GEOPH-526 : Signal Processing in Geophysics.

Department of Physics, University of Alberta, Fall 2021.

Professor: Mauricio D. Sacchi

Student: Joaquín Acedo

email: aacedo@ualberta.ca

In [5]:
```
using DSP, FFTW, PyPlot
```

Assignment 1:

1) Consider a Square Wave of period T = 100 sec:

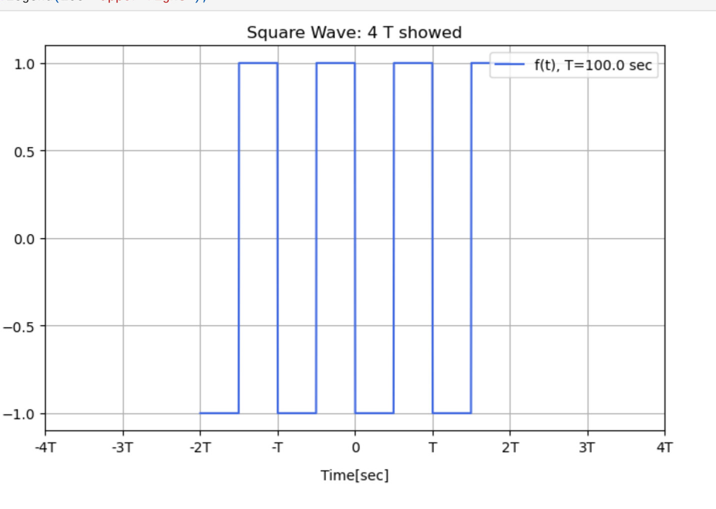
In [6]:
```
# Ploting the square wave:


T=100.0; # period length
dt=0.5; #
N= Int((T/dt)/2);
p=4; #number of periods showed


square=[-1ones(N);1*ones(N)]; # one period of the square wave

squarewave=repeat(square,p); # square wave of p periods.
t=collect(p*(-T/2)+dt/2:dt:p*(T/2)-dt/2); # time vector

# Display squarewave
figure(1,figsize=(8,5));
plt.plot(t,squarewave, c="RoyalBlue", label="f(t), T=$T sec");
plt.grid("True");
plt.title("Square Wave: $p T showed");
plt.xlabel("Time[sec]", labelpad=10.0);
plt.xticks(-4T:T:4T,["-4T","-3T","-2T","-T",0, "T", "2T", "3T", "4T"]);
plt.ylabel("Amplitude", labelpad=10.0);
plt.yticks(-1:0.5:1);
plt.legend(loc="upper right");
```



a. Find the coefficients F_n of the Fourier series:

The form of the square wave is given by:

$$f(t) = \begin{cases} 1, & t = [-T/2, 0]; \\ -1, & t = [0, T/2]; \end{cases} \tag{1}$$

To Fourier coefficients are computed with the following equation:

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f\,(t)\,e^{-i\omega_n t}\,dt$$

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{0} 1e^{-i\omega_n t}\,dt - \frac{1}{T}\int_{0}^{\frac{T}{2}} 1e^{-i\omega_n t}\,dt$$

$$F_n = -\frac{i}{T\omega_n}(e^{i\omega_n\frac{T}{2}} - 1) - \left[-\frac{i}{T\omega_n}(1 - e^{-i\omega_n\frac{T}{2}})\right]$$

$$F_n = \frac{2i}{T\omega_n} - \frac{i}{T\omega_n}(e^{i\omega_n\frac{T}{2}} + e^{-i\omega_n\frac{T}{2}})$$

$$F_n = \frac{2i}{T\omega_n} - \frac{i}{T\omega_n}(2cos(\omega_n\frac{T}{2}))$$

$$F_n = \frac{2i}{T\omega_n}(1 - cos(\omega_n\frac{T}{2}))$$

b. Plot the signal and the Fourier Synthesis for N= 1,3, 20, 40

In [7]:
```
# Function to compute Fourier Coefficients and Fourier Series of the square wave

function exercise1(nf :: Int64, T :: Float64)

    ω0=2*π/T;
    Fnp=zeros(Complex{Float64},nf); # Positive Coefficients
    Fnn=zeros(Complex{Float64},nf); # Negative Coefficients



#Fourier Coefficients: Lopp to compute Fn.

    for n=1:1:nf;
        ωn=ω0*n
        Fnp[n]=((2im)/(T*ωn))*(1 - cos(ωn*(T/2)));
        Fnn[n]=((2im)/(-1*T*ωn))*(1 - cos(-1*ωn*(T/2)));
    end

    f=zeros(length(t));


# Fourier Serie: Loop to compute Fourier Series.

    for n=1:nf;
        for i=1:length(f);
            ωn=ω0*n;
            f[i] = f[i] + Fnp[n]*exp(im*ωn*t[i])+ Fnn[n]*exp(-im*(ωn)*t[i]);
        end
    end

    return Fnp,Fnn,f
end
```

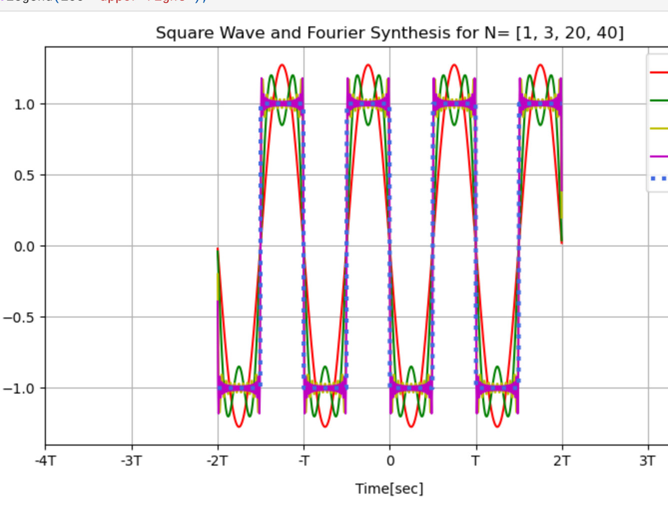
Out[7]: exercise1 (generic function with 1 method)

In [8]:
```
# Call the function to calculate different fourier synthesis for the square wave:

N=[1,3,20,40];
Fnp,Fnn,f1=exercise1(N[1],T);
Fnp2,Fnn2,f2=exercise1(N[2],T);
Fnp3,Fnn3,f3=exercise1(N[3],T);
Fnp4,Fnn4,f4=exercise1(N[4],T);
```

In [9]:
```
# Plot the results:

figure(16,figsize=(8,5),tight_layout=true);
plt.plot(t,f1, c="r",label=L"\hat f1(t)")
plt.plot(t,f2, c="g",label=L"\hat f2(t)")
plt.plot(t,f3, c="y",label=L"\hat f3(t)")
plt.plot(t,f4, c="m",label=L"\hat f4(t)")
plt.plot(t,squarewave, ":", c="RoyalBlue", label="f(t)", linewidth=3.0);
plt.grid("True");
plt.title("Square Wave and Fourier Synthesis for N= $N");
plt.xlabel("Time[sec]", labelpad=10.0);
plt.xticks(-4T:T:4T,["-4T","-3T","-2T","-T",0, "T", "2T", "3T", "4T"]);
plt.ylabel("Amplitude", labelpad=10.0);
plt.yticks(-1:0.5:1);
plt.legend(loc="upper right");
```



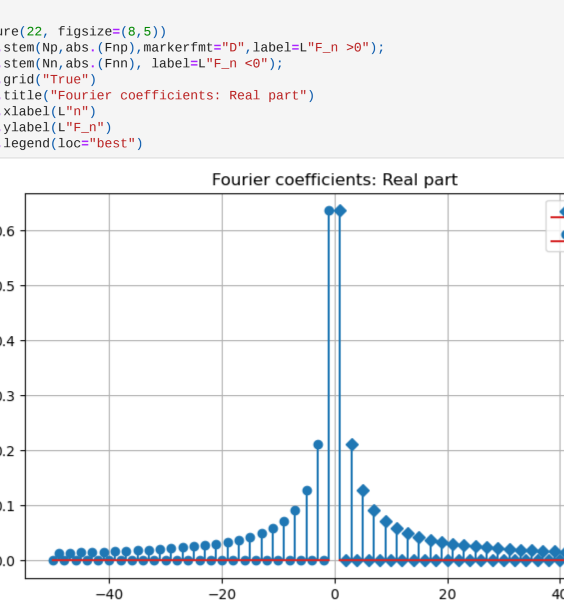
c. Plot the spectrum of discrete amplitudes for $n = [-50, 50]$

In [10]:
```
#Fourier Coefficient: # all the even coefficients are 0

Fnp,Fnn,f=exercise1(50,T);

Np=[i for i=1:50]
Nn=[i for i=-1:-1:-50]


figure(22, figsize=(8,5))
plt.stem(Np,abs.(Fnp),markerfmt="D",label=L"F_n >0");
plt.stem(Nn,abs.(Fnn), label=L"F_n <0");
plt.grid("True")
plt.title("Fourier coefficients: Real part")
plt.xlabel(L"n")
plt.ylabel(L"F_n")
plt.legend(loc="best")
```



Out[10]: PyObject <matplotlib.legend.Legend object at 0x0000000061AAA4F0>

2) Prove the following properties of the Fourier transform:

a. Linearity

b. Time delay

c. Time convolution

d. Symmetry

The **Fourier Transform** and the **Inverse Fourier Transform** of a signal are respectively given by:

$F\,(\omega) = \int_{-\infty}^{\infty} f\,(t)\,e^{-i\omega t}\,dt$

$f\,(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\,(\omega)\,e^{i\omega t}\,d\omega$

a. **Linearity**

If we apply the Fourier Transform ($\mathcal{F}$) to the sum of two scaled signals:

$\mathcal{F}\,(c_1 f_1\,(t) + c_2 f_2\,(t)) = \int_{-\infty}^{\infty}(c_1 f_1\,(t) + c_2 f_2\,(t))e^{-i\omega t}\,dt = c_1\int_{-\infty}^{\infty} f_1\,(t)\,e^{-i\omega t}\,dt + c_2\int_{-\infty}^{\infty} f_2\,(t)\,e^{-i\omega t}\,dt = c_1\mathcal{F}\,(f_1\,(t)) + c_2\mathcal{F}\,(f_2\,(t)) = c_1 F_1\,(\omega) + c_2 F_2\,(\omega)$

Therefore,**the Fourier Transform of the sum of two different signals is equal to the sum of the Fourier Transform of the signals. Moreover, the Fourier Transform of a signal which has a scalar applied is equal to the Fourier Transform of the signal scaled by the same factor.**

b. **Time delay property**

$\mathcal{F}\,(f\,(t - \tau)) = \int_{-\infty}^{\infty} f\,(t - \tau)\,e^{-i\omega t}\,dt$

if we replace $t' = t - \tau$ where $dt' = dt$ and we operate we obtain:

$\mathcal{F}\,(f\,(t - \tau)) = (e^{-i\omega\tau})\int_{-\infty}^{\infty} f\,(t)\,e^{-i\omega t}\,dt = F\,(\omega)\,e^{-i\omega\tau}.$

This demonstrates that a constant shift in time **represents a phase shift in the frequency domain, which in turn modify the structure of the signal, but it does not alter the amplitude spectrum or the energy content of the signal**. This property can be used to advance or delay a signal in time.

c. **Time convolution**

By definition:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f\,(t)\,g\,(u - t)\;dt$$

Taking the Fourier Transform of the convolution:

$\mathcal{F}(f(t) * g(t)) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f\,(t)\,g\,(u - t)\;dt\right]e^{-i\omega u}\,du$

Changing the ordder of integration:

$\mathcal{F}(f(t) * g(t)) = \int_{-\infty}^{\infty} f\,(t)\left[\int_{-\infty}^{\infty} g\,(u - t)\,e^{-i\omega u}\,du\right]\;dt$

By the shift property (**b. Time delay property**):

$\int_{-\infty}^{\infty} g\,(u - t)\,e^{-i\omega u}\,du = G\,(\omega)\,e^{-i\omega t}$

Therefore:

$\mathcal{F}(f(t) * g(t)) = \int_{-\infty}^{\infty} f\,(t)\left[\int_{-\infty}^{\infty} g\,(u - t)\,e^{-i\omega u}\,du\right]\;dt = \int_{-\infty}^{\infty} f\,(t)\,G\,(\omega)\,e^{-i\omega t}\,dt$

$\int_{-\infty}^{\infty} f\,(t)\,G\,(\omega)\,e^{-i\omega t}\,dt = G\,(\omega)\left[\int_{-\infty}^{\infty} f\,(t)\,e^{-i\omega t}\,dt\right] = F\,(\omega)\,G\,(\omega)$

Finally,

$$f(t) * g(t) = F\,(\omega)\,G\,(\omega)$$

The **FourierTransform of convolution in space/time domain is equivalent to multiplication in the frequency domain.**

d. **Symmetry**:

If $f\,(t)$ is real can be written as a sum of an even ($f_e\,(t)$) and an odd ($f_o\,(t)$) function:

$$f\,(t) = f_e\,(t) + f_o\,(t)$$

The Fourier Transform of $f\,(t)$ is:

$$F\,(\omega) = \int_{-\infty}^{\infty}(f_e\,(t) + f_o\,(t))e^{-i\omega t}\,dt$$

$$F\,(\omega) = \int_{-\infty}^{\infty}(f_e\,(t) + f_o\,(t))(cos(\omega t) - isen(\omega t))\,dt$$

Using the following properties:

1.**The product of two odd functions or two even functions is even.**

2.**The product of an even and an even function is odd.**

3.**The integral of an odd function from $-t_1$ to $+t_1$ is zero.**

$$F\,(\omega) = \int_{-\infty}^{\infty}(f_e\,(t)\,cos(\omega t) - if_o\,(t)\,sen(\omega t))\,dt = F_e\,(\omega) + F_o\,(\omega)$$

$F_e\,(\omega)$ and $F_o\,(\omega)$ are given by:

$$F_e\,(\omega) = \int_{-\infty}^{\infty} f_e\,(t)\,cos(\omega t)\,dt$$

$$F_o\,(\omega) = \int_{-\infty}^{\infty} if_o\,(t)\,sen(\omega t)\,dt$$

Therefore, if $f\,(t)$ is real $F\,(\omega)$ is complex, but the real part is even and the imaginary part is odd. Which means:

$$F\,(\omega) = F^*\,(-\omega) = F\,(-\omega)$$

$$Re(F\,(\omega)) = Re(F\,(-\omega))$$

$$Im(F\,(\omega)) = -Im(F\,(-\omega))$$

In the particular case in which $f\,(t)$ is real and even. The symmerty property shows:

$$Re(F\,(\omega)) = Re(F\,(-\omega))$$

$$Im(F\,(\omega)) = 0$$

3) By using the fact that $\mathcal{F}(e^{i\omega_0 t}) = 2\pi\delta(\omega - \omega_0)$ find the Fourier Transform of the following signals:

$c_{(t)} = cos(i\omega_0 t)$

$s_{(t)} = sin(i\omega_0 t)$

$p_{(t)} = sin(i\omega_0 t + \alpha)$

a.

$\mathcal{F}(cos(i\omega_0 t)) = \int_{-\infty}^{\infty} cos(i\omega_0 t)\,(t)\,e^{-i\omega t}\,dt$

$\int_{-\infty}^{\infty} cos(i\omega_0 t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty}\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}e^{-i\omega t}\,dt =$

$\frac{1}{2}\int_{-\infty}^{\infty} e^{i\omega_0 t}e^{-i\omega t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i\omega_0 t}e^{-i\omega t}\,dt =$

$\frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega - \omega_0)t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega + \omega_0)t}\,dt =$

$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$

b.

$\mathcal{F}(sin(i\omega_0 t)) = \int_{-\infty}^{\infty} sin(i\omega_0 t)\,(t)\,e^{-i\omega t}\,dt$

$\int_{-\infty}^{\infty} sin(i\omega_0 t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty}\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2}e^{-i\omega t}\,dt =$

$\frac{1}{2}\int_{-\infty}^{\infty} e^{i\omega_0 t}e^{-i\omega t}\,dt - \frac{1}{2}\int_{-\infty}^{\infty} e^{-i\omega_0 t}e^{-i\omega t}\,dt =$

$\frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega - \omega_0)t}\,dt - \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega + \omega_0)t}\,dt =$

$\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$

c.

$\mathcal{F}(sin(i\omega_0 t + \alpha)) = \int_{-\infty}^{\infty} sin(i\omega_0 t + \alpha)\,(t)\,e^{-i\omega t}\,dt$

$\int_{-\infty}^{\infty} sin(i\omega_0 t + \alpha)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty}\frac{e^{i\omega_0 t + \alpha} - e^{-i\omega_0 t + \alpha}}{2}e^{-i\omega t}\,dt =$

$\frac{1}{2}e^{i\alpha}\int_{-\infty}^{\infty} e^{i\omega_0 t}e^{-i\omega t}\,dt - \frac{1}{2}e^{-i\alpha}\int_{-\infty}^{\infty} e^{-i\omega_0 t}e^{-i\omega t}\,dt =$

$\frac{1}{2}e^{i\alpha}\int_{-\infty}^{\infty} e^{-i(\omega - \omega_0)t}\,dt - \frac{1}{2}e^{-i\alpha}\int_{-\infty}^{\infty} e^{-i(\omega + \omega_0)t}\,dt =$

$\pi(e^{i\alpha}\delta(\omega - \omega_0) - e^{-i\alpha}\delta(\omega + \omega_0))$

4) Compute the Fourier Transform of $f_{(t)} = e^{i\omega_0 t}$ $t = \left[\frac{-T}{2}, \frac{T}{2}\right]$ and $\omega_0 = 0.3$.

Function to calculate a complex exponential signal of frequency ω_0 whitin time range $t = \left[\frac{-T}{2}, \frac{T}{2}\right]$ and its Fourier Transform:

In [11]:
```
using DSP, FFTW, PyPlot

function exercise3(dt :: Float64, ω0 :: Float64, T:: Float64)

    t=collect(-T/2:dt:T/2);
    #f=cos.(ω0*t)
    f= exp.(1im*ω0*t);
    fr=real(f);
    dw= 2π/length(t);
    w= dw*(0:length(t)-1);
    F=fft(f);
    amp=abs.(F);
    phase=angle.(F);

    return f,amp,phase,t,w,T
end
```

Out[11]: exercise3 (generic function with 1 method)

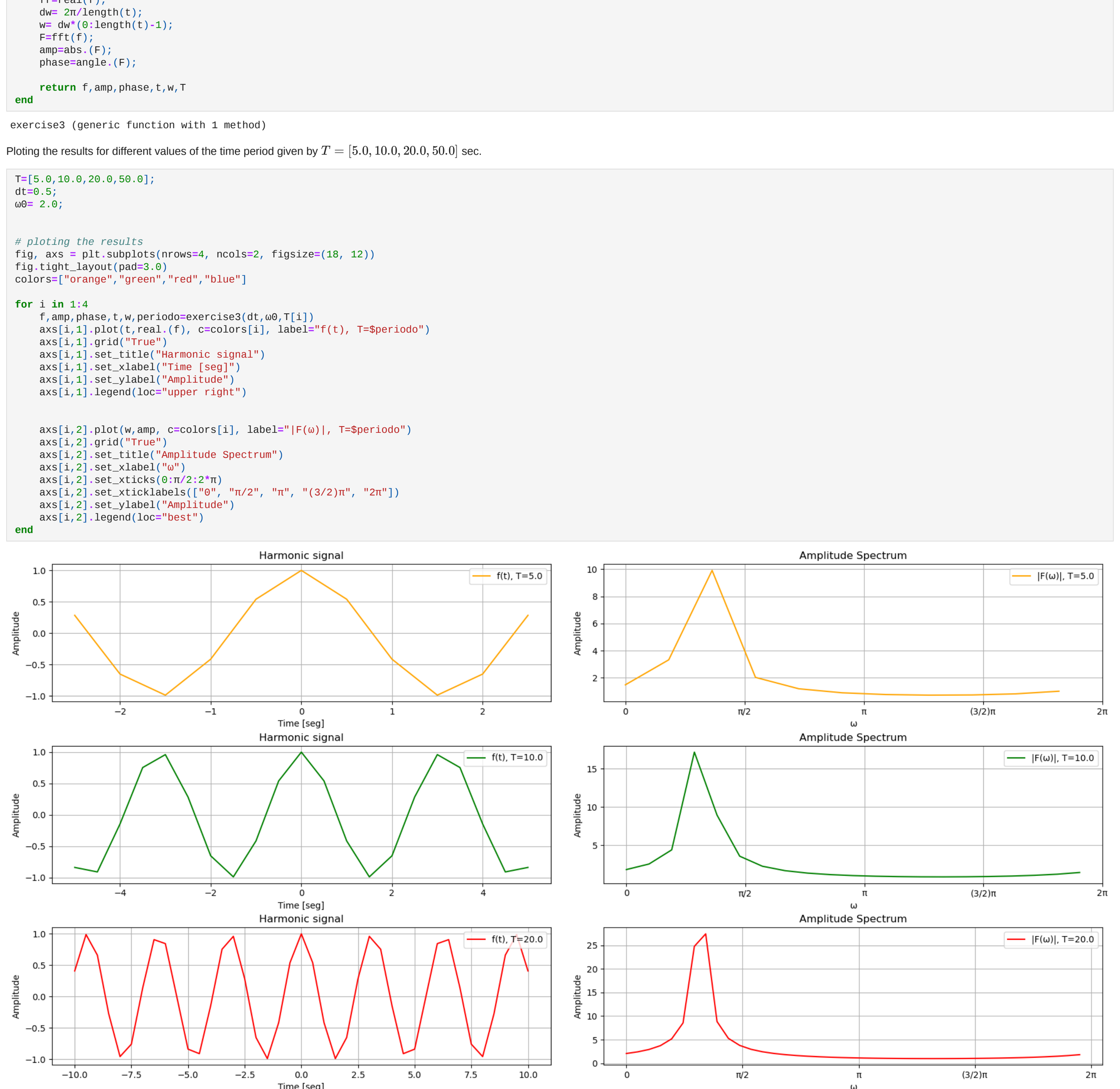
Ploting the results for different values of the time period given by $T = [5.0, 10.0, 20.0, 50.0]$ sec.

In [12]:
```
T=[5.0,10.0,20.0,50.0];
dt=0.5;
ω0= 2.0;


# ploting the results
fig, axs = plt.subplots(nrows=4, ncols=2, figsize=(18, 12))
fig.tight_layout(pad=3.0)
colors=["orange","green","red","blue"]

for i in 1:4
    f,amp,phase,t,w,periodo=exercise3(dt,ω0,T[i])
    axs[i,1].plot(t,real.(f), c=colors[i], label="f(t), T=$periodo")
    axs[i,1].grid("True")
    axs[i,1].set_title("Harmonic signal")
    axs[i,1].set_xlabel("Time [seg]")
    axs[i,1].set_ylabel("Amplitude")
    axs[i,1].legend(loc="upper right")


    axs[i,2].plot(w,amp, c=colors[i], label="|F(ω)|, T=$periodo")
    axs[i,2].grid("True")
    axs[i,2].set_title("Amplitude Spectrum")
    axs[i,2].set_xlabel("ω")
    axs[i,2].set_xticks(0:π/2:2*π)
    axs[i,2].set_xticklabels(["0", "π/2", "π", "(3/2)π", "2π"])
    axs[i,2].set_ylabel("Amplitude")
    axs[i,2].legend(loc="best")
end
```



The signal has only one frequency, but if the window length is not long enough, there is not enough resolution to see exactly where the peak is, and the energy seems to be scattered around a group of frequencies around the maximum of the spectrum.

5) Compute the Fourier Transform of the following singals:

$f\,(t) = A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t}$ $t = \left[\frac{-T}{2}, \frac{T}{2}\right]$ and $\omega_1 = 0.30$ rad/seg and $\omega_2 = 0.35$ rad/seg.

Function to calculate two different complex exponential signal of frequency ω_1 and ω_2 whitin time range $t = \left[\frac{-T}{2}, \frac{T}{2}\right]$ and their Fourier Transform:

In [16]:
```
function exercise4(dt :: Float64, ω1 :: Float64, ω2 :: Float64, T:: Float64)

    t=collect(-T/2:dt:T/2);
    #f=cos.(ω0*t)
    f1= exp.(1im*ω1*t);
    fr1=real(f1);
    f2=exp.(1im*ω2*t);
    fr2=real(f2);
    dw= 2π/length(t);
    w= dw*(0:length(t)-1);
    F1=fft(f1);
    F2=fft(f2);
    amp1=abs.(F1);
    amp2=abs.(F2);


    #phase=angle.(F1);

    return f1,f2,amp1,amp2,t,w,T
end
```

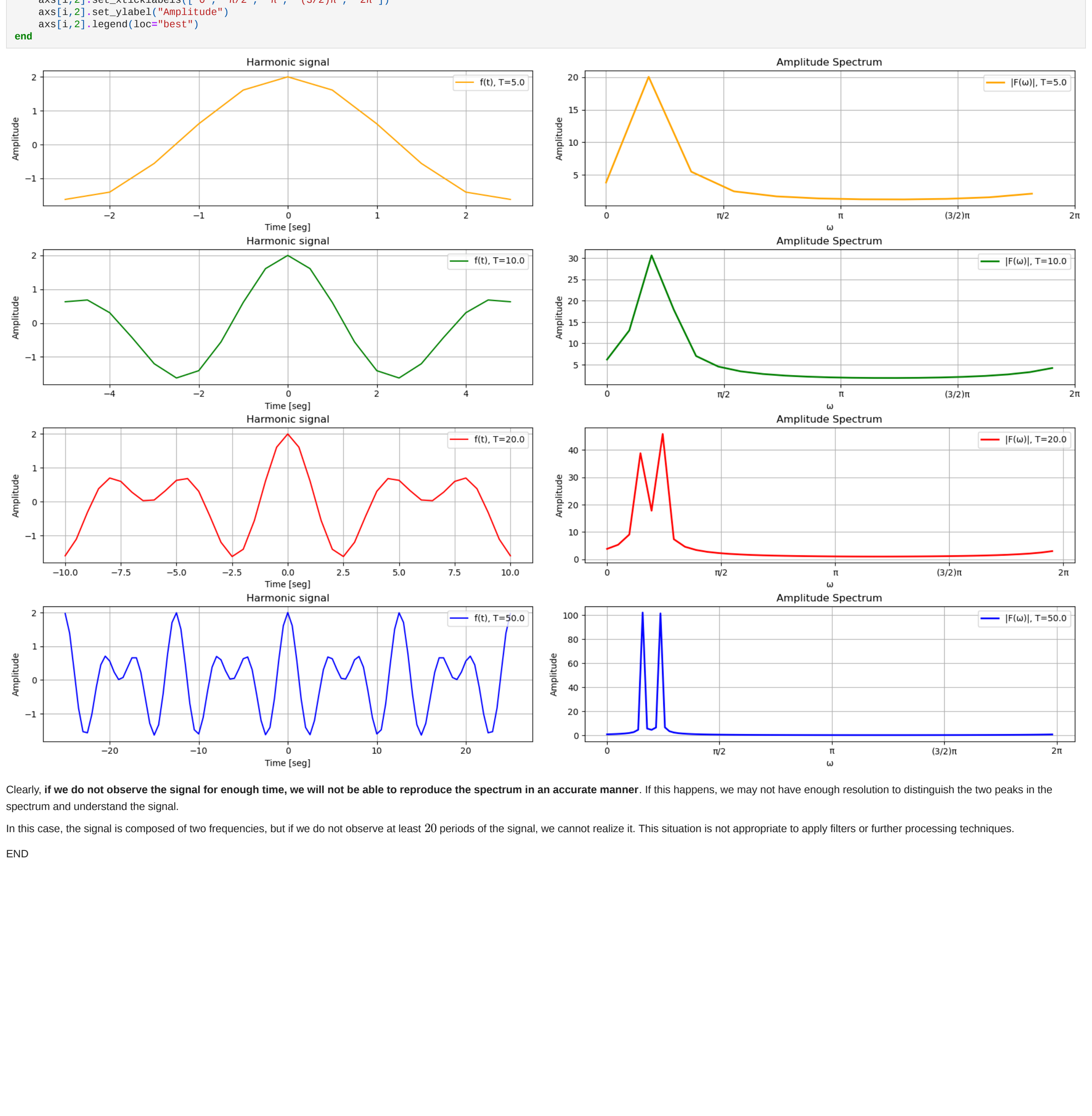
Out[16]: exercise4 (generic function with 1 method)

Ploting the amplitude spectrum of the sum of the signals for different values of the time period given by $T = [5.0, 10.0, 20.0, 50.0]$ sec.

In [17]:
```
T=[5.0,10.0,20.0,50.0];
dt=0.5;
ω1= 1.0;
ω2= 1.5;


fig, axs = plt.subplots(nrows=4, ncols=2, figsize=(18, 12))
fig.tight_layout(pad=3.0)
#colors=["RoyalBlue","LimeGreen","Sgibeet", "Salmon"];
colors=["orange","green","red","blue"]
for i in 1:4
    f1,f2,amp1,amp2,t,w,periodo=exercise4(dt,ω1,ω2,T[i])
    axs[i,1].plot(t,real.((f1+f2)), c=colors[i], label="f(t), T=$periodo")
    axs[i,1].grid("True")
    axs[i,1].set_title("Harmonic signal")
    axs[i,1].set_xlabel("Time [seg]")
    axs[i,1].set_ylabel("Amplitude")
    axs[i,1].legend(loc="upper right")


    axs[i,2].plot(w,(amp1+amp2), c=colors[i], linewidth=2, label="|F(ω)|, T=$periodo")
    axs[i,2].grid("True")
    axs[i,2].set_title("Amplitude Spectrum")
    axs[i,2].set_xlabel("ω")
    axs[i,2].set_xticks(0:π/2:2*π)
    axs[i,2].set_xticklabels(["0", "π/2", "π", "(3/2)π", "2π"])
    axs[i,2].set_ylabel("Amplitude")
    axs[i,2].legend(loc="best")
end
```



Clearly, **if we do not observe the signal for enough time, we will not be able to reproduce the spectrum in an accurate manner**. If this happens, we may not have enough resolution to distinguish the two peaks in the spectrum and understand the signal.

In this case, the signal is composed of two frequencies, but if we do not observe at least 20 periods of the signal, we cannot realize it. This situation is not appropriate to apply filters or further processing techniques.

END