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The general solution for an ellipsoid in low-Reynolds-number flow

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The general solution for low-Reynolds-number flow about an ellipsoid is derived by the singularity method and by representation in ellipsoidal harmonics. It is shown that, as in potential flow, the focal ellipse is the image system for the ellipsoid. A simple transformation which resembles a step in the derivation of the Dirichlet potential is introduced and its implications are explored. This transformation converts the velocity representation for an nth-order ambient field into that for the $(n+1)$th-order field. The method furnishes an explanation for the invariance of the domain of the singularity distribution (the focal ellipse) with respect to the ambient field. Faxén relations for all multipole moments for arbitrary Stokes flow are derived in both integral and symbolic operator forms.

1. Introduction

A strategy for calculating the velocity field about an ellipsoidal particle in low-Reynolds-number flow is presented here with a view to applications in modelling of particulate suspensions. The somewhat tedious nature of the computations, particularly when compared to analogous computations for spheres, may discourage some readers but we see the present effort as difficult but necessary extensions of the extensive literature on the hydrodynamics of spherical particles. This work was motivated by three concerns – two originating from modelling efforts for particulate suspensions and one originating from a fundamental question on the existence of certain velocity representations for Stokes flow.

The first motivation arises from the fact that non-spherical particles are encountered in many suspensions, with shapes ranging from disks (clay minerals) to slender bodies (stiff fibres and macromolecules). The rheological and dynamic behaviour of such suspensions, because of particle anisotropy, encompass a much richer class of phenomena than that exhibited by suspensions of spheres. Studies of model suspensions of spheroids (Giesekus 1962; Brenner 1972; Hinch & Leal 1972) have elucidated the role of hydrodynamic forces on the orientation distribution and consequently the bulk rheology.

The second motivation is a consequence of the observation that almost all models

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that incorporate non-sphericity, ranging from disks to needles, may be viewed as degenerate ellipsoids. Furthermore, the ellipsoid has received additional attention in the past as a relatively simple example of a non-axisymmetric shape (Hinch & Leal 1979) and also as a simple example of a particle with orthotropic symmetry. Past work involving ellipsoids of revolution and other degenerate shapes, has taken advantage of the simplifications due to axisymmetry. However, in problems where the general velocity representation is required in its full expansion, either because of the complexity of the ambient velocity field or because of the symmetry-destroying presence of multiparticle hydrodynamic interactions, it is more efficient to solve for the general ellipsoid and obtain the desired result by reduction. Somewhat ironically, this efficiency is gained, as shown in the text, because for the ellipsoid one has at hand the use of the full permutation symmetry group over the geometric parameters that define the particle shape. This technique is lost when degeneracy occurs in the geometry.

A general velocity representation is not essential in the development of the theory for dilute suspensions. Important entities such as the force, torque, translational and rotational velocities and stresslet for a single particle in arbitrary Stokes flow are all directly available or latent in the classical works of Oberbeck (1876) and Jeffery (1922). The procedure for extracting these entities by the Lorentz reciprocal theorem is discussed in Rallison (1978) and Brenner & Haber (1983). In contrast, a general representation is essential in the computation of hydrodynamic interactions in studies of non-dilute systems. Examples include studies of sedimentation rates, theories for the diffusion coefficient, calculation of stability ratios for colloidal coagulation (Zeichner & Schowalter 1977) and analyses of the secondary electro-viscous effect (Russel 1976). For example, the general representation could be used as the basis set for the boundary collocation method of Gluckman, Pfeffer & Weinbaum (1971). Problems involving arbitrary particle orientations are now solvable, thus extending a method that was originally implemented for stream functions and axisymmetric flows (see also Liao & Krueger 1980 for axisymmetric flow past two spheroids).

The final motivation for the present work originates from questions raised by Chwang & Wu (1974, 1975) in their seminal work on the singularity method for Stokes flow. The essence of their work is that for a surprisingly large class of problems, the disturbance velocity field produced by axisymmetric shapes such as prolate spheroids, may be constructed from rather simple line distributions of fundamental singularities (Green functions). These authors show by direct evaluation, the types and region of distribution required for various boundary-value problems and, by solving integral equations, the particle shape (if any) associated with given distributions. For a number of problems, an interesting observation was made that for a fixed shape, although the singularity type varied, the region of distribution did not depend on the boundary condition. Finally, for a given particle shape, the method does not answer the question of whether a simple singularity solution exists nor does the method provide an *ab initio* procedure for determining the region of distribution.† These questions remain as unsolved problems.

The solution strategy presented here for ellipsoids produces simultaneously both the singularity form and a representation in ellipsoidal harmonics with the latter recognizable as generalizations of the classical solutions of Oberbeck (1876) for

† In a private communication, Professor Wu has constructed specific counter-examples such as star-shaped regions for which it is readily shown that simple image systems do not exist.

translation, Edwardes (1892) for rotation and Jeffery (1922) for a linear field. (The equivalence of the singularity form and these three classical solutions has already been established in Kim 1986 by direct transformation of the classical solution into the singularity form.) The present work establishes for the general ellipsoid both the singularity type and the invariance of the region of distribution of singularities. The proof for an arbitrary shape eludes us at this moment but our approach offers some clues that will be pursued in future work.

The work is presented in the following order. The ambient field is represented in the vicinity of the ellipsoid by its Taylor series. Thus the general solution is obtained by first solving for the disturbance field induced by an nth-order ambient field. This solution is presented in both the singularity form and also in terms of the ellipsoidal harmonics. Clearly, singularity representations satisfy the governing equations – the only concern is the boundary condition at the surface of the ellipsoid. Not surprisingly, it turns out that boundary conditions can be evaluated more readily in ellipsoidal coordinates. Thus, the key step in the solution procedure consists of transformation between singularity and ellipsoidal-harmonic representations. The required expressions are presented for both the scalar potential and the velocity. (Thus our work is also applicable to the Laplace equation.)

In the third section, the solution for the $(n+1)$th-order ambient field is rederived by a homotopy on the solution for the nth-order field. We introduce this terminology because one solution is obtained from the other by performing an integral over a new parameter (the homotopy parameter) that rescales the particle size. It is shown that the homotopy changes the density function but not the region of the singularity distribution. The technique is also used to derive the Faxén relations for arbitrary Stokes flow for all multipole moments. In the fourth and final section, the utility of the method is demonstrated by solving for the disturbance velocity generated by an ellipsoid in a quadratic flow field.

2. The general solution

We consider the creeping motion of an ellipsoidal particle in a general ambient velocity field $\boldsymbol{v}^{\infty}(\boldsymbol{x})$. The coordinate system is chosen so that the equation for the surface of the ellipsoid is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1, \quad a \geqslant b \geqslant c. \tag{1}$$

The velocity field $\boldsymbol{v}$ and the pressure p satisfy the Stokes equation and the equation of continuity for incompressible flow

$$-\boldsymbol{\nabla} p+\mu\nabla^2\boldsymbol{v} = 0, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{3}$$

where μ is the viscosity. We shall employ the disturbance velocity, so consequently, the boundary conditions on $\boldsymbol{v}$ are that it vanish far away from the particle and at the particle surface,

$$\boldsymbol{v} = -\boldsymbol{v}^{\infty}(\boldsymbol{x}), \tag{4}$$

where $\boldsymbol{v}^{\infty}(\boldsymbol{x})$ satisfies the same equations of motion and is regular inside the particle.

Physically, $\boldsymbol{v}^{\infty}(\boldsymbol{x})$ is the ambient flow field about the particle and its value when $\boldsymbol{x}$ is within the particle is the velocity at that point in the absence of the particle.

In general, such fields are regular in the region occupied by the particle so we may represent them on the surface of the ellipsoid by a Taylor series. Since the governing equations (2) and (3) are linear, we proceed by solving for the disturbance field that satisfies the boundary condition for a specific term in the Taylor series and obtain the general solution by a summation over all such solutions. Thus, we need only examine the nth-order ambient velocity field,

$$v_i^\infty = H_{ik_1 k_2 \ldots k_n} x_{k_1} x_{k_2} \ldots x_{k_n},$$

where $\boldsymbol{x}$ is the position vector. The summation convention is used for repeated indices and bilevel indices are used since the number of indices varies with n. The tensor of rank $n+1$, is the result of taking the nth spatial derivative of v_i^∞ at the origin. (Caution: if the ambient field is that driven by a singularity placed nearer to the origin than the furthest point of the particle, the Taylor series will not converge at all points on the particle surface. For this case, an analytic continuation is needed – the reference point for the Taylor series should be chosen so that all points on the particle lie within the radius of convergence.)

The solution to this problem will be presented first in the singularity form and then in terms of ellipsoidal harmonics. The fundamental solution of the Stokes equation, i.e. the solution to

$$-\frac{\partial p_j}{\partial x_i} + \mu \nabla^2 v_{ij} = -\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta_{ij}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0,$$

is $\boldsymbol{p} = -\boldsymbol{\nabla}(4\pi|\boldsymbol{x}-\boldsymbol{x}'|)^{-1}$ for the pressure and a Green's dyadic $v_{ij} = I_{ij}(\boldsymbol{x}-\boldsymbol{x}')/(8\pi\mu)$ with the Oseen tensor defined by

$$I_{ij}(\boldsymbol{x}) = \frac{\delta_{ij}}{|\boldsymbol{x}|} + \frac{x_i x_j}{|\boldsymbol{x}|^3}.$$

Our singularity expressions may appear complicated, but their simple nature is revealed if they are considered in relation to the solutions derived by Chwang & Wu (1975) and Chwang (1975) for prolate spheroids. For spheroids, the image system is the line segment between the foci on the axis of symmetry. For ellipsoids, it is the focal ellipse. For spheroids, the required singularities for an nth-order field consist of the $(n-2m)$th derivatives of I and $\nabla^2 I$ where $m = 0, 1, \ldots, [\frac{1}{2}n]$.† Thus, an nth-order ambient field with n even (odd), induces a distribution of all even (odd) multipole moments up to the nth moment inclusively. The same set of singularities is used for the ellipsoid so the disturbance velocity for an $(n-1)$th order ambient field is explicitly

$$\sum_{m=0}^{[(n-1)/2]} L_{(n-2m)\,j} \int_E f_{(n-2m)}(\boldsymbol{x}') \left\{1 + \frac{c^2 q^2}{4(n-2m)-2}\nabla^2\right\} \frac{I_{ij}(\boldsymbol{x}-\boldsymbol{x}')}{8\pi\mu}\,\mathrm{d}A(\boldsymbol{x}'), \tag{5}$$

where

$$f_{(n)}(\boldsymbol{x}) = \frac{(2n-1)\,q^{(2n-3)}}{2\pi a_E b_E}, \quad q(\boldsymbol{x}) = (1 - x^2/a_E^2 - y^2/b_E^2)^{\frac{1}{2}},$$

$$a_E = (a^2-c^2)^{\frac{1}{2}}, \quad b_E = (b^2-c^2)^{\frac{1}{2}},$$

$$L_{(n)\,j} = \frac{(-1)^n}{(n-1)!} P_{jk_1 k_2 \ldots k_{n-1}} \frac{\partial}{\partial x_{k_1}} \frac{\partial}{\partial x_{k_2}} \cdots \frac{\partial}{\partial x_{k_{n-1}}}.$$

The pressure p is obtained by an identical distribution of the fundamental pressure

† This pattern may be inferred from the earlier results. The solutions in Chwang & Wu (1975) and Chwang (1975) correspond to $n = 0, 1, 2$. The result for arbitrary n is not given.

field. The constant tensors $\boldsymbol{P}$ give the strengths of the distributed multipole moments. The first two, P_j and P_{jk}, are the force and stress dipole. The precise relation between the $\boldsymbol{P}$'s and the multipole moments taken about the particle centre is examined in §3. For now, we simply note that the solution method includes a procedure for their evaluation. Finally, the focal ellipse $E(\boldsymbol{x}')$,

$$\frac{x'^2}{a_E^2}+\frac{y'^2}{b_E^2} = 1, \quad z = 0, \tag{6}$$

is the degenerate elliptical disk in the family of ellipsoids that are confocal to the particle ellipsoid. Their role as the image system in potential flow is discussed in Miloh (1974).

We shall also derive an alternative form of (5) expressed in terms of the ellipsoidal harmonic

$$G_n = \int_\lambda^\infty \left(\frac{x^2}{a^2+t}+\frac{y^2}{b^2+t}+\frac{z^2}{c^2+t}-1\right)^n \frac{\mathrm{d}t}{\Delta(t)},$$

with $\Delta(t) = [(a^2+t)(b^2+t)(c^2+t)]^{\frac{1}{2}}$, and the ellipsoidal coordinate λ defined as the positive root of

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda} = 1.$$

This alternative solution is

$$\sum_{m=0}^{[(n-1)/2]} \frac{(-1)^{n-2m-1}(n-2m)(2n-4m)!}{2^{2n-4m}(n-2m)!\,(n-2m)!} L_{(n-2m)\,j} \times \left\{\delta_{ij}\, G_{n-2m-1} - x_j \frac{\partial}{\partial x_i} G_{n-2m-1} + \frac{a_j^2}{2(n-2m)} \frac{\partial^2}{\partial x_i \partial x_j} G_{n-2m}\right\}. \tag{7}$$

There is a summation over j and the notation a_j for $j = 1,2,3$ has been introduced for a, b, c. (For $n = 1$ and 2, this form is precisely the classical solutions.)

The harmonics $\chi(\lambda)$ and Ω (the Dirichlet potential) that appear in the solution of Oberbeck and in Happel & Brenner (1973) are proportional to G_0 and G_1:

$$\chi = abcG_0, \quad \Omega = \pi abcG_1.$$

We now show that the two solutions, (5) and (7), are equivalent and that they satisfy the appropriate boundary conditions.

2.1. *Transformations for ellipsoidal harmonics*

We first establish the singularity representations for the ellipsoidal harmonic G_n. We define

$$H_n = \int_E \frac{q^{2n-1}}{2\pi a_E b_E} \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \mathrm{d}A(\boldsymbol{x}'), \tag{8}$$

and show that

$$H_n = \frac{(-1)^n (2n)!}{2^{2n+1}\, n!\, n!} G_n. \tag{9}$$

This also establishes that G_n is a harmonic.

(Equation (9) was conjectured by Mr S. Y. Lu of our department and established by the first author by mathematical induction.) For $n = 0$, (9) is equivalent to the singularity representation for $\chi(\lambda)$ and follows as an application of Gauss' law. The case $n = 1$ has also been derived previously (Kim 1986). Therefore, we assume that

(9) holds for n and show that this in turn implies that it holds for $n+1$. To do this, we first establish the following relations:

$$G_{n+1}(\boldsymbol{x};1) = -2(n+1)\int_0^1 u^{2n+2}G_n(\boldsymbol{x};u)\,\mathrm{d}u, \tag{10}$$

$$H_{n+1}(\boldsymbol{x};1) = (2n+1)\int_0^1 u^{2n+2}H_n(\boldsymbol{x};u)\,\mathrm{d}u. \tag{11}$$

The parameter u in $G_n(\boldsymbol{x};u)$ and $H_n(\boldsymbol{x};u)$ denotes that the particle size has been scaled with u, i.e. a, b, and c are replaced by ua, ub, and uc. The original ellipsoid corresponds to $u=1$.

These relations are verified by direct integration. For example, (10) is obtained in the following manner:

$$\int_0^1 u^{2n+2}G_n(\boldsymbol{x};u)\,\mathrm{d}u = \int_0^1 u\int_{v(u)}^{\infty}\left(\frac{x^2}{a^2+\tau}+\frac{y^2}{b^2+\tau}+\frac{z^2}{c^2+\tau}-u^2\right)^n\frac{\mathrm{d}\tau}{\Delta(\tau)}\mathrm{d}u$$

where $v(u)=\lambda/u^2$. The order of integration is exchanged so that the double integral is performed over the region $\lambda \leqslant \tau \leqslant \infty$ and $v^{-1}(\tau)\leqslant u \leqslant 1$, with

$$v^{-1}(\tau) = \left(\frac{x^2}{a^2+\tau}+\frac{y^2}{b^2+\tau}+\frac{z^2}{c^2+\tau}\right)^{\frac{1}{2}}.$$

Consequently,

$$\int_0^1 u^{2n+2}G_n(\boldsymbol{x};u)\,\mathrm{d}u = \int_{\lambda}^{\infty}\left[\int_{v^{-1}(\tau)}^{1}(v^{-1}(\tau)^2-u^2)^n\,u\,\mathrm{d}u\right]\frac{\mathrm{d}\tau}{\Delta(\tau)},$$

and the integration with respect to u leads to (10). Analogous steps may be used to derive (11).

The induction for the case $n+1$ from the case n is accomplished by u-integration of both sides of (9) with weights chosen as in (10) and (11). The result is

$$H_{n+1} = \frac{(-1)^{n+1}(2n+2)!}{2^{2n+3}(n+1)!\,(n+1)!}G_{n+1},$$

which completes the proof.

2.2. *Velocity representation in ellipsoidal coordinates*

The singularity solution (5) may be rearranged into (7) by using the following identities involving the Oseen tensor:

$$I_{ij}(\boldsymbol{x}-\boldsymbol{x}') = \delta_{ij}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}-(x-x')_j\frac{\partial}{\partial x_i}\left[\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right], \tag{12}$$

$$\nabla^2\boldsymbol{I}(\boldsymbol{x}-\boldsymbol{x}') = -2\nabla\nabla\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{13}$$

$$\int_E x_j' q^n\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}A(\boldsymbol{x}') = -\frac{(a_j^2-c^2)}{(n+2)}\frac{\partial}{\partial x_j}\int_E\frac{q^{n+2}}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}A(\boldsymbol{x}'). \tag{14}$$

The last equation follows from an integration by parts.

From (7) it is clear that the evaluation of $\boldsymbol{v}$ on the ellipsoid surface requires knowledge of the values taken by $\partial^n G_n$ and $\partial^{n+1}G_n$ on the surface. (When the tensorial subscripts are obvious the notation will be simplified by using ∂^n as the nth derivative.) These derivatives are obtained by successive application of Leibnitz' rule, with the additional simplification that the integrand of G_n evaluated at λ vanishes (by definition of λ). Thus

$$\partial G_n = 2nx_{k_1}\int_\lambda^\infty \frac{F^{n-1}}{(a_{k_1}^2+t)}\frac{dt}{\Delta(t)}, \quad \text{with } F = \frac{x^2}{a^2+t}+\frac{y^2}{b^2+t}+\frac{z^2}{c^2+t}-1,$$

$$\partial^2 G_n = 4n(n-1)\,x_{k_1}x_{k_2}\int_\lambda^\infty \frac{F^{n-2}}{(a_{k_1}^2+t)\,(a_{k_2}^2+t)}\frac{dt}{\Delta(t)}+2n\delta_{k_1k_2}\int_\lambda^\infty \frac{F^{n-1}}{(a_{k_1}^2+t)}\frac{dt}{\Delta(t)}.$$

until

$$\partial^n G_n = \sum_{m=0}^{[\frac{1}{2}n]} 2^{n-m}\frac{n!}{m!} \times\left\{\delta_{k_1k_2}\dots\delta_{k_{2m-1}k_{2m}}x_{k_{2m+1}}\dots x_{k_n}\int_\lambda^\infty \frac{F^m}{[\Pi_{\alpha=1}^n(a_{k_\alpha}^2+t)]}\frac{dt}{\Delta(t)}+\dots(\text{sym})\right\} \quad (15)$$

($\alpha \neq k_2, k_4, \dots, k_{2m}$ for term shown).

At the summation index m, there are $n!/(2^m m!(n-2m)!)$ terms corresponding to all possible permutations of $\{k_1 \dots k_n\}$ in the representative term. The key idea is that on the ellipsoid surface, $\lambda = 0$, the definite integrals become numerical constants that depend only on the shape of the ellipsoid. Thus $\partial^n G_n$ reduces to a polynomial in $\boldsymbol{x}$ of degree n.

The velocity expression also involves terms of the form $\partial^{n+1}G_n$ which we now proceed to evaluate at the ellipsoid surface:

$$\partial^{n+1}G_n = -2^n n!\,x_{k_1}\dots x_{k_n}\frac{1}{[\Pi_{\alpha=1}^n a_{k_\alpha}^2]\,\Delta(\lambda)}\frac{\partial\lambda}{\partial x_{k_{n+1}}}+\text{polynomial of degree } n-1. \quad (16)$$

At the surface $\lambda = 0$,

$$\frac{\partial\lambda}{\partial x_k} = \frac{2x_k}{a_k^2}\left[\sum_{l=1}^3 \frac{x_l^2}{a_l^4}\right]^{-1}. \quad (17)$$

Thus the leading-order term in (16) is not of the form specified in the boundary condition for the $(n-1)$th-order ambient field. In other words, the proposed velocity representation will have the proper behaviour at the ellipsoid surface if and only if this $(n+1)$th-order field is cancelled. There are only two such terms in the velocity representation of (7). They occur in the second and third terms when $m = 0$, i.e.

$$-x_j\,\partial^n G_n = 2^n(n-1)!\,x_i x_j x_{k_1}\dots x_{k_{n-1}}\left([\Pi_{\alpha=1}^n a_{k_\alpha}^2]\,\Delta(0)\left[\sum_{l=1}^3\frac{x_l^2}{a_l^4}\right]\right)^{-1}+\dots,$$

$$\frac{a_j^2}{2n}\partial^{n+1}G_n = -2^n(n-1)!\,x_i x_j x_{k_1}\dots x_{k_{n-1}}\,a_j^2\left([\Pi_{\alpha=1}^{n+1} a_{k_\alpha}^2]\,\Delta(0)\left[\sum_{l=1}^3\frac{x_l^2}{a_l^4}\right]\right)^{-1}+\dots.$$

In the second equation, a_j^2 may be eliminated so the two unwanted terms cancel each other. Thus, the velocity field evaluated at the surface of the ellipsoid is in fact a polynomial in $\boldsymbol{x}$ of degree $(n-1)$. The lower-order fields (with the order successively decreasing by two) may be eliminated by repeated use of the preceding argument, i.e. by mathematical induction. Thus the velocity may be expressed as in (5) or (7) as claimed.

The last step in the solution procedure requires the determination of the $\boldsymbol{P}$ in terms of the known $\boldsymbol{H}$, the gradients of the ambient field. This final step is accomplished by inverting the set of linear equations for the unknown tensorial coefficients. The symmetry in the ellipsoid geometry decouples the system so that the lower-order cases $n = 1,2,3$ may be inverted analytically, as shown later for the quadratic ambient flow field.

3. Homotopy

Equations (10) and (11) are key steps in the derivation of the velocity representation. Therefore, it is not surprising that the velocity field also satisfies a similar identity. If we define

$$J_n(x) = \int_E f_{(n)}(x')\left[1+\frac{c^2q^2}{4n-2}\nabla^2\right]\frac{\boldsymbol{I}(x-x')}{8\pi\mu}\,\mathrm{d}A(x'), \tag{18}$$

then

$$J_{n+1}(x;1) = (2n+1)\int_0^1 u^{2n}J_n(x;u)\,\mathrm{d}u \quad (n = 1,2,3,\ldots), \tag{19}$$

where again, the parameter u denotes that the ellipsoid dimensions have been rescaled by u. The derivation is analogous to the discussion subsequent to (11). The intermediate expression,

$$\int_0^1 u^{2n}J_n(x;u)\,\mathrm{d}u = \int_E\int_{u^{-1}(E)}^1 f_{(n)}(x';u)\,u\,\mathrm{d}u\frac{\boldsymbol{I}(x-x')}{8\pi\mu}\mathrm{d}A(x'), \tag{20}$$

with $u^{-1}(E) = (x'^2/a_E^2+y'^2/b_E^2)^{\frac{1}{2}}$ as before, leads to the final result after the u-integration is performed.

Equation (20) shows the explicit relation between $f_{(n+1)}$ and f_n. If the domain of the singularity distribution $E(u)$ at a given u is completely contained within $E(1)$, then the domain of the singularity distribution will be invariant in homotopic integrals such as that used in (19). This observation for ellipsoids raises the question of whether a general class of particle shapes may be constructed for which the velocity representation for Stokes flow may be generated by the homotopy method employed here. Conversely, one could also construct concave shapes for which the homotopy would alter the shape of the singularity domain thereby constructing shapes for which both the singularity type and domain would vary with the ambient field.

3.1. *The Faxén relations*

The Lorentz reciprocal theorem implies that the Faxén relations for the multipole moments have the functional form of the singularity distribution, (5). We introduce the resistance tensors $\boldsymbol{Z}$ which fix the linear relation between the $\boldsymbol{P}$ and $\boldsymbol{H}$:

$$P_{jk_1\ldots k_{n-2m-1}} = Z_{(jk_1\ldots k_{n-2m-1})(il_1\ldots l_{n-1})}H_{il_1\ldots l_{n-1}} \quad (m = 0,1,2,\ldots[\tfrac{1}{2}(n-1)])$$

As shown in Kim (1985) the singularity solution implies that the Faxén law for the multipole moments may then be expressed as

$$\int_S (\boldsymbol{\sigma}\cdot\boldsymbol{n})_i x_{l_1}\ldots x_{l_{n-1}}\,\mathrm{d}S = \sum_{m=0}^{[\frac{1}{2}(n-1)]}\frac{1}{(n-2m-1)!}Z_{(jk_1\ldots k_{n-2m-1})(il_1\ldots l_{n-1})}$$
$$\times\int_E f_{(n-2m)}(x')\left\{1+\frac{c^2q^2}{4(n-2m)-2}\nabla^2\right\}v^\infty_{j,k_1\ldots k_{n-2m-1}}(x')\,\mathrm{d}A(x'). \tag{21}$$

The comma after the subscript j denotes derivatives. Equation (21) for the Faxén laws for the arbitrary multipole moment extends the results presented in Kim (1986) with the earlier results for the force and dipole (torque and stresslet) corresponding to the cases $n = 1$ and $n = 2$ (antisymmetric and symmetric parts). Equation (21) may also be rewritten using the symbolic operator approach of Brenner & Haber (1983):

$$\int_S (\boldsymbol{\sigma}\cdot\boldsymbol{n})_i\, x_{l_1}\ldots x_{l_{n-1}}\,\mathrm{d}S = \sum_{m=0}^{[\frac{1}{2}(n-1)]} \frac{1}{(n-2m-1)!} Z_{(jk_1\ldots k_{n-2m-1})\,(il_1\ldots l_{n-1})}$$
$$\times \frac{(2n-4m)!}{2^{n-2m}(n-2m)!}\left\{\left(\frac{1}{\mathrm{D}}\frac{\partial}{\partial \mathrm{D}}\right)^{n-2m-1}\left(\frac{\sinh \mathrm{D}}{\mathrm{D}}\right)\right\} v^{\infty}_{j,\,k_1\ldots k_{n-2m-1}}(\boldsymbol{x}')|_{x'=x_c}, \quad (22)$$

where the operator is defined formally as a power series in D^2 with $\mathrm{D}^2 = a^2\,\partial^2/\partial x^2 + b^2\,\partial^2/\partial y^2 + c^2\,\partial^2/\partial z^2$. This alternative form is particularly useful in situations where the ambient field is expressed analytically.

The derivation of (22) from (21) follows the familiar theme – we derive the lowest-order case and finish the proof by using the homotopic integral and mathematical induction. Therefore, we first show that

$$\int_E f_{(1)}(\boldsymbol{x}')\{1+\tfrac{1}{2}c^2q^2\nabla^2\}\,\boldsymbol{v}^{\infty}(\boldsymbol{x}')\,\mathrm{d}A(\boldsymbol{x}') = \left(\frac{\sinh \mathrm{D}}{\mathrm{D}}\right)\boldsymbol{v}^{\infty}(\boldsymbol{x}')|_{x'=x_C}. \quad (23)$$

The right-hand side of (23) may be rearranged into two terms, the first expressed in terms of $\boldsymbol{v}^{\infty}$ and the second in terms of $\nabla^2\boldsymbol{v}^{\infty}$ as follows. The D^2 operator is rewritten as

$$\mathrm{D}^2 = \tilde{\mathrm{D}}^2 + c^2\nabla^2, \quad \text{with } \tilde{\mathrm{D}}^2 = a_E^2\frac{\partial^2}{\partial x^2} + b_E^2\frac{\partial^2}{\partial y^2}.$$

If we restrict the operand to biharmonic functions such as $\boldsymbol{v}^{\infty}$ and its gradients, then $\mathrm{D}^{2k} = \tilde{\mathrm{D}}^{2k} + c^2k\tilde{\mathrm{D}}^{2k-2}\nabla^2$ and

$$\frac{\sinh \mathrm{D}}{\mathrm{D}} = \frac{\sinh \tilde{\mathrm{D}}}{\tilde{\mathrm{D}}} + c^2\sum_{k=1}^{\infty}\frac{k\tilde{\mathrm{D}}^{2k-2}}{(2k+1)!}\nabla^2$$
$$= \frac{\sinh \tilde{\mathrm{D}}}{\tilde{\mathrm{D}}} + \frac{c^2}{2}\left(\frac{1}{\tilde{\mathrm{D}}}\frac{\partial}{\partial \tilde{\mathrm{D}}}\right)\left(\frac{\sinh \tilde{\mathrm{D}}}{\tilde{\mathrm{D}}}\right)\nabla^2.$$

Equation (23) now follows by matching the terms in $\boldsymbol{v}^{\infty}$ and $\nabla^2\boldsymbol{v}^{\infty}$, i.e. it can be shown that

$$\int_E f_{(1)}(\boldsymbol{x}')\,\boldsymbol{v}^{\infty}(\boldsymbol{x}')\,\mathrm{d}A(\boldsymbol{x}') = \left(\frac{\sinh \tilde{\mathrm{D}}}{\tilde{\mathrm{D}}}\right)\boldsymbol{v}^{\infty}(\boldsymbol{x}')|_{x'=x_c}, \quad (24)$$

$$\int_E f_{(1)}(\boldsymbol{x}')\,\tfrac{1}{2}c^2q^2\,\nabla^2\boldsymbol{v}^{\infty}(\boldsymbol{x}')\,\mathrm{d}A(\boldsymbol{x}') = \tfrac{1}{2}c^2\left\{\left(\frac{1}{\tilde{\mathrm{D}}}\frac{\partial}{\partial \tilde{\mathrm{D}}}\right)\left(\frac{\sinh \tilde{\mathrm{D}}}{\tilde{\mathrm{D}}}\right)\right\}\nabla^2\boldsymbol{v}^{\infty}(\boldsymbol{x}')|_{x'=x_c}. \quad (25)$$

These two identities follow as limiting forms (for an elliptical disk) of (26) and (27) of Brenner (1966).

The general result (22) is now derived by establishing that

$$\int_E f_{(n)}(\boldsymbol{x}')\left\{1+\frac{c^2q^2}{4n-2}\nabla^2\right\}(\nabla)^{n-1}\,\boldsymbol{v}^{\infty}(\boldsymbol{x}')\,\mathrm{d}A(\boldsymbol{x}')$$
$$= \frac{(2n)!}{2^n n!}\left\{\left(\frac{1}{\mathrm{D}}\frac{\partial}{\partial \mathrm{D}}\right)^{n-1}\left(\frac{\sinh \mathrm{D}}{\mathrm{D}}\right)\right\}(\nabla)^{n-1}\,\boldsymbol{v}^{\infty}(\boldsymbol{x}')|_{x'=x_c}$$
$$= \frac{(2n)!}{n!}\sum_{k=1}^{\infty}\frac{(k+n-1)!\,\mathrm{D}^{2k-2}}{(k-1)!\,(2k+2n-2)!}(\nabla)^{n-1}\,\boldsymbol{v}^{\infty}(\boldsymbol{x}')|_{x'=x_c}. \quad (26)$$

The left-hand side of (26) may be generated by successive sequences of homotopic integrations as in (19). The same procedure must be applied to the right-hand side. Since $D^2(u) = u^2 D^2(1)$, we find that

$$\int_0^1 u^{2n} \sum_{k=1}^{\infty} \frac{2^n (k+n-1)!\, D^{2k-2}(u)}{(k-1)!\,(2k+2n-2)!} du$$

$$= \sum_{k=1}^{\infty} \frac{2^n (k+n-1)!\, D^{2k-2}}{(k-1)!\,(2k+2n-2)!} \int_0^1 u^{2n+2k-2} du$$

$$= \sum_{k=1}^{\infty} \frac{2^{n+1} (k+n)!\, D^{2k-2}}{(k-1)!\,(2k+2n)!} = \left(\frac{1}{D}\frac{\partial}{\partial D}\right)^n \left(\frac{\sinh D}{D}\right)$$

as required.

In summary, the Faxén relations presented in the symbolic operator form in (22) extend the ideas in Brenner & Haber (1983) to higher-order multipole moments. Their results for the force and dipoles are recovered for $n = 1$ and $n = 2$ respectively. As noted by these authors, the solutions for the higher-order fields are not required in the derivation of the Faxén relations for the force (or translational velocity), torque (or rotational velocity) and stresslet. However, as shown in the present work, such solutions are useful in the derivation of the Faxén relations for the quadrupole and higher-order moments.

3.2. *Distributed and centred moments*

The $\boldsymbol{P}$ are related to but not identical with the multipole moments about the ellipsoid centre. The latter are defined as the coefficients in the multipole expansion at the centre. In fact, the distributed singularities of (5) may be rewritten as an infinite series expansion about the centre since (26) applies equally well with $\boldsymbol{v}^{\infty}(\boldsymbol{x}')$ replaced by $\boldsymbol{I}(\boldsymbol{x}-\boldsymbol{x}')$. We collect all contributions to the $(n-1)$th multipole moment in the form of

$$D^{2m} I_{ij,\, k_1 \ldots k_{n-2m-1}}(\boldsymbol{x}-\boldsymbol{x}').$$

Thus the $(n-1)$th multipole moment in the $(n-1)$th-order field is

$$\int_S (\boldsymbol{\sigma}\cdot\boldsymbol{n})_j x_{k_1} \ldots x_{k_{n-1}} dS = \sum_{m=0}^{[\frac{1}{2}(n-1)]} \frac{2^m (n-2m-1)\,(2n-4m)!\,(n-m)!}{(2n-2m)!} \times \{E_{k_{n-2m} \ldots k_{n-1}} P_{jk_1 \ldots k_{n-2m-1}} + \ldots (\mathrm{sym})\}, \tag{27}$$

with
$$E_{k_1 \ldots k_{2m}} = a_{k_1}^2 \delta_{k_1 k_2} a_{k_3}^2 \delta_{k_3 k_4} \ldots a_{k_{2m-1}}^2 \delta_{k_{2m-1} k_{2m}}$$

so that even (odd) multipole moments about the centre are equal to the corresponding distributed moment plus a correction term that contains all lower order even (odd) distributed moments.

4. Quadratic flow

We now consider the disturbance field produced by a stationary ellipsoid in a quadratic field, $v_i^{\infty} = H_{ijk} x_j x_k$, in order to illustrate explicitly some of the general statements of the preceding section. In particular, the use of more than one type of multipole moment first occurs at this level.

Without loss of generality, we assume that

$$H_{ijk} = H_{ikj}, \quad P_{ijk} = P_{ikj}, \tag{28}$$

and because of continuity take $H_{jij} = H_{jji} = 0$ and $P_{jij} = P_{jji} = 0$. We now insert $n = 3$ in the general expressions (5) and (7) and after some straightforward manipulations, arrive at the following expression of the boundary condition at the particle surface:

$$
\begin{aligned}
-8\pi\mu H_{ijk}\,x_j\,x_k \\
= \tfrac{15}{8}P_{jkl}\{&2\delta_{ij}\,x_k\,x_l(K_{kl}+a_j^2\,K_{ikl})+2\delta_{ik}\,x_j\,x_l(-K_{il}+a_j^2\,K_{ijl}) \\
&+2\delta_{il}\,x_j\,x_k(-K_{ik}+a_j^2\,K_{ijk})+2\delta_{jk}\,x_i\,x_l(-K_{il}+a_j^2\,K_{ijl}) \\
&+2\delta_{jl}\,x_i\,x_k(-K_{ik}+a_j^2\,K_{ijk})+2\delta_{kl}\,x_i\,x_j(-K_{ik}+a_j^2\,K_{ijk}) \\
&+\delta_{ij}\,\delta_{kl}(x^2(a_j^2\,K_{1ik}+K_{1k})+y^2(a_j^2\,K_{2ik}+K_{2k})+z^2(a_j^2\,K_{3ik}+K_{3k})-a_j^2\,K_{ik}-K_k) \\
&+\delta_{ik}\,\delta_{jl}(x^2(a_j^2\,K_{1ij}-K_{1i})+y^2(a_j^2\,K_{2ij}-K_{2i})+z^2(a_j^2\,K_{3ij}-K_{3i})-a_j^2\,K_{ij}+K_i)\} \\
&+\delta_{il}\,\delta_{jk}(x^2(a_j^2\,K_{1ij}-K_{1i})+y^2(a_j^2\,K_{2ij}-K_{2i})+z^2(a_j^2\,K_{3ij}-K_{3i})-a_j^2\,K_{ij}+K_i)\} \\
&+\tfrac{1}{2}P_i\left\{\int_0^\infty \frac{\mathrm{d}t}{\Delta(t)}+a_i^2\,K_i\right\}.
\end{aligned} \tag{29}
$$

The right-hand side of (29) is summed over the indices j, k and l but not i. The K's are elliptic integrals defined by

$$
K_i = \int_0^\infty (a_i^2+t)^{-1}\frac{\mathrm{d}t}{\Delta(t)},
$$

$$
K_{ij} = \int_0^\infty \{(a_i^2+t)\,(a_j^2+t)\}^{-1}\frac{\mathrm{d}t}{\Delta(t)},
$$

and so forth.

We now examine (29) in the light of earlier statements. The velocity representation is a quadratic on the ellipsoid surface. The unknown P_{jkl} are related to the known H_{ijk} by a linear system of equations which is obtained by matching the coefficients in $x_j\,x_k$. In principle, the P_i are determined after the P_{ijk}, using the fact that the terms of degree zero must vanish on the right-hand side. However, there is a more direct method since the force on the ellipsoid in a quadratic field must be (Brenner & Haber 1983)

$$
P_1 = -\frac{16\pi\mu}{3}\left(\int_0^\infty \frac{\mathrm{d}t}{\Delta(t)}+a^2K_1\right)^{-1}(a^2H_{111}+b^2H_{122}+c^2H_{133}),
$$

with P_2 and P_3 obtained by successive cycling of (a,b,c) and $(1,2,3)$.

The algebraic structure of (29) is as follows. Because of (28), and continuity, H_{ijk} and P_{ijk} have only 18 components of interest and these components are subject to three constraints. However, further reduction of the linear system may be achieved because of particle symmetry. The system is actually four decoupled systems – one 3×3 and three 5×5 systems.

The 3×3 system is obtained by considering the ambient field

$$
\boldsymbol{v}^\infty = (H_{123}+H_{132})\,yz\boldsymbol{e}_x+(H_{231}+H_{213})\,zx\boldsymbol{e}_y+(H_{312}+H_{321})\,xy\boldsymbol{e}_z. \tag{30}
$$

This field requires the use of just $(P_{123}+P_{132})$, $(P_{231}+P_{213})$ and $(P_{312}+P_{321})$. The three equations are obtained by equating the yz-, zx- and xy-terms in the x, y and z velocity components.

We now consider one of the 5×5 systems. The ambient field of interest is

$$\left.\begin{aligned} v_x^\infty &= H_{111}x^2+H_{122}y^2+H_{133}z^2, \\ v_y^\infty &= (H_{212}+H_{221})xy, \\ v_z^\infty &= (H_{313}+H_{331})xz, \end{aligned}\right\} \tag{31}$$

which requires just $P_{111}, P_{122}, P_{133}$, $(P_{212}+P_{221})$ and $(P_{313}+P_{331})$ plus P_1, the hydrodynamic force in the x-direction. The five equations are obtained by equating terms in x^2, y^2 and z^2 in the x-component of the velocity, terms in xy in the y-component of the velocity and terms in xz in the z-component of the velocity. Since $H_{111}+H_{212}+H_{313}=0$ and since x^2 may be related to y^2 and z^2 on the ellipsoid surface, the problem may be further reduced to a 4×4 system of equations.

The two remaining 5×5 problems are isomorphic to the one just considered and are obtained by cycling the dependence on (x, y, z) and (a, b, c) in the solution procedure for (31). The cyclic group of order 3 may be employed to formalize the symmetry argument used by Oberbeck (1876) for the translational problem. There, the solutions for translations in the y- and z-directions result from the same cyclic operations on the solution for translation in the x-direction. Thus the complete solution requires the solution of only one problem. Here, the algebraic structure is more intricate. One subsystem, the 3×3, is invariant under the group while the three 5×5 are transformed into each other. (Explicit solutions of (30) and (31) are available from the first author.)

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