Stokes flow in ellipsoidal geometry

Panayiotis Vafeas and George Dassios

Citation: Journal of Mathematical Physics **47**, 093102 (2006); doi: 10.1063/1.2345474
View online: http://dx.doi.org/10.1063/1.2345474
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/47/9?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **47**, 093102 (2006)

Stokes flow in ellipsoidal geometry

Panayiotis Vafeas[a)]
Division of Applied Mathematics & Mechanics, Department of Engineering Sciences, University of Patras, GR-265 04 Patras, Greece

George Dassios[b)]
DAMTP, University of Cambridge, Cambridge, United Kingdom

(Received 26 June 2006; accepted 4 August 2006; published online 28 September 2006)

Particle-in-cell models for Stokes flow through a relatively homogeneous swarm of particles are of substantial practical interest, because they provide a relatively simple platform for the analytical or semianalytical solution of heat and mass transport problems. Despite the fact that many practical applications involve relatively small particles (inorganic, organic, biological) with axisymmetric shapes, the general consideration consists of rigid particles of arbitrary shape. The present work is concerned with some interesting aspects of the theoretical analysis of creeping flow in ellipsoidal, hence nonaxisymmetric domains. More specifically, the low Reynolds number flow of a swarm of ellipsoidal particles in an otherwise quiescent Newtonian fluid, that move with constant uniform velocity in an arbitrary direction and rotate with an arbitrary constant angular velocity, is analyzed with an ellipsoid-in-cell model. The solid internal ellipsoid represents a particle of the swarm. The external ellipsoid contains the ellipsoidal particle and the amount of fluid required to match the fluid volume fraction of the swarm. The nonslip flow condition on the surface of the solid ellipsoid is supplemented by the boundary conditions on the external ellipsoidal surface which are similar to those of the sphere-in-cell model of Happel (self-sufficient in mechanical energy). This model requires zero normal velocity component and shear stress. The boundary value problem is solved with the aim of the potential representation theory. In particular, the Papkovich–Neuber complete differential representation of Stokes flow, valid for nonaxisymmetric geometries, is considered here, which provides the velocity and total pressure fields in terms of harmonic ellipsoidal eigenfunctions. The flexibility of the particular representation is demonstrated by imposing some conditions, which made the calculations possible. It turns out that the velocity of first degree, which represents the leading term of the series, is sufficient for most engineering applications, so long as the aspect ratios of the ellipsoids remains within moderate bounds. Analytical expressions for the leading terms of the velocity, the total pressure, the angular velocity, and the stress tensor fields are obtained. Corresponding results for the prolate and the oblate spheroid, the needle and the disk, as well as for the sphere are recovered as degenerate cases. Novel relations concerning the ellipsoidal harmonics are included in the Appendix. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2345474]

I. INTRODUCTION

The behavior of systems involving the motion of aggregates of small particles relative to viscous fluids, in which they are immersed, covers a wide range of heat and mass transfer phe-

[a)]Electronic-mail: vafeas@des.upatras.gr
[b)]On leave from: Division of Applied Mathematics, Department of Chemical Engineering, University of Patras and ICE/HT-FORTH, GR-265 04 Patras, Greece; electronic-mail: gdassios@chemeng.upatras.gr

nomena of great importance in practical applications. In order to construct tractable mathematical models of the flow systems involving particles, it is necessary to resort to a number of simplifications. A dimensionless criterion, which determines the relative importance of inertial and viscous effects, is the Reynolds number.[1] The steady and creeping flow (low Reynolds number) of an incompressible, viscous fluid is described by the well-known Stokes equations which have been known for over one and a half centuries (1851) and connect the biharmonic vector velocity with the harmonic scalar pressure field.[1] For many interior and exterior flow problems involving small particles, spherical and spheroidal (either prolate or oblate) geometry[2] provides a very good approximation for many important applications where the flow is considered to be axisymmetric. Nevertheless, more realistic models include particles of arbitrary shape where it is impossible to take advantage of the symmetry and the orientation of the particle must be taken into account. Ellipsoidal geometry[2] provides the most widely used framework for representing small particles of arbitrary shape embedded within a fluid that flows according to Stokes law. This nonaxisymmetric flow is governed by the genuine three-dimensional (3D) ellipsoidal geometry.

The introduction of a stream function[1] serves to unify the method on all two-dimensional incompressible fluid motions, as we shall see in the historical comments that follow. For these situations, the solutions of the equations of motion are reduced to the search for a single scalar function. Unfortunately, in the general case of three-dimensional motions this unified method of approach is not possible. Of course, it should be noted here that fully 3D flow could also be represented by a *pair* of stream functions, in which case the streamlines are the intersections of the two families that correspond to the two stream functions. However, this causes a number of difficulties and the appearance of certain indeterminacies in the solution of physical problems forces us to look for more general approaches. In order to avoid such problems the introduction of three-dimensional representations[3] of the flow fields is necessary. This is probably due to the fact that these representations use more than one potential to represent the physical fields, allowing for more flexibility. Papkovich (1932) and Neuber (1934) proposed a differential representation of the flow fields in terms of harmonic functions.[3,4] Their representation holds true also for nonaxisymmetric problems and is derivable from the well-known Naghdi–Hsu solution.[5] Since the last one is proved to be complete,[3] the Papkovich–Neuber differential representation also forms a complete solution of Stokes equations.

One of the important areas of applications concern the construction of particle-in-cell models which are useful in the development of simple but reliable analytical expressions for heat and mass transfer in swarms of particles in the case of concentrated suspensions. In applied type analysis it is not usually necessary to have detailed solution of the flow field over the entire swarm of particles taking into account the exact positions of the particles, since such solutions are cumbersome to use. Thus, the technique of cell models is adopted where the mathematical treatment of each problem is based on the assumption that a three-dimensional assemblage may be considered to consist of a number of identical unit cells. Each of these cells contains a particle surrounded by a fluid envelope, containing a volume of fluid sufficient to make the fractional void volume in the cell identical to that in the entire assemblage.

Of course, a good approximation of the solution of flow through a swarm of particles is also possible with the help of powerful numerical methods, notably Stokesian dynamics[6] or lattice—Boltzmann simulation.[7] In addition, Refs. 8 and 9 provide an excellent exposure of numerical approaches for the solution of physical problems concerning ellipsoidal particles. However, these methods involve the use of elaborate computer codes for each case. There is always room and need for analytical methods, which capture the essential features of the transport process under consideration in an analytic formula. It is to this end that particle-in-cell flow models serve as platforms both for theoretical investigation and for checking the reliability of complicated numerical codes.

Uchida[10] proposed a cell model where a spherical particle is surrounded by a fluid envelope with cubic outer boundary. The cubic shape offers the advantage that it is space filling but the nature of the boundaries leads to a three-dimensional flow problem.

Happel[11] and Kuwabara[12] proposed cell models in which both the particle and the outer envelope are spherical, having the significant advantage of preserving the axial symmetry of the flow and of providing a simple analytical solution in closed form. On the other hand, their models have the disadvantage that the outer envelope is not space filling, a difficulty that must be dealt with, when one tries to pass from the single unit cell to the assemblage of particles. The Happel and the Kuwabara formulations are slightly different in the sense that the Happel model assumes that the inner sphere moves with a constant velocity along the axis within a quiescent fluid, while the Kuwabara model assumes that the inner sphere is stationary and that the fluid passes through the unit cell with a constant velocity. Under the assumption of pseudosteady state, this difference effects the boundary conditions of each formulation. Hence, for the Happel model a nonslip flow condition in the inner sphere is imposed, and zero radial velocity and shear stress on the outer envelope. On the other hand, for the Kuwabara formulation the radial and the tangential velocity are assumed to be equal to zero on the inner sphere, while there exists a velocity with axial component equal to a constant approach velocity on the outer envelope. In addition, zero vorticity on the outer envelope is assumed. Despite the fact that both formulations give essentially the same velocity fields, with the appropriate change of frame of reference, it is the Happel model that is slightly superior. This is due to the fact that it does not require an exchange of mechanical energy between the cell and the environment. On the contrary, the Kuwabara model permits a small but significant exchange of mechanical energy with the environment.

Neale and Nader[13] improved the formulation of Happel and Kuwabara by considering that the unit cell under consideration is embedded in an unbounded, continuous, homogeneous, and isotropic permeable medium, which has the same permeability with that of the swarm of spheres. The Happel and the Kuwabara models also provide good agreement, but somewhat inferior to the Neale and Nader model.

Epstein and Masliyah[14] proposed a useful generalization by considering a spheroid-in-cell, instead of a sphere-in-cell, model for swarms of spheroidal particles. However, they had to solve the creeping flow problem numerically since the well-known equation of motion $E^4\psi=0$ (ψ: Stokes stream function) in spheroidal coordinates is not separable. This difficulty of nonseparation was resolved recently by Dassios *et al.*[15] with the use of semiseparation of solutions, which is based on an appropriate finite dimensional spectral decomposition of the operator E^4. This way an analytical solution for the Kuwabara model was obtained. Using this method, Dassios *et al.*[16] solved the Happel model in spheroidal coordinates analytically and the results were compared with those obtained by using the Kuwabara-type boundary conditions. Moreover, the problem of space filling, when we refer to the assemblage of particles as well as to the relation between the approach velocity and the mean interstitial velocity through the swarm, was discussed in detail in Ref. 16.

An indeterminacy appears when the Happel-type or the Kuwabara-type spheroidal models are solved in terms of the Stokes stream function. This indeterminacy does *not* appear in the case of a perfect sphere. The indeterminacy can be overcome through the imposition of an additional geometrical condition that secures the correct reduction to the perturbed-sphere case.[15,16] However, the introduction of the Papkovich–Neuber differential representation,[3,4] as we already mentioned, seems to offer certain advantages in solving such problems. More specific, despite the fact that a similar indeterminacy appears when the Papkovich–Neuber differential representation is used and it is handled in the same way, the degrees of freedom, which this representation offers,[17] make the Papkovich–Neuber solution a powerful tool. This representation was demonstrated recently by Dassios and Vafeas,[18] where the Kuwabara model is solved analytically in spheroidal domains and a comparison with the already solved flow problem with Kuwabara-type boundary conditions was made. The major advantage of the utilization of the Papkovich–Neuber differential representation is that it can be used to obtain solutions of creeping flow in cell models where the shape of the particles is genuine three-dimensional. Dassios and Vafeas[19] have demonstrated the practical efficiency of the above 3D representation by solving the three-dimensional Stokes flow problem in an assemblage of spherical particles, which translate and rotate, using Happel-type boundary conditions. The loss of symmetry is caused by the imposed rotation of the particles. The full solution is obtained in a closed form. The present work concerns the next step of our investigation where the sphere, that represents the isotropy, is replaced by a triaxial ellipsoid, which carries the complete anisotropy of the three-dimensional space.

The solution to the Stokes flow problem in an ellipsoid-in-cell with Happel-type boundary conditions is obtained here with the aim of the Papkovich–Neuber representation. The incentive for this is that the Happel-type boundary conditions are more compatible with the physics of the flow in a swarm, since they ensure that each unit cell is self-sufficient in mechanical energy. Under the assumption of very small Reynolds number and pseudosteady state, a three-dimensional Stokes flow in an ellipsoidal envelope of appropriate shape and dimensions is adopted as a fair approximation to the flow around a typical particle of the swarm, in accordance with the concept introduced by Happel.[11] The inner ellipsoid, which represents a particle in the assemblage, is solid, moves with a constant arbitrary velocity, and rotates arbitrarily with a constant angular velocity, whereas the outer ellipsoid represents a fictitious fluid envelope identifying the surface of a unit cell (ellipsoid-in-cell). The volume of the fluid cell is chosen so that the solid volume fraction in the cell coincides with the volume fraction of the swarm. The appropriate boundary conditions, resulting from these assumptions, are: nonslip flow on the inner ellipsoid, no normal flow, and zero tangential stresses on the outer ellipsoidal envelope.

In order to produce ready-to-use basic functions for Stokes flow in ellispoidal coordinates,[2] we calculate the Papkovich–Neuber eigensolutions, generated by the appropriate ellipsoidal eigenfunctions.[20,21] This way, we determine the flow fields as a full series expansion via the Papkovich–Neuber representation, which represents the velocity and the total pressure fields in terms of harmonic functions. The velocity, to the first degree, which represents the leading term of the series, is sufficient for most engineering applications and provides us, also, with the corresponding full 3D solution for the sphere given in Ref. 19 after a proper reduction. Besides, the first-order velocity field suits properly with the first-order nonslip flow condition on the surface of the ellipsoidal particle. Thus, this program offers us the opportunity to restrict to this appropriate degree of approximation. The application of the boundary conditions is accompanied by a set of extra conditions, which form the key to our work. They are based on the flexibility of the Papkovich–Neuber differential representation.[17] The imposition of these conditions is necessary in order to overcome certain difficulties caused by the geometry. Specifically, since we have two boundary surfaces to satisfy our conditions, we use twice the convenience that our representation offers[17] on each boundary. Hence, we adopt the so-called *techniques* mentioned during our analysis, as the result of the above-noted flexibility.

The whole analysis is based on the Lamé functions and the theory of ellipsoidal harmonics.[20,21] In fact, only harmonics of degree less than or equal to two are needed to obtain the velocity field of the first degree. Besides the velocity field, analytical expressions for the leading terms of the total pressure, the angular velocity, and the stress tensor fields are provided. Since the purely ellipsoidal expressions are not easy to handle, the results are given in the more tractable form where Cartesian coordinates are used for the interior harmonics plus the standard elliptic integrals that appear in the exterior Lamé products. Many relations involving the constants of the ellipsoidal harmonics as well as relations among the elliptic integrals had to be worked out in order to bring the result into its final form. The particular way the elliptic integrals are interconnected is provided in the Appendix, where one can also find useful relations that are used extensively and are necessary in order to transform from the Cartesian to the ellipsoidal system and vice versa. We must point out that our analytical method has been followed by the introduction of a new set of elliptic integrals, which do not differ from the aforementioned ones and which helped us to derive certain coefficients in a simple way as well as to overcome the difficulty of the corresponding boundary condition. Finally, the laborious task of reducing the results to the spheroidal and spherical geometry is included. The reduction of general results from the ellipsoidal to the spheroidal or spherical geometry is not a straightforward task because of the complicated indeterminacies that occur as the three semifocal distances of the ellipsoidal system approach zero. The only way to deal with these indeterminacies is to group appropriately the terms of the solution and to perform the algebraic manipulations, which eliminate the indeterminacies before the limiting process is applied.

Section II provides the mathematical formulation of the problem and Sec. III discusses the fundamentals of the ellipsoidal system and the eigenfunctions for the Papkovich–Neuber potentials in ellipsoidal coordinates. The Stokes flow fields are also provided as full series expansions and the boundary conditions are presented in the appropriate ellipsoidal shape. The Happel-type problem for an ellipsoid-in-cell model is solved explicitly in Sec. IV where the results are presented in a manageable Cartesian-ellipsoidal form. Section V is dedicated to the reduction of our expressions to the corresponding prolate–oblate spheroidal (including their limiting cases) and spherical ones. Section VI is devoted to a discussion of the obtained results. The necessary material from the theory of ellipsoidal harmonics as well as some useful formulas associated with ellipsoidal functions is collected in the Appendix. These formulas are the key identities of the present work.

II. MATHEMATICAL DEVELOPMENT

Under the assumption of pseudosteady, nonaxisymmetric, creeping flow (Reynolds number $\mathrm{Re} \ll 1$) which characterizes Stokes flow,[1] the governing equations of motion for an incompressible, viscous fluid in smooth, bounded domain $\Omega(\mathbb{R}^3)$, with dynamic viscosity μ_0 and mass density ρ_0, are a pair of partial differential equations connecting the biharmonic velocity field $\mathbf{v}$ with the harmonic total pressure field P,

$$\mu_0 \Delta \mathbf{v}(\mathbf{r}) = \nabla P(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{1}$$

$$\nabla \cdot \mathbf{v}(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{2}$$

Equation (1) states that, in creeping flow, the viscous force compensates for the force caused by the pressure gradient on any material point of the fluid, while Eq. (2) secures the incompressibility of the fluid. Once the velocity field is obtained, the harmonic vorticity field $\boldsymbol{\omega}$ is defined as

$$\boldsymbol{\omega}(\mathbf{r}) = \tfrac{1}{2} \nabla \times \mathbf{v}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{3}$$

Papkovich–Neuber[3,4] proposed the following differential representation of the solution for Stokes flow, in terms of the harmonic potentials $\boldsymbol{\Phi}$ and Φ_0,

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\Phi}(\mathbf{r}) - \tfrac{1}{2} \nabla (\mathbf{r} \cdot \boldsymbol{\Phi}(\mathbf{r}) + \Phi_0(\mathbf{r})), \quad \mathbf{r} \in \Omega(\mathbb{R}^3) \tag{4}$$

and

$$P(\mathbf{r}) = P_0 - \mu_0 \nabla \cdot \boldsymbol{\Phi}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{5}$$

whereas P_0 is a constant pressure of reference usually assigned at a convenient point. The potential functions $\boldsymbol{\Phi}$ and Φ_0 solve the equations

$$\Delta \boldsymbol{\Phi}(\mathbf{r}) = \mathbf{0}, \quad \Delta \Phi_0 = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{6}$$

If we define the thermodynamic pressure p, then the following relation gives the total pressure as a function of the thermodynamic pressure

$$P(\mathbf{r}) = p(\mathbf{r}) + \rho_0 g h, \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{7}$$

where the contribution of the term $\rho_0 g h$ (g is the acceleration of the gravity) refers to the gravitational pressure force, corresponding to a height of reference h.

The stress dyadic $\tilde{\boldsymbol{\Pi}}$ is defined as follows:

$$\tilde{\boldsymbol{\Pi}}(\mathbf{r}) = -p(\mathbf{r})\tilde{\mathbf{I}} + \mu_0[\nabla \otimes \mathbf{v}(\mathbf{r}) + (\nabla \otimes \mathbf{v}(\mathbf{r}))^{\mathsf{T}}], \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{8}$$

where $\tilde{\mathbf{I}}$ stands for the unit dyadic and the symbol "T" denotes transposition.

The Happel-type boundary conditions for a general 3D particle-in-cell model. By means of

Ref. 1, we consider a fluid–particle system consisting of any finite number of rigid particles of arbitrary shape. Introducing the particle-in-cell model, we examine the Stokes flow of one of the assemblage of particles neglecting the interaction with other particles or with the bounded walls of a container. Let S_i denote the surface of the particle of the swarm, which is solid, is moving with a constant translational velocity $\mathbf{U}$ in an arbitrary direction (the relation between the velocity $\mathbf{U}$ and the mean interstitial velocity through a swarm of spheroidal particles was discussed in Ref. 16 and is rotating, also arbitrarily, with a constant angular velocity $\boldsymbol{\Omega}$. It lives within an otherwise quiescent fluid layer, which is confined by the outer surface denoted by S_o. Following the formulation of Happel,[11] the velocity component normal to S_o and the tangential stresses are assumed to vanish on S_o. These boundary conditions are supplemented by the necessary nonslip flow conditions on the surface of the particle. Thus, the general BCs for a three-dimensional consideration of the Happel-type boundary value problem are:

$$\text{BC(1):} \quad \mathbf{v}(\mathbf{r}) = \mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r} \quad \text{for } \mathbf{r} \in S_i, \tag{9}$$

$$\text{BC(2):} \quad \hat{\mathbf{n}} \cdot \mathbf{v}(\mathbf{r}) = 0 \quad \text{for } \mathbf{r} \in S_o, \tag{10}$$

$$\text{BC(3):} \quad \hat{\mathbf{n}} \cdot \widetilde{\mathbf{\Pi}}(\mathbf{r}) \cdot (\widetilde{\mathbf{I}} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) = \mathbf{0} \quad \text{for } \mathbf{r} \in S_o, \tag{11}$$

where $\hat{\mathbf{n}}$ is the outer unit normal vector. Equations (1)–(11) define a well-posed Happel-type boundary value problem for 3D domains, $\mathbf{r} \in \Omega(\mathbb{R}^3)$, bounded by two arbitrary surfaces S_i and S_o.

Our goal is to solve the above-noted Happel problem with the appropriate boundary conditions given by Eqs. (9)–(11), with the aim of the Papkovich–Neuber differential representation using the ellipsoidal system, which represents the most general geometrical system, which is orthogonal and embodies the complete anisotropy of the three-dimensional space.

III. ELLIPSOIDAL GEOMETRY: FLOW FIELDS AND BOUNDARY CONDITIONS

The basic triaxial ellipsoid is defined by

$$\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1, \tag{12}$$

where $0 < \alpha_3 < \alpha_2 < \alpha_1 < +\infty$ are its semiaxes. The three positive numbers h_1, h_2, and h_3, which denote the semifocal distances of the system, are given by

$$h_1^2 = \alpha_2^2 - \alpha_3^2, \quad h_2^2 = \alpha_1^2 - \alpha_3^2, \quad h_3^2 = \alpha_1^2 - \alpha_2^2 = h_2^2 - h_1^2. \tag{13}$$

Define the system of ellipsoidal coordinates (ρ, μ, ν),[2,20] which are connected to the Cartesian ones (x_1, x_2, x_3), via

$$x_1 = \frac{\rho\mu\nu}{h_2 h_3}, \tag{14}$$

$$x_2 = \frac{\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3}, \tag{15}$$

$$x_3 = \frac{\sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2}. \tag{16}$$

The three families of second-degree surfaces, which are shown in Fig. 1, share the same set of foci at the points $\pm h_1$, $\pm h_2$, and $\pm h_3$.

In terms of the position vector $\mathbf{r} = x_1\hat{\mathbf{x}}_1 + x_2\hat{\mathbf{x}}_2 + x_3\hat{\mathbf{x}}_3$ expressed via the Cartesian basis $\hat{\mathbf{x}}_\kappa$, $\kappa = 1, 2, 3$, the variable ρ, $h_2 \leq \rho < +\infty$ specifies the ellipsoid

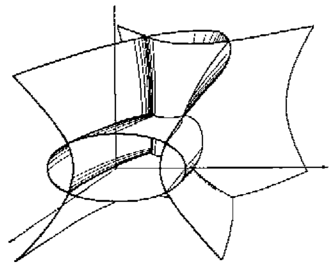

FIG. 1. Ellipsoidal geometry and coordinate surfaces.

$$\frac{x_1^2}{\rho^2}+\frac{x_2^2}{\rho^2-h_3^2}+\frac{x_3^2}{\rho^2-h_2^2}=\sum_{i=1}^{3}\frac{x_i^2}{\rho^2-\alpha_1^2+\alpha_i^2}=\mathbf{r}\cdot\sum_{i=1}^{3}\frac{\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i}{\rho^2-\alpha_1^2+\alpha_i^2}\cdot\mathbf{r}=1, \tag{17}$$

the variable μ, $h_3\leqslant\mu\leqslant h_2$ specifies the hyperboloid of one sheet

$$\frac{x_1^2}{\mu^2}+\frac{x_2^2}{\mu^2-h_3^2}-\frac{x_3^2}{h_2^2-\mu^2}=\sum_{i=1}^{3}\frac{x_i^2}{\mu^2-\alpha_1^2+\alpha_i^2}=1 \tag{18}$$

and the variable ν, $-h_3\leqslant\nu\leqslant h_3$ specifies the hyperboloid of two sheets

$$\frac{x_1^2}{\nu^2}-\frac{x_2^2}{h_3^2-\nu^2}-\frac{x_3^2}{h_2^2-\nu^2}=\sum_{i=1}^{3}\frac{x_i^2}{\nu^2-\alpha_1^2+\alpha_i^2}=1. \tag{19}$$

In the limit as the semifocal distances tend to zero $(h_1=h_2=h_3\to 0)$, our system degenerates to the corresponding spherical one with radial component r given by

$$\|\mathbf{r}\|=\sqrt{\rho^2+\mu^2+\nu^2-(h_2^2+h_3^2)}\to r,\quad 0\leqslant r<+\infty. \tag{20}$$

In terms of the metric coefficients of the system

$$h_\rho=\frac{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}{\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}},\quad h_\mu=\frac{\sqrt{\rho^2-\mu^2}\sqrt{\mu^2-\nu^2}}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}},\quad h_\nu=\frac{\sqrt{\rho^2-\nu^2}\sqrt{\mu^2-\nu^2}}{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}}, \tag{21}$$

the Jacobian determinant is $J=h_\rho h_\mu h_\nu$ and the differential operators ∇, Δ, assume the forms

$$\nabla=\sum_{i=1}^{3}\hat{\mathbf{x}}_i\frac{\partial}{\partial x_i}=\frac{\hat{\boldsymbol{\rho}}}{h_\rho}\frac{\partial}{\partial\rho}+\frac{\hat{\boldsymbol{\mu}}}{h_\mu}\frac{\partial}{\partial\mu}+\frac{\hat{\boldsymbol{\nu}}}{h_\nu}\frac{\partial}{\partial\nu} \tag{22}$$

and

$$\Delta=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}=\frac{1}{J}\left\{\frac{\partial}{\partial\rho}\left[\frac{J}{h_\rho^2}\frac{\partial}{\partial\rho}\right]+\frac{\partial}{\partial\mu}\left[\frac{J}{h_\mu^2}\frac{\partial}{\partial\mu}\right]+\frac{\partial}{\partial\nu}\left[\frac{J}{h_\nu^2}\frac{\partial}{\partial\nu}\right]\right\}, \tag{23}$$

where $\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\nu}}$ denote the orthonormal coordinate vectors of the system, i.e.,

$$\hat{\boldsymbol{\rho}}=\frac{\rho}{h_\rho}\sum_{i=1}^{3}\frac{x_i\hat{\mathbf{x}}_i}{\rho^2-\alpha_1^2+\alpha_i^2},\quad \hat{\boldsymbol{\mu}}=\frac{\mu}{h_\mu}\sum_{i=1}^{3}\frac{x_i\hat{\mathbf{x}}_i}{\mu^2-\alpha_1^2+\alpha_i^2},\quad \hat{\boldsymbol{\nu}}=\frac{\nu}{h_\nu}\sum_{i=1}^{3}\frac{x_i\hat{\mathbf{x}}_i}{\nu^2-\alpha_1^2+\alpha_i^2}. \tag{24}$$

The outward unit normal vector on the surface of the ellipsoid $\rho=$const coincides with the unit normal vector $\hat{\boldsymbol{\rho}}$, thus

$$\hat{\mathbf{n}}(\mathbf{r}) = \frac{\rho}{h_\rho}\left[\frac{x_1\hat{\mathbf{x}}_1}{\rho^2} + \frac{x_2\hat{\mathbf{x}}_2}{\rho^2 - h_3^2} + \frac{x_3\hat{\mathbf{x}}_3}{\rho^2 - h_2^2}\right] = \frac{\rho}{h_\rho}\sum_{i=1}^{3}\frac{\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i}{\rho^2 - \alpha_1^2 + \alpha_i^2}\cdot \mathbf{r} \equiv \hat{\boldsymbol{\rho}}, \tag{25}$$

while the unit dyadic assumes the ellipsoidal form

$$\tilde{\mathbf{I}} = \sum_{i=1}^{3}\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i = \hat{\boldsymbol{\rho}} \otimes \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\mu}} \otimes \hat{\boldsymbol{\mu}} + \hat{\boldsymbol{\nu}} \otimes \hat{\boldsymbol{\nu}}. \tag{26}$$

From now on we shall refer to ellipsoidal domains, in terms of which

$$\Omega(\mathbb{R}^3) \equiv \mathbb{R}^3 = \{(\rho,\mu,\nu):\rho \in [h_2,+\infty), \mu \in [h_3,h_2], \nu \in [-h_3,h_3]\}. \tag{27}$$

In order to construct the flow fields (3)–(5) and (8) in an appropriate form for the application of the boundary conditions (9)–(11), we need to represent the harmonic potentials $\mathbf{\Phi}$ and Φ_0 in this system by solving the corresponding Laplace's equations (6) in spectral form. This procedure leads to the Lamé equation[20]

$$(x^2 - h_3^2)(x^2 - h_2^2)E''(x) + x(2x^2 - h_3^2 - h_2^2)E'(x) + (Ax^2 + B)E(x) = 0, \tag{28}$$

for each one of the factors $E(\rho)$, $E(\mu)$, and $E(\nu)$ within the corresponding intervals $\rho \in [h_2,+\infty)$, $\mu \in [h_3,h_2]$, and $\nu \in [-h_3,h_3]$, where $A,B \in \mathbb{R}$ are constants. For each $n=0,1,\ldots$, which corresponds to the degree of the Lamé equation, and for each $m=1,2,\ldots,2n+1$, which stands for its order, Eq. (28) has two linearly independent solutions. The first one, E_n^m, is regular at the origin and it is known as the Lamé function of the first kind (interior solution), while the second one, F_n^m, is regular at infinity and gives the Lamé function of the second kind (exterior solution). In particular, the interior solution $E_n^m(\rho)$ is related to the exterior solution $F_n^m(\rho)$ via

$$F_n^m(\rho) = (2n+1)E_n^m(\rho)I_n^m(\rho), \quad \rho \in [h_2,+\infty), \tag{29}$$

where the elliptic integrals I_n^m are given by

$$I_n^m(\rho) = \int_\rho^{+\infty}\frac{\mathrm{d}u}{[E_n^m(u)]^2\sqrt{u^2 - h_2^2}\sqrt{u^2 - h_3^2}}, \quad \rho \in [h_2,+\infty) \tag{30}$$

for every value of $n=0,1,\ldots$ and $m=1,2,\ldots,2n+1$. In terms of the Lamé functions of the first and of the second kind, the Lamé products

$$\mathbb{E}_n^m(\mathbf{r}) = E_n^m(\rho)E_n^m(\mu)E_n^m(\nu), \quad n=0,1,\ldots, \quad m=1,2,\ldots,2n+1, \quad \mathbf{r} \in \Omega(\mathbb{R}^3) \tag{31}$$

define the interior solid ellipsoidal harmonics, while the products

$$\mathbb{F}_n^m(\mathbf{r}) = F_n^m(\rho)E_n^m(\mu)E_n^m(\nu), \quad n=0,1,\ldots, \quad m=1,2,\ldots,2n+1, \quad \mathbf{r} \in \Omega(\mathbb{R}^3) \tag{32}$$

define the exterior solid ellipsoidal harmonics. On the other hand, the complete orthogonal set $E_n^m(\mu)E_n^m(\nu)$ form the surface ellipsoidal harmonics on the surface of any ellipsoid $\rho=\rho_s$, which, with respect to the weighting function

$$l_{\rho_s}(\mu,\nu) = \frac{1}{\sqrt{(\rho_s^2 - \mu^2)(\rho_s^2 - \nu^2)}}, \quad \mu \in [h_3,h_2], \quad \nu \in [-h_3,h_3], \tag{33}$$

satisfy the orthogonality relation

$$\int\int_{\rho=\rho_s} E_n^m(\mu)E_n^m(\nu)E_{n'}^{m'}(\mu)E_{n'}^{m'}(\nu)l_{\rho_s}(\mu,\nu)\mathrm{d}S = \gamma_n^m \delta_{nn'}\delta_{mm'}, \tag{34}$$

for every $n,n'=0,1,\ldots$ and $m,m'=1,2,\ldots,2n+1$. Here, $\delta_{nn'}$ denotes the Kronecker delta function, whilst γ_n^m are the ellipsoidal normalization constants given by

$$\gamma_n^m = \int\int_{\rho=\rho_s} (E_n^m(\mu)E_n^m(\nu))^2 l_{\rho_s}(\mu,\nu)\mathrm{d}S, \quad n=0,1,\ldots, \quad m=1,2,\ldots,2n+1. \tag{35}$$

According to the aforementioned analysis of ellipsoidal harmonic functions, the complete representation of the Papkovich–Neuber potentials $\mathbf{\Phi}$ and Φ_0, which belong to the kernel space of Δ, assume the expressions

$$\mathbf{\Phi}(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} [\mathbf{e}_n^{(i)m}\mathbb{E}_n^m(\mathbf{r}) + \mathbf{e}_n^{(e)m}\mathbb{F}_n^m(\mathbf{r})], \quad \mathbf{r} \in \Omega(\mathbb{R}^3) \tag{36}$$

and

$$\Phi_0(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} [d_n^{(i)m}\mathbb{E}_n^m(\mathbf{r}) + d_n^{(e)m}\mathbb{F}_n^m(\mathbf{r})], \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{37}$$

Note that, for $n \geqslant 0$, the coefficients

$$\mathbf{e}_n^{(i/e)m} = a_n^{(i/e)m}\hat{\mathbf{x}}_1 + b_n^{(i/e)m}\hat{\mathbf{x}}_2 + c_n^{(i/e)m}\hat{\mathbf{x}}_3, \quad d_n^{(i/e)m}, \quad m=1,2,\ldots,2n+1 \tag{38}$$

denote the vector and the scalar coefficients of the harmonic potentials $\mathbf{\Phi}$ and Φ_0, respectively.

Let u, v and $\mathbf{f}$, $\mathbf{g}$ denote two scalar and two vector fields, respectively. Then, if we define by $\tilde{\mathbb{S}}$ a dyadic, the basic identities that we are using in the sequel concern the action of the gradient operator on the following expressions, i.e.,

$$\nabla \otimes (u\mathbf{f}) = u\nabla \otimes \mathbf{f} + \nabla u \otimes \mathbf{f}, \tag{39}$$

$$\nabla \cdot (u\mathbf{f}) = u\nabla \cdot \mathbf{f} + \nabla u \cdot \mathbf{f}, \tag{40}$$

$$\nabla \times (u\mathbf{f}) = u\nabla \times \mathbf{f} + \nabla u \times \mathbf{f}, \tag{41}$$

$$\nabla(\mathbf{f}\cdot\mathbf{g}) = (\nabla \otimes \mathbf{f})\cdot\mathbf{g} + (\nabla \otimes \mathbf{g})\cdot\mathbf{f}, \tag{42}$$

$$\nabla(uv) = u\nabla v + v\nabla u, \tag{43}$$

$$\nabla \otimes (\tilde{\mathbb{S}}\cdot\mathbf{f}) = (\nabla \otimes \tilde{\mathbb{S}})\cdot\mathbf{f} + (\nabla \otimes \mathbf{f})\cdot\tilde{\mathbb{S}}^{\mathrm{T}}, \tag{44}$$

$$\nabla \otimes (\mathbf{f} \otimes \mathbf{g}) = (\nabla \otimes \mathbf{f}) \otimes \mathbf{g} + [\mathbf{f} \otimes (\nabla \otimes \mathbf{g})]^{213}, \tag{45}$$

whereas $\tilde{\mathbb{S}}^{\mathsf{T}}$ is the inverted dyadic and the symbol $(\ )^{213}$ denotes left transposition for a triadic. Inserting the potentials (36) and (37) in the flow fields (4), (5), (3), (8) and making use of the above identities (39)–(45), we derive the relation

$$\begin{aligned}\mathbf{v}(\mathbf{r}) = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \{&\mathbf{e}_n^{(i)m}\mathbb{E}_n^m(\mathbf{r}) \\ &- [(\mathbf{e}_n^{(i)m}\cdot\mathbf{r}) + d_n^{(i)m}]\nabla\mathbb{E}_n^m(\mathbf{r}) + \mathbf{e}_n^{(e)m}\mathbb{F}_n^m(\mathbf{r}) - [(\mathbf{e}_n^{(e)m}\cdot\mathbf{r}) + d_n^{(e)m}]\nabla\mathbb{F}_n^m(\mathbf{r})\}, \quad \mathbf{r} \in \Omega(\mathbb{R}^3)\end{aligned} \tag{46}$$

for the velocity field, while for the total pressure field, taking into account relation (7), we obtain

$$P(\mathbf{r}) = P_0 - \mu_0 \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \{\mathbf{e}_n^{(i)m} \cdot \nabla \mathbb{E}_n^m(\mathbf{r}) + \mathbf{e}_n^{(e)m} \cdot \nabla \mathbb{F}_n^m(\mathbf{r})\} = p(\mathbf{r}) + \rho_0 g h, \tag{47}$$

for every $\mathbf{r} \in \Omega(\mathbb{R}^3)$. The vorticity field is then written as

$$\boldsymbol{\omega}(\mathbf{r}) = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \{\nabla \mathbb{E}_n^m(\mathbf{r}) \times \mathbf{e}_n^{(i)m} + \nabla \mathbb{F}_n^m(\mathbf{r}) \times \mathbf{e}_n^{(e)m}\}, \quad \mathbf{r} \in \Omega(\mathbb{R}^3) \tag{48}$$

and the stress tensor field is expressed as

$$\tilde{\mathbf{\Pi}}(\mathbf{r}) = -p(\mathbf{r})\tilde{\mathbf{I}} - \mu_0 \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \{[(\mathbf{e}_n^{(i)m} \cdot \mathbf{r}) + d_n^{(i)m}] \nabla \otimes \nabla \mathbb{E}_n^m(\mathbf{r}) + [(\mathbf{e}_n^{(e)m} \cdot \mathbf{r}) + d_n^{(e)m}] \nabla \otimes \nabla \mathbb{F}_n^m(\mathbf{r})\},$$
$$\mathbf{r} \in \Omega(\mathbb{R}^3). \tag{49}$$

The coefficients $\mathbf{e}_n^{(i)m}$, $\mathbf{e}_n^{(e)m}$, $d_n^{(i)m}$, and $d_n^{(e)m}$ for $n=0,1,\ldots$ and $m=1,2,\ldots,2n+1$ are to be determined from the boundary conditions (9)–(11), which also have to be expressed in ellipsoidal form. In accordance with the Happel formulation,[11] two confocal ellipsoids are considered. The inner one, indicated by S_a, at $\rho=\rho_a$ with semiaxes ρ_a, $\sqrt{\rho_a^2-h_3^2}$ and $\sqrt{\rho_a^2-h_2^2}$ is solid, it is moving with a constant translational velocity $\mathbf{U}$ in an arbitrary direction and is rotating arbitrarily with a constant angular velocity $\mathbf{\Omega}$, given by

$$\mathbf{U} = \sum_{i=1}^{3} U_i \hat{\mathbf{x}}_i, \quad \mathbf{\Omega} = \sum_{i=1}^{3} \Omega_i \hat{\mathbf{x}}_i, \tag{50}$$

where U_i and Ω_i, $i=1,2,3$ are the components at the Cartesian orthogonal basis. The outer ellipsoid indicated by S_b at $\rho=\rho_b$ with semiaxes ρ_b, $\sqrt{\rho_b^2-h_3^2}$ and $\sqrt{\rho_b^2-h_2^2}$ defines the quiescent fluid layer. Thus, the boundary conditions (9)–(11) are rewritten as follows:

$$\text{BC(1):} \quad \mathbf{v}(\mathbf{r}) = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r} \quad \text{for } \mathbf{r} \in S_a, \tag{51}$$

$$\text{BC(2):} \quad \hat{\boldsymbol{\rho}} \cdot \mathbf{v}(\mathbf{r}) = 0 \quad \text{for } \mathbf{r} \in S_b, \tag{52}$$

$$\text{BC(3):} \quad \hat{\boldsymbol{\rho}} \cdot \tilde{\mathbf{\Pi}}(\mathbf{r}) \cdot (\tilde{\mathbf{I}} - \hat{\boldsymbol{\rho}} \otimes \hat{\boldsymbol{\rho}}) = \mathbf{0} \quad \text{for } \mathbf{r} \in S_b, \tag{53}$$

where $\hat{\boldsymbol{\rho}}$ is the unit curvilinear vector relative to the variable ρ. These BCs must be applied to Eqs. (46) and (49) in order to calculate the coefficients and obtain the flow fields (46)–(49). Note that the domain $\Omega(\mathbb{R}^3)$ (27) is specified by the variable ρ that varies in the interval $[\rho_a, \rho_b]$.

IV. SOLUTION OF THE 3D HAPPEL-TYPE ELLIPSOID-IN-CELL MODEL

The purpose of this section is to solve the 3D Stokes ellipsoid-in-cell model with the Happel-type BCs (51)–(53). In our related work, for the corresponding complete isotropic Stokes flow,[19] the solution was obtained by using the full series representation for the velocity, the total pressure, the vorticity, and the stress tensor. Unfortunately, this is not possible when the present complex ellipsoidal geometry is used. The reason is that, although the form of ellipsoidal harmonics is known, the analytical expressions of those harmonics in terms of the semiaxes $\alpha_1, \alpha_2, \alpha_3$ are manageable only for degree $n \leqslant 3$. This difficulty restricts the analytical solutions of related physical problems to the 16th-dimensional harmonic subspace spanned by the harmonics of degree less than or equal to three for $m=1,2,\ldots,2n+1$.[20] The interior Lamé functions of degree less than or equal to 3 are given by

$$E_0^1(x) = 1, \tag{54}$$

$$E_1^{\kappa}(x) = \sqrt{|x^2 - \alpha_1^2 + \alpha_\kappa^2|}, \quad \kappa = 1,2,3, \tag{55}$$

$$E_2^1(x) = x^2 - \alpha_1^2 + \Lambda, \tag{56}$$

$$E_2^2(x) = x^2 - \alpha_1^2 + \Lambda', \tag{57}$$

$$E_2^{6-\kappa}(x) = \frac{x\sqrt{|x^2 - h_3^2|}\sqrt{|x^2 - h_2^2|}}{\sqrt{|x^2 - \alpha_1^2 + \alpha_\kappa^2|}}, \quad \kappa = 1,2,3 \tag{58}$$

or in terms of the semiaxes,

$$E_2^{\kappa+l}(x) = \sqrt{|x^2 - \alpha_1^2 + \alpha_\kappa^2|}\sqrt{|x^2 - \alpha_1^2 + \alpha_l^2|}, \quad \kappa, l = 1,2,3, \quad \kappa \neq l \tag{59}$$

and

$$E_3^{2\kappa-1}(x) = \sqrt{|x^2 - \alpha_1^2 + \alpha_\kappa^2|}(x^2 - \alpha_1^2 + \Lambda_\kappa), \quad \kappa = 1,2,3, \tag{60}$$

$$E_3^{2\kappa}(x) = \sqrt{|x^2 - \alpha_1^2 + \alpha_\kappa^2|}(x^2 - \alpha_1^2 + \Lambda'_\kappa), \quad \kappa = 1,2,3, \tag{61}$$

$$E_3^7(x) = x\sqrt{|x^2 - h_3^2|}\sqrt{|x^2 - h_2^2|}, \tag{62}$$

where the variable x represents the values of $\rho \in [h_2, +\infty)$, $\mu \in [h_3, h_2]$, $\nu \in [-h_3, h_3]$, the constants

$$\left.\begin{matrix}\Lambda \\ \Lambda'\end{matrix}\right\} = \frac{1}{3}\sum_{i=1}^{3} \alpha_i^2 \pm \frac{1}{3}\left[\sum_{i=1}^{3}\left(\alpha_i^4 - \frac{\alpha_1^2\alpha_2^2\alpha_3^2}{\alpha_i^2}\right)\right]^{1/2} \tag{63}$$

satisfy the quadratic equation

$$\sum_{i=1}^{3} \frac{1}{\Lambda - \alpha_i^2} = 0, \tag{64}$$

and the constants

$$\left.\begin{matrix}\Lambda_\kappa \\ \Lambda'_\kappa\end{matrix}\right\} = \frac{2}{5}\sum_{i=1}^{3} \alpha_i^2 - \frac{1}{5}\alpha_\kappa^2 \pm \frac{1}{5}\left\{4\sum_{i=1}^{3} \alpha_i^4 - 3\alpha_\kappa^4 - \alpha_1^2\alpha_2^2\alpha_3^2\left(\sum_{i=1}^{3}\frac{1}{\alpha_i^2} + \frac{6}{\alpha_\kappa^2}\right)\right\}^{1/2} \tag{65}$$

for $\kappa=1,2,3$ satisfy the relations

$$\sum_{i=1}^{3} \frac{1 + 2\delta_{i\kappa}}{(\Lambda_\kappa - \alpha_i^2)} = 0, \quad \kappa = 1,2,3. \tag{66}$$

The Cartesian representation of the ellipsoidal harmonics are given by

$$\mathbb{E}_0^1(\mathbf{r}) = 1, \tag{67}$$

$$\mathbb{E}_1^{\kappa}(\mathbf{r}) = \frac{h_1 h_2 h_3}{h_\kappa} x_\kappa, \quad \kappa = 1,2,3, \tag{68}$$

$$\mathbb{E}_2^1(\mathbf{r}) = (\Lambda - \alpha_1^2)(\Lambda - \alpha_2^2)(\Lambda - \alpha_3^2)\left[\sum_{i=1}^{3}\frac{x_i^2}{\Lambda - \alpha_i^2} + 1\right] \equiv \mathbb{E}_2^1(\Lambda), \tag{69}$$

$$\mathbb{E}_2^2(\mathbf{r}) = \mathbb{E}_2^1(\Lambda'), \tag{70}$$

$$\mathbb{E}_2^{6-\kappa}(\mathbf{r}) = h_1 h_2 h_3 h_\kappa \frac{x_1 x_2 x_3}{x_\kappa}, \quad \kappa = 1,2,3 \tag{71}$$

or equivalently,

$$\mathbb{E}_2^{\kappa+l}(\mathbf{r}) = \frac{h_1^2 h_2^2 h_3^2}{h_\kappa h_l} x_\kappa x_l, \quad \kappa, l = 1,2,3, \quad \kappa \neq l \tag{72}$$

and

$$\mathbb{E}_3^{2\kappa-1}(\mathbf{r}) = h_1 h_2 h_3 (\Lambda_\kappa - \alpha_1^2)(\Lambda_\kappa - \alpha_2^2)(\Lambda_\kappa - \alpha_3^2) \frac{x_\kappa}{h_\kappa} \left[\sum_{i=1}^{3} \frac{x_i^2}{\Lambda_\kappa - \alpha_i^2} + 1 \right] \equiv \mathbb{E}_3^{2\kappa-1}(\Lambda_\kappa), \quad \kappa = 1,2,3, \tag{73}$$

$$\mathbb{E}_3^{2\kappa}(\mathbf{r}) = \mathbb{E}_3^{2\kappa-1}(\Lambda_\kappa'), \quad \kappa = 1,2,3, \tag{74}$$

$$\mathbb{E}_3^7(\mathbf{r}) = h_1^2 h_2^2 h_3^2 x_1 x_2 x_3, \tag{75}$$

where $\mathbf{r} \in \Omega(\mathbb{R}^3)$. On the other hand, the Cartesian monomials of degree less than or equal to three are expressed via the ellipsoidal harmonic functions as

$$1 = \mathbb{E}_0^1(\mathbf{r}) = E_0^1(\rho) E_0^1(\mu) E_0^1(\nu), \tag{76}$$

$$x_\kappa = \frac{h_\kappa}{h_1 h_2 h_3} \mathbb{E}_1^\kappa(\mathbf{r}) = \frac{h_\kappa}{h_1 h_2 h_3} E_1^\kappa(\rho) E_1^\kappa(\mu) E_1^\kappa(\nu), \quad \kappa = 1,2,3, \tag{77}$$

$$x_\kappa^2 = \frac{\rho^2 - \alpha_1^2 + \alpha_\kappa^2}{3} \left[1 - \frac{E_2^1(\mu) E_2^1(\nu)}{(\Lambda - \Lambda')(\Lambda - \alpha_\kappa^2)} + \frac{E_2^2(\mu) E_2^2(\nu)}{(\Lambda - \Lambda')(\Lambda' - \alpha_\kappa^2)} \right], \tag{78}$$

$$\frac{x_1 x_2 x_3}{x_\kappa} = \frac{1}{h_1 h_2 h_3 h_\kappa} \mathbb{E}_2^{6-\kappa}(\mathbf{r}) = \frac{1}{h_1 h_2 h_3 h_\kappa} E_2^{6-\kappa}(\rho) E_2^{6-\kappa}(\mu) E_2^{6-\kappa}(\nu), \quad \kappa = 1,2,3 \tag{79}$$

or

$$x_\kappa x_l = \frac{h_\kappa h_l}{h_1^2 h_2^2 h_3^2} \mathbb{E}_2^{\kappa+l}(\mathbf{r}) = \frac{h_\kappa h_l}{h_1^2 h_2^2 h_3^2} E_2^{\kappa+l}(\rho) E_2^{\kappa+l}(\mu) E_2^{\kappa+l}(\nu), \quad \kappa, l = 1,2,3, \quad \kappa \neq l, \tag{80}$$

$$x_\kappa x_l^2 = \frac{\sqrt{\rho^2 - \alpha_1^2 + \alpha_\kappa^2}(\rho^2 - \alpha_1^2 + \alpha_l^2) h_\kappa}{5 h_1 h_2 h_3} \{ (1 + 2\delta_{\kappa l}) E_1^\kappa(\mu) E_1^\kappa(\nu) \tag{81}$$

$$+ \frac{5(-1)^l h_l^2}{h_1^2 h_2^2 h_3^2 (\Lambda_\kappa - \Lambda_\kappa')} [(\Lambda_\kappa' - \alpha_l^2) E_3^{2\kappa-1}(\mu) E_3^{2\kappa-1}(\nu) - (\Lambda_\kappa - \alpha_l^2) E_3^{2\kappa}(\mu) E_3^{2\kappa}(\nu)] \Big\}, \quad \kappa, l = 1,2,3, \tag{82}$$

$$x_1 x_2 x_3 = \frac{1}{h_1^2 h_2^2 h_3^2} \mathbb{E}_3^7(\mathbf{r}) = \frac{1}{h_1^2 h_2^2 h_3^2} E_3^7(\rho) E_3^7(\mu) E_3^7(\nu). \tag{83}$$

Relations (67)–(83) form the basis for moving from the Cartesian coordinates to the ellipsoidal ones and vice versa. Relations (76)–(83) are necessary for the application of the orthogonality relation (34). Note that

$$\mathbb{F}_n^m(\mathbf{r}) = (2n+1)I_n^m(\rho)\mathbb{E}_n^m(\mathbf{r}), \quad n=0,1,2,3, \quad m=1,2,\ldots,2n+1, \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{84}$$

while the set of

$$x_l^2 - x_\kappa^2 = \frac{\alpha_l^2-\alpha_\kappa^2}{3} - \frac{\mathbb{E}_2^1(\mathbf{r})}{3(\Lambda-\Lambda')}\left[\frac{1}{\Lambda-\alpha_l^2} - \frac{1}{\Lambda-\alpha_\kappa^2}\right] + \frac{\mathbb{E}_2^2(\mathbf{r})}{3(\Lambda-\Lambda')}\left[\frac{1}{\Lambda'-\alpha_l^2} - \frac{1}{\Lambda'-\alpha_\kappa^2}\right], \quad \mathbf{r} \in \Omega(\mathbb{R}^3) \tag{85}$$

for every $\kappa, l=1,2,3$, are harmonic functions.

Before we proceed to the full solution of our physical problem, it is necessary to write down the Cartesian-ellipsoidal representations for the single and the double action of the gradient operator (22) on the interior and on the exterior ellipsoidal harmonic eigenfunctions of degree $n \leqslant 3$. Consequently, some long but straightforward calculations lead to

$$\nabla\mathbb{E}_0^1(\mathbf{r}) = \mathbf{0}, \tag{86}$$

$$\nabla\mathbb{E}_1^\kappa(\mathbf{r}) = \frac{h_1h_2h_3}{h_\kappa}\hat{\mathbf{x}}_\kappa = \frac{h_1h_2h_3}{h_\kappa}\mathbb{E}_0^1(\mathbf{r})\hat{\mathbf{x}}_\kappa, \quad \kappa=1,2,3, \tag{87}$$

$$\begin{aligned}\nabla\mathbb{E}_2^1(\mathbf{r}) &= 2(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)\sum_{i=1}^{3}\frac{x_i\hat{\mathbf{x}}_i}{\Lambda-\alpha_i^2} \\ &= 2\frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{h_1h_2h_3}\sum_{i=1}^{3}\frac{h_i}{\Lambda-\alpha_i^2}\mathbb{E}_1^i(\mathbf{r})\hat{\mathbf{x}}_i \equiv \nabla\mathbb{E}_2^1(\Lambda),\end{aligned} \tag{88}$$

$$\nabla\mathbb{E}_2^2(\mathbf{r}) = \nabla\mathbb{E}_2^1(\Lambda'), \tag{89}$$

$$\nabla\mathbb{E}_2^{6-\kappa}(\mathbf{r}) = h_1h_2h_3h_\kappa\sum_{i=1}^{3}(1-\delta_{\kappa i})\frac{x_1x_2x_3}{x_\kappa x_i}\hat{\mathbf{x}}_i = h_1h_2h_3\sum_{i=1}^{3}\frac{(1-\delta_{\kappa i})}{h_i}\mathbb{E}_1^{6-(\kappa+i)}(\mathbf{r})\hat{\mathbf{x}}_i, \quad \kappa=1,2,3 \tag{90}$$

or with equivalent notation,

$$\nabla\mathbb{E}_2^{\kappa+l}(\mathbf{r}) = \frac{h_1^2h_2^2h_3^2}{h_\kappa h_l}(x_\kappa\hat{\mathbf{x}}_l + x_l\hat{\mathbf{x}}_\kappa) = h_1h_2h_3\left(\frac{1}{h_l}\mathbb{E}_1^\kappa(\mathbf{r})\hat{\mathbf{x}}_l + \frac{1}{h_\kappa}\mathbb{E}_1^l(\mathbf{r})\hat{\mathbf{x}}_\kappa\right), \quad \kappa,l=1,2,3, \quad \kappa \neq l, \tag{91}$$

and

$$\begin{aligned}\nabla\mathbb{E}_3^{2\kappa-1}(\mathbf{r}) &= \frac{h_1h_2h_3}{h_\kappa}(\Lambda_\kappa-\alpha_1^2)(\Lambda_\kappa-\alpha_2^2)(\Lambda_\kappa-\alpha_3^2)\left[\hat{\mathbf{x}}_\kappa\left(\sum_{i=1}^{3}\frac{x_i^2}{\Lambda_\kappa-\alpha_i^2}+1\right) + 2x_\kappa\sum_{i=1}^{3}\frac{x_i\hat{\mathbf{x}}_i}{\Lambda_\kappa-\alpha_i^2}\right] \\ &\equiv \nabla\mathbb{E}_3^{2\kappa-1}(\Lambda_\kappa), \quad \kappa=1,2,3,\end{aligned} \tag{92}$$

$$\nabla\mathbb{E}_3^{2\kappa}(\mathbf{r}) = \nabla\mathbb{E}_3^{2\kappa-1}(\Lambda_\kappa'), \quad \kappa=1,2,3, \tag{93}$$

$$\nabla \mathbb{E}_3^7(\mathbf{r}) = h_1^2 h_2^2 h_3^2 \sum_{i=1}^{3} \frac{x_1 x_2 x_3}{x_i} \hat{\mathbf{x}}_i = h_1 h_2 h_3 \sum_{i=1}^{3} \frac{1}{h_i} \mathbb{E}_2^{6-i}(\mathbf{r}) \hat{\mathbf{x}}_i. \tag{94}$$

The form of the stress tensor (49) requires the double action of the gradient operator on the following ellipsoidal harmonics:

$$\nabla \otimes \nabla \mathbb{E}_0^1(\mathbf{r}) = \tilde{\mathbf{0}}, \tag{95}$$

$$\nabla \otimes \nabla \mathbb{E}_1^\kappa(\mathbf{r}) = \tilde{\mathbf{0}}, \quad \kappa = 1,2,3, \tag{96}$$

$$\begin{aligned}\nabla \otimes \nabla \mathbb{E}_2^1(\mathbf{r}) &= 2(\Lambda - \alpha_1^2)(\Lambda - \alpha_2^2)(\Lambda - \alpha_3^2) \sum_{i=1}^{3} \frac{\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i}{\Lambda - \alpha_i^2} \\ &= 2(\Lambda - \alpha_1^2)(\Lambda - \alpha_2^2)(\Lambda - \alpha_3^2) \mathbb{E}_0^1(\mathbf{r}) \sum_{i=1}^{3} \frac{\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i}{\Lambda - \alpha_i^2} \equiv \nabla \otimes \nabla \mathbb{E}_2^1(\Lambda),\end{aligned} \tag{97}$$

$$\nabla \otimes \nabla \mathbb{E}_2^2(\mathbf{r}) = \nabla \otimes \nabla \mathbb{E}_2^1(\Lambda'), \tag{98}$$

$$\begin{aligned}\nabla \otimes \nabla \mathbb{E}_2^{6-\kappa}(\mathbf{r}) &= h_1 h_2 h_3 h_\kappa \sum_{i=1}^{3} (1 - \delta_{\kappa i})(\hat{\mathbf{x}}_{6-(\kappa+i)} \otimes \hat{\mathbf{x}}_i) \\ &= h_1 h_2 h_3 h_\kappa \mathbb{E}_0^1(\mathbf{r}) \sum_{i=1}^{3} (1 - \delta_{\kappa i})(\hat{\mathbf{x}}_{6-(\kappa+i)} \otimes \hat{\mathbf{x}}_i), \quad \kappa = 1,2,3\end{aligned} \tag{99}$$

and with equivalent notation for every $\kappa, l = 1,2,3$ and $\kappa \neq l$,

$$\nabla \otimes \nabla \mathbb{E}_2^{\kappa+l}(\mathbf{r}) = \frac{h_1^2 h_2^2 h_3^2}{h_\kappa h_l} (\hat{\mathbf{x}}_\kappa \otimes \hat{\mathbf{x}}_l + \hat{\mathbf{x}}_l \otimes \hat{\mathbf{x}}_\kappa) = \frac{h_1^2 h_2^2 h_3^2}{h_\kappa h_l} \mathbb{E}_0^1(\mathbf{r}) (\hat{\mathbf{x}}_\kappa \otimes \hat{\mathbf{x}}_l + \hat{\mathbf{x}}_l \otimes \hat{\mathbf{x}}_\kappa) \tag{100}$$

and

$$\begin{aligned}\nabla \otimes \nabla \mathbb{E}_3^{2\kappa-1}(\mathbf{r}) &= \frac{2 h_1 h_2 h_3}{h_\kappa} (\Lambda_\kappa - \alpha_1^2)(\Lambda_\kappa - \alpha_2^2)(\Lambda_\kappa - \alpha_3^2) \\ &\quad \times \left[\sum_{i=1}^{3} \frac{x_\kappa}{\Lambda_\kappa - \alpha_i^2} (\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i) + \sum_{i=1}^{3} \frac{x_i}{\Lambda_\kappa - \alpha_i^2} (\hat{\mathbf{x}}_\kappa \otimes \hat{\mathbf{x}}_i + \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_\kappa) \right] \\ &= \frac{2}{h_\kappa} (\Lambda_\kappa - \alpha_1^2)(\Lambda_\kappa - \alpha_2^2)(\Lambda_\kappa - \alpha_3^2) \\ &\quad \times \left[h_\kappa \mathbb{E}_1^\kappa(\mathbf{r}) \sum_{i=1}^{3} \frac{\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i}{\Lambda_\kappa - \alpha_i^2} + \sum_{i=1}^{3} \frac{h_i \mathbb{E}_1^i(\mathbf{r})}{\Lambda_\kappa - \alpha_i^2} (\hat{\mathbf{x}}_\kappa \otimes \hat{\mathbf{x}}_i + \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_\kappa) \right] \\ &\equiv \nabla \otimes \nabla \mathbb{E}_3^{2\kappa-1}(\Lambda_\kappa), \quad \kappa = 1,2,3,\end{aligned} \tag{101}$$

$$\nabla \otimes \nabla \mathbb{E}_3^{2\kappa}(\mathbf{r}) = \nabla \otimes \nabla \mathbb{E}_3^{2\kappa-1}(\Lambda'_\kappa), \quad \kappa = 1,2,3, \tag{102}$$

$$\nabla \otimes \nabla \mathbb{E}_3^7(\mathbf{r}) = h_1^2 h_2^2 h_3^2 \sum_{i,j=1}^{3} (1 - \delta_{ij}) \frac{x_1 x_2 x_3}{x_i x_j} (\hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i) = h_1^2 h_2^2 h_3^2 \sum_{i,j=1}^{3} (1 - \delta_{ij}) \frac{\hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i}{h_i h_j} \mathbb{E}_1^{6-(i+j)}(\mathbf{r}). \tag{103}$$

Unfortunately, it is not possible to write similar Cartesian expressions for the gradient of the external harmonic eigenmodes, because of the existence of the elliptic integrals (30) in relation (84). As we shall see later, this is the reason for the appearance of a number of difficulties in the application of the boundary conditions. Thus, we keep a mixed ellipsoidal-Cartesian formula of the above-mentioned expressions. In view of Eq. (30), we obtain

$$I_n^{m\prime}(\rho) = -\frac{1}{[E_n^m(\rho)]^2\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}, \quad \rho \in [h_2,+\infty) \tag{104}$$

and according to relations (21) and (22)

$$\nabla \mathbb{F}_n^m(\mathbf{r}) = (2n+1)\left[I_n^m(\rho)(\nabla \mathbb{E}_n^m(\mathbf{r})) - \frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} \frac{\mathbb{E}_n^m(\mathbf{r})}{[E_n^m(\rho)]^2}\right]. \tag{105}$$

Although expression (105) is the simplest one possible it is still problematic because of the appearance of the vectorial factor

$$\mathbf{R}^{\mathrm{el}}(\mathbf{r}) = \frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}, \quad \rho \in [h_2,+\infty), \quad \mu \in [h_3,h_2], \quad \nu \in [-h_3,h_3]. \tag{106}$$

This problem becomes even more complicated in the case of the double action of the gradient operator on the external harmonics, where

$$\begin{aligned}\nabla \otimes \nabla \mathbb{F}_n^m(\mathbf{r}) = (2n+1)[&I_n^m(\rho)(\nabla \otimes \nabla \mathbb{E}_n^m(\mathbf{r})) + \nabla I_n^m(\rho) \otimes \nabla \mathbb{E}_n^m(\mathbf{r}) + \nabla \mathbb{E}_n^m(\mathbf{r}) \otimes \nabla I_n^m(\rho) \\ &+ \mathbb{E}_n^m(\mathbf{r})(\nabla \otimes \nabla I_n^m(\rho))]\end{aligned} \tag{107}$$

for $n=0,1,2,\ldots$ and $m=1,2,\ldots,2n+1$. The dyadic $\nabla \otimes \nabla \mathbb{E}_n^m$ has been introduced earlier, and we can easily obtain the following relation:

$$\nabla I_n^m(\rho) = -\frac{\hat{\boldsymbol{\rho}}}{[E_n^m(\rho)]^2\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} = -\frac{1}{[E_n^m(\rho)]^2}\mathbf{R}^{\mathrm{el}}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{108}$$

As far as the factor $\nabla \otimes \nabla I_n^m$ is concerned that appears in (107), we actually need the formula

$$\nabla \otimes \hat{\boldsymbol{\rho}} = \frac{\rho}{h_\rho}\left[\frac{\hat{\boldsymbol{\mu}} \otimes \hat{\boldsymbol{\mu}}}{\rho^2-\mu^2} + \frac{\hat{\boldsymbol{\nu}} \otimes \hat{\boldsymbol{\nu}}}{\rho^2-\nu^2}\right] + \frac{\mu}{h_\mu}\frac{\hat{\boldsymbol{\rho}} \otimes \hat{\boldsymbol{\mu}}}{\rho^2-\mu^2} + \frac{\nu}{h_\nu}\frac{\hat{\boldsymbol{\rho}} \otimes \hat{\boldsymbol{\nu}}}{\rho^2-\nu^2} \tag{109}$$

with the metric coefficients given by Eq. (21) and

$$\frac{\partial}{\partial \rho}\mathbf{R}^{\mathrm{el}}(\mathbf{r}) = \frac{\partial}{\partial \rho}\left(\frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\right), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{110}$$

which is also written as

$$\begin{aligned}\frac{\partial}{\partial \rho}\mathbf{R}^{\mathrm{el}}(\mathbf{r}) = \frac{1}{(\rho^2-\mu^2)(\rho^2-\nu^2)}\Bigg\{&-\frac{2\rho(2\rho^2-\mu^2-\nu^2)}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\hat{\boldsymbol{\rho}} + \frac{1}{\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}[(2\rho^2-h_3^2-h_2^2)x_1\hat{\mathbf{x}}_1 \\ &+ (2\rho^2-h_2^2)x_2\hat{\mathbf{x}}_2 + (2\rho^2-h_3^2)x_3\hat{\mathbf{x}}_3]\Bigg\}\end{aligned} \tag{111}$$

or in ellipsoidal coordinates alone

$$\frac{\partial}{\partial\rho}\mathbf{R}^{\mathrm{el}}(\mathbf{r}) = \frac{1}{\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}(\rho^2-\mu^2)(\rho^2-\nu^2)} \times\left[-2\rho(2\rho^2-\mu^2-\nu^2)\frac{\rho}{h_\rho}\hat{\boldsymbol{\rho}}+(\rho^2-\nu^2)\frac{\mu}{h_\mu}\hat{\boldsymbol{\mu}}+(\rho^2-\mu^2)\frac{\nu}{h_\nu}\hat{\boldsymbol{\nu}}\right], \quad \mathbf{r}\in\Omega(\mathbb{R}^3). \tag{112}$$

These formulas seem to be complicated, but they are given in the appropriate form for the application of the boundary condition (53).

The nature of condition (51), restricts the degree of the velocity field (46) to one. Hence, we use ellipsoidal harmonics of degree $n\leqslant 2$, which provide the appropriate flow fields that are adequate for most applications. Besides, these terms are enough to calculate the most important term of the velocity. Consequently, we observe that the potential $\mathbf{\Phi}$ in (36) must be of degree one and the potential Φ_0 in (37) must be of degree two in the surface variables μ and ν. This implies that all the coefficients $\mathbf{e}_n^{(i)m}$, $\mathbf{e}_n^{(e)m}$ for $n>2$ and $d_n^{(i)m}$, $d_n^{(e)m}$ for $n>3$, vanish, i.e.,

$$\mathbf{e}_n^{(i)m}=\mathbf{e}_n^{(e)m}=\mathbf{0}, \quad n=2,3,\ldots, \quad m=1,2,\ldots,2n+1 \tag{113}$$

and

$$d_n^{(i)m}=d_n^{(e)m}=0, \quad n=3,4,\ldots, \quad m=1,2,\ldots,2n+1. \tag{114}$$

Hence, the expansions (46)–(49) degenerate to the following finite sums, where for particles of a particular size, the first term of the series is enough for most real applications. Our aim is to calculate the terms of the flow fields, which correspond to ellipsoidal harmonics of degree less or equal than two. Inserting the restrictions (113) and (114) into the flow fields, we conclude the finite dimensional projections that correspond to the first term of the velocity field, denoted by $\mathbf{v}^{(0)}$, $P^{(0)}$, $\boldsymbol{\omega}^{(0)}$, and $\tilde{\mathbf{\Pi}}^{(0)}$. Thus,

$$\mathbf{v}^{(0)}(\mathbf{r})=\frac{1}{2}\left\{\sum_{n=0}^{1}\sum_{m=1}^{2n+1}[\mathbf{e}_n^{(i)m}\cdot(\tilde{\mathbf{I}}\mathbb{E}_n^m(\mathbf{r})-\mathbf{r}\otimes\nabla\mathbb{E}_n^m(\mathbf{r}))+\mathbf{e}_n^{(e)m}\cdot(\tilde{\mathbf{I}}\mathbb{F}_n^m(\mathbf{r})-\mathbf{r}\otimes\nabla\mathbb{F}_n^m(\mathbf{r}))] -\sum_{n=0}^{2}\sum_{m=1}^{2n+1}[d_n^{(i)m}\nabla\mathbb{E}_n^m(\mathbf{r})+d_n^{(e)m}\nabla\mathbb{F}_n^m(\mathbf{r})]\right\}, \quad \mathbf{r}\in\Omega(\mathbb{R}^3) \tag{115}$$

defines the *first term* of the velocity field,

$$P^{(0)}(\mathbf{r})=P_0-\mu_0\sum_{n=0}^{1}\sum_{m=1}^{2n+1}\{\mathbf{e}_n^{(i)m}\cdot\nabla\mathbb{E}_n^m(\mathbf{r})+\mathbf{e}_n^{(e)m}\cdot\nabla\mathbb{F}_n^m(\mathbf{r})\}=p^{(0)}(\mathbf{r})+\rho_0 gh, \quad \mathbf{r}\in\Omega(\mathbb{R}^3) \tag{116}$$

defines the *first term* of the total pressure field,

$$\boldsymbol{\omega}^{(0)}(\mathbf{r})=\frac{1}{2}\sum_{n=0}^{1}\sum_{m=1}^{2n+1}\{\nabla\mathbb{E}_n^m(\mathbf{r})\times\mathbf{e}_n^{(i)m}+\nabla\mathbb{F}_n^m(\mathbf{r})\times\mathbf{e}_n^{(e)m}\}, \quad \mathbf{r}\in\Omega(\mathbb{R}^3) \tag{117}$$

defines the *first term* of the vorticity field, and

$$\tilde{\mathbf{\Pi}}^{(0)}(\mathbf{r})=-p^{(0)}(\mathbf{r})\tilde{\mathbf{I}}-\mu_0\left\{\sum_{n=0}^{1}\sum_{m=1}^{2n+1}[(\mathbf{e}_n^{(i)m}\cdot\mathbf{r})\nabla\otimes\nabla\mathbb{E}_n^m(\mathbf{r})+(\mathbf{e}_n^{(e)m}\cdot\mathbf{r})\nabla\otimes\nabla\mathbb{F}_n^m(\mathbf{r})] +\sum_{n=0}^{2}\sum_{m=1}^{2n+1}[d_n^{(i)m}\nabla\otimes\nabla\mathbb{E}_n^m(\mathbf{r})+d_n^{(e)m}\nabla\otimes\nabla\mathbb{F}_n^m(\mathbf{r})]\right\}, \quad \mathbf{r}\in\Omega(\mathbb{R}^3) \tag{118}$$

defines the *first term* of the stress tensor field. Of course, the remaining coefficients $\mathbf{e}_0^{(i)1}$, $\mathbf{e}_0^{(e)1}$ for $n=0$ and $\mathbf{e}_1^{(i)\kappa}$, $\mathbf{e}_1^{(e)\kappa}$, $\kappa=1,2,3$ for $n=1$, as well as the coefficients $d_0^{(i)1}$, $d_0^{(e)1}$ for $n=0$, $d_1^{(i)\kappa}$, $d_1^{(e)\kappa}$, $\kappa=1,2,3$ for $n=1$ and $d_2^{(i)1}$, $d_2^{(e)1}$, $d_2^{(i)2}$, $d_2^{(e)2}$, $d_2^{(i)6-\kappa}$, $d_2^{(e)6-\kappa}$, $\kappa=1,2,3$ or $d_2^{(i)\kappa+l}$, $d_2^{(e)\kappa+l}$, $\kappa,l=1,2,3$, $\kappa\neq l$ for $n=2$ have to be determined from the boundary conditions (51)–(53).

Application of the BCs (51)–(53) to the velocity (115) and to the stress tensor (118). By virtue of the initial definitions (50), we insert the velocity field (115) to the boundary condition (51) on the surface S_a of the solid ellipsoid and we conclude that

$$
\begin{aligned}
2(\mathbf{U}+\mathbf{\Omega}\times\mathbf{r}) = \Bigg\{ & \sum_{n=0}^{1}\sum_{m=1}^{2n+1}[(\mathbf{e}_n^{(i)m}+(2n+1)I_n^m(\rho)\mathbf{e}_n^{(e)m})\cdot(\tilde{\mathbf{I}}\mathbb{E}_n^m(\mathbf{r})-\mathbf{r}\otimes\nabla\mathbb{E}_n^m(\mathbf{r}))]-\sum_{n=0}^{2}\sum_{m=1}^{2n+1}[(d_n^{(i)m} \\
& +(2n+1)I_n^m(\rho)d_n^{(e)m})\nabla\mathbb{E}_n^m(\mathbf{r})]\Bigg\}+\mathbf{R}^{\mathrm{el}}(\mathbf{r})\Bigg\{\sum_{n=0}^{1}\sum_{m=1}^{2n+1}[(2n+1)(\mathbf{e}_n^{(e)m}\cdot\mathbf{r}) \\
& \times[E_n^m(\rho)]^{-2}\mathbb{E}_n^m(\mathbf{r})]+\sum_{n=0}^{2}\sum_{m=1}^{2n+1}[(2n+1)d_n^{(e)m}[E_n^m(\rho)]^{-2}\mathbb{E}_n^m(\mathbf{r})]\Bigg\}, \quad \rho=\rho_a, \qquad (119)
\end{aligned}
$$

where we have used Eqs. (84) and (105) as well as the complicated ellipsoidal factor (106) for every $\mu\in[h_3,h_2]$ and $\nu\in[-h_3,h_3]$. The three sums in the first curly brace on the right-hand side of (119) are of the first degree in the variables μ, ν. The two sums inside the second curly brace are of the second degree. We observe that due to the factor $\mathbf{R}^{\mathrm{el}}$ it is not possible to express the second curly brace in terms of a finite expression of surface ellipsoidal harmonics. Therefore, in practice, it is impossible to evaluate the coefficients from BC (119) explicitly. Nevertheless, since the harmonic potentials $\mathbf{\Phi}$ and Φ_0 of the general solution (4) and (5) are not independent,[17] the corresponding d's and $\mathbf{e}$'s coefficients are also not independent as well. Thus, based on the flexibility of the Papkovich–Neuber differential representation, we *choose* to express the $d_n^{(e)m}$ in terms of the $\mathbf{e}_n^{(e)m}$ in such a way that the two sums on the right-hand side of (119), which are multiplied by $\mathbf{R}^{\mathrm{el}}$, vanish for $\mathbf{r}\in S_a$. When this is done the "bad term" of the BC (119) will disappear and we can apply orthogonality to obtain relations between the rest unknown coefficients $d_n^{(i)m}$, $e_n^{(i)m}$, and $e_n^{(e)m}$. In that sense, the use of the aforementioned *technique* on $\rho=\rho_a$ forms the key to our method. For $\rho=\rho_a$ the vanishing of the ellipsoidal part of the boundary condition (119) implies that

$$
\Bigg\{\sum_{n=0}^{1}\sum_{m=1}^{2n+1}[(2n+1)(\mathbf{e}_n^{(e)m}\cdot\mathbf{r})[E_n^m(\rho)]^{-2}\mathbb{E}_n^m(\mathbf{r})]+\sum_{n=0}^{2}\sum_{m=1}^{2n+1}[(2n+1)d_n^{(e)m}[E_n^m(\rho)]^{-2}\mathbb{E}_n^m(\mathbf{r})]\Bigg\}=0, \quad \mathbf{r}\in S_a. \qquad (120)
$$

Representing now the factor $\mathbf{r}\mathbb{E}_n^m$ via surface ellipsoidal harmonics at $\rho=\rho_a$ with the aim of the formulas (67), (68), and (76)–(80), as well as relations (A20), (A32), and (A33) from the Appendix and using orthogonality arguments, the assumption (120) provides the following relations between the coefficients, i.e.,

$$d_0^{(e)1}=-h_1h_2h_3\sum_{i=1}^{3}\frac{(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i)}{h_i}, \qquad (121)$$

$$d_1^{(e)\kappa}=-\frac{h_\kappa(E_1^\kappa(\rho_a))^2}{3h_1h_2h_3}(\mathbf{e}_0^{(e)1}\cdot\hat{\mathbf{x}}_\kappa), \quad \kappa=1,2,3, \qquad (122)$$

$$d_2^{(e)1}=\frac{h_1h_2h_3}{5(\Lambda-\Lambda')}E_2^1(\rho_a)\sum_{i=1}^{3}\frac{(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i)}{h_i(\Lambda-\alpha_i^2)}, \qquad (123)$$

$$d_2^{(e)2} = -\frac{h_1h_2h_3}{5(\Lambda-\Lambda')}E_2^2(\rho_a)\sum_{i=1}^{3}\frac{(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i)}{h_i(\Lambda'-\alpha_i^2)}, \tag{124}$$

$$d_2^{(e)\kappa+l} = -\frac{3}{5h_1h_2h_3}[h_\kappa(E_1^\kappa(\rho_a))^2(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa)+h_l(E_1^l(\rho_a))^2(\mathbf{e}_1^{(e)\kappa}\cdot\hat{\mathbf{x}}_l)] \tag{125}$$

for $\kappa,l=1,2,3$, $\kappa\neq l$. Having expressed the external d-coefficients as a function of the external $\mathbf{e}$-coefficients and according to the *technique* (120), the ellipsoidal part of the boundary condition (119) is not present any more and eventually it becomes

$$2(\mathbf{U}+\mathbf{\Omega}\times\mathbf{r}) = \left\{\sum_{n=0}^{1}\sum_{m=1}^{2n+1}[(\mathbf{e}_n^{(i)m}+(2n+1)I_n^m(\rho)\mathbf{e}_n^{(e)m})\cdot(\tilde{\mathbf{I}}\mathbb{E}_n^m(\mathbf{r})-\mathbf{r}\otimes\nabla\mathbb{E}_n^m(\mathbf{r}))] - \sum_{n=0}^{2}\sum_{m=1}^{2n+1}[(d_n^{(i)m}+(2n+1)I_n^m(\rho)d_n^{(e)m})\nabla\mathbb{E}_n^m(\mathbf{r})]\right\}, \quad \mathbf{r}\in S_a. \tag{126}$$

In order to handle the condition (126) properly, we perform the following actions. First we use the Cartesian form of the solid ellipsoidal harmonics $\mathbb{E}_n^m$ and of the gradient acting on it $\nabla\mathbb{E}_n^m$ from (67), (68), and (86)–(91), respectively. Next we calculate the factor $\mathbf{r}\otimes\nabla\mathbb{E}_n^m$ in Cartesian form and finally, by virtue of the definitions (38), of Eqs. (76) and (77), of the connection formulas (121)–(124) and of the orthogonality relation (34), the condition (126), after long and tedious calculations, results

$$\frac{h_1h_2h_3}{h_\kappa}d_1^{(i)\kappa} = -2(\mathbf{U}\cdot\hat{\mathbf{x}}_\kappa)+(\mathbf{e}_0^{(i)1}\cdot\hat{\mathbf{x}}_\kappa)+[I_0^1(\rho_a)+(E_1^\kappa(\rho_a))^2I_1^\kappa(\rho_a)](\mathbf{e}_0^{(e)1}\cdot\hat{\mathbf{x}}_\kappa), \quad \kappa=1,2,3, \tag{127}$$

as well as

$$-\left[\frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{(\Lambda-\alpha_\kappa^2)}\frac{d_2^{(i)1}}{h_1h_2h_3}+\frac{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}{(\Lambda'-\alpha_\kappa^2)}\frac{d_2^{(i)2}}{h_1h_2h_3}\right] = \frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{(\Lambda-\alpha_\kappa^2)}\frac{E_2^1(\rho_a)I_2^1(\rho_a)}{(\Lambda-\Lambda')}\sum_{i=1}^{3}\frac{(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i)}{h_i(\Lambda-\alpha_i^2)} - \frac{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}{(\Lambda'-\alpha_\kappa^2)}\frac{E_2^2(\rho_a)I_2^2(\rho_a)}{(\Lambda-\Lambda')}\sum_{i=1}^{3}\frac{(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i)}{h_i(\Lambda'-\alpha_i^2)} \tag{128}$$

and

$$h_\kappa[(\mathbf{e}_1^{(i)l}\cdot\hat{\mathbf{x}}_\kappa)+3I_1^l(\rho_a)(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa)]-h_l[(\mathbf{e}_1^{(i)\kappa}\cdot\hat{\mathbf{x}}_l)+3I_1^\kappa(\rho_a)(\mathbf{e}_1^{(e)\kappa}\cdot\hat{\mathbf{x}}_l)]-h_1h_2h_3 \times[d_2^{(i)\kappa+l}+5I_2^{\kappa+l}(\rho_a)d_2^{(e)\kappa+l}] = \frac{2\lambda_{\kappa l}}{h_{6-(\kappa+l)}}(\mathbf{\Omega}\cdot\hat{\mathbf{x}}_{6-(\kappa+l)}), \quad \kappa,l=1,2,3, \quad \kappa\neq l, \tag{129}$$

where

$$\lambda_{\kappa l} = \begin{cases}+1, & (\kappa,l)=(1,3),(2,1),(3,2)\\ -1, & (\kappa,l)=(1,2),(2,3),(3,1)\end{cases}, \quad \kappa,l=1,2,3, \quad \kappa\neq l. \tag{130}$$

The next step now will be the investigation of the second condition on the boundary $\rho=\rho_b$ of the fictitious ellipsoidal surface S_b. In view of the boundary condition (52) we obtain

$$\left\{ \sum_{n=0}^{1} \sum_{m=1}^{2n+1} [(\hat{\boldsymbol{\rho}} \cdot \mathbf{e}_n^{(i)m}) \mathbb{E}_n^m(\mathbf{r}) - (\mathbf{e}_n^{(i)m} \cdot \mathbf{r})(\hat{\boldsymbol{\rho}} \cdot \nabla \mathbb{E}_n^m(\mathbf{r})) + (\hat{\boldsymbol{\rho}} \cdot \mathbf{e}_n^{(e)m}) \mathbb{F}_n^m(\mathbf{r}) - (\mathbf{e}_n^{(e)m} \cdot \mathbf{r})(\hat{\boldsymbol{\rho}} \cdot \nabla \mathbb{F}_n^m(\mathbf{r}))] \right.$$
$$\left. - \sum_{n=0}^{2} \sum_{m=1}^{2n+1} [d_n^{(i)m}(\hat{\boldsymbol{\rho}} \cdot \nabla \mathbb{E}_n^m(\mathbf{r})) + d_n^{(e)m}(\hat{\boldsymbol{\rho}} \cdot \nabla \mathbb{F}_n^m(\mathbf{r}))] \right\} = 0, \quad \mathbf{r} \in S_b, \tag{131}$$

where we utilized the assumption for the velocity field (115). It is obvious that the appearance of the unit normal vector $\hat{\boldsymbol{\rho}}$, provided by expression (25), increases the degree of the harmonic eigenmodes in Eq. (131) up to $n=2$. More specifically using the Cartesian-ellipsoidal formulas (67)–(72), (76)–(80), and (86)–(91), as well as

$$\hat{\boldsymbol{\rho}} \cdot \nabla \mathbb{F}_n^m(\mathbf{r}) = (2n+1)\left[I_n^m(\rho)(\hat{\boldsymbol{\rho}} \cdot \nabla \mathbb{E}_n^m(\mathbf{r})) - \frac{1}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} \frac{\mathbb{E}_n^m(\mathbf{r})}{[E_n^m(\rho)]^2} \right] \tag{132}$$

for $\rho=\rho_b$ and $n=0,1,2$, $m=1,2,\ldots,2n+1$, we convert, after long calculations, condition (131) to a suitable form, which is appropriate for applying orthogonality via (34). Thus, after having produced the ellipsoidal form of the factors $(\hat{\boldsymbol{\rho}} \cdot \nabla \mathbb{E}_n^m)$ for $n=0,1,2$, $m=1,2,\ldots,2n+1$ and $(\hat{\boldsymbol{\rho}} \cdot \mathbf{e}_n^{(i/e)m})$, $(\mathbf{r} \cdot \mathbf{e}_n^{(i/e)m})$ for $n=0,1$, $m=1,2,\ldots,2n+1$, we apply orthogonality and obtain nine more relations involving the still unknown coefficients. The number of equations we obtain is a result of the scalar character of the boundary condition (131) and, of course, of the number of the surface ellipsoidal harmonics for $n \leq 2$. We mention that no indeterminacy appeared in our program and all the calculations were easily performed. Actually, orthogonality of the surface ellipsoidal harmonic $E_0^1(\mu)E_0^1(\nu)$ provides the already known relation (121), which is independent of the boundary. On the other hand, by orthogonality arguments of $E_1^\kappa(\mu)E_1^\kappa(\nu)$, $\kappa=1,2,3$ and relation (127) we evaluate the coefficient $\mathbf{e}_0^{(e)1}$ in the form

$$(\mathbf{e}_0^{(e)1} \cdot \hat{\mathbf{x}}_\kappa) = -\frac{2E_3^7(\rho_b)}{N_\kappa}(\mathbf{U} \cdot \hat{\mathbf{x}}_\kappa), \quad \kappa = 1,2,3, \tag{133}$$

where

$$N_\kappa = E_3^7(\rho_b)[(I_0^1(\rho_b) - I_0^1(\rho_a)) + (E_1^\kappa(\rho_a))^2(I_1^\kappa(\rho_b) - I_1^\kappa(\rho_a))] + (E_1^\kappa(\rho_b))^2 - (E_1^\kappa(\rho_a))^2, \quad \kappa = 1,2,3. \tag{134}$$

Until now, four of the nine equations, which came up from boundary condition (131), have been used. The five remaining equations correspond to orthogonality on the surface ellipsoidal harmonics of degree $n=2$ for every $m=1,2,\ldots,5$. Hence, the surface ellipsoidal harmonics $E_2^1(\mu)E_2^1(\nu)$ and $E_2^2(\mu)E_2^2(\nu)$ lead to

$$-5d_2^{(e)1} + 2E_2^1(\rho_b)E_3^7(\rho_b)(d_2^{(i)1} + 5I_2^1(\rho_b)d_2^{(e)1}) + \frac{h_1h_2h_3E_2^1(\rho_b)}{(\Lambda-\Lambda')} \sum_{i=1}^{3} \frac{(\mathbf{e}_1^{(e)i} \cdot \hat{\mathbf{x}}_i)}{h_i(\Lambda-\alpha_i^2)} = 0 \tag{135}$$

and

$$-5d_2^{(e)2} + 2E_2^2(\rho_b)E_3^7(\rho_b)(d_2^{(i)2} + 5I_2^2(\rho_b)d_2^{(e)2}) - \frac{h_1h_2h_3E_2^2(\rho_b)}{(\Lambda-\Lambda')} \sum_{i=1}^{3} \frac{(\mathbf{e}_1^{(e)i} \cdot \hat{\mathbf{x}}_i)}{h_i(\Lambda'-\alpha_i^2)} = 0, \tag{136}$$

respectively, where the identity (A20) has been used. Inserting Eqs. (123) and (124) to relation (128), we obtain the following 2×2 homogeneous system of linear equations, in the variables $(d_2^{(i)1}+5I_2^1(\rho_a)d_2^{(e)1})$ and $(d_2^{(i)2}+5I_2^2(\rho_a)d_2^{(e)2})$. For example, for $\kappa=1,2$ we have

$$(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)(d_2^{(i)1} + 5I_2^1(\rho_a)d_2^{(e)1}) + (\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)(d_2^{(i)2} + 5I_2^2(\rho_a)d_2^{(e)2}) = 0 \tag{137}$$

and

$$(\Lambda - \alpha_1^2)(\Lambda - \alpha_3^2)(d_2^{(i)1} + 5I_2^1(\rho_a)d_2^{(e)1}) + (\Lambda' - \alpha_1^2)(\Lambda' - \alpha_3^2)(d_2^{(i)2} + 5I_2^2(\rho_a)d_2^{(e)2}) = 0. \quad (138)$$

Since the determinant is not zero, it follows that

$$d_2^{(i)1} + 5I_2^1(\rho_a)d_2^{(e)1} = d_2^{(i)2} + 5I_2^2(\rho_a)d_2^{(e)2} = 0 \quad (139)$$

or

$$d_2^{(i)1} = -5I_2^1(\rho_a)d_2^{(e)1}, \quad d_2^{(i)2} = -5I_2^2(\rho_a)d_2^{(e)2}. \quad (140)$$

We could come up with the same result if we had used expression (128) for $\kappa=1,3$ or for $\kappa=2,3$.

Our next step involves the substitution of the results (140) to (135) and (136), to obtain

$$\frac{h_1h_2h_3E_2^1(\rho_b)}{(\Lambda - \Lambda')}\sum_{i=1}^{3}\frac{(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i)}{h_i(\Lambda - \alpha_i^2)}\left[1 - \frac{E_2^1(\rho_a)}{E_2^1(\rho_b)} + 2E_3^7(\rho_b)E_2^1(\rho_a)(I_2^1(\rho_b) - I_2^1(\rho_a))\right] = 0 \quad (141)$$

and

$$\frac{h_1h_2h_3E_2^2(\rho_b)}{(\Lambda - \Lambda')}\sum_{i=1}^{3}\frac{(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i)}{h_i(\Lambda' - \alpha_i^2)}\left[1 - \frac{E_2^2(\rho_a)}{E_2^2(\rho_b)} + 2E_3^7(\rho_b)E_2^2(\rho_a)(I_2^2(\rho_b) - I_2^2(\rho_a))\right] = 0, \quad (142)$$

where the nonvanishing of the two square brackets implies

$$\sum_{i=1}^{3}\frac{(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i)}{h_i(\Lambda - \alpha_i^2)} = \sum_{i=1}^{3}\frac{(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i)}{h_i(\Lambda' - \alpha_i^2)} = 0. \quad (143)$$

As a consequence relations (123) and (124) force the coefficients $d_2^{(e)1}$ and $d_2^{(e)2}$ to vanish and then from (140) we obtain

$$d_2^{(i)1} = d_2^{(e)1} = 0 \quad (144)$$

and

$$d_2^{(i)2} = d_2^{(e)2} = 0. \quad (145)$$

In addition, working on the two summations (143) and with proper use of identities (A22) and (A23) we conclude that

$$(\mathbf{e}_1^{(e)3}\cdot\hat{\mathbf{x}}_3) = \frac{h_3}{h_1}(\mathbf{e}_1^{(e)1}\cdot\hat{\mathbf{x}}_1) = \frac{h_3}{h_2}(\mathbf{e}_1^{(e)2}\cdot\hat{\mathbf{x}}_2), \quad (146)$$

where two of the three diagonal scalar coefficients of $\mathbf{e}_1^{(e)\kappa}$, $\kappa=1,2,3$ have been evaluated via the third one. Finally, by means of orthogonality on the BC (131) of $E_2^{6-\kappa}(\mu)E_2^{6-\kappa}(\nu)$, $\kappa=1,2,3$, we derive the following elaborate relation, valid on the ellipsoidal surface S_b:

$$\begin{aligned}&E_3^7(\rho_b)h_{6-(\kappa+l)}^2\{h_\kappa[(\mathbf{e}_1^{(i)l}\cdot\hat{\mathbf{x}}_\kappa) + 3I_1^l(\rho_b)(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa)] - h_l[(\mathbf{e}_1^{(i)\kappa}\cdot\hat{\mathbf{x}}_l) + 3I_1^\kappa(\rho_b)(e_1^{(\mathbf{e})\kappa}\cdot\hat{\mathbf{x}}_l)]\}\\ &+ h_1h_2h_3[5d_2^{(e)\kappa+l} - E_3^7(\rho_b)((E_1^\kappa(\rho_b))^2 + (E_1^l(\rho_b))^2)(d_2^{(i)\kappa+l} + 5I_2^{\kappa+l}(\rho_b)d_2^{(e)\kappa+l})]\\ &+ 3[h_\kappa(E_1^\kappa(\rho_b))^2(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa) + h_l(E_1^l(\rho_b))^2(\mathbf{e}_1^{(e)\kappa}\cdot\hat{\mathbf{x}}_l)] = 0, \quad \kappa,l = 1,2,3, \quad \kappa > l, \end{aligned} \quad (147)$$

where by $\kappa > l$ we mean all the possible combinations such as $(\kappa,l)=(2,1),(3,1),(3,2)$. At this point, we take advantage of the particular form of the three pairs of equations in relation (129) by substracting one relation from the other of each pair that contains the same components of the applied vorticity $\mathbf{\Omega}$, to obtain the three equations

$$d_2^{(i)\kappa+l} + 5I_2^{\kappa+l}(\rho_a)d_2^{(e)\kappa+l} = 0 \Rightarrow d_2^{(i)\kappa+l} = -5I_2^{\kappa+l}(\rho_a)d_2^{(e)\kappa+l}, \quad \kappa,l = 1,2,3, \quad \kappa \neq l. \quad (148)$$

If we return to condition (147), we observe that we have to deal with three different relations for every value of $(\kappa,l)=(2,1),(3,1),(3,2)$. Therefore, by virtue of (148) and inserting (125) and (129) to (147), we conclude the symmetric expression

$$P_{\kappa}^{l}h_{\kappa}(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_{\kappa})+P_{l}^{\kappa}h_{l}(\mathbf{e}_1^{(e)\kappa}\cdot\hat{\mathbf{x}}_{l})=R^{\kappa+l},\quad \kappa,l=1,2,3,\quad \kappa\neq l \tag{149}$$

with

$$\begin{aligned}P_{\kappa}^{l}=&\frac{\rho_b^2-\rho_a^2}{(E_2^{\kappa+l}(\rho_b))^2}+E_3^7(\rho_b)\frac{\alpha_l^2-\alpha_\kappa^2}{(E_2^{\kappa+l}(\rho_b))^2}(I_1^l(\rho_b)-I_1^l(\rho_a))+E_3^7(\rho_b)(E_1^\kappa(\rho_a))^2\left(\frac{1}{(E_1^\kappa(\rho_b))^2}+\frac{1}{(E_1^l(\rho_b))^2}\right)\\&\times(I_2^{\kappa+l}(\rho_b)-I_2^{\kappa+l}(\rho_a))\end{aligned} \tag{150}$$

for every $\kappa,l=1,2,3$ and $\kappa\neq l$, and

$$\begin{aligned}R^{\kappa+l}&=\frac{2h_{6-(\kappa+l)}\Omega_{6-(\kappa+l)}}{3}(-1)^{\kappa+l}\frac{E_1^{6-(\kappa+l)}(\rho_b)}{E_2^{\kappa+l}(\rho_b)}\\&=\frac{2h_1h_2h_3\Omega_{6-(\kappa+l)}}{3}(-1)^{\kappa+l}\frac{E_3^7(\rho_b)}{(E_2^{\kappa+l}(\rho_b))^2},\quad \kappa,l=1,2,3,\quad \kappa\neq l.\end{aligned} \tag{151}$$

Recapitulating, we see that the effect of the two first boundary conditions (51) and (52) to the evaluation of the unknown coefficients, in the flow fields (115)–(118), is to obtain nine unknown coefficients $\mathbf{e}_1^{(e)\kappa}$, $\kappa=1,2,3$ from five relations, two from (146) and three from (149). Thus, there remain four coefficients that must be calculated explicitly from the final condition (53) on the ellipsoid $\rho=\rho_b$. However, the diagonal components of $\mathbf{e}_1^{(e)\kappa}$, $\kappa=1,2,3$, appearing in Eq. (146), will not enter the flow fields, as we shall see later on. This means that in practice we need to calculate three instead of four coefficients.

The application of the boundary condition (53), where in view of the stress tensor field (118), and according to the form of the unit dyadic (26) gives

$$\hat{\boldsymbol{\rho}}\cdot\tilde{\mathbf{I}}\cdot(\tilde{\mathbf{I}}-\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\rho}})=\hat{\boldsymbol{\rho}}\cdot\tilde{\mathbf{I}}\cdot(\hat{\boldsymbol{\mu}}\otimes\hat{\boldsymbol{\mu}}+\hat{\boldsymbol{\nu}}\otimes\hat{\boldsymbol{\nu}})=\mathbf{0}, \tag{152}$$

provides the following expression in terms of the unit normal vector $\hat{\boldsymbol{\rho}}$,

$$\begin{aligned}(\tilde{\mathbf{I}}-\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\rho}})\cdot\Bigg\{&\sum_{n=0}^{1}\sum_{m=1}^{2n+1}[(\mathbf{e}_n^{(i)m}\cdot\mathbf{r})(\hat{\boldsymbol{\rho}}\cdot\nabla\otimes\nabla\mathbb{E}_n^m(\mathbf{r}))+(\mathbf{e}_n^{(e)m}\cdot\mathbf{r})(\hat{\boldsymbol{\rho}}\cdot\nabla\otimes\nabla\mathbb{F}_n^m(\mathbf{r}))]\\&+\sum_{n=0}^{2}\sum_{m=1}^{2n+1}[d_n^{(i)m}(\hat{\boldsymbol{\rho}}\cdot\nabla\otimes\nabla\mathbb{E}_n^m(\mathbf{r}))+d_n^{(e)m}(\hat{\boldsymbol{\rho}}\cdot\nabla\otimes\nabla\mathbb{F}_n^m(\mathbf{r}))]\Bigg\}=\mathbf{0},\quad \mathbf{r}\in S_b.\end{aligned} \tag{153}$$

In order to deal with condition (153) we perform the following operations. First, we use the double action of the gradient operator on the interior solid ellipsoidal harmonics provided by relations (95)–(100) for the evaluation of the $\hat{\boldsymbol{\rho}}\cdot\nabla\otimes\nabla\mathbb{E}_n^m$, $n=0,1,2$, $m=1,2,\ldots,2n+1$. Then, using the expression

$$\hat{\boldsymbol{\rho}}=\frac{E_3^7(\rho)}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\sum_{i=1}^{3}\frac{x_i\hat{\mathbf{x}}_i}{(E_1^i(\rho))^2},\quad \rho=\rho_b \tag{154}$$

for the unit outward normal on S_b, we obtain

$$\hat{\boldsymbol{\rho}}\cdot\nabla I_n^m(\rho)=-\frac{1}{[E_n^m(\rho)]^2\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}},\quad \rho=\rho_b, \tag{155}$$

because

$$\mathbf{R}^{\mathrm{el}}(\mathbf{r}) = \frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}, \quad \rho=\rho_b. \tag{156}$$

Also, according to the definition (22), and with the help of relation (108), we reach at

$$\begin{aligned}\hat{\boldsymbol{\rho}}\cdot\nabla\nabla I_n^m(\rho) &= \frac{\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\frac{\partial}{\partial\rho}\nabla I_n^m(\rho) \\ &= -\frac{\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\left[\frac{\partial}{\partial\rho}[E_n^m(\rho)]^{-2}\mathbf{R}^{\mathrm{el}}(\mathbf{r}) + [E_n^m(\rho)]^{-2}\frac{\partial}{\partial\rho}\mathbf{R}^{\mathrm{el}}(\mathbf{r})\right]\end{aligned} \tag{157}$$

for $\rho=\rho_b$. Some easy algebra on Eq. (157) furnishes

$$\hat{\boldsymbol{\rho}}\cdot\nabla\nabla I_n^m(\rho) = \frac{\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}{[E_n^m(\rho)]^3\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\left[2E_n^{m\prime}(\rho)\mathbf{R}^{\mathrm{el}}(\mathbf{r}) - E_n^m(\rho)\frac{\partial\mathbf{R}^{\mathrm{el}}(\mathbf{r})}{\partial\rho}\right], \tag{158}$$

where $\rho=\rho_b$, whilst the ρ-derivative of the factor $\mathbf{R}^{\mathrm{el}}$ has been calculated through its Cartesian (111) and its ellipsoidal (112) representation. Substituting expressions (155) and (158) into the double gradient of the exterior solid ellipsoidal harmonics (107), inserting relations (86)–(91) and (95)–(100) at (107) and using

$$\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\rho}} = E_3^7(\rho)\sum_{i=1}^{3}\frac{x_i}{(E_1^i(\rho))^2}\hat{\mathbf{x}}_i\otimes\mathbf{R}^{\mathrm{el}}(\mathbf{r}) = E_3^7(\rho)\sum_{i=1}^{3}\frac{x_i}{(E_1^i(\rho))^2}\mathbf{R}^{\mathrm{el}}(\mathbf{r})\otimes\hat{\mathbf{x}}_i, \quad \rho=\rho_b, \tag{159}$$

the boundary condition (153), on $\rho=\rho_b$, is written as

$$\begin{aligned}(-\tilde{\mathbf{I}}+\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\rho}})\cdot\Bigg\{&\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}\frac{\partial\mathbf{R}^{\mathrm{el}}(\mathbf{r})}{\partial\rho}\left[\sum_{n=0}^{1}\sum_{m=1}^{2n+1}\frac{(2n+1)}{[E_n^m(\rho)]^2}\mathbb{E}_n^m(\mathbf{r})(\mathbf{e}_n^{(e)m}\cdot\mathbf{r})\right.\\ &\left.+\sum_{n=0}^{2}\sum_{m=1}^{2n+1}\frac{(2n+1)}{[E_n^m(\rho)]^2}\mathbb{E}_n^m(\mathbf{r})d_n^{(e)m}\right] - \sum_{m=1}^{5}[(d_2^{(i)m}+5I_2^m(\rho)d_2^{(e)m})\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}(\hat{\boldsymbol{\rho}}\cdot\nabla\otimes\nabla\mathbb{E}_n^m(\mathbf{r}))]\\ &+\left[\sum_{n=0}^{1}\sum_{m=1}^{2n+1}\frac{(2n+1)}{[E_n^m(\rho)]^2}\nabla\,\mathbb{E}_n^m(\mathbf{r})(\mathbf{e}_n^{(e)m}\cdot\mathbf{r}) + \sum_{n=0}^{2}\sum_{m=1}^{2n+1}\frac{(2n+1)}{[E_n^m(\rho)]^2}\nabla\,\mathbb{E}_n^m(\mathbf{r})d_n^{(e)m}\right]\Bigg\} = \mathbf{0}, \quad \rho=\rho_b.\end{aligned} \tag{160}$$

A careful observation of condition (160) reveals the difficulty to manipulate it in Cartesian coordinates and the impossibility of direct application of orthogonality arguments. Nevertheless, the use of purely ellipsoidal terms simplifies Eq. (160). In particular, we perform all the necessary calculations, use (67)–(72), (76)–(80), and (86)–(91), insert the already known coefficients, and use the fact that

$$\begin{aligned}(\hat{\boldsymbol{\mu}}\otimes\hat{\boldsymbol{\mu}}+\hat{\boldsymbol{\nu}}\otimes\hat{\boldsymbol{\nu}})\cdot\sum_{i=1}^{3}x_i\hat{\mathbf{x}}_i\frac{3h_1h_2h_3}{h_i(E_1^i(\rho_b))^2}(\mathbf{e}_1^{(e)i}\cdot\hat{\mathbf{x}}_i) &= 3h_1h_2(\mathbf{e}_1^{(e)3}\cdot\hat{\mathbf{x}}_3)(\hat{\boldsymbol{\mu}}\otimes\hat{\boldsymbol{\mu}}+\hat{\boldsymbol{\nu}}\otimes\hat{\boldsymbol{\nu}})\cdot\sum_{i=1}^{3}\frac{x_i\hat{\mathbf{x}}_i}{(E_1^i(\rho_b))^2}\\ &= 3h_1h_2(\mathbf{e}_1^{(e)3}\cdot\hat{\mathbf{x}}_3)(\hat{\boldsymbol{\mu}}\otimes\hat{\boldsymbol{\mu}}+\hat{\boldsymbol{\nu}}\otimes\hat{\boldsymbol{\nu}})\cdot\left(\frac{h_\rho}{\rho_b}\hat{\boldsymbol{\rho}}\right) = \mathbf{0},\\ &\rho=\rho_b,\end{aligned} \tag{161}$$

to rewrite the boundary condition (160) in the simple form

$$\mathrm{A}(\rho_b,\mu,\nu)\hat{\boldsymbol{\mu}}+\mathrm{B}(\rho_b,\mu,\nu)\hat{\boldsymbol{\nu}}=\mathbf{0}, \tag{162}$$

where the quantities A and B contain surface ellipsoidal harmonics, the coefficients under evaluation, and some known constants. The orthogonality of the unit normal vectors $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\nu}}$ implies that A and B are equal to zero, i.e.,

$$\sum_{i=1}^{3} B_i\left(\frac{(E_1^i(\rho_a))^2}{x^2-\alpha_1^2+\alpha_i^2}-1\right)E_1^i(\mu)E_1^i(\nu)+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\left(\frac{A_i^j+(E_1^i(\rho_b))^2C_i^j}{x^2-\alpha_1^2+\alpha_i^2}-C_i^j\right)E_2^{i+j}(\mu)E_2^{i+j}(\nu)=0, \tag{163}$$

where x assumes the values μ and ν, and

$$A_\kappa^l=\frac{3(E_1^\kappa(\rho_b))^2(\rho_b^2-\rho_a^2)h_l}{E_2^{\kappa+l}(\rho_b)}(\mathbf{e}_1^{(e)\kappa}\cdot\hat{\mathbf{x}}_l),\quad \kappa,l=1,2,3,\quad \kappa\neq l, \tag{164}$$

$$B_\kappa=\frac{2E_3^7(\rho_b)E_1^\kappa(\rho_b)h_\kappa U_\kappa}{N_\kappa},\quad \kappa=1,2,3, \tag{165}$$

with N_κ, $\kappa=1,2,3$ given in (134), while

$$\begin{aligned}C_\kappa^l=\frac{3}{E_2^{\kappa+l}(\rho_b)}[&-h_\kappa(E_1^\kappa(\rho_a))^2(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa)+E_3^7(\rho_b)(E_1^\kappa(\rho_b))^2(I_2^{\kappa+l}(\rho_b)-I_2^{\kappa+l}(\rho_a))(h_\kappa(E_1^\kappa(\rho_a))^2(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa)\\ &+h_l(E_1^l(\rho_a))^2(\mathbf{e}_1^{(e)\kappa}\cdot\hat{\mathbf{x}}_l))],\quad \kappa,l=1,2,3,\quad \kappa\neq l.\end{aligned} \tag{166}$$

On the other hand, $(\mathbf{e}_1^{(e)\kappa}\cdot\hat{\mathbf{x}}_l)$ for $\kappa,l=1,2,3$ and $\kappa\neq l$ provide the six unknown coefficients, which must be evaluated with the aid of (149).

At this stage we have to deal with two major problems. The first one reflects the difficulty of applying orthogonality of the surface ellipsoidal harmonics $E_n^m(\mu)E_n^m(\nu)$ in (163), and the second concerns the number of the equations that we have to satisfy. In fact, we are left with five relations (three from (149) and two from (163)) to evaluate the six coefficients mentioned earlier. As far as the first difficulty is concerned, we introduce the new set of elliptic integrals on the surface of the ellipsoid $\rho=\rho_s$,

$$J_{n,\kappa}^m(\rho_s)=\int\int_{\rho=\rho_s}\frac{E_n^m(\mu)E_n^m(\nu)}{(x^2-\alpha_1^2+\alpha_\kappa^2)\sqrt{\rho_s^2-\mu^2}\sqrt{\rho_s^2-\nu^2}}\mathrm{d}S,\quad \kappa=1,2,3 \tag{167}$$

for $\mu\in[h_3,h_2]$, $\nu\in[-h_3,h_3]$, where we define the following values of x:

$$\text{for } x=\mu\Rightarrow J_{n,\kappa}^m\equiv M_{n,\kappa}^m\quad\text{and for } x=\nu\Rightarrow J_{n,\kappa}^m\equiv N_{n,\kappa}^m. \tag{168}$$

Therefore, we multiply (163) by the weighting function (33) and use the orthogonality relation (34) and the zeroth degree eigenfunction $E_0^1(\mu)E_0^1(\nu)\equiv 1$ to obtain

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} X_i^j h_i(\mathbf{e}_1^{(e)j}\cdot\hat{\mathbf{x}}_i)=Q_x,\quad x=\mu,\nu,\quad \mu\in[h_3,h_2],\quad \nu\in[-h_3,h_3], \tag{169}$$

with

$$X_{\kappa}^{l}=\frac{3}{E_2^{\kappa+l}(\rho_b)}\{(E_1^l(\rho_b))^4J_{2,l}^{\kappa+l}(\rho_b)-(E_1^{\kappa}(\rho_a))^2[J_{2,l}^{\kappa+l}(\rho_b)(E_1^l(\rho_b))^2(1-E_3^7(\rho_b)(E_1^l(\rho_b))^2(I_2^{\kappa+l}(\rho_b)-I_2^{\kappa+l}(\rho_a)))+J_{2,\kappa}^{\kappa+l}(\rho_b)(E_1^{\kappa}(\rho_b))^2(1-E_3^7(\rho_b)(E_1^{\kappa}(\rho_b))^2(I_2^{\kappa+l}(\rho_b)-I_2^{\kappa+l}(\rho_a)))]\} \tag{170}$$

for every $\kappa,l=1,2,3$ and $\kappa\neq l$, and

$$Q_x=-\sum_{i=1}^{3}\frac{2E_3^7(\rho_b)E_1^i(\rho_b)(E_1^i(\rho_a))^2h_iU_i}{N_i}J_{1,i}^i(\rho_b) \tag{171}$$

for $x=\mu,\nu$. The constants N_κ, $\kappa=1,2,3$ are given in (134). Relation (169) refers to two different equations for the values of $x=\mu,\nu$ containing the two different types of elliptic integrals appearing in (168). Despite the simplification of the final boundary condition by the introduction of a new set of elliptic integrals, our already reduced potential problem remains undetermined as far as one coefficient is concerned. Indeed, combining (149) with relation (169) we see that we are missing one more condition. Hence, we adopt, once more, our previous analysis, which springs from the flexibility of the Papkovich–Neuber differential representation.[17] In fact, since we have two boundary surfaces to satisfy our conditions, we use our technique twice, one for each boundary. Therefore, in terms of the elliptic integrals

$$L_{n,\kappa}^m(\rho_s)=N_{n,\kappa}^m(\rho_s)-M_{n,\kappa}^m(\rho_s)=\int\int_{\rho=\rho_s}\frac{(\mu^2-\nu^2)E_n^m(\mu)E_n^m(\nu)}{(\mu^2-\alpha_1^2+\alpha_\kappa^2)(\nu^2-\alpha_1^2+\alpha_\kappa^2)}\frac{\mathrm{d}S}{\sqrt{\rho_s^2-\mu^2}\sqrt{\rho_s^2-\nu^2}},\quad \kappa=1,2,3, \tag{172}$$

we substract by parts Eq. (169) for the different values of $x=\mu,\nu$ and we *choose* the following condition for the unknown coefficients, i.e.,

$$Q_\kappa^l h_\kappa(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa)+Q_l^\kappa h_l(\mathbf{e}_1^{(e)\kappa}\cdot\hat{\mathbf{x}}_l)\equiv\tfrac{1}{3}(Q_\nu-Q_\mu)=Q,\quad \kappa,l=1,2,3,\quad \kappa\neq l, \tag{173}$$

where

$$Q_{\kappa}^{l}=\frac{3}{E_2^{\kappa+l}(\rho_b)}\{(E_1^l(\rho_b))^4L_{2,l}^{\kappa+l}(\rho_b)-(E_1^{\kappa}(\rho_a))^2[L_{2,l}^{\kappa+l}(\rho_b)(E_1^l(\rho_b))^2(1-E_3^7(\rho_b)(E_1^l(\rho_b))^2(I_2^{\kappa+l}(\rho_b)-I_2^{\kappa+l}(\rho_a)))+L_{2,\kappa}^{\kappa+l}(\rho_b)(E_1^{\kappa}(\rho_b))^2(1-E_3^7(\rho_b)(E_1^{\kappa}(\rho_b))^2(I_2^{\kappa+l}(\rho_b)-I_2^{\kappa+l}(\rho_a)))]\} \tag{174}$$

and

$$Q=-\frac{2E_3^7(\rho_b)}{3}\sum_{i=1}^{3}\frac{E_1^i(\rho_b)(E_1^i(\rho_a))^2h_iU_i}{N_i}L_{1,i}^i(\rho_b), \tag{175}$$

with the constants N_κ, $\kappa=1,2,3$ given by (134). This way we generate an additional relation for the evaluation of the coefficients through condition (173). This is the second and final restriction to our problem, which provided the three explicit equations (173).

We solve a system of six equations for the last six unknown coefficients $(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa)$, $\kappa,l=1,2,3$, $\kappa\neq l$, taking into account relations (149) and (173). Defining $(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa)\equiv e_\kappa^l$, we rewrite the solution as

$$(\mathbf{e}_1^{(e)l}\cdot\hat{\mathbf{x}}_\kappa)\equiv\mathbf{e}_\kappa^l=\frac{P_l^\kappa Q-Q_l^\kappa R^{\kappa+l}}{h_\kappa(P_l^\kappa Q_\kappa^l-P_\kappa^l Q_l^\kappa)},\quad \kappa,l=1,2,3,\quad \kappa\neq l, \tag{176}$$

where the constants P_κ^l, Q_κ^l, $R^{\kappa+l}$ for $\kappa,l=1,2,3$, $\kappa\neq l$, and Q are given by (150), (174), (151), and (175), respectively. It is easily proved that $P_l^\kappa Q_\kappa^l-P_\kappa^l Q_l^\kappa\neq 0$. Hence, having used all the boundary conditions (51)–(53) we evaluated all the coefficients.

The final step contains the presentation of a mixed Cartesian-ellipsoidal form of the *first terms* of the flow fields formulated by expressions (115)–(118), using the information (67)–(112). Introducing all the evaluated coefficients from the previous steps and using the Cartesian-ellipsoidal formulation of the interior and exterior ellipsoidal harmonics, as well as their gradients, some extended algebra on the velocity field (115) leads to

$$\mathbf{v}^{(0)}(\mathbf{r}) = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r} + \mathbf{Z}(\rho) + \sum_{j=1}^{3} \mathbf{H}_j(\rho)\mathbb{E}_1^j(\mathbf{r}) + \frac{\hat{\boldsymbol{\rho}}}{2\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} \times \left[\sum_{j=1}^{3} \Theta_j(\rho)\mathbb{E}_1^j(\mathbf{r}) + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \Phi_i^j(\rho)\mathbb{E}_2^{i+j}(\mathbf{r}) \right], \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{177}$$

where for every $\rho_a \leqslant \rho \leqslant \rho_b$ we define the vector quantities

$$\mathbf{Z}(\rho) = - E_3^7(\rho_b)\mathbf{U} \cdot \sum_{i=1}^{3} [(I_0^1(\rho) - I_0^1(\rho_a)) + (E_1^i(\rho_a))^2(I_1^i(\rho) - I_1^i(\rho_a))] \frac{\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i}{N_i} \tag{178}$$

and

$$\mathbf{H}_\kappa(\rho) = \frac{3}{2} \sum_{\substack{i=1\\ i\neq\kappa}}^{3} \{h_i e_i^\kappa(I_1^\kappa(\rho) - I_1^\kappa(\rho_a)) - h_\kappa e_\kappa^i(I_1^i(\rho) - I_1^i(\rho_a)) + [h_i e_i^\kappa(E_1^i(\rho_a))^2 + h_\kappa e_\kappa^i(E_1^\kappa(\rho_a))^2](I_2^{i+\kappa}(\rho) - I_2^{i+\kappa}(\rho_a))\} \frac{\hat{\mathbf{x}}_i}{h_i}, \quad \kappa = 1,2,3. \tag{179}$$

Furthermore, the scalar products for the same interval $\rho_a \leqslant \rho \leqslant \rho_b$ are

$$\Theta_\kappa(\rho) = - \frac{2h_\kappa E_3^7(\rho_b)}{h_1 h_2 h_3} \frac{(\mathbf{U} \cdot \hat{\mathbf{x}}_\kappa)}{N_\kappa} \left[1 - \left(\frac{E_1^\kappa(\rho_a)}{E_1^\kappa(\rho)} \right)^2 \right], \quad \kappa = 1,2,3 \tag{180}$$

and

$$\Phi_\kappa^l(\rho) = \frac{3h_\kappa}{h_1 h_2 h_3} e_\kappa^l \left[1 - \left(\frac{E_1^\kappa(\rho_a)}{E_1^\kappa(\rho)} \right)^2 \right] \frac{1}{(E_1^l(\rho))^2}, \quad \kappa,l = 1,2,3, \quad \kappa \neq l. \tag{181}$$

Once the constants N_κ for $\kappa=1,2,3$ in (134) and e_κ^l for $\kappa,l=1,2,3$, $\kappa \neq l$ in (176) are calculated, the velocity field is obtained in terms of the applied fields $\mathbf{U}$ and $\mathbf{\Omega}$ via Eq. (50). The total pressure field (116) assumes the expression

$$P^{(0)}(\mathbf{r}) = P_0 + \mu_0 \frac{E_3^7(\rho)}{(\rho^2-\mu^2)(\rho^2-\nu^2)} \left[\sum_{j=1}^{3} \frac{\Theta_j(\rho)}{\rho^2-\rho_a^2}\mathbb{E}_1^j(\mathbf{r}) + \frac{3}{h_1 h_2 h_3} \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{h_i e_i^j}{(E_2^{i+j}(\rho))^2}\mathbb{E}_2^{i+j}(\mathbf{r}) \right] = p^{(0)}(\mathbf{r}) + \rho_0 g h, \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{182}$$

where the coefficients e_κ^l for $\kappa,l=1,2,3$, $\kappa \neq l$ are given by (176), where the identity (A4) and formula

$$\sum_{i=1}^{3} \frac{x_i^2}{(E_1^i(\rho))^2} = \frac{h_\rho^2}{\rho^2}, \quad \rho_a \leqslant \rho \leqslant \rho_b \tag{183}$$

have been used. The reference constant h is appropriately chosen depending upon the physical requirements. Here, we must mention that the constant pressure of reference P_0 is written as

$$P_0 = -\mu_0 h_1 h_2 h_3 \sum_{j=1}^{3} \frac{1}{h_j} (\mathbf{e}_1^{(i)j} \cdot \hat{\mathbf{x}}_j), \tag{184}$$

which actually contains the coefficients $(\mathbf{e}_1^{(i)\kappa} \cdot \hat{\mathbf{x}}_\kappa)$, $\kappa=1,2,3$ that were not evaluated by the boundary conditions. Nevertheless, P_0 enter Stokes equations (1) and (2) under the action of the gradient operator, providing that way $\nabla P_0=\mathbf{0}$. Thus, we can use the very same physical arguments to specify this constant pressure. As far as the vorticity field is concerned we can expand relation (117) in view of (67)–(112), and substitute the obtained coefficients. Nevertheless, an easier root is followed with respect to the definition (3), by taking the rotation of the velocity field (177). After the application of certain identities we obtain

$$\nabla \times \mathbf{U} = \mathbf{0}, \quad \nabla \times (\boldsymbol{\Omega} \times \mathbf{r}) = 2\boldsymbol{\Omega} \tag{185}$$

and some extensive algebra leads to

$$\nabla \times \frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} = \mathbf{0}, \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{186}$$

Matching (185) and (186) and utilizing definitions (178)–(181), we arrive at

$$\begin{aligned} \boldsymbol{\omega}^{(0)}(\mathbf{r}) = \boldsymbol{\Omega} + \frac{h_1h_2h_3}{2}\sum_{j=1}^{3}\frac{1}{h_j}(\hat{\mathbf{x}}_j \times \mathbf{H}_j(\rho)) + \frac{\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}\hat{\boldsymbol{\rho}}}{2\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} \times \left[\mathbf{Z}'(\rho) + \sum_{j=1}^{3}\mathbf{H}_j'(\rho)\mathbb{E}_1^j(\mathbf{r})\right] \\ - \frac{h_1h_2h_3\hat{\boldsymbol{\rho}}}{4\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} \times \left[\sum_{j=1}^{3}\frac{\Theta_j(\rho)}{h_j}\hat{\mathbf{x}}_j + \sum_{\substack{i,j=1 \\ i\neq j}}^{3}\Phi_i^j(\rho)\left(\frac{\mathbb{E}_1^i(\mathbf{r})}{h_j}\hat{\mathbf{x}}_j + \frac{\mathbb{E}_1^j(\mathbf{r})}{h_i}\hat{\mathbf{x}}_i\right)\right] \end{aligned} \tag{187}$$

for every $\mathbf{r} \in \Omega(\mathbb{R}^3)$, where $\boldsymbol{\Omega}$ stands for the constant vorticity (50), with

$$\mathbf{Z}'(\rho) = \frac{E_3^7(\rho_b)\mathbf{U}}{\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}} \cdot \sum_{i=1}^{3}\frac{\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i}{N_i}\left[1 + \left(\frac{E_1^i(\rho_a)}{E_1^i(\rho)}\right)^2\right], \quad \rho_a \leqslant \rho \leqslant \rho_b \tag{188}$$

and

$$\begin{aligned} \mathbf{H}_\kappa'(\rho) = \frac{3}{2\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}\sum_{\substack{i=1 \\ i\neq\kappa}}^{3}\frac{\hat{\mathbf{x}}_i}{h_i}\left\{-\frac{h_ie_i^\kappa}{(E_1^\kappa(\rho))^2}\left[1 + \left(\frac{E_1^i(\rho_a)}{E_1^i(\rho)}\right)^2\right]\right. \\ \left. + \frac{h_\kappa e_\kappa^i}{(E_1^i(\rho))^2}\left[1 - \left(\frac{E_1^\kappa(\rho_a)}{E_1^\kappa(\rho)}\right)^2\right]\right\}, \quad \kappa = 1,2,3 \end{aligned} \tag{189}$$

for $\rho_a \leqslant \rho \leqslant \rho_b$. The constants N_κ for $\kappa=1,2,3$ are given by (134) and the coefficients e_κ^l for $\kappa, l=1,2,3$, $\kappa \neq l$ are provided by (176). Next, in order to obtain the expression for the stress tensor field, we use identities (39)–(45) so that

$$\nabla \otimes \mathbf{U} = (\nabla \otimes \mathbf{U})^{\mathsf{T}} = \tilde{\mathbf{0}}, \quad \nabla \otimes (\boldsymbol{\Omega} \times \mathbf{r}) = \boldsymbol{\Omega} \times \tilde{\mathbf{I}} \tag{190}$$

and we use relations (22) and (109) to obtain the dyadic $\tilde{\mathbf{S}}$ in the form

$$\widetilde{\mathbf{S}}(\mathbf{r}) \equiv \nabla \otimes \left(\frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\right) + \left[\nabla \otimes \left(\frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\right)\right]^{\mathsf{T}}$$
$$= \frac{2}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\left\{\frac{\rho}{h_\rho}\left[-\left(\frac{1}{\rho^2-\mu^2}+\frac{1}{\rho^2-\nu^2}\right)\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\rho}} + \frac{\hat{\boldsymbol{\mu}}\otimes\hat{\boldsymbol{\mu}}}{\rho^2-\mu^2} + \frac{\hat{\boldsymbol{\nu}}\otimes\hat{\boldsymbol{\nu}}}{\rho^2-\nu^2}\right]\right.$$
$$\left. + \frac{\mu}{h_\mu(\rho^2-\mu^2)}(\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\mu}}+\hat{\boldsymbol{\mu}}\otimes\hat{\boldsymbol{\rho}}) + \frac{\nu}{h_\nu(\rho^2-\nu^2)}(\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\nu}}+\hat{\boldsymbol{\nu}}\otimes\hat{\boldsymbol{\rho}})\right\}. \tag{191}$$

Further calculations lead to

$$\widetilde{\mathbf{\Pi}}^{(0)}(\mathbf{r}) = (\rho_0 g h - P^{(0)}(\mathbf{r}))\widetilde{\mathbf{I}} - \mu_0\left\{\frac{1}{h_\rho}(\hat{\boldsymbol{\rho}}\otimes\mathbf{Z}'(\rho)+\mathbf{Z}'(\rho)\otimes\hat{\boldsymbol{\rho}}) + \sum_{j=1}^{3}\left[\frac{\mathbb{E}_1^j(\mathbf{r})}{h_\rho}(\hat{\boldsymbol{\rho}}\otimes\mathbf{H}_j'(\rho)+\mathbf{H}_j'(\rho)\right.\right.$$
$$\left.\left.\otimes\hat{\boldsymbol{\rho}}) + \frac{h_1h_2h_3}{h_j}(\hat{\mathbf{x}}_j\otimes\mathbf{H}_j(\rho)+\mathbf{H}_j(\rho)\otimes\hat{\mathbf{x}}_j)\right]\right\} - \frac{\mu_0}{2}\widetilde{\mathbf{S}}(\mathbf{r})\left[\sum_{j=1}^{3}\Theta_j(\rho)\mathbb{E}_1^j(\mathbf{r})\right.$$
$$\left.+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\Phi_i^j(\rho)\mathbb{E}_2^{i+j}(\mathbf{r})\right] - \frac{\mu_0}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\left\{\frac{\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\rho}}}{h_\rho}\left[\sum_{j=1}^{3}\Theta_j'(\rho)\mathbb{E}_1^j(\mathbf{r}) + \sum_{\substack{i,j=1\\ i\neq j}}^{3}\Phi_i^{j'}(\rho)\mathbb{E}_2^{i+j}(\mathbf{r})\right]\right.$$
$$+\frac{h_1h_2h_3}{2}\left[\sum_{j=1}^{3}\frac{\Theta_j(\rho)}{h_j}(\hat{\boldsymbol{\rho}}\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\boldsymbol{\rho}}) + \sum_{\substack{i,j=1\\ i\neq j}}^{3}\Phi_i^j(\rho)\left(\frac{\mathbb{E}_1^i(\mathbf{r})}{h_j}(\hat{\boldsymbol{\rho}}\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\boldsymbol{\rho}})\right.\right.$$
$$\left.\left.\left.+\frac{\mathbb{E}_1^j(\mathbf{r})}{h_i}(\hat{\boldsymbol{\rho}}\otimes\hat{\mathbf{x}}_i+\hat{\mathbf{x}}_i\otimes\hat{\boldsymbol{\rho}})\right)\right]\right\} \tag{192}$$

for $\mathbf{r}\in\Omega(\mathbb{R}^3)$, where $P^{(0)}$ is the total pressure (182), while in view of the applied field $\mathbf{U}$ (50)

$$\Theta_\kappa'(\rho) = -\frac{4h_\kappa E_3^7(\rho_b)}{h_1h_2h_3}\frac{(\mathbf{U}\cdot\hat{\mathbf{x}}_\kappa)}{N_\kappa}\frac{(E_1^\kappa(\rho_a))^2}{(E_1^\kappa(\rho))^4}\rho, \quad \kappa=1,2,3 \tag{193}$$

and

$$\Phi_\kappa^{l'}(\rho) = -\frac{6h_\kappa}{h_1h_2h_3}\frac{\rho e_\kappa^l}{(E_1^l(\rho))^4}\left\{1-\left(\frac{E_1^\kappa(\rho_a)}{E_1^\kappa(\rho)}\right)^2\left[1+\left(\frac{E_1^l(\rho)}{E_1^\kappa(\rho)}\right)^2\right]\right\}, \quad \kappa,l=1,2,3, \quad \kappa\neq l \tag{194}$$

for every $\rho_a\leq\rho\leq\rho_b$. We remind that the coefficients e_κ^l for $\kappa,l=1,2,3$, $\kappa\neq l$ are satisfied by (176), the metric coefficient h_ρ is given by Eq. (21), N_κ for $\kappa=1,2,3$ are provided by (134) and expressions (178)–(181), (188), and (189) have been used. Recapitulating, the *first terms*, of the velocity field (177), of the total pressure field (182), of the vorticity field (187), and of the stress tensor field (192) have been analytically calculated on the basis of the Happel-type boundary conditions.

V. GEOMETRICALLY DEGENERATE CASES

The strict inequalities

$$0 < \alpha_3 < \alpha_2 < \alpha_1 < +\infty \tag{195}$$

form the basic reason why the triaxial ellipsoid reflects the general anisotropy of the three-dimensional space. As it is well known,[20] the reduction of general results from the ellipsoidal to the spheroidal, or to the spherical geometry is not straightforward, since certain indeterminacies appear during the limiting process. This is due to the fact that the spherical system springs from a zero-dimensional manifold (i.e., the center), while the ellipsoidal system springs from a two-dimensional manifold (i.e., the focal ellipse). Nevertheless, formulas (A42)–(A51) ensure the appropriate reductions. In particular, we consider every case separately.

The geometrically degenerate cases of spheroids: The equality of the two of the axis (195) of an ellipsoid degenerates it to a spheroid, whose axial symmetry coincides with the third axis. More specific, *a prolate spheroid* is obtained whenever

$$0 < \alpha_3 = \alpha_2 < \alpha_1 < +\infty, \tag{196}$$

while the case of an *oblate spheroid* corresponds to

$$0 < \alpha_3 < \alpha_2 = \alpha_1 < +\infty. \tag{197}$$

The axis of symmetry is the x_1 axis for the prolate spheroid and the x_3 axis for the oblate spheroid. The asymptotic case of the *needle* can be reached by a prolate spheroid where

$$0 < \alpha_3 = \alpha_2 \ll \alpha_1 < +\infty, \tag{198}$$

while in the case where

$$0 < \alpha_3 \ll \alpha_2 = \alpha_1 < +\infty, \tag{199}$$

the oblate spheroid takes the shape of a circular *disk*. As far as the semifocal distances are concerned, we have that

$$h_1 = 0, \quad h_2 = h_3 = c, \quad c > 0 \tag{200}$$

for the case of a prolate spheroid with semifocal distance c, and

$$h_3 = 0, \quad h_1 = h_2 = \bar{c}, \quad \bar{c} > 0 \tag{201}$$

for the case of an oblate spheroid with semifocal distance $\bar{c}$. The simple transformation

$$c \to -i\bar{c}, \quad c, \bar{c} > 0 \tag{202}$$

allows the transition from the prolate to the oblate spheroid, while the replacement

$$\bar{c} \to ic, \quad c, \bar{c} > 0 \tag{203}$$

secures the converse. In terms of the notation $\tau \equiv \cosh \eta$, $\zeta \equiv \cos \theta$, and φ for $1 \leqslant \tau < +\infty$, $-1 \leqslant \zeta \leqslant 1$ and $\varphi \in [0, 2\pi)$, respectively, for the prolate spheroid, the corresponding results for the oblate spheroid can be obtained through the transformation

$$\tau \to i\lambda, \quad 1 \leqslant \tau < +\infty, \tag{204}$$

where $0 \leqslant \lambda \equiv \sinh \eta < +\infty$ is the characteristic variable of the oblate system λ, ζ, and φ. Obviously the inverse transformation

$$\lambda \to -i\tau, \quad 0 \leqslant \lambda < +\infty \tag{205}$$

leads to the converse. Consequently, from this point on we shall refer to the prolate spheroidal geometry, since the oblate spheroidal geometry is recovered via

$$\tau \to i\lambda, \quad c \to -i\bar{c}. \tag{206}$$

In terms of the unit normal vector $\hat{\boldsymbol{\tau}}$ for the prolate spheroidal coordinates, the ellipsoidal variables are connected with the τ, ζ, and φ by

$$\rho = c\tau, \quad \hat{\boldsymbol{\rho}} \to \hat{\boldsymbol{\tau}}, \quad \rho \in [h_2, +\infty), \quad 1 \leqslant \tau < +\infty \tag{207}$$

and

$$\frac{\mu\nu}{h_2 h_3} = \zeta, \tag{208}$$

$$\frac{\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3} = \sqrt{1-\zeta^2}\cos\varphi, \tag{209}$$

$$\frac{\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2} = \sqrt{1-\zeta^2}\sin\varphi, \tag{210}$$

whereas $\mu \in [h_3, h_2]$, $\nu \in [-h_3, h_3]$ and $1 \leqslant \tau < +\infty$, $\varphi \in [0, 2\pi)$. For $\alpha_2 = \alpha_3$ (prolate spheroid) the constants (63) provide

$$\lim_{\alpha_2 \to \alpha_3} \Lambda = \frac{2\alpha_1^2 + \alpha_2^2}{3}, \quad \lim_{\alpha_2 \to \alpha_3} \Lambda' = \alpha_2^2, \tag{211}$$

while as a consequence of the limits (211) for $\alpha_2 = \alpha_3$,

$$\lim_{\alpha_2 \to \alpha_3} (\Lambda - \Lambda') = \lim_{\alpha_2 \to \alpha_3} (\Lambda - \alpha_2^2) = \lim_{\alpha_2 \to \alpha_3} (\Lambda - \alpha_3^2) = \frac{2c^2}{3}, \tag{212}$$

$$\lim_{\alpha_2 \to \alpha_3} (\Lambda - \alpha_1^2) = -\frac{c^2}{3}, \tag{213}$$

$$\lim_{\alpha_2 \to \alpha_3} (\Lambda' - \alpha_1^2) = -c^2, \tag{214}$$

$$\lim_{\alpha_2 \to \alpha_3} (\Lambda' - \alpha_2^2) = \lim_{\alpha_2 \to \alpha_3} (\Lambda' - \alpha_3^2) = 0. \tag{215}$$

Moreover, the constants (65) for $\alpha_2 = \alpha_3$ ($\Lambda_2 = \Lambda_3$, $\Lambda_2' = \Lambda_3'$, prolate spheroid) become

$$\lim_{\alpha_2 \to \alpha_3} \left.\begin{matrix} \Lambda_\kappa \\ \Lambda_\kappa' \end{matrix}\right\} = \frac{2}{5}(\alpha_1^2 + 2\alpha_2^2) - \frac{1}{5}\alpha_\kappa^2 \pm \frac{1}{5}\left\{4(\alpha_1^4 + 2\alpha_2^4) - 3\alpha_\kappa^4 - \alpha_1^2\alpha_2^4\left[\left(\frac{1}{\alpha_1^2} + \frac{2}{\alpha_2^2}\right) + \frac{6}{\alpha_\kappa^2}\right]\right\}^{1/2},$$

$$\kappa = 1, 2, \tag{216}$$

where one can calculate the $(\Lambda_\kappa - \alpha_l^2)$ and $(\Lambda_\kappa' - \alpha_l^2)$ for $\kappa, l = 1, 2, 3$. On the other hand, the integrals I_n^m in prolate spheroidal geometry ($\alpha_2 = \alpha_3$)

$$\lim_{\alpha_2\to\alpha_3} I_n^m(\rho) = \int_\rho^{+\infty} \frac{\mathrm{d}u}{[\lim_{\alpha_2\to\alpha_3} E_n^m(u)]^2 |u^2-\alpha_1^2+\alpha_2^2|}, \quad \rho \in [h_2, +\infty) \tag{217}$$

for $n=0,1,\ldots$ and $m=1,2,\ldots,2n+1$ are no more elliptic and they can be evaluated with analytic manipulations. Specifically, the determination of I_0^1 demands the calculation of the corresponding integral (217) for $n=0$ and $m=1$ in terms of the variable τ, that is

$$\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) = \frac{1}{2c} \ln \frac{\tau+1}{\tau-1}, \quad \rho \in [h_2, +\infty), \quad 1 \leqslant \tau < +\infty. \tag{218}$$

The other elliptic integrals, which concern the already known exterior ellipsoidal harmonic eigenfunctions for degree $n=1,2,3$ and order $m=1,2,\ldots,2n+1$, are given explicitly in terms of the $\lim_{\alpha_2\to\alpha_3} I_0^1$ and their prolate spheroidal expressions ($\alpha_2=\alpha_3$) give

$$\lim_{\alpha_2\to\alpha_3} I_1^1(\rho) = \frac{1}{c^2}\left(\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) - \frac{1}{c\tau}\right), \quad c>0, \tag{219}$$

$$\lim_{\alpha_2\to\alpha_3} I_1^2(\rho) = \lim_{\alpha_2\to\alpha_3} I_1^3(\rho) = -\frac{1}{2c^2}\left(\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) - \frac{\tau}{c(\tau^2-1)}\right), \quad c>0, \tag{220}$$

$$\lim_{\alpha_2\to\alpha_3} I_2^1(\rho) = \frac{9}{4c^4}\left(\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) - \frac{\tau}{c(3\tau^2-1)}\right), \quad c>0, \tag{221}$$

$$\lim_{\alpha_2\to\alpha_3} I_2^2(\rho) = \lim_{\alpha_2\to\alpha_3} I_2^5(\rho) = \frac{3}{8c^4}\left(\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) - \frac{\tau(3\tau^2-5)}{3c(\tau^2-1)^2}\right), \quad c>0, \tag{222}$$

$$\lim_{\alpha_2\to\alpha_3} I_2^3(\rho) = \lim_{\alpha_2\to\alpha_3} I_2^4(\rho) = -\frac{3}{2c^4}\left(\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) - \frac{3\tau^2-2}{3c\tau(\tau^2-1)}\right), \quad c>0. \tag{223}$$

Much more complicated calculations leads us to

$$\lim_{\alpha_2\to\alpha_3} I_3^1(\rho) = \frac{25}{c^6}\left(\frac{1}{4}\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) - \frac{25\tau}{36c(5\tau^2-3)} - \frac{1}{9c\tau}\right), \tag{224}$$

$$\lim_{\alpha_2\to\alpha_3} I_3^2(\rho) = \lim_{\alpha_2\to\alpha_3} I_3^7(\rho) = \frac{1}{c^6}\left(\frac{15}{8}\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) + \frac{\tau}{4c(\tau^2-1)^2} - \frac{7\tau}{8c(\tau^2-1)} - \frac{1}{c\tau}\right), \tag{225}$$

$$\lim_{\alpha_2\to\alpha_3} I_3^3(\rho) = \lim_{\alpha_2\to\alpha_3} I_3^5(\rho) = -\frac{25}{c^6}\left(\frac{3}{16}\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) - \frac{\tau}{32c(\tau^2-1)} - \frac{25\tau}{32c(5\tau^2-1)}\right), \tag{226}$$

$$\lim_{\alpha_2\to\alpha_3} I_3^4(\rho) = \lim_{\alpha_2\to\alpha_3} I_3^6(\rho) = -\frac{1}{c^6}\left(\frac{5}{16}\lim_{\alpha_2\to\alpha_3} I_0^1(\rho) - \frac{\tau}{6c(\tau^2-1)^3} + \frac{5\tau}{24c(\tau^2-1)^2} - \frac{5\tau}{16c(\tau^2-1)}\right) \tag{227}$$

for $c>0$. As far as the interior solid ellipsoidal harmonics as well as their gradients are concerned, their Cartesian representation easily implies the relative reductions. Finally, the limiting cases of the needle and of the disk are asymptotic reductions of the prolate and the oblate spheroidal geometry, respectively. For the *needle* we obtain

$$\alpha_1/\alpha_2 = \tau/\sqrt{\tau^2-1} \to +\infty, \quad 1 \leqslant \tau < +\infty, \tag{228}$$

and for the *disk* we obtain

$$\alpha_3/\alpha_2 = \lambda/\sqrt{\lambda^2+1} \to 0^+, \quad 0 \leqslant \lambda < +\infty. \tag{229}$$

The geometrically degenerate case of sphere: The sphere corresponds to

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha, \tag{230}$$

where α is the radius. In this case,

$$h_\kappa = 0, \quad \kappa = 1,2,3, \tag{231}$$

which means that all the semifocal distances of the ellipsoid coincide at the origin. Defining the limit from the ellipsoid to the sphere as "$\lim_{e\to s}$", the constants (63) give

$$\lim_{e\to s} \Lambda = \lim_{e\to s} \Lambda' = \alpha^2, \tag{232}$$

while from Eq. (65) we derive

$$\lim_{e\to s} \Lambda_\kappa = \lim_{e\to s} \Lambda'_\kappa = \alpha^2, \quad \kappa = 1,2,3 \tag{233}$$

so that

$$\lim_{e\to s}(\Lambda - \Lambda') = \lim_{e\to s}(\Lambda_\kappa - \Lambda'_\kappa) = 0, \quad \kappa = 1,2,3, \tag{234}$$

$$\lim_{e\to s}(\Lambda - \alpha_\kappa^2) = \lim_{e\to s}(\Lambda' - \alpha_\kappa^2) = 0, \quad \kappa = 1,2,3, \tag{235}$$

$$\lim_{e\to s}(\Lambda_\kappa - \alpha_l^2) = \lim_{e\to s}(\Lambda'_\kappa - \alpha_l^2) = 0, \quad \kappa, l = 1,2,3. \tag{236}$$

The intervals of variation of the variables μ and ν imply that

$$\lim_{e\to s} \mu = \lim_{e\to s} \nu = 0, \quad \kappa = 1,2,3. \tag{237}$$

In terms of the unit normal vector $\hat{\mathbf{r}}$ of the spherical system, the connection between the ellipsoidal variables and the corresponding spherical variables r, $\zeta \equiv \cos\theta$, and φ for $0 \leqslant r < +\infty$, $-1 \leqslant \zeta \leqslant 1$ and $\varphi \in [0, 2\pi)$ is

$$\lim_{e\to s} \rho = \lim_{e\to s}(\sqrt{\rho^2 - h_2^2}) = \lim_{e\to s}(\sqrt{\rho^2 - h_3^2}) = r, \quad \hat{\boldsymbol{\rho}} \to \hat{\mathbf{r}}, \quad \rho \in [h_2, +\infty), \quad r \geqslant 0 \tag{238}$$

for the radial component and

$$\frac{\mu\nu}{h_2 h_3} = \zeta, \tag{239}$$

$$\frac{\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3} = \sqrt{1-\zeta^2}\cos\varphi, \tag{240}$$

$$\frac{\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}}{h_1 h_2}=\sqrt{1-\zeta^2}\sin\varphi \tag{241}$$

for the angular dependence. The integrals I_n^m assume the values,

$$\lim_{e\to s} I_n^m(\rho)=\frac{1}{(2n+1)r^{2n+1}}, \quad n\geqslant 0, \quad m=1,2,\ldots,2n+1, \quad \kappa=1,2,3 \tag{242}$$

for every $\rho\in[h_2,+\infty)$ and $r\geqslant 0$.

We mention that a proper reduction of our results for the velocity field (177) gives the *first term* of the series. The full solution of the above-mentioned Happel-type boundary value problem in spherical coordinates is given in Ref. 19, where the three-dimensional complete solution is obtained. For the sake of completeness, we provide the solution from Ref. 19 in the following.

If we define the sphere

$$B_r=\{\mathbf{r}\in\mathbb{R}^3 | x_1^2+x_2^2+x_3^2\leqslant r^2\}, \tag{243}$$

where

$$\mathbf{r}=\sum_{i=1}^{3} x_i\hat{\mathbf{x}}_i = r\zeta\hat{\mathbf{x}}_1 + r\sqrt{1-\zeta^2}\cos\varphi\hat{\mathbf{x}}_2 + r\sqrt{1-\zeta^2}\sin\varphi\hat{\mathbf{x}}_3, \tag{244}$$

then the *total* velocity assumes the form

$$\mathbf{v}^{(\text{sphere})}(\mathbf{r})=\mathbf{U}+\mathbf{\Omega}\times\mathbf{r}+\mathbf{Z}(r)+f(r)(x_1\mathbf{r}), \quad \mathbf{r}\in\Omega(\mathbb{R}^3). \tag{245}$$

We introduce the expression

$$\mathbf{Z}(r)=\frac{\mathbf{U}\cdot(\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1)}{K}\left[-2(3\gamma^5+2)+4\gamma^5\left(\frac{r}{a}\right)^2+\left(\frac{a}{r}\right)^3+(2\gamma^5+3)\left(\frac{a}{r}\right)\right] \tag{246}$$

and

$$f(r)=\frac{1}{a^2}\left[(2\gamma^5+3)\left(\frac{a}{r}\right)^3-3\left(\frac{a}{r}\right)^5-2\gamma^5\right], \tag{247}$$

whereas

$$\gamma=\frac{a}{b}<1, \quad K=2-3\gamma+3\gamma^5-2\gamma^6, \quad \mathbf{U}=U_1\hat{\mathbf{x}}_1. \tag{248}$$

Note that, a,b are the radii of the concentric spheres of the corresponding Happel boundary value problem for Stokes flow around spherical particles.

VI. DISCUSSION

The physical interpretation of the mathematical problem analyzed in this work involves a method for solving three-dimensional Stokes flow problems with Happel-type boundary conditions. Based on this method we examine the flow in an ellipsoidal cell as a means of modeling flow through a swarm of ellipsoidal particles with the aim of the Papkovich–Neuber differential representation, which offers solutions for such problems in several orthogonal curvilinear geometries. The terms of major significance of the important physical flow fields such as the velocity, the total pressure, the vorticity, and the stress tensor are evaluated in closed form. The difficulty of this problem is focused on the determination of the coefficients, which characterize the nature of the corresponding potentials. This is caused by the appearance of certain indeterminacies, which reflect the complexity of the ellipsoidal geometry. The present work invoked a useful tool for dealing with nonaxisymmetric problems, which is the differential representation theory. The freedom that 3D representations offer, makes the solution of creeping flow problems within such domains feasible, a fact which is justified by the freedom, which this kind of representation offers. Consequently, a convenient handling of those extra conditions in order to cancel the singular behavior of the calculated expressions makes the calculations possible and leads to the explicit evaluation of the coefficients. It turns out that the velocity, to the first degree, which represents the leading term of the series, is sufficient for most real life applications, so long as the aspect ratios of the ellipsoids remains within moderate bounds. This is feasible, since the Stokes flow approximation requires a strict consideration of small particles and, then, our method stays valid. We conclude by showing the means for obtaining the corresponding results for the prolate and the oblate spheroid, the needle and the disk, and the sphere as degenerate ellipsoids.

ACKNOWLEDGMENT

Part of this work was performed while G.D. was at the University of Cambridge under the Marie Curie Chair of Excellence program ("BRAIN", EXC023928), funded by the European Union.

APPENDIX: RELATIONS ON ELLIPSOIDAL HARMONICS

For a detailed analysis of the ellipsoidal harmonics and related useful properties, one can refer to Ref. 20. However, in the interest of making this work complete and independent, we provide some useful material on ellipsoids and ellipsoidal harmonics, which were used in this work and most of them cannot be found in literature. They appear here for the first time.

We begin with the following connection formulas:

$$\rho^2+\mu^2+\nu^2=\sum_{i=1}^{3}x_i^2+h_2^2+h_3^2, \tag{A1}$$

$$\rho^2\mu^2+\rho^2\nu^2+\mu^2\nu^2=\sum_{i=1}^{3}h_i^2x_i^2+2x_1^2h_3^2+h_2^2h_3^2, \tag{A2}$$

$$\rho^2\mu^2\nu^2=x_1^2h_2^2h_3^2 \tag{A3}$$

for every $\rho\in[h_2,+\infty)$, $\mu\in[h_3,h_2]$, and $\nu\in[-h_3,h_3]$.

The elliptic integrals that enter the exterior ellipsoidal harmonics $\mathbb{F}_n^m$ are interconnected via the following relations:

$$\sum_{i=1}^{3}I_1^i(\rho)=\frac{1}{\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}=\frac{1}{E_3^7(\rho)}, \tag{A4}$$

$$\sum_{i=1}^{3}\alpha_i^2I_1^i(\rho)=I_0^1(\rho)-\frac{\rho^2-\alpha_1^2}{\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}, \tag{A5}$$

$$I_2^1(\rho)=\frac{1}{2(\Lambda-\alpha_1^2+\rho^2)\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}-\frac{1}{2}\sum_{i=1}^{3}\frac{I_1^i(\rho)}{\Lambda-\alpha_i^2}, \tag{A6}$$

$$I_2^2(\rho)=\frac{1}{2(\Lambda'-\alpha_1^2+\rho^2)\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}-\frac{1}{2}\sum_{i=1}^{3}\frac{I_1^i(\rho)}{\Lambda'-\alpha_i^2}, \tag{A7}$$

$$I_2^{\kappa+l}(\rho) = \frac{I_1^l(\rho) - I_1^\kappa(\rho)}{\alpha_\kappa^2 - \alpha_l^2}, \quad \kappa,l = 1,2,3, \quad \kappa \neq l, \tag{A8}$$

$$I_3^7(\rho) = \frac{1}{h_1^2 h_2^2 h_3^2} \sum_{i=1}^{3} (-1)^{i+1} h_i^2 I_1^i(\rho) = \frac{1}{h_1^2 h_2^2 h_3^2} \sum_{i=1}^{3} (-1)^{i+1} \alpha_i^2 h_i^2 I_2^{6-i}(\rho), \tag{A9}$$

$$\sum_{i=1}^{3} \frac{I_1^i(\rho)}{(\Lambda - \alpha_i^2)(\Lambda' - \alpha_i^2)} = 3 I_3^7(\rho), \tag{A10}$$

$$\sum_{i=1}^{3} \frac{\alpha_i^2 I_1^i(\rho)}{(\Lambda - \alpha_i^2)(\Lambda' - \alpha_i^2)} = \frac{3}{h_1^2}[I_1^2(\rho) - I_1^3(\rho)] + 3\alpha_1^2 I_3^7(\rho) = -3 I_2^5(\rho) + 3\alpha_1^2 I_3^7(\rho), \tag{A11}$$

$$[\alpha_l^2 I_1^\kappa(\rho) + \alpha_\kappa^2 I_1^l(\rho)] = [\alpha_l^2 I_1^l(\rho) + \alpha_\kappa^2 I_1^\kappa(\rho)] + (\alpha_l^2 - \alpha_\kappa^2)^2 I_2^{\kappa+l}(\rho), \quad \kappa,l = 1,2,3, \quad \kappa \neq l, \tag{A12}$$

$$\sum_{i=1}^{3} (-1)^i h_i^2 I_2^{6-i}(\rho) = 0. \tag{A13}$$

The essence of relations (A4)–(A13) is that between the ten integrals I_n^m, $n \leqslant 2$, and I_3^7 only two are linearly independent. For example, if I_0^1 and I_3^7 are known, then the other eight integrals can be written via these two.

The constants Λ and Λ', from Eq. (63), the semifocal distances h_κ, $\kappa=1,2,3$ and the semiaxes α_κ, $\kappa=1,2,3$ satisfy the following useful expressions:

$$\Lambda + \Lambda' = \frac{2}{3} \sum_{i=1}^{3} \alpha_i^2, \tag{A14}$$

$$\Lambda - \Lambda' = \frac{2}{3} \sqrt{h_1^4 + h_2^2 h_3^2}, \tag{A15}$$

$$\Lambda \Lambda' = \frac{\alpha_1^2 \alpha_2^2 \alpha_3^2}{3} \sum_{i=1}^{3} \frac{1}{\alpha_i^2} \tag{A16}$$

and

$$\sum_{i=1}^{3} (-1)^i (\Lambda - \alpha_i^2) h_i^2 = \sum_{i=1}^{3} (-1)^i (\Lambda' - \alpha_i^2) h_i^2 = 0, \tag{A17}$$

$$\sum_{i=1}^{3} (-1)^i (\Lambda - \alpha_i^2) h_i^2 \alpha_i^2 = \sum_{i=1}^{3} (-1)^i (\Lambda' - \alpha_i^2) h_i^2 \alpha_i^2 = h_1^2 h_2^2 h_3^2, \tag{A18}$$

$$\sum_{i=1}^{3} \frac{\alpha_i^2}{\Lambda - \alpha_i^2} = \sum_{i=1}^{3} \frac{\alpha_i^2}{\Lambda' - \alpha_i^2} = -3. \tag{A19}$$

Furthermore,

$$(\Lambda-\alpha_\kappa^2)(\Lambda'-\alpha_\kappa^2)=\frac{(-1)^{\kappa+1}h_1^2h_2^2h_3^2}{3h_\kappa^2}, \quad \kappa=1,2,3, \tag{A20}$$

which implies that

$$\tfrac{3}{2}(\Lambda-\Lambda')-3(\Lambda-\alpha_1^2)=-\tfrac{3}{2}(\Lambda-\Lambda')-3(\Lambda'-\alpha_1^2)=h_2^2+h_3^2, \tag{A21}$$

$$(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)+(\Lambda-\alpha_1^2)(\Lambda-\alpha_3^2)+(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)=0, \tag{A22}$$

$$(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)+(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_3^2)+(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)=0 \tag{A23}$$

and

$$\frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{(\Lambda-\alpha_\kappa^2)}=(-1)^{\kappa+1}\frac{h_1^2h_2^2h_3^2}{3h_\kappa^2}-\frac{(\Lambda-\alpha_\kappa^2)(\Lambda-\Lambda')}{2}, \tag{A24}$$

$$\frac{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}{(\Lambda'-\alpha_\kappa^2)}=(-1)^{\kappa+1}\frac{h_1^2h_2^2h_3^2}{3h_\kappa^2}+\frac{(\Lambda'-\alpha_\kappa^2)(\Lambda-\Lambda')}{2} \tag{A25}$$

for every $\kappa=1,2,3$. It is also easy to show that

$$\frac{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}{(\Lambda-\alpha_\kappa^2)(\Lambda-\alpha_l^2)}=\alpha_\kappa^2+\alpha_l^2-\frac{\Lambda+3\Lambda'}{2}+3(\Lambda'-\alpha_l^2)\delta_{\kappa l}, \tag{A26}$$

$$\frac{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}{(\Lambda'-\alpha_\kappa^2)(\Lambda'-\alpha_l^2)}=\alpha_\kappa^2+\alpha_l^2-\frac{3\Lambda+\Lambda'}{2}+3(\Lambda-\alpha_l^2)\delta_{\kappa l}. \tag{A27}$$

An important group of relations is given by

$$\sum_{i=1}^{3}\frac{\alpha_i^4}{\Lambda-\alpha_i^2}=-\frac{3}{2}(3\Lambda+\Lambda'), \tag{A28}$$

$$\sum_{i=1}^{3}\frac{\alpha_i^4}{\Lambda'-\alpha_i^2}=-\frac{3}{2}(3\Lambda'+\Lambda), \tag{A29}$$

$$\sum_{i=1}^{3}\frac{\alpha_i^4}{(\Lambda-\alpha_i^2)^2}=3-3\frac{\Lambda^2(\Lambda-\Lambda')}{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}, \tag{A30}$$

$$\sum_{i=1}^{3}\frac{\alpha_i^4}{(\Lambda'-\alpha_i^2)^2}=3+3\frac{\Lambda'^2(\Lambda-\Lambda')}{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}, \tag{A31}$$

$$\sum_{i=1}^{3}\frac{\alpha_i^2}{(\Lambda-\alpha_i^2)^2}=\Lambda\sum_{i=1}^{3}\frac{1}{(\Lambda-\alpha_i^2)^2}=-\frac{3\Lambda(\Lambda-\Lambda')}{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}, \tag{A32}$$

$$\sum_{i=1}^{3}\frac{\alpha_i^2}{(\Lambda'-\alpha_i^2)^2}=\Lambda'\sum_{i=1}^{3}\frac{1}{(\Lambda'-\alpha_i^2)^2}=\frac{3\Lambda'(\Lambda-\Lambda')}{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)} \tag{A33}$$

and

$$\sum_{i=1}^{3}\frac{(-1)^{i+1}h_i^2}{\Lambda-\alpha_i^2}=\frac{h_1^2h_2^2h_3^2}{(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)}, \tag{A34}$$

$$\sum_{i=1}^{3}\frac{(-1)^{i+1}h_i^2}{\Lambda'-\alpha_i^2}=\frac{h_1^2h_2^2h_3^2}{(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)}. \tag{A35}$$

The following relations connect the semifocal distances and the semiaxes of an ellipsoid:

$$\alpha_1^2h_1^2-\alpha_2^2h_2^2+\alpha_3^2h_3^2=0, \tag{A36}$$

$$\alpha_1^2h_1^2(h_2^2+h_3^2)-\alpha_2^2h_2^4+\alpha_3^2h_3^4=h_1^2h_2^2h_3^2, \tag{A37}$$

$$\alpha_1^4h_1^2-\alpha_2^4h_2^2+\alpha_3^4h_3^2=h_1^2h_2^2h_3^2, \tag{A38}$$

$$h_1^2\alpha_2^2\alpha_3^2-\alpha_1^2h_2^2\alpha_3^2+\alpha_1^2\alpha_2^2h_3^2=h_1^2h_2^2h_3^2, \tag{A39}$$

$$(\alpha_1^4\alpha_2^2-\alpha_1^2\alpha_2^4)+(\alpha_2^4\alpha_3^2-\alpha_2^2\alpha_3^4)+(\alpha_3^4\alpha_1^2-\alpha_3^2\alpha_1^4)=h_1^2h_2^2h_3^2, \tag{A40}$$

$$h_lh_\kappa h_{6-(\kappa+l)}=h_1h_2h_3, \quad \kappa,l=1,2,3, \quad \kappa\neq l. \tag{A41}$$

The tricky task of reducing the results from the ellipsoidal to the spheroidal and to the spherical geometry requires gathering terms containing ellipsoidal and Cartesian factors so as to eliminate the generated indeterminacies. We shall give some formulas, where these indeterminacies are suitably grouped. These are

$$(\alpha_\kappa^2-\alpha_l^2)(\Lambda-\alpha_s^2)\sum_{i=1}^{3}\frac{x_i^2}{\Lambda-\alpha_i^2}=3(\Lambda'-\alpha_\kappa^2)x_\kappa^2-3(\Lambda'-\alpha_l^2)x_l^2+(\alpha_\kappa^2-\alpha_l^2)\|\mathbf{r}\|^2, \tag{A42}$$

$$(\alpha_\kappa^2-\alpha_l^2)(\Lambda'-\alpha_s^2)\sum_{i=1}^{3}\frac{x_i^2}{\Lambda'-\alpha_i^2}=3(\Lambda-\alpha_\kappa^2)x_\kappa^2-3(\Lambda-\alpha_l^2)x_l^2+(\alpha_\kappa^2-\alpha_l^2)\|\mathbf{r}\|^2, \tag{A43}$$

where $\kappa,l,s=1,2,3$ and $\kappa\neq l$, $\kappa\neq s$, $l\neq s$, while

$$\frac{\mathbb{E}_2^1(\mathbf{r})}{(\Lambda-\Lambda')(\Lambda-\alpha_\kappa^2)}-\frac{\mathbb{E}_2^2(\mathbf{r})}{(\Lambda-\Lambda')(\Lambda'-\alpha_\kappa^2)}=\|\mathbf{r}\|^2-3x_\kappa^2+\alpha_\kappa^2-\frac{1}{3}\sum_{i=1}^{3}\alpha_i^2 \tag{A44}$$

for every $\kappa=1,2,3$. This is a useful relation, which with the aim of the identity

$$\frac{I_2^1(\rho)-I_2^2(\rho)}{\Lambda-\Lambda'}=\frac{3}{2}I_3^7(\rho)-\frac{1}{2\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}(\Lambda-\alpha_1^2+\rho^2)(\Lambda'-\alpha_1^2+\rho^2)} \tag{A45}$$

implies that

$$\frac{\mathbb{E}_2^1(\mathbf{r})I_2^1(\rho)}{(\Lambda-\Lambda')(\Lambda-\alpha_\kappa^2)}-\frac{\mathbb{E}_2^2(\mathbf{r})I_2^2(\rho)}{(\Lambda-\Lambda')(\Lambda'-\alpha_\kappa^2)}=\left(\|\mathbf{r}\|^2-3x_\kappa^2+\alpha_\kappa^2-\frac{1}{3}\sum_{i=1}^{3}\alpha_i^2\right)\times I_2^1(\rho)+\frac{\mathbb{E}_2^2(\mathbf{r})}{\Lambda'-\alpha_\kappa^2}\left[\frac{3}{2}I_3^7(\rho)-\frac{1}{2\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}(\Lambda-\alpha_1^2+\rho^2)(\Lambda'-\alpha_1^2+\rho^2)}\right] \tag{A46}$$

for $\kappa=1,2,3$. The factor that multiplies the square brackets on the second part of the right-hand side of Eq. (A46) has no indeterminacy, since

$$\frac{\mathbb{E}_2^1(\mathbf{r})}{\Lambda-\alpha_\kappa^2}=(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2)\left[\sum_{i=1}^{3}\frac{x_i^2-x_\kappa^2}{\Lambda-\alpha_i^2}+1\right]\frac{1}{\Lambda-\alpha_\kappa^2} \tag{A47}$$

and

$$\frac{\mathbb{E}_2^2(\mathbf{r})}{\Lambda'-\alpha_\kappa^2}=(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2)\left[\sum_{i=1}^{3}\frac{x_i^2-x_\kappa^2}{\Lambda'-\alpha_i^2}+1\right]\frac{1}{\Lambda'-\alpha_\kappa^2}, \tag{A48}$$

where $\kappa=1,2,3$. Two important expressions are

$$\frac{\Lambda'\mathbb{E}_2^1(\mathbf{r})}{(\Lambda-\Lambda')(\Lambda-\alpha_\kappa^2)}-\frac{\Lambda\mathbb{E}_2^2(\mathbf{r})}{(\Lambda-\Lambda')(\Lambda'-\alpha_\kappa^2)}=\sum_{i=1}^{3}x_i^2\left[\frac{3}{2}(\Lambda+\Lambda')-\alpha_i^2-\alpha_\kappa^2+3\delta_{i\kappa}(\alpha_i^2-\Lambda-\Lambda')\right]+\Lambda\Lambda'-\frac{\alpha_1^2\alpha_2^2\alpha_3^2}{\alpha_\kappa^2},\quad \kappa=1,2,3 \tag{A49}$$

and

$$\frac{(\alpha_\kappa^2-\alpha_l^2)\Lambda\mathbb{E}_2^1(\mathbf{r})I_2^1(\rho)}{(\Lambda-\Lambda')(\Lambda-\alpha_\kappa^2)(\Lambda-\alpha_l^2)}-\frac{(\alpha_\kappa^2-\alpha_l^2)\Lambda'\mathbb{E}_2^2(\mathbf{r})I_2^2(\rho)}{(\Lambda-\Lambda')(\Lambda'-\alpha_\kappa^2)(\Lambda'-\alpha_l^2)}$$
$$=\alpha_\kappa^2\left(\|\mathbf{r}\|^2-3x_\kappa^2+\alpha_\kappa^2-\frac{1}{3}\sum_{i=1}^{3}\alpha_i^2\right)I_2^1(\rho)-\alpha_l^2\left(\|\mathbf{r}\|^2-3x_l^2+\alpha_l^2-\frac{1}{3}\sum_{i=1}^{3}\alpha_i^2\right)I_2^1(\rho)+\left(\frac{\alpha_\kappa^2\mathbb{E}_2^2(\mathbf{r})}{\Lambda'-\alpha_\kappa^2}-\frac{\alpha_l^2\mathbb{E}_2^2(\mathbf{r})}{\Lambda'-\alpha_l^2}\right)\left[\frac{3}{2}I_3^7(\rho)-\frac{1}{2\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}(\Lambda-\alpha_1^2+\rho^2)(\Lambda'-\alpha_1^2+\rho^2)}\right] \tag{A50}$$

for every $\kappa,l=1,2,3$, $\kappa\neq l$. Finally, we provide the relation

$$\frac{\Lambda I_2^1(\rho)-\Lambda' I_2^2(\rho)}{\Lambda-\Lambda'}=\frac{3}{2}[\alpha_1^2I_3^7(\rho)-I_2^5(\rho)]+\frac{(\rho^2-\alpha_1^2)}{2\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}(\Lambda-\alpha_1^2+\rho^2)(\Lambda'-\alpha_1^2+\rho^2)},\quad \rho\in[h_2,+\infty), \tag{A51}$$

which also eliminates the indeterminacies in an appropriate way. We observe that relations (A42)–(A51) contain ellipsoidal functions of the second degree ($n=2$) and of order $m=1,2$. This is due to the fact that these ellipsoidal harmonics carry the indeterminacies, which enter the expressions, via the factors $\Lambda-\Lambda'$, $\Lambda-\alpha_\kappa^2$ and $\Lambda'-\alpha_\kappa^2$, $\kappa=1,2,3$ in the dominators. Similar expressions containing the factors $\Lambda_\kappa-\Lambda'_\kappa$, $\Lambda_\kappa-\alpha_l^2$ and $\Lambda'_\kappa-\alpha_l^2$ for $\kappa,l=1,2,3$ are much more complicated but they do not concern the present work.

[1] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics* (Prentice Hall, Englewood Cliffs, NJ, 1965); and (Nijholl, Dordrecht, 1986).
[2] P. Moon and E. Spencer, *Field Theory Handbook*, 2nd ed. (Springer, Berlin, 1971).
[3] X. Xu and M. Wang, Q. J. Mech. Appl. Math. **44**, 537 (1991).
[4] H. Neuber, Z. Angew. Math. Mech. **14**, 203 (1934).
[5] P. M. Naghdi and C. S. Hsu, J. Math. Mech. **10**, 233 (1961).
[6] J. F. Brady and G. Bossis, Annu. Rev. Fluid Mech. **20**, 111 (1988).
[7] A. N. Kalarakis, V. N. Burganos, and A. C. Payatakes, Phys. Rev. E **65**, 1 (2002).
[8] S. Kim, Int. J. Multiphase Flow **12**, 469 (1986).
[9] S. Kim and P. V. Arunachalam, J. Fluid Mech. **178**, 535 (1987).
[10] S. Uchida, Inst. Sci. Technol. Univ. Tokyo (in Japanese) **3**, 97 (1949); Ind. Eng. Chem. **46**, 1194 (1954) (translation by T. Motai).
[11] J. Happel, AIChE J. **4**, 197 (1958).
[12] S. Kuwabara, J. Phys. Soc. Jpn. **14**, 527 (1959).
[13] G. H. Neale and W. K. Nader, AIChE J. **20**, 530 (1974).
[14] N. Epstein and J. H. Masliyah, Chem. Eng. J. **3**, 169 (1972).
[15] G. Dassios, M. Hadjinicolaou, and A. C. Payatakes, Q. Appl. Math. **52**, 157 (1994).
[16] G. Dassios, M. Hadjinicolaou, F. A. Coutelieris, and A. C. Payatakes, Int. J. Eng. Sci. **33**, 1465 (1995).
[17] R. A. Eubanks and E. Sternberg, J. Rational Mech. Anal. **5**, 735 (1956).
[18] G. Dassios, A. C. Payatakes, and P. Vafeas, Q. J. Mech. Appl. Math. **57**, 181 (2004).
[19] G. Dassios and P. Vafeas, Int. J. Math. Math. Sci. **46**, 2429 (2004).
[20] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* (Chelsea, New York, 1965).
[21] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vols. I, II.