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Magnetoencephalography in ellipsoidal geometry

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An exact analytic solution for the forward problem in the theory of biomagnetics of the human brain is known only for the (1D) case of a sphere and the (2D) case of a spheroid, where the excitation field is due to an electric dipole within the corresponding homogeneous conductor. In the present work the corresponding problem for the more realistic ellipsoidal brain model is solved and the leading quadrupole approximation for the exterior magnetic field is obtained in a form that exhibits the anisotropic character of the ellipsoidal geometry. The results are obtained in a straightforward manner through the evaluation of the interior electric potential and a subsequent calculation of the surface integral over the ellipsoid, using Lamé functions and ellipsoidal harmonics. The basic formulas are expressed in terms of the standard elliptic integrals that enter the expressions for the exterior Lamé functions. The laborious task of reducing the results to the spherical geometry is also included. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1522135]

I. INTRODUCTION

Biomagnetics is by now a well-established interdisciplinary field extending from mathematics to electrical engineering, computer sciences, physics, and of course medicine. References 1, 3, 14, 19, 22 provide an excellent exposure of the different models, methods, and techniques in biomagnetics available today.

The actual size of the human organs such as the brain, the heart, or the lungs[14,19] justifies the use of the quasistatic approximation of the Maxwell system,[17] where the time derivative of the induction field in Faraday's law and the time derivative of the displacement field in the Biot–Savart–Maxwell's law are considered to be negligible. As it is well known,[1,14,19] a chemically stimulated electric source gives rise to an electric current within the conductive tissues, which in turn generates a weak magnetic field in the surroundings of the organ. The direct biomagnetic problem consists of the evaluation of the magnetic field caused by a given current distribution. The inverse biomagnetic problem then seeks the current distribution which generates a given (through measurements) magnetic field. When the human organ under investigation is the brain, then we refer to the above problems as the direct MEG (MagnetoEncephaloGraphy) and the inverse MEG problem.[14,19]

The inverse MEG problem is not uniquely solvable in the sense that an exteriously measured magnetic field does not specify uniquely the current that generates it. Fokas, Gelfand, and Kurylev[9] have specified the extent of nonuniqueness for the case of a spherical brain model, which includes an arbitrary current density function.

The basic mathematics governing the MEG problem were developed by Geselowitz[10,11] in the late 60's, while the complete solution for spherical geometry was given by Ilmoniemi, Hämäläinen, and Knuutila[16] as well as by Sarvas.[23] For related results in the case of spheroidal volume conductors we refer to Refs. 4 and 8. A systematic analysis of the dipole singularity in the Geselowitz formula reveals that, as far as the magnetic field is concerned, one third of the contribution from the volume current is canceled by one third of the contribution from the primary source current.[7] Sarvas' solution for the sphere is based on the radial component of the primary dipole field, and it is calculated via the use of a magnetic potential representing the irrotational magnetic field in the space exterior to the brain. For some closely related work we refer to Refs. 4, 8, 12, 13, 18, 20, 21.

The actual geometry of the human brain is that of an ellipsoid with semiaxes equal to 6, 6.5, and 9 cm.[24] In contrast to the complete isotropy that is represented by the sphere, the triaxial ellipsoid embodies the complete anisotropy of the three-dimensional space. As a consequence, the much more complicated theory of ellipsoidal harmonics, as opposed to the theory of spherical harmonics, is necessary to solve the direct MEG problem for a realistic brain model. This program is realized in the work at hand. The analysis is based on Lamé functions and ellipsoidal harmonics. In fact, only harmonics of degree less than or equal to 2 are needed to obtain the quadrupole term for the magnetic field. Besides the purely ellipsoidal expressions, the results are also given in the more tractable form where Cartesian coordinates are used for the interior harmonics plus the standard elliptic integrals that appear in the exterior Lamé products. The particular way these elliptic integrals are interconnected is provided in Appendix D.

Section II states the mathematical theory of the MEG problem. The solution of the interior boundary value problem that offers the electric potential within an ellipsoid due to an electric dipole is obtained in Sec. III, while Sec. IV involves the evaluation of the magnetic induction field in the exterior of the ellipsoid. The exact analytic form of the quadrupole term is given explicitly, while as it is expected, the dipole term vanishes. As it is well known,[5,6,15] the reduction of general results from the ellipsoidal to the spherical geometry is not a straightforward task because of the complicated indeterminacies that occur as the three semifocal distances of the ellipsoidal system approach zero. The only way to deal with these indeterminacies is to group appropriately the terms of the solution and to perform the algebraic manipulations that eliminate the indeterminacies before the limiting process is applied. In some cases this procedure is not much easier than the generation of the ellipsoidal solution itself. Section V is dedicated to this task and the corresponding result for the spherical case is recovered. The necessary material from the theory of ellipsoidal harmonics as well as some useful formulas associated with ellipsoidal functions are collected in the Appendices.

II. STATEMENT OF THE PROBLEM

In order to avoid technical complications and additional terminology, we will restrict attention to the single component model, which is actually what we need in the present work.

Let S denotes the triaxial ellipsoid

$$\frac{x_1^2}{\alpha_1^2}+\frac{x_2^2}{\alpha_2^2}+\frac{x_3^2}{\alpha_3^2}=1, \tag{1}$$

where $0<\alpha_3<\alpha_2<\alpha_1<+\infty$ are its semiaxes. The basic ellipsoid (1) specifies an ellipsoidal system[15] with coordinates $(\rho,\ \mu,\ \nu)$ which springs out of the focal ellipse

$$\frac{x_1^2}{h_2^2}+\frac{x_2^2}{h_1^2}=1, \tag{2}$$

with semifocal distance

$$h_3=\sqrt{h_2^2-h_1^2}, \tag{3}$$

where

$$\left.\begin{aligned} h_1&=\sqrt{\alpha_2^2-\alpha_3^2}\\ h_2&=\sqrt{\alpha_1^2-\alpha_3^2}\\ h_3&=\sqrt{\alpha_1^2-\alpha_2^2} \end{aligned}\right\} \tag{4}$$

are the semifocal distances of the principal ellipses of (1). The ellipsoidal coordinates (ρ, μ, ν), given in Appendix A, involve the ellipsoidal variable $\rho\in[h_2,+\infty)$ and the hyperboloidal variables $\mu\in[h_3,h_2]$ and $\nu\in[-h_3,h_3]$. The coordinate ρ plays the role of the radial variable r, while μ and ν correspond to the angular variable θ and φ in spherical coordinates. In particular, the value $\rho=h_2$ specifies the focal ellipse (2), the value $\rho=\alpha_1$ specifies the basic ellipsoid (1), and as $\rho\to+\infty$, the corresponding ellipsoid approaches a sphere of infinite radius. In what follows the $\rho=\alpha_1$ ellipsoid, given by (1), will represent the boundary of the ellipsoid representing the brain. Then, the brain fills the interior space $\rho\in[h_2,\alpha_1)$ while the exterior space is described by $\rho\in(\alpha_1,+\infty)$.

Having described the geometrical background, we turn now to the physics of our problem. Since the dielectric constant of the brain tissue is by five orders of magnitude higher than the dielectric constant of vacuum and its electric conductivity is approximately 0.3 $\Omega^{-1}\,\mathrm{m}^{-1}$,[14,19] a simple arithmetic shows that the physical realm of bioelectromagnetics is that of the quasistatic approximation of Maxwell's equations.[17] Hence, the set of governing equations, in the absence of electric charge, is taken to be

$$\nabla\times\mathbf{E}=\mathbf{0}, \tag{5}$$

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}, \tag{6}$$

$$\nabla\cdot\mathbf{E}=0, \tag{7}$$

$$\nabla\cdot\mathbf{B}=0, \tag{8}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction field, μ_0 is the magnetic permeability, which is assumed to be the same inside as well as outside the brain, while the expression

$$\mathbf{J}=\mathbf{J}^P+\sigma\mathbf{E}, \tag{9}$$

specifies the current density, with $\mathbf{J}^P$ the primarily imposed equivalent current and σ the conductivity of the brain tissue. The conductivity outside the brain is considered to be zero.

Since $\mathbf{E}$ is irrotational, there exist an electric potential u such that

$$\mathbf{E}(\mathbf{r})=-\nabla u(\mathbf{r}). \tag{10}$$

In particular, we denote the electric potential in the interior of the ellipsoid $\rho=\alpha_1$ by u^- and in the exterior of $\rho=\alpha_1$ by u^+.

Equation (6) implies that $\mathbf{J}$ is solenoidal, and consequently (9) and (10) force the electric potential to solve Poisson's equation

$$\Delta u^-(\mathbf{r})=\frac{1}{\sigma}\nabla\cdot\mathbf{J}^P(\mathbf{r}), \tag{11}$$

in the space V^- interior to the ellipsoid $\rho=\alpha_1$ and the Laplace's equation

$$\Delta u^+(\mathbf{r})=0, \tag{12}$$

in the space V^+ exterior to the ellipsoid $\rho=\alpha_1$.

It is easily shown[10,11,23] that in an unbounded, electrically homogeneous space with compactly supported primary current, the scalar field σu and the vector field $\mu_0^{-1}\mathbf{B}$ are the scalar and the vector invariants, respectively, of the dyadic field

$$\tilde{\mathbf{D}}(\mathbf{r})=\frac{1}{4\pi}\int_\Omega \mathbf{J}^p(\mathbf{r}')\otimes\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dv(\mathbf{r}'), \tag{13}$$

where Ω denotes the support of $\mathbf{J}^p$.

Indeed, the electric field is given by

$$u(\mathbf{r})=\frac{1}{4\pi\sigma}\int_{\Omega}\mathbf{J}^p(\mathbf{r}')\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dv(\mathbf{r}'), \tag{14}$$

while the magnetic field is given by

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_{\Omega}\mathbf{J}^p(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dv(\mathbf{r}'). \tag{15}$$

In general, for a single component model the electric potential solves the following transmission problem:

$$\Delta u^-(\mathbf{r})=\frac{1}{\sigma}\nabla\cdot\mathbf{J}^p(\mathbf{r}), \qquad \mathbf{r}\in V^- \tag{16}$$

$$\Delta u^+(\mathbf{r})=0, \qquad \mathbf{r}\in V^+ \tag{17}$$

$$u^-(\mathbf{r})=u^+(\mathbf{r}), \qquad \mathbf{r}\in S \tag{18}$$

$$\sigma^-\frac{\partial u^-(\mathbf{r})}{\partial n}=\sigma^+\frac{\partial u^+(\mathbf{r})}{\partial n}, \quad \mathbf{r}\in S \tag{19}$$

$$u^+(\mathbf{r})=O\left(\frac{1}{r}\right), \qquad r\to\infty \tag{20}$$

where the "$-$" and the "$+$" characterize the interior region V^- and the exterior region V^+, respectively. In the case of $\sigma^-=\sigma$, $\sigma^+=0$, as it is assumed in MEG, the above transmission problem splits into the interior Neumann problem

$$\Delta u^-(\mathbf{r})=\frac{1}{\sigma}\nabla\cdot\mathbf{J}^p(\mathbf{r}), \qquad \mathbf{r}\in V^- \tag{21}$$

$$\frac{\partial u^-(\mathbf{r})}{\partial n}=0, \qquad \mathbf{r}\in S \tag{22}$$

which can be solved independently, and the exterior Dirichlet problem

$$\Delta u^+(\mathbf{r})=0, \qquad \mathbf{r}\in V^+ \tag{23}$$

$$u^+(\mathbf{r})=u^-(\mathbf{r}), \qquad \mathbf{r}\in S \tag{24}$$

$$u^+(\mathbf{r})=O\left(\frac{1}{r}\right), \qquad r\to\infty \tag{25}$$

which is postulated via the trace of u^- on S.

Note that the interior problem (21), (22) involves an inhomogeneous equation with a homogeneous boundary condition, while the exterior problem (23)–(25) involves a homogeneous equation with an inhomogeneous boundary condition. In physical terms, the primary current $\mathbf{J}^p$ generates an electric field in V^- and the value of this field on S establishes the electric field in V^+. The asymptotic order of u^+ at infinity is dictated by (14).

The plan now is as follows: solve (21), (22) and then calculate the magnetic field from the integral form of the Biot–Savart–Maxwell law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V^-} [\mathbf{J}^p(\mathbf{r}') - \sigma \nabla u^-(\mathbf{r}')] \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} dv(\mathbf{r}'). \tag{26}$$

Following magnetoencephalographic practice[1,14,19,23] we represent the actual localized electromagnetic activity of the brain tissue by an equivalent electric dipole current, at a fixed point $\mathbf{r}_0$, with a dipole moment equal to $\mathbf{Q}$. In other words, the primary current is given by

$$\mathbf{J}^P(\mathbf{r}) = \mathbf{Q}\delta(\mathbf{r}-\mathbf{r}_0), \tag{27}$$

where δ stands for the Dirac measure at $\mathbf{r}_0$.

Introducing (27) in (14) and (15), it is obvious that this point current (27) furnishes the electric field

$$u_0(\mathbf{r}) = \frac{1}{4\pi\sigma} \mathbf{Q} \cdot \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}, \tag{28}$$

and the magnetic field

$$\mathbf{B}_0(\mathbf{r}) = \frac{\mu_0}{4\pi} \mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}. \tag{29}$$

Furthermore, in view of (27), Eq. (26) implies

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0\sigma}{4\pi} \int_{V^-} \nabla u^-(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} dv(\mathbf{r}'). \tag{30}$$

The volume integral in (30) can be transformed to a surface integral,[2,7,11] providing the formula

$$\int_{V^-} \nabla u^-(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} dv(\mathbf{r}') = \int_S u^-(\mathbf{r}') \hat{\mathbf{n}}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} ds(\mathbf{r}'), \tag{31}$$

where $\mathbf{r} \in \mathbb{R}^3 - S$ and $\hat{\mathbf{n}}$ stands for the outward unit normal on S at $\mathbf{r}'$. Formula (31) shows that the volume distribution of dipoles, with moments proportional to ∇u^-, can be replaced by a surface distribution of dipoles, with moments proportional to $u^- \hat{\mathbf{n}}'$. Its proof demands a careful treatment of the singularity at $\mathbf{r}_0$.[7]

In terms of the transformation (31), the magnetic field (30) is expressed as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0\sigma}{4\pi} \int_S u^-(\mathbf{r}') \hat{\mathbf{n}}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} ds(\mathbf{r}'). \tag{32}$$

The following section provides the evaluation of the interior electric field u^- when S is the ellipsoid (1).

III. THE INTERIOR ELECTRIC POTENTIAL

The goal of this section is to solve the interior problem (21), (22) where $\mathbf{J}^p$ is given by (27). Straightforward arguments conclude that the solution u^- assumes the form

$$u^-(\mathbf{r}) = \Phi(\mathbf{r}) + \frac{1}{4\pi\sigma} \mathbf{Q} \cdot \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} = \Phi(\mathbf{r}) + \frac{1}{4\pi\sigma} \mathbf{Q} \cdot \nabla_{\mathbf{r}_0} \frac{1}{|\mathbf{r}-\mathbf{r}_0|}, \tag{33}$$

where Φ is an interior harmonic function inside the ellipsoid $h_2 \leqslant \rho < \alpha_1$ which satisfies the Neumann boundary condition

$$\frac{\partial}{\partial \rho}\Phi(\mathbf{r}) = -\frac{1}{4\pi\sigma}\frac{\partial}{\partial \rho}\mathbf{Q}\cdot\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}, \tag{34}$$

on the ellipsoid $\rho=\alpha_1$.

The completeness of the ellipsoidal harmonics[15] secures the existence of a sequence $\{b_n^m\}$, $n=0,1,2,...,$ and $m=1,2,...,2n+1$ such that

$$\Phi(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} b_n^m \mathbb{E}_n^m(\rho,\mu,\nu), \tag{35}$$

for $\rho\in[h_2,\alpha_1)$, where $\mathbb{E}_n^m(\rho,\mu,\nu)$ denotes the interior solid ellipsoidal harmonic of degree n and order m (see Appendix B). The vector $\mathbf{r}$ is always assumed to be represented by the ellipsoidal triplet (ρ,μ,ν). In order to be able to use the orthogonality properties of the ellipsoidal eigenfunctions we need to express the particular solution of (21), which is the singular part of (33), in terms of surface ellipsoidal harmonics. To this end, we use the ellipsoidal expansion of the fundamental solution for the Laplace's operator[6] in the form

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{4\pi}{2n+1}\frac{1}{\gamma_n^m}\mathbb{E}_n^m(\rho_0,\mu_0,\nu_0)\mathbb{F}_n^m(\rho,\mu,\nu), \tag{36}$$

where (ρ_0,μ_0,ν_0) represents the position $\mathbf{r}_0$ of the dipole expressed in ellipsoidal coordinates, $\mathbb{F}_n^m$ are the exterior solid ellipsoidal harmonics (see Appendix B) and γ_n^m are the ellipsoidal normalization constants given by

$$\gamma_n^m = \int_{\rho=\alpha_1}[E_n^m(\mu)E_n^m(\nu)]^2\frac{1}{\sqrt{\alpha_1^2-\mu^2}\sqrt{\alpha_1^2-\nu^2}}\,ds. \tag{37}$$

Actually, the constant γ_n^m is the square of the L^2 norm of the surface ellipsoidal harmonic $E_n^m(\mu)E_n^m(\nu)$ with respect to the weighting function

$$l(\mu,\nu) = [(\alpha_1^2-\mu^2)(\alpha_1^2-\nu^2)]^{-1/2}, \tag{38}$$

which depends on the ellipsoidal surface $\rho=\alpha_1$.

Expansion (36) holds for $\rho>\rho_0$ and provides the appropriate form for the application of the boundary condition (34). In view of (33), (35), and (36), the field u^- is written as

$$u^-(\rho,\mu,\nu) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left[b_n^m\mathbb{E}_n^m(\rho,\mu,\nu) + \frac{1}{\sigma(2n+1)\gamma_n^m}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_n^m(\mathbf{r}_0))\mathbb{F}_n^m(\rho,\mu,\nu)\right]. \tag{39}$$

Using the form of $\mathbb{F}_n^m$, as it is given by (B6) as well as the fact the $\mathbb{E}_0^1(\rho,\mu,\nu)=1$, Eq. (39) is written as

$$u^-(\rho,\mu,\nu) = b_0^1 + \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\left[b_n^m + \frac{1}{\sigma\gamma_n^m}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_n^m(\mathbf{r}_0))I_n^m(\rho)\right]\mathbb{E}_n^m(\rho,\mu,\nu), \tag{40}$$

where $I_n^m(\rho)$ is given by (B4).

Applying the boundary condition (34) to (39) and using the orthogonality properties of the surface ellipsoidal harmonics[15] we are led to

$$b_n^m = \frac{1}{\sigma\gamma_n^m}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_n^m(\mathbf{r}_0))\left[\frac{1}{\alpha_2\alpha_3E_n^m(\alpha_1)E_n^{m\prime}(\alpha_1)} - I_n^m(\alpha_1)\right], \tag{41}$$

for each $n=0,1,2,...$ and $m=1,2,...,2n+1$, where E_n^m denote the Lamé functions of the first kind (see Appendix B) and the prime denotes differentiation with respect to the argument.

Using expression (41) for the coefficients the electric potential within the ellipsoid, $\rho=\alpha_1$ is then written as

$$u^-(\mathbf{r})=b_0^1+\frac{1}{4\pi\sigma}\mathbf{Q}\cdot\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}-\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\frac{1}{(2n+1)\sigma\gamma_n^m}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\mathbb{E}_n^m(\mathbf{r}_0))\frac{F_n^{m\prime}(\alpha_1)}{E_n^{m\prime}(\alpha_1)}\mathbb{E}_n^m(\mathbf{r}), \tag{42}$$

where F_n^m are the Lamé functions of the second kind (see Appendix B).

We observe that u^- is uniquely specified up to the additive constant b_0^1, a fact that is compatible with the wellposedness of the Neumann problem (21), (22).

In order to express the interior electric field u^- in a more tractable and useful form we use formulas (B16)–(B27) and restrict consideration to the leading two terms in the multipole ellipsoidal expansion of (42). Along these lines we introduce the following notation where the single wiggle on the top denotes a dyadic and the double wiggle on the top denotes a tetradic[2]

$$\tilde{\mathbf{M}}(\rho)=\sum_{m=1}^{3}(\rho^2-\alpha_1^2+\alpha_m^2)\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m, \tag{43}$$

$$\tilde{\mathbf{\Lambda}}=\sum_{m=1}^{3}\frac{\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m}{\Lambda-\alpha_m^2}, \tag{44}$$

$$\tilde{\mathbf{\Lambda}}'=\sum_{m=1}^{3}\frac{\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m}{\Lambda'-\alpha_m^2}, \tag{45}$$

$$\tilde{\mathbf{H}}_1(\rho)=\sum_{m=1}^{3}I_1^m(\rho)\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m, \tag{46}$$

$$\tilde{\tilde{\mathbf{H}}}_2(\rho)=\sum_{\substack{i,j=1\\ i\neq j}}^{3}I_2^{i+j}(\rho)\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j, \tag{47}$$

$$\tilde{\mathbf{N}}_1=\sum_{m=1}^{3}\frac{\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m}{\alpha_1^2+\alpha_2^2+\alpha_3^2-\alpha_m^2}, \tag{48}$$

$$\tilde{\tilde{\mathbf{N}}}_2=\sum_{\substack{i,j=1\\ i\neq j}}^{3}\frac{\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j}{\alpha_i^2+\alpha_j^2}, \tag{49}$$

where the constants Λ, Λ' are given by (B14) and the elliptic integrals that appear in (46)–(47) are given by (B4). The unit vectors $\hat{\mathbf{x}}_i$, $i=1,2,3$ stand for the Cartesian basis. The characteristic of the quantities (43)–(49) is that they are all modifications of the identity dyadic or the identity tetradic. If the (anisotropic) ellipsoid is replaced by the (isotropic) sphere all quantities (43)–(49) are reduced to multiples of the identity, while as they stand they incorporate the particular standards of each principal direction of the ellipsoid.

Furthermore, we define the dyadic functions

$$\tilde{\mathbf{A}}(\rho)=\frac{3}{4\pi\sigma}(\tilde{\mathbf{H}}_1(\rho)-\tilde{\mathbf{H}}_1(\alpha_1))+\frac{1}{\sigma V}\tilde{\mathbf{I}}, \tag{50}$$

$$\tilde{\mathbf{B}}(\mathbf{r}) = -\frac{5}{4\pi\sigma(\Lambda-\Lambda')}\Bigg[\left(I_2^1(\rho)-I_2^1(\alpha_1)+\frac{2\pi}{3V\Lambda}\right)\tilde{\boldsymbol{\Lambda}}\mathbb{E}_2^1(\mathbf{r}) - \left(I_2^2(\rho)-I_2^2(\alpha_1)+\frac{2\pi}{3V\Lambda'}\right)\tilde{\boldsymbol{\Lambda}}'\mathbb{E}_2^2(\mathbf{r})\Bigg], \tag{51}$$

where V denotes the volume of the ellipsoid $(3V=4\pi\alpha_1\alpha_2\alpha_3)$ and the tetradic function

$$\tilde{\tilde{\tilde{\boldsymbol{\Gamma}}}}(\rho) = \frac{15}{4\pi\sigma}(\tilde{\tilde{\tilde{\mathbf{H}}}}_2(\rho)-\tilde{\tilde{\tilde{\mathbf{H}}}}_2(\alpha_1)) + \frac{5}{\sigma V}\tilde{\tilde{\mathbf{N}}}_2, \tag{52}$$

in terms of which the interior electric field within the shell $\rho_0<\rho<\alpha_1$ assumes the compact form

$$u^-(\mathbf{r}) = b_0^1 + \mathbf{Q}\cdot\tilde{\mathbf{A}}(\rho)\cdot\mathbf{r} + \mathbf{Q}\otimes\mathbf{r}_0:\tilde{\mathbf{B}}(\mathbf{r}) + \mathbf{Q}\otimes\mathbf{r}_0:\tilde{\tilde{\tilde{\boldsymbol{\Gamma}}}}(\rho):\mathbf{r}\otimes\mathbf{r} + O(el_3), \tag{53}$$

where the notation $O(el_3)$ denotes terms in the multipole expansion that are of order greater or equal to 3 (octapole or higher terms) and the double contraction is defined as

$$\mathbf{a}\otimes\mathbf{b}:\mathbf{c}\otimes\mathbf{d} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}). \tag{54}$$

In obtaining expression (53) for u^-, we have also used the multipole expansion

$$\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} = 3\tilde{\mathbf{H}}_1(\rho)\cdot\mathbf{r} - \frac{5}{\Lambda-\Lambda'}\mathbf{r}_0\cdot[\tilde{\boldsymbol{\Lambda}}\mathbb{E}_2^1(\mathbf{r})I_2^1(\rho) - \tilde{\boldsymbol{\Lambda}}'\mathbb{E}_2^2(\mathbf{r})I_2^2(\rho)] + 15\mathbf{r}_0\cdot\tilde{\tilde{\tilde{\mathbf{H}}}}_2(\rho):\mathbf{r}\otimes\mathbf{r} + O(el_3). \tag{55}$$

Expressions (53) and (55) combine both the Cartesian coordinates of $\mathbf{r}$ and the elliptic integrals $I_n^m(\rho)$ that depend on the ellipsoidal variable ρ. This combination leads to the most compact way of expressing (53) and (55), and at the same time it minimizes the indeterminacies as the ellipsoid reduces to the sphere.

An equivalent expression for u^- which avoids the polyadic notation is furnished by

$$\begin{aligned} u^-(\rho,\mu,\nu) &= b_0^1 + \frac{3}{4\pi\sigma}\sum_{m=1}^{3} Q_m x_m\left[I_1^m(\rho)-I_1^m(\alpha_1)+\frac{1}{\alpha_1\alpha_2\alpha_3}\right] \\ &\quad - \frac{5}{4\pi\sigma(\Lambda-\Lambda')}\sum_{m=1}^{3} Q_m x_{0m}\Bigg[\left(I_2^1(\rho)-I_2^1(\alpha_1)+\frac{1}{2\alpha_1\alpha_2\alpha_3\Lambda}\right)\frac{\mathbb{E}_2^1(\mathbf{r})}{(\Lambda-\alpha_m^2)} \\ &\quad - \left(I_2^2(\rho)-I_2^2(\alpha_1)+\frac{1}{2\alpha_1\alpha_2\alpha_3\Lambda'}\right)\frac{\mathbb{E}_2^2(\mathbf{r})}{(\Lambda'-\alpha_m^2)}\Bigg] \\ &\quad + \frac{15}{4\pi\sigma}\sum_{\substack{i,j=1\\ i\neq j}}^{3} Q_i x_{0j} x_i x_j\left(I_2^{i+j}(\rho)-I_2^{i+j}(\alpha_1)+\frac{1}{\alpha_1\alpha_2\alpha_3(\alpha_i^2+\alpha_j^2)}\right) + O(el_3). \end{aligned} \tag{56}$$

IV. THE EXTERIOR MAGNETIC FIELD

The magnetic induction field $\mathbf{B}$ is obtained from (32) after we insert the values of u^- on the surface $\rho=\alpha_1$ and perform the indicated integration. Our plan is to focus on the dipole and the quadrupole terms, which provide the leading two approximations of u^-. To this end, we rewrite the integral in (32) in such a way as to be able to use orthogonality of the surface ellipsoidal harmonics.

First, we observe that

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\frac{4\pi}{\gamma_0^1}I_0^1(\rho)+\frac{4\pi}{3}\sum_{m=1}^{3}\frac{1}{\gamma_1^m}\mathbb{E}_1^m(\mathbf{r}_0)\mathbb{F}_1^m(\mathbf{r})+\frac{4\pi}{5}\sum_{m=1}^{5}\frac{1}{\gamma_2^m}\mathbb{E}_2^m(\mathbf{r}_0)\mathbb{F}_2^m(\mathbf{r})+O(el_3). \tag{57}$$

Taking the gradient of (57) and using (B17)–(B27), we obtain the following key formula in our work:

$$\begin{aligned}\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}=&\frac{3}{h_1h_2h_3}\sum_{m=1}^{3}h_mI_1^m(\rho)\hat{\mathbf{x}}_m\mathbb{E}_1^m(\mathbf{r})-\frac{5}{(\Lambda-\Lambda')h_1h_2h_3}\sum_{m=1}^{3}\frac{h_m}{\Lambda-\alpha_m^2}\mathbb{E}_1^m(\mathbf{r}_0)\hat{\mathbf{x}}_m\mathbb{E}_2^1(\mathbf{r})I_2^1(\rho)\\&+\frac{5}{(\Lambda-\Lambda')h_1h_2h_3}\sum_{m=1}^{3}\frac{h_m}{\Lambda'-\alpha_m^2}\mathbb{E}_1^m(\mathbf{r}_0)\hat{\mathbf{x}}_m\mathbb{E}_2^2(\mathbf{r})I_2^2(\rho)\\&+\frac{15}{h_1h_2h_3^3}\left[\frac{1}{h_1}\mathbb{E}_1^2(\mathbf{r}_0)\hat{\mathbf{x}}_1+\frac{1}{h_2}\mathbb{E}_1^1(\mathbf{r}_0)\hat{\mathbf{x}}_2\right]\mathbb{E}_2^3(\mathbf{r})I_2^3(\rho)\\&+\frac{15}{h_1h_2^3h_3}\left[\frac{1}{h_1}\mathbb{E}_1^3(\mathbf{r}_0)\hat{\mathbf{x}}_1+\frac{1}{h_3}\mathbb{E}_1^1(\mathbf{r}_0)\hat{\mathbf{x}}_3\right]\mathbb{E}_2^4(\mathbf{r})I_2^4(\rho)\\&+\frac{15}{h_1^3h_2h_3}\left[\frac{1}{h_2}\mathbb{E}_1^3(\mathbf{r}_0)\hat{\mathbf{x}}_2+\frac{1}{h_3}\mathbb{E}_1^2(\mathbf{r}_0)\hat{\mathbf{x}}_3\right]\mathbb{E}_2^5(\mathbf{r})I_2^5(\rho)+O(el_3).\end{aligned} \tag{58}$$

Obviously, expansion (58) is valid for $\rho>\rho_0$ and it can be used in (32) and in (42) to obtain the three dipole terms and the five quadrupole terms for the multipole expansion of the primary dipole field at $\mathbf{r}_0$, as well as for the induced dipole fields at the surface points $\mathbf{r}'$.

In order to be able to evaluate the surface integral in (32) we need to express the interior electric potential $u^-(\mathbf{r}')$, the outward unit normal $\hat{\boldsymbol{\rho}}'$, and the basic dipole field $|\mathbf{r}-\mathbf{r}'|^{-3}(\mathbf{r}-\mathbf{r}')$ in terms of surface ellipsoidal harmonics in the variable of integration $\mathbf{r}'$ on the surface $\rho=\alpha_1$. This is a long and tedious task which is developed in the following steps. First, we expand the interior electric field (53) to obtain

$$u^-(\alpha_1,\mu',\nu')=b_0^1+\sum_{m=1}^{3}\zeta_mE_1^m(\mu')E_1^m(\nu')+\sum_{m=1}^{5}\theta_mE_2^m(\mu')E_2^m(\nu')+O(el_3'), \tag{59}$$

where

$$\zeta_m=\frac{\alpha_mh_m}{\sigma Vh_1h_2h_3}(\mathbf{Q}\cdot\hat{\mathbf{x}}_m),\quad m=1,2,3 \tag{60}$$

and

$$\theta_1=-\frac{5}{6\sigma V(\Lambda-\Lambda')}(\mathbf{Q}\otimes\mathbf{r}_0:\tilde{\boldsymbol{\Lambda}}), \tag{61}$$

$$\theta_2=\frac{5}{6\sigma V(\Lambda-\Lambda')}(\mathbf{Q}\otimes\mathbf{r}_0:\tilde{\boldsymbol{\Lambda}}'), \tag{62}$$

$$\theta_3=\frac{5\alpha_1\alpha_2\alpha_3}{\sigma Vh_1h_2h_3}\frac{\mathbf{Q}\otimes\mathbf{r}_0:(\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_1)}{\alpha_3h_3(\alpha_1^2+\alpha_2^2)}, \tag{63}$$

$$\theta_4=\frac{5\alpha_1\alpha_2\alpha_3}{\sigma Vh_1h_2h_3}\frac{\mathbf{Q}\otimes\mathbf{r}_0:(\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_3+\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_1)}{\alpha_2h_2(\alpha_1^2+\alpha_3^2)}, \tag{64}$$

$$\theta_5=\frac{5\alpha_1\alpha_2\alpha_3}{\sigma V h_1 h_2 h_3}\,\frac{\mathbf{Q}\otimes\mathbf{r}_0:(\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_3+\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_2)}{\alpha_1 h_1(\alpha_2^2+\alpha_3^2)}. \tag{65}$$

The outward unit normal is written as[5]

$$\hat{\boldsymbol{\rho}}'=\alpha_1\alpha_2\alpha_3 l(\mu',\nu')\tilde{\mathbf{M}}^{-1}(\alpha_1)\cdot\mathbf{r}', \tag{66}$$

and if the basic dipole field is expanded for $\rho>\rho'$ it provides the form

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}=3\mathbf{r}\cdot\tilde{\mathbf{H}}_1(\rho)+\tilde{\mathbf{F}}(\mathbf{r})\cdot\mathbf{r}'+O(el_2'), \tag{67}$$

where

$$\tilde{\mathbf{F}}(\mathbf{r})=-\frac{\mathbb{F}_2^1(\mathbf{r})}{\Lambda-\Lambda'}\tilde{\boldsymbol{\Lambda}}+\frac{\mathbb{F}_2^2(\mathbf{r})}{\Lambda-\Lambda'}\tilde{\boldsymbol{\Lambda}}'+15\mathbf{r}\otimes\mathbf{r}:\tilde{\tilde{\mathbf{H}}}_2(\rho). \tag{68}$$

Then, we calculate the expression

$$\begin{aligned}\hat{\boldsymbol{\rho}}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}&=\alpha_1\alpha_2\alpha_3 l(\mu',\nu')[3\mathbf{r}'\cdot\tilde{\mathbf{M}}^{-1}(\alpha_1)\times\tilde{\mathbf{H}}_1(\rho)\cdot\mathbf{r}+\mathbf{r}'\cdot\tilde{\mathbf{M}}^{-1}(\alpha_1)\times\tilde{\mathbf{F}}(\mathbf{r})\cdot\mathbf{r}']+O(el_3')\\&=\alpha_1\alpha_2\alpha_3 l(\mu',\nu')\left[3\sum_{m=1}^{3}(\hat{\mathbf{x}}_m\cdot\tilde{\mathbf{M}}^{-1}(\alpha_1)\times\tilde{\mathbf{H}}_1(\rho)\cdot\mathbf{r})x_m'\right.\\&\quad\left.+\sum_{i,j=1}^{3}(\hat{\mathbf{x}}_i\cdot\tilde{\mathbf{M}}^{-1}(\alpha_1)\times\tilde{\mathbf{F}}(\mathbf{r})\cdot\hat{\mathbf{x}}_j)x_i'x_j'\right]+O(el_3').\end{aligned} \tag{69}$$

Expression (69) contains the variable of integration $\mathbf{r}'$ of (32) in Cartesian form. Since we want to use orthogonality properties over the ellipsoid $\rho'=\alpha_1$, we need to transform (69) to ellipsoidal coordinates. To end we use formulas (C7)–(C9) and perform some extensive algebraic manipulations that lead to the expression

$$\begin{aligned}\left.\hat{\boldsymbol{\rho}}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right|_{\rho'=\alpha_1}&=\alpha_1\alpha_2\alpha_3 l(\mu',\nu')\left\{\frac{1}{3}\sum_{m=1}^{3}\alpha_m^2(\hat{\mathbf{x}}_m\cdot\tilde{\mathbf{M}}^{-1}(\alpha_1)\times\tilde{\mathbf{F}}(\mathbf{r})\cdot\hat{\mathbf{x}}_m)\right.\\&\quad+\frac{3}{h_1h_2h_3}\sum_{m=1}^{3}\alpha_m h_m(\hat{\mathbf{x}}_m\cdot\tilde{\mathbf{M}}^{-1}(\alpha_1)\times\tilde{\mathbf{H}}_1(\rho)\cdot\mathbf{r})E_1^m(\mu')E_1^m(\nu')\\&\quad-\frac{1}{3(\Lambda-\Lambda')}\sum_{m=1}^{3}\frac{\alpha_m^2}{\Lambda-\alpha_m^2}(\hat{\mathbf{x}}_m\cdot\tilde{\mathbf{M}}^{-1}(\alpha_1)\times\tilde{\mathbf{F}}(\mathbf{r})\cdot\hat{\mathbf{x}}_m)E_2^1(\mu')E_2^1(\nu')\\&\quad+\frac{1}{3(\Lambda-\Lambda')}\sum_{m=1}^{3}\frac{\alpha_m^2}{\Lambda'-\alpha_m^2}(\hat{\mathbf{x}}_m\cdot\tilde{\mathbf{M}}^{-1}(\alpha_1)\times\tilde{\mathbf{F}}(\mathbf{r})\cdot\hat{\mathbf{x}}_m)E_2^2(\mu')E_2^2(\nu')\\&\quad\left.+\frac{1}{h_1h_2h_3}\sum_{\substack{i,j=1\\ i\neq j}}^{3}\frac{\alpha_i\alpha_j}{h_{6-(i+j)}}(\hat{\mathbf{x}}_i\cdot\tilde{\mathbf{M}}^{-1}(\alpha_1)\times\tilde{\mathbf{F}}(\mathbf{r})\cdot\hat{\mathbf{x}}_j)E_2^{i+j}(\mu')E_2^{i+j}(\nu')\right\}\\&\quad+O(el_3').\end{aligned} \tag{70}$$

At this stage we observe that the monopole term in (70) vanishes. Indeed, from (43)–(45), (47), and (68) we observe that

$$\sum_{m=1}^{3} \alpha_m^2 (\hat{\mathbf{x}}_m \cdot \tilde{\mathbf{M}}^{-1}(\alpha_1) \times \tilde{\mathbf{F}}(\mathbf{r}) \cdot \hat{\mathbf{x}}_m)$$

$$= \sum_{m=1}^{3} \hat{\mathbf{x}}_m \times \tilde{\mathbf{F}}(\mathbf{r}) \cdot \hat{\mathbf{x}}_m$$

$$= -\frac{\mathbb{F}_2^1(\mathbf{r})}{\Lambda - \Lambda'} \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \times \hat{\mathbf{x}}_m}{\Lambda - \alpha_m^2} + \frac{\mathbb{F}_2^2(\mathbf{r})}{\Lambda - \Lambda'} \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \times \hat{\mathbf{x}}_m}{\Lambda' - \alpha_m^2} + 15 \sum_{m=1}^{3} \hat{\mathbf{x}}_m \times (\mathbf{r} \otimes \mathbf{r} : \tilde{\tilde{\mathbf{H}}}_2(\rho)) \cdot \hat{\mathbf{x}}_m$$

$$= 15 \sum_{\substack{i,j=1 \\ i \neq j}}^{3} x_i x_j I_2^{i+j}(\rho) \hat{\mathbf{x}}_j \times \hat{\mathbf{x}}_i = \mathbf{0}. \tag{71}$$

Then (70) is written as

$$\hat{\boldsymbol{\rho}}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\bigg|_{\rho' = \alpha_1} = l(\mu', \nu') \left[\sum_{m=1}^{3} \boldsymbol{\beta}_m E_1^m(\mu') E_1^m(\nu') + \sum_{m=1}^{5} \boldsymbol{\delta}_m E_2^m(\mu') E_2^m(\nu') \right] + O(el_3'), \tag{72}$$

where

$$\boldsymbol{\beta}_m = 3 \frac{\alpha_1 \alpha_2 \alpha_3}{h_1 h_2 h_3} \frac{h_m}{\alpha_m} \hat{\mathbf{x}}_m \otimes \mathbf{r} \dot{\times} \tilde{\mathbf{H}}_1(\rho), \qquad m = 1,2,3 \tag{73}$$

$$\boldsymbol{\delta}_1 = -\frac{\alpha_1 \alpha_2 \alpha_3}{3(\Lambda - \Lambda')} \tilde{\boldsymbol{\Lambda}} \dot{\times} \tilde{\mathbf{F}}(\mathbf{r}), \tag{74}$$

$$\boldsymbol{\delta}_2 = \frac{\alpha_1 \alpha_2 \alpha_3}{3(\Lambda - \Lambda')} \tilde{\boldsymbol{\Lambda}}' \dot{\times} \tilde{\mathbf{F}}(\mathbf{r}), \tag{75}$$

$$\boldsymbol{\delta}_3 = \frac{\alpha_1 \alpha_2 \alpha_3}{h_1 h_2 h_3^2} \left[\frac{\alpha_2}{\alpha_1} \hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_2 + \frac{\alpha_1}{\alpha_2} \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_1 \right] \dot{\times} \tilde{\mathbf{F}}(\mathbf{r}), \tag{76}$$

$$\boldsymbol{\delta}_4 = \frac{\alpha_1 \alpha_2 \alpha_3}{h_1 h_2^2 h_3} \left[\frac{\alpha_3}{\alpha_1} \hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_3 + \frac{\alpha_1}{\alpha_3} \hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_1 \right] \dot{\times} \tilde{\mathbf{F}}(\mathbf{r}), \tag{77}$$

$$\boldsymbol{\delta}_5 = \frac{\alpha_1 \alpha_2 \alpha_3}{h_1^2 h_2 h_3} \left[\frac{\alpha_3}{\alpha_2} \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_3 + \frac{\alpha_2}{\alpha_3} \hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_2 \right] \dot{\times} \tilde{\mathbf{F}}(\mathbf{r}), \tag{78}$$

and the cross-dot product is defined as

$$(\mathbf{a} \otimes \mathbf{b}) \dot{\times} (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \times \mathbf{c})(\mathbf{b} \cdot \mathbf{d}). \tag{79}$$

Finally, the surface integral in (32) can be evaluated after (37), (59), (72), and (B8) are appropriately used to conclude

$$\int_s u^-(\mathbf{r}') \hat{\boldsymbol{\rho}}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} ds(\mathbf{r}') = \sum_{m=1}^{3} \zeta_m \boldsymbol{\beta}_m \gamma_1^m + \sum_{m=1}^{5} \theta_m \boldsymbol{\delta}_m \gamma_2^m + O(el_3), \tag{80}$$

where the constants γ_n^m are given by (B17)–(B20).

Next, we analyze the dipole terms in (80), which in view of (60), (73), and (B17) provide

$$\sum_{m=1}^{3} \zeta_m \boldsymbol{\beta}_m \gamma_1^m = \frac{3}{\sigma} \sum_{m=1}^{3} (\mathbf{Q}\cdot\hat{\mathbf{x}}_m)\hat{\mathbf{x}}_m \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho) = \frac{3}{\sigma} \mathbf{Q}\otimes\mathbf{r}\underset{\cdot}{\times}\tilde{\mathbf{H}}_1(\rho). \tag{81}$$

If we substitute the dipole contribution (81) into (32) and use expansion (55), we immediately see that the dipole contribution to the exterior magnetic field vanishes, a conclusion that is compatible with the theory of magnetostatics.[7,17]

In the sequel we investigate further the form of the leading (quadrupole) contribution to the exterior magnetic field as it is given by (32) and (80). Indeed, formulas (55)–(65), (68), (74)–(78), (80), and (B18)–(B20) yield

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) = {} & \frac{\mu_0}{4\pi}\mathbf{Q}\otimes\mathbf{r}_0 \underset{\cdot}{\times} \tilde{\mathbf{F}}(\mathbf{r}) + \frac{\mu_0}{12\pi}\frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{\Lambda-\Lambda'}\mathbf{Q}\otimes\mathbf{r}_0 : \tilde{\boldsymbol{\Lambda}}\otimes\tilde{\boldsymbol{\Lambda}}\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r}) \\
& - \frac{\mu_0}{12\pi}\frac{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}{\Lambda-\Lambda'}\mathbf{Q}\otimes\mathbf{r}_0 : \tilde{\boldsymbol{\Lambda}}'\otimes\tilde{\boldsymbol{\Lambda}}'\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r}) \\
& - \frac{\mu_0}{4\pi}\mathbf{Q}\otimes\mathbf{r}_0 : \frac{(\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_1)\otimes(\alpha_2^2\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_2+\alpha_1^2\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_1)}{\alpha_1^2+\alpha_2^2}\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r}) \\
& - \frac{\mu_0}{4\pi}\mathbf{Q}\otimes\mathbf{r}_0 : \frac{(\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_3+\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_1)\otimes(\alpha_3^2\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_3+\alpha_1^2\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_1)}{\alpha_1^2+\alpha_3^2}\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r}) \\
& - \frac{\mu_0}{4\pi}\mathbf{Q}\otimes\mathbf{r}_0 : \frac{(\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_3+\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_2)\otimes(\alpha_3^2\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_3+\alpha_2^2\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_2)}{\alpha_2^2+\alpha_3^2}\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r}) + O(el_3).
\end{aligned} \tag{82}$$

By means of the scalar identities

$$\frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{(\Lambda-\alpha_i^2)^2} = (\Lambda-\alpha_i^2) - \sum_{m=1}^{3}(\Lambda-\alpha_m^2), \quad i=1,2,3 \tag{83}$$

and

$$\frac{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}{(\Lambda'-\alpha_i^2)^2} = (\Lambda'-\alpha_i^2) - \sum_{m=1}^{3}(\Lambda'-\alpha_m^2), \quad i=1,2,3 \tag{84}$$

we can easily prove the tetradic formula

$$\begin{aligned}
& \frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{3(\Lambda-\Lambda')}\tilde{\boldsymbol{\Lambda}}\otimes\tilde{\boldsymbol{\Lambda}} - \frac{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}{3(\Lambda-\Lambda')}\tilde{\boldsymbol{\Lambda}}'\otimes\tilde{\boldsymbol{\Lambda}}' \\
& = \frac{1}{3}\tilde{\mathbf{I}}\otimes\tilde{\mathbf{I}} - \sum_{i=1}^{3}\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i .
\end{aligned} \tag{85}$$

Furthermore, the identities

$$\begin{aligned}
& \frac{(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)\otimes(\alpha_j^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\alpha_i^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)}{\alpha_i^2+\alpha_j^2} \\
& = \hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j + \hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i + \frac{(\alpha_i^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j-\alpha_j^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)\otimes(\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i-\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j)}{\alpha_i^2+\alpha_j^2}, \quad i\neq j
\end{aligned} \tag{86}$$

and

$$\hat{\mathbf{x}}_1\times\tilde{\mathbf{I}}=\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_3, \tag{87}$$

$$\hat{\mathbf{x}}_2\times\tilde{\mathbf{I}}=\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_3-\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_1, \tag{88}$$

$$\hat{\mathbf{x}}_3\times\tilde{\mathbf{I}}=\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_2, \tag{89}$$

can be used to reduce (82) to

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{12\pi}(\mathbf{Q}\cdot\mathbf{r}_0)\hat{\mathbf{I}}\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r})-\frac{\mu_0}{4\pi}(\mathbf{d}\times\tilde{\mathbf{I}})\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r})+O(el_3), \tag{90}$$

where

$$\mathbf{d}=(\mathbf{Q}\cdot\tilde{\mathbf{M}}(\alpha_1)\times\mathbf{r}_0)\cdot\tilde{\mathbf{N}}_1. \tag{91}$$

Some further algebra reveals that

$$\tilde{\mathbf{I}}\underset{\cdot}{\times}\tilde{\boldsymbol{\Lambda}}=\tilde{\mathbf{I}}\underset{\cdot}{\times}\tilde{\boldsymbol{\Lambda}}'=\mathbf{0}, \tag{92}$$

and

$$\tilde{\mathbf{I}}\underset{\cdot}{\times}(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)=\mathbf{0} \tag{93}$$

for every pair i, j with $i\neq j$ which in view of (68) implies that

$$\tilde{\mathbf{I}}\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r})=\mathbf{0}. \tag{94}$$

Finally, relation (B15) confirms that

$$(\mathbf{d}\times\tilde{\mathbf{I}})\underset{\cdot}{\times}\tilde{\boldsymbol{\Lambda}}=\mathbf{d}\cdot\tilde{\boldsymbol{\Lambda}}, \tag{95}$$

$$(\mathbf{d}\times\tilde{\mathbf{I}})\underset{\cdot}{\times}\tilde{\boldsymbol{\Lambda}}'=\mathbf{d}\cdot\tilde{\boldsymbol{\Lambda}}', \tag{96}$$

and since for $i\neq j$

$$(\mathbf{d}\times\tilde{\mathbf{I}})\underset{\cdot}{\times}(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)=\mathbf{d}\cdot(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i), \tag{97}$$

we arrive at the expression

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\mathbf{d}\cdot\left[\frac{\mathbb{F}_2^1(\mathbf{r})}{\Lambda-\Lambda'}\tilde{\boldsymbol{\Lambda}}-\frac{\mathbb{F}_2^2(\mathbf{r})}{\Lambda-\Lambda'}\tilde{\boldsymbol{\Lambda}}'-\frac{3}{h_1^2h_2^2h_3^2}\sum_{\substack{i,j=1\\ i\neq j}}^{3}h_ih_j\mathbb{F}_2^{i+j}(\mathbf{r})\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\right]+O(el_3), \tag{98}$$

which provides the quadrupole approximation of the magnetic field exterior to the ellipsoid (1). Note that **B** is a harmonic function, a property that follows from the fact that **B** is both irrotational and solenoidal in the exterior to the ellipsoid space.

The vector

$$\mathbf{Q}\cdot\tilde{\mathbf{M}}(\alpha_1)=\sum_{i=1}^{3}\alpha_i^2Q_i\hat{\mathbf{x}}_i, \tag{99}$$

represents the dipole moment as it is modified by the spatial effects of the anisotropy imposed by the ellipsoid. It actually incorporates the effects of the geometry on the physics of the problem.

Relations (C3)–(C5) can be invoked to rewrite (98) in Cartesian form as follows:

$$\mathbf{B}(\mathbf{r}) = \frac{5\mu_0}{4\pi}\frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{\Lambda-\Lambda'}(\mathbf{d}\cdot\tilde{\mathbf{\Lambda}})(\tilde{\mathbf{\Lambda}}:\mathbf{r}\otimes\mathbf{r}+1)I_2^1(\rho)$$
$$-\frac{5\mu_0}{4\pi}\frac{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}{\Lambda-\Lambda'}(\mathbf{d}\cdot\tilde{\mathbf{\Lambda}}')(\tilde{\mathbf{\Lambda}}':\mathbf{r}\otimes\mathbf{r}+1)I_2^2(\rho)$$
$$-\frac{15\mu_0}{4\pi}\mathbf{d}\cdot\sum_{\substack{i,j=1\\ i\neq j}}^{3} x_i x_j I_2^{i+j}(\rho)\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j + O(el_3), \tag{100}$$

where the elliptic integrals $I_n^m(\rho)$ are given by (B4) and

$$\frac{3}{h_1^2h_2^2h_3^2}\sum_{\substack{i,j=1\\ i\neq j}}^{3} h_i h_j \mathbb{F}_2^{i+j}(\mathbf{r})\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j = 15\sum_{\substack{i,j=1\\ i\neq j}}^{3} x_i x_j I_2^{i+j}(\rho)\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j = 15\tilde{\tilde{\mathbf{H}}}_2(\rho):\mathbf{r}\otimes\mathbf{r}. \tag{101}$$

As in the case of the electric potential, we also provide an expression of the magnetic field which avoids the use of polyadics. Indeed, if

$$\mathbf{d} = \frac{\alpha_2^2Q_2x_{03}-\alpha_3^2Q_3x_{02}}{\alpha_2^2+\alpha_3^2}\hat{\mathbf{x}}_1 + \frac{\alpha_3^2Q_3x_{01}-\alpha_1^2Q_1x_{03}}{\alpha_1^2+\alpha_3^2}\hat{\mathbf{x}}_2 + \frac{\alpha_1^2Q_1x_{02}-\alpha_2^2Q_2x_{01}}{\alpha_1^2+\alpha_2^2}\hat{\mathbf{x}}_3, \tag{102}$$

then

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbb{F}_2^1(\rho,\mu,\nu)}{\Lambda-\Lambda'}\sum_{i=1}^{3}\frac{d_i}{\Lambda-\alpha_i^2}\hat{\mathbf{x}}_i - \frac{\mu_0}{4\pi}\frac{\mathbb{F}_2^2(\rho,\mu,\nu)}{\Lambda-\Lambda'}\sum_{i=1}^{3}\frac{d_i}{\Lambda'-\alpha_i^2}\hat{\mathbf{x}}_i$$
$$-\frac{15\mu_0}{4\pi}\sum_{\substack{i,j=1\\ i\neq j}}^{3} d_i x_i x_j I_2^{i+j}(\rho)\hat{\mathbf{x}}_j + O\left(\frac{1}{\rho^4}\right). \tag{103}$$

V. REDUCTION TO THE SPHERE

The magnetic field outside a spherical conductor[23] is given by

$$\mathbf{B}_s(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q}\times\mathbf{r}_0\cdot[\tilde{\mathbf{I}}+\mathbf{r}\otimes\nabla]\frac{1}{F(\mathbf{r})}, \tag{104}$$

where

$$F(\mathbf{r}) = r|\mathbf{r}-\mathbf{r}_0|^2 + \mathbf{r}\cdot(\mathbf{r}-\mathbf{r}_0)|\mathbf{r}-\mathbf{r}_0|. \tag{105}$$

In the interest of obtaining the quadrupole term of (104) we have to expand asymptotically (104) and calculate the leading terms of this expansion. This program furnishes

$$\frac{1}{F(\mathbf{r})} = \frac{1}{2r^3} + \frac{\mathbf{r}_0\cdot\hat{\mathbf{r}}}{r^4} + O\left(\frac{1}{r^5}\right), \tag{106}$$

$$\nabla\frac{1}{F(\mathbf{r})} = -\frac{3\hat{\mathbf{r}}}{2r^4} + \frac{\mathbf{r}_0}{r^5}\cdot(\tilde{\mathbf{I}}-5\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}) + O\left(\frac{1}{r^6}\right), \tag{107}$$

and finally

$$\mathbf{B}_s(\mathbf{r}) = \frac{\mu_0}{8\pi}\mathbf{Q}\times\mathbf{r}_0\cdot\frac{\tilde{\mathbf{I}}-3\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{r^3} + \frac{\mu_0}{4\pi}\mathbf{Q}\times\mathbf{r}_0\cdot\frac{\tilde{\mathbf{I}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\tilde{\mathbf{I}}-5\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{r^4}\cdot\mathbf{r}_0 + O\left(\frac{1}{r^5}\right). \tag{108}$$

We turn now to our ellipsoidal result (98) or (100) and we consider the limit as

$$(\alpha_1,\alpha_2,\alpha_3)\underset{e\to s}{\longrightarrow}(\alpha,\alpha,\alpha), \tag{109}$$

where $e\to s$ indicates the limit as the ellipsoid becomes a sphere of radius α.

It is easily shown that

$$\lim_{e\to s}\Lambda=\lim_{e\to s}\Lambda'=\alpha^2, \tag{110}$$

$$\lim_{e\to s}h_i=0,\quad i=1,2,3 \tag{111}$$

$$\lim_{e\to s}\mu=\lim_{e\to s}\nu=0, \tag{112}$$

and

$$\lim_{e\to s}\rho=r, \tag{113}$$

where r denotes the spherical radial variable.

Furthermore

$$\lim_{e\to s}I_2^m(\rho)=\frac{1}{5r^5},\quad m=1,2,3,4,5. \tag{114}$$

The last term on the right-hand side of (100) is continuous in the spherical limit, and it provides the limit

$$\lim_{e\to s}\sum_{\substack{i,j=1\\ i\neq j}}^{3}x_ix_jI_2^{i+j}(\rho)\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j=\frac{1}{5r^5}\sum_{\substack{i,j=1\\ i\neq j}}^{3}x_ix_j\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j=\frac{\mathbf{r}\otimes\mathbf{r}}{5r^5}-\frac{1}{5r^5}\sum_{i=1}^{3}x_i^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i\,. \tag{115}$$

On the other hand, the corresponding limit of the first two terms on the right-hand side of (100) exhibit an indeterminant behavior and they need to be handled in the following special way.

Long but straightforward calculations are needed to prove the identities

$$\frac{\mathbb{E}_2^1(\rho,\mu,\nu)}{(\Lambda-\Lambda')(\Lambda-\alpha_m^2)}-\frac{\mathbb{E}_2^2(\rho,\mu,\nu)}{(\Lambda-\Lambda')(\Lambda'-\alpha_m^2)}=r^2-3x_m^2+\alpha_m^2-\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2}{3}, \tag{116}$$

and

$$\frac{I_2^1(\rho)-I_2^2(\rho)}{\Lambda-\Lambda'}=\frac{3}{2}I_3^7(\rho)-\frac{1}{2\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}(\Lambda-\alpha_1^2+\rho^2)(\Lambda'-\alpha_1^2+\rho^2)}, \tag{117}$$

where

$$I_3^7(\rho)=\int_\rho^{+\infty}\frac{dt}{t^2(t^2-h_2^2)^{3/2}(t^2-h_3^2)^{3/2}}. \tag{118}$$

Then (116) and (117) are used to show that

$$
\begin{aligned}
&\frac{\mathbb{E}_2^1(\rho,\mu,\nu)I_2^1(\rho)}{(\Lambda-\Lambda')(\Lambda-\alpha_m^2)}-\frac{\mathbb{E}_2^2(\rho,\mu,\nu)I_2^2(\rho)}{(\Lambda-\Lambda')(\Lambda'-\alpha_m^2)} \\
&\quad=\left(r^2-3x_m^2+\alpha_m^2-\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2}{3}\right)I_2^1(\rho) \\
&\qquad+\frac{\mathbb{E}_2^2(\rho,\mu,\nu)}{\Lambda'-\alpha_m^2}\left[\frac{3}{2}I_2^7(\rho)-\frac{1}{2\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}(\Lambda-\alpha_1^2+\rho^2)(\Lambda'-\alpha_1^2+\rho^2)}\right].
\end{aligned}
\tag{119}
$$

Furthermore

$$\frac{\mathbb{E}_2^2(\rho,\mu,\nu)}{\Lambda'-\alpha_1^2}=(\Lambda'-\alpha_2^2)(x_3^2-x_1^2)+(\Lambda'-\alpha_3^2)(x_2^2-x_1^2)+(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2), \tag{120}$$

$$\frac{\mathbb{E}_2^2(\rho,\mu,\nu)}{\Lambda'-\alpha_2^2}=(\Lambda'-\alpha_1^2)(x_3^2-x_2^2)+(\Lambda'-\alpha_3^2)(x_1^2-x_2^2)+(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_3^2), \tag{121}$$

$$\frac{\mathbb{E}_2^2(\rho,\mu,\nu)}{\Lambda'-\alpha_3^2}=(\Lambda'-\alpha_1^2)(x_2^2-x_3^2)+(\Lambda'-\alpha_2^2)(x_1^2-x_3^2)+(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2), \tag{122}$$

which imply that

$$\lim_{e\to s}\frac{\mathbb{E}_2^2(\rho,\mu,\nu)}{\Lambda'-\alpha_m^2}=0, \quad m=1,2,3. \tag{123}$$

Consequently

$$\lim_{e\to s}\left[\frac{\mathbb{E}_2^1(\rho,\mu,\nu)I_2^1(\rho)}{(\Lambda-\Lambda')(\Lambda-\alpha_m^2)}-\frac{\mathbb{E}_2^2(\rho,\mu,\nu)I_2^2(\rho)}{(\Lambda-\Lambda')(\Lambda'-\alpha_m^2)}\right]=\frac{r^2-3x_m^2}{5r^5}, \tag{124}$$

and

$$\lim_{e\to s}\left[\frac{\mathbb{F}_2^1(\mathbf{r})}{\Lambda-\Lambda'}\tilde{\mathbf{\Lambda}}-\frac{\mathbb{F}_2^2(\mathbf{r})}{\Lambda-\Lambda'}\tilde{\mathbf{\Lambda}}'\right]=5\sum_{m=1}^{3}\frac{r^2-3x_m^2}{5r^5}\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m=\frac{\tilde{\mathbf{I}}}{r^3}-\frac{3}{r^5}\sum_{m=1}^{3}x_m^2\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m. \tag{125}$$

By virtue of the reduction formulas (115) and (125), expression (98) provides the limit

$$
\begin{aligned}
\lim_{e\to s}\mathbf{B}(\mathbf{r})&=\frac{\mu_0}{4\pi}(\lim_{e\to s}\mathbf{d})\cdot\left[\frac{\tilde{\mathbf{I}}}{r^3}-\frac{3}{r^5}\sum_{m=1}^{3}x_m^2\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m\right]-\frac{\mu_0}{4\pi}(\lim_{e\to s}\mathbf{d})\cdot\left[3\frac{\mathbf{r}\otimes\mathbf{r}}{r^5}-\frac{3}{r^5}\sum_{m=1}^{3}x_m^2\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m\right] \\
&=\frac{\mu_0}{8\pi}\mathbf{Q}\times\mathbf{r}_0\cdot\frac{\tilde{\mathbf{I}}-3\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{r^3}+O\left(\frac{1}{r^4}\right)=\mathbf{B}_s(\mathbf{r}),
\end{aligned}
\tag{126}
$$

where $\mathbf{B}_s$ is the sphere solution given by (108).

APPENDIX A: THE ELLIPSOIDAL SYSTEM

The ellipsoidal coordinates (ρ,μ,ν) are connected to the Cartesian coordinates (x_1,x_2,x_3) via the expressions

$$x_1=\frac{\rho\mu\nu}{h_2h_3}, \tag{A1}$$

$$x_2=\frac{\sqrt{\rho^2-h_3^2}\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}}{h_1h_3}, \tag{A2}$$

$$x_3=\frac{\sqrt{\rho^2-h_2^2}\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}}{h_1h_2}, \tag{A3}$$

where the variable $\rho\in[h_2,+\infty)$ specifies the ellipsoid

$$\frac{x_1^2}{\rho^2}+\frac{x_2^2}{\rho^2-h_3^2}+\frac{x_3^2}{\rho^2-h_2^2}=1, \tag{A4}$$

the variable $\mu\in[h_3,h_2]$ specifies the hyperboloid of one sheet

$$\frac{x_1^2}{\mu^2}+\frac{x_2^2}{\mu^2-h_3^2}-\frac{x_3^2}{h_2^2-\mu^2}=1, \tag{A5}$$

and the variable $\nu\in[-h_3,h_3]$ specifies the hyperboloid of two sheets

$$\frac{x_1^2}{\nu^2}-\frac{x_2^2}{h_3^2-\nu^2}-\frac{x_3^2}{h_2^2-\nu^2}=1. \tag{A6}$$

The three families of second-degree surfaces (A4), (A5), (A6) share the same set of foci at the points $\pm h_1,\ \pm h_2,\ \pm h_3$. The outward unit normal on the ellipsoid $\rho=\alpha_1$ is given by

$$\frac{\partial}{\partial n}=\hat{\boldsymbol{\rho}}\cdot\nabla=\frac{\alpha_2\alpha_3}{\sqrt{\alpha_1^2-\mu^2}\sqrt{\alpha_1^2-\nu^2}}\frac{\partial}{\partial\rho}. \tag{A7}$$

APPENDIX B: ELLIPSOIDAL HARMONICS

Separation of variables for the Laplace's equation in the ellipsoidal coordinate system leads to the Lamé equation[15]

$$(x^2-h_2^2)(x^2-h_3^2)E''(x)+x(2x^2-h_2^2-h_3^2)E'(x)+(Ax^2+B)E(x)=0, \tag{B1}$$

for each one of the factors $E(\rho)$, $E(\mu)$, and $E(\nu)$ that form the harmonic function

$$\mathbb{E}(\rho,\mu,\nu)=E(\rho)E(\mu)E(\nu), \tag{B2}$$

where A and B are constants.

The only difference between these functions is that $E(\rho)$ satisfies (B1) in the interval $[h_2,+\infty)$, $E(\mu)$ in the interval $[h_2,h_3]$, and $E(\nu)$ in the interval $[-h_3,h_3]$.

Lamé equation (B1) has a long history that dominated differential equations the whole of the 19th century.[15] A complicated analysis shows that the constants A and B are appropriately associated with two integers n and m, where just like the spherical harmonics, n specifies the degree, and m specifies the order, of the different Lamé functions of the same degree. For each $n=0,1,2,...$ and each $m=1,2,...,2n+1$ Eq. (B1) has two linearly independent solutions, one regular at the origin and one regular at infinity. For fixed values of n and m, a solution of Eq. (B1) is called a Lamé function of degree n and order m. In particular, the solution E_n^m that is regular at the origin is the Lamé function of the first kind (interior solution), while the solution F_n^m that is regular at infinity is the Lamé function of the second kind (exterior solution). The interior solutions $E_n^m(\rho)$ are connected to the exterior solutions $F_n^m(\rho)$ via the expression

$$F_n^m(\rho)=(2n+1)E_n^m(\rho)I_n^m(\rho), \tag{B3}$$

where the elliptic integrals $I_n^m(\rho)$ are given by

$$I_n^m(\rho)=\int_\rho^{+\infty}\frac{dt}{[E_n^m(t)]^2\sqrt{t^2-h_2^2}\sqrt{t^2-h_3^2}}=\frac{1}{2}\int_{\rho^2-\alpha_1^2}^{+\infty}\frac{dx}{[E_n^m(\sqrt{x+\alpha_1^2})]^2\sqrt{x+\alpha_1^2}\sqrt{x+\alpha_2^2}\sqrt{x+\alpha_3^2}}. \tag{B4}$$

The Lamé products

$$\mathbb{E}_n^m(\rho,\mu,\nu)=E_n^m(\rho)E_n^m(\mu)E_n^m(\nu), \tag{B5}$$

define the interior solid ellipsoidal harmonics and the Lamé products

$$\mathbb{F}_n^m(\rho,\mu,\nu)=F_n^m(\rho)E_n^m(\mu)E_n^m(\nu)=(2n+1)\mathbb{E}_n^m(\rho,\mu,\nu)I_n^m(\rho), \tag{B6}$$

define the exterior solid ellipsoidal harmonics. The surface ellipsoidal harmonics are defined by the product $E_n^m(\mu)E_n^m(\nu)$ and they form a complete orthogonal set of ''angular'' eigenfunctions on the surface of any ellipsoid from the confocal family (A4). In fact, the orthogonality is defined via the weighting function

$$l(\mu,\nu)=[(\rho_0^2-\mu^2)(\rho_0^2-\nu^2)]^{-1/2}, \tag{B7}$$

on the ellipsoid $\rho=\rho_0$ and provides the relations

$$\int\int_{\rho=\rho_0} E_n^m(\mu)E_n^m(\nu)E_{n'}^{m'}(\mu)E_{n'}^{m'}(\nu)l(\mu,\nu)ds=0, \tag{B8}$$

unless $n=n'$ and $m=m'$, in which case the normalization constants γ_n^m are given by (37).

Although the form of the ellipsoidal harmonics is known, the exact values of the parameters they involve are not expressed in terms of the semiaxes $\alpha_1,\alpha_2,\alpha_3$ when the degree n is higher than 3. This difficulty restricts the analytical solutions of related boundary value problems to the 16th-dimensional harmonic subspace spanned by the harmonics of degree less than or equal to 3.[15] The needs of the present work are restricted to the ellipsoidal harmonics of degree less than or equal to 2 which are given explicitly below. These harmonics are enough to obtain an analytic expression for the dipole as well as for the quadrupole term for the exterior magnetic field.

The interior Lamé functions of degree less than or equal to 2 are given by

$$E_0^1(x)=1, \tag{B9}$$

$$E_1^m(x)=\sqrt{|x^2-\alpha_1^2+\alpha_m^2|},\qquad m=1,2,3 \tag{B10}$$

$$E_2^1(x)=x^2-\alpha_1^2+\Lambda, \tag{B11}$$

$$E_2^2(x)=x^2-\alpha_1^2+\Lambda' \tag{B12}$$

$$E_2^{6-m}(x)=\frac{E_1^1(x)E_1^2(x)E_1^3(x)}{E_1^m(x)},\qquad m=1,2,3 \tag{B13}$$

where the constants

$$\left.\begin{matrix}\Lambda\\ \Lambda'\end{matrix}\right\}=\frac{1}{3}\sum_{n=1}^{3}\alpha_n^2\pm\frac{1}{3}\sqrt{\sum_{n=1}^{3}\left(\alpha_n^4-\frac{\alpha_1^2\alpha_2^2\alpha_3^2}{\alpha_n^2}\right)},\tag{B14}$$

are the two roots of the quadradic equation

$$\sum_{n=1}^{3}\frac{1}{\Lambda-\alpha_n^2}=0.\tag{B15}$$

Once the interior Lamé functions are known, the corresponding exterior ones are obtained via formulas (B3), (B4). Interior and exterior solid ellipsoidal harmonics are then constructed via (B5) and (B6), respectively. Finally, the normalization constants used in the present work are the following:

$$\gamma_0^1=4\pi,\tag{B16}$$

$$\gamma_1^m=\frac{4\pi}{3}\frac{h_1^2h_2^2h_3^2}{h_m^2},\qquad m=1,2,3\tag{B17}$$

$$\gamma_2^1=-\frac{8\pi}{5}(\Lambda-\Lambda')(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2),\tag{B18}$$

$$\gamma_2^2=\frac{8\pi}{5}(\Lambda-\Lambda')(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2),\tag{B19}$$

and

$$\gamma_2^{6-m}=\frac{4\pi}{15}h_1^2h_2^2h_3^2h_m^2,\qquad m=1,2,3.\tag{B20}$$

It can be shown that all γ_n^m's are positive as formula (36) demands. The following relations express the gradients of ellipsoidal harmonics in terms of ellipsoidal harmonics as well:

$$\nabla\mathbb{E}_0^1(\rho,\mu,\nu)=0,\tag{B21}$$

$$\nabla\mathbb{E}_1^m(\rho,\mu,\nu)=\frac{h_1h_2h_3}{h_m}\hat{\mathbf{x}}_m,\qquad m=1,2,3\tag{B22}$$

where $\hat{\mathbf{x}}_m$ stand for the Cartesian orthonormal basis, and

$$\nabla\mathbb{E}_2^1(\rho,\mu,\nu)=2\,\frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{h_1h_2h_3}\sum_{m=1}^{3}\frac{h_m}{\Lambda-\alpha_m^2}\mathbb{E}_1^m(\rho,\mu,\nu)\hat{\mathbf{x}}_m,\tag{B23}$$

$$\nabla\mathbb{E}_2^2(\rho,\mu,\nu)=2\,\frac{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}{h_1h_2h_3}\sum_{m=1}^{3}\frac{h_m}{\Lambda'-\alpha_m^2}\mathbb{E}_1^m(\rho,\mu,\nu)\hat{\mathbf{x}}_m,\tag{B24}$$

$$\nabla\mathbb{E}_2^3(\rho,\mu,\nu)=h_1h_2h_3\left[\frac{1}{h_1}\mathbb{E}_1^2(\rho,\mu,\nu)\hat{\mathbf{x}}_1+\frac{1}{h_2}\mathbb{E}_1^1(\rho,\mu,\nu)\hat{\mathbf{x}}_2\right],\tag{B25}$$

$$\nabla\mathbb{E}_2^4(\rho,\mu,\nu)=h_1h_2h_3\left[\frac{1}{h_1}\mathbb{E}_1^3(\rho,\mu,\nu)\hat{\mathbf{x}}_1+\frac{1}{h_3}\mathbb{E}_1^1(\rho,\mu,\nu)\hat{\mathbf{x}}_3\right],\tag{B26}$$

$$\nabla \mathbb{E}_2^5(\rho,\mu,\nu) = h_1 h_2 h_3 \left[\frac{1}{h_2} \mathbb{E}_1^3(\rho,\mu,\nu) \hat{\mathbf{x}}_2 + \frac{1}{h_3} \mathbb{E}_1^2(\rho,\mu,\nu) \hat{\mathbf{x}}_3 \right]. \tag{B27}$$

APPENDIX C: CONNECTION FORMULAS

The solid ellipsoidal harmonics are expressed in terms of Cartesian coordinates as follows:[5]

$$\mathbb{E}_0^1(\rho,\mu,\nu) = 1, \tag{C1}$$

$$\mathbb{E}_1^m(\rho,\mu,\nu) = \frac{h_1 h_2 h_3}{h_m} x_m, \qquad m = 1,2,3 \tag{C2}$$

$$\mathbb{E}_2^1(\rho,\mu,\nu) = (\Lambda - \alpha_1^2)(\Lambda - \alpha_2^2)(\Lambda - \alpha_3^2) \left(\sum_{n=1}^{3} \frac{x_n^2}{\Lambda - \alpha_n^2} + 1 \right), \tag{C3}$$

$$\mathbb{E}_2^2(\rho,\mu,\nu) = (\Lambda' - \alpha_1^2)(\Lambda' - \alpha_2^2)(\Lambda' - \alpha_3^2) \left(\sum_{n=1}^{3} \frac{x_n^2}{\Lambda' - \alpha_n^2} + 1 \right), \tag{C4}$$

$$\mathbb{E}_2^{6-m}(\rho,\mu,\nu) = h_1 h_2 h_3 h_m \frac{x_1 x_2 x_3}{x_m}, \quad m = 1,2,3. \tag{C5}$$

Furthermore, the Cartesian monomials of degree less than or equal to 2 are expressed in terms of surface ellipsoidal harmonics as follows:

$$1 = \mathbb{E}_0^1(\rho,\mu,\nu), \tag{C6}$$

$$x_m = \frac{h_m}{h_1 h_2 h_3} \mathbb{E}_1^m(\rho,\mu,\nu), \quad m = 1,2,3 \tag{C7}$$

$$x_m^2 = \frac{\rho^2 - \alpha_1^2 + \alpha_m^2}{3} \left[1 - \frac{E_2^1(\mu) E_2^1(\nu)}{(\Lambda - \Lambda')(\Lambda - \alpha_m^2)} + \frac{E_2^2(\mu) E_2^2(\nu)}{(\Lambda - \Lambda')(\Lambda' - \alpha_m^2)} \right], \quad m = 1,2,3 \tag{C8}$$

$$\frac{x_1 x_2 x_3}{x_m} = \frac{1}{h_1 h_2 h_3 h_m} \mathbb{E}_2^{6-m}(\rho,\mu,\nu), \quad m = 1,2,3. \tag{C9}$$

APPENDIX D: USEFUL RELATIONS

The constants Λ, Λ' given in (B14), the semifocal distances h_1, h_2, h_3 given by (4), and the semiaxes $\alpha_1, \alpha_2, \alpha_3$, satisfy the following useful expressions:

$$3(\Lambda + \Lambda') = 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2), \tag{D1}$$

$$3\Lambda\Lambda' = \alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2, \tag{D2}$$

$$\sum_{n=1}^{3} (-1)^n (\Lambda - \alpha_n^2) h_n^2 = \sum_{n=1}^{3} (-1)^n (\Lambda' - \alpha_n^2) h_n^2 = 0, \tag{D3}$$

$$\sum_{n=1}^{3} (-1)^n (\Lambda - \alpha_n^2) h_n^2 \alpha_n^2 = \sum_{n=1}^{3} (-1)^n (\Lambda' - \alpha_n^2) h_n^2 \alpha_n^2 = h_1^2 h_2^2 h_3^2, \tag{D4}$$

$$\sum_{n=1}^{3} \frac{\alpha_n^2}{\alpha_n^2-\Lambda}=\sum_{n=1}^{3} \frac{\alpha_n^2}{\alpha_n^2-\Lambda'}=3, \tag{D5}$$

$$3h_n^2(\Lambda-\alpha_n^2)(\Lambda'-\alpha_n^2)=(-1)^{n+1}h_1^2h_2^2h_3^2, \tag{D6}$$

for each $n=1,2,3$.

The elliptic integrals that enter the exterior ellipsoidal harmonics $\mathbb{F}_n^m$, $n\leqslant 2$ are connected via the following relations:

$$I_1^1(\rho)+I_1^2(\rho)+I_1^3(\rho)=\frac{1}{\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}, \tag{D7}$$

$$\alpha_1^2 I_1^1(\rho)+\alpha_2^2 I_1^2(\rho)+\alpha_3^2 I_1^3(\rho)=I_0^1(\rho)-\frac{\rho^2-\alpha_1^2}{\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}, \tag{D8}$$

$$I_2^1(\rho)=\frac{1}{2(\Lambda-\alpha_1^2+\rho^2)\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}-\frac{1}{2}\left(\frac{I_1^1(\rho)}{\Lambda-\alpha_1^2}+\frac{I_1^2(\rho)}{\Lambda-\alpha_2^2}+\frac{I_1^3(\rho)}{\Lambda-\alpha_3^2}\right), \tag{D9}$$

$$I_2^2(\rho)=\frac{1}{2(\Lambda'-\alpha_1^2+\rho^2)\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}-\frac{1}{2}\left(\frac{I_1^1(\rho)}{\Lambda'-\alpha_1^2}+\frac{I_1^2(\rho)}{\Lambda'-\alpha_2^2}+\frac{I_1^3(\rho)}{\Lambda'-\alpha_3^2}\right), \tag{D10}$$

$$I_2^3(\rho)=\frac{1}{h_3^2}(I_1^2(\rho)-I_1^1(\rho)), \tag{D11}$$

$$I_2^4(\rho)=\frac{1}{h_2^2}(I_1^3(\rho)-I_1^1(\rho)), \tag{D12}$$

$$I_2^5(\rho)=\frac{1}{h_1^2}(I_1^3(\rho)-I_1^2(\rho)). \tag{D13}$$

The expressions can be established through long and tedious manipulations and they actually show that among the nine integrals $I_n^m(\rho)$ with $n\leqslant 2$, only two are independent. For instance, if $I_0^1(\rho)$ and $I_1^1(\rho)$ are known the other seven integrals can be expressed, via (D7)–(D13), in terms of these two.

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