13- Dados 2 conjuntos numéricos limitados A e B, definimos o conjunto $A+B = \{a+b : a \in A \text{ e } b \in B\}$ Prove que:

$$\sup(A+B) = \sup A + \sup B$$

$$\operatorname{Inf}(A+B) = \operatorname{Inf} A + \operatorname{Inf} B$$

Infimo:

a) $s \leq c \ \forall c \in C$

b) $\varepsilon > 0$, $\exists c$ tq $c < s + \varepsilon$

Supremo:

a) $c \leq s \ \forall c \in C$

b) $\varepsilon > 0 \ \exists c$ tq $s - \varepsilon < c$

I - $\sup(A+B) = \sup(A) + \sup(B)$

- Provando que $\sup(A) + \sup(B) \geq \sup(A+B)$:

$\sup(A) \geq a$

$\sup(B) \geq b$

$\sup(A) + \sup(B) \geq a+b$

Então $\sup(A) + \sup(B)$ é limite superior de $A+B$:

$$\sup(A) + \sup(B) \geq \sup(A+B) \geq a+b$$

∴ Provando que $\sup(A+B) > \sup(A) + \sup(B)$

$$\sup(A+B) \geq a + b$$

$$\sup(A+B) - b \geq a$$

Dessa forma, $\sup(A+b) - b$ é Limite superior de a:

$$\sup(A+B) - b \geq \sup(A)$$

$$\sup(A+B) - \sup(A) \geq b$$

Dado b arbitrário, $\sup(A+B) - \sup(A)$ é Limite superior de B:

$$\sup(A+B) - \sup(A) \geq \sup(B)$$

$$\sup(A+B) \geq \sup(A) + \sup(B)$$

∴ Como as 2 afirmações acima são verdadeiras

$$\sup(A+B) = \sup(A) + \sup(B)$$

II - $\operatorname{Inf}(A+B) = \operatorname{Inf}(A) + \operatorname{Inf}(B)$

- Provando $\operatorname{Inf}(A+B) \geq \operatorname{Inf}(A) + \operatorname{Inf}(B)$

$$a \geq \operatorname{Inf}(A)$$

$$b \geq \operatorname{Inf}(B)$$

$$a+b \geq \operatorname{Inf}(A) + \operatorname{Inf}(B)$$

$$\Downarrow$$

$$\operatorname{Inf}(A+B) \geq \operatorname{Inf}(A) + \operatorname{Inf}(B).$$

- Provando $\operatorname{Inf}(A) + \operatorname{Inf}(B) \geq \operatorname{Inf}(A+B)$

$$a + b \geq \operatorname{Inf}(A+B)$$

$$a \geq \operatorname{Inf}(A+B) - b$$

$\operatorname{Inf}(A) + \operatorname{Inf}(B) - b$ é limite inferior de a:

$$\operatorname{Inf}(A) \geq \operatorname{Inf}(A+B) - b$$

$$b \geq \operatorname{Inf}(A+B) - \operatorname{Inf}(A)$$

$\operatorname{Inf}(A+B) - \operatorname{Inf}(A)$ é limite inferior de B:

$$\operatorname{Inf}(B) \geq \operatorname{Inf}(A+B) - \operatorname{Inf}(A)$$

$$\operatorname{Inf}(A) + \operatorname{Inf}(B) \geq \operatorname{Inf}(A+B)$$

$\therefore$ como as duas afirmações acima são verdadeiras $\operatorname{Inf}(A+B) = \operatorname{Inf}(A) + \operatorname{Inf}(B)$