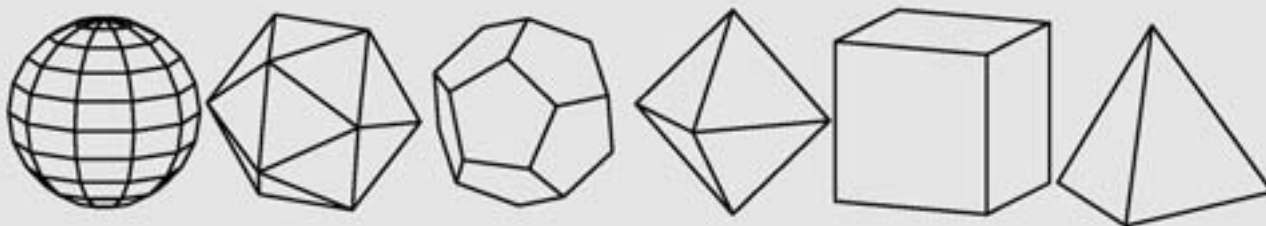


## POLARIZABILITY ANALYSIS OF CANONICAL DIELECTRIC AND BI-ANISOTROPIC SCATTERERS

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## Abstract

Solutions for simple, canonical problems are important in electromagnetics, because they can be often utilized in more complicated problems. This thesis consists of analyses of some basic canonical objects and fundamental principles of electromagnetic theory. One of the fundamental objects analyzed in this thesis is an ellipsoid, and especially a layered ellipsoid. Although it is a basic and classical object in electromagnetics, some new properties of the layered ellipsoid can still be found. Very important concept in static electromagnetics is polarizability, which simply is the connection between the incident field and the dipole moment that is induced in an object. The polarizability of a dielectric sphere and ellipsoid is well known and can be calculated with simple formulas, but for a more complicated object the evaluation of the polarizability requires more effort. This thesis presents an analysis of one particular class of objects, namely the Platonic polyhedra. The thesis also describes how inhomogeneous materials can be modelled with mixing formulas, in which the polarizability is a key parameter.

**Key words:** canonical problems, mixing theories, electrostatic polarizability, the Method of Moments, numerical methods in electromagnetics.

## Esipuhe

Tämä väitöskirja sisältää tärkeimmät tulokset vuosina 1998–2003 Teknillisen korkeakoulun Sähkömagnetiikan laboratoriossa tekemstäni tutkimustyöstä. Kiitän laboratorion henkilökuntaa avusta ja neuvoista, joita olen saanut työni eri vaiheissa. Erityisesti haluan kiittää ohjaajaani, professori Ari Sihvolaa ystävällisestä tuesta ja kärsivällisyydestä sekä professori Jukka Sarvasta ja tohtori Perttu Puskaa hedelmällisistä keskusteluista ja neuvoista. Kiitän myös Sovelletun sähkömagnetiikan tutkijakoulua jatko-opintojeni rahoituksesta.

## Preface

This thesis includes the most important results of research that I have made in the Electromagnetics Laboratory of the Helsinki University of Technology so far. I thank the personnel of the laboratory for help and guidance. Especially I want to thank my tutor, Professor Ari Sihvola for his kind support and patience, and Professor Jukka Sarvas and Doctor Perttu Puska for fruitful conversations and advice. I thank also the Graduate School of Applied Electromagnetics for funding of my graduate studies.

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## List of publications

- [P1] **J. Avelin, A. Sihvola.** Internal field of a hollow dielectric ellipsoid: the amplification effect, *Journal of Electrostatics*, vol. 56, no. 1, 2002, pp. 19–27
- [P2] **J. Avelin, T. Sengör, A. Sihvola.** Shielding effects of bi-isotropic ellipsoidal shells, *Proceedings of the 2001 International Symposium on Electromagnetic Theory (URSI)*, May 13–17, 2001, Victoria, Canada, pp. 276–278
- [P3] **J. Avelin, A. Sihvola.** Polarisability analysis of layered bi-anisotropic ellipsoids, *Bianisotropics 2000, 8th international conference on complex media*, September 2000, Lisbon, pp. 297–300
- [P4] **J. Avelin, R. Sharma, I. Hänninen, A. Sihvola.** Polarizability analysis of cubical and square-shaped dielectric scatterers, *IEEE Transactions on Antennas and Propagation*, vol. 49, no. 3, 2001, pp. 451–457
- [P5] **J. Avelin, A. Sihvola.** Polarizability of polyhedral dielectric scatterers, *Microwave and Optical Technology Letters*, vol. 32, no. 1, 2002, pp. 60–64.
- [P6] **A. Sihvola, P. Ylä-Oijala, S. Järvenpää, J. Avelin.** Polarizabilities of Platonic solids, Report 402, Electromagnetics Laboratory, Helsinki University of Technology, Espoo, November 2002. Accepted for publication in *IEEE Transactions on Antennas and Propagation*.
- [P7] **J. Avelin, A. Nadir Arslan, J. Brännback, M. Flykt, C. Icheln, J. Jun-tunen, K. Kärkkäinen, T. Niemi, O. Nieminen, T. Tares, C. Toma, T. Uusitupa, A. Sihvola.** Electric fields in the source region: the depolarization dyadic for a cubic cavity, *Electrical Engineering*, vol. 81, no. 4, 1998, pp. 199–202
- [P8] **J. Avelin, A. Sihvola.** Software for enumeration of effective parameters for chiral and bi-anisotropic materials, *Bianisotropics '98, 7th International Conference on Complex Media*, Braunschweig, 3–6 June, 1998, pp. 293–296

## Contribution of the author

The first author was responsible of the writing and preparation of the manuscript with each paper. Papers [P1]...[P5] and [P8] were mainly contributed by the author. In [P6] the author helped with the analysis of the results and developed the approximation formulas in collaboration with the first author of the paper. [P7] was the result of a post-graduate course on electromagnetic mixing formulas. Each student was supposed to contribute to the project of calculating the depolarization dyadic of the cube. The first author was mainly responsible for solving the problem and illustrating the results.



# List of symbols and acronyms

$\alpha$	polarizability
$\overline{\overline{\alpha}}$	polarizability dyadic
<b>A</b>	polarizability six-dyadic
<b>B</b>	magnetic flux density vector
$c$	speed of light in vacuum
<b>D</b>	electric flux density vector
<b>E</b>	electric field vector
$f$	volume fraction (in mixing formulas)
<b>H</b>	magnetic field vector
$\overline{\overline{I}}$	unit dyadic
<b>I</b>	unit six-dyadic
$\overline{\overline{L}}$	depolarization dyadic
<b>L</b>	depolarization six-dyadic
<b>M</b>	material parameter six-dyadic
$N_x, N_y, N_z$	depolarization factors of an ellipsoid
<b>P</b>	polarization
$\langle \mathbf{P} \rangle$	average polarization
<b>p</b>	dipole moment
<b>r</b>	position vector
$V$	volume
$\epsilon$	permittivity
$\epsilon_0$	permittivity of vacuum
$\epsilon_{\text{eff}}$	effective permittivity
$\eta$	wave impedance $\sqrt{\mu/\epsilon}$
$\chi$	nonreciprocity (or Tellegen) parameter
$\phi$	electric potential
$\phi^m$	magnetic potential
$\kappa$	chirality parameter
$\overline{\overline{\kappa}}^T$	transpose of chirality dyadic
$\tau$	permittivity contrast, ratio of permittivities $\epsilon_i/\epsilon_e$
$\mu$	permeability
$\xi, \zeta$	bi-anisotropic cross-coupling parameters
$(\xi, \eta, \zeta)$	ellipsoidal coordinates
$\Omega$	space angle
$\frac{\partial}{\partial n}$	normal component of the partial derivative
<b>BEM</b>	<i>Boundary Element Method</i>
<b>FDTD</b>	<i>Finite Difference Time Domain</i>
<b>FEM</b>	<i>Finite Element Method</i>
<b>MoM</b>	<i>Method of Moments</i>
<b>RCS</b>	<i>Radar Cross Section</i>

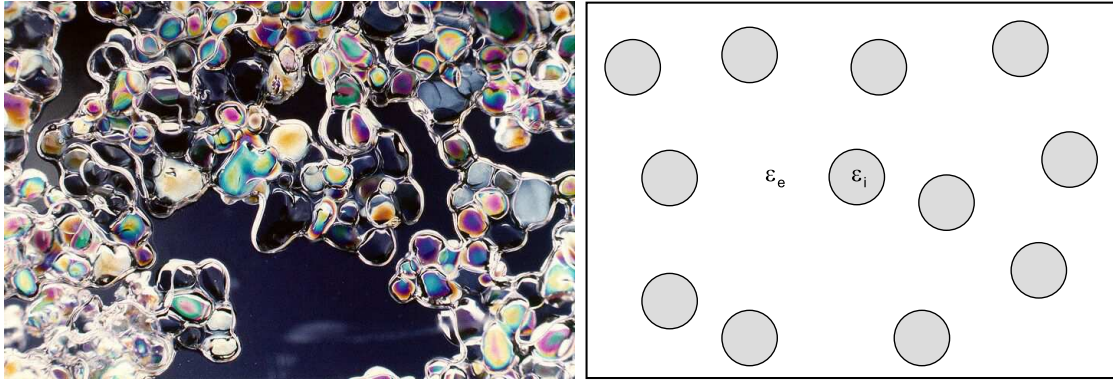


Figure 1: An example of a natural mixture and how a mixture is modelled by the classical mixing theories. The photo (taken by A. Denoth) represents Austrian snow.

## 1 Introduction

Analysis of mixtures of materials started at the early days of electromagnetics by modelling the inhomogeneities as spherical inclusions in a homogeneous material. Sphere was a natural choice to start with, since it is the simplest canonical shape.<sup>1</sup> The sphere was then followed by ellipsoid, but after that it became clear that the electromagnetic properties of even more complex shapes are too intricate to allow simple mixing rules to be formulated.

The purpose of this thesis is to analyze the properties of inhomogeneous matter, a mixture, and more specially, the properties of single particles of which the mixture consists. Some basic principles and notations which are needed in the articles consisting this thesis are explained in the following sections.

There are two important parameters for conducting or dielectric objects in electrostatics: capacitance and polarizability. Mathematicians and physicists have searched analytic solutions for them for years, but only quite few have been found. Capacitance is probably a well-known parameter, but polarizability might be quite unfamiliar.

Polarizability is defined as the relation between the dipole moment induced in an object, and the incident field which induces the dipole moment. Polarizability is an important parameter in mixing theories. The value of polarizability depends on the shape and the material of the object, and the contrast of materials of the object and the environment. The determination of the polarizability of an object requires solving the electrostatic problem where the object is located in a uniform external field. For spherical and ellipsoidal objects there are quite simple analytical solutions. For a few other canonical shapes there exist complicated analytical expressions, but generally the problem must be solved numerically, with great effort.

---

<sup>1</sup>Canonical shapes are standard, basic geometrical shapes.

In classical mixing theories the inhomogeneous inclusions are modelled as dipoles, whose density gives the approximate, effective response of the mixture. And since the polarizability parameter gives us the dipole moment of the inclusion, it is very important concept in mixing theories.

The text is organized as follows: the second section introduces the Maxwell equations and explains some important concepts in electromagnetics. The third section concentrates on one of the most important equations in static electromagnetics: the Laplace equation, especially in one important coordinate system, the confocal ellipsoidal coordinates. Section 4 introduces an important parameter in electromagnetics: polarizability. Section 5 describes another important concept: the depolarization dyadic. Section 6 focuses on homogenization of inhomogeneous matter or mixing theories. Section 7 concludes the text, and the last section includes summaries of the articles of the thesis.

## 2 Basics of electromagnetics

Maxwell equations

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) \quad (1)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) + \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t) \quad (2)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \varrho(\mathbf{r}, t) \quad (3)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \quad (4)$$

originating from 19th century, certainly belong to the most important equations in physics. But what do they mean, and how can they be used? A short explanation might be a good idea.

The equations include four field variables (electric field  $\mathbf{E}$ , electric flux density  $\mathbf{D}$ , magnetic field  $\mathbf{H}$ , and magnetic flux density  $\mathbf{B}$ ), and two source variables (electric current density  $\mathbf{J}$  and electric charge density  $\varrho$ ), from which the last one is a scalar quantity and the other ones vectors.<sup>2</sup> Classically in physics literature [1, 2]  $\mathbf{E}$  and  $\mathbf{B}$  are considered as primary fields, which in electromagnetics texts usually are  $\mathbf{E}$  and  $\mathbf{H}$ . In fact, the Maxwell equations can be written without  $\mathbf{D}$  and  $\mathbf{H}$  :

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) \quad (5)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{J}_a(\mathbf{r}, t) + \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E}(\mathbf{r}, t) \quad (6)$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\varrho_a(\mathbf{r}, t)}{\epsilon_0} \quad (7)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0. \quad (8)$$

---

<sup>2</sup> $\mathbf{B}$  is called also the magnetic induction.

In (2-3),  $\varrho$  represents free, unbound charges, and  $\mathbf{J}$  free currents, whereas in the latter form of the Maxwell equations,  $\varrho_a$  represents all charges, and  $\mathbf{J}_a$  all currents.

Although many physicists regard (5-8) as more fundamental than (1-4), in electromagnetics the Maxwell equations are most often represented as (1-4).

What can one say about the sources  $\varrho$  and  $\mathbf{J}$  ? Their connection (the continuity equation)

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (9)$$

is implicit in the Maxwell equations (2,3) and is demanded by conservation of charge.

Often it is not desirable to analyze the problem with the time-dependent Maxwell equations (1-4). A common practice is then to transform Maxwell equations to *time-harmonic* form, i.e. the time dependence is present as sinusoidal functions with a single frequency. The analysis can be simplified even more by expressing the time dependence as complex exponential function  $e^{j\omega t}$ , where  $\omega$  is the angular frequency.<sup>3</sup> Now the time-harmonic Maxwell equations can be written as

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega \mathbf{B}(\mathbf{r}) \quad (10)$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) + j\omega \mathbf{D}(\mathbf{r}) \quad (11)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \varrho(\mathbf{r}) \quad (12)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0, \quad (13)$$

from which the time dependence has been omitted. In the time-harmonic Maxwell equations the bold-faced variables are not anymore physical fields, but complex vectors or phasors.

One can see that first two equations connect electric and magnetic fields:  $\mathbf{E}$  and  $\mathbf{B}$  in (1), and  $\mathbf{H}$  and  $\mathbf{D}$  in (2). In Maxwell equations there is no relationship between  $\mathbf{E}$  and  $\mathbf{D}$ , nor between  $\mathbf{H}$  and  $\mathbf{B}$ . Now an important point arises: those relations come from the *material*. In vacuum the relations are just

$$\mathbf{D} = \epsilon_0 \mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu_0 \mathbf{H}, \quad (14)$$

where  $\epsilon_0$  and  $\mu_0$  are the free-space permittivity and permeability, respectively. In a general material the connections (or constitutive relations) are

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H}) \quad \text{and} \quad \mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H}) \quad (15)$$

In electromagnetics the view is usually macroscopic, and the response of material (the constitutive relations) appears in an averaged sense, i.e. all microscopic, molecular interactions are included in the average. If the material is isotropic, the constitutive relations can be written as

$$\mathbf{D}(\mathbf{r}) = \epsilon \mathbf{E}(\mathbf{r}) \quad \text{and} \quad \mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r}), \quad (16)$$

---

<sup>3</sup>Unfortunately physicists tend to use complex notation with opposite sign:  $e^{-i\omega t}$ .

where  $\epsilon$  and  $\mu$  now include the whole response of the material.<sup>4</sup> For an inhomogeneous material the material parameters have spatial dependence:  $\epsilon = \epsilon(\mathbf{r})$  and  $\mu = \mu(\mathbf{r})$ .

More generally, if the material is anisotropic, the permittivity and permeability appear as tensors (or dyadics):

$$\mathbf{D}(\mathbf{r}) = \bar{\bar{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) \quad \text{and} \quad \mathbf{B}(\mathbf{r}) = \bar{\bar{\mu}} \cdot \mathbf{H}(\mathbf{r}). \quad (17)$$

Again, the material parameter dyadics can have spatial dependence.

The material can also be nonlinear. A good example is ferromagnetic material, for which the permeability depends on the magnetic field.

The constitutive relations written as in (16,17) relate the electric and magnetic polarization to the electric and magnetic excitation, respectively. In addition to that, *magnetoelectric* effect can be observed in many materials, and so one is forced to define more general, bi-anisotropic constitutive relations:

$$\mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E} + \sqrt{\epsilon_0 \mu_0} (\bar{\bar{\chi}}^T - j \bar{\bar{\kappa}}^T) \cdot \mathbf{H} \quad (18)$$

$$\mathbf{B} = \sqrt{\epsilon_0 \mu_0} (\bar{\bar{\chi}} + j \bar{\bar{\kappa}}) \cdot \mathbf{E} + \bar{\bar{\mu}} \cdot \mathbf{H} \quad (19)$$

In (19),  $\bar{\bar{\kappa}}$  and  $\bar{\bar{\chi}}$  are dimensionless dyadics representing chirality and nonreciprocity of the material. In bi-isotropic, lossless medium chirality and nonreciprocity are real-valued scalars (or actually multiples of unit dyadic:  $\chi \bar{\bar{I}}$  and  $\kappa \bar{\bar{I}}$ ), and their effect on propagating electric field can be described with the following behaviour [3]:

- The nonreciprocity parameter (or Tellegen parameter)  $\chi$  affects the phase of the propagating electric field, not its polarization.
- The chirality parameter  $\kappa$  affects the polarization, not the phase.

This behaviour explains the imaginary unit  $j$  in front of  $\kappa$ . Or, more precisely, the imaginary unit is a consequence of the phase difference between the charge density and current.

With *six-dyadic* notation [4] the bi-anisotropic constitutive relations can be written in a more compact form. The electric and magnetic vector quantities are collected into two six-vectors, and the relation between them is the material six-dyadic:<sup>5</sup>

$$\mathbf{d} = \mathbf{M} \cdot \mathbf{e} \quad (20)$$

with

$$\mathbf{d} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} \bar{\bar{\epsilon}} & \sqrt{\epsilon_0 \mu_0} (\bar{\bar{\chi}}^T - j \bar{\bar{\kappa}}^T) \\ \sqrt{\epsilon_0 \mu_0} (\bar{\bar{\chi}} + j \bar{\bar{\kappa}}) & \bar{\bar{\mu}} \end{pmatrix} \quad (21)$$

<sup>4</sup>The vector  $\mathbf{D}$ , a *macroscopic* quantity, is often called the electric displacement.

<sup>5</sup>See Appendix for more discussion concerning six-vectors and six-dyadics.

At this point it might be necessary to say something about *magnetic* charges and currents: they are considered unphysical (because no one has detected magnetic charges), but sometimes they are useful in calculations. Magnetic current density is expressed as  $\mathbf{J}_m$  and magnetic charge density as  $\varrho_m$ . Vector  $-\mathbf{J}_m$  should be added to the right hand side of (1), and  $\varrho_m$  added to the right hand side of (4).

Regarding the solution of Maxwell equations, it must be first noted that they form a system of differential equations in which boundary conditions are needed to make the solution unique. Obviously the geometry of the problem gives the needed boundary conditions for the solution of the equations.

It is clear that the central problem in electromagnetics is to solve Maxwell equations in the given geometry with the knowledge of sources and the response of the material. Often the problems in electromagnetics require intricate evaluations, hence it might be advantageous to restrict the analysis to simple, *canonical* problems, and to try to use the results in more complicated problems.

### 3 Laplace equation in ellipsoidal coordinates

In electrostatics there is no time dependence, so  $\nabla \times \mathbf{E} = 0$ . And because of that, the electric field can be expressed with a potential function,  $\mathbf{E} = -\nabla\phi$ . In a source-free region, where no free charges exist,  $\nabla \cdot \mathbf{D} = 0$ . And finally, if the material is isotropic and homogeneous,

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = \epsilon \nabla \cdot \mathbf{E} = 0 \longrightarrow \nabla \cdot \mathbf{E} = 0.$$

From these assumptions arises one of the most important equations in electrostatics: Laplace equation

$$\nabla^2 \phi = 0. \tag{22}$$

Solutions of Laplace equation are called harmonic functions. Every separable coordinate system<sup>6</sup> has its own set of solutions, eigensolutions for Laplace equation [5]. The three commonly used coordinate systems are Cartesian, spherical, and cylindrical coordinates. The forms of the solutions in these coordinate systems [6] can be seen in Table 1.

Still one important coordinate system exists: confocal ellipsoidal coordinates. Its importance arises from the fact that one can find quite simple closed-form results for many problems involving ellipsoidal geometry.

Ellipsoidal coordinates can be related to Cartesian coordinates in many ways. A classical book about ellipsoidal coordinate system is [7], and the definition presented

---

<sup>6</sup>There are 11 coordinate systems, in which the Laplace equation is separable. In two additional systems (bispherical and toroidal) Laplace equation can be separated by introducing a multiplicative factor. Thus the total number is 13. For details see [5].

Cartesian	$\phi_{k_x k_y k_z} =$	$\begin{Bmatrix} e^{k_x x} \\ e^{-k_x x} \end{Bmatrix}$	$\begin{Bmatrix} e^{k_y y} \\ e^{-k_y y} \end{Bmatrix}$	$\begin{Bmatrix} e^{k_z z} \\ e^{-k_z z} \end{Bmatrix}$
Spherical	$\phi_{lm} =$	$\begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix}$	$\begin{Bmatrix} P_l^m(\cos \theta) \\ Q_l^m(\cos \theta) \end{Bmatrix}$	$\begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}$
Cylindrical	$\phi_{m\alpha} =$	$\begin{Bmatrix} J_m(\alpha\rho) \\ N_m(\alpha\rho) \end{Bmatrix}$	$\begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix}$	$\begin{Bmatrix} e^{-\alpha z} \\ e^{\alpha z} \end{Bmatrix}$

Table 1: Eigensolutions of Laplace equation for three most important coordinate systems. The parameters in exponential and Bessel functions can also be imaginary, leading to trigonometric and Hankel functions. For solutions of Cartesian coordinates,  $k_x^2 + k_y^2 + k_z^2 = 0$ .

there is commonly used in mathematics and physics. However, the definition used here is from [8,9], where ellipsoidal coordinates are related to Cartesian coordinates by the equation

$$\frac{x^2}{u+a^2} + \frac{y^2}{u+b^2} + \frac{z^2}{u+c^2} = 1, \quad (a > b > c) \quad (23)$$

which has three different real roots  $\xi, \eta, \zeta$ . The roots lie in the following ranges:

$$\xi \geq -c^2, \quad -c^2 \geq \eta \geq -b^2, \quad -b^2 \geq \zeta \geq -a^2.$$

The surfaces of constant  $\xi$  are ellipsoids, and  $\eta$  and  $\zeta$  are respectively hyperboloids of one and two sheets. All three surfaces are confocal to a reference ellipsoid of semiaxes  $(a, b, c)$  :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (24)$$

Confocality means that the semiaxes  $a_1, b_1, c_1$  of another ellipsoid, confocal to the reference ellipsoid, have to satisfy

$$a_1^2 - a^2 = b_1^2 - b^2 = c_1^2 - c^2. \quad (25)$$

Separation of variables for Laplace equation in the ellipsoidal system leads to the Lamé differential equation for each of the variables  $\xi, \eta$  and  $\zeta$ . The Lamé functions are solutions to these equations, but unfortunately they are exactly known only for degrees less than or equal to three [7]. Thus the ellipsoidal harmonic functions cannot be expressed as compactly as in Table 1.

For example the potential of a uniform,  $x$ -directed incident field of amplitude  $E$  can be written in ellipsoidal coordinates, as an ellipsoidal harmonic function:

$$\phi_0(\mathbf{r}) = \phi_0(x) = -Ex = -E\sqrt{\frac{(\xi + a^2)(\eta + a^2)(\zeta + a^2)}{(b^2 - a^2)(c^2 - a^2)}}. \quad (26)$$

### 3.1 Field inside and outside of a dielectric ellipsoid

One important application of Laplace equation in ellipsoidal coordinates is to determine the field inside and outside of a dielectric ellipsoid which is immersed in a uniform incident field. In order to solve the field inside the ellipsoid and the “scattered” field, one has to apply boundary conditions: the tangential electric field is continuous across the boundary, which is the same as the potential being continuous, and the normal displacement is continuous across the boundary.

And the result is that the field inside a dielectric ellipsoid with relative permittivity  $\epsilon_i$  is uniform and its potential is related to the incident potential as:

$$\phi_i(x) = \frac{\epsilon_e}{\epsilon_e + (\epsilon_i - \epsilon_e)N_x} \phi_0(x). \quad (27)$$

$N_x$  is an important quantity related to the ellipsoid: the *depolarization factor* in  $x$ -direction.<sup>7</sup> Outside the ellipsoid the relative permittivity is  $\epsilon_e$ .

The potential outside the ellipsoid is of the form

$$\phi_e(x) = \phi_0(x) + \phi_s(x) = \phi_0(x) - \frac{abc(\epsilon_i - \epsilon_e)}{2(\epsilon_e + N_x(\epsilon_i - \epsilon_e))} \phi_0(x) F(\xi), \quad (28)$$

in which the  $\eta$  and  $\zeta$  dependence is the same as in the incident potential  $\phi_0(x)$ . The function  $F(\xi)$  in (28) is

$$F(\xi) = \int_{\xi}^{\infty} \frac{ds}{(s + a^2)\sqrt{(s + a^2)(s + b^2)(s + c^2)}}. \quad (29)$$

At large distances  $\xi$  is large and  $\xi \approx r^2$ , hence  $\phi_e(x)$  acts like the potential of a dipole:

$$\phi_s(x) \approx -\frac{abc(\epsilon_i - \epsilon_e)}{2(\epsilon_e + N_x(\epsilon_i - \epsilon_e))} \phi_0(x) \int_{r^2}^{\infty} \frac{ds}{s^{5/2}} = -\frac{abc(\epsilon_i - \epsilon_e)}{\epsilon_e + N_x(\epsilon_i - \epsilon_e)} \frac{x}{3r^3} E. \quad (30)$$

This potential equals the potential of a dipole with dipole moment

$$p = \frac{4\pi abc}{3} \frac{\epsilon_0 \epsilon_e (\epsilon_i - \epsilon_e)}{\epsilon_e + N_x(\epsilon_i - \epsilon_e)} E. \quad (31)$$

---

<sup>7</sup>The depolarization factor is a mathematical concept. Later, in (3.3), it will be used in calculating the internal field and polarization of dielectric ellipsoids.



### 3.2 Depolarization factors

The field solution includes a set of important parameters, which can be used to simplify the expressions for the fields: the depolarization factors

$$N_p = \frac{abc}{2} \int_0^\infty \frac{ds}{(s+q^2)\sqrt{(s+a^2)(s+b^2)(s+c^2)}}, \quad (32)$$

with  $p = x, y, z$  and  $q = a, b, c$ , respectively. The depolarization factors satisfy

$$N_x + N_y + N_z = 1. \quad (33)$$

With incomplete elliptic integrals

$$F(k, \varphi) = \int_0^\varphi \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}} = \int_0^{\sin \varphi} \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}} \quad (34)$$

$$E(k, \varphi) = \int_0^\varphi \sqrt{1-k^2 \sin^2 \psi} d\psi = \int_0^{\sin \varphi} \frac{\sqrt{1-k^2 z^2}}{\sqrt{1-z^2}} dz \quad (35)$$

the depolarization factors can be calculated as [10, 11]:

$$N_x = \frac{abc}{(a^2-b^2)\sqrt{a^2-c^2}} (F(k, \varphi) - E(k, \varphi)) \quad (36)$$

$$N_y = 1 - N_x - N_z \quad (37)$$

$$N_z = \frac{b}{b^2-c^2} \left( b - \frac{ac}{\sqrt{a^2-c^2}} E(k, \varphi) \right), \quad (38)$$

where  $a > b > c$  and  $k = \sqrt{\frac{a^2-b^2}{a^2-c^2}}$  and  $\varphi = \arccos(\frac{c}{a})$ .

### 3.3 Layered ellipsoid

The analysis can be extended to a *layered* dielectric ellipsoid with confocal ellipsoidal boundaries. It is found that in each layer (except in the innermost layer, or the core of the ellipsoid) the field is a combination of a uniform field and a dipole-type field, with the potential of the form

$$\phi_k(x) = -Ex \left[ A_k - \frac{B_k}{2} \int_\xi^\infty \frac{ds}{(s+a_1^2)\sqrt{(s+a_1^2)(s+b_1^2)(s+c_1^2)}} \right]. \quad (39)$$

Note that only the range of the lower integration limit  $\xi$  in (39) is different in each layer, but the integrand remains the same in all layers. In the inner layers  $\xi < 0$ ,

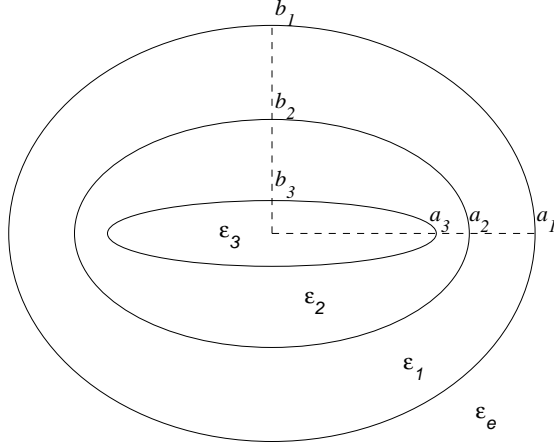


Figure 2: Cross-section of a layered ellipsoid with confocal boundaries between layers.

at the outer boundary of the whole ellipsoid  $\xi = 0$ , and outside the whole ellipsoid  $\xi > 0$ . The relative permittivity in the  $k$ th layer is  $\epsilon_k$ .

In the core (the  $N$ th layer) of an ellipsoid with  $N$  layers the field is uniform, and so  $B_N = 0$ . Coefficients  $A_k$  and  $B_k$  for the  $k$ th layer are solved using the boundary conditions, and they can be evaluated with the matrix equation<sup>8</sup> [12]

$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = \frac{1}{\epsilon_{k-1}} \begin{bmatrix} \epsilon_k + N_k^x(\epsilon_{k-1} - \epsilon_k) & \frac{N_k^x(1-N_k^x)}{a_k b_k c_k}(\epsilon_{k-1} - \epsilon_k) \\ a_k b_k c_k(\epsilon_{k-1} - \epsilon_k) & \epsilon_{k-1} + N_k^x(\epsilon_k - \epsilon_{k-1}) \end{bmatrix} \begin{bmatrix} A_{k-1} \\ B_{k-1} \end{bmatrix}, \quad (40)$$

with  $A_0 = 1$  and  $B_N = 0$ . Detailed analysis is in Appendix.

Note that the depolarization factors are not the same for all ellipsoids, although they are confocal.

Similar analysis can be done to the  $y$ - and  $z$ -components of the field, and the resultant field is a superposition of the field components. The results of this analysis are used in [P1], to calculate the field in a hollow dielectric ellipsoid. It is also possible to derive dyadic expressions for the field amplitude coefficients.

Although this treatment considered a layered *dielectric* ellipsoid, the same analysis can be done in magnetostatic case leading to the same matrix equations, with permittivity changed to permeability and electric potential to magnetic potential.

In quasistatic case with chiral or bi-anisotropic materials the situation changes, since electric and magnetic fields couple in the material. By careful analysis it is possible to derive six-dyadic expressions for the field amplitude ratios, which is done in [P2,P3]. Detailed derivation for the six-dyadic boundary conditions is in Appendix.

<sup>8</sup>For index  $k = 0$  the permittivity  $\epsilon_k = \epsilon_e$ .

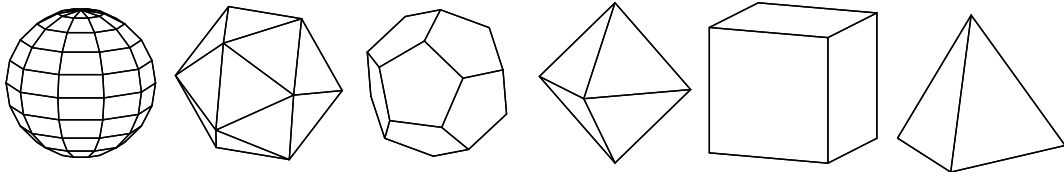


Figure 3: Three-dimensional shapes for which the polarizability is a scalar: sphere, icosahedron, dodecahedron, octahedron, hexahedron (cube), and tetrahedron.

## 4 Polarizability

Capacitance and polarizability are important concepts in electromagnetics. Capacitance is not a subject in this thesis, but some remarks on capacitance are given further. Polarizability is defined as the relation between the dipole moment induced in an object, and the incident field which induces the dipole moment.

More precisely: let there be uniform, static electric field  $\mathbf{E}$  in empty space. If a dielectric (or conducting) object is embedded in the space, the object produces a perturbation to the field. Far away from the object this perturbation behaves like the field of an electric dipole, if the net charge in the object is zero. So, from a distant viewpoint, the object can be replaced by a dipole. The polarizability  $\alpha$  is the relation between the electric field  $\mathbf{E}$  and the dipole moment  $\mathbf{p}$  that is “induced” in the object by this field<sup>9</sup> [2, 13]:

$$\mathbf{p} = \alpha \mathbf{E}. \quad (41)$$

In (41) the polarizability is a scalar variable. For a class of simple, symmetric objects the polarizability is a scalar: sphere and regular<sup>10</sup> polyhedra, see Figure 3. But generally the polarizability should be expressed as a tensor or dyadic, so that the relation is

$$\mathbf{p} = \overline{\overline{\alpha}} \cdot \mathbf{E}. \quad (42)$$

In other words, the induced dipole moment depends on the orientation of the inclusion with respect to the incident field. A good example of such behaviour is obviously an ellipsoid.

Capacitance and polarizability are related to the first terms of multipole expansion; capacitance is related to the monopole field amplitude (first term) of the multipole expansion, and polarizability to the dipole field amplitude terms [14].

Capacitance and polarizability are important parameters in electrostatics, but they are useful also in scattering theory: polarizability can be used to estimate the far field in the Rayleigh (low-frequency) scattering regime [15–17], and capacitance (or *capacity*) is important in low-frequency scattering in other domains of physics, for

<sup>9</sup>Static field is considered here, but the field can be time-dependent, if the size of the object is much smaller than wavelength (quasistatic consideration).

<sup>10</sup>There are only five *regular* polyhedra, the Platonic polyhedra, see Figure 3.

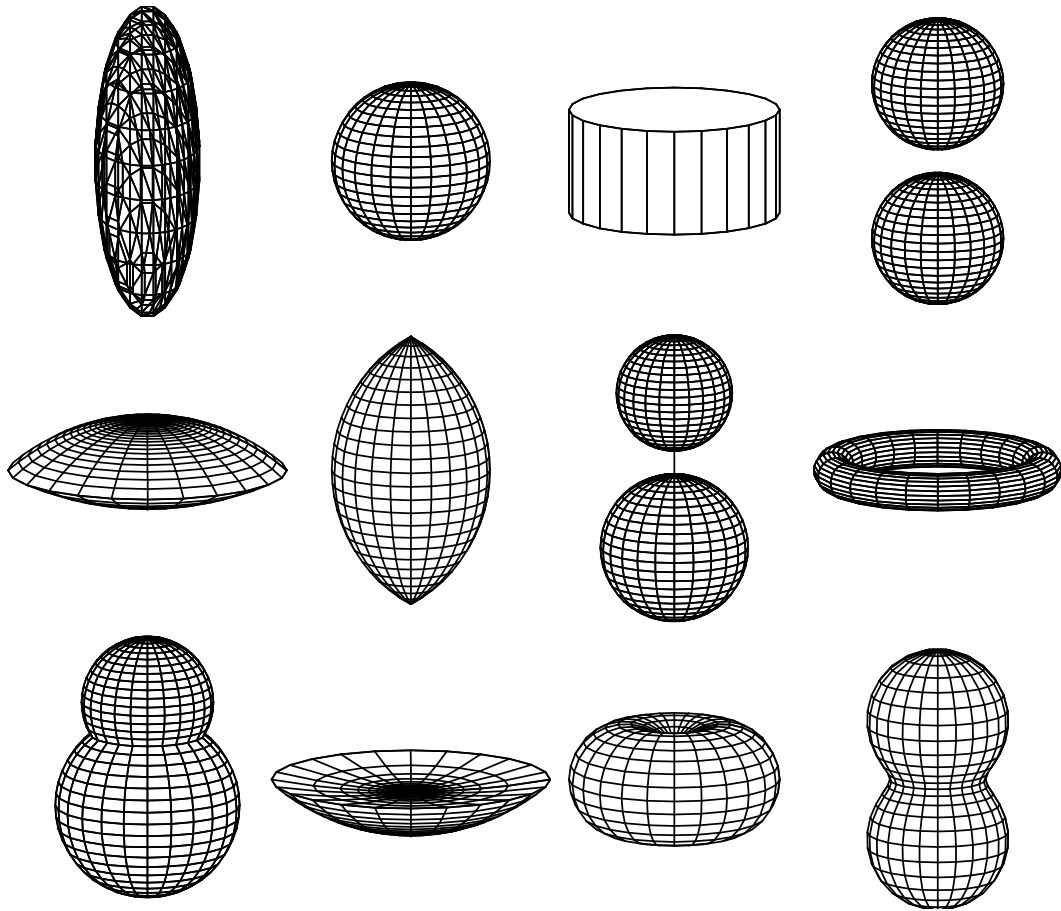


Figure 4: Shapes for which exact solutions for capacitance and/or polarizability are known. Top row: ellipsoid, sphere, short right circular cylinder, and two spheres. Middle row: lens, spindle, two spheres at the same potential, and torus. Bottom row: two intersecting spheres, bowl, blood cell, and peanut.

example in acoustics [18]. Polarizability is important also in the homogenization theories [13]

There are a few shapes, for which closed-form solution<sup>11</sup> exists for capacitance and/or polarizability: ellipsoid (sphere), torus, lense, spindle, bowl, pair of spheres at arbitrary separation, and right circular cylinder [19–29]. Quite recently an exact solution for an *Kelvin-inverted* ellipsoid was derived in [30]; “blood cell” and “peanut” are special cases for it [31]. Most solutions are for conducting objects, but a closed-form solution exists for the polarizability of a dielectric ellipsoid (sphere) and two separated (or touching) spheres [24, 25].

The sphere and ellipsoid have the simplest solutions for capacitance and polarizability. This fact arises from several reasons: the surfaces of these objects are whole constant-coordinate surfaces, the field is uniform in a dielectric sphere or ellipsoid

<sup>11</sup>Usually the solutions are in the form of an integral or a slowly converging series.

(if the incident field is uniform), and the capacitance and polarizability are closely connected to the lowest-order spherical and ellipsoidal harmonics, which are quite simple functions.

Attempts have been made to derive closed-form solutions for other basic shapes, but without success. For example, cube is the “principal object” of Cartesian coordinate system, yet there is no closed-form solution for capacitance or polarizability of a cube. More precisely, a *conjecture* [32] for the capacitance of a cube exists:

$$C = \frac{32\pi^3 a \epsilon_0}{\sqrt{6}} \frac{1}{\Gamma(1/24)\Gamma(5/24)\Gamma(7/24)\Gamma(11/24)} \approx 0.659463 a \epsilon_0, \quad (43)$$

where  $a$  is the length of cube side. It is intriguing to see if any one can present even a conjecture for a closed-form solution of *polarizability* of a cube.

An interesting historical note is that famous mathematicians Jacobi and Kirchhoff have claimed that Dirichlet, shortly before his death had a solution for the capacitance of a rectangular parallelepiped, but that it was lost [33].

## 4.1 Polarizability of a sphere

Sphere is a special case of an ellipsoid, and it (or circle in two-dimensional space) is geometrically the simplest three-dimensional object, and indeed many electromagnetic properties of a sphere can be expressed exactly with simple formulas. For a sphere with relative permittivity  $\epsilon_i$  in an environment with relative permittivity  $\epsilon_e$  the polarizability is well known [1]:

$$\alpha = 3\epsilon_e \epsilon_0 V \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e}, \quad (44)$$

where  $V$  is the volume of the sphere.

For a conducting sphere, let  $\epsilon_i/\epsilon_e \rightarrow \infty$ , thus  $\alpha = 3\epsilon_e \epsilon_0 V$ .

## 4.2 Ellipsoid

Let us assume a dielectric ellipsoid whose semiaxes  $a_x$ ,  $a_y$ , and  $a_z$  fix the Cartesian coordinate system. The permittivities of the material of the ellipsoid and the environment are  $\epsilon_i$  and  $\epsilon_e$ , respectively. Now the polarizability is a tensor or dyadic. In dyadic notation the polarizability of a dielectric ellipsoid can be expressed (see Section 3) as

$$\bar{\bar{\alpha}} = V(\epsilon_i - \epsilon_e) \sum_{j=x,y,z} \frac{\epsilon_e \epsilon_0}{\epsilon_e + N_j(\epsilon_i - \epsilon_e)} \mathbf{u}_j \mathbf{u}_j = V \epsilon_e \epsilon_0 (\epsilon_i - \epsilon_e) \left[ \epsilon_e \bar{\bar{\mathbf{I}}} + \bar{\bar{\mathbf{L}}}(\epsilon_i - \epsilon_e) \right]^{-1}, \quad (45)$$

where  $N_j$ 's are the depolarization factors and  $V = \frac{4}{3}\pi a_x a_y a_z$  is the volume of the ellipsoid. The depolarization dyadic  $\bar{\bar{L}}$  contains the depolarization factors as eigenvalues:

$$\bar{\bar{L}} = N_x \mathbf{u}_x \mathbf{u}_x + N_y \mathbf{u}_y \mathbf{u}_y + N_z \mathbf{u}_z \mathbf{u}_z. \quad (46)$$

For a conducting ellipsoid, let again  $\epsilon_i/\epsilon_e \rightarrow \infty$ , and so

$$\bar{\bar{\alpha}} = V \epsilon_e \epsilon_0 \sum_{j=x,y,z} N_j^{-1} \mathbf{u}_j \mathbf{u}_j = V \epsilon_e \epsilon_0 \bar{\bar{L}}^{-1}. \quad (47)$$

### Layered ellipsoid

It is quite straightforward to derive the polarizability dyadic  $\bar{\bar{\alpha}}$  for a layered ellipsoid with any number of layers. The ratios of the fields in the layers can be calculated using the recursive scheme presented in (3.3) and in Appendix. All three components of the polarizability dyadic can then be calculated from the corresponding dipole field amplitudes.

## 4.3 Polarizability of an ellipsoid of complex material

Let us briefly discuss a more complicated situation: the material in ellipsoid is anisotropic. It can be shown that the internal field of the ellipsoid is still uniform, but the polarizability should be expressed as a dyadic even if the ellipsoid is actually a sphere [13].

Let the ellipsoid material be anisotropic so that the relative permittivity is a dyadic,  $\bar{\bar{\epsilon}}_i$ . Now the polarizability is

$$\bar{\bar{\alpha}} = V \epsilon_e \epsilon_0 (\bar{\bar{\epsilon}}_i - \epsilon_e \bar{\bar{I}}) \cdot \left[ \epsilon_e \bar{\bar{I}} + \bar{\bar{L}} \cdot (\bar{\bar{\epsilon}}_i - \epsilon_e \bar{\bar{I}}) \right]^{-1}. \quad (48)$$

Closed-form solutions can be written for the polarizability of an ellipsoid of even more complex (bi-anisotropic, inhomogeneous, etc.) materials [12, 13, 34].

## 4.4 Polarizability of more complicated objects

For a scatterer with arbitrary shape, the relation between induced field and dipole moment is not as simple as for a sphere or an ellipsoid. The internal field is not uniform, and hence in such a geometry, evaluation of dipole moment and polarizability requires more effort. If for example the object is a polyhedron, there might not exist an analytical solution for the internal field.

The determination of the polarizability of an object requires solving an electrostatic problem where an object is located in a uniform external field. Classical methods, like solving a boundary value problem with the separation of variables, can be used with simple geometries, but often it is necessary to use numerical methods.

One popular method is to derive an integral equation and solve it numerically. Various volume and surface integral equations are derived and presented in the literature. In this case the integral equation should obviously be static, but it is possible to calculate the (static) polarizability from a limit (frequency goes to zero) of frequency-dependent scattering behaviour of the object.

Statistical methods, like random walk simulation, have been recently used in the determination of capacitance and polarizability, and with good accuracy [19].

#### 4.4.1 Surface integral equation method

The polarizability of a dielectric object can be evaluated from the potential on the surface of the object. In this subsection the derivation of the integral equation for the potential on the surface is shortly presented. More detailed derivation is in [2].

The derivation uses the fact that for one important harmonic function, the potential of a point source, the following is true:

$$\nabla'^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r}') \quad (49)$$

Thus the Green's theorem can be written as

$$\int_V \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla'^2 \phi - \phi \nabla'^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] dV' = \int_S \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial}{\partial n'} \phi - \phi \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] dS', \quad (50)$$

where  $\phi$  is an arbitrary function which is single-valued in  $V$  and on its boundary  $S$ , and has continuous second partial derivatives in  $V$ .

Let us assume a dielectric object of volume  $V$  and surface area  $S$  with relative permittivity  $\epsilon_i$  in a uniform field with potential  $\phi_0$ . Outside of the object the relative permittivity is  $\epsilon_e$ .

First the Green's theorem is applied to  $1/|\mathbf{r} - \mathbf{r}'|$  and to an arbitrary harmonic function  $\phi$ , leading to two equations:

$$\frac{1}{4\pi} \int_S \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial}{\partial n'} \phi(\mathbf{r}') - \phi(\mathbf{r}') \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] dS' = \begin{cases} \phi(\mathbf{r}) & \text{if } \mathbf{r} \in V \\ 0 & \text{if } \mathbf{r} \notin V \end{cases} \quad (51)$$

Next the Green's theorem is applied to the *exterior* volume bounded by  $S$  and a sphere at infinity. For a harmonic function  $\phi$  which is regular at infinity, the con-

tribution from the spherical surface at infinity vanishes. As a result, two equations are again attained as a result:

$$\frac{1}{4\pi} \int_S \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial}{\partial n'} \phi(\mathbf{r}') - \phi(\mathbf{r}') \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] dS' = \begin{cases} -\phi(\mathbf{r}) & \text{if } \mathbf{r} \notin V \\ 0 & \text{if } \mathbf{r} \in V \end{cases} \quad (52)$$

The next step is to insert  $\phi = \epsilon_i \phi_2 - \phi_0$  in (51) and  $\phi = \epsilon_e \phi_1 - \phi_0$  in (52). Remember that  $\epsilon_i$  and  $\epsilon_e$  are relative, dimensionless quantities.

If one now subtracts the last one of the two equations in (52) from the first one in (51) and uses boundary conditions for the potential and normal component of the gradient of the potential, one gets an equation:

$$\epsilon_i \phi_2(\mathbf{r}) = \phi_0(\mathbf{r}) - \frac{\epsilon_i - \epsilon_e}{4\pi} \int_S \phi(\mathbf{r}') \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dS', \quad \mathbf{r} \text{ inside } V. \quad (53)$$

And subtracting the first equation in (52) from the last one in (51) leads to

$$\epsilon_e \phi_1(\mathbf{r}) = \phi_0(\mathbf{r}) - \frac{\epsilon_i - \epsilon_e}{4\pi} \int_S \phi(\mathbf{r}') \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dS', \quad \mathbf{r} \text{ outside } V. \quad (54)$$

Note that  $\phi$  in (53,54) is the common limit of  $\phi_1$  and  $\phi_2$ . With (53) one can calculate the potential  $\phi_2$  inside  $V$  once the potential  $\phi$  on the surface is known, and with (54) the potential  $\phi_1$  outside  $V$ .

### Integral equation for the potential

The final step is to combine the two equations (53)-(54) into one integral equation.

The integral in (53) and (54) has the form of a double-layer potential which is not continuous on the surface. The jump condition for the integral at a point  $\mathbf{r}_s$  on the surface is [35]

$$\begin{aligned} \lim_{\mathbf{r} \rightarrow \mathbf{r}_s} \frac{1}{4\pi} \int_S \phi(\mathbf{r}') \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dS' = \\ \frac{1}{4\pi} \int_S \phi(\mathbf{r}'_s) \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dS' + \begin{cases} \phi(\mathbf{r}_s) \left( 1 - \frac{\Omega}{4\pi} \right), & \text{P}_1 : \mathbf{r} \rightarrow \mathbf{r}_s \text{ from outside} \\ -\phi(\mathbf{r}_s) \frac{\Omega}{4\pi}, & \text{P}_2 : \mathbf{r} \rightarrow \mathbf{r}_s \text{ from inside} \end{cases} \end{aligned} \quad (55)$$

where the  $\Omega$  is the solid angle for the boundary seen from point  $\mathbf{P}_s$ .





Figure 5: Field point  $P_s$  on an edge or vertex, and on a smooth surface.

Let  $\mathbf{r}$  in (53,54) approach  $\mathbf{r}_s$ , as in Fig. 5. Using jump condition (55), one can combine (53) and (54), and the result is an integral equation

$$\phi_0(\mathbf{r}) = \frac{\epsilon_i + \epsilon_e}{2} \phi(\mathbf{r}) + \frac{(\epsilon_i - \epsilon_e)}{2} \left(1 - \frac{\Omega}{2\pi}\right) \phi(\mathbf{r}) + \frac{\epsilon_i - \epsilon_e}{4\pi} \int_S \phi(\mathbf{r}') \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dS', \quad (56)$$

in which  $\mathbf{r}$  is on  $S$ .

If  $\mathbf{r}_s$  is on a smooth surface (i.e.  $\mathbf{r}_s$  is not an edge or vertex point), the solid angle  $\Omega = 2\pi$ , and the jump term reduces to  $\pm\phi(\mathbf{r}_s)/2$ . Thus the integral equation reduces to

$$\phi_0(\mathbf{r}) = \frac{\epsilon_i + \epsilon_e}{2} \phi(\mathbf{r}) + \frac{\epsilon_i - \epsilon_e}{4\pi} \int_S \phi(\mathbf{r}') \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dS', \quad \mathbf{r} \text{ on } S. \quad (57)$$

In the equations,  $\phi_0$  is the potential of the incident field, and  $\phi$  is the unknown total potential on the surface of the object.

Equations (56,57) are Fredholm integral equations of the second kind, since they are of the form

$$\phi(\mathbf{r}) + \lambda \int_S K(\mathbf{r}, \mathbf{r}') \phi(\mathbf{r}') dS' = \psi(\mathbf{r}), \quad (58)$$

where

$$\lambda = \frac{\epsilon_i - \epsilon_e}{\epsilon_i + \epsilon_e + (\epsilon_i - \epsilon_e)(1 - \frac{\Omega}{2\pi})} \quad \text{or} \quad \lambda = \frac{\epsilon_i - \epsilon_e}{\epsilon_i + \epsilon_e}$$

The integration has to be performed over the surface of the object, and the integral is in general a convergent improper integral in which a small subarea, where  $\mathbf{r} = \mathbf{r}'$ , is excluded and shrunk to zero. And since the kernel

$$K(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$

of the integrals is only weakly singular, the contribution of the singularity vanishes:

$$\lim_{\delta S \rightarrow 0} \int_{\delta S} \phi(\mathbf{r}') \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dS' = 0, \quad (59)$$

where  $\delta S$  is a small subarea enclosing  $\mathbf{r}$ .

Fredholm theory assures that the integral equation (58) is convergent, and it has a unique solution, since  $|\lambda| < 1$ , i.e.  $\lambda$  is not an eigenvalue of the corresponding homogeneous equation [2, 36]. As a conclusion, the integral equations (56, 57) suit well for the calculation of the potential on the surface of the dielectric object.

The induced dipole moment of the object is calculated by integrating the induced polarization  $\mathbf{P}$  in the object:

$$\mathbf{p} = \int_V \mathbf{P} dV = (\epsilon_i - \epsilon_e) \epsilon_0 \int_V \mathbf{E} dV = -(\epsilon_i - \epsilon_e) \epsilon_0 \int_V \nabla \phi(\mathbf{r}) dV = (1 - \tau) \epsilon_e \epsilon_0 \int_V \nabla \phi(\mathbf{r}) dV, \quad (60)$$

where  $\tau$  is the contrast parameter  $\tau = \epsilon_i / \epsilon_e$ .

Using Gauss' identity the dipole moment can be calculated from the potential on the surface of the object:

$$\mathbf{p} = (1 - \tau) \epsilon_e \epsilon_0 \int_S \phi(\mathbf{r}) \mathbf{u}_n dS. \quad (61)$$

The polarizability  $\alpha$  can now be calculated, since  $\mathbf{p} = \alpha \mathbf{E}_e$ , with  $\mathbf{E}_e$  being the incident uniform field:  $\mathbf{E}_0 = -\nabla \phi_0 = E_0 \mathbf{u}_z$ .

In two-dimensional case the corresponding integral equation is

$$\phi_0(\mathbf{r}) = \frac{\tau + 1}{2} \phi(\mathbf{r}) - \frac{\tau - 1}{2\pi} \int_C \phi(\mathbf{r}') \frac{\partial}{\partial n'} \ln |\mathbf{r} - \mathbf{r}'| dc', \quad \mathbf{r} \text{ on } C, \quad (62)$$

where  $C$  is the contour enclosing the surface. The dipole moment is now

$$\mathbf{p} = (1 - \tau) \epsilon_e \epsilon_0 \int_C \phi(\mathbf{r}) \mathbf{u}_n dC. \quad (63)$$

#### 4.4.2 Numerical implementation

One method to solve the unknown potential in (57) is to use the Method of Moments (MoM), which was also originally used in [2, 37] to calculate the polarizability of a dielectric cube.<sup>12</sup> In those days the performance of computers was quite poor and the results were not accurate; the first published results [37] were even erroneous. The first reasonably accurate results were published in [P4, P5].

When looking how to apply the MoM to (57), the equation is first written as a linear operator equation:

$$(a\mathbf{L} + \mathbf{I})\phi = b\phi_0, \quad (64)$$

---

<sup>12</sup>More accurate results are in [38] Different approach, based on expansion of the potential and the Green function as spherical harmonics, is in [39], but it seems to be rather cumbersome.

where  $\mathbf{L}$  is the integral operator,  $\mathbf{I}$  is an identity operator, and

$$a = \frac{1}{2\pi} \left( \frac{\epsilon_i - \epsilon_e}{\epsilon_i + \epsilon_e} \right) \quad \text{and} \quad b = \frac{2}{\epsilon_i + \epsilon_e}.$$

The first task is to choose a set of basis functions, which are used to express the unknown function  $\phi$ . One possibility is to use so-called entire-domain basis functions, which are continuous in the whole domain of integration. The other possibility is to choose subsectional basis functions, which are defined in subsections of the integration domain.

Now the unknown function  $\phi$  is approximated with the set of basis functions as a sum

$$\hat{\phi} = \sum_{k=1}^M \phi_k B_k, \quad (65)$$

where  $B_k$  is the  $k$ :th basis function and  $\phi_k$  is the unknown coefficient to be solved.

After inserting the expanded potential (65) to the integral equation, the residual is

$$R = b\phi_0 - (a\mathbf{L} + \mathbf{I}) \hat{\phi} = b\phi_0 - (a\mathbf{L} + \mathbf{I}) \sum_{k=1}^M \phi_k B_k. \quad (66)$$

Next task is to minimize the residual  $R$ , which can be done by weighting the  $R$  with properly chosen set of weight functions<sup>13</sup>  $T$ , and setting these “weighted residuals” to zero. Mathematically this means that the weight functions should be orthogonal to the residual, i.e. the inner products<sup>14</sup> of the weight functions and the residual are zero:

$$\langle R, T_j \rangle = \int R T_j = 0, \quad j = 1 \dots M. \quad (67)$$

The result is a matrix equation

$$\sum_{k=1}^M \phi_k \left( a \int T_j \mathbf{L} B_k + \int T_j B_k \right) = b \int T_j \phi_0, \quad j = 1 \dots M, \quad (68)$$

from which the unknown coefficients  $\phi_k$  have to be solved. If the weight functions are the same as the basis functions, the method is called the Galerkin method, but in the case where the weight functions are delta functions, the method is usually called point collocation.

The unknowns  $\phi_k$  can be solved by inverting the matrix, but it is more efficient to solve the equation iteratively, especially when the matrix is large. An iteration

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<sup>13</sup>Usually there are the same number of basis and weight functions, but in principle the set of weight functions could be larger than the set of basis functions.

<sup>14</sup>One of the terms in the integrand should be complex-conjugate, but in this case (static, real-valued potential) the conjugation can be neglected.

formula can be easily constructed from (68) by rearranging the terms:

$$\sum_{k=1}^M \phi_k^{[n+1]} \int T_j B_k = b \int T_j \phi_0 - a \sum_{k=1}^M \phi_k^{[n]} \int T_j \mathcal{L} B_k, \quad j = 1 \dots M. \quad (69)$$

The inner-product terms in the left side of (69) can be collected to a matrix<sup>15</sup>  $D$ . Multiplying (69) with the inverse of  $D$  leads to a matrix equation of the iteration algorithm. This algorithm usually converges fairly quickly; it is actually a Neumann series solution [36], and the iteration formula simply consists of one matrix-vector product and a sum of two vectors. If one wants to solve the matrix equation efficiently, more powerful methods like the Generalized Minimum Residual method (GMRES) have to be used.

After the approximation for the potential is calculated, it is time to calculate the estimated relative polarizability  $\hat{\alpha}_r$  of the object:

$$\hat{\alpha}_r = \frac{\hat{\alpha}}{\epsilon_e V} = \frac{1}{\epsilon_e V} \mathbf{P} \cdot \mathbf{u}_z = -\frac{\tau - 1}{V} \mathbf{u}_z \cdot \int_S \hat{\phi}(\mathbf{r}) \mathbf{u}_n dS. \quad (70)$$

#### 4.4.3 Polarizability of dielectric polyhedra from the MoM solution

The usual method in MoM is to use subsectional basis and weight functions with piecewise-continuous polynomials. The simplest ones of them are piecewise-constant functions (pulses) and piecewise-linear (triangular) functions. Pulses were used in the early calculations of the polarizability of a cube [37,38], and in a quite straightforward way. More accurate results could be achieved by using pulses more efficiently or using higher-order basis functions.

Pulse basis functions are easy to use, and they have one additional advantage: there is no need to place weight points or pulses on the edges and vertices, and the equation (56) can be avoided. The solution with pulse basis functions can be made better by using variable-sized pulses: pulses are denser in those places where the potential is expected to have rapid changes; obviously the edges and vertices of the polyhedra are those places.

Variable-sized pulses were used in [P4,P5], and in [40] as basis functions  $B_k$ . The aim was to keep the code simple, thus rectangular pulses were used as much as possible. However, a few additional triangular pulses<sup>16</sup> were needed on the other polyhedra than cube (see Fig. 1-2 in [P5]).

<sup>15</sup>The matrix  $D$  is diagonal in the case of orthogonal basis functions, for example pulses.

<sup>16</sup>Triangular pulse basis functions, or piecewise-constant triangular basis functions have a constant value on a triangular “patch”; they are not the same as piecewise-linear triangular basis functions.

The pulse basis function  $B_k$  is defined as:

$$B_k = \Pi_k(\xi, \eta) = \begin{cases} 1 & (\xi, \eta) \in \text{Supp}(B_k) \\ 0 & \text{otherwise} \end{cases}, \quad (71)$$

where  $\xi$  and  $\eta$  are the appropriate coordinates on the surface.

When using piecewise constant basis functions, the integrand for the integral operator in (68)-(69) consists only of the normal derivative of the Green function:

$$\mathbb{L}B_k = \int_{(\xi', \eta') \in \text{Supp}(B_k)} \frac{\partial}{\partial n'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d\xi' d\eta'. \quad (72)$$

This integral can be evaluated analytically; the resulting equation is somewhat lengthy, but still usable in computer solution.

The inner product integral is also possible to calculate analytically, but it is advisable to calculate it with suitable numerical integration method (for instance Gaussian quadrature method with  $5 \cdot 5 \dots 7 \cdot 7$  points). This is because in this case it is inefficient to utilize the closed-form solution. However, the closed-form solution was tested and it worked, but did not bring additional accuracy.

For higher-order basis functions it is advisable to use more advanced methods in performing the integration corresponding to (72), as is done in [P6].

In writing the code, the symmetry of polyhedra should be exploited carefully. It is illustrative to take a closer look at a dielectric cube. Let us have a dielectric cube with the normal of one face parallel to the incident field. The surface of the cube is divided into  $N^2$  subsections (patches) so that the total number of patches (and basis functions) is  $6N^2$ . By using symmetry, the number of *different* potential coefficients  $\phi_k$  can be reduced<sup>17</sup> to  $N^2/2$ , and the number of weight functions is then also  $N^2/2$ . In (65) the basis functions  $B_k$  are grouped so that several basis functions (eight or sixteen) “share” the same potential coefficient<sup>18</sup>  $\phi_k$ . As a consequence, the size of the matrix is only  $(\frac{N^2}{2}) \cdot (\frac{N^2}{2}) = \frac{N^4}{4}$ , not  $36N^4$ .

Octahedron is even more convenient object, since the same potential values can be used on its all eight faces (with different signs for the potential values above and below the  $xy$ -plane).

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<sup>17</sup>The number could be even smaller, since the top and bottom faces could be divided into eight symmetric pieces.

<sup>18</sup>Of course the potential coefficient above the  $xy$ -plane has different sign than the corresponding potential coefficient below the  $xy$ -plane.

#### 4.4.4 Convergence and accuracy

The iteration formula (69) is used in [P4] and [P5]. It works well and is tolerant to the initial guess  $\phi^{[0]}$ : the solution converges well even with quite badly selected  $\phi^{[0]}$ . A suitable choice is  $\phi^{[0]} = \phi_0$ . Within the range  $0.1 < \tau < 10$  the variation error in the values of the normalized polarizability of a dielectric cube is less than  $10^{-7}$  after about 20 iteration rounds, whereas more than 30 rounds are needed if  $\tau$  is very large or close to zero. The polarizability of a dielectric octahedron behaves in quite the same manner, but tetrahedron is the most difficult object, more than 80 iteration rounds are needed with large or small  $\tau$ .

Another story is the convergence (or the accuracy) of the polarizability versus the discretization density. The MoM often leads to large matrices and relatively slow convergence, which is true in this case, too. This can be clearly seen in the case of cube. If the gridding of the faces of a cube is uniform (the subsections are of equal size), the error is assumed<sup>19</sup> to be about  $O(N^{-1.5})$ . However, with properly chosen variation of size of subsections the error seems to be about  $O(N^{-2.0})$ . This means one decimal digit more accuracy, when<sup>20</sup>  $N = 160$ . Since the analysis of accuracy is based on the convergence, a conservative estimate for the error of the results is less than 0.1 %.

With piecewise linear (or in three-dimensional case, piecewise bilinear) basis functions, the error might be of the order  $O(N^{-3.0})$  or less, but the method could be less stable. In the two-dimensional case the piecewise linear basis functions were tested, but some stability problems appeared with them.<sup>21</sup> Therefore no tests were made in the three-dimensional cases with other basis functions than pulses.

#### 4.4.5 Boundary element method

In [P6] a special computer program was tested, designed for analyzing acoustic scattering problems. The program appeared to be suitable for calculating the potential on the surface of a dielectric object with (56,57). This program utilizes the MoM, although mathematicians tend to use the name Boundary Element Method (BEM) for MoM applied to a boundary integral equation. Polarizability of dielectric tetrahedron, cube, and octahedron was solved once again in [P6] with basis functions of higher degree than in [P4] and [P5], leading to more accuracy in the results. Probably one digit more accuracy could have been gained, if symmetry were taken advantage of like in [P4] and [P5], but unfortunately the program was not designed to exploit symmetries in the problem. Polarizability of the two remaining regular polyhedra, icosahedron and dodecahedron, was calculated as well.

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<sup>19</sup>The error in the MoM is difficult to analyze, so it is estimated from the convergence of the results.

<sup>20</sup>Please remember that each face of the cube is divided into  $N \cdot N$  subsections.

<sup>21</sup>Afterwards it became clear that there were actually some errors in the computer code.

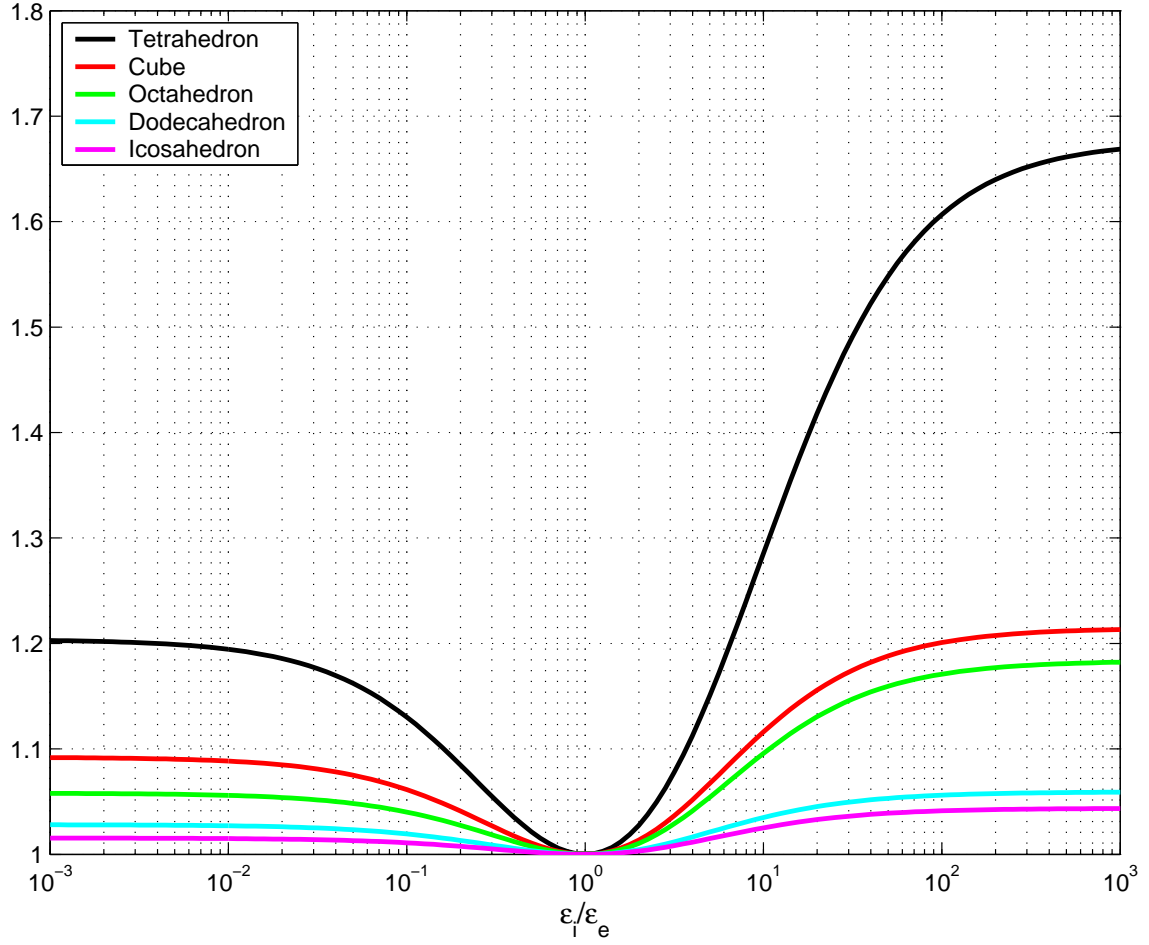


Figure 6: Polarizabilities of the five Platonic solids as functions of the permittivity, normalized to the polarizability of a sphere with the same volume and permittivity.

#### 4.4.6 An example: polarizabilities of polyhedra

As an important and illustrative example from the results in [P6], the polarizabilities of five Platonic polyhedra are shown in Figure 6. And as an example of the practical results, an approximation formula for the normalized polarizability of cube is presented here. The approximation formula is based on the most accurate calculation, made in [P6].

$$\alpha_n = \frac{\alpha}{\epsilon_e V} \approx \alpha_\infty (\tau - 1) \frac{\tau^3 + 4.83981\tau^2 + 5.54742\tau - \alpha_0}{\tau^4 + 8.0341\tau^3 + 19.3534\tau^2 + 15.4349\tau + \alpha_\infty} \quad (73)$$

where  $\alpha_\infty = 3.6442$  and  $\alpha_0 = -1.6383$ . The permittivity contrast  $\tau = \epsilon_i/\epsilon_e$ .

#### 4.4.7 RCS method

As it was described in Section 3, the scattered field of a sphere in an electrostatic field is equivalent to a dipole field. One can replace a sphere which is electrically small by an ideal dipole<sup>22</sup> with a dipole moment  $\mathbf{p}$  even when the applied field is a plane wave. It is then possible to relate the polarizability  $\alpha$  of the sphere to its radar cross section (RCS)  $\sigma_{\text{rcs}}$  by the following equation [41]:

$$\frac{\alpha}{\epsilon_e V} = \frac{\sqrt{4\pi\sigma_{\text{rcs}}}}{k_o^2 V}, \quad (74)$$

where  $k_o$  is the free space wavenumber and  $V$  is the volume of the sphere. The polarizability of a dielectric cube can be calculated with the same equation, if the radar cross section of the cube is known. The dimensions of the cube should be much smaller than the operating wavelength. This method is used in [P4] as a comparison method for the surface integral equation method.

The radar cross section of the cube was calculated numerically using volume integral equation method [42, 43].

## 5 Depolarization dyadic

One basic task in electromagnetics is to calculate the field from known sources, for instance the electric field  $\mathbf{E}$  in a homogeneous medium from a known electric current distribution  $\mathbf{J}$ :

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu_0 \int_{V_j} \bar{\bar{\mathbf{G}}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') dV', \quad \mathbf{r} \notin V_j, \quad (75)$$

where  $\bar{\bar{\mathbf{G}}}(\mathbf{r}, \mathbf{r}')$  is the free-space Green dyadic

$$\bar{\bar{\mathbf{G}}}(\mathbf{r}, \mathbf{r}') = \left( \bar{\bar{\mathbf{I}}} + \frac{\nabla\nabla}{k^2} \right) \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (76)$$

and  $\mathbf{J}$  is the source current density.

Now what happens if the field point  $\mathbf{r}$  is inside the source region? Then the integral includes a singularity, which arises when the source point and the field point coincide. It is possible to exclude a small region from the integration domain, and calculate a principal value integral by shrinking the exclusion region  $V_\delta$  to zero. Usually the exclusion region is a sphere, but it can have another shape. A.D. Yaghjian did a detailed derivation for a field equation in which the exclusion region has an arbitrary shape [44, 45]:

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<sup>22</sup>This is true only outside the sphere, inside the sphere the field is uniform.



$$\mathbf{E}(\mathbf{r}) = -j\omega\mu_0 \lim_{\delta \rightarrow 0} \int_{V_j - V_\delta} \overline{\overline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') dV' - \frac{1}{j\omega\epsilon_0} \overline{\overline{\mathbf{L}}} \cdot \mathbf{J}(\mathbf{r}), \quad \mathbf{r} \in V_\delta, \quad (77)$$

where

$$\overline{\overline{\mathbf{L}}} = \oint \frac{\mathbf{n} \mathbf{u}'_R}{4\pi R'^2} dS' \quad (78)$$

is a source dyadic or a generalized depolarization dyadic. The dyadic  $\overline{\overline{\mathbf{L}}}$  depends only on the shape of the exclusion region, not on its size. The source current density  $\mathbf{J}$  in (77) is assumed ‘sufficiently well-behaved’<sup>23</sup>.

Electrostatic field can be expressed in terms of electric polarization  $\mathbf{P}$  within dielectric material, if in (77)  $j\omega\mathbf{P}$  replaces  $\mathbf{J}$ , and  $\omega = 0$ , and  $k = 0$ :

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \lim_{\delta \rightarrow 0} \int_{V_j - V_\delta} \nabla \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \cdot \mathbf{P}(\mathbf{r}') dV' - \frac{\overline{\overline{\mathbf{L}}} \cdot \mathbf{P}(\mathbf{r})}{\epsilon_0}, \quad \mathbf{r} \in V_\delta. \quad (79)$$

Now the same requirement for  $\mathbf{P}$  in (79) is assumed, as for  $\mathbf{J}$  above.

The depolarization dyadic needs some explanation. Let an external electric field  $\mathbf{E}$  excite a uniform polarization  $\mathbf{P}$  into dielectric material. If a cavity is carved within the material, the cavity field<sup>24</sup>  $\mathbf{E}_L$  can be calculated using the depolarization dyadic  $\overline{\overline{\mathbf{L}}}$ :

$$\mathbf{E}_L = \mathbf{E} + \frac{1}{\epsilon_0} \overline{\overline{\mathbf{L}}} \cdot \mathbf{P}. \quad (80)$$

The depolarization dyadic  $\overline{\overline{\mathbf{L}}}$  is just the same as in (79). For an ellipsoidal geometry it has appeared earlier, see (46). The depolarization dyadic of a sphere is  $\overline{\overline{\mathbf{L}}} = \overline{\overline{\mathbf{I}}}/3$ .

For spherical and ellipsoidal geometry the field is uniform in the cavity, and so (80) gives the correct local field in the whole cavity. But for example in cubical cavity the local field in (80) is correct only in the center point of the cube, where  $\overline{\overline{\mathbf{L}}} = \overline{\overline{\mathbf{I}}}/3$ , like in the sphere. A procedure for calculating depolarization dyadics of polyhedra is presented in [48], with the field point (origin) at the center of the object.

The assumption above was that the exclusion volume shrinks to zero, and the origin (the field point) lies in the center of the volume. The integral in (78) does not depend on the size of the exclusion volume, only on the *geometry* of it. So it is possible to put the field point anywhere in the exclusion volume in (78), which was done in [P7], where the depolarizing dyadic of a cube was calculated. The result could be used in analyzing the field of a dielectric cube, at least as a first approximation.

<sup>23</sup>This was indeed the original expression used in [44]. The author probably means that  $\mathbf{J} \in C_0^2$ , or, more precisely that  $\mathbf{J}$  is Hölder continuous in  $V_j$  (see, for example [46, 47]).

<sup>24</sup>It is called also local field.

A minor flaw in [P7] is that the diagonal elements of  $\bar{\bar{L}} : L_{xx}, L_{yy},$  and  $L_{zz}$  are not uniquely defined at the edges of the cube.<sup>25</sup> The limit value depends on the direction from which the edge is approached. Thus the values for  $L_{xx}$  at the edge and corner of the cube, given in section 2.2 and 2.3 of [P7], are not unique. On the other hand, the reported value  $L_{zz}^{edge} \approx 0.148$  is well-defined. As a conclusion: all formulas and curves in [P7] are still valid, but the values for the diagonal elements should not have been mentioned in the text.

The formulation presented above is not the only one, for other formulations exist, too. Quite similar approach was taken by J.G. Fikioris [46], but the exclusion region in his formulation can be finite-sized; diminishing the exclusion region to zero leads to the same result as in [44]. The Fikioris' approach has been recently developed further, see for instance [47, 49].

## 6 Homogenization of mixtures

Now it is time to use some of the previously defined concepts. They can be applied to model mixtures or heterogeneous materials. A mixture is generally difficult to analyze, but certain, simple models can be constructed using rather simple procedures [13]. The very first models were introduced in the 19th century. Analysis of the inhomogeneous materials and mixing models can be found for example in [50–53]. Tabular data about electromagnetic properties of materials is collected for example in [54, 55].

The models or mixing rules, typically give an effective parameter as an output: effective refractive index, permittivity, permeability, or conductivity, depending on the context of the problem. The inhomogeneity of the mixture is averaged according to an appropriate procedure. Mixing rules can be used in various applications involving heterogeneous materials, for example in [56], where scattering of microwaves by an ensemble of nonspherical ice particles is studied. Ice particle inhomogeneity (consisting of air, water, and ice) is taken into account using three different classical mixing rules.

One of the most famous models is the *Maxwell Garnett* mixing formula [57], which can be derived by a quite straightforward procedure. Let us start with uniform electric field  $\mathbf{E}$ , which permeates infinite, homogeneous material with permittivity  $\epsilon_e$ . If a homogeneous dielectric sphere (with permittivity  $\epsilon_i$ ) is introduced in the material, a dipole moment  $\mathbf{p} = \alpha \mathbf{E}$  is induced in the sphere. As is already known, the polarizability of the sphere is

$$\alpha = 3\epsilon_e V \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e}.$$

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<sup>25</sup>Dr. Martin Norgren from the Royal Institute of Technology, Sweden, kindly reported about this minor flaw.

If there are several such spheres sparsely in the environment,  $n$  in the unit volume, the average dipole moment density (or the average polarization) is

$$\langle \mathbf{P} \rangle = n\mathbf{p} = n\alpha\mathbf{E}. \quad (81)$$

On the other hand, inhomogeneous material can be described with an effective permittivity  $\epsilon_{\text{eff}}$  :

$$\mathbf{D} = \epsilon_{\text{eff}}\mathbf{E} \implies \langle \mathbf{P} \rangle = (\epsilon_{\text{eff}} - \epsilon_e)\mathbf{E}.$$

So the effective permittivity of a dilute mixture is

$$\epsilon_{\text{eff}} \approx \epsilon_e + n\alpha. \quad (82)$$

If the mixture is not dilute, the interaction of the spheres with each other should be taken into account. Thus the local field that is felt by any one of the spheres can be described as an “effective local field”, which includes the interactions from all other spheres averaged in  $\langle \mathbf{P} \rangle$  (see (80)):

$$\mathbf{E}_L = \mathbf{E} + \frac{1}{\epsilon_e} \bar{\mathbf{L}} \cdot \langle \mathbf{P} \rangle = \mathbf{E} + \frac{1}{3\epsilon_e} \langle \mathbf{P} \rangle. \quad (83)$$

Now the dipole moment induced in one sphere is

$$\mathbf{p}_{\text{mix}} = \alpha\mathbf{E}_L,$$

and since

$$\langle \mathbf{P} \rangle = n\mathbf{p}_{\text{mix}} \quad \text{and} \quad \langle \mathbf{P} \rangle = (\epsilon_{\text{eff}} - \epsilon_e)\mathbf{E},$$

the effective permittivity can be written as

$$\epsilon_{\text{eff}} = \epsilon_e + \frac{n\alpha}{1 - \frac{n\alpha}{3\epsilon_e}} \quad \text{or} \quad \frac{\epsilon_{\text{eff}} - \epsilon_e}{\epsilon_{\text{eff}} + 2\epsilon_e} = \frac{n\alpha}{3\epsilon_e} \quad (84)$$

which is called the *Clausius-Mossotti* or *Lorenz-Lorentz* formula.

The polarizability of a sphere includes the volume  $V$  of a sphere. The volume fraction  $f$  of the spheres in the mixture is  $f = nV$ , thus the formula can be written also as

$$\epsilon_{\text{eff}} = \epsilon_e + 3f\epsilon_e \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e - f(\epsilon_i - \epsilon_e)} \quad (85)$$

which is just the Maxwell Garnett mixing formula. Note that in (85) the inclusion spheres can have different sizes, as long as all of them are small compared to the wavelength.

There are several other mixing formulas in use, for example the Bruggeman<sup>26</sup> formula [58, 59] and the Coherent potential formula [60]. The so-called *unified mixing formula* [13, 61]

$$\frac{\epsilon_{\text{eff}} - \epsilon_e}{\epsilon_{\text{eff}} + 2\epsilon_e + \nu(\epsilon_{\text{eff}} - \epsilon_e)} = f \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e + \nu(\epsilon_{\text{eff}} - \epsilon_e)} \quad (86)$$

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<sup>26</sup>Actually there are several Bruggeman formulas, the most popular one being the “symmetric Bruggeman” formula.

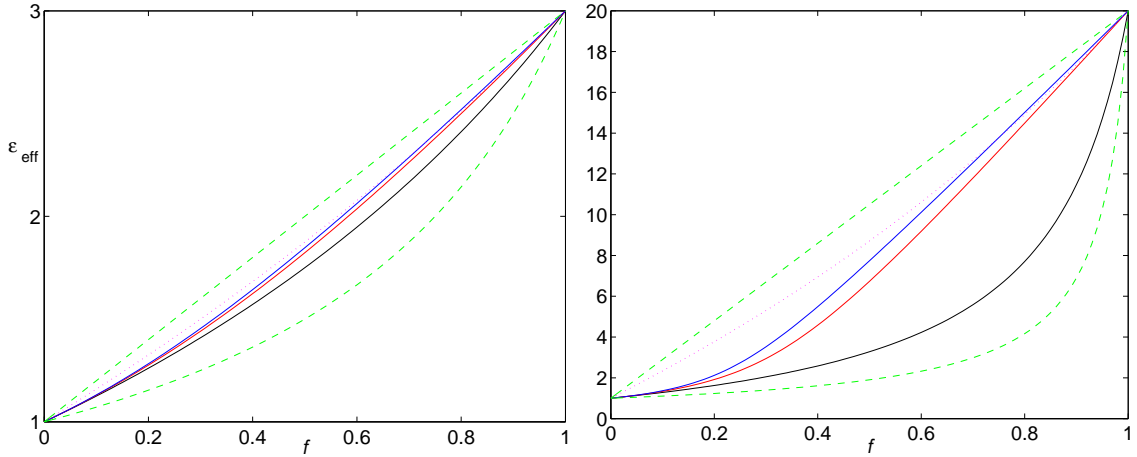


Figure 7: Effective relative permittivity with Maxwell Garnett (lowest solid curve), Bruggeman (solid curve at the middle), and Coherent potential (uppermost solid curve) mixing rules. Left: permittivity contrast  $\epsilon_i/\epsilon_e = 3$ , right:  $\epsilon_i/\epsilon_e = 20$ . Dashed curves are the Wiener bounds, dotted curve is the higher Hashin-Shtrikman bound.

includes all three formulas: with  $\nu = 0$ , the Maxwell Garnett rule is recovered,  $\nu = 2$  gives the Bruggeman formula, and  $\nu = 3$  gives the Coherent potential formula. The parameter  $\nu$  can be chosen freely in the range  $0 \leq \nu \leq 3$ . There are also other generalized mixing formulas, for example one due to McLachlan [62], which was disputed many years, but has nowadays generated renewed interest [63, 64].

The above mentioned mixing rules assume spherical inclusions with isotropic, homogeneous inclusion material, but mixing rules can be generalized to allow ellipsoidal inclusions. Mixing rules for more complex materials, like anisotropic or bi-anisotropic material, have been derived, too.

## 6.1 Validity of the mixing rules

It is intuitively obvious that  $\epsilon_{\text{eff}}$  should fall between  $\epsilon_i$  and  $\epsilon_e$ , and that is true for real- and scalar-valued material parameters. But there are stricter bounds [65], for example the Wiener bounds [58, 66],

$$\epsilon_{\text{eff,max}} = f\epsilon_i + (1 - f)\epsilon_e \quad (87)$$

and

$$\epsilon_{\text{eff,min}} = \frac{\epsilon_i\epsilon_e}{f\epsilon_e + (1 - f)\epsilon_i} \quad (88)$$

which are in one sense “absolute” bounds, and the Hashin-Shtrikman bounds [67]. Figure 7 gives an idea, how strict the bounds are, and in which relation the three well-known mixing rules are to them. It should be noted that the lower Hashin-Shtrikman bound is given by the Maxwell Garnett rule for spherical inclusions, and

the higher Hashin-Shtrikman bound is actually also the Maxwell Garnett rule, but with the complementary change  $\epsilon_i \rightarrow \epsilon_e$ ,  $\epsilon_e \rightarrow \epsilon_i$ , and  $f \rightarrow 1 - f$ .

It is difficult to say which mixing rule comes closest to reality. The mixing rules involve many approximations, for example the Maxwell Garnett rule includes only the dipole-dipole interactions between the spherical inclusions. One possibility to evaluate the mixing rules is to analyze the mixtures numerically [68]. The Bruggeman rule seems to be closer to the results obtained from numerical simulations, if clustering of spherical inclusions is allowed. On the other hand, the simulations are better in agreement with the Maxwell Garnett rule, if the spherical inclusions are all separated.

The three mixing rules have still similarities, which can be seen by analyzing the unified mixing formula. The Taylor expansion of (86) reveals that for dilute mixtures ( $f \ll 1$ ) the value of  $\nu$  does not have very significant effect on  $\epsilon_{\text{eff}}$ :

$$\epsilon_{\text{eff}} = \epsilon_e + 3\epsilon_e \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e} f + 3\epsilon_e \left( \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e} \right)^2 \left( 1 + \nu \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e} \right) f^2 + \dots \quad (89)$$

The expansion is independent of  $\nu$  up to the first order in  $f$ :

$$\epsilon_{\text{eff}} \approx \epsilon_e + 3f\epsilon_e \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e} = \epsilon_e + n\alpha \quad (90)$$

which is the same as (82). Now it is clear that the Maxwell Garnett, Bruggeman, and Coherent Potential formula, as well as the unified mixing formula, all give the same prediction for  $\epsilon_{\text{eff}}$  up to the first order in  $f$ . This behaviour can be clearly seen in Fig. 7: all three mixing formulas have the same slope at  $f = 0$ .

The coefficient of the *second* order term in  $f$  depends on  $\nu$ . One might ask what the proper choice for  $\nu$  would be. Coefficient of the second order term in  $f$  is calculated analytically in [25] by calculating the polarizability of a double sphere, but unfortunately the result is a very slowly converging series.<sup>27</sup> Thus it remains unclear if one can derive a higher-order mixing formula with the same simplicity as the classical formulas.

The trend seems to be to use modern, numerical methods also in the analysis of heterogeneous materials, but the classical mixing rules are still useful because of their simple nature.

## 6.2 Recent progress in mixing theories

The classical mixing rules have been useful for decades. The software package *Mix-tool*, presented in paper [P8], is based on the three classical models which are generalized to ellipsoidal inclusions and bi-anisotropic materials. *Mixtool* is a useful

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<sup>27</sup>Numerical analysis leads to an estimate of  $\nu \approx 0.4$  for a mixture of separated spherical inclusions. The value of  $\nu$  depends slightly on the permittivity contrast  $\epsilon_i/\epsilon_e$ .

tool for demonstrating and teaching various aspects of the classical mixing rules. It shows clearly what kind of effects for example chirality brings to the electromagnetic response of the mixture.

In material science it is of great importance to use the best available methods in order to model the response of media. The recent progress of computing power and of numerical methods has allowed the use of computers with numerically very intensive calculations. The Finite Element Method (FEM) has become more and more popular in the last decades, and is used also for mixture analysis [64, 69, 70]. MoM (or BEM) was very popular method in the early days of numerical computation, and has recently attained back its popularity. It has been also used to analyze mixtures [71–75]. Also the Finite difference method [68] has been used in mixture analysis, and even Finite Difference, Time Domain method (FDTD) [76]. It should be noted that all these methods approximate the real randomness of the sample by a periodic model since there cannot be an infinite simulation space. Thus the mixtures are actually pseudorandom. On the other hand, the artificial material itself can have a periodic structure, like in [69, 74, 75]. An approximate method to analyze random media using MoM is presented in. [77]

## 7 Conclusions

This thesis presents various results and analyses in static and quasi-static electromagnetics. The research can be divided into two research lines. The first one started with the development of Mixtool which is a useful Matlab program for demonstrating and teaching the various properties of mixtures. The development work of Mixtool led to the analysis of layered ellipsoids. As a result, a six-dyadic expression for the polarizability of a layered bi-anisotropic ellipsoid was formulated. This expression is easily adopted for example in Matlab. As a by-product, a six-dyadic expression was formulated for the field in the core of a two-layer ellipsoid. The analysis of this expression revealed a remarkable behaviour: the amplitude of the field in the core can be larger than the amplitude of the incident field! This was not expected, since for example for a layered *sphere* the shell acts always as a “shield” that attenuates the field so that the amplitude of the field in the core cannot be larger than the amplitude of the incident field. The “amplification effect” was studied more thoroughly with a layered dielectric ellipsoid.

Another line of research started with analysis of the depolarization dyadic of a cube. The approach was intriguing, since no one had thought that the field point could be at another place than at the center of the object. The analysis required performing quite straightforward integrations, but the result, the depolarizing dyadic, is very complicated. The interest to the cubical shape led to the analysis of another property: the polarizability of the cube. It was soon realized that former calculations were not accurate. Numerical analysis for the polarizability of a dielectric cube

was performed, to achieve a result that is as accurate as possible. Later on, also other important shapes were studied: the other four Platonic polyhedra. As a result, our analysis has provided very accurate numerical values as well as practical approximation formulas for the polarizabilities of Platonic polyhedra.

As can be seen from the two research lines, there are many possibilities for the future work. A possible choice is to continue the analysis of polarizability of various canonical shapes, for example cylinders of various cross sections, clusters of spheres, as well as objects with anisotropic and bianisotropic materials.

## 8 Summary of the publications

[P1] analyzes the electric field in a hollow dielectric ellipsoid, as the ellipsoid is exposed to a static and uniform external field. The amplitude of the internal field depends on the permittivity contrast of the shell, the volume ratio of the core and the whole ellipsoid, and the orientation of the ellipsoid with respect to the external incident field. Analysis of the field ratio reveals an interesting fact: the field in an ellipsoidal cavity inside the ellipsoidal object can be larger than the incident field. The internal field attains its maximum in a tiny disc-like cavity, the normal of which is parallel to the incident field, as the whole ellipsoid becomes a sphere. The maximum field ratio can be as large as 3. A geometrical interpretation for this is that the amplification effect comes from the different curvatures of the two boundaries that the electric field experiences: the outer one and the inner one. They are different because the two ellipsoidal boundaries are of different axis ratios.

[P2] adds chirality to the material of the ellipsoidal shell in [P1]. Quasi-static analysis leads to a six-dyadic formula for the field ratio. Of course, the same ‘amplification effect’ can be found as in [P1], but chirality has an additional effect on the field ratio. A six-dyadic is an operator which acts on six-vectors in the same manner as a dyadic acts on vectors. The six-vectors are combinations of field vectors: electric and magnetic field vectors combined to an electromagnetic field six-vector, and electric and magnetic flux density vectors to an electromagnetic flux density six-vector. In a special case, with no chirality, the six-dyadic expression could be reduced to a dyadic expression, with help of which the dielectric case in [P1] could be analyzed more generally.

[P3] analyzes the polarizability problem of a layered bi-anisotropic ellipsoid. Using six-vector formalism, the six-dyadic expression for the polarizability is solved. The polarizability six-dyadic is explicitly calculated for the case of a two-component ellipsoid where the core is fully bi-anisotropic but the shell and the environment are bi-isotropic at most. It is reasonably straightforward to include more bi-isotropic layers on the ellipsoid. The limitation of the analysis is that all ellipsoidal boundaries for the composite structure have to be confocal. A numerical example is given where the inclusions are layered spheres with core and shell of opposite handedness.

[P4] describes the polarizability characteristics of a dielectric cube. Polarizability is the relation between the induced dipole moment in the cube, and the incident electric field. The electrostatic potential is calculated, solving a surface integral equation numerically with the MoM, in which the potential is expressed as piecewise constant basis functions with rectangular grid on the surface. The dipole moment is then calculated from the potential on the surface of the cube. Other authors have calculated polarizability with quite sparse and uniform grid. Modern computers with large memory allow much denser grid, which leads to more accurate results. But more cleverly designed, non-uniform grid, which is denser near the edges and vertices of the cube, brings additional accuracy in the solution. The results obtained



for the polarizability of cubes using the surface integral equation are compared with those obtained from the volume integral equation, and they are in good agreement. The work presents simple approximation formulas for the polarizability as functions of the permittivity of the cube. The results show an improvement over results presented by earlier authors. The polarizability of an infinite cylinder with square-shaped cross-section, which is the two-dimensional counterpart of the cube is also described.

[P5] is an extension to [P4]. It presents numerical results for the dielectric polarizability of canonical homogeneous scatterers: cube, tetrahedron, and octahedron. The same computational principle and numerical method as was applied in [P4] is used here to these solids, and symmetry of the objects has been taken advantage of just as in [P4]. Rectangular, nonuniform grid is used on all solids, with a few triangular patches on tetrahedron and octahedron. For inclusions with the same volume and permittivity, it is seen that the polarizabilities of these inclusions fall in to the following order: tetrahedron has the largest polarizability, then cube and octahedron with the sphere as the one with smallest polarizability. Interesting fact is that the polarizability of a tetrahedron is substantially larger than the one of a cube.

[P6] extends the polarizability analysis to all Platonic polyhedra: tetrahedron, hexahedron (cube), octahedron, dodecahedron, and icosahedron. The polarizability is calculated with the same integral equation as in [P4] and [P5], but now a more elaborate computational scheme is used and the results are more accurate, although the software implementation does not utilize symmetry. The basis functions used in this case are higher-order piecewise-continuous polynomials, and the gridding is totally triangular. It is found that the polarizability correlates more with the number of edges of the polyhedron than the number of vertices or faces.

[P7] gives a closed-form expression for the depolarization dyadic for a cubic volume. With this dyadic, the singularity associated with the electromagnetic field calculation within the source region can be properly accounted for. The depolarization dyadic is a classical concept, with which the cavity field (or local field) in a cavity in dielectric material can be calculated. For a cubic volume, the dyadic depends on the position of the field point, unlike in the case of ellipsoidal or spherical volumes. The behaviour of the dyadic along the cube axes and diagonals is illustrated graphically.

[P8] demonstrates a software package, *Mixtool*, written in MATLAB<sup>TM</sup>, and developed for the analysis of complex heterogeneous materials, including chiral, bi-isotropic, and bi-anisotropic media. The program calculates the effective magnetoelectric material parameters as functions of the structure and geometry of the mixture. Three popular mixing models are used: Maxwell Garnett, Bruggeman, and Coherent Potential formulations. The output of the program is given numerically and as illustrative curves. *Mixtool* was presented in the Bianisotropics 1998 conference, where it won the honorary award.

## 9 Appendix: derivation of expressions for layered bi-anisotropic ellipsoids

The expressions for the internal field and polarizability of layered bianisotropic ellipsoid are derived in this Appendix. The expressions are in six-vector form, thus the six-vector formalism is first briefly described.

### Six-vectors and six-dyadics

In problems involving bi-anisotropic materials it is convenient to use six-vector formalism [4], in order to simplify the often quite complicated field expressions. The electric and magnetic vector quantities are collected into a six-vector, and the relation between two six-vectors is a six-dyadic. A six-dyadic can be treated as a  $2 \times 2$  matrix, the components of which are ordinary three-dyadics.

Thus, for example, the bi-anisotropic material can be described with a six-dyadic (consisting of four ordinary three-dyadics, or 36 scalar parameters):

$$\mathbf{M} = \begin{pmatrix} \bar{\bar{\epsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} & \bar{\bar{\mu}} \end{pmatrix}, \quad \text{where} \quad \bar{\bar{\xi}} = \sqrt{\epsilon_0 \mu_0} (\bar{\bar{\chi}}^T - \mathbf{j} \bar{\bar{\kappa}}^T) \quad \text{and} \quad \bar{\bar{\zeta}} = \sqrt{\epsilon_0 \mu_0} (\bar{\bar{\chi}} + \mathbf{j} \bar{\bar{\kappa}}). \quad (91)$$

The constitutive relations can be expressed with

$$\mathbf{d} = \mathbf{M} \cdot \mathbf{e} \iff \begin{cases} \mathbf{D} &= \bar{\bar{\epsilon}} \cdot \mathbf{E} + \bar{\bar{\xi}} \cdot \mathbf{H} \\ \mathbf{B} &= \bar{\bar{\zeta}} \cdot \mathbf{E} + \bar{\bar{\mu}} \cdot \mathbf{H}, \end{cases} \quad (92)$$

where the field six-vectors are

$$\mathbf{d} = \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}, \quad \text{and} \quad \mathbf{e} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \quad (93)$$

### Layered dielectric ellipsoid

Let us start by briefly describing the boundary conditions in a layered *dielectric* ellipsoid [12], and then adopt the analysis in the bianisotropic case.

Assume first that the semiaxes of the ellipsoid fix the Cartesian coordinate system, and the semiaxes of the  $k$ th boundary ellipsoid are  $(a_k, b_k, c_k)$  in  $(x, y, z)$  direction. In the  $k$ th layer the potential is (39):

$$\phi_k(\mathbf{r}) = -Ex \left( A_k - \frac{B_k}{2} \int_{\xi}^{\infty} \frac{ds}{(s + a_1^2) R_1(s)} \right), \quad (94)$$

where  $R_1(s) = \sqrt{(s + a_1^2)(s + b_1^2)(s + c_1^2)}$ .

The potential and the normal component of the displacement have to be continuous across the boundary of the  $k$ th and  $(k + 1)$ th layers. The normal component of the displacement is

$$\mathbf{n} \cdot \mathbf{D} = -\epsilon \frac{1}{h_\xi} \frac{\partial \phi}{\partial \xi}, \quad (95)$$

where  $h_\xi$  is the metric coefficient

$$h_\xi = \frac{\sqrt{(\xi - \eta)(\xi - \zeta)}}{2R_1(\xi)}. \quad (96)$$

Since the dependence of  $\phi_k$  and  $\phi_{k+1}$  on the coordinates  $\eta$  and  $\zeta$  is the same on the boundary,  $h_\xi$  cancels out.

Performing the differentiation for the potentials leads to

$$\begin{aligned} \frac{\partial \phi_n}{\partial \xi} &= \frac{\partial}{\partial \xi} \left[ \sqrt{\xi + a_1^2} \left( A_n - \frac{B_n}{2} \int_{\xi}^{\infty} \frac{ds}{(s + a_1^2)R_1(s)} \right) \right] \\ &= \frac{1}{2\sqrt{\xi + a_1^2}} \left( A_n + \frac{B_n}{R_1(\xi)} - \frac{B_n}{2} \int_{\xi}^{\infty} \frac{ds}{(s + a_1^2)R_1(s)} \right), \quad n = k \text{ or } k + 1 \end{aligned} \quad (97)$$

The term  $\sqrt{\xi + a_1^2}$  comes from the  $x$ -dependence of the  $\phi_0$  potential.

The boundary separating layers  $k$  and  $k + 1$  is ellipsoid with semiaxes  $a_{k+1}$ ,  $b_{k+1}$ , and  $c_{k+1}$ , where the coordinate  $\xi$  has the value  $\xi_{k+1}$ . The boundary conditions can be written as

$$A_k - \frac{B_k}{2} \int_{\xi_{k+1}}^{\infty} \frac{ds}{(s + a_1^2)R_1(s)} = A_{k+1} - \frac{B_{k+1}}{2} \int_{\xi_{k+1}}^{\infty} \frac{ds}{(s + a_1^2)R_1(s)} \quad (98)$$

$$\begin{aligned} \epsilon_k \left( A_k + \frac{B_k}{R_1(\xi_{k+1})} - \frac{B_k}{2} \int_{\xi_{k+1}}^{\infty} \frac{ds}{(s + a_1^2)R_1(s)} \right) = \\ \epsilon_{k+1} \left( A_{k+1} + \frac{B_{k+1}}{R_1(\xi_{k+1})} - \frac{B_{k+1}}{2} \int_{\xi_{k+1}}^{\infty} \frac{ds}{(s + a_1^2)R_1(s)} \right) \end{aligned} \quad (99)$$

Since the semiaxes  $a_1, b_1, c_1$  of the outermost ellipsoid are the basis of the coordinate system, the value of  $\xi$  is  $\xi_1 = 0$  at the boundary of the whole ellipsoid. Thus

$$\int_{\xi_1}^{\infty} \frac{ds}{(s + a_1^2)R_1(s)} = \int_0^{\infty} \frac{ds}{(s + a_1^2)\sqrt{(s + a_1^2)(s + b_1^2)(s + c_1^2)}} = \frac{2}{a_1 b_1 c_1} N_1^x, \quad (100)$$

where  $N_1^x$  is the depolarization factor of the outermost ellipsoid in the  $x$ -direction.

At the other boundaries, the integral can be transformed by setting  $s' = s - \xi_{k+1}$  and noting that  $a_1^2 - a_{k+1}^2 = b_1^2 - b_{k+1}^2 = c_1^2 - c_{k+1}^2 = -\xi_{k+1}$  :

$$\begin{aligned} \int_{\xi_{k+1}}^{\infty} \frac{ds}{(s + a_1^2)R_1(s)} &= \int_0^{\infty} \frac{ds'}{(s' + \xi_{k+1} + a_1^2)\sqrt{(s' + \xi_{k+1} + a_1^2)(s' + \xi_{k+1} + b_1^2)(s' + \xi_{k+1} + c_1^2)}} \\ &= \int_0^{\infty} \frac{ds'}{(s' + a_{k+1}^2)\sqrt{(s' + a_{k+1}^2)(s' + b_{k+1}^2)(s' + c_{k+1}^2)}} = \frac{2}{a_{k+1}b_{k+1}c_{k+1}}N_{k+1}^x. \end{aligned} \quad (101)$$

Note also that in (99),  $R_1(\xi_{k+1}) = a_{k+1}b_{k+1}c_{k+1}$ .

Therefore the equations (98-99) can be expressed as:

$$A_k - \frac{B_k}{a_{k+1}b_{k+1}c_{k+1}}N_{k+1}^x = A_{k+1} - \frac{B_{k+1}}{a_{k+1}b_{k+1}c_{k+1}}N_{k+1}^x \quad (102)$$

$$\epsilon_k \left[ A_k + \frac{B_k}{a_{k+1}b_{k+1}c_{k+1}}(1 - N_{k+1}^x) \right] = \epsilon_{k+1} \left[ A_{k+1} + \frac{B_{k+1}}{a_{k+1}b_{k+1}c_{k+1}}(1 - N_{k+1}^x) \right] \quad (103)$$

From these equations it is quite straightforward to form the matrix equation (40). Note that in the innermost layer (the core) there can be only a uniform field, thus the coefficient  $B = 0$ .

## Six-dyadic expressions for boundary conditions of bi-anisotropic layered ellipsoid

Assume now that the layers are made of a more general material: bi-isotropic or even bi-anisotropic,<sup>28</sup> and let the material six-dyadic for the  $k$ th layer be

$$\mathbf{M}_k = \begin{pmatrix} \bar{\bar{\epsilon}}_k & \bar{\bar{\xi}}_k \\ \bar{\bar{\xi}}_k & \bar{\bar{\mu}}_k \end{pmatrix}$$

Assume again that the semiaxes of the ellipsoid fix the Cartesian coordinate system, and the semiaxes of the  $k$ th boundary ellipsoid are  $(a_k, b_k, c_k)$  in  $(x, y, z)$  direction. The depolarization factors are included in depolarization six-dyadic

$$\mathbf{L}_k = \begin{pmatrix} \bar{\bar{L}}_k & 0 \\ 0 & \bar{\bar{L}}_k \end{pmatrix}, \quad \bar{\bar{L}}_k = \begin{pmatrix} N_k^x & 0 & 0 \\ 0 & N_k^y & 0 \\ 0 & 0 & N_k^z \end{pmatrix}$$

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<sup>28</sup>The analysis allows the core to be bi-anisotropic, but the other layers can be at most bi-isotropic.

An important assumption in the quasi-static analysis is that the ellipsoid is much smaller than the wavelength. The analysis is accomplished here by the use of electric and magnetic potentials:

$$\mathbf{E} = -\nabla\phi \quad \text{and} \quad \mathbf{H} = -\nabla\phi^m \quad (104)$$

By superposition principle the fields are divided into three orthogonal components, which leads to the following superposition of the electric potential:

$$\begin{aligned} \phi_k(\mathbf{r}) = \phi_{kx}(\mathbf{r}) + \phi_{ky}(\mathbf{r}) + \phi_{kz}(\mathbf{r}) = -E \left\{ x \left[ A_{kx} - \frac{B_{kx}}{2} \int_{\xi}^{\infty} \frac{ds}{(s+a_1^2)R_1(s)} \right] \right. \\ \left. - y \left[ A_{ky} - \frac{B_{ky}}{2} \int_{\xi}^{\infty} \frac{ds}{(s+b_1^2)R_1(s)} \right] - z \left[ A_{kz} - \frac{B_{kz}}{2} \int_{\xi}^{\infty} \frac{ds}{(s+c_1^2)R_1(s)} \right] \right\}, \quad (105) \end{aligned}$$

and similarly for the magnetic potential

$$\begin{aligned} \phi_k^m(\mathbf{r}) = \phi_{kx}^m(\mathbf{r}) + \phi_{ky}^m(\mathbf{r}) + \phi_{kz}^m(\mathbf{r}) = -H \left\{ x \left[ A_{kx}^m - \frac{B_{kx}^m}{2} \int_{\xi}^{\infty} \frac{ds}{(s+a_1^2)R_1(s)} \right] \right. \\ \left. - y \left[ A_{ky}^m - \frac{B_{ky}^m}{2} \int_{\xi}^{\infty} \frac{ds}{(s+b_1^2)R_1(s)} \right] - z \left[ A_{kz}^m - \frac{B_{kz}^m}{2} \int_{\xi}^{\infty} \frac{ds}{(s+c_1^2)R_1(s)} \right] \right\} \quad (106) \end{aligned}$$

The continuity of potential means, of course, the continuity of the total potential ( $\phi_k$  and  $\phi_k^m$ ), but the components ( $\phi_{kx}, \phi_{ky}, \phi_{kz}$ ) and ( $\phi_{kx}^m, \phi_{ky}^m, \phi_{kz}^m$ ) are orthogonal in such a way that the components are continuous as well, separately.

The coefficients of the potential components are collected into coefficient vectors:

$$\mathbf{A}_k = \begin{pmatrix} A_{kx} \\ A_{ky} \\ A_{kz} \end{pmatrix}, \quad \mathbf{A}_k^m = \begin{pmatrix} A_{kx}^m \\ A_{ky}^m \\ A_{kz}^m \end{pmatrix}, \quad \mathbf{B}_k = \begin{pmatrix} B_{kx} \\ B_{ky} \\ B_{kz} \end{pmatrix}, \quad \text{and} \quad \mathbf{B}_k^m = \begin{pmatrix} B_{kx}^m \\ B_{ky}^m \\ B_{kz}^m \end{pmatrix}, \quad (107)$$

which are then collected into coefficient six-vectors:

$$\mathbf{c}_k = \begin{pmatrix} \mathbf{A}_k \\ \mathbf{A}_k^m \end{pmatrix} \quad \text{and} \quad \mathbf{d}_k = \begin{pmatrix} \mathbf{B}_k \\ \mathbf{B}_k^m \end{pmatrix}. \quad (108)$$

Let us define the connection of the coefficient six-vector and field six-vector as

$$\mathbf{e}_k = \begin{pmatrix} \mathbf{E}_k \\ \mathbf{H}_k \end{pmatrix} = E \mathbf{c}_k = \begin{pmatrix} E \mathbf{A}_k \\ E \mathbf{A}_k^m \end{pmatrix} = \begin{pmatrix} E \mathbf{A}_k \\ H \eta \mathbf{A}_k^m \end{pmatrix} \quad (109)$$

Since the incident field six-vector

$$\mathbf{e} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = E \mathbf{c}_0 = \begin{pmatrix} E \mathbf{A}_0 \\ E \mathbf{A}_0^m \end{pmatrix} = \begin{pmatrix} E \mathbf{A}_0 \\ H \eta \mathbf{A}_0^m \end{pmatrix} \implies |\mathbf{A}_0| = 1, \quad \text{and} \quad |\mathbf{A}_0^m| = \frac{1}{\eta}, \quad (110)$$

where  $\eta = \sqrt{\mu_e/\epsilon_e}$ .

Now it is clear that the continuity of all three electric potential and three magnetic potential components can be expressed as a single six-vector equation, which corresponds to (102):

$$\mathbf{c}_k - \frac{1}{a_{k+1}b_{k+1}c_{k+1}}\mathbf{L}_{k+1} \cdot \mathbf{d}_k = \mathbf{c}_{k+1} - \frac{1}{a_{k+1}b_{k+1}c_{k+1}}\mathbf{L}_{k+1} \cdot \mathbf{d}_{k+1} \quad (111)$$

The continuity of the normal components of electric and magnetic flux densities is more complicated and should be treated carefully, since electric and magnetic fields and responses are coupled in the material. It should be obvious that the six-dyadic equation has to be written as

$$\mathbf{M}_k \cdot \mathbf{c}_k + \frac{1}{a_{k+1}b_{k+1}c_{k+1}}(\mathbf{I} - \mathbf{L}_{k+1}) \cdot \mathbf{M}_k \cdot \mathbf{d}_k = \mathbf{M}_{k+1} \cdot \mathbf{c}_{k+1} + \frac{1}{a_{k+1}b_{k+1}c_{k+1}}(\mathbf{I} - \mathbf{L}_{k+1}) \cdot \mathbf{M}_{k+1} \cdot \mathbf{d}_{k+1} \quad (112)$$

where  $\mathbf{I}$  is the unit six-dyadic

$$\mathbf{I} = \begin{pmatrix} \bar{\bar{\mathbf{I}}} & 0 \\ 0 & \bar{\bar{\mathbf{I}}} \end{pmatrix},$$

and  $\bar{\bar{\mathbf{I}}}$  is the unit dyadic. The motivation for (112) is that  $(\mathbf{I} - \mathbf{L})$  takes the normal component of flux density six-vector  $\mathbf{M} \cdot \mathbf{d}$ , thus the order is  $(\mathbf{I} - \mathbf{L}) \cdot \mathbf{M} \cdot \mathbf{d}$ .

## Two-layer bi-anisotropic ellipsoid

Consider now a two-layer ellipsoid with material parameters  $\mathbf{M}_e, \mathbf{M}_1, \mathbf{M}_2$  :

$$\mathbf{M}_e = \begin{pmatrix} \epsilon_e \bar{\bar{\mathbf{I}}} & \xi_e \bar{\bar{\mathbf{I}}} \\ \zeta_e \bar{\bar{\mathbf{I}}} & \mu_e \bar{\bar{\mathbf{I}}} \end{pmatrix}, \quad \mathbf{M}_1 = \begin{pmatrix} \epsilon_1 \bar{\bar{\mathbf{I}}} & \xi_1 \bar{\bar{\mathbf{I}}} \\ \zeta_1 \bar{\bar{\mathbf{I}}} & \mu_1 \bar{\bar{\mathbf{I}}} \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} \bar{\bar{\epsilon}}_2 & \bar{\bar{\xi}}_2 \\ \bar{\bar{\zeta}}_2 & \bar{\bar{\mu}}_2 \end{pmatrix}, \quad (113)$$

In this case there are two boundaries, which means four six-dyadic equations from boundary conditions:

$$\begin{cases} \mathbf{c}_0 - \frac{1}{a_1 b_1 c_1} \mathbf{L}_1 \cdot \mathbf{d}_0 = \mathbf{c}_1 - \frac{1}{a_1 b_1 c_1} \mathbf{L}_1 \cdot \mathbf{d}_1 \\ \mathbf{M}_e \cdot \mathbf{c}_0 + \frac{1}{a_1 b_1 c_1} (\mathbf{I} - \mathbf{L}_1) \cdot \mathbf{M}_e \cdot \mathbf{d}_0 = \mathbf{M}_1 \cdot \mathbf{c}_1 + \frac{1}{a_1 b_1 c_1} (\mathbf{I} - \mathbf{L}_1) \cdot \mathbf{M}_1 \cdot \mathbf{d}_1 \\ \mathbf{c}_1 - \frac{1}{a_2 b_2 c_2} \mathbf{L}_2 \cdot \mathbf{d}_1 = \mathbf{c}_2 \\ \mathbf{M}_1 \cdot \mathbf{c}_1 + \frac{1}{a_2 b_2 c_2} (\mathbf{I} - \mathbf{L}_2) \cdot \mathbf{M}_1 \cdot \mathbf{d}_1 = \mathbf{M}_2 \cdot \mathbf{c}_2 \end{cases} \quad (114)$$

Note that  $\mathbf{d}_2 = 0$ , thus it is not included in (114). The six-vector equation array (114) includes five coefficient vectors, thus by proper elimination one can derive an expression for one coefficient vector, if an other coefficient vector is known.

## Internal field

In order to get the internal field amplitude vector  $\mathbf{c}_2$  with respect to the incident field amplitude vector  $\mathbf{c}_0$ , one should eliminate  $\mathbf{d}_0$ ,  $\mathbf{c}_1$ , and  $\mathbf{d}_1$  from equations (114).

Vector  $\mathbf{c}_1$  is eliminated from the first two equations in (114), which gives an expression for  $\mathbf{d}_0$  :

$$\mathbf{d}_0 = [\mathbf{M}_1 \cdot \mathbf{L}_2 + (\mathbf{I} - \mathbf{L}_2) \cdot \mathbf{M}_e]^{-1} \cdot \{a_1 b_1 c_1 (\mathbf{M}_1 - \mathbf{M}_e) \cdot \mathbf{c}_0 + [\mathbf{M}_1 \cdot \mathbf{L}_1 + (\mathbf{I} - \mathbf{L}_1) \cdot \mathbf{M}_1] \cdot \mathbf{d}_1\} \quad (115)$$

From the last two equations in (114), one gets expressions for  $\mathbf{c}_1$  and  $\mathbf{d}_1$  :

$$\mathbf{c}_1 = \mathbf{L}_2 \cdot [\mathbf{M}_1 \cdot \mathbf{L}_2 + (\mathbf{I} - \mathbf{L}_2) \cdot \mathbf{M}_1]^{-1} \cdot [\mathbf{M}_2 \cdot \mathbf{L}_2 + (\mathbf{I} - \mathbf{L}_2) \cdot \mathbf{M}_1] \cdot \mathbf{L}_2^{-1} \cdot \mathbf{c}_2 \quad \text{and} \quad (116)$$

$$\mathbf{d}_1 = a_2 b_2 c_2 (\mathbf{M}_1 \cdot \mathbf{L}_2 + (\mathbf{I} - \mathbf{L}_2) \cdot \mathbf{M}_1)^{-1} \cdot (\mathbf{M}_2 - \mathbf{M}_1) \cdot \mathbf{c}_2. \quad (117)$$

After inserting (115-117) in the first two equations in (114), and eliminating  $\mathbf{d}_0$ , one can write

$$\mathbf{c}_2 = \{\mathbf{M}_e + (\mathbf{M}_1 - \mathbf{M}_e) \cdot \mathbf{L}_1 - \{(\mathbf{M}_1 - \mathbf{M}_e) \cdot \mathbf{L}_1 \cdot [w(\mathbf{L}_1 - \mathbf{I}) - \mathbf{L}_2] - \mathbf{M}_e \cdot \mathbf{L}_2\} \cdot \mathbf{M}_1^{-1} \cdot (\mathbf{M}_2 - \mathbf{M}_1)\}^{-1} \cdot \mathbf{M}_e \cdot \mathbf{c}_0 \quad (118)$$

The equation (118) is general: the shell material can be chiral or bi-isotropic, but the core material is allowed to be bianisotropic.

If the ellipsoid is hollow, and

$$\mathbf{M}_2 = \mathbf{M}_e = \begin{pmatrix} \epsilon_0 \bar{\bar{\mathbf{I}}} & 0 \\ 0 & \mu_0 \bar{\bar{\mathbf{I}}} \end{pmatrix}, \quad (119)$$

(118) can be simplified:

$$\mathbf{c}_2 = \mathbf{C}^{-1} \cdot \mathbf{M}_r \cdot \{\mathbf{M}_r + [(w\mathbf{L}_1 \cdot (\mathbf{M}_r - \mathbf{I}) + \mathbf{L}_2) \cdot (\mathbf{L}_1 - \mathbf{I}) - \mathbf{M}_r \cdot \mathbf{L}_1 \cdot (\mathbf{L}_2 - \mathbf{I})] \cdot (\mathbf{M}_r - \mathbf{I})\}^{-1} \cdot \mathbf{C} \cdot \mathbf{c}_0, \quad (120)$$

where  $\mathbf{M}_r$  is the relative material six-dyadic

$$\mathbf{M}_r = \begin{pmatrix} \epsilon_r \bar{\bar{\mathbf{I}}} & (\chi - j\kappa) \bar{\bar{\mathbf{I}}} \\ (\chi + j\kappa) \bar{\bar{\mathbf{I}}} & \mu_r \bar{\bar{\mathbf{I}}} \end{pmatrix} \quad (121)$$

and

$$\mathbf{C} = \begin{pmatrix} \sqrt{\epsilon_0} \bar{\bar{\mathbf{I}}} & 0 \\ 0 & \sqrt{\mu_0} \bar{\bar{\mathbf{I}}} \end{pmatrix}. \quad (122)$$

The fraction of the core from the total volume of the object is  $w = \frac{V_2}{V_1} = \frac{a_2 b_2 c_2}{a_1 b_1 c_1}$ .

Equation (120) represents the six-dyadic expression for the internal field of hollow bi-isotropic ellipsoid, and it is presented in [P2].

## Polarizability

The polarizability six-dyadic is defined as  $\mathbf{p} = \mathbf{A} \cdot \mathbf{e}$ , where the field and dipole moment six-vectors are

$$\mathbf{p} = \begin{pmatrix} \mathbf{p}_e \\ \mathbf{p}_m \end{pmatrix} \quad \text{and} \quad \mathbf{e} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \quad (123)$$

The polarizability six-dyadic consists of four ordinary three-dyadics, or 36 scalar parameters:

$$\mathbf{A} = \begin{pmatrix} \bar{\bar{\alpha}}_{ee} & \bar{\bar{\alpha}}_{em} \\ \bar{\bar{\alpha}}_{me} & \bar{\bar{\alpha}}_{mm} \end{pmatrix} \quad (124)$$

The dipole moment  $\mathbf{p}$  is related to the dipole field amplitude vector  $\mathbf{d}_0$ . From the last two equations in (114) one gets, after eliminating  $\mathbf{c}_2$  :

$$\mathbf{c}_1 = \frac{1}{a_2 b_2 c_2} (\mathbf{M}_2 - \mathbf{M}_1)^{-1} \cdot [\mathbf{M}_2 \cdot \mathbf{L}_2 + (\mathbf{I} - \mathbf{L}_2) \cdot \mathbf{M}_1] \cdot \mathbf{d}_1. \quad (125)$$

Inserting this to the first equation in (114) leads to

$$\mathbf{d}_1 = \left( \frac{1}{a_1 b_1 c_1} (\mathbf{M}_1 - \mathbf{M}_2) \cdot \mathbf{L}_1 + \frac{1}{a_2 b_2 c_2} \mathbf{M}_2 \cdot \mathbf{L}_2 + \frac{1}{a_2 b_2 c_2} (\mathbf{I} - \mathbf{L}_2) \cdot \mathbf{M}_1 \right)^{-1} \cdot (\mathbf{M}_2 - \mathbf{M}_1) \cdot \left( \mathbf{c}_0 - \frac{1}{a_1 b_1 c_1} \mathbf{L}_1 \cdot \mathbf{d}_0 \right) \quad (126)$$

On the other hand, from the first two equations in (114) one gets another expression for coefficient vector  $\mathbf{d}_1$  :

$$\mathbf{d}_1 = a_1 b_1 c_1 [\mathbf{M}_1 \cdot \mathbf{L}_1 + (\mathbf{I} - \mathbf{L}_1) \cdot \mathbf{M}_1]^{-1} \cdot [(\mathbf{M}_e - \mathbf{M}_1) \cdot \mathbf{c}_0 + \frac{1}{a_1 b_1 c_1} (\mathbf{M}_e - \mathbf{L}_1 \cdot \mathbf{M}_e + \mathbf{M}_1 \cdot \mathbf{L}_1) \cdot \mathbf{d}_0] \quad (127)$$

Vector  $\mathbf{d}_1$  can now be eliminated from these two expressions, and one can write, after some algebra, an expression for  $\mathbf{d}_0$  in terms of  $\mathbf{c}_0$  :

$$\mathbf{d}_0 = a_1 b_1 c_1 \left\{ \mathbf{K} \cdot [(\mathbf{M}_1 - \mathbf{M}_e) \cdot \mathbf{L}_1 + \mathbf{M}_e] + w (\mathbf{M}_2 - \mathbf{M}_1) \cdot \mathbf{L}_1 \right\}^{-1} \cdot [\mathbf{K} \cdot (\mathbf{M}_1 - \mathbf{M}_e) + w (\mathbf{M}_2 - \mathbf{M}_1)] \cdot \mathbf{c}_0 \quad (128)$$

with the auxiliary dyadic

$$\mathbf{K} = (\mathbf{M}_2 - \mathbf{M}_1) \cdot (\mathbf{L}_2 - w \mathbf{L}_1) \cdot \mathbf{M}_1^{-1} + \mathbf{I}$$

and, again  $w = \frac{V_2}{V_1} = \frac{a_2 b_2 c_2}{a_1 b_1 c_1}$ .

The dipole moment six-dyadic is related to the dipole field amplitude  $\mathbf{d}_0$  and the total volume of the ellipsoid:

$$\mathbf{p} = \frac{4\pi}{3} \mathbf{M}_e \cdot \mathbf{d}_0 \quad (129)$$

Thus the polarizability six-dyadic can be written as

$$\mathbf{A} = V_1 \mathbf{M}_e \cdot \left\{ \mathbf{K} \cdot [(\mathbf{M}_1 - \mathbf{M}_e) \cdot \mathbf{L}_1 + \mathbf{M}_e] + w (\mathbf{M}_2 - \mathbf{M}_1) \cdot \mathbf{L}_1 \right\}^{-1} \cdot [\mathbf{K} \cdot (\mathbf{M}_1 - \mathbf{M}_e) + w (\mathbf{M}_2 - \mathbf{M}_1)], \quad (130)$$

since the total volume of the ellipsoid is  $V_1 = \frac{4\pi}{3} a_1 b_1 c_1$ .

Equation (130) represents the six-dyadic expression for the polarizability of a hollow bi-anisotropic ellipsoid, and it is analyzed in [P3].



### Anisotropic case

It should be noted that expressions (118) and (130) are “backwards compatible”: they can be easily transformed for example to an anisotropic case by replacing  $\mathbf{M}_2$  with  $\bar{\bar{\epsilon}}_2$ ,  $\mathbf{M}_1$  with  $\epsilon_1 \bar{\bar{\mathbf{I}}}$ , and  $\mathbf{M}_e$  with  $\epsilon_e \bar{\bar{\mathbf{I}}}$ . They should be treated then, of course, as ordinary three-dyadic expressions.

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## 10 Errata



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