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Ellipsoidal harmonics and their application in electrostatics

Johan C.-E. Sten

Technical Research Centre of Finland, P.O. Box 1202, FIN 02044 VTT, Espoo, Finland

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Abstract

Ellipsoidal functions and ellipsoidal co-ordinates, naturally adapted to the treatment of potential problems involving ellipsoidal boundaries, are due to a relatively complicated mathematical appearance avoided in the majority of textbooks on electromagnetism, leading to an unjustified ignorance of the whole system. The paper at hand is devoted to showing that the system is worthwhile using and investigating. By first recollecting the key results of the co-ordinate system and the related potential functions, including the indispensable orthogonality result and the reciprocal distance formula, basic potential problems of electrostatics are reviewed. Closed-form expressions are thereupon derived for the induced charge and its centroid in the case of a point charge influencing a grounded ellipsoid. An image source interpretation, generalising the image solution for a sphere, follows easily from these expressions, as well as an approximate expression for the mutual capacitance between two ellipsoids or a parallel disk capacitor. An expression is also given for the total polarisation induced in a dielectric ellipsoid by the influence of a point dipole.

Keywords: Ellipsodial particles; Ellipsodial harmonics; Capacitance; Polarizability

1. Introduction

The ellipsoidal system of co-ordinates belongs to one of the most general three-dimensional systems in which the Laplace equation is separable [1]. Accordingly, the ellipsoid is held as the most general three-dimensional body which lends itself to rigorous field analysis [2–5]. In spite of this, the theory of ellipsoidal harmonics, developed in the late 19th century in the same pattern as the spherical harmonics [6,7], is rarely applied in electricity and magnetism. Beyond the classical problems of a charged [3–5,8] and a uniformly polarised [2,4,5] ellipsoid, the use of ellipsoidal harmonics in electromagnetics seems to be limited to calculations of forces near aspherical molecular charges, see e.g., [9].

This paradoxical circumstance is undoubtedly due to the co-ordinate system and the related potential functions, whose prima facie impression is both elaborate and cumbersome. In many other areas of potential theory, however, ellipsoidal harmonics have been successfully applied: e.g., for calculating gravitational forces near aspherical planetary objects [10] and hydrodynamic forces exerted on water vessels [11] as well as for computing scattering of elastic and acoustic waves by ellipsoidal inhomogeneities [12,13].

The objective of this paper is the following: Section 2 provides a concise introduction to the ellipsoidal system of co-ordinates and related functions, needed for solving the ellipsoidal boundary value problems of electrostatics dealt with in Section 3. This introduction is also required for emphasising the parallelism that exists between the ellipsoidal system and its more familiar spherical counterpart. Having reviewed the case of a charged and a uniformly polarised ellipsoid by means of ellipsoidal harmonics, we also consider the response of a grounded conducting ellipsoid to the influence of a point charge, and derive closed form expressions for the amount of induced charge and the charge centroid.

These expressions are first used to estimate the capacitance between two ellipsoids in two instances: 1:o for distant ellipsoids and 2:o for two thin disks, lying above one another in parallel. The latter solution is compared with Kirchhoff's formula for the parallel plate capacitor [5,8]. Using the aforementioned expressions for the induced charge and the charge centroid, a part of the potential

E-mail address: johan.sten@vtt.fi.

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arising from (or 'scattered' by) the ellipsoid is interpreted in terms of a generalised Kelvin's image in a sphere. Finally, the dipolar response of a dielectric ellipsoid to the action of an arbitrarily directed dipole is given, which is directly applicable to modelling multiple interactions of two or several ellipsoids.

2. Laplace equation in ellipsoidal co-ordinates

Potential problems involving ellipsoidal boundaries are best analysed by means of the ellipsoidal system of co-ordinates (ξ, μ, ν),[1] generated by a reference ellipsoid

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1, \tag{1}$$

where $a > b > c$ are its semidiameters. The co-ordinates corresponding to the point (x, y, z) then satisfy

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - h^2} + \frac{z^2}{\lambda^2 - k^2} = 1 \quad \text{with } k = \sqrt{a^2 - c^2}, \quad h = \sqrt{a^2 - b^2}, \tag{2}$$

where λ stands for either ξ, μ, or ν. Every co-ordinate thus has the physical dimension of length, each one being defined in the intervals $0 \leqslant \nu^2 \leqslant h^2 \leqslant \mu^2 \leqslant k^2 \leqslant \xi^2 < \infty$. Surfaces of $\xi =$ constant are ellipsoids, in particular, $\xi = \xi_1 = a$ is the ellipsoid (1), those of $\mu =$ constant are hyperboloids of one sheet and those of $\nu =$ constant hyperboloids of two sheets. The variable ξ assumes only positive values, while μ and ν may also be negative.

In the limit $\xi \to k$ the ellipsoid (1) degenerates into a disk

$$(x/k)^2 + y^2/(k^2 - h^2) = 1, \tag{3}$$

in the xy-plane. In the other extreme, $\xi \gg k$, the surface approaches a sphere, and $\xi \to r$, the distance to the origin. Conversion formulae and other properties of the system are summarised in Appendix A.

The assumption of separability of the Laplace equation in the ellipsoidal system requires that the functions describing the dependence of each variable satisfy the Lamé equation [7]

$$(\lambda^2 - h^2)(\lambda^2 - k^2)\frac{\mathrm{d}^2 E(\lambda)}{\mathrm{d}\lambda^2} + \lambda(2\lambda^2 - h^2 - k^2)\frac{\mathrm{d}E(\lambda)}{\mathrm{d}\lambda} + \{(h^2 + k^2)q - n(n+1)\lambda^2\}E(\lambda) = 0, \tag{4}$$

where q is an arbitrary constant to be fixed appropriately, cf. [1]. It is noteworthy that in every other curvilinear co-ordinate system, each variable is governed by its own differential equation.

[1] To avoid confusion with the transverse distance of the usual cylindrical co-ordinates (ρ, z, φ) we have replaced the symbol ρ in Hobson's original notation (ρ, μ, ν) [7] into ξ.

2.1. Lamé functions

The solutions of (4), E_n^p, are called Lamé functions of the first kind, of degree n and order p. They appear in four different classes [7], the characteristics of which are

$$E_n^p(\lambda) = \begin{cases} K(\lambda) = P(\lambda), \\ L(\lambda) = P(\lambda)\sqrt{\lambda^2 - h^2}, \\ M(\lambda) = P(\lambda)\sqrt{\lambda^2 - k^2}, \\ N(\lambda) = P(\lambda)\sqrt{(\lambda^2 - k^2)(\lambda^2 - h^2)}, \end{cases} \tag{5}$$

where $P(\lambda)$ is a polynomial to be given later. The index n is the degree of the expression in λ, and for each n, there are $2n+1$ different solutions altogether, enumerated by the index p. For the determination of Lamé functions, see e.g., [1], and for more complete algorithms [6,7,10]. A short list of Lamé functions is given in Appendix B.

2.2. Ellipsoidal harmonics

An ellipsoidal harmonic which is regular at the origin (internal harmonic) may be written in terms of the Lamé product

$$\mathscr{E}_n^p(\mathbf{r}) = E_n^p(\xi)E_n^p(\mu)E_n^p(\nu), \tag{6}$$

which is even in z when E_n^m is of the class K or L and odd in z, when E_n^m is of the class M or N. Correspondingly, an ellipsoidal harmonic regular at infinity (external harmonic) is

$$\mathscr{F}_n^p(\mathbf{r}) = F_n^p(\xi)E_n^p(\mu)E_n^p(\nu), \tag{7}$$

where $F_n^p(\xi)$ is the Lamé function of the second kind, derivable from the corresponding function of the first kind through [6]

$$F_n^p(\xi) = (2n+1)E_n^p(\xi)\int_\xi^\infty \frac{\mathrm{d}\xi'}{[E_n^p(\xi')]^2\sqrt{(\xi'^2 - h^2)(\xi'^2 - k^2)}}. \tag{8}$$

When $\xi \to \infty$, $E_n^p \sim c_0\xi^n$ and $F_n^p(\xi) \sim E_n^p(\xi)/\xi^{2n+1} \sim c_0/\xi^{n+1}$ which corresponds to the r^{-n-1}-potential term in the spherical system. For evaluation of the functions F_n^p, a Gauss-Laguerre type quadrature can be used [10].

Thus, an expansion of a scalar potential in ellipsoidal harmonics reads

$$\Phi = \sum_{n=0}^{\infty}\sum_{p=1}^{2n+1} A_{np} \begin{cases} \dfrac{\mathscr{E}_n^p(\mathbf{r})}{E_n^p(\xi_1)}, & \xi \leqslant \xi_1, \\ \dfrac{\mathscr{F}_n^p(\mathbf{r})}{F_n^p(\xi_1)}, & \xi \geqslant \xi_1, \end{cases} \tag{9}$$

where ξ_1 is the co-ordinate of the surface (1) of an ellipsoid. It may be noted that for large ξ, the $n = 0$ term is proportional to $\xi^{-1} \sim r^{-1}$ and thus corresponds to the total charge. Correspondingly, the $n = 1$ terms ($p = 1, 2, 3$) are proportional to $\xi^{-2} \sim r^{-2}$ and thus correspond to dipole moments.

2.3. Static Green's function

An expansion for the reciprocal distance (the Green's function) can be constructed following [1, Eq. (7.2.63)] by the aid of an orthogonality relation for the pertinent eigenfunctions—in the present case the double integral:

$$\int_0^h \int_h^k \frac{E_n^p(\mu)E_n^p(\nu)E_{n'}^{p'}(\mu)E_{n'}^{p'}(\nu)}{\sqrt{(\mu^2-h^2)(k^2-\mu^2)(h^2-\nu^2)(k^2-\nu^2)}} \times(\mu^2-\nu^2)\,\mathrm{d}\mu\,\mathrm{d}\nu = \gamma_n^p \delta_{nn'}\delta_{pp'}, \tag{10}$$

where $\delta_{nn'} = 1$ when $n = n'$ and is zero otherwise. Also needed are the scale factors (54) and the Wronskian (55) to be able to write

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sum_{p=1}^{2n+1}\frac{1}{\gamma_n^p}\begin{cases}\mathscr{E}_n^p(\mathbf{r})\mathscr{F}_n^p(\mathbf{r}'), & \xi'\geqslant\xi,\\ \mathscr{F}_n^p(\mathbf{r})\mathscr{E}_n^p(\mathbf{r}'), & \xi'\leqslant\xi.\end{cases} \tag{11}$$

This expression was first given by Heine [6, Band II, p. 172]. The coefficients

$$\gamma_n^p = \int_0^h \int_h^k \frac{[E_n^p(\mu)E_n^p(\nu)]^2(\mu^2-\nu^2)\,\mathrm{d}\mu\,\mathrm{d}\nu}{\sqrt{(\mu^2-h^2)(k^2-\mu^2)(h^2-\nu^2)(k^2-\nu^2)}}, \tag{12}$$

have the dimensions of length to the power $4n$, and accordingly, the first one is a dimensionless constant; $\gamma_0^1 = \pi/2$. Appendix B provides a list of γ_n^p functions of low order.

3. Electrostatic field problems

3.1. Isolated ellipsoid

Let an ellipsoid in free space be centred at the origin and defined by Eq. (1). When the surface of the ellipsoid, $\xi = \xi_1$, is held at the potential $\Phi(\xi_1, \mu, \nu)$ the potential for $\xi \geqslant \xi_1$ can be written as

$$\Phi = \sum_{n=0}^{\infty}\sum_{p=1}^{2n+1} B_{np}\frac{\mathscr{F}_n^p(\mathbf{r})}{F_n^p(\xi_1)}, \tag{13}$$

where

$$B_{np} = \int_0^h \int_h^k \Phi(\xi_1,\mu,\nu) \times \frac{E_n^p(\mu)E_n^p(\nu)(\mu^2-\nu^2)\,\mathrm{d}\mu\,\mathrm{d}\nu}{\gamma_n^p\sqrt{(\mu^2-h^2)(k^2-\mu^2)(h^2-\nu^2)(k^2-\nu^2)}}. \tag{14}$$

If ξ_1 is an equipotential surface, then $\Phi(\xi_1, \mu, \nu)$ is constant and $= U_1$. This happens when the ellipsoid is perfectly conducting, or, equivalently, when $\varepsilon_r \to \infty$. In this case, the only nonzero coefficient is $B_{01} = U_1$, whence we are left with a single term

$$\Phi = U_1\frac{\mathscr{F}_0^1(\mathbf{r})}{F_0^1(\xi_1)}, \tag{15}$$

approaching $U_1/(rF_0^1(\xi_1))$ at a large distance r from the ellipsoid. By recalling the expression for the potential of a charge (Q) centred at the origin, namely; $Q/(4\pi\varepsilon_0 r)$, one may identify the capacitance $C = Q/U_1$ as [3, Section 5.02]

$$C = \frac{4\pi\varepsilon_0}{F_0^1(\xi_1)} = \frac{4\pi\varepsilon_0 k}{\mathbb{F}\,(\arcsin(k/\xi_1, h^2/k^2))}, \tag{16}$$

in terms of Legendre's incomplete elliptic integral $\mathbb{F}$. Hence, for instance, when $a = b$ and $c \to 0$ (or $h = 0$ and $\xi_1 \to k = a$) one obtains $C = 8\varepsilon_0 a$, which is the familiar expression for the capacitance of a circular disk of radius a; a result found empirically as early as in the 1770s by Cavendish [14, p. 54].

3.2. Dielectric ellipsoid in uniform field

Now let the ellipsoid be characterised by the relative permittivity ε_r, the outside medium being air, and let an external homogeneous field be applied. To find the field disturbance produced by the ellipsoid the applied field is expressed as

$$\Phi_o = -E_{ox}x - E_{oy}z - E_{oz}z = -\frac{E_{ox}\mathscr{E}_1^1(\mathbf{r})}{kh} - \frac{E_{oy}\mathscr{E}_1^2(\mathbf{r})}{h\sqrt{k^2-h^2}} - \frac{E_{oz}\mathscr{E}_1^3(\mathbf{r})}{k\sqrt{k^2-h^2}}, \tag{17}$$

where E_{ox}, E_{oy}, E_{oz} are the field strengths in the x, y, and z directions, respectively. In consideration of Eq. (10) the total potential must then be of the form

$$\Phi_- = \sum_{p=1}^{3} A_{1p}\mathscr{E}_1^p(\mathbf{r}) \quad \text{inside and } \Phi_+ = \Phi_o + \sum_{p=1}^{3} B_{1p}\mathscr{F}_1^p(\mathbf{r}), \tag{18}$$

outside the ellipsoid. The continuity conditions

$$\Phi_+ = \Phi_- \quad \text{and} \quad \frac{\partial\Phi_+}{\partial\xi} = \varepsilon_r\frac{\partial\Phi_-}{\partial\xi}, \tag{19}$$

when applied at the interface $\xi = \xi_1$, render the coefficients for the internal potential

$$A_{11} = -\frac{E_{ox}}{kh}\frac{1}{1+L_1^1(\xi_1)(\varepsilon_r-1)}, \tag{20}$$

$$A_{12} = -\frac{E_{oy}}{h\sqrt{k^2-h^2}}\frac{1}{1+L_1^2(\xi_1)(\varepsilon_r-1)}, \tag{21}$$

$$A_{13} = -\frac{E_{oz}}{k\sqrt{k^2-h^2}}\frac{1}{1+L_1^3(\xi_1)(\varepsilon_r-1)} \tag{22}$$

and those of the potential 'scattered' by the ellipsoid

$$B_{11} = \frac{(1-\varepsilon_r)L_1^1(\xi_1)E_1^1(\xi_1)}{F_1^1(\xi_1)}A_{11} = (1-\varepsilon_r)\frac{abcA_{11}}{3}, \tag{23}$$

$$B_{12} = \frac{(1-\varepsilon_r)L_1^2(\xi_1)E_1^2(\xi_1)}{F_1^2(\xi_1)}A_{12} = (1-\varepsilon_r)\frac{abcA_{12}}{3}, \tag{24}$$

$$B_{13} = \frac{(1-\varepsilon_r)L_1^3(\xi_1)E_1^3(\xi_1)}{F_1^3(\xi_1)}A_{13} = (1-\varepsilon_r)\frac{abcA_{13}}{3}. \tag{25}$$

Here

$$L_1^{1,2,3}(\xi_1) = \frac{\dot{E}_1^{1,2,3}(\xi_1)F_1^{1,2,3}(\xi_1)}{\dot{E}_1^{1,2,3}(\xi_1)F_1^{1,2,3}(\xi_1) - E_1^{1,2,3}(\xi_1)\dot{F}_1^{1,2,3}(\xi_1)} = \xi_1\sqrt{(\xi_1^2-h^2)(\xi_1^2-k^2)} \times \int_{\xi_1}^{\infty} \frac{\mathrm{d}\xi'}{[E_1^{1,2,3}(\xi')]^2\sqrt{(\xi'^2-h^2)(\xi'^2-k^2)}}, \quad (26)$$

are the so-called *depolarisation, or geometric factors* of the ellipsoid, e.g., [2, Section 437; 4, Section 3.27; 5, Section 4]. In deriving (26) as well as in the simplifications of (23)–(25), we have used (55) and the relationship $\dot{E}_1^p(x)E_1^p(x) = x$ for all $p = 1, 2, 3$, where the dot on top signifies differentiation with respect to the argument. It is not very difficult to show that $L_1^1 = L_1^2 = L_1^3 = 1/3$ for a sphere ($a = b = c$), nor that $L_1^1 + L_1^2 + L_1^3 = 1$ in general.

The most important parameter, however, is the dipole moment $\mathbf{p}_o$, whose components in the three co-ordinate directions [4]

$$p_{ox} = 4\pi\varepsilon_0 khB_{11}, \quad p_{oy} = 4\pi\varepsilon_0 h\sqrt{k^2-h^2}B_{12}, \quad p_{oz} = 4\pi\varepsilon_0 k\sqrt{k^2-h^2}B_{13} \quad (27)$$

are directly visible in terms of the $n = 1$ coefficients. They may alternatively be deduced by integrating the polarisation density of the dielectric; $\mathbf{P} = -\varepsilon_0(\varepsilon_r - 1)\nabla\Phi_-$, over the volume of the ellipsoid. The whole procedure can be made straightforward by recognising that $\mathbf{P}$, like the forcing field, is constant (though, in general, not parallel with the field) and may be taken out from the integral, which thus yields the volume of the ellipsoid; $4\pi abc/3$.

3.3. Conducting ellipsoid influenced by a point charge

Let the same ellipsoid now be influenced by an external point charge Q_o at $\mathbf{r}_o = (\xi_o, \mu_o, \nu_o)$. When the ellipsoid is conducting and grounded, the electric potential in the range $\xi_1 \leqslant \xi \leqslant \xi_o$ can be expressed as

$$\Phi = \sum_{n=0}^{\infty}\sum_{p=1}^{2n+1}\left[B_{np}F_n^p(\xi) + Q_o\frac{\mathscr{F}_n^p(\mathbf{r})E_n^p(\xi)}{8\varepsilon_0\gamma_n^p(2n+1)}\right]E_n^p(\mu)E_n^p(\nu), \quad (28)$$

with

$$B_{np} = -Q_o\frac{\mathscr{F}_n^p(\mathbf{r}_o)E_n^p(\xi_1)}{8\varepsilon_0\gamma_n^p(2n+1)F_n^p(\xi_1)}, \quad (29)$$

found by requiring that $\Phi = 0$ at $\xi = \xi_1$. Resting on the conclusions drawn from the discussion after Eq. (9), the total charge induced can be seen to be

$$Q_{\mathrm{ind}} = -Q_o\frac{F_0^1(\xi_o)}{F_0^1(\xi_1)} = -Q_o\frac{\mathbb{F}\left(\arcsin\dfrac{k}{\xi_o}, \dfrac{h^2}{k^2}\right)}{\mathbb{F}\left(\arcsin\dfrac{k}{\xi_1}, \dfrac{h^2}{k^2}\right)}. \quad (30)$$

The same result may of course be found by integrating the charge density $q_{\mathrm{ind}}(\mu, \nu) = -\varepsilon_0\mathbf{u}_\xi \cdot (\nabla\Phi_+)|_{\xi=\xi_1} = -\varepsilon_0((1/h_\xi)(\partial\Phi_+/\partial\xi))|_{\xi=\xi_1}$ over the surface of the ellipsoid:

$$Q_{\mathrm{ind}} = 8\int_0^h\int_h^k q_{\mathrm{ind}}(\mu, \nu)\, h_\mu h_\nu\, \mathrm{d}\mu\, \mathrm{d}\nu. \quad (31)$$

Here, the factor 8 in front of the integral stems from the fact that exactly the same integration is to be performed over each of the eight octants having the same values for ξ^2, μ^2 and ν^2. Note, however, that if the ellipsoid is isolated (= not grounded) one must enforce $B_{01} = 0$ to ensure a vanishing net charge.

3.4. Image-source interpretation

The centroid of the induced charge distribution is obtained as

$$\mathbf{r}_{\mathrm{ind}} = \frac{8}{Q_{\mathrm{ind}}}\int_0^h\int_h^k \mathbf{r}\; q_{\mathrm{ind}}(\mu,\nu)h_\mu h_\nu\,\mathrm{d}\mu\,\mathrm{d}\nu = \frac{F_0^1(\xi_1)}{F_0^1(\xi_o)}\left[\frac{F_1^1(\xi_o)E_1^1(\xi_1)}{F_1^1(\xi_1)E_1^1(\xi_o)}x_o\mathbf{u}_x + \frac{F_1^2(\xi_o)E_1^2(\xi_1)}{F_1^2(\xi_1)E_1^2(\xi_o)}y_o\mathbf{u}_y + \frac{F_1^3(\xi_o)E_1^3(\xi_1)}{F_1^3(\xi_1)E_1^3(\xi_o)}z_o\mathbf{u}_z\right]. \quad (32)$$

This point undoubtedly is located inside the ellipsoid. This expression is very useful, because $\mathbf{r}_{\mathrm{ind}}$ serves as the first approximation for the location of the induced charge, when regarded at a distance. Thus, it allows us to interprete the reflected potential as arising from an image-source.

Assume that a point charge of magnitude (30) is placed at $\mathbf{r}_{\mathrm{ind}}$. In order not to change the potential outside the ellipsoid, one obviously has to subtract from the expression for the 'scattered' potential the effect of a similar charge, expressed in terms of ellipsoidal harmonics. Thus, the potential 'scattered' by the ellipsoid is developed as

$$\sum_{n=0}^{\infty}\sum_{p=1}^{2n+1}B_{np}\mathscr{F}_n^p(\mathbf{r}) = \frac{Q_{\mathrm{ind}}}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_{\mathrm{ind}}|} + \sum_{n=2}^{\infty}\sum_{p=1}^{2n+1}\underbrace{\left\{B_{np} - \frac{Q_{\mathrm{ind}}\mathscr{E}_n^p(\mathbf{r}_{\mathrm{ind}})}{8\varepsilon_0\gamma_n^p(2n+1)}\right\}}_{B'_{nm}}\mathscr{F}_n^p(\mathbf{r}). \quad (33)$$

The induced charge being thus separated, the residual series obviously must represent the potential due to a neutral charge distribution, lacking the $n = 0$ term. However, as a consequence of the location of the image charge at the centre of charge of the ellipsoid, also the coefficients for $n = 1$ (= the dipole terms) will be identically zero, which is clearly seen by carefully expanding B'_{np} for each $p = 1, 2, 3$.

The faster the modified coefficients, B'_{np}, for $n \geqslant 2$ converge, the closer the shape of the body resembles a sphere. In fact, in the spherical limit B'_{np} approaches zero for all n, p, while the induced charge and its centre are seen to approach $-Q_o a/r_o$ and $\mathbf{r}_o(a/r_o)^2$, respectively, where a is the radius of the sphere and r_o the distance of the source and the centre of the sphere. The extracted charge thus coincides with the image for a sphere found by Thomson (Kelvin) in 1845 [2, Chapter 11; 3, Section 5.06]. Fig. 1 shows an example of the distribution of the electric potential around a grounded ellipsoid influenced by a point charge.

If one wishes to further speed up the calculation of the potential—which may be desired if the charge lies very close to the surface—it is possible to interpret the expansion terms as being due to certain image distributions of superficial dipoles [11] on the fundamental elliptic disk (3), the dipoles of these distributions being either tangential or normal to the xy-plane. Furthermore, these distribution are nonsingular and well behaved over the entire domain and thus easily approximated by a finite number of points.

3.5. Capacitance problems

Using the previous results, let us estimate the mutual capacitance between two ellipsoids, labelled A and B. The coefficient of induction is $C_{AB} = C_{AA} Q_{\text{ind},B}/Q_A$, C_{AA} being the self-capacitance of ellipsoid A carrying the charge Q_A, and $Q_{\text{ind},B}$ the charge induced upon ellipsoid B being grounded. Furthermore, reciprocity asserts that $C_{\bar{A}B} = C_{\bar{B}A}$. Thus, the problem of mutual capacitance is converted into the problem of estimating the self-capacitance. In the following, two approximations are offered for its calculation.

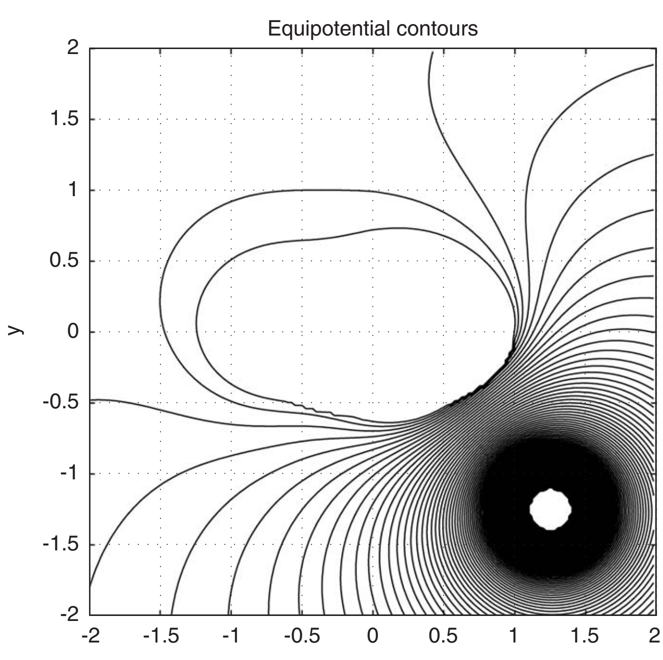

Fig. 1. Equipotential lines in the xy-plane around an ellipsoid of normalised dimension $a = 1$, $b = 0.6$, $c = 0.4$, with a unit point charge lying at $x_o = 1.25$, $y_o = -1.25$, $z_o = 0$. A point image and expansion terms for $n = 2, 3$ were used.

When the two ellipsoids are separated by a large distance [3, Section 2.18], the self-capacitance of each ellipsoid is known to be roughly the same as in the isolated case (16). Thus,

$$C_{AB} = C_{BA} \approx -\frac{4\pi\varepsilon_0}{F_0^1(\xi_A)F_0^1(\xi_B)\mathrm{dist}(A,B)}, \tag{34}$$

where $\mathrm{dist}(A,B)$ is the distance between the centres of ellipsoid A and B, ξ_A and ξ_B denoting the bounding co-ordinates of each ellipsoid.

A useful approximation may also be derived for a circular parallel plate capacitor, or a system of two overlapping circular disks of radius a and zero thickness, separated by a distance H. Evidently, the self-capacitance of the system; $2(C_{AA} + C_{AB})$, is comparable with that of a single ellipsoid, which by its form approximates the overall shape of the pair. Practicable estimates for the equatorial and polar radii of the fictitious ellipsoid are a and $H/2$, respectively. The capacitance between the two plates; $(C_{AA} - C_{AB})/2$, can then be written as

$$C \approx \frac{\varepsilon_0 \pi a\sqrt{1 - H^2/(2a)^2}}{\arcsin\left(\sqrt{1 - H^2/(2a)^2}\right)} \frac{(\pi/2 + \arcsin(a/\xi_{\text{ave}}))}{(\pi/2 - \arcsin(a/\xi_{\text{ave}}))}, \tag{35}$$

where ξ_{ave} is the average ξ co-ordinate of the charge distribution $\varrho(\rho)$ on disk B (where ρ denotes the transverse distance $\sqrt{x^2 + y^2}$), viewed from disk A (and vice versa), i.e.,

$$\xi_{\text{ave}} = \int_0^a \varrho(\rho)\ \xi(\rho)\ \rho\,\mathrm{d}\rho \left[\int_0^a \varrho(\rho)\ \rho\,\mathrm{d}\rho\right]^{-1}, \tag{36}$$

where, using (49)

$$\xi(\rho) = \frac{s}{\sqrt{3}}\sqrt{1 + 2\sqrt{1 - 3\left(\frac{a\rho}{s^2}\right)^2}\cos\left[\frac{1}{3}\arccos\frac{s^6 - 4.5(a\rho s)^2}{(s^4 - 3(a\rho)^2)^{3/2}}\right]}, \tag{37}$$

with the abbreviation $s = \sqrt{a^2 + H^2 + \rho^2}$.

To calculate ξ_{ave}, an estimate for the charge distribution $\varrho(\rho)$ is needed. When the two disks are far from touching, say $H > 0.4a$, the charge distribution is nearly that of a disk in free space, $\varrho(\rho) \sim 1/\sqrt{1 - (\rho/a)^2}$. In that case (36) gives $\xi_{\text{ave}} \approx \xi(0.84\,a)$, by which the capacitance (35) can be evaluated. In Fig. 2, this value, labelled 'approx', is compared with that given by the formula

$$C \approx \varepsilon_0 \frac{\pi a^2}{H} + \varepsilon_0 a\left(\ln\left(16\pi\frac{a}{H}\right) - 1\right), \tag{38}$$

due to Kirchhoff [5, p. 19; 8, p. 491], valid for small separations, and an exact result obtained by integral equation techniques [15]. The overall behaviour of the approximation is quite similar to the exact curve, the discrepancy being partly due to a slightly underestimated self-capacitance of the two plates.

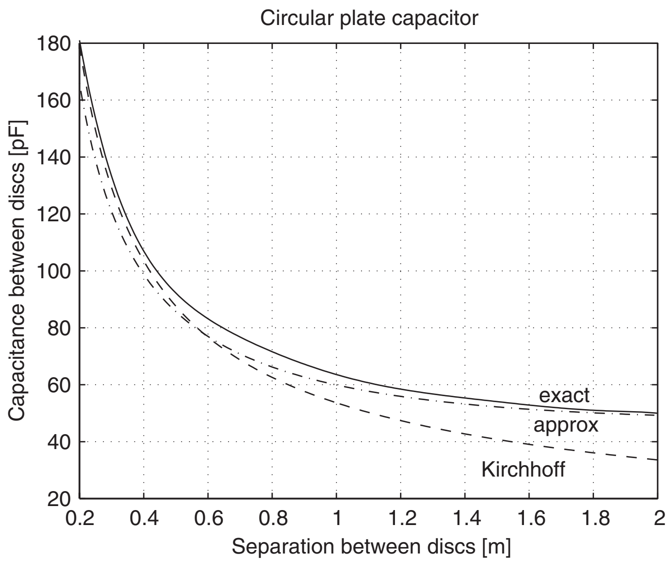

Fig. 2. Approximate capacitance between two circular disks of radius 1 m compared with a numerical solution ('exact') and Kirchhoff's approximation.

3.6. Dielectric ellipsoid influenced by a dipole

In order to describe the interaction between two or several dielectric ellipsoids in a uniform field, knowledge of the response of a dielectric ellipsoid to the action of a point dipole is required. Thus, let a doublet of dipole moments p_{ox}, p_{oy}, and p_{oz}, pointing in the x, y, and z directions, respectively, be placed at (x_o, y_o, z_o). Then, the total potential over the range $\xi_1 \leqslant \xi \leqslant \xi_o$ becomes

$$\Phi_+ = \sum_{n=1}^{\infty} \sum_{p=1}^{2n+1} \left[B_{np} \mathscr{F}_n^p(\mathbf{r}) + \frac{\mathscr{E}_n^p(\mathbf{r})}{8\varepsilon_0 \gamma_n^p (2n+1)} \times \sum_{\zeta=x,y,z} p_{o\zeta} \frac{\mathrm{d}}{\mathrm{d}\zeta} [\mathscr{F}_n^p(\mathbf{r}_o)] \right]. \tag{39}$$

We note that the term $n = 0$ has been discarded so as to keep the system free from charge. For the same reason, the potential inside the ellipsoid $\xi \leqslant \xi_1$

$$\Phi_- = \sum_{n=1}^{\infty} \sum_{p=1}^{2n+1} A_{np} \mathscr{E}_n^p(\mathbf{r}), \tag{40}$$

does not include an $n = 0$ term.

On application of the continuity conditions (19) at the interface $\xi = \xi_1$ and the orthogonality property (10), the unknowns can be written

$$A_{np} = \frac{\sum_\zeta p_{o\zeta} \frac{\mathrm{d}}{\mathrm{d}\zeta} [\mathscr{F}_n^p(r_o)]}{8\varepsilon_0 \gamma_n^p (2n+1)(1 + L_n^p(\xi_1)(\varepsilon_r - 1))}, \tag{41}$$

$$B_{np} = \frac{(1-\varepsilon_r) L_n^p(\xi_1) E_n^p(\xi_1)}{F_n^p(\xi_1)} A_{np}, \tag{42}$$

where, for convenience, we denote

$$L_n^p(\xi_1) = \frac{\dot{E}_n^p(\xi_1) F_n^p(\xi_1)}{\dot{E}_n^p(\xi_1) F_n^p(\xi_1) - E_n^p(\xi_1) \dot{F}_n^p(\xi_1)} = \frac{\sqrt{(\xi^2 - h^2)(\xi^2 - k^2)}}{2n+1} \dot{E}_n^p(\xi_1) F_n^p(\xi_1), \tag{43}$$

in analogy with the depolarisation factors for a homogeneous field (26). The induced dipole moment $\mathbf{p}_{\text{ind}}$ is again proportional to the $n = 1$ terms, and given in analogy with (27), in the three directions as

$$\frac{p_{\text{ind},x}}{4\pi\varepsilon_0} = khB_{11}, \quad \frac{p_{\text{ind},y}}{4\pi\varepsilon_0} = h\sqrt{k^2 - h^2} B_{12}, \quad \frac{p_{\text{ind},z}}{4\pi\varepsilon_0} = k\sqrt{k^2 - h^2} B_{13}. \tag{44}$$

The electromagnetic scattering of a low-frequency plane-wave field by an assemblage of dielectric ellipsoids may be formulated by writing the dipole densities of each ellipsoid in two components; (i) the incident field and (ii) the influence of the polarisation induced by the neighbouring ellipsoid(s). Considering two ellipsoids labelled A and B, the interaction mechanism may be written symbolically by the pair of equations

$$\mathbf{P}_{\text{tot,A}} = \mathbf{P}_{\text{o,A}} + \mathbf{P}_{\text{ind,A}} \mathbf{P}_{\text{tot,B}}$$
$$\mathbf{P}_{\text{tot,B}} = \mathbf{P}_{\text{o,B}} + \mathbf{P}_{\text{ind,B}} (\mathbf{P}_{\text{tot,A}}). \tag{45}$$

In the dipole–dipole approximation, applicable when the ellipsoids are distant to each other, the dipole densities can be replaced by appropriate dipole moments of each ellipsoids (the integral over the whole volume), leading to a coupled system of equations that can be solved, for example, by successive iteration.

4. Conclusion

In this paper, a review of ellipsoidal harmonics, rarely encountered in textbooks on electromagnetic theory, has been presented along with examples of their most useful applications in the solution of elementary electrostatic problems involving ellipsoidal boundaries. Analogies with the spherical system of co-ordinates have been emphasised. Orthogonality between the ellipsoidal modes is shown to be very important. Regrettably, this property is almost systematically overlooked or left unmentioned when the classical results for the capacitance and polarisability of an ellipsoid are derived in textbooks.

This article has endeavoured to show that an analytical approach to problems involving ellipsoids or resembling bodies is conceivable, and sometimes more straightforward, than a numerical solution of relevant integral or differential equations. Benchmark solutions as well as simple approximations for problems of engineering interest can be constructed without the need of software programs employing large-scale electromagnetic field solutions.

Appendix A

The equations linking the co-ordinate systems are

$$x^2 = \frac{\xi^2\mu^2\nu^2}{k^2h^2}, \tag{46}$$

$$y^2 = \frac{(\xi^2-h^2)(\mu^2-h^2)(h^2-\nu^2)}{h^2(k^2-h^2)}, \tag{47}$$

$$z^2 = \frac{(\xi^2-k^2)(k^2-\mu^2)(k^2-\nu^2)}{k^2(k^2-h^2)}. \tag{48}$$

A conversion in the converse direction is given by [10]

$$\xi^2 = 2\sqrt{Q}\cos\frac{\vartheta}{3} - \frac{a_1}{3}, \tag{49}$$

$$\mu^2 = 2\sqrt{Q}\cos\left(\frac{\vartheta}{3}+\frac{4\pi}{3}\right) - \frac{a_1}{3}, \tag{50}$$

$$\nu^2 = 2\sqrt{Q}\cos\left(\frac{\vartheta}{3}+\frac{2\pi}{3}\right) - \frac{a_1}{3}, \tag{51}$$

where

$$Q = \frac{a_1^2-3a_2}{9}, \quad R = \frac{9a_1a_2-27a_3-2a_1^3}{54}, \quad \cos\vartheta = RQ^{-3/2}. \tag{52}$$

$$a_1 = -(x^2+y^2+z^2+k^2+h^2),$$
$$a_2 = x^2(k^2+h^2)+y^2k^2+z^2h^2+h^2k^2 \tag{53}$$

and $a_3 = -(khx)^2$. Since every coordinate above is appearing squared, the relations are not one-to-one: in fact, for every (ξ,μ,ν), there will be eight combinations of (x,y,z). To select the right sign for μ and ν is not easy, but in the present applications of potential theory, we only have to observe whether the appropriate Lamé product, $\mathscr{E}_n^p(\mathbf{r})$ or $\mathscr{F}_n^p(\mathbf{r})$, is an odd function of x, y or z, and choose the sign accordingly.

The linearising factors of the ellipsoidal system are

$$h_\xi = \sqrt{\frac{(\xi^2-\mu^2)(\xi^2-\nu^2)}{(\xi^2-h^2)(\xi^2-k^2)}}, \quad h_\mu = \sqrt{\frac{(\xi^2-\mu^2)(\mu^2-\nu^2)}{(k^2-\mu^2)(\mu^2-h^2)}},$$
$$h_\nu = \sqrt{\frac{(\xi^2-\nu^2)(\mu^2-\nu^2)}{(h^2-\nu^2)(k^2-\nu^2)}} \tag{54}$$

and the Wronskian relation

$$F_n^p(\xi_1)\frac{\mathrm{d}}{\mathrm{d}\xi}E_n^p(\xi)|_{\xi=\xi_1} - E_n^p(\xi_1)\frac{\mathrm{d}}{\mathrm{d}\xi}F_n^p(\xi)|_{\xi=\xi_1} = \frac{2n+1}{\sqrt{(\xi^2-h^2)(\xi^2-k^2)}}. \tag{55}$$

The ellipsoidal unit vectors are resolved in cartesian vectors as follows:

$$\mathbf{u}_\xi = \mathbf{u}_x\frac{\mu\nu\sqrt{(\xi^2-h^2)(\xi^2-k^2)}}{kh\sqrt{(\xi^2-\mu^2)(\xi^2-\nu^2)}} + \mathbf{u}_y\frac{\xi\sqrt{(\xi^2-k^2)(\mu^2-h^2)(h^2-\nu^2)}}{h\sqrt{(k^2-h^2)(\xi^2-\mu^2)(\xi^2-\nu^2)}} + \mathbf{u}_z\frac{\xi\sqrt{(\xi^2-h^2)(k^2-\mu^2)(k^2-\nu^2)}}{k\sqrt{(k^2-h^2)(\xi^2-\mu^2)(\xi^2-\nu^2)}}, \tag{56}$$

$$\mathbf{u}_\mu = \mathbf{u}_x\frac{\xi\nu\sqrt{(k^2-\mu^2)(\mu^2-h^2)}}{kh\sqrt{(\mu^2-\nu^2)(\xi^2-\mu^2)}} + \mathbf{u}_y\frac{\mu\sqrt{(k^2-\mu^2)(h^2-\nu^2)(\xi^2-h^2)}}{h\sqrt{(k^2-h^2)(\mu^2-\nu^2)(\xi^2-\mu^2)}} - \mathbf{u}_z\frac{\mu\sqrt{(\mu^2-h^2)(k^2-\nu^2)(\xi^2-k^2)}}{k\sqrt{(k^2-h^2)(\mu^2-\nu^2)(\xi^2-\mu^2)}}. \tag{57}$$

and $\mathbf{u}_\nu = \mathbf{u}_\xi \times \mathbf{u}_\mu$. In the spherical limit $\xi \gg k$ the unit vector $\mathbf{u}_\xi \to (x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_z)/r = \mathbf{u}_r$ or the unit position vector, while both $\mathbf{u}_\mu$ and $\mathbf{u}_\nu$ become tangent to the sphere (however, not equal to $\mathbf{u}_\theta$ or $\mathbf{u}_\varphi$ of the spherical system).

Appendix B

The few lowest and most used Lamé functions are given below.

$$E_0^1(\lambda) = K_0(\lambda) = 1,$$

$$E_1^1(\lambda) = K_1(\lambda) = \lambda,$$

$$E_1^2(\lambda) = L_1(\lambda) = \sqrt{\lambda^2-h^2},$$

$$E_1^3(\lambda) = M_1(\lambda) = \sqrt{\lambda^2-k^2},$$

$$E_2^{1,2}(\lambda) = K_2^{\mp}(\lambda) = \lambda^2 - \left\{k^2+h^2 \mp \sqrt{k^4+h^4-k^2h^2}\right\}/3,$$

$$E_2^3(\lambda) = L_2(\lambda) = \lambda\sqrt{\lambda^2-h^2},$$

$$E_2^4(\lambda) = M_2(\lambda) = \lambda\sqrt{\lambda^2-k^2},$$

$$E_2^5(\lambda) = N_1(\lambda) = \sqrt{(\lambda^2-h^2)(\lambda^2-k^2)}. \tag{58}$$

The associated normalisation constants (12) can be found analytically by the aid of the properties of the elliptic integrals [16]. By the aid of integral relations [17, Eq. 3.152 et seq.] integral (12) for each combination of n, p can be seen to reduce to a rational function in k and h times an expression which, on account of Legendre's relation [16, Eq. 17.7.13] equals $\pi/2$. Thus, the coefficients corresponding to (58) are

$$\gamma_0^1 = \frac{\pi}{2}, \quad \gamma_1^1 = \frac{\pi}{6}k^2h^2, \quad \gamma_1^2 = \frac{\pi}{6}(h^2 - k^2)h^2,$$
$$\gamma_1^3 = \frac{\pi}{6}(k^2 - h^2)k^2,$$
$$\gamma_2^{1,2} = \frac{2\pi}{405}\Big[2(k^4 + h^4 - k^2h^2)^2 \mp \sqrt{k^4 + h^4 - k^2h^2}(2(k^6 + h^6) - 3k^2h^2(k^2 + h^2))\Big],$$
$$\gamma_2^3 = \frac{\pi}{30}(h^2 - k^2)h^4k^2, \quad \gamma_2^4 = \frac{\pi}{30}(k^2 - h^2)k^4h^2,$$
$$\gamma_2^5 = -\frac{\pi}{30}(k^2 - h^2)^2k^2h^2. \tag{59}$$

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