_{\mathsf{Exit}} := \left[\hat{\gamma}_1^E, \ldots, \hat{\gamma}_B^E\right]$$

Indirect Inference IIb: Entry

1. Compute

$$\widehat{\mathbb{P}}(\text{Entry} \mid \boldsymbol{\xi}; \{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho, \boldsymbol{\theta}_\phi\}) = F_\phi\left(\int_\nu \widehat{V}_E^A(\boldsymbol{\xi}_{-1}; \{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho\}) \mathrm{d}F_\nu; \boldsymbol{\theta}_\phi \right)$$

2. Estimate, as in Collard-Wexler (2013),

$$\widehat{\mathbb{P}}(\text{Entry} \mid \boldsymbol{\xi}; \{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho, \boldsymbol{\theta}_\phi\}) = \sum_{k=1}^{B} \lambda_k^N \Psi_k^N(\boldsymbol{\xi}_i) + \zeta_i^N \quad \Rightarrow \quad \hat{\boldsymbol{\lambda}}_{\text{Entry}}(\{\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho, \boldsymbol{\theta}_\phi\}; \hat{\Phi})$$

3. Estimate

$$\mathbb{1}\{\xi_m' > -\infty | \xi_m = -\infty\} = \sum_{k=1}^{B} \gamma_k^N \Psi_k^N(\boldsymbol{\xi}_m) + \eta_m^N \quad \Rightarrow \quad \hat{\boldsymbol{\gamma}}_{\text{Entry}} := \left[\hat{\gamma}_1^N, \ldots, \hat{\gamma}_B^N\right]$$

Indirect Inference III: Estimator

Let

$$\hat{\boldsymbol{\lambda}}(\boldsymbol{\theta};\hat{\Phi}) = \left[\hat{\boldsymbol{\lambda}}_{\text{Investment}}(\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho;\hat{\Phi}) \quad \hat{\boldsymbol{\lambda}}_{\text{Entry}}(\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho,\boldsymbol{\theta}_\phi;\hat{\Phi}) \quad \hat{\boldsymbol{\lambda}}_{\text{Exit}}(\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho;\hat{\Phi})\right]$$

and

$$\hat{\gamma} = \left[\hat{\gamma}_{\text{Investment}} \quad \hat{\gamma}_{\text{Entry}} \quad \hat{\gamma}_{\text{Exit}}\right]$$

The Indirect Inference estimator solves

$$\min_{\boldsymbol{\theta}} \left[\hat{\boldsymbol{\lambda}}(\boldsymbol{\theta};\hat{\Phi}) - \hat{\gamma}\right]^\top \boldsymbol{\Omega} \left[\hat{\boldsymbol{\lambda}}(\boldsymbol{\theta};\hat{\Phi}) - \hat{\gamma}\right]$$

Consistency conditions (dependence on $\hat{\Phi}$ omitted):

- $B_{MK}(\boldsymbol{\lambda},\boldsymbol{\theta}) \to \mathcal{B}(\boldsymbol{\lambda},\boldsymbol{\theta})$ uniformly in $\boldsymbol{\lambda},\boldsymbol{\theta}$.
- For any $\boldsymbol{\theta}$, $\mathcal{B}(\boldsymbol{\lambda},\boldsymbol{\theta})$ has a unique optimum in $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}(\boldsymbol{\theta})$.
- $\lim_{n\to\infty} \hat{\boldsymbol{\lambda}}(\boldsymbol{\theta}) = \boldsymbol{\lambda}(\boldsymbol{\theta})$.
- $\gamma = \boldsymbol{\lambda}(\boldsymbol{\theta})$ has a unique solution, i.e. $\boldsymbol{\lambda}^{-1}$ is well-defined.

Back to Indirect Inference.

Alternative Recursive Estimators

Other objective functions could be considered:

- Nonlinear Least Squares:

$$\min_{\{\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho\}} \sum_{i=1}^{N} \left(x_i - \mathbb{E}_\nu [T_{\{\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho\}}(\boldsymbol{\xi}_i, \nu; \hat{\Phi}) \mid \boldsymbol{\xi}] \right)^2$$

- Based on Asymptotic Least Squares:

$$\min_{\{\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho\}} \sum_{\boldsymbol{\xi}} \left(\hat{\mathbb{E}}[\sigma(\boldsymbol{\xi}, \nu) \mid \boldsymbol{\xi}] - \mathbb{E}_\nu [T_{\{\boldsymbol{\theta}_x,\boldsymbol{\theta}_\rho\}}(\boldsymbol{\xi}_i, \nu; \hat{\Phi}) \mid \boldsymbol{\xi}] \right)^2$$

Indirect Inference:

- Performed better than NLLS/ALS in early experimentation.
- Is cheaper to compute (NLLS and ALS solve as many problems as ν nodes)
- ALS + entry/exit moments $\approx$ BBL “moment matching” estimator

One could also use Simulated Maximum Likelihood (to do list).

Back to Indirect Inference.

Bajari et al. (2007): Non-Linearity in $\boldsymbol{\theta}$

Note the model is not linear in parameters:

$$\bar{V}_I(\boldsymbol{\xi}) = \int_\rho \max\left\{\pi(\boldsymbol{\xi}) + \rho, \int V_I^A(\boldsymbol{\xi}, \nu) dF_\nu\right\} dF_\rho$$

Back to BBL.

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Back to BBL.

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$$\bar{V}_I(\boldsymbol{\xi}) = \pi(\boldsymbol{\xi}) + \int_{\int V_I^A(\boldsymbol{\xi},\nu)\mathrm{d}F_\nu - \pi(\boldsymbol{\xi})}^{\infty} \rho \mathrm{d}F_\rho + F_\rho\left(\int V_I^A(\boldsymbol{\xi},\nu)\mathrm{d}F_\nu - \pi(\boldsymbol{\xi})\right)\left[\int V_I^A(\boldsymbol{\xi},\nu)\mathrm{d}F_\nu - \pi(\boldsymbol{\xi})\right]$$

Back to BBL.

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Note the model is not linear in parameters:

$$\bar{V}_I(\boldsymbol{\xi}) = \pi(\boldsymbol{\xi}) + \int_{F_\rho^{-1}[\mathbb{P}(\alpha^I(\boldsymbol{\xi},\rho)=1|\boldsymbol{\xi})]}^{\infty} \rho \mathrm{d} F_\rho + \mathbb{P}\big(\alpha^I(\boldsymbol{\xi},\rho) = 1 \mid \boldsymbol{\xi}\big) F_\rho^{-1}\big[\mathbb{P}\big(\alpha^I(\boldsymbol{\xi},\rho) = 1 \mid \boldsymbol{\xi}\big)\big]$$

$\int_a^\infty \rho \mathrm{d} F_\rho$ is not generally linear in parameters (e.g. exponential is, lognormal isn't).

Back to BBL.

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$\int_a^\infty \rho \mathrm{d}F_\rho$ is not generally linear in parameters (e.g. exponential is, lognormal isn't).

Nonetheless, forward simulation can still be performed once.

- Draw $u \sim U[0,1]$
- Exit: $\mathbb{1}\{u \leq \mathbb{P}(\alpha^I(\boldsymbol{\xi},\rho) = 1 \mid \boldsymbol{\xi})\} \Leftrightarrow \mathbb{1}\{F_\rho^{-1}(u) \leq \int V_I^A(\boldsymbol{\xi},\nu)\mathrm{d}F_\nu - \pi(\boldsymbol{\xi})\}$

Back to BBL.

Bajari et al. (2007): Non-Linearity in $\boldsymbol{\theta}$

Note the model is not linear in parameters:

$$\bar{V}_I(\boldsymbol{\xi}) = \pi(\boldsymbol{\xi}) + \int_{F_\rho^{-1}[\mathbb{P}(\alpha^I(\boldsymbol{\xi},\rho)=1|\boldsymbol{\xi})]}^{\infty} \rho \mathrm{d}F_\rho + \mathbb{P}\big(\alpha^I(\boldsymbol{\xi},\rho) = 1 \mid \boldsymbol{\xi}\big) F_\rho^{-1}\big[\mathbb{P}\big(\alpha^I(\boldsymbol{\xi},\rho) = 1 \mid \boldsymbol{\xi}\big)\big]$$

$\int_a^\infty \rho \mathrm{d}F_\rho$ is not generally linear in parameters (e.g. exponential is, lognormal isn't).

Nonetheless, forward simulation can still be performed once.

- Draw $u \sim U[0,1]$
- Exit: $\mathbb{1}\{u \leq \mathbb{P}(\alpha^I(\boldsymbol{\xi},\rho) = 1 \mid \boldsymbol{\xi})\} \Leftrightarrow \alpha^I(\boldsymbol{\xi}, F_\rho^{-1}(u))$
- Forward simulate, storing exit u_{τ_e}. As $\boldsymbol{\theta}_\rho$ changes, only recompute $F_\rho^{-1}(u_{\tau_e}; \boldsymbol{\theta}_\rho)$.
- Generalizes to investment, entry choices if wanted.

Back to BBL.

Computational Considerations

- $[I - \beta \boldsymbol{M}(\boldsymbol{P})]\bar{\boldsymbol{V}}_I = \boldsymbol{\pi} - \boldsymbol{K}(\boldsymbol{\theta}_x) + \boldsymbol{\Sigma}(F_\rho)$ must be solved for each $\boldsymbol{\theta}$.
- *However*, the matrix $I - \beta \boldsymbol{M}(\boldsymbol{P})$ is *banded*:

- The LU decomposition of a banded matrix has banded components.
  - Speeds up computation considerably.
  - Decomposition computed once and stored in memory.

Back to Monte Carlo Setup.

Monte Carlo: First Stage and Deviation Details

- Empirical policy functions:

$$\hat{\sigma}^x(\boldsymbol{\xi}, \nu_\tau) = f^x(\boldsymbol{\xi})^\top \hat{\chi}^x_\tau \qquad \hat{\mathbb{P}}^E(\boldsymbol{\xi}_{-1}) = \Lambda\big(f^E(\boldsymbol{\xi})^\top \hat{\chi}^E\big) \qquad \hat{\mathbb{P}}^I(\boldsymbol{\xi}) = \Lambda\big(f^I(\boldsymbol{\xi})^\top \hat{\chi}^I\big).$$

- BBL deviations: asymptotic,

We draw $\tilde{\boldsymbol{\chi}}_i \sim \mathrm{N}(\hat{\chi}, \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\chi}})$ for each deviation i. We then form deviations as

$$\tilde{\sigma}^x_i(\boldsymbol{\xi}, \nu_\tau) = f^x(\boldsymbol{\xi})^\top \hat{\chi}^x_{i\tau}; \quad \tilde{\mathbb{P}}^E_i(\boldsymbol{\xi}_{-1}) = \Lambda\big(f^E(\boldsymbol{\xi})^\top \hat{\chi}^E_i\big); \quad \tilde{\mathbb{P}}^I_i(\boldsymbol{\xi}) = \Lambda\big(f^I(\boldsymbol{\xi})^\top \hat{\chi}^I_i\big).$$

Back to Monte Carlo Results.

Monte Carlo: First Stage and Deviation Details

- Empirical policy functions:

$$\hat{\sigma}^x(\boldsymbol{\xi}, \nu_\tau) = f^x(\boldsymbol{\xi})^\top \hat{\chi}^x_\tau \qquad \hat{\mathbb{P}}^E(\boldsymbol{\xi}_{-1}) = \Lambda\big(f^E(\boldsymbol{\xi})^\top \hat{\chi}^E\big) \qquad \hat{\mathbb{P}}^I(\boldsymbol{\xi}) = \Lambda\big(f^I(\boldsymbol{\xi})^\top \hat{\chi}^I\big).$$

- BBL deviations: multiplicative,

We draw $\iota_i \in \{.95, .975, 1.025, 1.05, 1.075\}$ for each deviation i. We then form deviations as

$$\tilde{\sigma}^x_i(\boldsymbol{\xi}, \nu_\tau) = \iota_i f^x(\boldsymbol{\xi})^\top \hat{\chi}^x_\tau; \quad \tilde{\mathbb{P}}^E_i(\boldsymbol{\xi}_{-1}) = \Lambda\big(\iota_i f^E(\boldsymbol{\xi})^\top \hat{\chi}^E\big); \quad \tilde{\mathbb{P}}^I_i(\boldsymbol{\xi}) = \Lambda\big(\iota_i f^I(\boldsymbol{\xi})^\top \hat{\chi}^I\big)\ ,$$

Back to Monte Carlo Results.

Monte Carlo: First Stage and Deviation Details

- Empirical policy functions:

$$\hat{\sigma}^x(\xi, \nu_\tau) = f^x(\xi)^\top \hat{\chi}^x_\tau \qquad \hat{\mathbb{P}}^E(\xi_{-1}) = \Lambda\big(f^E(\xi)^\top \hat{\chi}^E\big) \qquad \hat{\mathbb{P}}^I(\xi) = \Lambda\big(f^I(\xi)^\top \hat{\chi}^I\big).$$

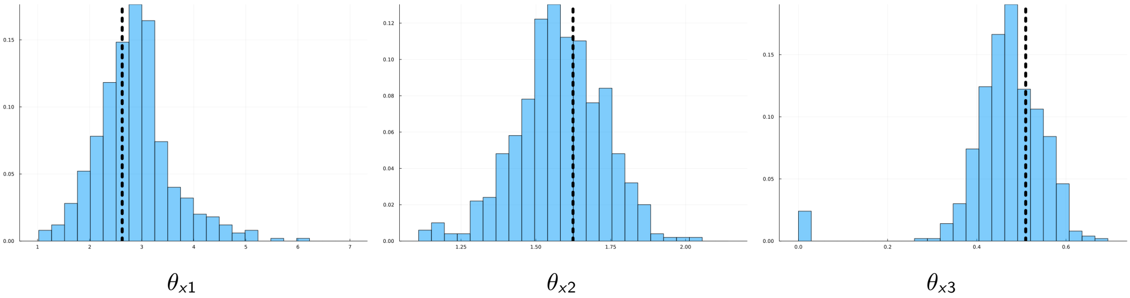
- BBL deviations: and additive.

We draw $o_{is} \sim N(0, 0.5)$ for each deviation i and simulated decision s.
We then form deviations as

$$\tilde{\sigma}^x_{is}(\xi, \nu_\tau) = f^x(\xi)^\top \hat{\chi}^x_\tau + o^x_{is}; \quad \tilde{\mathbb{P}}^E_{is}(\xi_{-1}) = \Lambda\big(f^E(\xi)^\top \hat{\chi}^E + o^{\text{Exit}}_{is}\big); \quad \tilde{\mathbb{P}}^I_{is}(\xi) = \Lambda\big(f^I(\xi)^\top \hat{\chi}^I + o^{\text{Entry}}_{is}\big).$$

Back to Monte Carlo Results.

Investment Cost Parameters II Estimates

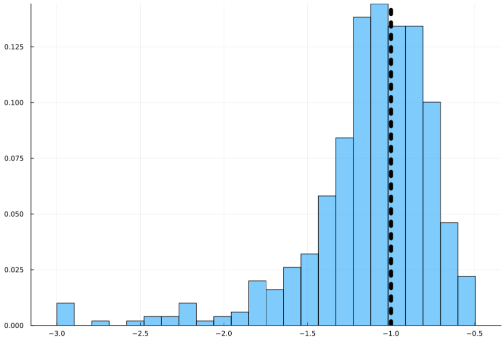


$$c(x,\nu) = \theta_{x1}x + \theta_{x2}x^2 + \theta_{x3}x\nu$$

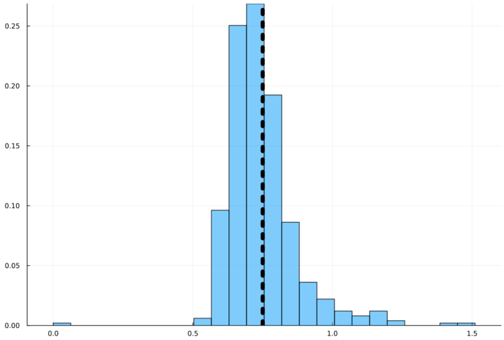
Indirect Inference   BBL - Asymptotic   BBL - Multiplicative   BBL - Additive

▸ Back to Investment Cost Parameters Estimates

Scrap Value Parameters II Estimates



μ_ρ



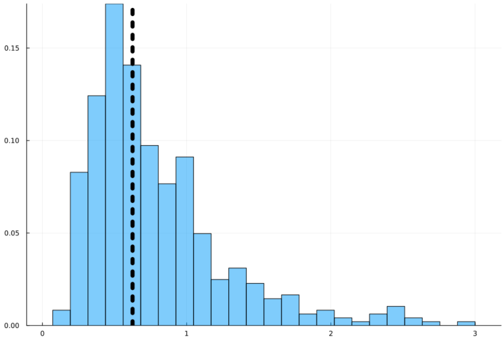
σ_ρ

$$\rho \sim \log N(\mu_\rho, \sigma_\rho)$$

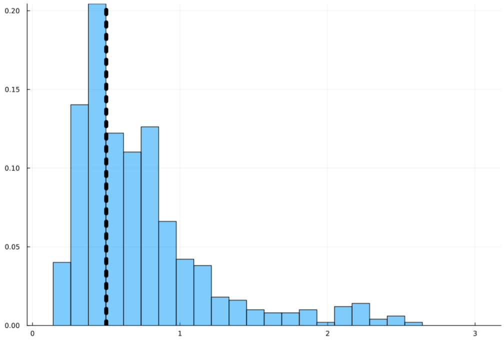
Indirect Inference | BBL - Asymptotic | BBL - Multiplicative | BBL - Additive

Back to Investment Cost Parameters Estimates

Entry Cost Parameters II Estimates



μ_ϕ

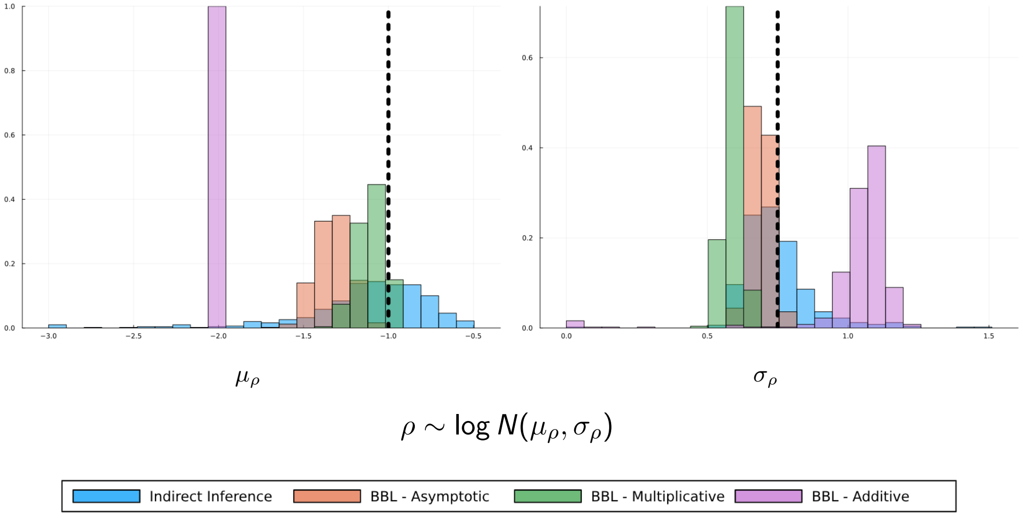


σ_ϕ

$$\phi \sim \log N(\mu_\phi, \sigma_\phi)$$

Back to Investment Cost Parameters Estimates

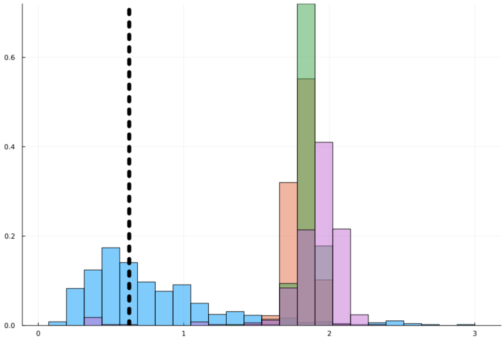
Scrap Value Parameters Estimates



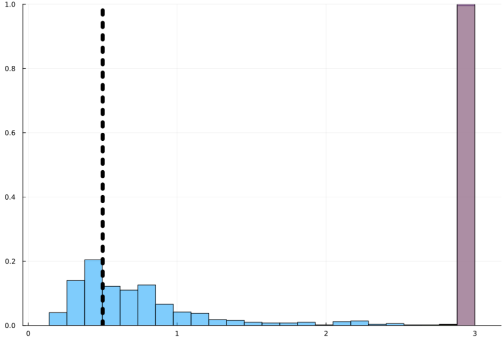
$$\rho \sim \log N(\mu_\rho, \sigma_\rho)$$

Back to Investment Cost Parameters Estimates

Entry Cost Parameters Estimates



μ_ϕ



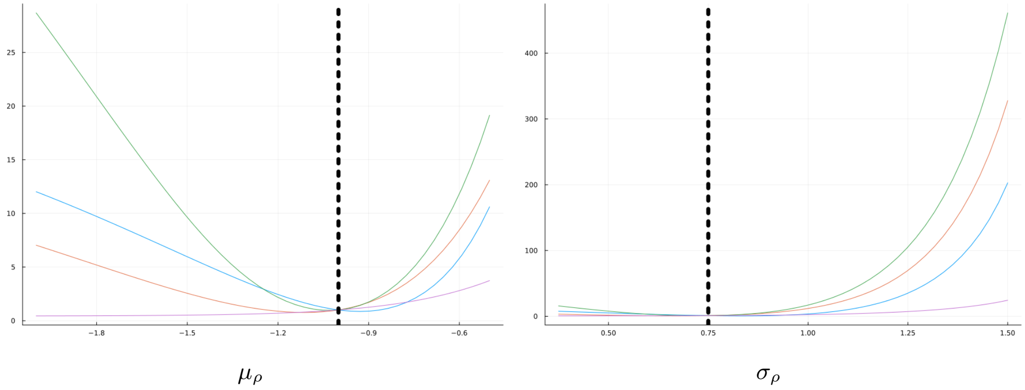
σ_ϕ

$$\phi \sim \log N(\mu_\phi, \sigma_\phi)$$

Indirect Inference | BBL - Asymptotic | BBL - Multiplicative | BBL - Additive

Back to Investment Cost Parameters Estimates

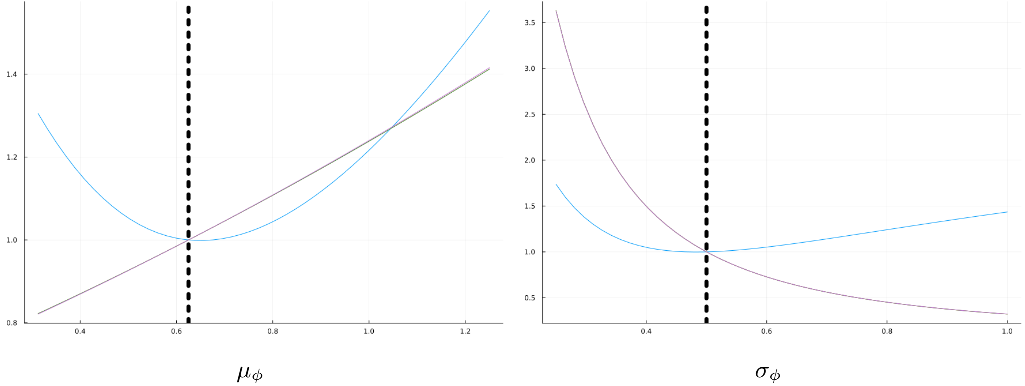
Objective Function: Scrap Value Parameters



$$\rho \sim \log N(\mu_\rho, \sigma_\rho)$$

Back to Objective Function

Objective Function: Entry Cost Parameters



$$\phi \sim \log N(\mu_\phi, \sigma_\phi)$$

Indirect Inference | BBL - Asymptotic | BBL - Multiplicative | BBL - Additive

Back to Objective Function