1

The ellipsoidal system and its geometry

1.1 Confocal families of second-degree surfaces

In Cartesian coordinates, a point is specified by the intersection of three planes, that is, of three first-degree surfaces. In almost all other coordinate systems, a point is specified by a combination of first- and second-degree surfaces. The ellipsoidal coordinate system is characterized by the fact that it specifies a point by using solely non-degenerate second-degree surfaces. There are three non-degenerate second-degree surfaces, that of an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets, and these three surfaces are used to define the coordinate surfaces of the ellipsoidal system.

A second-degree surface in three-dimensional Euclidean space is the geometrical object defined by the general quadratic form

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_i x_j + \sum_{i=1}^{3} b_i x_i + c = 0, \tag{1.1}$$

or, in matrix form,

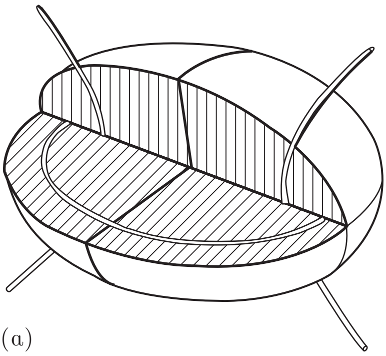
$$\boldsymbol{x}^\top \mathbb{A}\boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{x} + c = 0, \tag{1.2}$$

where $\mathbb{A}$ is a real symmetric 3×3 matrix, $\boldsymbol{b}$ is a real vector in $\mathbb{R}^3$, $\boldsymbol{x}^\top = (x_1, x_2, x_3)$, and the upper index $\top$ denotes transposition. Since $\mathbb{A}$ is real and symmetric, it has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and three orthogonal eigenvectors. The form (1.2) is non-degenerate if none of the eigenvalues is equal to zero, that is, if $\mathbb{A}$ is not singular. Diagonalizing $\mathbb{A}$, translating the origin, and normalizing the resulting constant term (given that it is not zero) to unity we end up with the canonical form

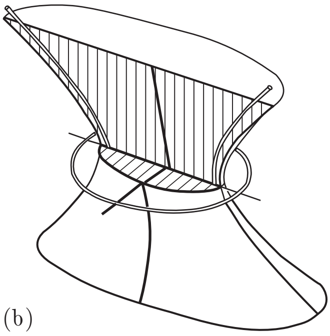
$$\mu_1 x_1^2 + \mu_2 x_2^2 + \mu_3 x_3^2 = 1. \tag{1.3}$$

The form (1.3) yields the following three generic cases, depicted in Figure 1.1:

(i) $\mu_1 > 0, \mu_2 > 0, \mu_3 > 0$, in which case (1.3) defines a triaxial ellipsoid;
(ii) $\mu_1 > 0, \mu_2 > 0, \mu_3 < 0$, in which case (1.3) defines a hyperboloid of one sheet;
(iii) $\mu_1 > 0, \mu_2 < 0, \mu_3 < 0$, in which case (1.3) defines a hyperboloid of two sheets.



(a)



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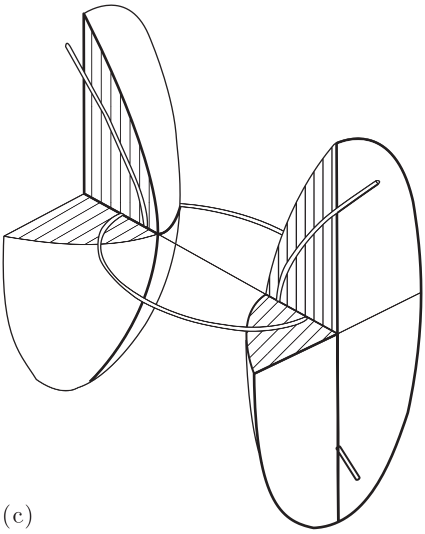

(c)

Figure 1.1 An ellipsoid (a), a hyperboloid of one sheet (b), and a hyperboloid of two sheets (c). From [179].

Every other case leads to a degenerate form. From now on, whenever we express a quadratic surface in a Cartesian system we will choose the system to be the one that reduces the representation of the surface to its canonical form (1.3). The axes and the planes of this system are called *principal axes* and *principal planes* of the surface, or of the relative quadratic form.

The spherical coordinate system, where all directions are equivalent, is completely specified by choosing a unit sphere. The orientation angles, being dimensionless, are uniquely specified. On the other hand, in the ellipsoidal system, where every direction has its own character, the specification of a coordinate system is based on a reference ellipsoid which establishes the variations in angular dependence.

The *reference ellipsoid* is given by the equation

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1, \quad 0 < a_3 < a_2 < a_1 < \infty, \tag{1.4}$$

where a_3, a_2, a_1 are three fixed parameters determining the *reference semi-axes*. The constants

$$h_1^2 = a_2^2 - a_3^2, \quad h_2^2 = a_1^2 - a_3^2, \quad h_3^2 = a_1^2 - a_2^2 \tag{1.5}$$

are the squares of the *semi-focal distances* of the system. The role of the reference ellipsoid corresponds to the role of the unit sphere in the case of the spherical system. Obviously, $h_1 < h_2$ and $h_3 < h_2$, but no order relation between h_1 and h_3 is implied. Since

$$h_1^2 - h_2^2 + h_3^2 = 0, \tag{1.6}$$

it follows that the ellipsoidal system is characterized by two independent semi-focal distances, which we take to be h_2 and h_3. The six foci of the ellipsoidal system are located at the points $(\pm h_2, 0, 0)$, $(\pm h_3, 0, 0)$, $(0, \pm h_1, 0)$ and they are fixed. For this reason the system is characterized as *confocal*. The ellipse

$$\frac{x_1^2}{h_2^2} + \frac{x_2^2}{h_1^2} = 1, \quad x_3 = 0 \tag{1.7}$$

has its foci at $(\pm h_3, 0, 0)$ and defines the *focal ellipse* of the system. Similarly, the hyperbola

$$\frac{x_1^2}{h_3^2} - \frac{x_3^2}{h_1^2} = 1, \quad x_2 = 0 \tag{1.8}$$

has its foci at $(\pm h_2, 0, 0)$ and defines the *focal hyperbola* of the system. Therefore, the focal hyperbola and the focal ellipse lie on planes that are perpendicular to each other. Furthermore, the focal ellipse passes through the foci of the focal hyperbola and the focal hyperbola passes through the foci of the focal ellipse. The focal ellipse and the

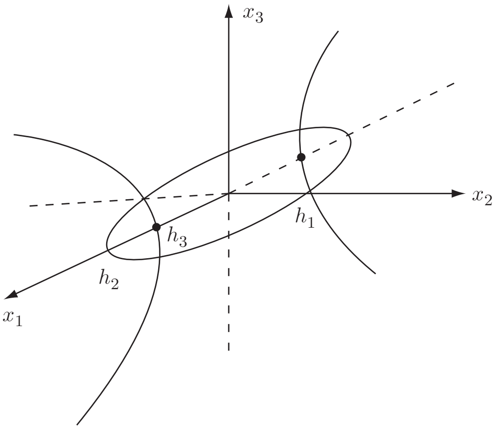

Figure 1.2 The focal ellipse (on the x_1x_2-plane) and the focal hyperbola (on the x_1x_3-plane).

focal hyperbola provide the backbone of the ellipsoidal coordinate system. Both the focal ellipse and the focal hyperbola are depicted in Figure 1.2. They correspond to the center of the spherical system.

The degeneracies $a_3 = a_2 < a_1$, $a_3 < a_2 = a_1$, and $a_3 = a_2 = a_1$ yield the prolate spheroidal system, the oblate spheroidal system, and the spherical system, respectively.

Let us now introduce the confocal family of second-degree surfaces

$$\frac{x_1^2}{a_1^2 - \lambda} + \frac{x_2^2}{a_2^2 - \lambda} + \frac{x_3^2}{a_3^2 - \lambda} = 1, \quad \lambda \in \mathbb{R}, \tag{1.9}$$

which represents for:

(i) $-\infty < \lambda < a_3^2$, a family of ellipsoids;
(ii) $\lambda = a_3^2$, the focal ellipse;
(iii) $a_3^2 < \lambda < a_2^2$, a family of hyperboloids of one sheet (1-hyperboloids);
(iv) $\lambda = a_2^2$, the focal hyperbola;
(v) $a_2^2 < \lambda < a_1^2$, a family of hyperboloids of two sheets (2-hyperboloids),

while for $\lambda \geqslant a_1^2$, it does not represent a real surface.

Proposition 1.1 *For every point (x_1, x_2, x_3), with $x_1x_2x_3 \neq 0$, the cubic polynomial in λ (1.9) has three real roots λ_1, λ_2, λ_3, which are ordered as follows:*

$$-\infty < \lambda_3 < a_3^2 < \lambda_2 < a_2^2 < \lambda_1 < a_1^2. \tag{1.10}$$

Proof Consider the function

$$f(\lambda) = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 - \lambda} - 1, \tag{1.11}$$

Cambridge University Press
978-0-521-11309-0 - Ellipsoidal Harmonics: Theory and Applications
George Dassios
Excerpt
More information

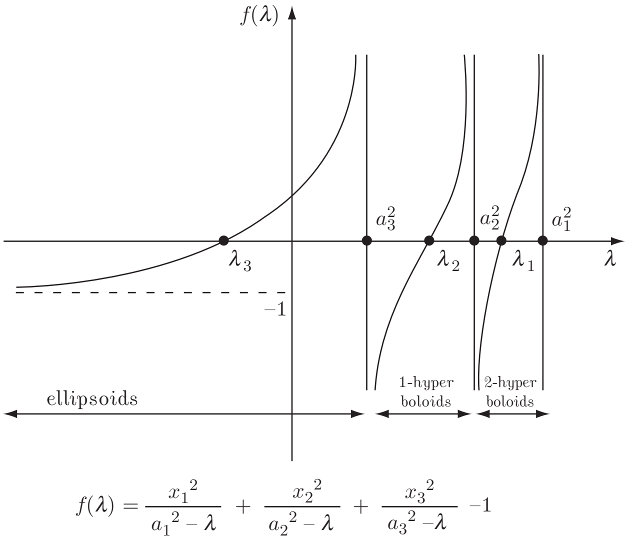

Figure 1.3 Graph of the function $f(\lambda)$. There is one such graph for every fixed vector $(a_1, a_2, a_3, x_1, x_2, x_3)$.

which is continuously differentiable in the domain $\mathrm{D} = (-\infty, a_3^2) \cup (a_3^2, a_2^2) \cup (a_2^2, a_1^2)$. For $\boldsymbol{x} \neq \mathbf{0}$, we obtain

$$\frac{d}{d\lambda} f(\lambda) = \sum_{i=1}^{3} \frac{x_i^2}{(a_i^2 - \lambda)^2} > 0, \tag{1.12}$$

and therefore the function f (see Figure 1.3) is strictly increasing in each one of the three intervals in D. Furthermore, we can easily show that

$$\lim_{\lambda \to -\infty} f(\lambda) = -1, \tag{1.13}$$

$$\lim_{\lambda \to a_3^2-} f(\lambda) = +\infty, \tag{1.14}$$

$$\lim_{\lambda \to a_3^2+} f(\lambda) = -\infty, \tag{1.15}$$

$$\lim_{\lambda \to a_2^2-} f(\lambda) = +\infty, \tag{1.16}$$

$$\lim_{\lambda \to a_2^2+} f(\lambda) = -\infty, \tag{1.17}$$

$$\lim_{\lambda \to a_1^2-} f(\lambda) = +\infty. \tag{1.18}$$

Consequently, the function f has exactly one root λ_3 in the interval $(-\infty, a_3^2)$, exactly one root λ_2 in the interval (a_3^2, a_2^2) and exactly one root λ_1 in the interval

(a_2^2, a_1^2). Since f is a cubic polynomial in λ, these three are the only roots it can have. □

Proposition 1.1 establishes an one-to-one correspondence between the sets of vectors

$$\mathbb{R}_0 = \{(x_1, x_2, x_3) | x_1 x_2 x_3 \neq 0\} \quad \text{and} \quad \mathbb{P} = (-\infty, a_3^2) \times (a_3^2, a_2^2) \times (a_2^2, a_1^2),$$

which allows to parametrize $\mathbb{R}_0$ by the vector $(\lambda_1, \lambda_2, \lambda_3)$. This parametrization provides the basis for the introduction of the ellipsoidal coordinate system. The non-generic points of the Cartesian planes, where $x_1 x_2 x_3 = 0$, will be discussed in Section 1.3.

In their famous book *Geometry and the Imagination* [179], Hilbert and Cohn-Vossen describe the ellipsoidal system in the following geometrical way. As λ_3 approaches from $-\infty$ we start with a large almost spherical surface of the family of ellipsoids, which gradually becomes a more and more pronounced ellipsoid as λ_3 tends to a_3^2 from the left. For $\lambda_3 = a_3^2$ the ellipsoid collapses down to the focal ellipse (1.7). This way, the whole space is swept out by the family of ellipsoids once. Then, we consider the complement of the focal ellipse on the $x_1 x_2$-plane, represented by $\lambda_2 = a_3^2$. As λ_2 increases from a_3^2 to a_2^2, the complement of the focal ellipse is gradually inflated and generates the family of 1-hyperboloids, which ultimately collapses down to the interior (the part of the plane extended between the two branches) of the focal hyperbola (1.8) on the $x_1 x_3$-plane, represented by $\lambda_2 = a_2^2$. Hence, the family of 1-hyperboloids also covers once the whole space. Finally, we consider the two exteriors (the two parts of the plane that are bounded by the two branches) of the focal hyperbola on the $x_1 x_3$-plane, represented by $\lambda_1 = a_2^2$. As λ_1 increases from a_2^2 to a_1^2 these two exteriors of the focal hyperbola are continuously inflated and form the family of the 2-hyperboloids, which ultimately collapses down to the $x_2 x_3$-plane, represented by $\lambda_1 = a_1^2$. Then, the whole space is covered once more by the family of 2-hyperboloids.

Since every family fills up the whole space simply (with the exception of the Cartesian planes), it follows that from every point in space passes exactly one ellipsoid, exactly one 1-hyperboloid, and exactly one 2-hyperboloid. These three surfaces constitute the coordinate surfaces of the ellipsoidal system. All three families of quadrics have the same foci. Note that the three Cartesian planes are singular sets of the ellipsoidal system since they are covered twice. Indeed, the family of ellipsoids starts at infinity and ends up at the focal ellipse which is a singular set, the family of 1-hyperboloids starts from the singular set of the complement of the focal ellipse and ends up at the singular set of the interior of the focal hyperbola, and the family of 2-hyperboloids starts from the singular set of the exterior of the focal ellipse and ends up at the singular set of the $x_2 x_3$-plane. This is the reason why in Proposition 1.1 these three planes are excluded from the one-to-one correspondence introduced by the mapping f. Hence, for the ellipsoidal system, the three Cartesian planes correspond to the x_3-axis of the spherical system.

The intersections of any coordinate surface by the three Cartesian planes give either ellipses, which are called *principal ellipses*, or hyperbolas, which are called *principal hyperbolas*. Obviously, the reason why no parabolas are obtained is because the defining quadrics are non-degenerate.

Proposition 1.2 *The confocal ellipsoidal system is orthogonal.*

Proof Consider the arbitrary point $\boldsymbol{r} = (x_1, x_2, x_3)$ and let

$$E(\boldsymbol{r}) = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 - \lambda_3} - 1, \quad \lambda_3 \in (-\infty, a_3^2) \tag{1.19}$$

be the ellipsoid,

$$H_1(\boldsymbol{r}) = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 - \lambda_2} - 1, \quad \lambda_2 \in (a_3^2, a_2^2) \tag{1.20}$$

be the 1-hyperboloid, and

$$H_2(\boldsymbol{r}) = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 - \lambda_1} - 1, \quad \lambda_1 \in (a_2^2, a_1^2) \tag{1.21}$$

be the 2-hyperboloid that passes through $\boldsymbol{r}$. The normal vectors to each one of these surfaces are given by their gradients. Using the fact that $\boldsymbol{r}$ lies on the ellipsoid E and on the 1-hyperboloid H_1, we obtain

$$\begin{aligned}
\nabla E(\boldsymbol{r}) \cdot \nabla H_1(\boldsymbol{r}) &= \frac{4x_1^2}{(a_1^2 - \lambda_3)(a_1^2 - \lambda_2)} + \frac{4x_2^2}{(a_2^2 - \lambda_3)(a_2^2 - \lambda_2)} + \frac{4x_3^2}{(a_3^2 - \lambda_3)(a_3^2 - \lambda_2)} \\
&= \frac{4}{\lambda_3 - \lambda_2} \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 - \lambda_3} - \frac{4}{\lambda_3 - \lambda_2} \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 - \lambda_2} \\
&= \frac{4}{\lambda_3 - \lambda_2}(1 - 1) \\
&= 0.
\end{aligned} \tag{1.22}$$

Similarly, we can show that

$$\nabla E(\boldsymbol{r}) \cdot \nabla H_2(\boldsymbol{r}) = \nabla H_1(\boldsymbol{r}) \cdot \nabla H_2(\boldsymbol{r}) = 0. \tag{1.23}$$

Hence, at every point in space the three quadrics E, H_1, H_2 are mutually perpendicular. □

1.2 Ellipsoidal coordinates

In the previous section we demonstrated that it is possible to introduce an orthogonal coordinate system having as coordinate surfaces non-degenerate quadrics. Here we define this system and analyze its basic characteristics. Following the original Lamé notation [223], we introduce the *ellipsoidal coordinates* (ρ, μ, ν) via the transformation

$$\rho^2 = a_1^2 - \lambda_3, \quad \mu^2 = a_1^2 - \lambda_2, \quad \nu^2 = a_1^2 - \lambda_1, \tag{1.24}$$

where

$$0 \leqslant \nu^2 \leqslant h_3^2 \leqslant \mu^2 \leqslant h_2^2 \leqslant \rho^2 < +\infty. \tag{1.25}$$

Then, the family of ellipsoids is given by

$$\frac{x_1^2}{\rho^2} + \frac{x_2^2}{\rho^2 - h_3^2} + \frac{x_3^2}{\rho^2 - h_2^2} = 1, \quad \rho^2 \in (h_2^2, +\infty), \tag{1.26}$$

the family of 1-hyperboloids is given by

$$\frac{x_1^2}{\mu^2} + \frac{x_2^2}{\mu^2 - h_3^2} + \frac{x_3^2}{\mu^2 - h_2^2} = 1, \quad \mu^2 \in (h_3^2, h_2^2), \tag{1.27}$$

and the family of 2-hyperboloids is given by

$$\frac{x_1^2}{\nu^2} + \frac{x_2^2}{\nu^2 - h_3^2} + \frac{x_3^2}{\nu^2 - h_2^2} = 1, \quad \nu^2 \in (0, h_3^2). \tag{1.28}$$

Equations (1.26)–(1.28) form a linear system for the quantities x_1^2, x_2^2, x_3^2, which we solve to obtain

$$x_1^2 = \frac{\rho^2\mu^2\nu^2}{h_2^2 h_3^2}, \tag{1.29}$$

$$x_2^2 = \frac{(\rho^2 - h_3^2)(\mu^2 - h_3^2)(h_3^2 - \nu^2)}{h_1^2 h_3^2}, \tag{1.30}$$

$$x_3^2 = \frac{(\rho^2 - h_2^2)(h_2^2 - \mu^2)(h_2^2 - \nu^2)}{h_1^2 h_2^2}. \tag{1.31}$$

The successive variations in (1.25) demonstrate the triple covering of the space in the opposite direction to the one followed by λ, as described in (1.10). Indeed, at $\nu^2 = 0$ we start with the $x_1 = 0$ plane. As ν^2 increases, the $x_1 = 0$ plane splits into a plane that bends toward the positive x_1-axis and a plane that bends toward the negative x_1-plane. These two sheets form the family of the 2-hyperboloids that finally collapse to the exterior of the focal hyperbola at $\nu^2 = h_3^2$. At $\mu^2 = h_3^2$ we start with the interior of the focal hyperbola, which is inflated with increasing μ^2 forming the family of 1-hyperboloids and ends up at the exterior of the focal ellipse at $\mu^2 = h_2^2$. The interior of the focal ellipse corresponds to the value $\rho^2 = h_2^2$. As ρ^2 increases the focal ellipse inflates to an ellipsoid which gradually deforms to a sphere at infinity.

Equations (1.29)–(1.31) express the squares of the Cartesian coordinates of a point in terms of the squares of its ellipsoidal coordinates. Because of this quadratic symmetry, the geometric structure of the ellipsoidal system in each one of the eight Cartesian octants is identical. Taking the positive and negative square root branches of (1.29)–(1.31), we end up with eight expressions, which provide the coordinates of the geometrically equivalent eight points within the eight Cartesian octants. Nevertheless, we need an analytic convention to identify these eight equivalent points in space. The convention-rules we adopt here are the following:

(a) ρ varies from h_2 to infinity and therefore it is always positive;
(b) μ varies initially from h_3 to h_2 along which the positive branch of $\sqrt{h_2^2 - \mu^2}$ is taken, and then it varies back from h_2 to h_3 along which the negative branch of $\sqrt{h_2^2 - \mu^2}$ is taken;
(c) ν varies initially from $-h_3$ to $+h_3$ along which the positive branch of $\sqrt{h_3^2 - \nu^2}$ is taken, and then it varies back from $+h_3$ to $-h_3$ along which the negative branch of $\sqrt{h_3^2 - \nu^2}$ is taken.

Therefore, the sign of x_1 is controlled by ν, the sign of x_2 is controlled by $\sqrt{h_3^2 - \nu^2}$, and the sign of x_3 is controlled by $\sqrt{h_2^2 - \mu^2}$. A symbolic description of the ellipsoidal angular variations is provided in Figure 1.4.

With the above understanding, the ellipsoidal to Cartesian transformation of points in the first octant, where all three components x_i are positive, is given by

$$x_1 = \frac{\rho\mu\nu}{h_2 h_3}, \qquad h_2 < \rho < +\infty, \tag{1.32}$$

$$x_2 = \frac{\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3}, \qquad h_3 < \mu < h_2, \tag{1.33}$$

$$x_3 = \frac{\sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2}, \qquad 0 < \nu < h_3. \tag{1.34}$$

On the x_1x_2-plane, the part that is inside the focal ellipse corresponds to $\rho = h_2$, and the part that is outside the focal ellipse corresponds to $\mu = h_2$. On the x_1x_3-plane, the interior of the focal hyperbola corresponds to $\mu = h_3$, and the exterior to the focal hyperbola corresponds to $\nu = h_3$. The x_2x_3-plane corresponds to $\nu = 0$. The x_1-axis corresponds to $\rho = h_2$, $\mu = h_3$ for $x_1 \in (-h_2, h_2)$, and to $\mu = h_2$, $\nu = h_3$ for x_1

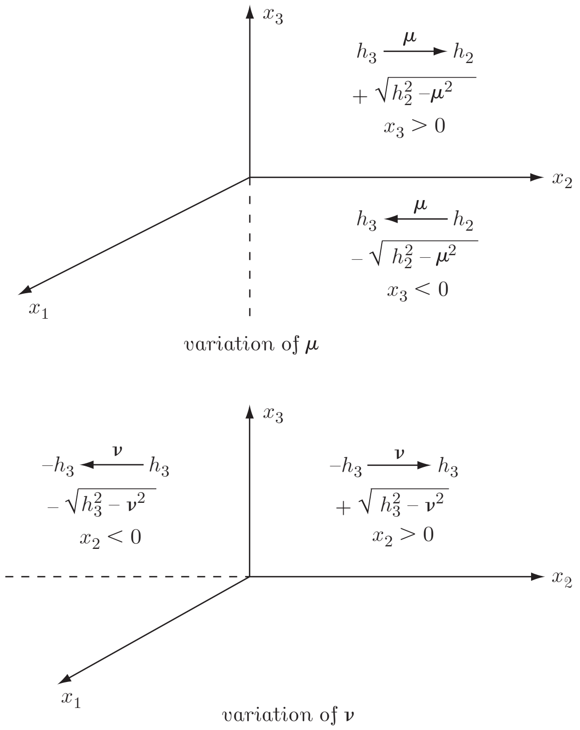

Figure 1.4 The variation of ν and the variation of μ.

outside this interval. The x_2-axis corresponds to $\rho = h_2$, $\nu = 0$ for $x_2 \in (-h_1, h_1)$, and to $\mu = h_2$, $\nu = 0$ for x_2 outside this interval. Finally, the x_3-axis corresponds to $\mu = h_3$, $\nu = 0$.

Note that, in contrast to spherical coordinates, where one coordinate has units of length and the other two are angles, the three ellipsoidal coordinates are all measured in units of length. The local system of ellipsoidal coordinates is shown in Figure 1.5.

Although the transformation from ellipsoidal to Cartesian coordinates, given by (1.29)–(1.31), is obtained relatively easy, the inversion of this transformation is by no means trivial. This is due to the fact that the system of equations (1.26)–(1.28) is non-linear in the variables ρ^2, μ^2, ν^2. In the process of inverting this transformation, in order to express the ellipsoidal in terms of the Cartesian coordinates, we first observe that the three equations (1.26)–(1.28) are identical with the equation

$$\frac{x_1^2}{\kappa} + \frac{x_2^2}{\kappa - h_3^2} + \frac{x_3^2}{\kappa - h_2^2} = 1, \tag{1.35}$$