Appendix C

Modal Analysis of a Uniform Cantilever with a Tip Mass

C.1 Transverse Vibrations

The following analytical modal analysis is given for the linear transverse vibrations of an undamped Euler–Bernoulli beam with clamped–free boundary conditions and a tip mass rigidly attached at the free end. The expressions for the undamped natural frequencies and mode shapes are obtained and the normalization conditions of the eigenfunctions are given. The procedure for reducing the partial differential equation of motion to an infinite set of ordinary differential equations is summarized, which is applicable to proportionally damped systems as well.

C.1.1 Boundary-Value Problem

Using the Newtonian or the Hamiltonian approach, the governing equation of motion for undamped free vibrations of a uniform Euler–Bernoulli beam can be obtained as [1]

$$YI\frac{\partial^4 w(x,t)}{\partial x^4} + m\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{C.1}$$

where $w(x, t)$ is the transverse displacement of the neutral axis (at point x and time t) due to bending, YI is the bending stiffness, and m is the mass per unit length of the beam.[1]

The clamped–free boundary conditions with a tip mass attachment (Figure C.1) can be expressed as

$$w(0,t) = 0, \qquad \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0 \tag{C.2}$$

[1] Note that the discussion here is given for free vibrations. Hence the clamped end does not move and $w(x, t)$ is identical to $w_{rel}(x, t)$ as far as the notation of Chapters 2 and 3 is concerned.

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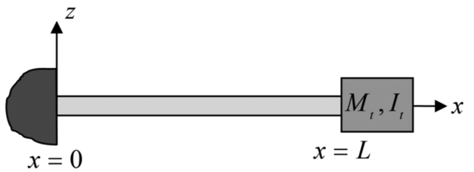

Figure C.1 Cantilevered uniform beam with a tip mass attachment

$$\left[YI \frac{\partial^2 w(x,t)}{\partial x^2} + I_t \frac{\partial^3 w(x,t)}{\partial t^2 \partial x} \right]_{x=L} = 0, \quad \left[YI \frac{\partial^3 w(x,t)}{\partial x^3} - M_t \frac{\partial^2 w(x,t)}{\partial t^2} \right]_{x=L} = 0 \quad \text{(C.3)}$$

where M_t is the tip mass and I_t is the mass moment of inertia of the tip mass about $x = L$.[2] Therefore the geometric boundary conditions at $x = 0$ are given by Equations (C.2) while the natural boundary conditions at $x = L$ are given by Equations (C.3). Equations (C.1)–(C.3) define the *boundary-value problem* for the transverse vibrations of a cantilevered uniform Euler–Bernoulli beam with a tip mass attachment.

C.1.2 Solution Using the Method of Separation of Variables

The method of separation of variables [3] can be used to solve Equation (C.1) by separating the spatial and temporal functions as

$$w(x,t) = \phi(x)\eta(t) \quad \text{(C.4)}$$

which can be substituted into Equation (C.1) to give

$$\frac{YI}{m} \frac{1}{\phi(x)} \frac{d^4\phi(x)}{dx^4} = -\frac{1}{\eta(t)} \frac{d^2\eta(t)}{dt^2} \quad \text{(C.5)}$$

The left hand side of Equation (C.5) depends on x alone while the right hand side depends on t alone. Since x and t are independent variables, the standard argument in the method of separation of variables states that both sides of Equation (C.5) must be equal to the same constant γ:

$$\frac{YI}{m} \frac{1}{\phi(x)} \frac{d^4\phi(x)}{dx^4} = -\frac{1}{\eta(t)} \frac{d^2\eta(t)}{dt^2} = \gamma \quad \text{(C.6)}$$

[2] Knowing the mass moment of inertia of the tip mass attachment about its centroid, one can use the *parallel axis theorem* [2] to obtain its mass moment of inertia about the end point of the elastic beam where the boundary condition is expressed.

yielding

$$\frac{d^4\phi(x)}{dx^4} - \gamma\frac{m}{YI}\phi(x) = 0 \tag{C.7}$$

$$\frac{d^2\eta(t)}{dt^2} + \gamma\eta(t) = 0 \tag{C.8}$$

It follows from Equation (C.8) that γ is a positive constant so that the response is oscillatory (not growing or decaying) since one is after the vibratory response with small oscillations. Therefore this positive constant can be expressed as the square of another constant: $\gamma = \omega^2$.

The solution forms of Equations (C.7) and (C.8) are then

$$\phi(x) = A\cos\left(\frac{\lambda}{L}x\right) + B\cosh\left(\frac{\lambda}{L}x\right) + C\sin\left(\frac{\lambda}{L}x\right) + D\sinh\left(\frac{\lambda}{L}x\right) \tag{C.9}$$

$$\eta(t) = E\cos\omega t + F\sin\omega t \tag{C.10}$$

where A, B, C, D, E, and F are unknown constants and

$$\lambda^4 = \omega^2\frac{mL^4}{YI} \tag{C.11}$$

Equation (C.4) can be employed in Equations (C.2) and (C.3) to obtain

$$\phi(0) = 0, \qquad \left.\frac{d\phi_r(x)}{dx}\right|_{x=0} = 0 \tag{C.12}$$

$$\left[YI\frac{d^2\phi(x)}{dx^2} - \omega^2 I_t\frac{d\phi(x)}{dx}\right]_{x=L} = 0, \qquad \left[YI\frac{d^3\phi(x)}{dx^3} + \omega^2 M_t\phi(x)\right]_{x=L} = 0 \tag{C.13}$$

The spatial form of the boundary conditions, Equations (C.12) and (C.13), should then be used to find the values of λ which give non-trivial $\phi(x)$. This process is called the *differential eigenvalue problem* and it is covered next.

C.1.3 Differential Eigenvalue Problem

Using Equation (C.9) in Equations (C.12) gives

$$A + B = 0 \tag{C.14}$$

$$C + D = 0 \tag{C.15}$$

Equation (C.9) then becomes

$$\phi(x) = A\left[\cos\left(\frac{\lambda}{L}x\right) - \cosh\left(\frac{\lambda}{L}x\right)\right] + C\left[\sin\left(\frac{\lambda}{L}x\right) - \sinh\left(\frac{\lambda}{L}x\right)\right] \tag{C.16}$$

Hence the unknown constants are A and C only. Using Equation (C.16) in the remaining two boundary conditions given by Equations (C.13) yields

$$\begin{bmatrix} \cos\lambda + \cosh\lambda - \dfrac{\lambda^3 I_t}{mL^3}(\sin\lambda + \sinh\lambda) & \sin\lambda + \sinh\lambda + \dfrac{\lambda^3 I_t}{mL^3}(\cos\lambda - \cosh\lambda) \\ \sin\lambda - \sinh\lambda + \dfrac{\lambda M_t}{mL}(\cos\lambda - \cosh\lambda) & -\cos\lambda - \cosh\lambda + \dfrac{\lambda M_t}{mL}(\sin\lambda - \sinh\lambda) \end{bmatrix} \begin{Bmatrix} A \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{C.17}$$

The coefficient matrix in Equation (C.17) has to be singular in order to obtain non-trivial values of A and C (hence non-trivial $\phi(x)$). Setting the determinant of the above coefficient matrix gives the *characteristic equation* of the differential eigenvalue problem as

$$1 + \cos\lambda\cosh\lambda + \lambda\frac{M_t}{mL}(\cos\lambda\sinh\lambda - \sin\lambda\cosh\lambda) - \frac{\lambda^3 I_t}{mL^3}(\cosh\lambda\sin\lambda + \sinh\lambda\cos\lambda)$$
$$+\frac{\lambda^4 M_t I_t}{m^2 L^4}(1 - \cos\lambda\cosh\lambda) = 0 \tag{C.18}$$

For the given system parameters m, L, M_t, and I_t, one can solve for the roots of Equation (C.18), which are the *eigenvalues* of the system (note that M_t/mL and I_t/mL^3 are dimensionless in the above equation).[3] The boundary-value problem defined in Section C.1.1 is positive definite [1], hence the system has infinitely many positive eigenvalues for the infinitely many *natural modes* of vibration. The eigenvalue (or the dimensionless *frequency parameter*) of the rth vibration mode is denoted here as λ_r (where r is a positive integer), and it is associated with the rth *eigenfunction* denoted by $\phi_r(x)$:

$$\phi_r(x) = A_r\left[\cos\frac{\lambda_r}{L}x - \cosh\frac{\lambda_r}{L}x + \varsigma_r\left(\sin\frac{\lambda_r}{L}x - \sinh\frac{\lambda_r}{L}x\right)\right] \tag{C.19}$$

where the second row of Equation (C.17) is used in Equation (C.16) to keep only a single modal constant (A_r) while ς_r is obtained from

$$\varsigma_r = \frac{\sin\lambda_r - \sinh\lambda_r + \lambda_r\dfrac{M_t}{mL}(\cos\lambda_r - \cosh\lambda_r)}{\cos\lambda_r + \cosh\lambda_r - \lambda_r\dfrac{M_t}{mL}(\sin\lambda_r - \sinh\lambda_r)} \tag{C.20}$$

[3] If the tip mass can be modeled as a point mass, that is, if the rotary inertia of the tip mass is negligible, $I_t = 0$ can be used in these expressions. In the absence of a tip mass, simply $M_t = I_t = 0$.

The *undamped natural frequency* (or the eigenfrequency) of free oscillations for the rth vibration mode is obtained from Equation (C.11) as

$$\omega_r = \lambda_r^2 \sqrt{\frac{YI}{mL^4}} \tag{C.21}$$

and it is associated with the natural *mode shape* of the rth vibration mode given by Equation (C.19).

C.1.4 Response to Initial Conditions

From Equation (C.4), the natural motion of the rth vibration mode becomes

$$w_r(x,t) = \phi_r(x)\eta_r(t) \tag{C.22}$$

where $\eta_r(t)$ is called the *modal coordinate* (or the *modal response*). Since the distributed-parameter system has infinitely many vibration modes, the general response is a linear combination of the contributions from all vibration modes:

$$w(x,t) = \sum_{r=1}^{\infty} \phi_r(x)\eta_r(t) \tag{C.23}$$

Using Equations (C.10) and (C.19) in (C.23), the response to initial conditions becomes

$$w(x,t) = \sum_{r=1}^{\infty} \left[\cos\frac{\lambda_r}{L}x - \cosh\frac{\lambda_r}{L}x + \varsigma_r \left(\sin\frac{\lambda_r}{L}x - \sinh\frac{\lambda_r}{L}x \right) \right] (A_r \cos\omega_r t + B_r \sin\omega_r t) \tag{C.24}$$

where A_r and B_r are the unknown constants which can be solved for using the initial conditions $w(x,0)$ and $\partial w(x,t)/\partial t|_{t=0}$.

C.1.5 Orthogonality of the Eigenfunctions

Substituting two distinct solutions (modes r and s) of the eigenvalue problem into the spatial Equation (C.7) gives [1]

$$YI\frac{d^4\phi_r(x)}{dx^4} = \omega_r^2 m\phi_r(x) \tag{C.25}$$

$$YI\frac{d^4\phi_s(x)}{dx^4} = \omega_s^2 m\phi_s(x) \tag{C.26}$$

Multiplying Equation (C.25) by $\phi_s(x)$ and integrating over the domain gives

$$\int_0^L \phi_s(x) YI \frac{d^4\phi_r(x)}{dx^4} dx = \omega_r^2 \int_0^L \phi_s(x) m \phi_r(x)\, dx \tag{C.27}$$

Applying integration by parts twice to the left hand side of Equation (C.27) and using Equations (C.12) and (C.13), one can obtain

$$\begin{aligned}&\int_0^L \frac{d^2\phi_s(x)}{dx^2} YI \frac{d^2\phi_r(x)}{dx^2} dx\\ &\quad = \omega_r^2 \left\{ \int_0^L \phi_s(x) m \phi_r(x)\, dx + \phi_s(L) M_t \phi_r(L) + \left[\frac{d\phi_s(x)}{dx} I_t \frac{d\phi_r(x)}{dx} \right]_{x=L} \right\}\end{aligned} \tag{C.28}$$

Multiplying Equation (C.26) by $\phi_r(x)$ and following a similar procedure gives

$$\begin{aligned}&\int_0^L \frac{d^2\phi_s(x)}{dx^2} YI \frac{d^2\phi_r(x)}{dx^2} dx\\ &\quad = \omega_s^2 \left\{ \int_0^L \phi_s(x) m \phi_r(x)\, dx + \phi_s(L) M_t \phi_r(L) + \left[\frac{d\phi_s(x)}{dx} I_t \frac{d\phi_r(x)}{dx} \right]_{x=L} \right\}\end{aligned} \tag{C.29}$$

Subtracting Equation (C.29) from (C.28) and recalling that r and s are distinct modes ($\omega_r^2 \neq \omega_s^2$), the orthogonality condition of the eigenfunctions is obtained as

$$\int_0^L \phi_s(x) m \phi_r(x)\, dx + \phi_s(L) M_t \phi_r(L) + \left[\frac{d\phi_s(x)}{dx} I_t \frac{d\phi_r(x)}{dx} \right]_{x=L} = 0, \quad \omega_r^2 \neq \omega_s^2 \tag{C.30}$$

C.1.6 Normalization of the Eigenfunctions

It follows from the orthogonality condition given by Equation (C.30) that

$$\int_0^L \phi_s(x) m \phi_r(x)\, dx + \phi_s(L) M_t \phi_r(L) + \left[\frac{d\phi_s(x)}{dx} I_t \frac{d\phi_r(x)}{dx} \right]_{x=L} = \delta_{rs} \tag{C.31}$$

where δ_{rs} is the Kronecker delta. Equation (C.31) is the orthogonality condition of the eigenfunctions and $s = r$ can be used to solve for the modal amplitude A_r of the eigenfunctions (i.e., to normalize the eigenfunctions). The eigenfunctions normalized according to Equation (C.31) are called the *mass-normalized* eigenfunctions.

Equations (C.27)–(C.29) can be used along with Equations (C.12) and (C.13) to give the companion orthogonality condition as

$$\int_0^L \phi_s(x) YI \frac{d^4\phi_r(x)}{dx^4} dx - \left[\phi_s(x) YI \frac{d^3\phi_r(x)}{dx^3}\right]_{x=L} + \left[\frac{d\phi_s(x)}{dx} YI \frac{d^2\phi_r(x)}{dx^2}\right]_{x=L} = \omega_r^2 \delta_{rs} \tag{C.32}$$

Finally, from Equations (C.29) and (C.31), one extracts

$$\int_0^L \frac{d^2\phi_s(x)}{dx^2} YI \frac{d^2\phi_r(x)}{dx^2} dx = \omega_r^2 \delta_{rs} \tag{C.33}$$

which is the alternative companion form. Once any one of Equations (C.31)–(C.33) is satisfied, the other two are automatically satisfied. Therefore, either one of these three expressions can be used for mass normalizing the eigenfunctions (i.e., to solve for A_r).

C.1.7 Response to External Forcing

Consider the undamped forced vibrations of the beam shown in Figure C.1 under the excitation of the distributed transverse force per length $f(x,t)$ acting in the positive z-direction. The governing equation of motion is then

$$YI \frac{\partial^4 w(x,t)}{\partial x^4} + m \frac{\partial^2 w(x,t)}{\partial t^2} = f(x,t) \tag{C.34}$$

The solution of Equation (C.34) can be expressed in the form of Equation (C.23). Substituting Equation (C.23) into Equation (C.34) gives

$$YI \frac{\partial^4}{\partial x^4}\left[\sum_{r=1}^{\infty} \phi_r(x)\eta_r(t)\right] + m \frac{\partial^2}{\partial t^2}\left[\sum_{r=1}^{\infty} \phi_r(x)\eta_r(t)\right] = f(x,t) \tag{C.35}$$

Multiplying both sides of Equation (C.35) by the mass-normalized eigenfunction $\phi_s(x)$ and integrating over the length of the beam, one obtains

$$\sum_{r=1}^{\infty}\left[\eta_r(t)\int_0^L \phi_s(x) YI \frac{d^4\phi_r(x)}{dx^4} dx\right] + \sum_{r=1}^{\infty}\left[\frac{d^2\eta_r(t)}{dt^2}\int_0^L \phi_s(x) m \phi_r(x)\, dx\right] = \int_0^L \phi_s(x) f(x,t)\, dx \tag{C.36}$$

Using the orthogonality conditions given by Equations (C.31) and (C.32), one can then express the following:

$$\frac{d^2\eta_s(t)}{dt^2}+\omega_s^2\eta_s(t)-\sum_{r=1}^{\infty}\left\{\phi_s(x)\left[M_t\phi_r(x)\frac{d^2\eta_r(t)}{dt^2}-YI\frac{d^3\phi_r(x)}{dx^3}\eta_r(t)\right]\right\}_{x=L}$$
$$-\sum_{r=1}^{\infty}\left\{\frac{d\phi_s(x)}{dx}\left[I_t\frac{d\phi_r(x)}{dx}\frac{d^2\eta_r(t)}{dt^2}+YI\frac{d^2\phi_r(x)}{dx^2}\eta_r(t)\right]\right\}_{x=L}$$
$$=\int_0^L\phi_s(x)f(x,t)\,dx \tag{C.37}$$

In view of the boundary conditions given by Equations (C.2) and (C.3), Equation (C.37) reduces to

$$\frac{d^2\eta_r(t)}{dt^2}+\omega_r^2\eta_r(t)=F_r(t) \tag{C.38}$$

where

$$F_r(t)=\int_0^L\phi_r(x)f(x,t)\,dx \tag{C.39}$$

If the applied force is arbitrary, the Duhamel integral (undamped version) can be used to solve for the modal coordinate, which can then be used in Equation (C.23) to give the physical response for zero initial conditions as

$$w(x,t)=\sum_{r=1}^{\infty}\frac{\phi_r(x)}{\omega_r}\int_0^t F_r(\tau)\sin\omega_r(t-\tau)d\tau \tag{C.40}$$

If the applied force is harmonic of the form

$$f(x,t)=F(x)\cos\omega t \tag{C.41}$$

The steady-state response can be obtained as

$$w(x,t)=\sum_{r=1}^{\infty}\frac{\phi_r(x)}{\omega_r^2-\omega^2}F_r\cos\omega t \tag{C.42}$$

where

$$F_r=\int_0^L\phi_r(x)F(x)\,dx \tag{C.43}$$

C.2 Longitudinal Vibrations

The analytical modal analysis for the linear longitudinal vibrations of a uniform clamped–free bar[4] with a tip mass attachment is summarized in this section. The steps are identical to those followed in the transverse vibrations case; however, the governing equation of motion, hence the resulting expressions, are completely different.

C.2.1 Boundary-Value Problem

Consider the free vibrations of the uniform bar shown in Figure C.1 in the longitudinal direction (i.e., x-direction). The governing equation of motion can be obtained as [1]

$$YA\frac{\partial^2 u(x,t)}{\partial x^2} - m\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \tag{C.44}$$

where $u(x,t)$ is the longitudinal displacement at any point x and time t, YA is the axial stiffness (Y is the elastic modulus and A is the cross-sectional area), and m is the mass per unit length of the bar. Equation (C.44) is in the one-dimensional form of the celebrated *wave equation*.

The displacement at the clamped end ($x = 0$) is zero, while the force resultant of the internal dynamic stresses is in equilibrium with the inertia of the tip mass attachment at the free end ($x = L$). Therefore, the boundary conditions are

$$u(0,t) = 0, \quad \left[YA\frac{\partial u(x,t)}{\partial x} + M_t\frac{\partial^2 u(x,t)}{\partial t^2}\right]\Bigg|_{x=L} = 0 \tag{C.45}$$

Equations (C.44) and (C.45) define the boundary-value problem for the longitudinal vibrations of a uniform clamped–free bar with a tip mass attachment.

C.2.2 Solution Using the Method of Separation of Variables

Separating the space and the time variables yields

$$u(x,t) = \varphi(x)\chi(t) \tag{C.46}$$

which can be employed in Equation (C.44) to give

$$\frac{YA}{m}\frac{1}{\varphi(x)}\frac{d^2\varphi(x)}{dx^2} = \frac{1}{\chi(t)}\frac{d^2\chi(t)}{dt^2} = \upsilon \tag{C.47}$$

[4] Following the conventional technical jargon, the structure is called a bar (or a rod) when longitudinal vibrations are discussed.

where the space-dependent and time-dependent sides are equal to each other and, following the standard argument of the separation of variables solution procedure, they are equal to a constant, which is denoted in Equation (C.47) by υ, yielding

$$\frac{d^2\varphi(x)}{dx^2} - \upsilon\frac{m}{YA}\varphi(x) = 0 \tag{C.48}$$

$$\frac{d^2\chi(t)}{dt^2} - \upsilon\chi(t) = 0 \tag{C.49}$$

It is clear from Equation (C.49) that non-growing or non-decaying solutions in time require that $\upsilon < 0$; therefore, one can set $\upsilon = -\omega^2$, where ω is another constant.

The solution forms of Equations (C.48) and (C.49) are then

$$\varphi(x) = A\cos\left(\frac{\alpha}{L}x\right) + B\sin\left(\frac{\alpha}{L}x\right) \tag{C.50}$$

$$\chi(t) = C\cos\omega t + D\sin\omega t \tag{C.51}$$

where A, B, C, and D are unknown constants and

$$\alpha^2 = \omega^2\frac{mL^2}{YA} \tag{C.52}$$

Equation (C.46) can be employed in Equation (C.45) to obtain the spatial form of the boundary conditions as

$$\varphi(0) = 0, \quad \left[YA\frac{d\varphi(x)}{dx} - \omega^2 M_t\varphi(x)\right]_{x=L} = 0 \tag{C.53}$$

C.2.3 Differential Eigenvalue Problem

Using Equation (C.50) in the first one of Equations (C.53) gives

$$A = 0 \tag{C.54}$$

which reduces Equation (C.50) to

$$\varphi(x) = B\sin\left(\frac{\alpha}{L}x\right) \tag{C.55}$$

Equation (C.55) can be used in the second boundary condition to give the characteristic equation as

$$\frac{M_t}{mL}\alpha\sin\alpha - \cos\alpha = 0 \tag{C.56}$$

Given the bar mass (mL) and the tip mass (M_t), Equation (C.56) can be used in order to solve for the eigenvalue (or the dimensionless frequency parameter) α_r of the rth vibration mode (the system is again positive definite and one seeks positive-valued eigenvalues). From Equation (C.55), the eigenfunction of the rth vibration mode is

$$\varphi_r(x) = B_r \sin\left(\frac{\alpha_r}{L}x\right) \tag{C.57}$$

where B_r is the modal amplitude constant (obtained from the normalization conditions in the solution process). Rearranging Equation (C.52), the undamped natural frequency of the rth vibration mode is obtained from[5]

$$\omega_r = \alpha_r\sqrt{\frac{YA}{mL^2}} \tag{C.58}$$

C.2.4 Response to Initial Conditions

The natural motion of the rth vibration mode can be expressed using Equation (C.46) as

$$u_r(x,t) = \varphi_r(x)\chi_r(t) \tag{C.59}$$

where $\chi_r(t)$ is the modal coordinate of the rth vibration mode. The general response is then expressed as a linear combination of the contributions from all vibration modes:

$$u(x,t) = \sum_{r=1}^{\infty}\varphi_r(x)\chi_r(t) \tag{C.60}$$

Using Equations (C.51) and (C.57) in Equation (C.60), the response to initial conditions becomes

$$u(x,t) = \sum_{r=1}^{\infty}(C_r\cos\omega_r t + D_r\sin\omega_r t)\sin\left(\frac{\alpha_r}{L}x\right) \tag{C.61}$$

where C_r and D_r are the unknown constants which can be solved for using the initial conditions $u(x,0)$ and $\partial u(x,t)/\partial t|_{t=0}$.

C.2.5 Orthogonality of the Eigenfunctions

Two distinct solutions of the differential eigenvalue problem (for modes r and s) can be substituted into Equation (C.48) to give

$$-YA\frac{d^2\varphi_r(x)}{dx^2} = \omega_r^2 m\varphi_r(x) \tag{C.62}$$

$$-YA\frac{d^2\varphi_s(x)}{dx^2} = \omega_s^2 m\varphi_s(x) \tag{C.63}$$

[5] The undamped natural frequency is denoted by ω_r in both the transverse and the longitudinal vibration problems for ease of notation, but they are not identical.

Multiplying Equation (C.62) by $\varphi_s(x)$ and integrating the resulting equation over the length of the bar, one can obtain

$$-\int_0^L \varphi_s(x)YA\frac{d^2\varphi_r(x)}{dx^2}dx = \omega_r^2\int_0^L \varphi_s(x)m\varphi_r(x)\,dx \tag{C.64}$$

Integrating by parts the left hand side of Equation (C.64) and using Equations (C.53) leads to

$$\int_0^L \frac{d\varphi_s(x)}{dx}YA\frac{d\varphi_r(x)}{dx}dx = \omega_s^2\left[\int_0^L \varphi_s(x)m\varphi_r(x)\,dx + \varphi_s(L)M_t\varphi_r(L)\right] \tag{C.65}$$

Likewise, multiplying Equation (C.63) by $\varphi_r(x)$ and following the same steps, one can obtain

$$\int_0^L \frac{d\varphi_r(x)}{dx}YA\frac{d\varphi_s(x)}{dx}dx = \omega_r^2\left[\int_0^L \varphi_r(x)m\varphi_s(x)\,dx + \varphi_r(L)M_t\varphi_s(L)\right] \tag{C.66}$$

Subtracting Equation (C.66) from Equation (C.65) and recalling that r and s are distinct modes (hence $\omega_r^2 \neq \omega_s^2$), one concludes

$$\int_0^L \varphi_s(x)m\varphi_r(x)\,dx + \varphi_s(L)M_t\varphi_r(L) = 0, \quad \omega_r^2 \neq \omega_s^2 \tag{C.67}$$

C.2.6 Normalization of the Eigenfunctions

The foregoing orthogonality condition can then be used to express the following mass normalization condition of the eigenfunctions:

$$\int_0^L \varphi_s(x)m\varphi_r(x)\,dx + \varphi_s(L)M_t\varphi_r(L) = \delta_{rs} \tag{C.68}$$

Equations (C.64)–(C.66) can be used with Equations (C.53) to give the companion orthogonality relation as

$$-\int_0^L \varphi_s(x)YA\frac{d^2\varphi_r(x)}{dx^2}dx + \left[\varphi_s(x)YA\frac{d\varphi_r(x)}{dx}\right]_{x=L} = \omega_r^2\delta_{rs} \tag{C.69}$$

and the alternative form is

$$\int_0^L \frac{d\varphi_r(x)}{dx} YA \frac{d\varphi_s(x)}{dx} dx = \omega_r^2 \delta_{rs} \tag{C.70}$$

which is obtained simply from Equations (C.66) and (C.68).

Equation (C.57) can be substituted into Equation (C.68) to solve for B_r (by setting $s = r$). Then the mass-normalized form of the eigenfunctions becomes

$$\varphi_r(x) = \frac{1}{\sqrt{\frac{mL}{2}\left(1 - \frac{\sin 2\alpha_r}{2\alpha_r}\right) + M_t \sin^2 \alpha_r}} \sin \frac{\alpha_r}{L} x \tag{C.71}$$

which satisfies Equations (C.69) and (C.70) as well.

C.2.7 Response to External Forcing

If the undamped bar shown in Figure C.1 is excited by the distributed axial force per length $p(x, t)$ acting in the positive x-direction, the governing equation of motion becomes

$$-YA \frac{\partial^2 u(x, t)}{\partial x^2} + m \frac{\partial^2 u(x, t)}{\partial t^2} = p(x, t) \tag{C.72}$$

The solution of Equation (C.72) can be expressed in the form of Equation (C.60) and substituting the latter into the former one obtains

$$-YA \frac{\partial^2}{\partial x^2} \left[\sum_{r=1}^{\infty} \varphi_r(x) \chi_r(t) \right] + m \frac{\partial^2}{\partial t^2} \left[\sum_{r=1}^{\infty} \varphi_r(x) \chi_r(t) \right] = p(x, t) \tag{C.73}$$

Multiplying both sides of Equation (C.73) by the mass-normalized eigenfunctions $\varphi_s(x)$ and integrating over the length of the bar gives

$$\begin{aligned} -\sum_{r=1}^{\infty} \left[\chi_r(t) \int_0^L \varphi_s(x) YA \frac{d^2\varphi_r(x)}{dx^2} dx \right] + \sum_{r=1}^{\infty} \left[\frac{d^2\chi_r(t)}{dt^2} \int_0^L \varphi_s(x) m \varphi_r(x)\, dx \right] \\ = \int_0^L \varphi_s(x) p(x, t)\, dx \end{aligned} \tag{C.74}$$

Then, using Equations (C.68) and (C.69), Equation (C.74) can be rearranged to give

$$\frac{d^2\chi_s(t)}{dt^2} + \omega_s^2\chi_s(t) - \sum_{r=1}^{\infty}\left\{\varphi_s(x)\left[M_t\varphi_r(x)\frac{d^2\chi_r(t)}{dt^2} + YA\frac{d\varphi_r(x)}{dx}\chi_r(t)\right]\right\}_{x=L}$$
$$= \int_0^L \varphi_s(x)p(x,t)\,dx \tag{C.75}$$

Considering the boundary conditions given by Equations (C.45), Equation (C.75) reduces to

$$\frac{d^2\eta_r(t)}{dt^2} + \omega_r^2\eta_r(t) = P_r(t) \tag{C.76}$$

where

$$P_r(t) = \int_0^L \varphi_r(x)p(x,t)\,dx \tag{C.77}$$

For an arbitrary forcing function, the physical response $u(x,t)$ can be obtained using the undamped version of the Duhamel integral (with zero initial conditions) as

$$u(x,t) = \sum_{r=1}^{\infty}\frac{\varphi_r(x)}{\omega_r}\int_0^t P_r(\tau)\sin\omega_r(t-\tau)\,d\tau \tag{C.78}$$

If the applied force is harmonic of the form

$$p(x,t) = P(x)\cos\omega t \tag{C.79}$$

The steady-state response can be obtained as

$$u(x,t) = \sum_{r=1}^{\infty}\frac{\varphi_r(x)}{\omega_r^2-\omega^2}P_r\cos\omega t \tag{C.80}$$

where

$$P_r = \int_0^L \varphi_r(x)P(x)\,dx \tag{C.81}$$

References

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