1.1.20.1 An arbitrary set $\mathcal{S}$ may be considered as a (discreet) category with elements of the set as objects and arrows the identity functions mapping set elements to themselves. Under this construction, identity and composition properties follows trivially.

Alternatively, let the arrows be functions from any one element of the set to any other.

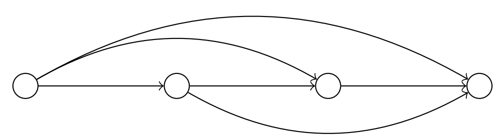
1.1.20.2 An arbitrary group G can be considered as a category with a single object. Arrows are the group homomorphisms. Associativity and composition follow from the group properties.

This category isn't very exciting, but all the arrows are isomorphisms.

1.1.20.3 The category **1** is trivially a poset; the identity arrow is the only one, so reflexivity and transitivity of $\leq$ follow.

Categories **2** and **3** satisfy reflexivity by their identity arrows, transitivity by arrow composition, and asymmetry by construction.

The category **4** looks like:



The category **5** would add one additional object X and 5 additional arrows. For every other object Y, one arrow would connect Y to X; the identity arrow from X to itself is the fifth.

The category N would consist of an infinite number of objects with one arrow connecting each object to its successor. Other arrows follow by composition.

1.1.20.4 Points (c) and (d) below complete the specification of the category **M**

(a) the objects of **M** are the natural numbers;

(b) an **M**-arrow $f : m \to n$ is an m-by-n matrix of real numbers;

(c) the composite $f \circ g$ of two arrows $f : m \to n$ and $g : n \to$ is the matrix product of f and g.

(d) the identity arrow of an object m is an m-by-m matrix of real numbers

(e) composition follows by the associativity of matrix multiplication.

1.1.20.5 We first present the diagram with generic labels for edges. Definitions of the arrow families represented by the labels are given below.

$$
\begin{tikzcd}
& \mathit{Int} \arrow[loop above, "B"] \arrow[dl, "F"'] \arrow[dr, "G"] & \\
\mathit{Bool} \arrow[loop left, "C"] & & \mathit{Real} \arrow[loop right, "D"] \\
& \mathit{Unit} \arrow[loop below, "A"] \arrow[uu, "E"'] \arrow[ul, shift left, "H"] \arrow[ul, shift right, "I"'] \arrow[ur, dashed, "J"'] &
\end{tikzcd}
$$

$$
\begin{aligned}
A &= (unit \mid id_{unit})* \\
B &= (succ_{int} \mid id_{int})* \\
C &= (not \mid id_{bool})* \\
D &= (succ_{read} \mid id_{real})* \\
E &= A;\ zero;\ B \\
F &= B;\ iszero;\ C \\
G &= B;\ toreal;\ D \\
H &= A;\ true;\ C \\
I &= A;\ false;\ C \\
J &= A;\ E;\ B;\ G;\ D
\end{aligned}
$$

1.3.1 To show that the epimorphisms in **Set** are the surjective functions, let $f : A \to B$ be a surjective function and let $g : B \to C$ and $h : B \to C$ be other arrows in **Set**. First, assume both that g and h are equal and that $g \circ f \neq h \circ f$ to derive a contradiction. Let x be an element of A such that $g(f(x)) \neq h(f(x))$. If $f(x) = y$ we have that $g(y) \neq h(y)$, but this contradicts the assumption that g and h are equal.

For the converse, assume that $g \circ f = h \circ f$ but that $g \neq h$. The first equality states that $\forall x \in A \, . \, g(f(x)) = h(f(x)) \to g(y) = h(y)$ if $y = f(x)$. Because f is surjective, this derivation shows that g and h agree on all elements of their common domain, thus contradicting the assumption that $g \neq h$.

1.3.2 To show that the composition of two monic arrows is monic, let $f : B \to C$ and $g : C \to D$ be monic arrows. First, let $a : A \to B$ and $b : A \to B$ be two equal arrows. We derive $(g \circ f) \circ a = (g \circ f) \circ b$ by assuming there is some element $x \in A$ for which the equality does not hold.

By the equality of a and b, we know that $(g \circ f)(a(x)) \neq (g \circ f)(b(x)) \Rightarrow (g \circ f)(y) \neq (g \circ f)(y)$ for a $y = a(x) = b(x)$. This implies $g(f(y)) \neq g(f(y))$, a contradiction.

Next, assume that $(g \circ f) \circ a = (g \circ f) \circ b$ but that $a \neq b$. Then we have some element $x \in A$ such that $a(x) = y_1 \neq y_2 = b(x)$. This implies that $g(f(y_1)) = g(f(y_2))$. However, g is a monomorphism and f is equal to itself, so we arrive at a contradiction.

For the second part, let $f : B \to C$ and $g : C \to D$ be functions such that $(g \circ f) : B \to D$ is monic. Then for functions $a, b : A \to B$, $(g \circ f) \circ a = (g \circ f) \circ b$ implies $a = b$. We see that f must be monic because the domain of f is equal to the domain of $(g \circ f)$ and the codomains of a and b. If f were not monic then our assumption could not hold.

1.3.3 Let f and g be epic functions. We see that their composition $g \circ f$ must also be epic by associativity of $\circ$. The equality $a \circ (g \circ f) = b \circ (g \circ f)$ may be parenthesized as $(a \circ g) \circ f = (b \circ g) \circ f$. Because f is epic, equality proves that $a \circ g = b \circ g$. Because g is epic, equality here proves that $a = b$. Thus we have that $a \circ (g \circ f) = b \circ (g \circ f)$ implies $a = b$.

Proving that if $(g \circ f)$ is epic then g is epic follows a similar argument as used in the second part of 1.3.2. Namely, the codomain of g is the same as the codomain of $(g \circ f)$. This observation leads to a similar contradiction as before; if g were not epic then the statement $a \circ (g \circ f) = b \circ (g \circ f)$ implies $a = b$ would be false.

1.3.4 To show that the inverse $f^{-1} : B \to A$ of an isomorphism $f : A \to B$ is unique, assume the existence of a function $f_2^{-1} : B \to A$ such that $f^{-1} \circ f = id_A = f_2^{-1} \circ f$ and $f^{-1} \neq f_2^{-1}$. Let y be an element of B

such that $f^{-1}(b) \neq f_2^{-1}(b)$, and let x be the preimage of b in f. Then we have $f^{-1}(f(x)) \neq f_2^{-1}(f(x)) \Rightarrow f^{-1}(y) \neq f_2^{-1}(y) \to x_1 \neq x_2$ for unequal elements $x_1, x_2 \in A$. However, if $x_1 \neq x_2$ then either (or both) of x_1 and x_2 cannot equal x. If $x_i = x$ does not hold, then the "inverse" function was incorrect; an isomorphism preserves identity.

1.3.5 We show that if $f : A \to B$ and $g : B \to C$ are isomorphisms then $g \circ f$ is an isomorphism with $f^{-1} \circ g^{-1}$ as its inverse.

First, define objects A, B, and C.

$$\begin{tikzcd} A & B & C \end{tikzcd}$$

By definition, we have arrows $f : A \to B$ and $f^{-1} : B \to A$ such that $f^{-1} \circ f = id_A$.

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{-1}"] & C \end{tikzcd}$$

Likewise, we have $g : B \to C$ and $g^{-1} : C \to B$ such that $g^{-1} \circ g = id_B$.

$$\begin{tikzcd} A & B \arrow[r, bend left, "g"] & C \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

Together, these arrows yield the diagram:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & C \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

From which it is clear that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = id_A$.

1.3.6 The category **2** has an arrow, f, that is both a monomorphism and an epimorphism but not an isomorphism.

$$A \xrightarrow{f} B$$

This holds because of the limited number of compositions we can make. Ignoring composition of identities, there are only two possibilities: $id_A;\ f$ and $f;\ id_B$. Yet there is no arrow $f^{-1} : B \to A$, so f is not an isomorphism.

1.4.6.1 *Terminal objects are unique up to isomorphism*

Proof. Assume $t1$ and $t2$ are terminal objects in a category $\mathcal{C}$. Since for every object $c \in \mathcal{C}$ there exists a unique arrow from c to each terminal object, there exist arrows $f : t1 \to t2$ and $g : t2 \to t1$. Thus $g \circ f = id_{t1}$ and $f \circ g = id_{t2}$, and f and g define our isomorphism. $\square$

A nearly-identical argument proves initial objects are unique up to isomorphism. Just flip the arrows.

1.4.6.2 **Set** $\times$ **Set:**

Initial Object: $(\{\} \times \{\})$

Terminal Objects: $\{(\{A\} \times \{B\}) \mid A, B \in$ **Set**$\}$ The arrows in **Set** $\times$ **Set** are pairs of **Set** arrows. So the arrows leaving the initial object for other objects in the category are the pairs of arrows leaving the initial object in **Set**. Similarly for the terminal objects.

Set$^{\rightarrow}$**:**

Initial Object: $\{f \mid f : \{\} \to \{\}\}$

There is exactly one way to map the empty set to any other domain or range.

Terminal Objects: $\{f \mid f : \{A\} \to \{B\},\ A, B \in$ **Set**$\}$

Any domain maps uniquely to a singleton domain. Likewise for ranges.

poset $\mathcal{P}$**:**

Initial Objects: The min objects $\{p \mid \forall p' \in \mathcal{P},\ p \leq p'\}$

Terminal Objects: The max objects $\{p \mid \forall p' \in \mathcal{P},\ p' \leq p\}$

1.4.6.3 The initial and terminal objects in a single-object category (with only the identity arrow) are the same.

Any discrete category with more than one element has no initial or terminal objects. More generally, any category with disconnected components has no initial or terminal objects.

A simple functional language with types UNIT and BOOL and arrows $true$: UNIT $\to$ BOOL and $false$: UNIT $\to$ BOOL can be formalized as a category with no initial objects. Initial objects must have one unique arrow to every other object in the category, and neither UNIT nor BOOL satisfy this criteria.

This category also has no terminal objects, for there are no arrows from BOOL to UNIT and two arrows from UNIT to BOOL.

1.5.2 We have an object X with projection arrows $\pi_1 : X \to A$ and $\pi_2 : X \to B$ and know that "X is a product of A & B". The goal is to show that X is isomorphic to $A \times B$.

By the informal declaration, we know that for every object C with arrows $f : C \to A$ and $g : C \to B$, we have an arrow $\langle f, g\rangle : C \to X$. Likewise for $A \times B$, there exists an induced arrow $\langle f', g'\rangle : C' \to A \times B$.

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dashed, "{\langle f, g\rangle}"] \arrow[dr, "g"] & \\
A & X \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C' \arrow[dl, "f'"'] \arrow[d, dashed, "{\langle f', g'\rangle}"] \arrow[dr, "g'"] & \\
A & A \times B \arrow[l, "\pi_1'"'] \arrow[r, "\pi_2'"] & B
\end{tikzcd}$$

Taking $f = \pi_1'$, $g = \pi_2'$, $f' = \pi_1$, and $g' = \pi_2$, we have an isomorphism between X and $A \times B$ from the arrows $\langle f, g\rangle$ and $\langle f', g'\rangle$.

$$\begin{tikzcd}
 & A & \\
X \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rr, dashed, shift left, "{\langle \pi_1, \pi_2\rangle}"] & & A \times B \arrow[ul, "\pi_1'"'] \arrow[dl, "\pi_2'"] \arrow[ll, dashed, shift left, "{\langle \pi_1', \pi_2'\rangle}"] \\
 & B &
\end{tikzcd}$$

For the second half of the problem, we want to show that an object C isomorphic to a product object $A \times B$ is a product of A and B. The isomorphism and definition of $A \times B$ give us the diagram:

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, shift right, "{\langle f, g\rangle}"'] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, shift right, "h"'] & B
\end{tikzcd}$$

From which it is clear that h is $\langle \pi_1, \pi_2\rangle$ and f and g are the projections from C.

1.5.6.1 We want to show that $\langle f \circ h, g \circ h \rangle = \langle f, g \rangle \circ h$. The arrow $\langle f, g \rangle$ implies the existence of a product object:

$$
\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dashed, "\langle f, g \rangle"'] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}
$$

The composition $\langle f, g \rangle \circ h$ for a function $h : D \to C$ means we have another object out in space:

$$
\begin{tikzcd}
 & D \arrow[d, "h"'] & \\
 & C \arrow[dl, "f"'] \arrow[d, dashed, "\langle f, g \rangle"'] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}
$$

By composition, we obtain the arrows $f \circ h$ and $g \circ h$:

$$
\begin{tikzcd}
 & D \arrow[d, "h"'] \arrow[ddl, dashed, "f \circ h"'] \arrow[ddr, dashed, "g \circ h"] & \\
 & C \arrow[dl, "f"'] \arrow[d, dashed, "\langle f, g \rangle"'] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}
$$

In turn, this satisfies the product construction, thus the arrows $\langle f \circ h, g \circ h \rangle$ and $\langle f, g \rangle \circ h$ are equal.

$$
\begin{tikzcd}
 & D \arrow[dl, "f \circ h"'] \arrow[d, "\langle f \circ h, g \circ h \rangle"'] \arrow[dr, "g \circ h"] & \\
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}
$$

1.5.6.2 Definition 1.5.3 states that a product map $(f \times h)$ is the arrow $\langle f \circ \pi_1, h \circ \pi_2 \rangle$. Because the mediating arrow $\langle g, k \rangle$ defines the left and right elements of a product (aka the inputs for f and h, respectively), we may compose and skip the projection arrows to obtain $\langle f \circ g, h \circ k \rangle$.

1.5.6.3 We want to show that $(f \times h) \circ (g \times k) = (f \circ g) \times (h \circ k)$. The terms on either side of the equality yield the following diagrams:

$$
\begin{tikzcd}
 & A \times B \arrow[dl, "g \circ \pi_1"'] \arrow[d, dashed] \arrow[dr, "k \circ \pi_2"] & \\
C & C \times D \arrow[l] \arrow[r] \arrow[dl, "f \circ \pi_1"'] \arrow[d, dashed] \arrow[dr, "h \circ \pi_2"] & D \\
E & E \times F \arrow[l] \arrow[r] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \times B \arrow[ddl, "f \circ g \circ \pi_1"'] \arrow[dd, dashed] \arrow[ddr, "h \circ k \circ \pi_2"] & \\
 & & \\
E & E \times F \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & F
\end{tikzcd}
$$

Our result follows by the uniqueness of the dashed arrows.

1.5.6.4 If X and Y are objects in a poset P considered as a category, a product of X and Y is an object Z such that:

- There exist arrows $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$.
- For every other object C with arrows $f : C \to X$ and $g : C \to Y$, there is the unique arrow $\langle f, g \rangle : C \to X \times Y$.

In a poset considered as a category, this means that $X \times Y \leq X$ and $X \times Y \leq Y$. Additionally, we have that for all objects C satisfying $C \leq X \wedge C \leq Y$ we have that $C \leq X \times Y$.

In other words, the product of X and Y is X is $X \leq Y$ holds and Y otherwise.

1.5.6.5 If X and Y are objects in a poset P considered as a category, a coproduct of X and Y has the properties:

- $X \leq X + Y$
- $Y \leq X + Y$
- For every Z such that $X \leq Z$ and $Y \leq Z$, we have $X + Y \leq Z$.

The coproduct of X and Y is the greater of X and Y. That is, Y if $X \leq Y$ and X otherwise.

1.5.6.6 The category poset has pairs of objects that lack a product (unless the poset is linear and finite).

The discrete category has no products, nor does the category **1**.

1.5.6.7 The definition 1.5.5 of a product of a family $(A_i)_{i \in I}$ of objects simplifies to a terminal object. Each element of the category must have a unique arrow to the product object if it has an arrow to all elements of the product. This requirement is vacuously true if the product has no elements, thus each object of the category must have a unique arrow to the product object.

Contrariwise, the empty coproduct would be an initial object in a category.

1.7.2 An equalizer $e : X \to A$ for arrows $f : A \to B$ and $g : A \to B$ has the properties:

(a) $e;\ f = e;\ g$

(b) For every arrow $e' : X' \to A$ satisfying 1, there exists a unique arrow $k : X' \to X$ such that $k;\ e = e'$.

Universal constructions describe a class of objects and arrows sharing a common property and pick out the terminal objects when this class is considered as a category. For equalizers, the common property is that $e'; f = e'; g$ for a class of arrows, and the terminal objects are the equalizer arrows; that is, the e with a unique arrow from every other e'.

1.7.4.1 First, we observe that a identity arrows are equalizers in a poset considered as a category because the dashed arrow $k = id_A$ is unique.

$$\begin{tikzcd}
A \arrow[r, "id_A"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
A \arrow[u, dashed, "id_A"] \arrow[ur, "id_A"'] & &
\end{tikzcd}$$

To prove that identity arrows are the only equalizers, assume have non-identity equalizer $e : A \to B$ for two functions $f, g : B \to C$. We know that id_B is another equalizer of f and g, so we must prove that there is a unique arrow $k : B \to A$ such that $e \circ k = id_B$.

$$\begin{tikzcd}
A \arrow[r, "e"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\
B \arrow[u, dashed, red, "k"] \arrow[ur, "id_B"'] & &
\end{tikzcd}$$

However, there is no such arrow. All arrows from an object A to an object B in a poset considered as a category have the property that $A \leq B$ in the original poset. Hence if $e : A \to B$ exists then $k : B \to A$ cannot exist unless A and B are the same object. This is impossible unless e is an identity arrow. Therefore the only equalizers in a poset considered as a category *are* identity arrows.

1.7.4.2 Every equalizer $e : B \to C$ is monic because if we have arrows $k : A \to B$ and $k' : A \to B$ such that $e \circ k = e \circ k'$ then we are guaranteed that $k = k'$ because there is exactly one unique arrow from the object A to the domain of the equalizer e. This follows because the arrows $k \circ e$ and $k' \circ e$ must also be equalizers (and any other equalizer must have a unique arrow to e).

1.7.4.3 Let $e : X \to A$ be an epic equalizer of $f : A \to B$ and $g : A \to B$. An epic equalizer has the property that $f \circ e = g \circ e$, the normal equalizer condition, now implies that $f = g$. This is stronger than normal—instead of restricting the domains of f and g to a subset on which they're equal (nevermind if they disagree on certain inputs), we know that the codomain of e is the entire domain of f and g. That is, e is surjective. It follows that every other equalizer e' such that $f \circ e' = g \circ e'$ is also epic.

To construct an isomorphism from X to A, consider the equalizer id_A as shown in the following diagram:

$$\begin{tikzcd}
X \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
A \arrow[u, dashed, "k"] \arrow[ur, "id_A"']
\end{tikzcd}$$

By definition, there is a unique arrow $k : A \to X$. Furthermore, $e \circ k = id_A$.

This gives us half of the isomorphism. But since the identity arrow is it's own inverse, we may rewrite the triangle from the diagram to clearly show that $k \circ e = id_X$.

$$\begin{tikzcd}
X \arrow[r, "e"] & A \arrow[dl, "id_A"] \\
A \arrow[u, dashed, "k"]
\end{tikzcd}$$

The end.

1.8.7.1 The pullback lemma states that if both squares of the diagram below are pullbacks then the entire rectangle is a pullback. In the other direction, it states that if the overall rectangle and right square are pullbacks, the left rectangle is a pullback.

$$
\begin{tikzcd}
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}
$$

First, we show that if the squares are pullbacks, the entire rectangle is a pullback. That is, we know the following:

$$
\begin{tikzcd}
{} \arrow[dr, dashed, "k_1"] \arrow[drr, "i_1"] \arrow[ddr, "j_1"'] & & \\
& {} \arrow[r, "f_1"] \arrow[d, "g_1"] & {} \arrow[d, "g_2"] \\
& {} \arrow[r, "f_2"'] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[dr, dashed, "k_2"] \arrow[drr, "i_2"] \arrow[ddr, "j_2"'] & & \\
& {} \arrow[r, "f_3"] \arrow[d, "g_2"] & {} \arrow[d, "g_3"] \\
& {} \arrow[r, "f_4"'] & {}
\end{tikzcd}
$$

and want to show that the pair of arrows $(f_1; f_3)$, g is a pullback for $(f_2; f_4)$ and g_1.

$$
\begin{tikzcd}
{} \arrow[dr, dashed, "k"] \arrow[drr, "i"] \arrow[ddr, "j"'] & & \\
& {} \arrow[r, "f_1; f_3"] \arrow[d, "g_1"] & {} \arrow[d, "g_3"] \\
& {} \arrow[r, "f_2; f_4"'] & {}
\end{tikzcd}
$$

From the diagram, it is immediately clear that g_1 is the pullback arrow of g_3 along $(f_2; f_4)$. That side of the rectangle is the same as that side of the square. More formally, the codomain of g_1 is the same as the domain of $(f_2; f_4)$, hence for any arrow j that we could replace g_1 with and still have a commutative diagram we can map its domain to $\mathbf{dom}(g_1)$ using the unique arrow k_1.

The case for $(f_1; f_3)$ follows because f_1 and f_3 are the unique pullbacks of f_2 and f_4 (along g_2 and g_3), respectively. If the arrows are unique then their composition is unique, thus $(f_1; f_3)$ must be the pullback of $(f_2; f_4)$ along g_3.

Next, we show that if the overall rectangle and right square are pullbacks, the left square is a pullback.

That is, we have:

$$\begin{tikzcd}
{} \arrow[drr, "i"] \arrow[dr, dashed, "k"] \arrow[ddr, "j"'] & & \\
& {} \arrow[r, "f_1; f_3"] \arrow[d, "g_1"] & {} \arrow[d, "g_3"] \\
& {} \arrow[r, "f_2; f_4"'] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[drr, "i_2"] \arrow[dr, dashed, "k_2"] \arrow[ddr, "j_2"'] & & \\
& {} \arrow[r, "f_3"] \arrow[d, "g_2"] & {} \arrow[d, "g_3"] \\
& {} \arrow[r, "f_4"'] & {}
\end{tikzcd}$$

and want to show:

$$\begin{tikzcd}
{} \arrow[drr, "i_1"] \arrow[dr, dashed, "k_1"] \arrow[ddr, "j_1"'] & & \\
& {} \arrow[r, "f_1"] \arrow[d, "g_1"] & {} \arrow[d, "g_2"] \\
& {} \arrow[r, "f_2"'] & {}
\end{tikzcd}$$

Again, we know the arrow g_1 is the pullback of g_2 along f_2. This is uniquely determined by the rectangle being a pullback. Then to show that f_1 is the pullback of f_2 along g_2, we use the uniqueness of g_2 and f_2 (which comes from the right square being a pullback). Because they are uniquely determined and because the arrow $(f_1; f_2)$ of the overall rectangle is uniquely determined, f_1 must be uniquely determined for the left square.

1.8.7.2 Goal: prove that pullbacks preserve monomorphisms.

Let the object P and arrows $f' : P \to B$, $g' : P \to C$ consitute a pullback of arrows $g : B \to D$ and $f : C \to D$.

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

Furthermore, let f be monic. Thus for any arrows $x : A \to C$ and $y : A \to C$ we have that $f \circ x = f \circ y$ implies $x = y$.

$$\begin{tikzcd} X \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & C \arrow[r, "f"] & D \end{tikzcd}$$

We want to show that f' is monic; that is, for any arrows $x : A \to P$, $y : A \to P$, a proof that $f' \circ x = f' \circ y$ implies $x = y$.

$$\begin{tikzcd} X \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & P \arrow[r, "f'"] & B \end{tikzcd}$$

Assume that $f' \circ x = f' \circ y$. Because we have the function $g : B \to D$ we have two equal arrows from X to D.

$$\begin{tikzcd} X \arrow[r, "f' \circ x = f' \circ y"] & B \arrow[d, "g"] \\ & D \end{tikzcd}$$

Because g' is a pullback of g along f and because x and y share the pullback object P as their codomain, we see that $f \circ g' \circ x$ and $f \circ g' \circ y$ are also arrows from X to D that make the diagram commute.

$$
\begin{tikzcd}
X \arrow[r, "f' \circ x = f' \circ y"] \arrow[d, "g' \circ x"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f' \circ x = f' \circ y"] \arrow[d, "g' \circ y"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"] & D
\end{tikzcd}
$$

Furthermore, because f is monic, we know that since $f \circ g' \circ x = f \circ g' \circ y$ we have that $g' \circ x = g' \circ y$. Finally the universal property of P guarantees a unique arrow k between X and P.

$$
\begin{tikzcd}
X \arrow[dr, dashed, "k"] \arrow[drr, bend left, "f' \circ x = f' \circ y"] \arrow[ddr, bend right, "g' \circ x = g' \circ y"'] & & \\
& P \arrow[r, "g'"'] \arrow[d, "f'"'] & B \arrow[d, "g"] \\
& C \arrow[r, "f"] & D
\end{tikzcd}
$$

The commutativity conditions of the pullback show that both $x = k$ and $y = k$. Thus by transitivity, $x = y$.

1.8.7.3 Given a category where every pair of objects has a product and every pair of arrows with a common codomain has an equalizer, we construct pullbacks for each pair of arrows with a common codomain. We begin with our pair of arrows, $f : B \to D$ and $g : C \to D$:

$$\begin{tikzcd}
 & B \arrow[d, "f"] \\
C \arrow[r, "g"] & D
\end{tikzcd}$$

Because every pair of objects has a product, we can connect the domains of f and g via their product and its projection arrows:

$$\begin{tikzcd}
B \times C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \\
C &
\end{tikzcd}$$

From here the goal is to connect the diagrams so that they commute. However, we do not have that $\pi_1; g = \pi_2; f$; they may not agree on all values. But each composition is itself an arrow and we know by assumption that every pair of arrows has an equalizer. Therefore we have the equalizer e making $e; \pi_1; g = e; \pi_2; g$ true. Moreover this is our pullback. For any other arrow from an object X to the domains of f and g, we have a unique arrow connecting X to the domain of e. This is the pullback condition.

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
 & P \arrow[r, "e; \pi_2"'] \arrow[d, "e; \pi_1"] & B \arrow[d, "f"] \\
 & C \arrow[r, "g"] & D
\end{tikzcd}$$

1.8.7.4 A pushout of arrows $f : A \to B$ and $g : A \to C$ is a pair of arrows $f' : C \to D$ and $g' : B \to D$ with a common codomain such that for any other pair of arrows from the domains of f and g to a common codomain X there is a unique arrow $k : D \to X$ making the following diagram commute.

$$\begin{tikzcd}
X & & \\
& D \arrow[ul, dashed, "k"] & C \arrow[l, "f'"'] \arrow[ull] \\
& B \arrow[u, "g'"] \arrow[uul] & A \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

This is just a pullback with all arrows reversed. Also, in the same way Pierce showed products as pullbacks we can show coproducts as pushouts.

$$\begin{tikzcd}
X & & \\
& B+C \arrow[ul, dashed] & C \arrow[l] \arrow[ull] \\
& B \arrow[u] \arrow[uul] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

In the category **Set**, the arrow from B to $B+C$ should be the left injection ι_1 and likewise the arrow from C to $B+C$ should be ι_2. These arrows must satisfy the condition that any other pair of arrows from B and C to a codomain X uniquely determine an arrow $k : B+C \to X$. Provided ι_1 and ι_2 are coequalizers of f and g, this condition holds.

$$\begin{tikzcd}
B+C \arrow[d, dashed] & B \arrow[l, "\iota_1"'] \arrow[dl] & A \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"] \\
X & &
\end{tikzcd}$$

1.9.10.1 The goal is to show that the limit of the two object diagram with arrows $f : A \to B$ and $g : A \to B$ is an equalizer of f and g.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

To begin, a cone for this diagram consists of an object X and arrows $a : X \to A$ and $b : X \to B$ such that the diagram commutes.

$$\begin{tikzcd} X \arrow[d, "a"'] \arrow[dr, "b"] & \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

However, b is uniquely determined by the commutativity of the diagram to be the common composition $f \circ a = g \circ a$. (This is the same argument Pierce uses in his example of pullbacks as the limit of L-shaped diagrams.)

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "a"'] \arrow[dr, "b"] & A \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "a"] \arrow[d, "a"'] & A \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

A limit of these diagrams has the property that any other X' with an arrow a' making the diagram commute has a unique arrow $k : X' \to X$. This gives us a commutative diagram identical to the equalizer diagram. Therefore our limit is an equalizer of f and g.

$$\begin{tikzcd} X \arrow[r, "a"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ X' \arrow[u, dashed, "k"] \arrow[ur, "a'"'] & & \end{tikzcd}$$

1.9.10.2 The limit is the infinium $i \in \mathcal{C}$ of the elements $D_i \in \mathcal{D}$.

The colimit is the suprenam in $\mathcal{C}$ of the same elements, $D_i \in \mathcal{D}$.

1.9.10.3 We can construct limits of arbitrary diagrams D in $\mathbf{Set}$ or $\mathbf{Poset}$ by taking as limit the product object $\prod D_i$ of all objects in the diagram. Firstly, the object $\prod D_i$ exists because $\mathbf{Set}$ and $\mathbf{Poset}$ are categories with products. Commutativity of all mediating triangles $f_{ij} \circ \pi_i = \pi_j$ follows because all arrows f_{ij} in $\mathbf{Set}$ and $\mathbf{Poset}$ are total maps. Uniqueness of the limit $\prod D_i$ comes from the universal property of product objects.

$$
\begin{tikzcd}
 & \prod D_i \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
D_i \arrow[rr, "f_{ij}"] & & D_j
\end{tikzcd}
$$

1.9.10.4 Dually, colimits in $\mathbf{Set}$ and $\mathbf{Poset}$ are the coproduct of all objects in the diagram. Again, the coproduct always exists because $\mathbf{Set}$ and $\mathbf{Poset}$ are categories with products, the mediating triangles commute because the arrows in these categories are total maps, and uniqueness follows from the universal property of coproducts.

Going further, we can dualize the limit theorem by using coproducts and coequalizers. Construct the coproduct $\sum D_i$ of all objects in the diagram and the coproduct $\sum e$ of all edge destinations. The colimit is the codomain of the coequalizer of the unique arrows from $\sum e$ to $\sum D_i$.

1.10.5.1 An example of a small, finite category with binary products and a terminal object but no exponentials is:

$$\begin{tikzcd} A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & B \arrow[r, dash, "g"] & C \end{tikzcd}$$

This category is clearly small and finite; the hom-set of each pair of objects is a set and there are a finite number of objects. We require that C is a terminal object by separating $f_1 \neq f_2$ and identifying $g \circ f_1 = g \circ f_2$ (i.e., g is not a monomorphism). The product of any two objects is their lower bound, just like in a poset.

$$\begin{array}{rcl} A \times B & = & A \\ A \times C & = & A \\ B \times C & = & B \end{array}$$

Now for the exponentials. We show that the object B^A does not exist.

By contradiction, first assume that $B^A = A$, hence there exists an arrow $i : A \times B^A \to B$. Then for all objects X such that $A \times X$ has an arrow j to B, there must exist a unique arrow $\langle id_A, j^* \rangle$ such that $j = i \circ \langle id_A, j^* \rangle$. One choice of X is C, because $j = f_1 \circ \pi_1 : A \times C \to B$, however there is no corresponding arrow $j^* : C \to A$. So B^A cannot be A. A similar argument shows that B^A cannot be B: there is no arrow j^* from $X = C$ to B.

We are left with $B^A = C$. There are two choices for $i : A \times C \to B$. We can use $f_1 \circ \pi_1$ or $f_2 \circ \pi_1$. But no matter which we choose, the other will be an arrow from $A \times X \to B$ that we cannot build a commutative diagram for because $f_1 \neq f_2$. These counterexamples, shown below, demonstrate that $B^A \neq C$, which completes the proof of non-existence.

$$\begin{tikzcd} A \times C \arrow[r, "f_1 \circ \pi_1"] & B \\ A \times X \arrow[u, dashed, "\langle id_A, ? \rangle"] \arrow[ur, "f_2 \circ \pi_1"'] & \end{tikzcd} \qquad \begin{tikzcd} A \times C \arrow[r, "f_2 \circ \pi_1"] & B \\ A \times X \arrow[u, dashed, "\langle id_A, ? \rangle"] \arrow[ur, "f_1 \circ \pi_1"'] & \end{tikzcd}$$

1.10.5.2 Exponentation in $\mathbf{Set} \times \mathbf{Set}$ is componentwise. That is, every pair of $\mathbf{Set} \times \mathbf{Set}$ objects determines an exponential.

Let $A \times B$ and $C \times D$ be two objects in $\mathbf{Set} \times \mathbf{Set}$. The exponential of this pair is an object $C^A \times D^B$. The function *eval* has domain $(C^A \times D^B) \times (A \times B)$ and codomain $C \times D$. Application is componentwise. The result of *eval*$((f, g), (a, b))$ is the pair (fa, gb).

Then for every arrow $g : E \times (A \times B) \to C \times D$, we have a unique arrow *curry*(g) that maps the $\mathbf{Set} \times \mathbf{Set}$ object E to the exponential $C^A \times D^B$. This arrow is uniquely determined comoponentwise; because E is itself a pair of $\mathbf{Set}$ objects, each with an exponential, we obtain the $csetset$ object by taking the exponential of the first and second member of the pair.

The entire construction is summarized in the below diagram. Parenthesis are inserted for readability.

$$\begin{tikzcd}
(C^A \times D^B) \times (A \times B) \arrow[r, "eval"] & C \times D \\
(X \times Y) \times (A \times B) \arrow[u, dashed, "curry(g)"] \arrow[ur, "g"'] &
\end{tikzcd}$$

1.10.5.3 An exponential object in $\mathbf{Set}^{\rightarrow}$ is a collection of arrows between two $\mathbf{Set}^{\rightarrow}$ objects.

An object in $\mathbf{Set}^{\rightarrow}$ is an arrow $f : A \rightarrow B$ between two sets. An arrow between objects $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ in $\mathbf{Set}^{\rightarrow}$ is a pair of arrows (a, b) where $a : A \rightarrow A'$ and $b : B \rightarrow B'$ are such that $f' \circ a = b \circ f$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "b"] & B' \end{tikzcd}$$

Hence an exponential object f'^f is the collection of all such $\mathbf{Set}^{\rightarrow}$ arrows (a, b). Filling in the canonical diagram for exponentials, we have the following, where $f'' : A'' \rightarrow B''$ is any other object in $\mathbf{Set}^{\rightarrow}$.

$$\begin{tikzcd} f'^f \times f \arrow[r, "eval_{ff'}"] & f' \\ f'' \times f \arrow[u, dashed, "curry(g) \times id_f"] \arrow[ur, "g"'] & \end{tikzcd}$$

Here *curry*(g) is the uniquely determined function that brings a $\mathbf{Set}^{\rightarrow}$ object f'' to one of the arrows $f \rightarrow f'$ in the exponential f'^f.

1.10.5.4 To show that *curry*($eval_{AB}$) $= id_{(B^A)}$, we first consider what we know about $eval_{AB} : B^A \to B$.

$$B^A \times A \xrightarrow{eval_{AB}} B$$

Assuming B^A is an exponential object and that $eval_{AB}$ is the arrow associated with this exponential, we know that for any other object C and arrow $g : C \times A \to B$ we have a unique mediating arrow $curry(g) : C \to B^A$ such that the following diagram commutes.

$$\begin{tikzcd}
B^A \times A \arrow[r, "eval_{AB}"] & B \\
C \times A \arrow[u, dashed, "curry(g) \times id_A"] \arrow[ur, "g"'] &
\end{tikzcd}$$

One such possible g is $eval_{AB}$. It takes an object $C = B^A$ to B. Therefore, there must be a unique arrow $curry(eval_{AB}) : B^A \to B^A$ that makes the diagram commute.

$$\begin{tikzcd}
B^A \times A \arrow[r, "eval_{AB}"] & B \\
B^A \times A \arrow[u, dashed, "curry(eval_{AB}) \times id_A"] \arrow[ur, "eval_{AB}"'] &
\end{tikzcd}$$

However, id_{B^A} is another unique arrow that also makes this diagram commute. This is more clearly shown if we collapse identical nodes in the diagram.

$$\begin{tikzcd}
B^A \times A \arrow[loop, dashed, "id_{B^A} \times id_A"'] \arrow[r, "eval_{AB}"] & B
\end{tikzcd}$$

Therefore $curry(eval_{AB}) = id_{B^A}$.

1.10.5.5 The goal is to show that $(B^A)^{A'}$ is isomorphic to $B^{A \times A'}$ in any cartesian-closed category.

One direction of this argument is fairly straightforward. The exponential $B^{A \times A'}$ defines a function $\mathit{eval}_1 : B^{A \times A'} \times (A \times A') \to B$, which in turn uniquely determines a function $\mathit{curry}(g)$ for any function $g : C \times (A \times A') \to B$. Taking C to be $(B^A)^{A'}$, we can define $g : (B^A)^{A'} \to (A \times A') \to B$ as **fun** $f\ (a, a') = f\ a'\ a$.

$$\begin{tikzcd}
B^{A \times A'} \times (A \times A') \arrow[r, "\mathit{eval}_1"] & B \\
(B^A)^{A'} \times (A \times A') \arrow[u, dashed, "\mathit{curry}(g) \times id_{A \times A'}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Now the uniquely determined arrow $\mathit{curry}(g) : (B^A)^{A'} \to B^{A \times A'}$ is half our isomorphism.

For the reverse, we need to unroll the definition of $(B^A)^{A'}$. As is, it defines an evaluation function that accepts one function $f : A' \to A$ to return a B. Rather than taking a function on objects, we want to take just objects.

Let $\mathit{eval}_2 : (B^A)^{A'} \times A' \to B^A$ and $\mathit{eval}_3 : B^A \times A \to B$ in the obvious way. Then with a slight abuse of notation we can use their composition to obtain a B from a pair $A' \times A$. This composition $\mathit{eval}_3 \circ \mathit{eval}_2$ then uniquely determines a function $\mathit{curry}(g') : C \to B^{A \times A'}$ for each function $g' : C \times (A' \times A) \to B$. Let C be $B^{A \times A'}$ and let g' permute its arguments then apply eval_1 from above. Now $\mathit{curry}(g')$ is the unique arrow taking $B^{A \times A'}$ to $(B^A)^{A'}$, thus completing the isomorphism.

$$\begin{tikzcd}
(B^A)^{A'} \times (A' \times A) \arrow[r, "\mathit{eval}_3 \circ \mathit{eval}_2"] & B \\
B^{A \times A'} \times (A' \times A) \arrow[u, dashed, "\mathit{curry}(g') \times id_{A' \times A}"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

1.10.5.6 We show the category defined by the powerset of a set S and the subset relation $\subseteq$ is cartesian closed by showing it has binary products and exponentials.

The binary product of two elements A, B of the powerset $\mathcal{P}(S)$ is the intersection of the elements. This ensures that the product object is a subset of both A and of B. Moreover, it induces a unique arrow from any other object X that is a subset of both A and B because $A \cap B$ is the largest subset of both.

Two examples are below. One has an empty intersection and the other a non-empty intersection.

$$\{1\} \xleftarrow{\subseteq} \{\} \xrightarrow{\subseteq} \{2\} \quad \{1,2\} \xleftarrow{id} \{1,2\} \xrightarrow{\subseteq} \{1,2,3\}$$

The exponent of two elements A, B of $\mathcal{P}S$ can be encoded as B if $B \subseteq A$, otherwise it must be S. I suspect there is a cleaner way to convey this, but the obvious formula $\neg A \cup B$ and similar variants fail.

If $B \subseteq A$, then $B^A \subseteq B$ must hold. We choose $B^A = B$ because it is the largest set for which the equality is true. This, combined with the fact that our category is thin, gives B^A the desired universal property.

However if $A \subseteq B$, then for any object C, the unique arrow $f : A \to B$ composed with π_2 gives a map from the product $C \times A$ to B. Therefore C can be any object in the set and therefore B must be the set S for the universal property to hold.

1.10.5.7 If we consider S, the set of sentences of propositional logic, as a preorder $(S, \leq)$ where $p \leq q$ means "from p we can derive q", then we can show that the preorder forms a cartesian closed category. Objects are elements of S; that is, sentences of propositional logic. An arrow between two objects p and q exists if and only if $p \leq q$. Composition then follows by transitivity of $\leq$ and identity arrows follow from the reflexivity of $\leq$.

Pierce defines the binary product $p \times q$ as the conjunction $p \wedge q$, which makes sense because given $p \wedge q$ we can derive either p or q. Likewise for a product object we can extract either p or q using the right projection arrow. The universality property is satisfied by observing that if sentence x lets us derive both p and q then it will allow us to derive $p \wedge q$ in propositional logic. Therefore $x \leq p \wedge q$ and we have the desired arrow from x to $p \times q$.

For exponents, Pierce claims the sentence "p implies q" should correspond to the object q^p. To validate this claim, we first observe that *eval* corresponds to *modus ponens*. Hence $q^p \wedge p \leq q$ and we have an arrow from $q^p \times p \to q$. Then supposing there existed an arrow $g : r \times p \to q$ we could derive a unique arrow $\mathit{curry}(g) : r \to q^p$. In terms of propositional logic, assuming we have p the weakest precondition for deriving q is "q implies p". Therefore any other proposition r that we can replace for q^p and still derive the same conclusion must itself imply q^p. This implies that $r \leq q$, thus yielding our curry arrow.

1.10.5.8 We can extend **FPL** with higher-order functions by adding exponentials. This would make it possible to reason about all functions of a certain type, say *bool* $\to$ *int* in the domain or codomain of an arrow.

2.1.4 We want to show that the definitions *maplist*:

$$\mathit{maplist}(f)([]) = []$$
$$\mathit{maplist}(f)([x]) = [fx]$$
$$\mathit{maplist}(f)(L * L') = \mathit{maplist}(f)(L) * \mathit{maplist}(f)(L')$$

and *maplist′*

$$\mathit{maplist}'(f)([]) = []$$
$$\mathit{maplist}'(f)([x] * L) = [fx] * \mathit{maplist}'(f)(L)$$

are equivalent. This follows by the associativity of *. The second definition forces right associativity and the first allows any parenthesization.

2.1.10.1
- Example 2.1.5, the forgetful functor on monoids, satisfies the definition of a functor because it takes the identity homomorphism to the identity on the underlying set and the result of $F(f; g)$ for homomorphisms $f : (M, \cdot, e) \to (M', \cdot', e')$ and $g : (M', \cdot', e') \to (M'', \cdot'', e'')$ is $F(f); F(g)$ where $F(f) : M \to M'$ and $F(g) : M' \to M''$ are defined on the underlying sets. Thus identities and composition are preserved.
- Example 2.1.6, the identity functor, trivially preserves identities and composition.
- Example 2.1.7, the right product functor, takes the identity arrow for an object B to the product (id_B, id_A), which is the identity arrow for the resulting pair, and maps compositions to the composition of product arrows $(f, id_A); (g, id_A)$, which is well-defined in the product category.

2.1.10.2 We define the powerset operator $\mathcal{P} : S \to 2^S$, as an endofunctor $\mathcal{P}$ on the category $\mathbf{Set}$.

- For objects $A \in \mathbf{Set}$, $\mathcal{P}(A)$ is the powerset of A, $\mathcal{P}(A) \in \mathbf{Set}$.
- Identity arrows on sets A are lifted to identity arrows on the corresponding powerset, $\mathcal{P}(A)$. This the functor conserves identities.
- For morphisms $f : A \to B$, $\mathcal{P}(f) : \mathcal{P}(A) \to \mathcal{P}(B)$ is the result of applying f to all sets $a \in \mathcal{P}(A)$. Composition $g \circ f$ is then preserved because $f : A \to B$ maps to $\mathcal{P}(f) : \mathcal{P}(A) \to \mathcal{P}(B)$ and $g : B \to C$ maps to $\mathcal{P}(g) : \mathcal{P}(B) \to \mathcal{P}(C)$.

At first I wanted to define $\mathcal{P}(f)$ as extracting the highest-cardinality element A from the set $\mathcal{P}(A)$, applying $f(A)$, then taking $\mathcal{P}(f(A))$. But if A is the domain of f then we can take $f(A')$ for any $A' \subseteq A$, so it's easier this way.

2.1.10.3 The functors between monoids considered as one-object categories are the structure-preserving maps between the two monoids. These are monoid homomorphisms.

2.1.10.4 Pierce defines the diagonal functor $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ on a category $\mathbf{C}$ as taking each $\mathbf{C}$-object A to the object $(A,\ A)$.

The corresponding action of Δ on arrows takes morphisms f to the pair of morphism $(f,\ f)$. Identities are preserved because $(\mathbf{id}_A,\ \mathbf{id}_A)$ is the identity arrow for the object $(A,\ A) \in \mathbf{C} \times \mathbf{C}$ and composition is similarly the composition of arrows applied pointwise in $\mathbf{C} \times \mathbf{C}$.

2.1.12 In the contravariant hom-functor for an object $B \in \mathbf{C}$, called $\mathbf{C}(-, B)$:

- Each object $A \in \mathbf{C}$ maps to $\mathbf{C}(A, B)$, the set of all arrows from A to B.
- Each arrow $f : A \to C$ maps to the function $\mathbf{C}(f, B) : \mathbf{C}(C, B) \to \mathbf{C}(A, B)$ defined for a function $g : C \to B$ as $\mathbf{C}(f, B)(g) = g \circ f$.

This hom-functor combined with the covariant one combine to form a bifunctor where:

- Each pair of objects A, B map to $\mathbf{C}(A, B)$, the set of all morphisms from A to B.
- Each pair of arrows $f : C \to A,\ g : B \to D$ define a functor $\mathbf{C}(f, g) : \mathbf{C}(A, B) \to \mathbf{C}(C, D)$
- An object, arrow pair or an arrow, object pair behave like the covariant and contravariant hom-functors, respectively.

The bifunctor arises because the following diagram commutes for functions $f : C \to A$ and $g : B \to D$.

$$\begin{tikzcd}
\mathbf{C}(A, B) \arrow[r, "\mathbf{C}(f,\ B)"] \arrow[d, "\mathbf{C}(A,\ g)"] & \mathbf{C}(C, B) \arrow[d, "\mathbf{C}(C,\ g)"] \\
\mathbf{C}(A, D) \arrow[r, "\mathbf{C}(f,\ D)"'] & \mathbf{C}(C, D)
\end{tikzcd}$$

2.2.1 We want to show that the definition of a homomorphism in an Ω-algebra is equivalent to the definition of a homomorphism in an F-algebra with signature Ω.

First, let $\mathbf{ar}(\omega)$ denote the arity of each operator symbol $\omega \in \Omega$. Additionally let a_ω be the interpretation of the operator symbol a in the Ω-algebra A. A homomorphism between Ω-algebras A and B is a function $h : |A| \to |B|$ such that for each $\omega \in \Omega$ and $(x_1, \ldots, x_{\mathbf{ar}(\omega)}) \in |A|^{\mathbf{ar}(\omega)}$,

$$h(a_\omega\ (x_1, \ldots, x_{\mathbf{ar}(\omega)})) = b_\omega\ (h(x_1), \ldots, h(x_{\mathbf{ar}(\omega)})) \tag{1}$$

Next, an F-homomorphism is defined as a function $h : |A| \to |B|$ making the following diagram commute.

$$\begin{tikzcd}
A \arrow[r, "h"] & B \\
F\,A \arrow[u, "a"] \arrow[r, "F\,h"'] & F\,B \arrow[u, "b"']
\end{tikzcd}$$

To prove equivalence, we first show that if h is an Ω-homomorphism then the diagram must commute. This follows immediately from the definition of an Ω-homomorphism; the composition $h \circ a$ applied to the object $(\omega, (x_1, \ldots, x_{\mathbf{ar}(\omega)})) \in F\,A$ is exactly $h(a_\omega, (x_1, \ldots, x_{\mathbf{ar}(\omega)}))$. We know by definition that this is equivalent to $b_\omega\ (h(x_1), \ldots, h(x_{\mathbf{ar}(\omega)}))$, which is exactly the composition $b \circ F\,h$ applied to the object $(\omega, (x_1, \ldots, x_{\mathbf{ar}(\omega)})) \in F\,B$.

Conversely, if the diagram commutes then h must be an Ω-homomorphism because the compositions $h \circ a$ and $b \circ F\,h$ are equal. Thus the equation (1) proving that h is an Ω-homomorphism holds.

2.2.4.1 If $\mathbf{K}$ is a category and $F : \mathbf{K} \to \mathbf{K}$ is an endofunctor on $\mathbf{K}$ then the F-algebras and F-homomorphisms over $\mathbf{K}$ form a category **F-Alg** as follows:

- Objects are F-algebras.
- Morphisms are F-homomorphisms.
- Identities are lifted from $\mathbf{K}$.

  If $id_A : A \to A$ is the identity on an object $A \in \mathbf{K}$ then we get a morphism $F\, id_A : F\, A \to F\, A$ because id_A is trivially a structure-preserving map (homomorphism) between objects in $\mathbf{K}$ and therefore F must have an arrow $F\, id_A$ making the analog of the above diagram commute. It follows that $F\, id_A = id_{F\, A}$ by definition of the functor F.
- Composition is similarly lifted from $\mathbf{K}$.

  If there exist homomorphisms $h_{AB} : A \to B$ and $h_{BC} : B \to C$ then the composition $h_{BC} \circ h_{AB}$ must exist in the category $\mathbf{K}$. This composition is also trivially a homomorphism h_{AC} from A to B and therefore the arrow $F\, h_{AC} = F\, h_{BC} \circ F\, h_{AB}$ must exist in the category **F-Alg**. Again, by definition of the functor F we have that $F\, hBC \circ F\, h_{AB} = F\, (h_{BC} \circ h_{AB})$ and we're done.

2.2.4.2 The goal is to show that if an F-algebra (A, a) is initial in the category **F-Alg** then a is an isomorphism.

Consider the $\mathbf{K}$-object $F\,A$. We know it exists because $F : \mathbf{K} \to \mathbf{K}$ must be defined on all objects $A \in \mathbf{K}$. Furthermore, we know it must be the carrier of some object $F\,F\,A$ because the pair $(F\,A, F\,a)$ form an F-algebra. Moreover, the initiality of $F\,A$ uniquely determines an arrow $F\,h : F\,A \to F\,F\,A$. Thus the h in $F\,h$ must be an arrow $h : A \to F\,A$.

Pictorally, we have the following diagram where $F\,h$ is determined by the initiality of $F\,A$ and h follows as a consequence:

$$\begin{tikzcd}
A \arrow[r, dashed, "h"] & F\,A \\
F\,A \arrow[u, "a"] \arrow[r, dashed, "F\,h"'] & F\,F\,A \arrow[u, "F\,a"']
\end{tikzcd}$$

What remains is to show that $h; a = id_A$ and that $a; h = id_{FA}$. We extend the above diagram with more applications of $F\,a$ and a to help reach this conclusion.

$$\begin{tikzcd}
A \arrow[r, dashed, "h"] & F\,A \arrow[r, "a"] & A \\
F\,A \arrow[u, "a"'] \arrow[r, dashed, "F\,h"'] & F\,F\,A \arrow[u, "F\,a"'] \arrow[r, "F\,a"'] & F\,A \arrow[u, "a"']
\end{tikzcd}$$

Immediately, we have $h; a = id_A$ by initiality. To show $a; h = id_{FA}$, we first note that the left square gives $a; h = F\,h; F\,a$. Because F is a functor, we may rewrite the right side to obtain $F\,h; F\,a = F\,(h; a)$. However, this further rewrites to $F\,id_A$, and by definition of a functor $F\,id_A = id_{FA}$, thus completing the proof.

Lastly, this initial object (A, a) corresponds to a fixed point of F because there is an isomorphism between $F\,A$ and A and similarly between $F\,a$ and a.

2.3.11.1 The diagram from Pierce's example 2.3.4 argued that $\mathit{eval} : F_A \dot{\to} I_{\mathbf{Set}}$ was a natural transformation. Proof follows if the following diagram commutes for a fixed set A and every arrow $g : C \to B$.

$$\begin{tikzcd}
F_A(C) = C^A \times A \arrow[r, "\mathit{eval}_{AC}"] \arrow[d, "F_A(g) = (g \circ -) \times \mathit{id}_A"'] & C = I_{\mathbf{Set}}(C) \arrow[d, "g = I_{\mathbf{Set}}(g)"] \\
F_A(B) = B^A \times A \arrow[r, "\mathit{eval}_{AB}"] & B = I_{\mathbf{Set}}(B)
\end{tikzcd}$$

We need to show that $g \circ \mathit{eval}_{AC} = \mathit{eval}_{AB} \circ (g \circ -) \times \mathit{id}_A$, and we can prove this by using the universal property of eval_{AB}.

The exponential object B^A and arrow eval_{AB} guarantee that for every arrow $h : C^A \times A \to B$, we can derive an arrow $C^A \times A \to B^A \times A$ by currying h. Take $h = g \circ \mathit{eval}_{AC} : C^A \times A \to B$. We see that the following diagram commutes.

$$\begin{tikzcd}
B^A \times A \arrow[r, "\mathit{eval}_{AB}"] & B \\
C^A \times A \arrow[u, dashed, "\mathit{curry}(g \circ \mathit{eval}_{AC}) \times \mathit{id}_A"] \arrow[ur, "g \circ \mathit{eval}_{AB}"'] &
\end{tikzcd}$$

Filling in the blank above in the arrow $F_A(g) = (g \circ -) \times \mathit{id}_A$, we see that $(g \circ -) = (g \circ \mathit{eval}_{AC})$. Therefore the diagram from example 2.3.4 commutes, proving that $\mathit{eval} : F_A \dot{\to} I_{\mathbf{Set}}$ is a natural transformation.

2.3.11.2 Let $\mathbf{P}$ be a preorder regarded as a category, let $\mathbf{C}$ be an arbitrary category, and let $S, T : \mathbf{C} \to \mathbf{P}$ be functors. We show that there is a unique natural transformation $\tau : S \dot{\to} T$ if and only if for all $\mathbf{C}$ objects C, the relationship $S(C) \leq T(C)$ holds.

First assume that τ is a unique natural transformation. This implies that for any $\mathbf{C}$ arrow $f : A \to B$, there exist arrows $\tau_A : S(A) \to T(A)$ and $\tau_B : S(B) \to T(B)$. Let f stand for an arbitrary identity arrow $id_C : C \to C$ in $\mathbf{C}$. We see that there exists $\tau_C : S(C) \to T(C)$. This is an arrow in $\mathbf{P}$, thus the inequality $S(C) \leq T(C)$ holds for arbitrary objects C in $\mathbf{C}$.

Next assume that for every $\mathbf{C}$-object C, we have that $S(C) \leq T(C)$. This inequality implies that there exists a $\mathbf{P}$-arrow $f : S(C) \to T(C)$. Now consider a $\mathbf{C}$-arrow $f : A \to B$. We know that there exist arrows $S(A) \to T(A)$ and $S(B) \to T(B)$ in $\mathbf{P}$. Furthermore, these arrows are uniquely determined by f. Hence τ in the following diagram is a unique natural transformation between S and T.

$$\begin{array}{ccc} S(A) & \xrightarrow{\tau_A} & T(A) \\ {\scriptstyle S(f)}\big\downarrow & & \big\downarrow{\scriptstyle T(f)} \\ S(B) & \xrightarrow{\tau_B} & T(B) \end{array}$$

2.3.11.3 The identity functor $I_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ is represented by any singleton set because a singleton set is an initial object in the category $\mathbf{Set}$. Let R stand for an arbitrary singleton set and consider the functors $I_{\mathbf{Set}}$ and $\mathbf{hom}(R, -)$, the identity functor on $\mathbf{Set}$ and the hom-functor defined by the singleton R. These functors define the following arrows for each $\mathbf{Set}$-arrow $f : A \to B$.

$$\begin{array}{ccc} I_{\mathbf{Set}}(A) & \qquad & \mathbf{hom}(R, A) \\ {\scriptstyle I_{\mathbf{Set}}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{hom}(R,f) : \mathbf{hom}(R,A) \to \mathbf{hom}(R,B)} \\ I_{\mathbf{Set}}(B) & & \mathbf{hom}(R, B) \end{array}$$

Because R is a singleton, there exists one unique arrow in each hom-set $\hom(R, A)$. Furthermore, there is one hom-set for each $\mathbf{Set}$-object A, therefore we have an isomorphism between $\mathbf{Set}$ objects and the unique arrows to these objects from the singleton R. This natural isomorphism proves that $I_{\mathbf{Set}}$ is represented by the singleton set R.

2.3.11.4 Not sure exactly how the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ is represented by the monoid $R = (\mathbb{N}, +, 0)$, but it's important that R is a free monoid.

A free monoid (in abstract algebra) is a monoid whose elements are strings. Multiplication is string concatenation and the identity is the empty string. The monoid $(\mathbb{N}, +, 0)$ fits this description if we consider numbers as iterated applications of the successor function. Then the number 1 is equally $1 + 0 = 0 + 1 = 1$, because 0 is the additive identity, and 4 would be represented as $1 + 1 + 1 + 1$. Thus we can view a number as a string, $+$ as string concatenation, and 0 as the empty string.

Because R is free, we know there exists a unique arrow from it to any other monoid in $\mathbf{Mon}$. So any monoids A, B are uniquely represented by their image in the hom-functor $\mathbf{hom}(R, -)$.

There should be an isomorphism between the underlying set of A to the arrow from R to A, but I'm not sure what it is.

2.3.11.5 To extend $\mathbf{FPL}$ with lists and polymorphic functions on lists, we would need to create an endofunctor $F : \mathbf{FPL} \to \mathbf{FPL}$ that constructed lists from $\mathbf{FPL}$ objects. Then the polymorphic functions would be natural transformations between F and F.

2.4.5 **Unit:** The unit diagram operates on the distinguished object $*$ of the category $\mathbf{1}$ and arbitrary objects y of the category $\mathbf{C}$. Like Pierce said, the functor $F : \mathbf{1} \to \mathbf{C}$ is the left adjoint of the constant functor $T : \mathbf{C} \to \mathbf{1}$. Additionally the unit $\iota : \mathbf{id_1} \to \mathbf{id_1}$ is fixed.

$$\begin{tikzcd} * \arrow[r, "\iota"] \arrow[dr, "f"'] & T(F(*)) \arrow[d, "T\ f^{\#}"] \\ & T(y) \end{tikzcd}$$

For each object of $\mathbf{1}$, each object of $\mathbf{C}$, and each $\mathbf{1}$-arrow $f : X \to T(y)$ there is a unique $\mathbf{C}$ arrow $f^{\#} : F(*) \to y$. Because there is only one object in the category $\mathbf{1}$, we have that for each object in $\mathbf{C}$ there is a unique arrow $f^{\#}$ taking the image of $*$ to this object y. Thus the image of $*$ must be an initial object.

Existence and uniqueness follow from the existence and uniqueness of $f = \mathbf{id}_*$ in the category $\mathbf{1}$.

Co-unit: Similarly, the counit diagram works in the opposite direction.

$$\begin{tikzcd} F\ T\ y \arrow[r, "\epsilon"] & y \\ F\ * \arrow[u, "F\ g^*"] \arrow[ur, "g"'] & \end{tikzcd}$$

We have that for each $y \in \mathbf{C}$ the counit ϵ picks out the unique arrow from the initial object $F\ *$ to y. Existence and uniqueness follow because $F*$ is an initial object in $\mathbf{C}$.

2.4.7 We have a category $\mathbf{C}$ with products, a product functor $\mathbf{\Pi} : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and a diagonal functor $\mathbf{\Delta} : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$. The diagram for the unit is:

$$\begin{tikzcd} C \arrow[r, "\langle id_C, id_C \rangle"] \arrow[dr, "\langle f_1, f_2 \rangle"'] & \langle C, C \rangle \arrow[d, "f_1 \times f_2"] \\ & \langle A, B \rangle \end{tikzcd}$$

The important thing to remember is that C, $\langle C, C \rangle$, and $\langle A, B \rangle$ are all objects in $\mathbf{C}$. The difference between the $\mathbf{C}$-arrow $\langle f_1, f_2 \rangle$ and the $\mathbf{C} \times \mathbf{C}$-arrow $f_1 \times f_2$ is that the latter is the image of the former under the product functor $\mathbf{\Pi}$. Existence and uniqueness follow by the existence and uniqueness of the arrow $\langle f_1, f_2 \rangle$.

The diagram for the counit is conversely between objects of $\mathbf{C} \times \mathbf{C}$. We represent these with parenthesis rather than angle braces, so $(C, C) \in \mathbf{C} \times \mathbf{C}$ and $\langle C, C \rangle \in \mathbf{C}$.

$$\begin{tikzcd} (\langle A, B \rangle, \langle A, B \rangle) \arrow[r, "\epsilon"] & (A, B) \\ (C, C) \arrow[u, "(\langle g_1, g_2 \rangle, \langle g_1, g_2 \rangle)"] \arrow[ur, "(g_1, g_2)"'] & \end{tikzcd}$$

The counit ϵ must be the pair of projections $(\pi_1, \pi_2) \in \mathbf{C} \times \mathbf{C}$. The existence and uniqueness of the arrow $(\langle g_1, g_2 \rangle, \langle g_1, g_2 \rangle)$ follows from the universal mapping property of the product $A \times B$. If C is such that we have arrows $g_1 : C \to A$ and $g_2 : C \to B$:

$$A \xleftarrow{g_1} C \xrightarrow{g_2} B$$

Then there exists a unique mediating arrow $\langle g_1, g_2 \rangle$ such that the following diagram commutes.

$$\begin{tikzcd} & C \arrow[dl, "g_1"'] \arrow[d, dashed, "\langle g_1, g_2 \rangle"] \arrow[dr, "g_2"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

2.4.12.1 Exercise **2.4.5** showed how an initial object in a category $\mathbf{C}$ arises as the image of the unique object $*$ of the category $\mathbf{1}$ under a left adjoint to the constant functor $T : \mathbf{C} \to \mathbf{1}$.

Dually, a final object 1 in a category $\mathbf{C}$ arises as the *preimage* of the unique object $*$ of the category $\mathbf{1}$ under a *right* adjoint to the constant functor T. The unit picks out the unique $\mathbf{C}$-arrow from any $\mathbf{C}$-object to the final object, 1, and the counit is fixed as $I_{\mathbf{1}} \to I_{\mathbf{1}}$.

The unit and counit diagrams are below. The functor $G : \mathbf{1} \to \mathbf{C}$ maps the object $*$ to the final object 1 in $\mathbf{C}$.

$$\begin{tikzcd} * \arrow[r, "\eta"] \arrow[dr, "f"'] & G(T(*)) \arrow[d, "G\ f^{\#}"] \\ & G(y) \end{tikzcd} \qquad \begin{tikzcd} T\ F\ y \arrow[r, "\iota"] & y \\ T\ * \arrow[u, "T\ g^*"] \arrow[ur, "g"'] & \end{tikzcd}$$

The existence and uniqueness of the arrow $f^{\#}$ follows because 1 is a final object in $\mathbf{C}$. Hence there exists a unique arrow for each $C \in \mathbf{C}$ mapping C to 1.

The existence and uniqueness of the arrow g^* follows by the existence and uniqueness of the identity arrow $\mathbf{id}_*$ in the category $\mathbf{1}$.

2.4.12.2 The categorical coproduct is the left adjoint to the diagonal functor $\mathbf{\Delta}$. The unit of the adjunction carries a $\mathbf{C}$-object to its product and the counit of the adjuction collapses the identity coproduct into a $\mathbf{C}$-object.

$$\begin{tikzcd} C \arrow[r, "\langle \mathbf{id}_C, \mathbf{id}_C \rangle"] \arrow[dr, "\langle f, g \rangle"'] & C \times C \arrow[d, "f \times g"] \\ & A \times B \end{tikzcd} \qquad \begin{tikzcd} Y + Y \arrow[r, "{[\mathbf{id}_Y, \mathbf{id}_Y]}"] & Y \\ X + X \arrow[u, "{[\langle f, f \rangle, \langle f, f \rangle]}"] \arrow[ur, "{[g, g]}"'] & \end{tikzcd}$$

2.4.12.3 Example **2.4.8** showed that $eval_{AB}$ was the counit of the adjunction between the functors $F = (- \times A)$ and $G = (-)^A$. The counit diagram Pierce gave was the diagram for an exponential object B^A.

The unit of this adjunction is the class of arrows that take a fixed **C**-object C to the pair of a **C**-object A and **C**-arrows from A to C. The mapping from f to $f^{\#}$ is the process of uncurrying a function $f : C \to A \to B$ into a function $f^{\#} : C \times A \to B$.

$$
\begin{tikzcd}
C \arrow[r, "\eta"] \arrow[dr, "f"'] & C^A \times A \arrow[d, "(f^{\#})^A \times A"] \\
& B^A \times A
\end{tikzcd}
$$

2.4.12.4 Just as the ceiling functor $\lceil\rceil : \mathbb{Z} \to \mathbb{R}$ is the left adjoint of the inclusion functor $U : \mathbb{Z} \to \mathbb{R}$, we can show that the floor functor $\lfloor\rfloor : \mathbb{R} \to \mathbb{Z}$ is the right adjoint to the inclusion U. The unit of the adjunction represents the fact that for all integers N, the inequality $N \leq \lfloor U(N) \rfloor$ holds.

$$
\begin{tikzcd}
N \arrow[r, "\eta"] \arrow[dr, "f"'] & \lfloor U(N) \rfloor \arrow[d, "\lfloor f^{\#} \rfloor"] \\
& \lfloor M \rfloor
\end{tikzcd}
$$

Here, the function f is any map between integers. The unique f^* arises from the definition of the floor function: namely that the floor of a real number r is the *greatest* integer less than r.

2.4.12.5 We first show how the unit and counit of an adjunction uniquely determine one another.

Assume first the we have $\eta : X \to G(F(X))$, the unit of the adjunction, and want to derive the counit ϵ. Diagramatically, the unit gives the following picture.

$$\begin{tikzcd} X \arrow[rr, "\eta"] \arrow[drr, "f"'] & & G(F(X)) \arrow[d, "G(f^{\#})"] \\ & & G(Y) \end{tikzcd}$$

The object X stands for any arbitrary object in the domain of G (or equally, the codomain of F). Fix X to be $G(Y)$. This yields a new picture where f is the identity $\mathbf{id}_{G(Y)}$.

$$\begin{tikzcd} G(Y) \arrow[rr, "\eta"] \arrow[drr, "f"'] & & G(F(G(Y))) \arrow[d, "G(f^{\#})"] \\ & & G(Y) \end{tikzcd}$$

The unit η uniquely determines an arrow $f^{\#} : F(G(Y)) \to Y$. This is the counit of the adjunction.

Conversely, if we assume we have the counit $\epsilon : F(G(Y)) \to Y$ then by fixing $Y = F(X)$ we uniquely determine $g*$ as the unit of the adjunction. In the corresponding diagram, g is the identity on $F(X)$ and g^* is uniquely determined by the counit.

If we look at the adjoint structure as a natural transformation between hom-functors, Pierce explains that we can represent the adjoint schematically as:

$$\frac{F(X) \to Y}{X \to G(Y)}$$

Filling in $Y = F(X)$ on the left and $X = G(Y)$ on the right, it's easier to see how the unit and counit are determined.

$$\frac{F(X) \to F(X)}{X \to G(F(X))}$$

$$\frac{F(G(Y)) \to Y}{G(Y) \to G(Y)}$$

The arrow from X to $G(F(X))$ is the unit and the arrow from $F(G(Y))$ to Y is the counit. Thanks to Professor Awodey's lectures from OPLSS for showing this.

Because the unit and counit uniquely determine one another, it follows that the functors F and G determine one another to within a natural isomorphism. If we know F is the left adjoint of some other functor, there is only one choice (up to isomorphism) of G that will satisfy the necessary requirements. Likewise if we know G and derive F.

3.4.7 The "category" **CPO** has ω-complete partial orders as objects and ω-continuous functions as arrows. We prove **CPO** is a category by showing the identity and associative laws hold.

assoc We show that for any two arrows $f : A \to B$ and $g : B \to C$ in **CPO**, their composition $g \circ f$ is also in **CPO**. First, f's monotonicity implies that for all $p_1 \sqsubseteq_A p_2$, we have $f(p_1) \sqsubseteq_B f(p_2)$. Now using g's monotonicity, we have that $g(f(p_1)) \sqsubseteq_C g(f(p_2))$, or rather, $(gcircf)(p_1) \sqsubseteq (g \circ f)(p_2)$. This shows that $g \circ f$ is monotone.

To see that $g \circ f$ is continuous, we observe:

$$\begin{aligned} (g \circ f) \sqcup p_n &= g(f(\sqcup p_n)) \\ \langle \textit{f continuous} \rangle &= g(\sqcup f(p_n)) \\ \langle \textit{g continuous} \rangle &= \sqcup g(f(p_n)) \\ &= \sqcup (g \circ f)(p_n) \end{aligned}$$

identity For any partial order $P \in$ **CPO**, define the function $id_P : P \to P$ as $id_P(p) = p$ for all elements p of the chain P. Clearly $p_1 \sqsubseteq p_2$ implies $id_P(p_1) \sqsubseteq id_P(p_2)$. Also $id_P \sqcup p_n = \sqcup p_n = \sqcup id_P(p_n)$, so id_P is continuous and therefore in **CPO**.

It follows by definition that id_P is a unit for composition.

3.4.9 Suppose the hom-sets of **CPO** are ordered pointwise, such that $f \sqsubseteq f' \iff \forall a \in A.\ f(a) \sqsubseteq f'(a)$. We have that:

- For every pair of **CPO** objects A and B, their hom-sets are equipped with a partial ordering. This partial ordering is ω-complete because $\sqsubseteq_A$ and $\sqsubseteq_B$ are ω-complete partial orderings.
- Assuming $f \sqsubseteq f'$ and $g \sqsubseteq g'$, we can derive $g \circ f \sqsubseteq g' \circ f'$ by proving $(g \circ f)(a) \sqsubseteq (g' \circ f')(a)$ for all a in the domain of f. We show this by unfolding the compositions to get $g(f(a)) \sqsubseteq g'(f'(a))$. Then since $f(a)$ and $f'(a)$ are in the domain of g the assumption $g \sqsubseteq g'$ finishes the proof.
- We can derive $\sqcup(g_n \circ f_n) = \sqcup g_n \circ \sqcup f_n$ for ω-sequences of functions $\{f_i\}$ and $\{g_i\}$ by chain continuity.

  First, $\sqcup g_n \circ \sqcup f_n \sqsubseteq \sqcup(g_n \circ f_n)$ holds if and only if for all a in the domain of $\sqcup f_n$, we can prove $(\sqcup g_n \circ \sqcup f_n)(a) \sqsubseteq (\sqcup(g_n \circ f_n))(a)$. By chain continuity, we can expand left hand side:

$$\begin{aligned} (\sqcup g_n \circ \sqcup f_n)(a) &= (\sqcup g_n)((\sqcup f_n)a) \\ &= g_n \sqcup (f_n(\sqcup a)) \\ &= g_n(f_n(\sqcup \sqcup a)) \\ &= g_n(f_n(\sqcup a)) \\ &= g_n \circ f_n \sqcup a \\ &= \sqcup(g_n \circ f_n)a \end{aligned}$$

Second, $\sqcup g_n \circ \sqcup f_n \sqsubseteq \sqcup(g_n \circ f_n)$ because $\sqcup(g_n \circ f_n)$ is an upper bound of any composition of functions in $\{g_i\}$ and $\{f_i\}$. In particular, $\sqcup g_n \in \{g_i\}$ and $\sqcup f_n \in \{f_i\}$ are two such functions, so their composition is bounded above by $\sqcup(g_n \circ f_n)$.

Therefore $\mathbf{CPO}$ is an $\mathbf{O}$ category when the hom-sets are ordered pointwise.

3.4.11 If f is an embedding, there must be an f^R such that $f^R \circ f = id_A$. This f^R must be unique because id_A is unique. Likewise, if we have a projection f^R then by definition there exists an embedding f, and uniqueness follows because id_A is unique.