Appendix A

Piezoelectric Constitutive Equations

A.1 Three-Dimensional Form of the Linear Piezoelectric Constitutive Equations

In general, poled piezoceramics (such as PZT-5A and PZT-5H) are transversely isotropic materials. To be in agreement with the IEEE Standard on Piezoelectricity [1], the plane of isotropy is defined here as the 12-plane (or the *xy*-plane). The piezoelectric material therefore exhibits symmetry about the 3-axis (or the *z*-axis), which is the poling axis of the material. The field variables are the stress components (T_{ij}), strain components (S_{ij}), electric field components (E_k), and the electric displacement components (D_k).

The standard form of the piezoelectric constitutive equations can be given in four different forms by taking either two of the four field variables as the independent variables. Consider the tensorial representation of the strain–electric displacement form [1] where the independent variables are the stress components and the electric field components (and the remaining terms are as defined in Section 1.4):

$$S_{ij} = s^E_{ijkl} T_{kl} + d_{kij} E_k \tag{A.1}$$

$$D_i = d_{ikl} T_{kl} + \varepsilon^T_{ik} E_k \tag{A.2}$$

which is the preferred form of the piezoelectric constitutive equations for bounded media (to eliminate some of the stress components depending on the geometry and some of the electric field components depending on the placement of the electrodes). Equations (A.1) and (A.2) can be given in matrix form as

$$\begin{bmatrix} \mathbf{S} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{s}^E & \mathbf{d}^t \\ \mathbf{d} & \boldsymbol{\varepsilon}^T \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{E} \end{bmatrix} \tag{A.3}$$

where the superscripts E and T denote that the respective constants are evaluated at constant electric field and constant stress, respectively, and the superscript t stands for the transpose.

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The expanded form of Equation (A.3) is

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} s_{11}^E & s_{12}^E & s_{13}^E & 0 & 0 & 0 & 0 & 0 & d_{31} \\ s_{12}^E & s_{11}^E & s_{13}^E & 0 & 0 & 0 & 0 & 0 & d_{31} \\ s_{13}^E & s_{13}^E & s_{33}^E & 0 & 0 & 0 & 0 & 0 & d_{33} \\ 0 & 0 & 0 & s_{55}^E & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & 0 & s_{55}^E & 0 & d_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66}^E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{15} & 0 & \varepsilon_{11}^T & 0 & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 & 0 & \varepsilon_{11}^T & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 & 0 & 0 & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{A.4}$$

where the contracted notation (i.e., Voigt's notation: $11 \to 1$, $22 \to 2$, $33 \to 3$, $23 \to 4$, $13 \to 5$, $12 \to 6$) is used so that the vectors of strain and stress components are

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{13} \\ 2S_{12} \end{bmatrix}, \quad \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} \tag{A.5}$$

Therefore the shear strain components in the contracted notation are the engineering shear strains. It should be noted from the elastic, piezoelectric, and dielectric constants in Equation (A.4) that the symmetries of transversely isotropic material behavior ($s_{11}^E = s_{22}^E$, $d_{31} = d_{32}$, etc.) are directly applied.

A.2 Reduced Equations for a Thin Beam

If the piezoelastic behavior of the thin structure is to be modeled as a thin beam based on the Euler–Bernoulli beam theory or Rayleigh beam theory, the stress components other than the one-dimensional bending stress T_1 are negligible so that

$$T_2 = T_3 = T_4 = T_5 = T_6 = 0 \tag{A.6}$$

Along with this simplification, if an electrode pair covers the faces perpendicular to the 3-direction, Equation (A.4) becomes

$$\begin{bmatrix} S_1 \\ D_3 \end{bmatrix} = \begin{bmatrix} s_{11}^E & d_{31} \\ d_{31} & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} T_1 \\ E_3 \end{bmatrix} \tag{A.7}$$

which can be written as

$$\begin{bmatrix} s_{11}^E & 0 \\ -d_{31} & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ D_3 \end{bmatrix} = \begin{bmatrix} 1 & -d_{31} \\ 0 & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} S_1 \\ E_3 \end{bmatrix} \tag{A.8}$$

Therefore the stress–electric displacement form of the reduced constitutive equations for a thin beam is

$$\begin{bmatrix} T_1 \\ D_3 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11}^E & -\bar{e}_{31} \\ \bar{e}_{31} & \bar{\varepsilon}_{33}^S \end{bmatrix} \begin{bmatrix} S_1 \\ E_3 \end{bmatrix} \tag{A.9}$$

where the reduced matrix of the elastic, piezoelectric, and dielectric constants is

$$\bar{\mathbf{C}} = \begin{bmatrix} \bar{c}_{11}^E & -\bar{e}_{31} \\ \bar{e}_{31} & \bar{\varepsilon}_{33}^S \end{bmatrix} = \begin{bmatrix} s_{11}^E & 0 \\ -d_{31} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -d_{31} \\ 0 & \varepsilon_{33}^T \end{bmatrix} \tag{A.10}$$

Here and hereafter, an overbar denotes that the respective constant is reduced from the three-dimensional form to the plane-stress condition. In Equation (A.10),

$$\bar{c}_{11}^E = \frac{1}{s_{11}^E}, \quad \bar{e}_{31} = \frac{d_{31}}{s_{11}^E}, \quad \bar{\varepsilon}_{33}^S = \varepsilon_{33}^T - \frac{d_{31}^2}{s_{11}^E} \tag{A.11}$$

where the superscript S denotes that the respective constant is evaluated at constant strain.

A.3 Reduced Equations for a Moderately Thick Beam

If the piezoelasticity of the structure is to be modeled as a moderately thick beam based on the Timoshenko beam theory, the stress components other than T_1 (the stress component in the axial direction) and T_5 (the transverse shear stress) are negligible so that

$$T_2 = T_3 = T_4 = T_6 = 0 \tag{A.12}$$

is applied in Equation (A.4). Then,

$$\begin{bmatrix} S_1 \\ S_5 \\ D_3 \end{bmatrix} = \begin{bmatrix} s_{11}^E & 0 & d_{31} \\ 0 & s_{55}^E & 0 \\ d_{31} & 0 & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} T_1 \\ T_5 \\ E_3 \end{bmatrix} \tag{A.13}$$

which can be written as

$$\begin{bmatrix} s_{11}^E & 0 & 0 \\ 0 & s_{55}^E & 0 \\ -d_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_5 \\ D_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -d_{31} \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} S_1 \\ S_5 \\ E_3 \end{bmatrix} \tag{A.14}$$

Therefore the stress–electric displacement form of the reduced constitutive equations is

$$\begin{bmatrix} T_1 \\ T_5 \\ D_3 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11}^E & 0 & -\bar{e}_{31} \\ 0 & \bar{c}_{55}^E & 0 \\ \bar{e}_{31} & 0 & \bar{\varepsilon}_{33}^S \end{bmatrix} \begin{bmatrix} S_1 \\ S_5 \\ E_3 \end{bmatrix} \tag{A.15}$$

Here, the reduced matrix of the elastic, piezoelectric, and permittivity constants is

$$\bar{\mathbf{C}} = \begin{bmatrix} \bar{c}_{11}^E & 0 & -\bar{e}_{31} \\ 0 & \bar{c}_{55}^E & 0 \\ \bar{e}_{31} & 0 & \bar{\varepsilon}_{33}^S \end{bmatrix} = \begin{bmatrix} s_{11}^E & 0 & 0 \\ 0 & s_{55}^E & 0 \\ -d_{31} & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -d_{31} \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_{33}^T \end{bmatrix} \tag{A.16}$$

where

$$\bar{c}_{11}^E = \frac{1}{s_{11}^E}, \quad \bar{c}_{55}^E = \frac{1}{s_{55}^E}, \quad \bar{e}_{31} = \frac{d_{31}}{s_{11}^E}, \quad \bar{\varepsilon}_{33}^S = \varepsilon_{33}^T - \frac{d_{31}^2}{s_{11}^E} \tag{A.17}$$

Note that the transverse shear stress in Equation (A.15) is corrected due to Timoshenko [2,3]

$$T_5 = \kappa \bar{c}_{55}^E S_5 \tag{A.18}$$

where κ is the shear correction factor [2–12].

A.4 Reduced Equations for a Thin Plate

If the thin structure is to be modeled as a thin plate (i.e., Kirchhoff plate) due to two-dimensional strain fluctuations, the normal stress in the thickness direction of the piezoceramic and the respective transverse shear stress components are negligible:

$$T_3 = T_4 = T_5 = 0 \tag{A.19}$$

Equation (A.4) becomes

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \\ D_3 \end{bmatrix} = \begin{bmatrix} s_{11}^E & s_{12}^E & 0 & d_{31} \\ s_{12}^E & s_{11}^E & 0 & d_{31} \\ 0 & 0 & s_{66}^E & 0 \\ d_{31} & d_{31} & 0 & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_6 \\ E_3 \end{bmatrix} \tag{A.20}$$

which can be rearranged to give

$$\begin{bmatrix} s_{11}^E & s_{12}^E & 0 & 0 \\ s_{12}^E & s_{11}^E & 0 & 0 \\ 0 & 0 & s_{66}^E & 0 \\ -d_{31} & -d_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_6 \\ D_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -d_{31} \\ 0 & 1 & 0 & -d_{31} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_6 \\ E_3 \end{bmatrix} \tag{A.21}$$

The stress–electric displacement form of the reduced constitutive equations becomes

$$\begin{bmatrix} T_1 \\ T_2 \\ T_6 \\ D_3 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11}^E & \bar{c}_{12}^E & 0 & -\bar{e}_{31} \\ \bar{c}_{12}^E & \bar{c}_{11}^E & 0 & -\bar{e}_{31} \\ 0 & 0 & \bar{c}_{66}^E & 0 \\ \bar{e}_{31} & \bar{e}_{31} & 0 & \bar{\varepsilon}_{33}^S \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_6 \\ E_3 \end{bmatrix} \tag{A.22}$$

where

$$\bar{\mathbf{C}} = \begin{bmatrix} \bar{c}_{11}^E & \bar{c}_{12}^E & 0 & -\bar{e}_{31} \\ \bar{c}_{12}^E & \bar{c}_{11}^E & 0 & -\bar{e}_{31} \\ 0 & 0 & \bar{c}_{66}^E & 0 \\ \bar{e}_{31} & \bar{e}_{31} & 0 & \bar{\varepsilon}_{33}^S \end{bmatrix} = \begin{bmatrix} s_{11}^E & s_{12}^E & 0 & 0 \\ s_{12}^E & s_{11}^E & 0 & 0 \\ 0 & 0 & s_{66}^E & 0 \\ -d_{31} & -d_{31} & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & -d_{31} \\ 0 & 1 & 0 & -d_{31} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon_{33}^T \end{bmatrix} \tag{A.23}$$

Here, the reduced elastic, piezoelectric, and permittivity constants are

$$\bar{c}_{11}^E = \frac{s_{11}^E}{\left(s_{11}^E + s_{12}^E\right)\left(s_{11}^E - s_{12}^E\right)} \tag{A.24}$$

$$\bar{c}_{12}^E = \frac{-s_{12}^E}{\left(s_{11}^E + s_{12}^E\right)\left(s_{11}^E - s_{12}^E\right)} \tag{A.25}$$

$$\bar{c}_{66}^E = \frac{1}{s_{66}^E} \tag{A.26}$$

$$\bar{e}_{31} = \frac{d_{31}}{s_{11}^E + s_{12}^E} \tag{A.27}$$

$$\bar{\varepsilon}_{33}^S = \bar{\varepsilon}_{33}^T - \frac{2d_{31}^2}{s_{11}^E + s_{12}^E} \tag{A.28}$$

where the first term (Equation (A.24)) is the elastic constant that is related to the bending stiffness of a piezoelectric plate (accounting for the Poisson effect) in the absence of torsion.

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